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Stochastic Optimal Control and MPC – Dissipativity, Performance and Risk

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Zusammenfassung

Gegenstand dieser Dissertation ist die dissipativitätsbasierte Analyse stochastischer Optimalsteuerungsprobleme sowie stochastischer modellprädiktiver Regelung (MPC). Dabei liegt ein besonderer Fokus auf der Analyse von Stabilität, Regelgüte und dem Langzeitverhalten von Lösungen unter Unsicherheit. Entscheidungsfindung unter Unsicherheit stellt eine zentrale Herausforderung in vielen Bereichen der Ingenieurwissenschaft, Wirtschaftswissenschaft und angewandten Mathematik dar. Stochastische Einflüsse in der Systemdynamik, in Messung und in der Kostenfunktion erfordern Kontrollmethoden, die Unsicherheit explizit berücksichtigen. Die stochastische Optimalsteuerung liefert hierfür einen grundlegenden theoretischen Rahmen und bildet insbesondere die Basis für MPC-Ansätze für stochastische Systeme.

Ein wesentliches Ziel dieser Arbeit ist die Entwicklung eines einheitlichen dissipativitätsbasierten Analyseansatzes für stochastische Optimalsteuerung. Dissipativität wird dabei als strukturelles Werkzeug genutzt, um Langzeiteigenschaften stochastischer Systeme zu untersuchen und deren asymptotisches Verhalten zu charakterisieren. Im Unterschied zu bestehenden Ansätzen, die überwiegend auf Erwartungswerten oder verteilungsbasierten Größen beruhen, ermöglicht diese Vorgehensweise eine strukturelle Analyse stochastischer Prozesse über rein verteilungsbasierte Beschreibungen hinaus.

Auf dieser Grundlage werden stochastische Formen von Dissipativität sowie stochastische Turnpike-Eigenschaften eingeführt und systematisch analysiert. Dabei zeigt sich, dass stationäre stochastische Prozesse eine zentrale Rolle als asymptotische Objekte spielen und über zugehörige stationäre Optimierungsprobleme charakterisiert werden können. Zudem werden hierarchische Zusammenhänge zwischen unterschiedlichen stochastischen Dissipativitäts- und Turnpike-Konzepten hergeleitet.

Die gewonnenen Resultate werden zur Herleitung von Garantien für die Stabilität und Regelgüte stochastischer MPC-Verfahren genutzt. Dissipativität dient dabei als Werkzeug zur Analyse des asymptotischen Verhaltens des geschlossenen Regelkreises. Dies führt zu verschiedenen stochastischen Konvergenzaussagen, insbesondere in Verteilung und auf der Ebenene der Zufallsvariablen, sowie zu präzisen Abschätzungen der Regelgüte.

Abschließend wird der entwickelte Ansatz auf risikosensitive stochastische MPC-Formulierungen erweitert. Hierbei werden sowohl risikoadaptive Zielfunktionen als auch risikosensitive Nebenbedingungen berücksichtigt, wodurch eine einheitliche Behandlung erwartungsbasierter und risikosensitiver MPC-Ansätze ermöglicht wird.

Abstract

The subject of this thesis is the dissipativity-based analysis of stochastic optimal control problems and stochastic model predictive control (MPC), with a particular focus on stability, performance, and long-horizon behavior under uncertainty. Decision-making under uncertainty is a fundamental challenge in modern engineering, economics, and applied mathematics. Many real-world systems are inherently affected by stochastic disturbances, incomplete information, and random environmental dynamics, which makes it necessary to develop control methodologies that explicitly account for uncertainty in both system evolution and performance criteria. Stochastic optimal control provides a natural framework for this purpose and forms the basis for a wide range of theoretical and applied developments, including stochastic model predictive control as a key methodology for closed-loop operation of stochastic systems.

In this work, a comprehensive dissipativity-based framework for stochastic optimal control is developed. The main idea is to use dissipativity as a structural tool for analyzing long-horizon properties of stochastic systems and for characterizing their stationary behavior. In contrast to existing approaches, which are mainly based on expectations or distributional quantities, the proposed framework allows for a more refined analysis that also captures structural properties of the underlying stochastic processes.

Based on this framework, stochastic notions of dissipativity and turnpike properties are introduced and systematically studied. It is shown that stationary stochastic processes play a central role as asymptotic objects for stochastic optimal control problems and can be characterized via associated stationary optimization problems. Furthermore, hierarchical relations between different notions of dissipativity and turnpike behavior are established.

The theoretical results are then used to derive closed-loop stability and performance guarantees for stochastic economic MPC. In particular, dissipativity is employed as a tool to analyze the asymptotic behavior of the closed-loop system. This leads to convergence results in several stochastic senses, including convergence results in distribution and on the level of random variables, as well as tight performance estimates.

Finally, the framework is extended to risk-averse stochastic MPC formulations. Both risk-averse objective functions and risk-averse constraint formulations are considered, leading to a unified treatment of expectation-based and risk-averse stochastic MPC within the same dissipativity-based framework.

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Contents

Zusammenfassung	i
Abstract	iii
Contents	vii
1 Introduction	1
1.1 Contribution and outline	2
1.2 Notation	3
2 Fundamentals of stochastic optimal control	5
2.1 Basic concepts of probability theory	5
2.2 Stochastic control systems and stationarity concepts	7
2.3 Stochastic optimal control problems	11
2.4 Dynamic programming principles	15
3 Dissipativity and turnpike in stochastic optimal control	19
3.1 Stochastic dissipativity concepts	20
3.2 Optimal stationary solutions	25
3.3 Stochastic turnpike properties	29
3.4 Implications of strict dissipativity and its relation to stochastic turn- pike properties	31
3.5 Hierarchy of stochastic turnpike and dissipativity notions	38
3.6 Examples	43
3.6.1 The generalized stochastic linear-quadratic optimal control problem	43
3.6.2 A nonlinear example	60
4 Closed-loop analysis of stochastic economic MPC with an expected cost criterion	65
4.1 Setting and preliminaries	66
4.2 Practical asymptotic stability	70
4.3 Near-optimal performance results	83
4.3.1 Non-averaged performance estimates	83
4.3.2 Averaged performance estimates	91
4.4 Numerical results	92
5 First results on closed-loop guarantees for stochastic MPC with risk- averse formulations	103
5.1 Fundamentals of risk measures	104
5.2 Asymptotic estimates for problems with risk-averse costs	108
5.2.1 Averaged performance estimates	110
5.2.2 Trajectory convergence results	113

5.2.3	Numerical simulations	115
5.3	Stochastic linear MPC with risk-averse constraints	120
5.3.1	The indirect feedback approach	122
5.3.2	Closed-loop guarantees	125
5.3.3	Moment-based reformulation for linear-quadratic problems . .	131
5.3.4	Numerical simulations	137
6	Outlook and future research	141
	List of Figures and Algorithms	145
	Bibliography	149
	Publications	157
	Eidesstattliche Versicherung	159

1 Introduction

Stochastic optimal control provides a fundamental framework for decision-making under uncertainty and forms the basis for a wide range of applications in engineering, economics, and applied mathematics. In particular, stochastic model predictive control (MPC) has emerged as a widespread tool for controlling such uncertain systems, cf. [70].

In MPC, at each time step an optimal control problem on a finite horizon is solved to construct a controller for a very long or possibly infinite horizon. Since this yields only a suboptimal approximation of the true optimal solution, stability and performance analysis are of central interest in both deterministic and stochastic settings. In the deterministic case, a comprehensive theory has been developed, providing rigorous guarantees for stability and performance even in economic formulations; see, e.g., [5, 22, 27, 34]. In the stochastic setting, however, the situation is considerably more limited. Existing results often rely on restrictive assumptions, terminal ingredients, or provide guarantees only in terms of expected values; see, e.g., [65, 68, 70]. In particular, many available approaches do not fully capture the asymptotic behavior of the closed-loop system and typically do not characterize its limiting stochastic structure in a precise way.

These limitations become even more pronounced in applications where rare but significant events play an important role. This has led to increasing interest in risk-averse formulations of stochastic control and MPC, where uncertainty is not only treated in terms of expectations but also through risk measures that capture tail behavior; see, e.g., [83, 94]. Such formulations include coherent risk measures and conditional value-at-risk type objectives and have been studied in stochastic optimal control and MPC contexts; see, e.g., [96, 97]. At the same time, risk-sensitive constraint formulations and chance-constrained approaches further extend the class of stochastic MPC problems; see, e.g., [25, 49, 55]. However, these extensions complicate stability and performance analysis, even more, and a unified framework that treats expectation-based and risk-averse stochastic MPC on the same structural level is still missing.

In the deterministic setting, dissipativity theory and the turnpike property provide a fundamental framework for understanding long-horizon optimal control problems. Dissipativity, originally introduced by Willems [105, 106], has become a key tool in the analysis of economic MPC and its stability properties; see, e.g., [27, 34]. The turnpike phenomenon, first observed in classical economic growth theory by Ramsey [82] and von Neumann [75], has since been systematically studied in optimal control; see, e.g., [13, 28, 36, 40]. In stochastic settings, however, corresponding results remain significantly less developed. Existing approaches mainly focus on distributional or moment-based notions of stochastic dissipativity and turnpike behavior; see, e.g., [33, 56, 57, 66, 100]. More refined pathwise interpretations are still largely missing, although numerical evidence suggests turnpike phenomena at the level of realizations [77].

This gap motivates the development of a unified dissipativity-based framework for stochastic optimal control and MPC. The central idea of this thesis is to use dissipativity as a structural tool for the analysis of stochastic MPC systems and their closed-loop behavior.

Within this framework, stochastic dissipativity and turnpike concepts are developed beyond measure-based formulations, allowing for a more refined description of stationary stochastic behavior. Based on this, dissipativity-based arguments are used to derive stability and performance guarantees for stochastic MPC schemes. This includes both expectation-based formulations and risk-averse extensions, thereby providing a unified treatment of stochastic MPC.

1.1 Contribution and outline

Across all parts of this thesis, dissipativity serves as the central analytical tool for understanding stability, performance, and asymptotic behavior in stochastic model predictive control systems. The main objective of this work is to develop a dissipativity-based theory for stochastic MPC that enables the systematic analysis of stability, performance, and long-horizon behavior under uncertainty, including both expectation-based and risk-averse formulations.

This thesis is organized as follows.

Chapter 1 – Introduction. After motivating the scope of this thesis, the remainder of the chapter introduces the notation used throughout the manuscript. More specialized concepts and problem formulations are introduced when they first arise in the subsequent chapters.

Chapter 2 – Fundamentals of stochastic optimal control. This chapter introduces the foundations of probability theory and stochastic control required throughout the thesis. After reviewing the necessary probabilistic concepts, stochastic control systems and different notions of stationarity are introduced. Building on these concepts, the class of stochastic optimal control problems considered in this thesis is formulated. The chapter concludes with the dynamic programming principle in finite- and infinite-horizon settings, which serves as a fundamental tool for the subsequent analysis.

Chapter 3 – Dissipativity and turnpike in stochastic optimal control. This chapter develops a unified framework for dissipativity and turnpike properties in discrete-time stochastic optimal control. Different notions of stochastic dissipativity and turnpike behavior are introduced and related to stationary stochastic processes. Optimality characterizations of the associated stationary solutions are established and the implications of strict dissipativity for qualitative properties of optimal trajectories are investigated. In particular, conditions under which dissipativity implies turnpike behavior are derived. Furthermore, hierarchical relations between the various dissipativity and turnpike notions are analyzed. The theoretical results are complemented by explicit constructions of storage functions for nonlinear and linear-quadratic systems as well as numerical illustrations of stochastic turnpike phenomena.

Chapter 4 – Closed-loop analysis of stochastic economic MPC with an expected cost criterion. This chapter studies stability and performance properties of stochastic economic model predictive control with an expected cost criterion and without probabilistic constraints. An abstract stochastic MPC formulation and an equivalent implementable MPC scheme are introduced. Based on the dissipativity concepts developed in Chapter 3, practical asymptotic stability results are derived for the MPC closed loop without the use of terminal ingredients. The asymptotic behavior of the closed-loop system is characterized by stationary stochastic processes, and convergence results are established with respect to several stochastic notions, including convergence in probability, convergence in distribution, and convergence in the p -th mean. In addition, averaged and non-averaged performance estimates are derived, providing stochastic counterparts of classical performance guarantees from deterministic economic MPC. The theoretical findings are illustrated by numerical examples.

Chapter 5 – First results on closed-loop guarantees for risk-averse stochastic MPC. This chapter extends the dissipativity-based analysis to risk-averse stochastic MPC formulations. The first part considers risk-averse objective functions and derives performance guarantees and closed-loop convergence results based on parameterized risk measures. The second part investigates stochastic MPC with expectation-based objectives and risk-averse constraints. Extending the indirect feedback approach known from chance-constrained MPC, closed-loop performance guarantees are established for systems subject to general risk-measure-based constraints. The resulting analysis provides a unified dissipativity-based perspective on expectation-based and risk-averse stochastic MPC formulations.

Chapter 6 – Outlook and future research. The final chapter concludes the thesis and discusses possible directions for future research arising from the results presented in this work.

1.2 Notation

We denote by $\mathbb{N} := \{1, 2, \dots\}$ the set of natural numbers and by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the set of nonnegative integers. Furthermore, we define the extended natural numbers by

$$\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}.$$

The set of real numbers is denoted by $\mathbb{R}$, and

$$\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$$

denotes the extended real line.

For a set $\mathcal{Q}$, we denote the number of elements contained in $\mathcal{Q}$ by $\#\mathcal{Q}$.

Throughout this thesis, random variables representing states and controls are typically denoted by uppercase letters, whereas deterministic quantities are denoted by lowercase letters. Sequences are usually indicated by bold symbols.

For vectors $x, y \in \mathbb{R}^n$, the inequality $x \leq y$ is understood componentwise, i.e.,

$$x_i \leq y_i \quad \text{for all } i = 1, \dots, n.$$

The relation $x \geq y$ is defined analogously.

For a positive semidefinite matrix $Q \in \mathbb{R}^{n \times n}$ and a vector $x \in \mathbb{R}^n$, we define

$$\|x\|_Q^2 := x^T Q x.$$

If Q is positive definite, then $\|\cdot\|_Q$ defines a norm on $\mathbb{R}^n$.

Furthermore, for a symmetric and positive semidefinite matrix $Q \in \mathbb{R}^{n \times n}$, we denote by $Q^{1/2} \in \mathbb{R}^{n \times n}$ its symmetric positive semidefinite square root, i.e.,

$$Q = Q^{1/2} Q^{1/2}.$$

The spectral radius of $A \in \mathbb{R}^{n \times n}$ is defined by

$$\varrho(A) := \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } A\}.$$

A matrix $A \in \mathbb{R}^{n \times n}$ is called stable if

$$\varrho(A) < 1.$$

A pair (A, B) is called stabilizable if there exists a matrix K such that $A + BK$ is stable. Such a matrix K is called a stabilizing feedback matrix. A pair (A, Q) is called detectable if (A^T, Q^T) is stabilizable.

Moreover, we employ the standard classes of comparison functions; see, e.g., [52]:

$$\mathcal{K} := \left\{ \alpha : [0, \infty) \rightarrow [0, \infty) \mid \alpha \text{ is continuous, strictly increasing, and } \alpha(0) = 0 \right\},$$

$$\mathcal{K}_\infty := \left\{ \alpha : [0, \infty) \rightarrow [0, \infty) \mid \alpha \in \mathcal{K} \text{ and } \alpha \text{ is unbounded} \right\},$$

$$\mathcal{L} := \left\{ \delta : [0, \infty) \rightarrow [0, \infty) \mid \delta \text{ is continuous, strictly decreasing, and } \lim_{t \rightarrow \infty} \delta(t) = 0 \right\},$$

$$\mathcal{KL} := \left\{ \beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \mid \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K} \forall t \geq 0, \right. \\ \left. \beta(r, \cdot) \in \mathcal{L} \forall r \geq 0 \right\}.$$

2 Fundamentals of stochastic optimal control

In this chapter, we introduce the fundamental concepts of stochastic optimal control that are required for the subsequent analysis. We begin by recalling basic notions from probability theory that are used throughout this thesis, following standard references such as [11, 20, 51, 53]. We then define stochastic control systems and introduce different notions of stationarity, which generalize the concept of a steady state in deterministic systems. Based on this, we extend these systems to stochastic optimal control problems by incorporating suitable cost functionals and constraint formulations. Finally, we present the dynamic programming principle for stochastic optimal control problems in both finite and infinite horizon settings, which serves as a key tool for the closed-loop analysis of the stochastic model predictive control schemes considered in this thesis.

2.1 Basic concepts of probability theory

Let Ω be a set. A collection $\mathcal{F}$ of subsets of Ω is called a σ -algebra on Ω if it satisfies the following properties:

- $\Omega \in \mathcal{F}$.
- For every $A \in \mathcal{F}$, its complement $A^c := \Omega \setminus A$ belongs to $\mathcal{F}$.
- For every countable sequence $A_1, A_2, \dots$ with $A_i \in \mathcal{F}$, we have $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$.

For any subset $M \subseteq \Omega$, the smallest σ -algebra containing M is denoted by $\sigma(M)$ and is called the σ -algebra generated by M .

A function $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is called a *probability measure* if it satisfies:

- $\mathbb{P}(\Omega) = 1$.
- For every countable sequence $A_1, A_2, \dots$ of pairwise disjoint sets $A_i \in \mathcal{F}$, it holds that

$$\mathbb{P}\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mathbb{P}(A_i).$$

The set of all probability measures on Ω is denoted by $\mathcal{P}(\Omega)$.

A triple $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of a sample space Ω , a σ -algebra $\mathcal{F}$, and a probability measure $\mathbb{P}$ is called a *probability space*. If not stated otherwise, we assume throughout that $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed but arbitrary.

An event $E \in \mathcal{F}$ is said to hold *almost surely* (a.s.) if $\mathbb{P}(E) = 1$.

Let $\mathcal{X}$ be a separable Banach space equipped with the norm $\|\cdot\|$. A *random variable* is a measurable mapping

$$X : (\Omega, \mathcal{F}) \rightarrow (\mathcal{X}, \mathcal{B}(\mathcal{X})),$$

i.e.,

$$X^{-1}(B) := \{\omega \in \Omega \mid X(\omega) \in B\} \in \mathcal{F}$$

for all $B \in \mathcal{B}(\mathcal{X})$, where $\mathcal{B}(\mathcal{X})$ denotes the Borel σ -algebra on $\mathcal{X}$.

The *distribution* $\mathbb{P}_X \in \mathcal{P}(\mathcal{X})$ of X is defined by

$$\mathbb{P}_X(B) := \mathbb{P}(X^{-1}(B)), \quad B \in \mathcal{B}(\mathcal{X}).$$

The set of all such random variables is denoted by $\mathcal{R}(\mathcal{X})$.

The σ -algebra generated by a random variable $X \in \mathcal{R}(\mathcal{X})$ is

$$\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}(\mathcal{X})\}.$$

For $X_1, \dots, X_n \in \mathcal{R}(\mathcal{X})$, $n \in \mathbb{N}$, we define

$$\begin{aligned} \sigma(X_1, \dots, X_n) &= \{(X_1, \dots, X_n)^{-1}(B) : B \in \mathcal{B}(\mathcal{X}^n)\} \\ &= \sigma(\{X_1^{-1}(B_1), \dots, X_n^{-1}(B_n) : B_i \in \mathcal{B}(\mathcal{X})\}). \end{aligned}$$

Random variables $X_1, \dots, X_n \in \mathcal{R}(\mathcal{X})$ are called (*stochastically*) *independent* if the σ -algebras $\sigma(X_1), \dots, \sigma(X_n)$ are independent, i.e.,

$$\mathbb{P}\left(\bigcap_{i=1}^n A_i\right) = \prod_{i=1}^n \mathbb{P}(A_i) \quad \text{for all } A_i \in \sigma(X_i).$$

A sequence $X_1, X_2, \dots$ is called *independent and identically distributed (i.i.d.)* if for every $n \in \mathbb{N}$ the random variables $X_1, \dots, X_n$ are independent and $\mathbb{P}_{X_l} = \mathbb{P}_{X_m}$ for all $l, m \in \mathbb{N}$.

A (possibly infinite) sequence $\mathbf{X} := (X_1, \dots, X_n)$, $n \in \bar{\mathbb{N}}$, of random variables is called a *stochastic process*.

The *expected value* of $X \in \mathcal{R}(\mathcal{X})$ is defined as

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{\mathcal{X}} x \mathbb{P}_X(dx),$$

provided that $\mathbb{E}[\|X\|] < \infty$.

For $p \in \mathbb{N}$, the quantity $\mathbb{E}[\|X\|^p]$ is called the *p-th moment* of X . The space of all random variables with finite p -th moment is denoted by $L^p(\mathcal{X})$.

The *variance* of $X \in L^2(\mathcal{X})$ is defined by

$$\mathbb{V}[X] = \mathbb{E}[\|X - \mathbb{E}[X]\|^2].$$

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then, for given $X, Y \in L^2(\mathcal{H})$ the *covariance operator*

$$\text{Cov}(X, Y) : \mathcal{H} \rightarrow \mathcal{H}$$

is a linear operator defined by

$$\text{Cov}(X, Y)(h) = \mathbb{E}[\langle (X - \mathbb{E}[X]), h \rangle (Y - \mathbb{E}[Y])] \quad \text{for } h \in \mathcal{H}.$$

For $X = Y$ we write $\text{Cov}(X) := \text{Cov}(X, X)$.

For random variables $X \in \mathcal{R}(\mathbb{R}^n)$, we allow $\mathbb{E}[X] \in \bar{\mathbb{R}}^n$. More precisely, the expectation is said to exist if

$$\mathbb{E}[\max\{X, 0\}]_i < \infty \quad \text{or} \quad \mathbb{E}[-\min\{X, 0\}]_i < \infty$$

for all $i = 1, \dots, n$, and it is finite if both expressions are finite.

From now on, we equip $\mathbb{R}^n$ with the standard Euclidean scalar product and the associated norm. Then, for random variables $X, Y \in \mathcal{R}(\mathbb{R}^n)$ with existing expected values, the variance of X is given by

$$\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^T(X - \mathbb{E}[X])] = \mathbb{E}[X^T X] - \mathbb{E}[X]^T \mathbb{E}[X] \in \bar{\mathbb{R}},$$

and the covariance operator is represented by the *covariance matrix*

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])^T] \in \bar{\mathbb{R}}^{n \times n}.$$

Furthermore, for any positive semidefinite matrix $Q \in \mathbb{R}^{n \times n}$, we have

$$\mathbb{V}[Q^{1/2}X] = \mathbb{E}[X^T Q X] - \mathbb{E}[X]^T Q \mathbb{E}[X] = \text{Tr}(Q \text{Cov}(X)). \quad (2.1)$$

2.2 Stochastic control systems and stationarity concepts

For separable Banach spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{W}$, a discrete-time stochastic control system is given by a continuous function

$$f : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \rightarrow \mathcal{X}, \quad (x, u, w) \mapsto f(x, u, w).$$

Then, for a given state $x(k) \in \mathcal{X}$ and control input $u(k) \in \mathcal{U}$ at time $k \in \mathbb{N}_0$, a realization of the next state $x(k+1) \in \mathcal{X}$ is computed as

$$x(k+1) = f(x(k), u(k), w(k)). \quad (2.2)$$

Here, the stochastic influence $w(k) \in \mathcal{W}$ on the system is sampled according to a given distribution $\mathbb{P}_W$, which is assumed to be independent of the state $x(k)$ and control $u(k)$.

While this computation yields a single realization of the state at each time $k \in \mathbb{N}_0$, the stochastic influence can alternatively be modeled by an *i.i.d.* sequence of random variables $\mathbf{W} := (W(0), W(1), \dots)$, with $W(k) \sim \mathbb{P}_W$ and $W(k)$ independent of $X(k)$ and $U(k)$ for all $k \in \mathbb{N}_0$. This allows us to represent all possible realizations of (2.2) simultaneously via the iteration

$$X(k+1) = f(X(k), U(k), W(k)), \quad (2.3)$$

where the state $X(k) \in \mathcal{R}(\mathcal{X})$, the control $U(k) \in \mathcal{R}(\mathcal{U})$, and the disturbance $W(k) \in \mathcal{R}(\mathcal{W})$ are random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Although using (2.3) instead of (2.2) requires the analysis of stochastic systems on infinite-dimensional spaces of random variables—even if $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{W}$ are finite-dimensional—the advantage is that, by fixing the *i.i.d.* sequence $\mathbf{W} =$

$(W(0), W(1), \dots)$, we can interpret (2.3) as a time-varying system on the space of random variables. This viewpoint allows us to transfer ideas and proof techniques from time-varying deterministic optimal and predictive control to the stochastic setting.

Accordingly, for an initial state $X_j \in \mathcal{R}(\mathcal{X})$ at time $j \in \mathbb{N}_0$ and a control sequence $\mathbf{U} := (U(0), \dots, U(N-1))$, we adopt the notation of [39] and denote the solution of (2.3) by

$$X_{\mathbf{U}}(k+1; j, X_j) = f(X_{\mathbf{U}}(k; j, X_j), U(k), W(k+j)), \quad X_{\mathbf{U}}(0; j, X_j) = X_j. \quad (2.4)$$

Remark 2.1. *Note that although we assume that the disturbance at time k is stochastically independent of the states and controls at time k , the state at time k will generally depend on the past disturbances $(W(0), \dots, W(k-1))$ through the dynamics (2.3). Hence, strictly speaking, the solution $X_{\mathbf{U}}(k; j, X_j)$ from (2.4) also depends on the disturbance sequence $\mathbf{W}$. However, for the sake of readability, we do not highlight this dependence in our notation.*

Furthermore, in the following we will not repeatedly state that the disturbance at time k is stochastically independent of the states and controls at time k . Instead, we indicate the relevant time instance through subscripts or arguments, thereby making clear with respect to which element of the disturbance sequence independence is assumed. That is, in the following, for $X_k \in \mathcal{R}(\mathcal{X})$ will implicitly mean that $X_k \in \mathcal{R}(\mathcal{X})$ is independent of $W(k)$.

Since the controls admissible in (2.4) are not yet restricted, the control design may, in principle, exploit information about future events. However, as the future is random and therefore unknown, it is necessary to specify the information available at each time instant in order to design admissible control laws. To this end, we introduce stochastic filtrations; see also [30, 81] for a detailed discussion.

Definition 2.2 (Stochastic filtrations). *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A filtration $(\mathcal{F}_k)_{k \in \mathbb{N}_0}$ is a sequence of σ -algebras with $\mathcal{F}_k \subseteq \mathcal{F}$, $k \in \mathcal{I} \subseteq \mathbb{N}_0$, such that*

$$\mathcal{F}_s \subseteq \mathcal{F}_k \quad \text{for all } 0 \leq s \leq k \in \mathcal{I}.$$

The natural (or induced) filtration of a stochastic process $\mathbf{Z} = (Z(0), \dots, Z(N))$, $N \in \mathbb{N}$ is given by

$$\mathcal{F}_k = \sigma(Z(0), \dots, Z(k)), \quad 0 \leq k \leq N.$$

Using the natural filtration, we can now mathematically formalize that an admissible control sequence may only use information available through the history of past states.

Definition 2.3 (Admissible control sequences). *Given an initial state X_j at time $j \in \mathbb{N}_0$, a control sequence $\mathbf{U} = (U(0), \dots, U(N-1))$ is called admissible for the system (2.3) if it is adapted to the natural filtration of the corresponding state process, i.e.,*

$$\sigma(U(k)) \subseteq \sigma(X_{\mathbf{U}}(0; j, X_j), \dots, X_{\mathbf{U}}(k; j, X_j)), \quad 0 \leq k \leq N-1.$$

The set of all admissible control sequences with $N \in \bar{\mathbb{N}}$ control instances for the initial state X_j at time $j \in \mathbb{N}_0$ is denoted by $\mathcal{U}_{\text{ad}}^N(j, X_j)$.

An important observation is that any control sequence adapted to the natural filtration can always be written in the form of a feedback policy, as stated in the following lemma.

Lemma 2.4 (cf. [51, Lemma 1.14]). *Given an initial state X_j at time $j \in \mathbb{N}_0$ a control sequence $\mathbf{U} = (U(0), \dots, U(N-1))$ is admissible for system (2.3) if and only if there exists a sequence of measurable feedback policies $\boldsymbol{\pi} := (\pi_0, \dots, \pi_{N-1})$ with $\pi_k : \mathcal{X}^{k+1} \rightarrow \mathcal{U}$ such that*

$$U(k) = \pi_k(X_{\mathbf{U}}(0; j, X_j), \dots, X_{\mathbf{U}}(k; j, X_j))$$

holds for all $0 \leq k \leq N-1$.

The main consequence of the above lemma is that we can always switch between controls formulated as random variables and feedback policies, and thus also between a distributional and a random-variable perspective, as will be shown later.

Remark 2.5 (Disturbance policies). *Another common choice for defining admissibility is to require the controls to be adapted to the natural filtration generated by the disturbance sequence. This would lead to disturbance feedback policies instead of state feedback policies and could potentially result in a larger set of admissible controls, since two different disturbance realizations may lead to the same state. However, since state measurements are typically available whereas direct disturbance measurements are not, we adhere to our definition of admissibility for the remainder of this thesis.*

Although an admissible control sequence can depend on the entire history of states, in many cases the information available through the current state is sufficient for our analysis, cf. Remark 2.10 and Theorem 2.13. The following definition formalizes this class of controls, which are referred to as Markovian.

Definition 2.6 (Markov controls). *A control sequence $\mathbf{U} = (U(0), \dots, U(N-1))$ is called Markovian if it is adapted to the filtration $\mathcal{F} = (\mathcal{F}_k)_{0 \leq k \leq N-1}$ with*

$$\mathcal{F}_k := \sigma(X_{\mathbf{U}}(k; j, X_j)),$$

or equivalently, if there exists a sequence of measurable policies $\boldsymbol{\pi} = (\pi_0, \dots, \pi_{N-1})$ with $\pi_k : \mathcal{X} \rightarrow \mathcal{U}$ such that

$$U(k) = \pi_k(X_{\mathbf{U}}(k; j, X_j)) \quad \text{for all } 0 \leq k \leq N-1.$$

In this case, π_k is called a Markov policy, and the set of all admissible Markov control sequences with $N \in \bar{\mathbb{N}}$ control instances for the initial state X_j at time $j \in \mathbb{N}_0$ is denoted by $\tilde{\mathcal{U}}_{\text{ad}}^N(j, X_j)$.

Note that $\tilde{\mathcal{U}}_{\text{ad}}^N(j, X_j) \subseteq \mathcal{U}_{\text{ad}}^N(j, X_j)$ holds for all $N \in \bar{\mathbb{N}}$, and thus, considering only Markov controls generally restricts the set of admissible control sequences. However, if we restrict ourselves to Markov policies, we can directly forward-propagate the distribution of the states in a deterministic manner using the so-called transition operator.

Definition 2.7 (Transition operator). *Consider a Markov policy $\pi : \mathcal{X} \rightarrow \mathcal{U}$ and a state $X \sim \mathbb{P}_X$. Then, for all times $j \in \mathbb{N}_0$, the distribution $\mathbb{P}_Y$ of the state $Y = f(X, \pi(X), W(j))$ can be calculated using the transition operator (or Markov operator) $\mathcal{T}$ given by*

$$\mathbb{P}_Y = \mathcal{T}(\pi, \mathbb{P}_X) := \int_{\mathcal{X}} p_{\pi}(x) \mathbb{P}_X(dx). \quad (2.5)$$

Here, for a given $x \in \mathcal{X}$, the measure $p_{\pi}(x) \in \mathcal{P}(\mathcal{X})$ denotes the transition probability defined by

$$p_{\pi}(x)(A) := \mathbb{P}(Y(\omega) \in A \mid X = x), \quad \text{for every } A \in \mathcal{B}(\mathcal{X}).$$

Hence, we can simulate the evolution of the state distributions of system (2.3) via the dynamics

$$\mathbb{P}_{X(k+1)} = \mathcal{T}(\pi_k, \mathbb{P}_{X(k)}), \quad (2.6)$$

if we restrict ourselves to admissible controls given by Markov policies. Moreover, since the dynamics (2.6) define a deterministic, time-invariant system, we can use the usual definition of a steady state to introduce the first stationarity concept for stochastic systems on the layer of distributions and policies.

Definition 2.8 (Stationarity in distribution). *A distribution-policy pair $(\mathbb{P}_X^s, \pi^s)$ is called stationary if it is a steady state of the transition operator (2.5), i.e.,*

$$\mathbb{P}_X^s = \mathcal{T}(\pi^s, \mathbb{P}_X^s).$$

The above stationarity concept for stochastic systems is well established in stochastic control and the study of Markov decision processes, cf. [3, 47, 71, 72], where the stationary distribution is also often called invariant measure. However, since this concept only captures the distributional behavior of the system, information about specific realization paths of the stochastic system is lost. Consequently, we cannot analyze the behavior of individual realization paths when switching from random variables to the layer of distributions and policies, even though numerical simulations suggest structural behavior from a pathwise perspective, cf. [77]. We therefore extend the distributional notion of stationarity to stochastic processes through the following definition, which is equivalent to the one given in [24] but stated in a form that is more convenient for our pathwise considerations.

Definition 2.9 (Stationary pair of stochastic processes). *A pair of stochastic processes $(\mathbf{X}^s, \mathbf{U}^s)$, given by*

$$X^s(k+1) = f(X^s(k), U^s(k), W(k)), \quad (2.7)$$

with $U^s(k) = \pi^s(X^s(k))$ for some Markov policy $\pi^s : \mathcal{X} \rightarrow \mathcal{U}$, is called stationary for system (2.3) if there exist probability distributions $\mathbb{P}_X^s, \mathbb{P}_U^s$, and $\mathbb{P}_{X,U}^s$ such that

$$X^s(k) \sim \mathbb{P}_X^s, \quad U^s(k) \sim \mathbb{P}_U^s, \quad (X^s(k), U^s(k)) \sim \mathbb{P}_{X,U}^s \quad (2.8)$$

holds for all $k \in \mathbb{N}_0$.

It should be noted that every stationary distribution-policy pair $(\mathbb{P}_X^s, \pi^s)$ from Definition 2.8 induces a stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ in the sense of Definition 2.9. This stationary pair can be constructed via

$$X^s(k+1) = f(X^s(k), U^s(k), W^s(k)), \quad U^s(k) = \pi^s(X^s(k)), \quad X^s(0) = X_0^s,$$

for an arbitrary initial condition $X_0^s \sim \mathbb{P}_X^s$. Note that stationarity of the distributions of the control process and the joint state-control process follows directly from the feedback structure of $\mathbf{U}^s$. Hence, it is sufficient to verify stationarity of the distribution of the state process in order to conclude that $(\mathbf{X}^s, \mathbf{U}^s)$ is a stationary pair. Furthermore, this construction shows that stationarity is independent of the specific realization of the initial state as a random variable and depends only on its distribution. However, in Theorem 3.10 we will see that for optimal stationary solutions the influence of the initial condition vanishes over time and thus, for long-time properties this ambiguity is not relevant.

Summarizing, Definition 2.9 provides a natural extension of the notion of stationarity defined via the transition operator (2.5) to stochastic processes. The main difference between stationarity in distribution and Definition 2.9 is the additional requirement (2.7) that the state process representing the stationary distribution evolves according to the stochastic system (2.3).

Remark 2.10 (Restriction to Markovian controls). *One may wonder how restrictive the consideration of only Markov policies in Definition 2.8 and Definition 2.9 is. To address this question, consider a stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ in the sense of Definition 2.9, where $\mathbf{U}^s$ is generated by a sequence of feedback policies $\pi^s = (\pi_0^s, \pi_1^s, \dots)$ with $\pi_k^s : \mathcal{X}^{k+1} \rightarrow \mathcal{U}$ satisfying (2.8). By stationarity, the distribution of $X^s(k)$ is invariant in k . In particular, the pair $(\mathbb{P}_X^s, \pi_0^s)$ defines a stationary distribution-policy pair in the sense of Definition 2.8. Consequently, the Markov control sequence $\mathbf{U}^s := (\pi_0^s, \pi_0^s, \dots)$ induces a stationary stochastic process according to Definition 2.9 with the same invariant state distribution. Hence, while non-Markovian stationary control sequences may exist, they do not generate additional stationary distributions beyond those induced by Markov policies. Therefore, the restriction to Markovian controls entails no loss of generality when studying stationarity properties of stochastic systems.*

Remark 2.11 (Ergodicity). *Another common concept in stochastic control, often considered in connection with stationarity, is ergodicity. In contrast to the stationarity concepts introduced above, ergodicity is a strictly stronger notion. In an ergodic system, time averages along almost every realization coincide with expectations taken with respect to the stationary distribution. While ergodicity implies uniqueness of the stationary distribution under mild assumptions, stationarity alone neither guarantees convergence nor equivalence between time and ensemble averages. Since stationarity is sufficient for the purposes of this thesis, ergodic properties are not considered further.*

2.3 Stochastic optimal control problems

The goal of optimal control is to find a control sequence that minimizes a given cost criterion subject to the underlying dynamical system. This cost criterion is typically

given by the sum of stage costs evaluated at the current state and control at each time step. Hence, in order to extend the stochastic systems introduced in Section 2.2 to optimal control problems, we need to introduce appropriate stage costs.

However, since both states and controls are random variables, a deterministic stage cost function $g : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$ cannot be applied directly, as this would again result in a random variable. Instead, an additional mapping $\mathbb{L} : \mathcal{R}(\mathbb{R}) \rightarrow \bar{\mathbb{R}}$ must be applied to obtain a real-valued quantity that can be optimized, leading to the stochastic stage cost

$$\ell : \mathcal{R}(\mathcal{X}) \times \mathcal{R}(\mathcal{U}) \rightarrow \bar{\mathbb{R}}, \quad \ell(X, U) := \mathbb{L}[g(X, U)].$$

Here, and in the following, we assume that $g : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$ is lower semicontinuous and bounded from below, and that $\mathbb{L} \circ g$ is bounded from below, which is typically guaranteed by the boundedness of g .

A common choice for the mapping $\mathbb{L} : \mathcal{R}(\mathbb{R}) \rightarrow \bar{\mathbb{R}}$ is the expected value, leading to the standard formulation

$$\ell(X, U) := \mathbb{E}[g(X, U)].$$

This choice is widely used in stochastic optimal control and Markov decision processes. While our turnpike and dissipativity notions do not explicitly depend on a particular choice of the mapping $\mathbb{L}$, the results obtained in Chapter 4 rely on properties of stage costs in expectation form and their implications for dynamic programming, cf. Theorem 2.22. Beyond the risk-neutral formulation, risk-averse stage costs can be obtained by choosing alternative aggregation mappings. A prominent example is the conditional value-at-risk (CVaR), which penalizes unfavorable outcomes more strongly and allows for explicit control of tail risks. Another extreme choice is the essential supremum, resulting in a worst-case or robust formulation that focuses on the most adverse realizations. First insights into such risk-averse formulations are provided in Chapter 5.

Furthermore, we impose additional constraints on the states and controls, distinguishing between pathwise and probabilistic constraints. Pathwise constraints are of the form

$$h(X(\omega), U(\omega)) \leq 0 \text{ almost surely.}$$

Here, $h : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^{n_c}$ is a Borel-measurable function, and we assume that h satisfies the viability condition that for every $x \in \mathcal{X}$ there exists a $u \in \mathcal{U}$ satisfying $h(x, u) \leq 0$. This ensures consistency of our pathwise setting with [10].

In contrast, probabilistic constraints are of the form

$$\mathbb{T}[c(X, U)] \leq 0,$$

where $c : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^{n_c}$ is a Borel-measurable function and $\mathbb{T} : \mathcal{R}(\mathbb{R}^{n_c}) \rightarrow \bar{\mathbb{R}}^n$ is a law-invariant mapping. An important choice for $\mathbb{T}$ is the componentwise application of risk measures such as the conditional value-at-risk (CVaR), which is studied in Chapter 5. Moreover, probabilistic constraints also encompass chance constraints commonly used in stochastic optimal control.

The conceptual difference between pathwise and probabilistic constraints lies in the level at which feasibility is enforced. Pathwise constraints require that the constraint is satisfied for almost every realization of the underlying uncertainty. As

a consequence, they guarantee feasibility along each individual realization of the stochastic system without relying on distributional information of the states and controls, but may lead to conservative control laws. In contrast, probabilistic constraints restrict only aggregated properties of the random variables induced by the system dynamics. Rather than enforcing feasibility for every realization, they allow for constraint violations on individual paths as long as a prescribed statistical criterion is satisfied. This typically leads to less conservative formulations and provides a natural mechanism to balance performance and risk in stochastic optimal control.

Hence, taking the above cost structure and the imposed constraints into account, the stochastic optimal control problem considered in this work on a finite horizon $N \in \mathbb{N}$ is given by

$$\begin{aligned} \underset{\mathbf{U} \in \mathcal{U}_{\text{ad}}^N(j, X_j)}{\text{minimize}} \quad & J_N(X_j, \mathbf{U}) := \sum_{k=0}^{N-1} \ell(X_{\mathbf{U}}(k; j, X_j), U(k)) \\ \text{s.t.} \quad & X_{\mathbf{U}}(k+1; j, X_j) = f(X_{\mathbf{U}}(k; j, X_j), U(k), W(k+j)), \\ & X_{\mathbf{U}}(0; j, X_j) = X_j, \\ & \mathbb{T}[c(X_{\mathbf{U}}(k; j, X_j), U(k))] \leq 0, \\ & h(X_{\mathbf{U}}(k; j, X_j)(\omega), U(k)(\omega)) \leq 0, \quad \text{almost surely.} \end{aligned} \quad (2.9)$$

The corresponding infinite-horizon stochastic optimal control problem is obtained by replacing the finite-horizon cost functional $J_N(X_j, \mathbf{U})$ in (2.9) with

$$J_{\infty}(X_j, \mathbf{U}) := \limsup_{N \rightarrow \infty} \sum_{k=0}^{N-1} \ell(X_{\mathbf{U}}(k; j, X_j), U(k)).$$

From Section 2.2 we already know that Markovian controls are sufficient to characterize the stationary behavior of stochastic systems. This naturally raises the question of whether a similar statement applies to optimal control problems. In other words, can optimal control sequences also be chosen to be Markovian?

To this end, for a given state $X \in \mathcal{R}(\mathcal{X})$, we define the set of controls satisfying the pathwise and probabilistic constraints (2.3) and (2.3) as

$$\mathbb{U}(X) := \{U \in \mathcal{R}(\mathcal{U}) \mid \mathbb{T}[c(X, U)] \leq 0, \ h(X(\omega), U(\omega)) \leq 0 \text{ a.s.}\}. \quad (2.10)$$

The set of all control sequences on a horizon $N \in \mathbb{N}$ that are admissible for the stochastic optimal control problem (2.9), i.e., admissible for the stochastic system (2.3) and satisfying the constraints, is given by

$$\mathbb{U}_{\text{ad}}^N(j, X_j) := \{\mathbf{U} \in \mathcal{U}_{\text{ad}}^N(j, X_j) \mid U(k) \in \mathbb{U}(X_{\mathbf{U}}(k; j, X_j)), \ 0 \leq k \leq N-1\}. \quad (2.11)$$

Analogously, the set of all Markovian controls satisfying the constraints for a given state $X \in \mathcal{R}(\mathcal{X})$ is given by

$$\check{\mathbb{U}}(X) := \{U \in \mathbb{U}(X) \mid \sigma(U) \subseteq \sigma(X)\}$$

and the set of admissible Markovian control sequences for the stochastic optimal control problem (2.9) is defined as

$$\check{\mathbb{U}}_{\text{ad}}^N(j, X_j) := \{\mathbf{U} \in \check{\mathcal{U}}_{\text{ad}}^N(j, X_j) \mid U(k) \in \mathbb{U}(X_{\mathbf{U}}(k; j, X_j)), \ 0 \leq k \leq N-1\}. \quad (2.12)$$

Remark 2.12 (Numerical impact of time dependence). *The time-dependent formulation of this thesis mainly serves as an analytical tool ensuring that realization paths are compared with respect to the same disturbance realizations. From a numerical point of view, this time dependence has no direct impact on the implementation of the optimization problems. In practice, optimal controls are typically obtained in a feedback form, i.e., the applied control depends on the current realization of the state rather than explicitly on the initial time index. Consequently, the numerical algorithms themselves remain unchanged. The only relevant aspect for simulations is that realization paths intended for comparison are generated using the same disturbance sequence. Apart from this consistency requirement, the explicit time dependence does not need to be incorporated into the numerical implementation. Therefore, in the following examples, we restrict ourselves to problems with initial condition $(j, X_j) = (0, X_0)$ whenever open-loop problems are considered in concrete examples, since this does not affect the numerical results or the observed qualitative behavior.*

The following theorem shows that the optimal cost can always be achieved by a Markovian control sequence.

Theorem 2.13 ([3, Theorem 6.2]). *For all $N \in \bar{\mathbb{N}}$, all $j \in \mathbb{N}_0$, and all $X_j \in \mathcal{R}(\mathcal{X})$ with $\mathbb{U}_{\text{ad}}^N(j, X_j) \neq \emptyset$, it holds that*

$$\inf_{\mathbf{U} \in \mathbb{U}_{\text{ad}}^N(j, X_j)} J_N(X_j, \mathbf{U}) = \inf_{\mathbf{U} \in \tilde{\mathbb{U}}_{\text{ad}}^N(j, X_j)} J_N(X_j, \mathbf{U}).$$

Remark 2.14. *Theorem 2.13 applies to our stochastic control problem because the cost and constraint structure is fully law-invariant. Pathwise constraints are inherently law-invariant, and the mappings $\mathbb{L}$ and $\mathbb{T}$ are assumed to be law-invariant as well. Furthermore, the dynamics (2.3) are defined pointwise by the function $f : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \rightarrow \mathcal{X}$ through the iteration (2.2), forming a controlled Markov chain that preserves law invariance. Importantly, this result can be extended to other law-invariant systems, since the proof relies mainly on law invariance. For instance, systems incorporating the expectation or variance of the state, as in mean-field type control problems, cf. [12], can also be treated within this framework.*

Theorem 2.13 implies that restricting the minimization in (2.9) to Markovian control sequences is not restrictive with respect to optimality. Hence, unless stated otherwise, we consider the stochastic optimal control problem

$$\underset{\mathbf{U} \in \tilde{\mathbb{U}}_{\text{ad}}^N(j, X_j)}{\text{minimize}} \quad J_N(X_j, \mathbf{U}), \quad (2.13)$$

which corresponds to problem (2.9) with the minimization restricted to Markovian control sequences.

The associated optimal value function is given by

$$V_N(j, X_j) := \inf_{\mathbf{U} \in \tilde{\mathbb{U}}_{\text{ad}}^N(j, X_j)} J_N(X_j, \mathbf{U}). \quad (2.14)$$

Moreover, if $|V_N(j, X_j)| < \infty$ holds, we assume that a minimizer of problem (2.13) exists. We denote such a minimizer by $\mathbf{U}_N^*$, or by $\mathbf{U}_{N,j,X_j}^*$ when we wish to emphasize its dependence on the initial condition. Note that $|V_N(j, X_j)| = \infty$ may occur, since we allow the mapping $\mathbb{L}$ in the stage cost to attain the values $\pm\infty$. As will be shown later, this issue can be circumvented by only considering suitable initial values X_j .

In contrast, on the infinite horizon the optimal cost will in general not be finite, even if $|V_N(j, X_j)| < \infty$ holds for all $N \in \mathbb{N}$. Consequently, a different notion of optimality is required in order to compare control sequences whose accumulated costs both diverge.

In this work, we therefore adopt the concept of *overtaking optimality*, originally introduced in [31], to define optimal control sequences on the infinite horizon.

Definition 2.15 (Overtaking optimality). *Let $X_j \in \mathcal{R}(\mathcal{X})$ and consider a control sequence $\mathbf{U}_\infty^* \in \check{\mathcal{U}}_{\text{ad}}^\infty(j, X_j)$ with corresponding state trajectory $X_{\mathbf{U}_\infty^*}(\cdot; j, X_j)$. The pair $(X_{\mathbf{U}_\infty^*}(\cdot; j, X_j), \mathbf{U}_\infty^*)$ is called overtaking optimal if*

$$\liminf_{K \rightarrow \infty} \sum_{k=0}^{K-1} \left(\ell(X_{\mathbf{U}}(k; j, X_j), U(k)) - \ell(X_{\mathbf{U}_\infty^*}(k; j, X_j), U_\infty^*(k)) \right) \geq 0$$

holds for all $\mathbf{U} \in \check{\mathcal{U}}_{\text{ad}}^\infty(j, X_j)$.

In the remainder of this work, we assume that such an overtaking optimal control sequence always exists whenever it is needed. We denote it by $\mathbf{U}_\infty^*$, or by $\mathbf{U}_{\infty, j, X_j}^*$ when emphasizing its dependence on the initial condition, analogously to the finite-horizon case.

Remark 2.16 (Time-dependence of the constraint sets and costs). *Note that the constraint set (2.10) is time-invariant, since the underlying pathwise and probabilistic constraints (2.3) and (2.3) do not depend on time.*

However, when defining admissibility on horizon $N \in \bar{\mathbb{N}}$ for the stochastic optimal control problem (2.9), the filtration and the system dynamics must also be taken into account. As a consequence, admissibility depends on the initial time through the available information and the history of the disturbance process. Therefore, in contrast to the pointwise constraint set (2.10), the admissible control sets (2.11) and (2.12) generally depend on the initial time. This time dependence is also inherited by the optimal value function, since the set of admissible controls depends on the initial time. In contrast, the cost itself is, in general, independent of the initial time. Due to the law invariance of the problem formulation, it depends only on the initial state, or more precisely on its distribution. For this reason, we write $J_N(X_0, \mathbf{U})$ rather than explicitly indicating a dependence on the initial time.

2.4 Dynamic programming principles

Dynamic programming provides a fundamental tool for the analysis of stochastic optimal control problems. Its core idea is that minimizing the accumulated cost over a shorter horizon $M < N$, augmented by the optimal value of the remaining horizon, yields the same optimal value as directly minimizing the cost over the entire horizon. This principle enables the decomposition of long—or even infinite—horizon optimization problems into a sequence of smaller subproblems.

In the following, we formulate dynamic programming principles for the stochastic optimal control problem (2.13). The next two theorems formalize this principle for finite and infinite horizons, respectively, within the random-variable framework adopted throughout this work.

Theorem 2.17 (Finite-horizon DPP, cf. [37, Theorem 3.15]). *Consider the stochastic optimal control problem (2.13) with $j, N \in \mathbb{N}_0$ and an initial condition $X_j \in \mathcal{R}(\mathcal{X})$ such that $|V_N(j, X_j)| < \infty$. Let $\mathbf{U}_N^* \in \check{\mathbb{U}}_{\text{ad}}^N(j, X_j)$ be an optimal control sequence on the horizon N , and define $V_0 \equiv 0$. Then, for all $M = 1, \dots, N$, it holds that*

$$V_N(j, X_j) = \sum_{k=0}^{M-1} \ell(X_{\mathbf{U}_N^*}(k; j, X_j), U^N(k)) + V_{N-M}(j+M, X_{\mathbf{U}_N^*}(M; j, X_j)).$$

Theorem 2.18 (Infinite-horizon DPP, cf. [37, Theorem 4.4]). *Consider the stochastic optimal control problem (2.13) with $j \in \mathbb{N}_0$, $N = \infty$, and an initial condition $X_j \in \mathcal{R}(\mathcal{X})$ such that $|V_\infty(j, X_j)| < \infty$. Let $\mathbf{U}_\infty^* \in \check{\mathbb{U}}_{\text{ad}}^\infty(j, X_j)$ be an optimal control sequence on the infinite horizon. Then, for all $M \in \mathbb{N}$, it holds that*

$$V_\infty(j, X_j) = \sum_{k=0}^{M-1} \ell(X_{\mathbf{U}_\infty^*}(k; j, X_j), U^\infty(k)) + V_\infty(j+M, X_{\mathbf{U}_\infty^*}(M; j, X_j)).$$

Note that both results can be proven analogously to the time-varying deterministic case, cf. [37], by working directly on the space of random variables. Alternatively, one may exploit the Markovian structure of the problem and switch to a formulation in terms of policies and state distributions, cf. [10]. Both approaches are equivalent, since the optimal value function depends only on the distribution of the state and not on its specific realization as a random variable.

Remark 2.19 (Finiteness of the optimal value function). *As already mentioned after equation (2.14), in Theorem 2.17 and Theorem 2.18 we have to assume finiteness of the optimal value function, since we allow the mapping $\mathbb{L}$ in the stage cost to attain the values $\pm\infty$.*

While in the finite-horizon case this issue can be resolved by only considering suitable initial values (j, X_j) , such an assumption is, in general, not sufficient in the infinite-horizon setting. However, in Chapter 3, we will show that, under suitable assumptions, one can apply a constant shift to the stage cost such that the resulting auxiliary infinite-horizon problem yields a finite optimal value, which makes the infinite-horizon dynamic programming principle applicable.

While Theorems 2.17 and 2.18 formulate the dynamic programming principle on the level of random variables, they do not provide a pointwise characterization of optimal controls in terms of the current state realization. However, for practical purposes it is desirable to express the dynamic programming principle in a pointwise form, since in applications typically only the current state realization is available through measurements, while the full state distribution is unknown.

In order to formulate a dynamic programming principle with respect to deterministic state realizations $x \in \mathcal{X}$, an appropriate notion of measurability is required. In general, Borel measurability is not sufficient to guarantee that a pointwise-defined control policy is compatible with *arbitrary* state distributions $X(k) \in \mathcal{R}(\mathcal{X})$. Since no prior knowledge of the state distribution is assumed, such compatibility must be ensured uniformly over all possible distributions.

To this end, we introduce the notion of universal measurability, which guarantees that the resulting feedback policies are admissible for any probability measure on the state space.

Definition 2.20 (Universal measurability). *A function $\pi : \mathcal{X} \rightarrow \mathcal{U}$ is called universally measurable if it is measurable from the universal σ -algebra on $\mathcal{X}$ into the Borel σ -algebra on $\mathcal{U}$, i.e.,*

$$\pi^{-1}(B) \in \bigcap_{\mathbb{Q} \in \mathcal{P}(\mathcal{X})} \mathcal{B}_{\mathbb{Q}}(\mathcal{X}) \quad \text{for all } B \in \mathcal{B}(\mathcal{U}).$$

Here,

$$\mathcal{B}_{\mathbb{Q}}(\mathcal{X}) := \{E \subseteq \mathcal{X} \mid \mathbb{Q}(E) + \mathbb{Q}^*(E^c) = 1\}$$

denotes the completion of $\mathcal{B}(\mathcal{X})$ with respect to $\mathbb{Q}$, and $\mathbb{Q}^*$ is the $\mathbb{Q}$ -outer measure defined by

$$\mathbb{Q}^*(E) := \inf\{\mathbb{Q}(B) \mid E \subseteq B, B \in \mathcal{B}(\mathcal{X})\}.$$

However, simply switching from Borel measurability to universal measurability is not sufficient to guarantee that a pointwise dynamic programming principle can always be applied. In addition, it must be ensured that the stochastic optimal control problem does not couple individual sample paths through the chosen cost and constraint structure, which would otherwise make knowledge of the full state distribution unavoidable when solving the problem on a shifted horizon consistently with the original one.

To this end, we impose the following assumption, which guarantees that only pathwise information is required in order to solve the stochastic optimal control problems on shifted horizons consistently.

Assumption 2.21. *The mapping $\mathbb{L} : \mathcal{R}(\mathbb{R}) \rightarrow \mathbb{R}$ in the stage costs is the expectation, i.e.,*

$$\ell(X, U) = \mathbb{E}[g(X, U)],$$

for some lower semicontinuous function $g : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$ bounded from below. Furthermore, no probabilistic constraints of the form (2.3) are imposed; that is, only pathwise constraints of the form (2.3) are considered in problem (2.13).

Based on the notion of universal measurability and Assumption 2.21, we can conclude this section by stating a pathwise stochastic dynamic programming principle.

Theorem 2.22 (cf. [10, Proposition 8.5]). *Let Assumption 2.21 hold. Consider $N \in \mathbb{N}$, $j \in \mathbb{N}_0$ and assume that for $k = 0, \dots, N-1$ the infimum in*

$$\inf_{u_k \in \mathcal{U}} \{g(x_k, u_k) + V_{N-k-1}(j+k, f(x_k, u_k, W(j+k)))\} \quad (2.15)$$

is attained for each $x_k \in \mathcal{X}$. Define $\pi_k^*(x_k) := u_k^*$ for each $x_k \in \mathcal{X}$, where $u_k^* \in \mathcal{U}$ forms a measurable selection of minimizers of (2.15).

Let Π denotes the set of all universally measurable functions $\pi : \mathcal{X} \rightarrow \mathcal{U}$. Then, $\pi^* := (\pi_0^*, \dots, \pi_{N-1}^*)$ defines a sequence of universally measurable functions and

$$\pi_k^* \in \arg \min_{\pi_k \in \Pi} \{ \ell(X, \pi_k(X)) + V_{N-k-1}(j+k, f(X, \pi_k(X), W(j+k))) \}$$

holds for all $X \in \mathcal{R}(\Omega, \mathcal{X})$ with $|V_N(j, X)| < \infty$ and $k = 0, \dots, N-1$.

The remaining issue is that Theorem 2.22 only guarantees that the control sequence constructed in this way is *universally* measurable. Hence, it may fail to be a feasible solution of problem (2.13), since in general $U(k) = \pi_k^*(X(k)) \in \mathcal{R}(\mathcal{U})$ does not hold for such a policy.

To ensure that we can take advantage of Theorem 2.22, we therefore need to guarantee that a measurable control sequence can be constructed from the pathwise solutions provided by Theorem 2.22. This is covered by the following lemma.

Lemma 2.23. *For every universally measurable function $\pi : \mathcal{X} \rightarrow \mathcal{U}$ and every $\mathbb{Q} \in \mathcal{P}(\mathcal{X})$, there exists a measurable function $\pi_{\mathbb{Q}} : \mathcal{X} \rightarrow \mathcal{U}$ such that $\pi = \pi_{\mathbb{Q}}$ holds $\mathbb{Q}$ -almost surely.*

Proof. Since π is universally measurable, it is measurable with respect to the completed Borel σ -algebra $\mathcal{B}_{\mathbb{Q}}(\mathcal{X})$ for any $\mathbb{Q} \in \mathcal{P}(\mathcal{X})$. The claim therefore follows directly from [19, Lemma 1.2]. $\square$

3 Dissipativity and turnpike in stochastic optimal control

Dissipativity and turnpike properties are fundamental concepts in optimal control theory for the qualitative analysis of optimal solutions. Dissipativity, originally introduced by Willems [105, 106], provides an energy-based framework for characterizing system behavior, while the turnpike property, first observed by Ramsey [82] and von Neumann [75], describes the phenomenon that optimal trajectories of finite-horizon problems remain close to a stationary solution for most of the optimization horizon. Both concepts play an important role in the analysis of long-horizon optimal control problems and have become particularly relevant through applications in model predictive control [13, 26, 27, 34, 35, 54], optimal control of partial differential equations [42, 43, 103, 109], shape optimization [60], and mean field games [17, 80].

For deterministic optimal control problems, the relation between strict dissipativity and turnpike behavior is well understood. Under suitable assumptions, strict dissipativity implies turnpike properties, while converse implications can also be established; see, e.g., [28, 36, 40]. These results provide a comprehensive framework for analyzing the asymptotic structure of optimal solutions in both nonlinear and linear-quadratic settings.

In stochastic optimal control, considerably fewer results are available. The presence of uncertainty raises the question of which mathematical objects should serve as the basis for dissipativity and turnpike analysis in place of deterministic steady states. Existing approaches mainly focus on probability distributions or moments of stochastic processes, see [56, 66, 100], or are limited to special classes of linear-quadratic problems, see [101]. Likewise, available notions of stochastic dissipativity are predominantly formulated on spaces of probability measures [33, 57]. While such approaches yield information on distributional properties of optimal solutions, they generally do not permit conclusions about individual realization paths, although numerical evidence suggests that stochastic optimal control problems may also exhibit turnpike phenomena at this level [77]. Capturing such behavior therefore requires a framework that goes beyond purely measure-theoretic formulations.

This chapter develops such a framework for discrete-time stochastic optimal control problems. The presented results unify and extend previous work on stochastic dissipativity and turnpike analysis, covering both general nonlinear systems and linear-quadratic problems.

The main contribution of this chapter is the development of unified notions of stochastic dissipativity and turnpike properties that yield different implications for solution behavior depending on the statistical properties under consideration. We investigate the connections between these notions by establishing a hierarchy of implications and provide optimality characterizations of the corresponding stationary solutions at which dissipativity holds. In particular, the stationary distributions underlying the dissipativity and turnpike formulations are characterized through

associated stationary optimization problems.

In addition to this general framework, we explicitly construct storage functions for particular classes of problems, including nonlinear and linear-quadratic systems, and illustrate the resulting stochastic turnpike properties by numerical simulations.

The chapter is organized as follows. Section 3.1 introduces the proposed notions of dissipativity in the stochastic setting. Section 3.2 provides optimality characterizations of dissipative stationary solutions, while Section 3.3 introduces turnpike properties for stochastic optimal control problems. In Section 3.4, we investigate which qualitative properties can be derived from strict dissipativity and, in particular, under which conditions strict dissipativity implies turnpike behavior in the stochastic setting. Section 3.5 is devoted to the relations between the different notions of stochastic dissipativity and turnpike properties and establishes the hierarchy induced by these concepts. Finally, Section 3.6 presents explicit constructions of storage functions together with numerical illustrations of stochastic turnpike phenomena for a class of linear-quadratic optimal control problems and a nonlinear example.

The material presented in this chapter is based on the results of [84, 87, 89], which are extended here to a slightly more general framework based on arbitrary pseudometrics.

3.1 Stochastic dissipativity concepts

In this section, we aim to extend the classical notion of dissipativity to stochastic optimal control problems. Dissipativity was originally introduced by Willems in [105, 106] with an energy-based interpretation, asserting that a system cannot generate energy internally but can only store energy supplied from the outside. Strict dissipativity additionally requires that a certain amount of stored energy is irreversibly dissipated to the environment. Willems' original motivation was to extend Lyapunov concepts to systems with inputs and outputs, described as “A generalization of Lyapunov functions to open systems, to systems with inputs and outputs” [107].

In optimal control, dissipativity establishes a connection between the running cost and a storage function that captures the system's ability to accumulate and release energy over time. In the strict case, it additionally penalizes persistent deviations from an optimal steady state through a dissipation term. This interpretation naturally extends to the stochastic setting by replacing the deterministic steady state with a stationary stochastic process, reflecting the fact that optimal operation is characterized not by a single state but by invariant statistical behavior.

However, replacing the deterministic steady state by a stochastic quantity makes it more involved to measure the deviation from the stationary regime. To address this issue, we introduce the notion of a pseudometric on the space of random variables, which provides a flexible framework for quantifying the distance between different stochastic states.

Definition 3.1 (Pseudometric). *A function*

$$d : \mathcal{R}(\mathcal{X}) \times \mathcal{R}(\mathcal{X}) \rightarrow [0, \infty]$$

is called a metric on the space of random variables $\mathcal{R}(\mathcal{X})$ if for all $X, Y, Z \in \mathcal{R}(\mathcal{X})$ the following properties hold:

(i) Positive semidefiniteness:

$$(a) \ d(X, Y) \geq 0, \quad (b) \ d(X, Y) = 0 \iff X = Y \text{ a.s.};$$

(ii) Symmetry:

$$d(X, Y) = d(Y, X);$$

(iii) Triangle inequality:

$$d(X, Z) \leq d(X, Y) + d(Y, Z).$$

The function d is called a pseudometric if properties (i)(a), (iii), and (iv) hold, while condition (ii)(b) is relaxed to

$$X = Y \text{ a.s.} \implies d(X, Y) = 0.$$

In contrast to a metric, a pseudometric allows for $d(X, Y) = 0$ even if $X \neq Y$ almost surely. Such situations arise naturally in stochastic settings, for instance when only distributional properties are of interest and random variables are considered equivalent whenever they share the same distribution. For further details and examples, we refer to [98], where such pseudometrics are studied under the term *probability metrics*.

Based on the notion of a pseudometric introduced in Definition 3.1, we formalize dissipativity for stochastic optimal control problems as follows.

We denote by

$$\ell(\mathbf{X}^s, \mathbf{U}^s) = \ell(X^s(k), U^s(k)), \quad k \in \mathbb{N}_0,$$

the constant stage cost along the stationary process, which is independent of k due to the law-invariance assumption. Furthermore, throughout this thesis, we adopt the conventions $\alpha(\infty) = \infty$ for $\alpha \in \mathcal{K}_\infty$ and $\infty \geq \infty$.

Definition 3.2 (Stochastic dissipativity). *Consider a stationary pair of stochastic processes $(\mathbf{X}^s, \mathbf{U}^s)$ according to Definition 2.9 with $|\ell(\mathbf{X}^s, \mathbf{U}^s)| < \infty$, and let d be a pseudometric on $\mathcal{R}(\mathcal{X})$.*

The stochastic optimal control problem (2.13) is called strictly stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to the pseudometric d if there exist a storage function

$$\lambda : \mathbb{N}_0 \times \mathcal{R}(\mathcal{X}) \rightarrow \mathbb{R},$$

which is uniformly bounded from below, and a function $\alpha \in \mathcal{K}_\infty$ such that

$$\ell(X_k, U_k) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(k, X_k) - \lambda(k+1, f(X_k, U_k, W(k))) \geq \alpha(d(X_k, X^s(k))) \quad (3.1)$$

holds for all $k \in \mathbb{N}_0$ and all $X_k \in \mathcal{R}(\mathcal{X})$ and $U_k \in \check{\mathcal{U}}(X_k)$.

If (3.1) holds with $\alpha \equiv 0$, the problem is called stochastically dissipative.

Remark 3.3. *For all pairs (X_k, U_k) satisfying $\ell(X_k, U_k) = \infty$, the dissipativity inequality (3.1) always holds independently of the choice of the storage function λ . Hence, the inequality must only be verified for all (X_k, U_k) with $\ell(X_k, U_k) < \infty$. On the other hand, if $d(X_k, X^s(k)) = \infty$ holds, this directly implies that the cost $\ell(X_k, U_k)$ must be infinite for all Markovian controls $U_k \in \check{\mathcal{U}}(X_k)$ if the problem is dissipative, leading to a kind of compatibility property between the proposed pseudometric and the stage costs.*

The choice of pseudometric in the dissipation inequality (3.1) determines which aspects of the deviation from the stationary process are considered relevant. The following definition introduces specific notions of strict stochastic dissipativity corresponding to different choices of pseudometrics.

A separate distinction for *non-strict* stochastic dissipativity is not required, since in that case $\alpha \equiv 0$ and thus, there is no dependence on the pseudometric d . Accordingly, we simply refer to this property as stochastic dissipativity.

Definition 3.4 (Types of stochastic dissipativity). *Assume that the stochastic optimal control problem is strictly stochastically dissipative according to Definition 3.2. Then, it is called*

(i) strictly L^p -dissipative if d is the L^p -norm, i.e.,

$$d(X, Y) = \|X - Y\|_{L^p} := \begin{cases} (\mathbb{E}[\|X - Y\|^p])^{\frac{1}{p}} & \text{for } 1 \leq p < \infty \\ \text{ess sup } \|X - Y\| & \text{for } p = \infty, \end{cases} \quad (3.2)$$

where

$$\text{ess sup } Z := \inf\{M \in \mathbb{R} : Z \leq M \text{ almost sure}\};$$

(ii) strictly pathwise-in-probability dissipative if d is the Ky-Fan metric, i.e.,

$$d(X, Y) = d_{\text{KF}}(X, Y) := \inf\{\varepsilon > 0 \mid \mathbb{P}(\|X - Y\| > \varepsilon) \leq \varepsilon\}; \quad (3.3)$$

(iii) strictly distributionally dissipative if d is a metric on the space of probability measures $\mathcal{P}(\mathcal{X})$, more precisely,

(a) Wasserstein dissipative of order p if d is the Wasserstein distance of order p , i.e.,

$$d(X, Y) = d_{W^p}(X, Y) := \inf\{\|\bar{X} - \bar{Y}\|_{L^p} \mid \bar{X} \sim X, \bar{Y} \sim Y\}; \quad (3.4)$$

(b) strictly weakly distributionally dissipative if d is the Lévy-Prokhorov metric, i.e.,

$$d(X, Y) = d_{LP}(X, Y) := \inf\{d_{\text{KF}}(\bar{X}, \bar{Y}) \mid \bar{X} \sim X, \bar{Y} \sim Y\}; \quad (3.5)$$

(iv) p -th moment dissipative if d is the p -th absolute moments metric, i.e.,

$$d(X, Y) = d_{M^p}(X, Y) := \left| (\mathbb{E}[\|X\|^p])^{\frac{1}{p}} - (\mathbb{E}[\|Y\|^p])^{\frac{1}{p}} \right|. \quad (3.6)$$

Remark 3.5. Note that all distances introduced in Definition 3.4 (i)–(iii) are well-defined mappings from $\mathcal{R}(\mathcal{X}) \times \mathcal{R}(\mathcal{X})$ to $[0, \infty]$ as required in Definition 3.1, since probabilities lie in $[0, 1]$ and expectations of nonnegative random variables are in $[0, \infty]$.

For the absolute moments metric, a problem occurs if both $\mathbb{E}[\|X\|^p]$ and $\mathbb{E}[\|Y\|^p]$ are infinite, since in this case $d_{M^p}(X, Y)$ is not well-defined. In this situation, we adopt the convention

$$d_{M^p}(X, Y) = \begin{cases} 0, & \text{if } X \sim Y, \\ \infty, & \text{otherwise,} \end{cases} \quad \text{whenever } \mathbb{E}[\|X\|^p] = \mathbb{E}[\|Y\|^p] = \infty.$$

This choice is consistent with the interpretation of d_{M^p} as a law-invariant pseudometric and preserves its dominance by the Wasserstein metric d_{W^p} , cf. Lemma 3.29.

However, since we are typically only interested in the distance to the stationary process $\mathbf{X}^s$ it is usually sufficient to assume that $\mathbb{E}[\|X^s(k)\|^p]$ is finite for all $k \in \mathbb{N}_0$ in order to ensure that all expressions involving this pseudometric are well-defined.

Since we interpret the stochastic dynamics as a time-varying system on the space of random variables, the notion of stochastic dissipativity from Definition 3.2 becomes time-varying as well, i.e., the storage function λ depends explicitly on time. This is consistent with deterministic time-varying settings, where storage functions are likewise allowed to depend on time, cf. [40].

However, when passing to the level of distributions, the problem is no longer time-dependent, as explained in Section 2.2. If we consider law-invariant pseudometrics d as the Wasserstein distance or the p -th absolute moments metric, and thus, only take distributional properties into account, the notion of dissipativity can and should be considered time-independent in order to remain consistent with the deterministic setting.

This is also in line with [33], who introduced dissipativity concepts for stochastic systems at the level of distributions, in particular for Markov decision processes, where the corresponding storage functions are indeed time-invariant.

The following lemma shows that, in our setting, the storage function can always be chosen to be both time- and law-invariant on the layer of distributions. Nevertheless, we will continue to use the time-varying notion from Definition 3.2 in this case as well, in order to maintain a unified notation and avoid having to distinguish between working on the space of random variables or on the layer of distributions.

Lemma 3.6. *Assume that the stochastic optimal control problem (2.13) is stochastically dissipative or strictly stochastically dissipative with respect to a law-invariant pseudometric d_D . Then, there exists a law-invariant and time-independent storage function*

$$\lambda_D : \mathcal{R}(\mathcal{X}) \rightarrow \mathbb{R}$$

satisfying the dissipation inequality (3.1).

Proof. Let the problem be strictly stochastically dissipative. Then, there exists a storage function

$$\lambda : \mathbb{N}_0 \times \mathcal{R}(\mathcal{X}) \rightarrow \mathbb{R}$$

which is uniformly bounded from below and $\alpha \in \mathcal{K}_\infty$ such that

$$\ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(k, X) - \lambda(k+1, f(X, U, W(k))) \geq \alpha(d_D(X, X^s(k))) \quad (3.7)$$

for all $k \in \mathbb{N}_0$ and all admissible Markovian controls $U \in \mathbb{U}(X)$.

We construct a law-invariant storage function by defining

$$\tilde{\lambda}(k, X) := \inf \{ \lambda(k, Y) \mid Y \in \mathcal{R}(\mathcal{X}), Y \sim X \}.$$

By construction, $\tilde{\lambda}$ is law-invariant and uniformly bounded from below due to the uniform lower bound on λ .

Now fix $k \in \mathbb{N}_0$, $X \in \mathcal{R}(\mathcal{X})$, and an admissible Markovian control $U = \pi(X) \in \mathbb{U}(X)$. Let

$$X^+ := f(X, U, W(k)).$$

By the definition of $\tilde{\lambda}$, for every $\varepsilon > 0$, there exists a random variable $\bar{X}_\varepsilon$ with $\bar{X}_\varepsilon \sim X$ such that

$$\lambda(k, \bar{X}_\varepsilon) \leq \tilde{\lambda}(k, X) + \varepsilon.$$

Since the dynamics are Markovian and distributionally consistent, we get

$$f(\bar{X}_\varepsilon, \bar{U}_\varepsilon, W(k)) \sim X^+, \quad \text{for } \bar{U}_\varepsilon = \pi(\bar{X}_\varepsilon) \sim U$$

and hence, due to the law-invariance of $\tilde{\lambda}$ that

$$\tilde{\lambda}(k+1, X^+) = \tilde{\lambda}(k+1, f(\bar{X}_\varepsilon, \bar{U}_\varepsilon, W(k))) \leq \lambda(k+1, f(\bar{X}_\varepsilon, \bar{U}_\varepsilon, W(k))).$$

Note that $\bar{U}_\varepsilon \in \mathbb{U}(\bar{X}_\varepsilon)$ holds due to the law-invariance of the constraints.

Inserting $(\bar{X}_\varepsilon, \bar{U}_\varepsilon)$ into (3.7) yields

$$\ell(\bar{X}_\varepsilon, \bar{U}_\varepsilon) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(k, \bar{X}_\varepsilon) - \lambda(k+1, f(\bar{X}_\varepsilon, \bar{U}_\varepsilon, W(k))) \geq \alpha(d_D(\bar{X}_\varepsilon, X^s(k))).$$

Since ℓ and d_D are law-invariant and $(\bar{X}_\varepsilon, \bar{U}_\varepsilon) \sim (X, U)$ holds, we have

$$d_D(\bar{X}_\varepsilon, X^s(k)) = d_D(X, X^s(k)), \quad \ell(\bar{X}_\varepsilon, \bar{U}_\varepsilon) = \ell(X, U).$$

Using the estimates from above gives

$$\begin{aligned} \ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \tilde{\lambda}(k, X) - \tilde{\lambda}(k+1, X^+) + \varepsilon \\ \geq \ell(\bar{X}_\varepsilon, \bar{U}_\varepsilon) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(k, \bar{X}_\varepsilon) - \lambda(k+1, f(\bar{X}_\varepsilon, \bar{U}_\varepsilon, W(k))) \\ \geq \alpha(d_D(\bar{X}_\varepsilon, X^s(k))) = \alpha(d_D(X, X^s(k))), \end{aligned}$$

which by letting $\varepsilon \rightarrow 0$ shows that $\tilde{\lambda}$ is a law-invariant storage function satisfying the dissipativity inequality.

It remains to show that the storage function can also be chosen as a time-invariant function. To this end, let us consider a state $X \in \mathcal{R}(\mathcal{X})$ and a control $U \in \mathbb{U}(X)$. Then, due to the stationarity of $\mathbf{X}^s$ and the fact that $\mathbf{W}$ is an *i.i.d.* sequence, we obtain

$$d_D(X, X^s(k)) = d_D(X, X^s(j))$$

and

$$f(X, U, W(k)) \sim f(X, U, W(j))$$

for every $k, j \in \mathbb{N}_0$. Hence, due to the law invariance of $\tilde{\lambda}$ it holds that

$$\begin{aligned} \ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \tilde{\lambda}(k, X) - \tilde{\lambda}(k+1, f(X, U, W(j))) \\ = \ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \tilde{\lambda}(k, X) - \tilde{\lambda}(k+1, f(X, U, W(k))) \\ \geq \alpha(d_D(X, X^s(k))) = \alpha(d_D(X, X^s(j))) \end{aligned}$$

Now set

$$\lambda_D(X) := \inf_{k \in \mathbb{N}_0} \tilde{\lambda}(k, X).$$

Then, for every $\varepsilon > 0$ there exists a $k_\varepsilon \in \mathbb{N}_0$ such that

$$\tilde{\lambda}(k_\varepsilon, X) \leq \lambda_D(X) + \varepsilon.$$

This yields

$$\begin{aligned}
& \ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda_D(X) - \lambda_D(f(X, U, W(k))) + \varepsilon \\
& \geq \ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \tilde{\lambda}(k_\varepsilon, X) - \tilde{\lambda}(k_\varepsilon + 1, f(X, U, W(k))) \\
& = \ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \tilde{\lambda}(k_\varepsilon, X) - \tilde{\lambda}(k_\varepsilon + 1, f(X, U, W(k_\varepsilon))) \\
& \geq \alpha(d_D(X, X^s(k_\varepsilon))) = \alpha(d_D(X, X^s(k))),
\end{aligned}$$

which shows that λ_D is a time-independent, law-invariant storage function.

The proof for stochastic dissipativity follows by setting $\alpha \equiv 0$. $\square$

As a direct consequence of Lemma 3.6 and the law-invariance assumptions, the dissipation inequality (3.1) can be reformulated on the distributional level as

$$\bar{\ell}(\mathbb{P}_X, \pi) - \bar{\ell}(\mathbb{P}_X^s, \pi^s) + \bar{\lambda}_D(\mathbb{P}_X) - \bar{\lambda}_D(\mathcal{T}(\mathbb{P}_X, \pi)) \geq \alpha(\bar{d}_D(\mathbb{P}_X, \mathbb{P}_X^s)). \quad (3.8)$$

Here, we use the definitions

$$\begin{aligned}
\bar{\ell}(\mathbb{P}_X, \pi) &:= \ell(X, U) \quad \text{for } X \sim \mathbb{P}_X, U = \pi(X), \\
\bar{\lambda}_D(\mathbb{P}_X) &:= \lambda_D(X) \quad \text{for } X \sim \mathbb{P}_X, \\
\bar{d}_D(\mathbb{P}_X, \mathbb{P}_Y) &:= d_D(X, Y) \quad \text{for } X \sim \mathbb{P}_X, Y \sim \mathbb{P}_Y.
\end{aligned} \quad (3.9)$$

Furthermore, $(\mathbb{P}_X^s, \pi^s)$ denotes the stationary distribution-policy pair associated with the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ and $\mathcal{T}$ is the transition operator for the dynamics f from equation 2.5. Since this corresponds to the notion of dissipativity based on probability measures introduced in [33], the dissipativity concept proposed in Definition 3.2 can be seen as a generalization of [33], in particular to the space of random variables.

Remark 3.7 (Non-Markovian extensions). *The concept of a stationary pair, cf. Definition 2.9, as well as the dissipativity properties from Definitions 3.2, can be extended to a non-Markovian setting by considering the admissible set of control sequences $\mathbb{U}_{\text{ad}}^N(j, X_j)$ instead of $\tilde{\mathbb{U}}_{\text{ad}}^N(j, X_j)$. In this case, however, the storage function λ in Definition 3.2 may implicitly depend on the entire history of the state process through the underlying filtration. We do not pursue this generalization here, since such history-dependent storage functions substantially complicate the analysis while providing no additional insight for the class of problems considered in this work, as the optimal control policies are Markovian. Moreover, in the non-Markovian setting the implications of Lemma 3.6 fail to hold in general, as the dependence of λ on the filtration introduces an implicit time dependence that cannot be removed by law-invariance arguments.*

3.2 Optimal stationary solutions

In this section, we investigate structural properties of stationary solutions of the stochastic optimal control problem (2.13) under dissipativity assumptions. In particular, we aim to characterize optimality within the class of stationary pairs and to analyze their uniqueness and convergence properties.

To this end, we first show that stochastic dissipativity already implies a fundamental performance bound, referred to as *optimal operation*. This property can be

interpreted as an asymptotic optimality condition, stating that $(\mathbf{X}^s, \mathbf{U}^s)$ provides a lower bound on the achievable average performance over infinite horizons. Remark that the following lemma holds for all control sequences $\mathbf{U} \in \mathbb{U}_{\text{ad}}^\infty(j, X_j)$ and is not restricted to Markovian controls $\mathbf{U} \in \check{\mathbb{U}}_{\text{ad}}^\infty(j, X_j)$.

Lemma 3.8 (Optimal operation). *Assume that the stochastic optimal control problem (2.13) is stochastically dissipative. Then, the system is optimally operated at the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$, i.e.,*

$$\liminf_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(X_{\mathbf{U}}(k; j, X_j), U(k)) \geq \ell(\mathbf{X}^s, \mathbf{U}^s)$$

holds for all $X_j \in \mathcal{R}(\mathcal{X})$ and all $\mathbf{U} \in \mathbb{U}_{\text{ad}}^\infty(j, X_j)$.

Proof. Since the problem is stochastically dissipative, there exists a storage function λ which is uniformly bounded from below, i.e., there exists a constant $C_\lambda^l > 0$ such that

$$\lambda(k, X) \geq -C_\lambda^l \quad \text{for all } k \in \mathbb{N}_0, X \in \mathcal{R}(\mathcal{X}).$$

By the dissipativity inequality (3.1), we obtain

$$\begin{aligned} \ell(X_{\mathbf{U}}(k; j, X_j), U(k)) &\geq \ell(\mathbf{X}^s, \mathbf{U}^s) - \lambda(j+k, X_{\mathbf{U}}(k; j, X_j)) \\ &\quad + \lambda(j+k+1, X_{\mathbf{U}}(k+1; j, X_j)) \end{aligned}$$

for all $k \in \mathbb{N}_0$ and $\mathbf{U} \in \check{\mathbb{U}}_{\text{ad}}^\infty(j, X_j)$. Summing this inequality from $k = 0$ to $K-1$ and dividing by K yields

$$\begin{aligned} \frac{1}{K} \sum_{k=0}^{K-1} \ell(X_{\mathbf{U}}(k; j, X_j), U(k)) &\geq \ell(\mathbf{X}^s, \mathbf{U}^s) - \frac{1}{K} \lambda(j, X_j) + \frac{1}{K} \lambda(j+K, X_{\mathbf{U}}(K; j, X_j)) \\ &\geq \ell(\mathbf{X}^s, \mathbf{U}^s) - \frac{\lambda(j, X_j)}{K} - \frac{C_\lambda^l}{K}. \end{aligned}$$

Taking the $\liminf_{K \rightarrow \infty}$ then yields

$$\liminf_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(X_{\mathbf{U}}(k; j, X_j), U(k)) \geq \ell(\mathbf{X}^s, \mathbf{U}^s)$$

for all $\mathbf{U} \in \check{\mathbb{U}}_{\text{ad}}^\infty(j, X_j)$.

Moreover, by Theorem 2.13, optimal control sequences can always be chosen to be Markovian. Hence, for every admissible control sequence $\tilde{\mathbf{U}} \in \mathbb{U}_{\text{ad}}^\infty(j, X_j)$, there exists a Markovian control sequence $\check{\mathbf{U}} \in \check{\mathbb{U}}_{\text{ad}}^\infty(j, X_j)$ such that

$$\liminf_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(X_{\tilde{\mathbf{U}}}(k; j, X_j), \tilde{U}(k)) \geq \liminf_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(X_{\check{\mathbf{U}}}(k; j, X_j), \check{U}(k)).$$

This proves the claim. $\square$

The previous lemma shows that $(\mathbf{X}^s, \mathbf{U}^s)$ is optimal in an averaged sense over infinite horizons. Building on this result, we now compare $(\mathbf{X}^s, \mathbf{U}^s)$ with arbitrary stationary pairs. The following theorem shows that $(\mathbf{X}^s, \mathbf{U}^s)$ is not only averaged optimal among all admissible control sequences, but in fact minimizes the stage cost over all stationary pairs.

Theorem 3.9. *Assume that the stochastic optimal control problem (2.13) is stochastically dissipative at the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$. Then, for every other stationary pair $(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s)$ it holds that*

$$\ell(\mathbf{X}^s, \mathbf{U}^s) \leq \ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s). \quad (3.10)$$

Moreover, if the stochastic optimal control problem (2.13) is strictly distributionally dissipative with respect to the pseudometric d_D , then

$$\ell(\mathbf{X}^s, \mathbf{U}^s) = \ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s) \Rightarrow d_D(X^s(k), \tilde{X}^s(k)) = 0 \quad \text{for all } k \in \mathbb{N}_0, \quad (3.11)$$

i.e., the stationary processes $\mathbf{X}^s$ and $\tilde{\mathbf{X}}^s$ share the same stationary distribution with respect to d_D .

Proof. Since $(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s)$ is a stationary pair, the value $\ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s)$ is constant in time. Hence, by Lemma 3.8,

$$\ell(\mathbf{X}^s, \mathbf{U}^s) \leq \liminf_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(\tilde{X}^s(k), \tilde{U}^s(k)) = \ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s),$$

which proves (3.10).

Now assume that the problem is strictly distributionally dissipative and that

$$d_D(X^s(k), \tilde{X}^s(k)) > 0$$

for some $k \in \mathbb{N}_0$, which by stationarity implies that $d_D(X^s(k), \tilde{X}^s(k)) > 0$ for all $k \in \mathbb{N}_0$. Applying the strict dissipativity inequality (3.1) to the stationary pair $(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s)$ yields

$$\begin{aligned} \ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s) &= \frac{1}{K} \sum_{k=0}^{K-1} \ell(\tilde{X}^s(k), \tilde{U}^s(k)) \\ &\geq \ell(\mathbf{X}^s, \mathbf{U}^s) - \frac{\lambda(0, \tilde{X}^s(0))}{K} - \frac{C_\lambda^l}{K} + \frac{1}{K} \sum_{k=0}^{K-1} \alpha(d_D(X^s(k), \tilde{X}^s(k))). \end{aligned}$$

Since $d_D(X^s(k), \tilde{X}^s(k))$ is constant in k and strictly positive, we obtain

$$\frac{1}{K} \sum_{k=0}^{K-1} \alpha(d_D(X^s(k), \tilde{X}^s(k))) = \alpha(d_D(X^s(0), \tilde{X}^s(0))) > 0$$

for all $K \in \mathbb{N}$. Hence, taking the $\liminf$ for $K \rightarrow \infty$ therefore yields

$$\ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s) > \ell(\mathbf{X}^s, \mathbf{U}^s),$$

which proves (3.11). □

The first statement of Theorem 3.9 shows that dissipativity at $(\mathbf{X}^s, \mathbf{U}^s)$ implies that the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ attains the minimal stationary cost. The second statement strengthens this result under a *strict* dissipativity assumption. In this case, equality of stationary costs implies a uniqueness property: any stationary pair achieving the same performance must coincide with $(\mathbf{X}^s, \mathbf{U}^s)$ in distribution with

respect to the law-invariant pseudometric d_D . Thus, strict distributional dissipativity enforces uniqueness of the optimal stationary behavior at the level of stationary distributions.

In particular, as a consequence of Theorem 3.9, the stationary distribution–policy pair can be characterized as a solution of the following stationary optimal control problem:

$$\begin{aligned} \min_{\mathbb{P}_X, \pi} \quad & \bar{\ell}(\mathbb{P}_X, \pi) \\ \text{s.t.} \quad & \mathbb{P}_X = \mathcal{T}(\mathbb{P}_X, \pi), \\ & \pi \in \mathbb{F}(\mathbb{P}_X), \end{aligned} \tag{3.12}$$

where $\bar{\ell}(\mathbb{P}_X, \pi) = \ell(X, U)$ for $X \sim \mathbb{P}_X$ and $U = \pi(X)$, cf. (3.9), and

$$\mathbb{F}(\mathbb{P}_X) := \{\pi : \mathcal{X} \rightarrow \mathcal{U} \mid \pi \text{ measurable, } \pi(X) \in \mathbb{U}(X) \text{ for } X \sim \mathbb{P}_X\}$$

denotes the set of admissible Markov policies for the distribution $\mathbb{P}_X$. Note that this reformulation of the stage cost and constraints on the level of distributions and policies is justified by the law-invariant structure of the problem. Moreover, under the strict dissipativity assumption, the (partial) solution $\mathbb{P}_X$ of this problem is unique with respect to the law-invariant pseudometric d_D .

While Theorem 3.9 ensures uniqueness in a distributional sense, it does not provide information about the temporal behavior of the associated stationary process. The following result addresses this issue by establishing convergence of stationary processes when their corresponding costs coincide.

Theorem 3.10. *Assume that the stochastic optimal control problem is strictly stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to a pseudometric d . Then, for every other stationary pair $(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s)$ satisfying*

$$\ell(\mathbf{X}^s, \mathbf{U}^s) = \ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s),$$

it holds that

$$\lim_{k \rightarrow \infty} d(X^s(k), \tilde{X}^s(k)) = 0.$$

Proof. By strict stochastic dissipativity, for all $k \in \mathbb{N}_0$,

$$\ell(\tilde{X}^s(k), \tilde{U}^s(k)) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(k, \tilde{X}^s(k)) - \lambda(k+1, \tilde{X}^s(k+1)) \geq \alpha(d(X^s(k), \tilde{X}^s(k))).$$

Summing from $k = 0$ to $K - 1$ and using $\ell(\mathbf{X}^s, \mathbf{U}^s) = \ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s)$ yields

$$\sum_{k=0}^{K-1} \alpha(d(X^s(k), \tilde{X}^s(k))) \leq \lambda(0, \tilde{X}^s(0)) - \lambda(K, \tilde{X}^s(K)) \leq \lambda(0, \tilde{X}^s(0)) + C_\lambda^l =: M$$

for all $K \in \mathbb{N}$. Since $\alpha(d(X^s(k), \tilde{X}^s(k))) \geq 0$, these partial sums are monotone and uniformly bounded. Hence

$$\sum_{k=0}^{\infty} \alpha(d(X^s(k), \tilde{X}^s(k))) < \infty.$$

This implies

$$\lim_{k \rightarrow \infty} \alpha(d(X^s(k), \tilde{X}^s(k))) = 0.$$

As $\alpha \in \mathcal{K}_\infty$ is continuous and strictly increasing, we conclude

$$\lim_{k \rightarrow \infty} d(X^s(k), \tilde{X}^s(k)) = 0.$$

□

Theorem 3.10 shows that strict stochastic dissipativity implies asymptotic convergence of stationary trajectories with respect to the pseudometric d . More precisely, if two stationary pairs achieve the same optimal cost, then their corresponding state processes exhibit identical asymptotic behavior in the pseudometric d , which is sufficient since the primary interest typically lies in the long-term behavior of the system.

3.3 Stochastic turnpike properties

Another property closely related to dissipativity is the so-called turnpike property. It describes the phenomenon that optimal and near-optimal trajectories spend most of their time close to a particular reference solution, which, in deterministic settings, is typically a steady state of the system. In the stochastic setting considered in this work, Section 2.2 suggests a stationary stochastic process as the natural analogue of a deterministic steady state.

However, in contrast to the deterministic case, it is not immediately clear how closeness to a stationary process should be quantified for stochastic trajectories. In particular, pointwise notions of distance are no longer meaningful, and the appropriate notion of proximity depends on whether one is interested in pathwise, moment-based, or distributional behavior. To address this issue, and in analogy to the definition of strict stochastic dissipativity, we employ the concept of a pseudometric, cf. Definition 3.1, to quantify distances between random variables in our stochastic framework.

Furthermore, since most properties considered in this thesis are required to hold only locally, we introduce the following notion of a neighborhood of a stationary process $\mathbf{X}^s$.

Definition 3.11 (Neighborhood of a stationary process). *A neighborhood of a stationary process $\mathbf{X}^s$ is a set*

$$\mathcal{S} := \{(j, X_j) \mid j \in \mathbb{N}_0, X_j \in S(j)\},$$

where $S(j) \subseteq \mathcal{R}(\mathcal{X})$ satisfies $X^s(j) \in S(j)$ for all $j \in \mathbb{N}_0$.

A neighborhood $\mathcal{S}$ is called an r -tube around $\mathbf{X}^s$ with respect to a pseudometric d if

$$\mathcal{S} = \bar{B}_r^d(\mathbf{X}^s) := \{(j, X_j) \mid j \in \mathbb{N}_0, X_j \in \bar{B}_r^d(X^s(j))\},$$

where

$$\bar{B}_r^d(X^s(j)) := \{X \in \mathcal{R}(\mathcal{X}) \mid d(X, X^s(j)) \leq r\}$$

denotes the closed ball around $X^s(j)$ with radius $r > 0$.

Using the concepts of a pseudometric and a neighborhood, the following definition provides a general formulation of the turnpike property for stochastic optimal control problems on both finite and infinite horizons.

Definition 3.12 (Stochastic turnpike properties). *Let $(\mathbf{X}^s, \mathbf{U}^s)$ be a stationary pair with $|\ell((\mathbf{X}^s, \mathbf{U}^s))| < \infty$ and consider a pseudometric d on the space $\mathcal{R}(\mathcal{X})$ and a neighborhood $\mathcal{S}$ of $\mathbf{X}^s$.*

We say that the stochastic optimal control problem (2.13) has the stochastic turnpike property locally on $\mathcal{S}$

- (i) *on finite horizon if for each $j \in \mathbb{N}_0$ and each optimal trajectory $X_{\mathbf{U}_N^*}(\cdot; j, X_j)$ with $X_j \in S(j)$ there exists a function $\vartheta \in \mathcal{L}$ and a set*

$$\mathcal{Q}(j, X_j, L, N) \subseteq \{0, \dots, N\}, \quad \#\mathcal{Q}(j, X_j, L, N) \leq L,$$

such that

$$d(X_{\mathbf{U}_N^*}(k; j, X_j), X^s(j+k)) \leq \vartheta(L) \quad (3.13)$$

for all $k \in \{0, \dots, N\} \setminus \mathcal{Q}(j, X_j, L, N)$ and $N, L \in \mathbb{N}$;

- (ii) *on infinite horizon if for each $j \in \mathbb{N}_0$ and each optimal trajectory $X_{\mathbf{U}_\infty^*}(\cdot; j, X_j)$ with $X_j \in S(j)$ there exists a function $\vartheta_\infty \in \mathcal{L}$ and a set*

$$\mathcal{Q}(j, X_j, L, \infty) \subseteq \mathbb{N}_0, \quad \#\mathcal{Q}(j, X_j, L, \infty) \leq L,$$

such that

$$d(X_{\mathbf{U}_\infty^*}(k; j, X_j), X^s(j+k)) \leq \vartheta_\infty(L) \quad (3.14)$$

for all $k \in \mathbb{N}_0 \setminus \mathcal{Q}(j, X_j, L, \infty)$ and $L \in \mathbb{N}$.

The stochastic turnpike property is called uniform if (3.13) (respectively (3.14)) holds with the same function $\vartheta \in \mathcal{L}$ (respectively $\vartheta_\infty \in \mathcal{L}$) for all $(j, X_j) \in \mathcal{S}$.

The term *locally* in Definition 3.12 indicates that the turnpike property is only required to hold for trajectories whose initial conditions (j, X_j) are sufficiently close to the stationary process at the initial time j .

Moreover, the stochastic turnpike property may, in general, depend on the initial condition $(j, X_j) \in \mathcal{S}$ in the non-uniform case, even when assumed locally, cf. Theorem 3.19. If the sets $S(j)$ are compact and ϑ (or ϑ_∞) depends continuously on the initial condition, local turnpike properties can be strengthened to uniform ones. However, as we work on an infinite-dimensional space of random variables endowed with different pseudometrics inducing different topologies, compactness arguments are not used in this thesis.

Remark 3.13. *In [84], a slightly different notion of the stochastic turnpike property was introduced, formulated for near steady-state trajectories rather than optimal ones. Apart from this difference in the class of trajectories under consideration, the definition in [84] is equivalent to the non-uniform turnpike property from Definition 3.12 with $\mathcal{S} = \mathcal{R}(\mathcal{X})$, that is, the turnpike property was assumed to hold globally.*

Since the MPC analysis in Chapter 4 and Chapter 5 rely crucially on uniformity with respect to the initial condition, in this thesis we adopt the above definition of the stochastic turnpike property, as it is tailored to this purpose.

Remark 3.14 (Local dissipativity). *Similarly to Definition 3.12, it is also possible to define stochastic dissipativity locally on a neighborhood of $\mathbf{X}^s$ and not globally as in Definition 3.2. However, for the proofs in Chapter 4, it is necessary that the*

closed-loop trajectories remain in a forward-invariant family of sets, which we ensure by assuming global strict dissipativity. Hence, we do not consider a local form of dissipativity in this work.

Remark 3.15 (Input-state and exponential turnpike). *Our notions of dissipativity and turnpike properties are based on measuring distances between state processes. Hence, they may also be interpreted as state dissipativity and state turnpike properties, respectively. A natural extension consists in additionally incorporating the distance between controls or policies into these definitions. This leads to corresponding notions of input-state dissipativity and input-state turnpike properties, which provide more detailed information about the behavior of optimal controls. Furthermore, one may consider stronger formulations of the turnpike property based on exponential estimates. In contrast to the notion used in this chapter—sometimes also referred to as a measure turnpike property—an exponential turnpike property guarantees that optimal solutions approach the turnpike exponentially fast and are close to the turnpike in the middle of the horizon. For linear-quadratic problems, such stronger exponential input-state turnpike properties have been established in [100, 101].*

As in the case of dissipativity, the choice of the metric or pseudometric in Definition 3.12 determines which aspects of the deviation from the optimal stationary process are considered relevant, such as moment discrepancies, distributional distances, or pathwise fluctuations. This leads to stronger or weaker notions of stochastic turnpike properties, which will be discussed later in Section 3.5. The following definition introduces specific types of stochastic turnpike properties associated with different choices of pseudometrics, analogously to Definition 3.4.

Definition 3.16 (Types of stochastic turnpike properties). *Assume that the stochastic optimal control problem has the stochastic turnpike property according to Definition 3.12. Then, we say that it has*

- (i) *the L^p turnpike property if d is the L^p -norm from (3.2);*
- (ii) *the pathwise-in-probability turnpike property if d is the Ky–Fan metric from (3.3);*
- (iii) *the distributional turnpike property if d is a metric on the space of probability measures $\mathcal{P}(\mathcal{X})$, or more precisely,*
 - (a) *the Wasserstein turnpike property of order p if d is the Wasserstein distance of order p from (3.4);*
 - (b) *the weak distributional turnpike property if d is the Lévy–Prokhorov metric from (3.5);*
- (iv) *the p -th moment turnpike property if d is the p -th absolute moments metric from equation 3.6.*

3.4 Implications of strict dissipativity and its relation to stochastic turnpike properties

In this section, we investigate the implications of strict dissipativity for the stochastic optimal control problem. In particular, we show that, under suitable assumptions,

strict dissipativity implies the stochastic turnpike properties from Definition 3.12 on both finite and infinite horizons. To this end, we introduce the following definition of *cheap reachability*, which formalizes the property that the system can be steered toward an optimal stationary process at a bounded additional cost relative to the cost of maintaining the state in the stationary regime, independently of the horizon N .

Definition 3.17 (Cheap reachability). *Consider a stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$, and a neighborhood $\mathcal{S}$ of $\mathbf{X}^s$. Then, the stationary process $\mathbf{X}^s$ is called cheaply reachable on $\mathcal{S}$ if there exist a constant $C_V > 0$ such that for all $(j, X_j) \in \mathcal{S}$ and all $N \in \mathbb{N}$ it holds that*

$$V_N(j, X_j) - N \ell(\mathbf{X}^s, \mathbf{U}^s) \leq C_V. \quad (3.15)$$

For example, cheap reachability holds if the stage costs ℓ are bounded from above and the stationary process $\mathbf{X}^s$ can be reached in a fixed finite number of steps from every initial condition $(j, X_j) \in \mathcal{S}$, see also [36, Lemma 3.6].

If a stationary process $\mathbf{X}^s$ is cheaply reachable and stochastic dissipativity holds at $(\mathbf{X}^s, \mathbf{U}^s)$, then there cannot exist another stationary process with lower cost that also satisfies the cheap reachability assumption. This follows from Theorem 3.9, which establishes optimality of $\ell(\mathbf{X}^s, \mathbf{U}^s)$ among stationary pairs. Moreover, due to this fact inequality (3.15) holds for any other stationary pair with the same constant C_V , although this bound is generally not tight.

However, the following lemma shows that, under stochastic dissipativity, the situation is stronger: not only does no stationary pair with lower cost satisfy (3.15), but the cheap reachability inequality cannot hold for any constant $C_\ell < \ell(\mathbf{X}^s, \mathbf{U}^s)$.

Lemma 3.18. *Assume that the stochastic optimal control problem (2.13) is stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to a pseudometric d . Furthermore, assume that $\mathbf{X}^s$ is cheaply reachable on a neighborhood $\mathcal{S}$ of $\mathbf{X}^s$. Then, there exists no constant $C_\ell \in \mathbb{R}$ with $C_\ell < \ell(\mathbf{X}^s, \mathbf{U}^s)$ such that*

$$V_N(j, X_j) - N C_\ell \leq C_V \quad (3.16)$$

holds for some $C_V > 0$, all $(j, X_j) \in \mathcal{S}$ and all $N \in \mathbb{N}$.

Proof. Consider $(j, X_j) \in \mathcal{S}$. By stochastic dissipativity, there exist a function $\alpha \in \mathcal{K}_\infty$, a constant $C_\lambda^l > 0$, and a storage function λ satisfying

$$\lambda(j, X_j) \geq -C_\lambda^l \quad \text{for all } j \in \mathbb{N}_0, X_j \in \mathcal{R}(\mathcal{X}),$$

such that

$$\begin{aligned} V_N(j, X_j) - N \ell(\mathbf{X}^s, \mathbf{U}^s) &= \sum_{k=0}^{N-1} \left(\ell(X_{\mathbf{U}_N^*}(k; j, X_j), U_N^*(k)) - \ell(\mathbf{X}^s, \mathbf{U}^s) \right) \\ &\geq -\lambda(j, X_j) + \lambda(j + N, X_{\mathbf{U}_N^*}(N; j, X_j)). \end{aligned}$$

Hence, we obtain

$$V_N(j, X_j) \geq N \ell(\mathbf{X}^s, \mathbf{U}^s) - C^s \quad (3.17)$$

with $C^s = \lambda(j, X_j) + C_\lambda^l > 0$.

Now assume that equation 3.16 holds for some $C_\ell \in \mathbb{R}$ with $C_\ell < \ell(\mathbf{X}^s, \mathbf{U}^s)$. Then, using equation (3.17) we obtain

$$N\ell(\mathbf{X}^s, \mathbf{U}^s) - C^s \leq V_N(j, X_j) \leq C_V + N C_\ell$$

and thus,

$$N(\ell(\mathbf{X}^s, \mathbf{U}^s) - C_\ell) \leq C_V + C^s.$$

Since $C_\ell < \ell(\mathbf{X}^s, \mathbf{U}^s)$, the left-hand side of the inequality becomes unbounded as $N \rightarrow \infty$ and hence cannot be uniformly bounded in N , which shows the claim by contradiction. $\square$

The following theorem shows that, under the additional assumption of cheap reachability, strict stochastic dissipativity implies the non-uniform stochastic turnpike property on finite horizon. Furthermore, if the storage function is bounded, the function ϑ from equation 3.13 can be constructed independently of the initial condition $(j, X_j) \in \mathcal{S}$, which implies uniformity of the finite-horizon turnpike property.

Theorem 3.19. *Assume that the stochastic optimal control problem (2.13) is strictly stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to some pseudometric d . Furthermore, assume that $\mathbf{X}^s$ is cheaply reachable on a neighborhood $\mathcal{S}$ of $\mathbf{X}^s$. Then, the stochastic optimal control problem (2.13) has the finite-horizon turnpike property on $\mathcal{S}$ with respect to the pseudometric d .*

Moreover, the stochastic turnpike property is uniform on $\mathcal{S}$, if the storage function λ is uniformly bounded on $\mathcal{S}$, i.e., there exists $C_\lambda > 0$ such that

$$|\lambda(j, X_j)| \leq C_\lambda \quad \text{for all } (j, X_j) \in \mathcal{S}. \quad (3.18)$$

Proof. By strict stochastic dissipativity, there exist a function $\alpha \in \mathcal{K}_\infty$, a constant $C_\lambda^l > 0$, and a storage function λ satisfying

$$\lambda(j, X_j) \geq -C_\lambda^l \quad \text{for all } j \in \mathbb{N}_0, X_j \in \mathcal{R}(\mathcal{X}),$$

such that

$$\begin{aligned} V_N(j, X_j) - N\ell(\mathbf{X}^s, \mathbf{U}^s) &= \sum_{k=0}^{N-1} \left(\ell(X_{\mathbf{U}_N^*}(k; j, X_j), U_N^*(k)) - \ell(\mathbf{X}^s, \mathbf{U}^s) \right) \\ &\geq \sum_{k=0}^{N-1} \alpha(d(X_{\mathbf{U}_N^*}(k; j, X_j), X^s(j+k))) \\ &\quad - \lambda(j, X_j) + \lambda(j+N, X_{\mathbf{U}_N^*}(N; j, X_j)). \end{aligned} \quad (3.19)$$

Now consider $(j, X_j) \in \mathcal{S}$ and suppose that equation (3.13) from the finite-horizon turnpike property from Definition 3.12 does not hold with

$$\vartheta(L) := \alpha^{-1} \left(\frac{C_\lambda^j + C_\lambda^l + C_V}{L} \right) \in \mathcal{L}, \quad (3.20)$$

where $C_V > 0$ is the constant from Definition 3.17 and $C_\lambda^j := \lambda(j, X_j)$.

Then, there exist $N \in \mathbb{N}$, a number $L \in \mathbb{N}$, and a set $\mathcal{M} \subseteq \{0, \dots, N-1\}$ with $\#\mathcal{M} \geq L+1$ such that

$$d(X_{\mathbf{U}_N^*}(k; j, X_j), X^s(j+k)) > \vartheta(L) \quad \text{for all } k \in \mathcal{M}.$$

Using (3.19), this implies

$$\begin{aligned} V_N(j, X_j) - N\ell(\mathbf{X}^s, \mathbf{U}^s) &\geq -C_\lambda^j - C_\lambda^l + \sum_{k \in \mathcal{M}} \alpha(\vartheta(L)) \\ &> -C_\lambda^j - C_\lambda^l + L\alpha(\vartheta(L)) \\ &= C_V, \end{aligned} \tag{3.21}$$

which contradicts the cheap reachability assumption. Hence, the stochastic finite-horizon turnpike property holds.

Moreover, since the uniform bound $C_\lambda > 0$ in contrast to C_λ^j is independent of the initial condition and $C_\lambda^j \geq C_\lambda$ holds for all $(j, X_j) \in \mathcal{S}$, the uniform version follows directly by replacing C_λ^j by C_λ in equation (3.20). $\square$

Next, we turn to the infinite-horizon case. To this end, we need to determine an appropriate formulation of cheap reachability on the infinite horizon.

We therefore introduce the shifted cost

$$\hat{J}_N(X_j, \mathbf{U}) := \sum_{k=0}^{N-1} \hat{\ell}(X_{\mathbf{U}}(k; j, X_j), U(k)), \tag{3.22}$$

where

$$\hat{\ell}(X, U) := \ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s).$$

Denoting the corresponding optimal value function by

$$\hat{V}_N(j, X_j) := \inf_{\mathbf{U} \in \tilde{\mathcal{U}}_{\text{ad}}^N(j, X_j)} \hat{J}_N(X_j, \mathbf{U}), \tag{3.23}$$

the cheap reachability condition (3.15) can equivalently be written as

$$\hat{V}_N(j, X_j) \leq C_V.$$

This motivates the consideration of the shifted infinite-horizon problem with cost

$$\hat{J}_\infty(X_j, \mathbf{U}) := \limsup_{N \rightarrow \infty} \sum_{k=0}^{N-1} \hat{\ell}(X_{\mathbf{U}}(k; j, X_j), U(k)),$$

and corresponding optimal value function

$$\hat{V}_\infty(j, X_j) := \inf_{\mathbf{U} \in \tilde{\mathcal{U}}_{\text{ad}}^\infty(j, X_j)} \hat{J}_\infty(X_j, \mathbf{U})$$

for which infinite-horizon version of cheap reachability could then be defined as

$$\hat{V}_\infty(j, X_j) \leq C_{V_\infty} \quad \text{for all } (j, X_j) \in \mathcal{S}. \tag{3.24}$$

However, rather than imposing this property as an additional assumption, we will show that finite-horizon cheap reachability carries over to the infinite-horizon setting under suitable assumptions. Moreover, we prove that the shifted infinite-horizon optimal value is not only bounded from above but also from below and hence finite.

This is crucial, since it allows us to apply the infinite-horizon dynamic programming principle from Theorem 2.18, which will serve as a key tool in the closed-loop analysis of Chapter 4.

To this end, we begin by showing that, under stochastic dissipativity, the infinite-horizon shifted optimal cost is finite along the stationary process $\mathbf{X}^s$.

Lemma 3.20. *Assume that the stochastic optimal control problem (2.13) is stochastically dissipative. Then, it holds that there exists a constant $C_{V_\infty}^s > 0$ such that*

$$-C_{V_\infty}^s \leq \hat{V}_\infty(j, X^s(j)) \leq 0$$

for all $j \in \mathbb{N}_0$.

Proof. Since the stationary control $\mathbf{U}^s$ yields zero shifted stage cost by definition, we immediately obtain

$$\hat{V}_\infty(j, X^s(j)) \leq 0$$

for all $j \in \mathbb{N}_0$.

On the other hand, stochastic dissipativity implies that there exists a law-invariant storage function $\lambda : \mathcal{R}(\mathcal{X}) \rightarrow \mathbb{R}$, cf. Lemma 3.6, such that for all horizons $N \in \mathbb{N}$,

$$\hat{V}_N(j, X^s(j)) \geq -\lambda(X^s(j)) + \lambda(X_{\mathbf{U}_N^s}(N; j, X^s(j))).$$

Since λ is law-invariant

$$\lambda(X^s(j)) = \lambda(X^s(0)) =: C_\lambda^s$$

holds due to the stationarity of $\mathbf{X}^s$. Using this together with the uniform lower bound $-C_\lambda^l$ of λ , we obtain

$$\hat{V}_N(j, X^s(j)) \geq -(C_\lambda^s + C_\lambda^l) =: C_{V_\infty}^s$$

for all $N \in \mathbb{N}$, which yields the claim by taking the lim sup for $N \rightarrow \infty$. $\square$

Lemma 3.20 shows in particular that $|\hat{V}_\infty(j, X^s(j))| < \infty$ holds for all $j \in \mathbb{N}_0$. However, in order to ensure that this finiteness property also holds for all $(j, X_j) \in \mathcal{S}$ under dissipativity and turnpike assumptions, we need an additional requirement: namely, that finiteness persists at least in an arbitrarily small neighborhood of the stationary process.

To this end, we make use of the following definition of continuity of the infinite-horizon shifted optimal value function.

Definition 3.21 (Continuity of the infinite-horizon shifted optimal value function). *Consider a stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ with $|\ell(\mathbf{X}^s, \mathbf{U}^s)| < \infty$ and a pseudometric d on $\mathcal{R}(\mathcal{X})$. The shifted infinite-horizon optimal value function $\hat{V}_\infty$ is called continuous at $\mathbf{X}^s$ with respect to the pseudometric d if there exist $\varepsilon > 0$ and a function $\gamma_{V_\infty} \in \mathcal{K}_\infty$ such that*

$$\left| \hat{V}_\infty(j, X_j) - \hat{V}_\infty(j, X^s(j)) \right| \leq \gamma_{V_\infty}(d(X_j, X^s(j)))$$

holds for all $(j, X_j) \in \bar{B}_\varepsilon^d(\mathbf{X}^s)$.

The following theorem shows that, under this continuity property on an arbitrarily small tube around the stationary process, strict stochastic dissipativity and finite-horizon cheap reachability on $\mathcal{S}$ imply finiteness of the shifted infinite-horizon optimal value function for all $(j, X_j) \in \mathcal{S}$.

Theorem 3.22 (Finiteness of the shifted infinite-horizon optimal value function). *Assume that the stochastic optimal control problem (2.13) is strictly stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to some pseudometric d and that $\mathbf{X}^s$ is cheaply reachable on $\mathcal{S}$. Furthermore, assume that the shifted infinite-horizon optimal value function $\hat{V}_\infty$ is continuous at $\mathbf{X}^s$ with respect to the pseudometric d . Then, there exists a constant $C_{V_\infty} > 0$ such that*

$$|\hat{V}_\infty(j, X_j)| \leq C_{V_\infty}$$

holds for all $(j, X_j) \in \mathcal{S}$.

Proof. Let $(j, X_j) \in \mathcal{S}$ and let $\varepsilon > 0$ be given by the infinite-horizon continuity of $\hat{V}_\infty$ at $\mathbf{X}^s$ from Definition 3.21.

By Theorem 3.19 we know that the stochastic optimal control problem (2.13) has the stochastic finite-horizon turnpike property on $\mathcal{S}$ with respect to the pseudometric d . Hence, choose $L \in \mathbb{N}$ such that $\vartheta(L) < \varepsilon$, where $\vartheta \in \mathcal{L}$ is the function from the stochastic turnpike property in Definition 3.12. Then, by the finite-horizon turnpike property, for all $N > L$ there exists an index $M \in \{0, \dots, N\}$ such that

$$X_{\mathbf{U}_N^*}(M; j, X_j) \in \bar{B}_\varepsilon^d(X^s(j+M)).$$

By the continuity of $\hat{V}_\infty$, this implies

$$\begin{aligned} & |\hat{V}_\infty(j+M, X_{\mathbf{U}_N^*}(M; j, X_j)) - \hat{V}_\infty(j+M, X^s(j+M))| \\ & \leq \gamma_{V_\infty}(d(X_{\mathbf{U}_N^*}(M; j, X_j), X^s(j+M))) \\ & < \gamma_{V_\infty}(\varepsilon). \end{aligned}$$

By optimality of $\mathbf{U}_N^*$ and the dynamic programming principle, we obtain

$$\begin{aligned} |\hat{V}_\infty(j, X_j)| &= |\hat{J}_M(X_j, \mathbf{U}_N^*) + \hat{V}_\infty(j+M, X_{\mathbf{U}_N^*}(M; j, X_j))| \\ &\leq |\hat{J}_M(X_j, \mathbf{U}_N^*)| + |\hat{V}_\infty(j+M, X_{\mathbf{U}_N^*}(M; j, X_j)) - \hat{V}_\infty(j+M, X^s(j+M))| \\ &\quad + |\hat{V}_\infty(j+M, X^s(j+M))|. \end{aligned}$$

Note that the dynamic programming principle can be applied since $\hat{V}_N(j, X_j)$ is finite due to the cheap reachability assumption and the boundedness of the stage costs ℓ from below

Moreover, by Lemma 3.20 we can conclude that there exists $C_{V_\infty}^s > 0$ such that $|\hat{V}_\infty(j, X^s(j))| \leq C_{V_\infty}^s$.

Finally, using the finite-horizon DPP and cheap reachability, we obtain along the optimal trajectory

$$|\hat{J}_M(X_j, \mathbf{U}_N^*)| = |\hat{V}_N(j, X_j) - \hat{V}_{N-M}(j+M, X_{\mathbf{U}_N^*}(M; j, X_j))| \leq 2C_V.$$

Collecting all estimates yields

$$|\hat{V}_\infty(j, X_j)| \leq 2C_V + \gamma_{V_\infty}(\varepsilon) + C_{V_\infty}^s =: C_{V_\infty},$$

which proves the claim. $\square$

Remark 3.23. For the conclusion on the finiteness of the shifted optimal value function $\hat{V}(j, X_j)$ in Theorem 3.22, it is not important with respect to which pseudometric d the assumptions are formulated. What is essential, however, is that strict stochastic dissipativity and the continuity property from Definition 3.21 are satisfied with respect to the same pseudometric.

While for finite horizons $N \in \mathbb{N}$ it is clear that an optimal trajectory of the original stochastic optimal control problem also minimizes the shifted cost, since the constant shift $\ell(\mathbf{X}^s, \mathbf{U}^s)$ does not affect optimality, for the infinite-horizon case it is not obvious that an overtaking optimal control sequence of the original problem in the sense of Definition 2.15 also minimizes the shifted cost.

The following lemma shows that, indeed, every overtaking optimal control sequence for problem (2.13) on the infinite horizon also solves the corresponding shifted optimal control problem.

Lemma 3.24. Let the assumptions of Theorem 3.22 hold and consider $(j, X_j) \in \mathcal{S}$. Then, every overtaking optimal control sequence $\mathbf{U}_\infty^* \in \check{\mathcal{U}}_{\text{ad}}^\infty(j, X_j)$ for the stochastic optimal control problem (2.13) also minimizes the shifted infinite-horizon cost, i.e.,

$$\hat{V}_\infty(j, X_j) = \hat{J}_\infty(X_j, \mathbf{U}_\infty^*).$$

Proof. Consider $(j, X_j) \in \mathcal{S}$. Due to the definition of the shifted optimal value function and since $\hat{V}_\infty(j, X_j)$ is finite for all $(j, X_j) \in \mathcal{S}$, we know that for every $\varepsilon > 0$ there exists a control sequence $\hat{\mathbf{U}}_\infty^\varepsilon \in \check{\mathcal{U}}_{\text{ad}}^\infty(j, X_j)$ such that

$$\hat{V}_\infty(j, X_j) = \hat{J}_\infty(X_j, \hat{\mathbf{U}}_\infty^\varepsilon) - \varepsilon$$

holds.

By the definition of overtaking optimality (Definition 2.15), the control sequence $\mathbf{U}_\infty^*$ satisfies

$$\liminf_{K \rightarrow \infty} \sum_{k=0}^{K-1} \left(\ell(X_{\mathbf{U}}(k; j, X_j), \hat{U}_\infty^*(k)) - \ell(X_{\mathbf{U}_\infty^*}(k; j, X_j), U_\infty^*(k)) \right) \geq 0$$

for all admissible $\mathbf{U} \in \check{\mathcal{U}}_{\text{ad}}^\infty(j, X_j)$.

Using the shifted cost from equation (3.22) this yields

$$\limsup_{N \rightarrow \infty} (\hat{J}_N(X_j, \mathbf{U}_\infty^*) - \hat{J}_N(X_j, \hat{\mathbf{U}}_\infty^\varepsilon)) \leq 0.$$

Since

$$\limsup_{N \rightarrow \infty} \hat{J}_N(X_j, \mathbf{U}_\infty^*) - \limsup_{N \rightarrow \infty} \hat{J}_N(X_j, \hat{\mathbf{U}}_\infty^\varepsilon) \leq \limsup_{N \rightarrow \infty} (\hat{J}_N(X_j, \mathbf{U}_\infty^*) - \hat{J}_N(X_j, \hat{\mathbf{U}}_\infty^\varepsilon))$$

we can hence conclude that

$$\hat{J}_\infty(X_j, \mathbf{U}_\infty^*) \leq \hat{J}_\infty(X_j, \hat{\mathbf{U}}_\infty^\varepsilon) = \hat{V}_\infty(j, X_j) + \varepsilon$$

holds.

Since $\hat{V}_\infty(j, X_j) \geq \hat{J}_\infty(X_j, \mathbf{U}_\infty^*)$ holds due to the definition of the shifted optimal value function, this proves the claim by letting $\varepsilon \rightarrow 0$. $\square$

Lemma 3.24 guarantees that, even if we characterize the infinite-horizon optimal control sequence of the original stochastic optimal control problem (2.13) by overtaking optimality, we can still apply the infinite-horizon dynamic programming principle to the shifted optimal control problem using this overtaking optimal control sequence.

Moreover, we can now conclude that the infinite-horizon cheap-reachability condition (3.24) holds for this overtaking optimal control sequence. This, in turn, enables us to derive the turnpike property on the infinite horizon from strict stochastic dissipativity, as shown in the following theorem.

Theorem 3.25. *Let the assumptions of Theorem 3.22 hold. Then, the stochastic optimal control problem (2.13) has the stochastic infinite-horizon turnpike property on $\mathcal{S}$ with respect to the pseudometric d . Moreover, the stochastic turnpike property is uniform on $\mathcal{S}$, if the storage function λ is uniformly bounded on $\mathcal{S}$.*

Proof. By Theorem 3.22, the shifted infinite-horizon optimal value function $\hat{V}_\infty$ is uniformly bounded on $\mathcal{S}$. In particular due to Lemma 3.24, for all $(j, X_j) \in \mathcal{S}$,

$$\hat{V}_\infty(j, X_j) = \limsup_{N \rightarrow \infty} \sum_{k=0}^{N-1} \left(\ell(X_{\mathbf{U}_\infty^*}(k; j, X_j), U_\infty^*(k)) - \ell(\mathbf{X}^s, \mathbf{U}^s) \right) \leq C_{V_\infty}.$$

As already discussed, cf. equation (3.24), this bound can be interpreted as an infinite-horizon cheap reachability condition in the sense of Definition 3.17.

Consequently, the proof follows analogously to the finite-horizon case in Theorem 3.19 by using

$$\vartheta_\infty(L) := \alpha^{-1} \left(\frac{C_\lambda^j + C_\lambda^l + C_{V_\infty}}{L} \right) \in \mathcal{L}, \quad (3.25)$$

instead of $\vartheta \in \mathcal{L}$ and taking the $\limsup$ for $N \rightarrow \infty$ in equation (3.21). $\square$

3.5 Hierarchy of stochastic turnpike and dissipativity notions

Since we have defined stochastic dissipativity and stochastic turnpike properties with respect to general pseudometrics on $\mathcal{R}(\mathcal{X})$, we now investigate the role of the particular choice of pseudometric. In particular, we analyze whether certain pseudometrics are “stronger” or “weaker” than others, thereby inducing a hierarchy among the corresponding notions of stochastic dissipativity and turnpike properties. To formalize this idea, we introduce the following concept of dominance.

Definition 3.26 (Dominance of pseudometrics). *Let d_1, d_2 be pseudometrics on $\mathcal{R}(\mathcal{X})$. We say that d_1 is dominated by d_2 , and write $d_1 \preceq d_2$, if there exists a function $\gamma \in \mathcal{K}_\infty$ such that*

$$d_1(X, Y) \leq \gamma(d_2(X, Y))$$

holds for all $X, Y \in \mathcal{R}(\mathcal{X})$. Furthermore, if $d_2 \preceq d_1$, we say that d_2 is a weaker metric than d_1 , while d_1 is stronger than d_2 .

Using the above notion of dominance, the following theorem shows that strict stochastic dissipativity with respect to a pseudometric d implies strict stochastic dissipativity with respect to any pseudometric that is dominated by d .

Theorem 3.27 (Hierarchy of stochastic dissipativity). *Let d_1, d_2 be pseudometrics on $\mathcal{R}(\mathcal{X})$ such that $d_1 \preceq d_2$. If the stochastic optimal control problem (2.13) is strictly stochastically dissipative at the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to the pseudometric d_2 , then it is also strictly stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to the pseudometric d_1 .*

Proof. By strict stochastic dissipativity with respect to d_2 , there exist a storage function

$$\lambda : \mathbb{N}_0 \times \mathcal{R}(\mathcal{X}) \rightarrow \mathbb{R},$$

uniformly bounded from below, and a function $\alpha \in \mathcal{K}_\infty$ such that

$$\ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(k, X) - \lambda(k+1, f(X, U, W(k))) \geq \alpha(d_2(X, X^s(k))) \quad (3.26)$$

holds for all $k \in \mathbb{N}_0$ and all Markovian controls $U \in \mathbb{U}(X)$. Since $d_1 \preceq d_2$, there exists $\gamma \in \mathcal{K}_\infty$ such that

$$d_1(X, X^s(k)) \leq \gamma(d_2(X, X^s(k))) \quad \text{for all } X \in \mathcal{R}(\mathcal{X}).$$

Defining

$$\tilde{\alpha} := \alpha \circ \gamma^{-1} \in \mathcal{K}_\infty,$$

we obtain for all $X \in \mathcal{R}(\mathcal{X})$

$$\alpha(d_2(X, X^s(k))) \geq \tilde{\alpha}(d_1(X, X^s(k))).$$

Substituting this estimate into (3.26) yields

$$\ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(k, X) - \lambda(k+1, f(X, U, W(k))) \geq \tilde{\alpha}(d_1(X, X^s(k))),$$

which shows that the stochastic optimal control problem is strictly stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to d_1 . $\square$

Analogously to the case of strict stochastic dissipativity, the same dominance implications also hold for the stochastic turnpike properties introduced in Definition 3.12, as shown in the following theorem.

Theorem 3.28 (Hierarchy of stochastic turnpike properties). *Let d_1, d_2 be pseudometrics on $\mathcal{R}(\mathcal{X})$ such that $d_1 \preceq d_2$. Assume that the stochastic optimal control problem (2.13) has the stochastic turnpike property with respect to the pseudometric d_2 locally on $\mathcal{S}$ on finite-horizon (or infinite-horizon, respectively). Then, it also has the stochastic turnpike property with respect to the pseudometric d_1 locally on $\mathcal{S}$ on finite-horizon (or infinite-horizon, respectively).*

Moreover, if the stochastic turnpike property with respect to d_2 is uniform on $\mathcal{S}$, then the stochastic turnpike property with respect to d_1 is uniform on $\mathcal{S}$ as well.

Proof. Since $d_1 \preceq d_2$, there exists a function $\gamma \in \mathcal{K}_\infty$ such that

$$d_1(X, Y) \leq \gamma(d_2(X, Y)) \quad \forall X, Y \in \mathcal{R}(\mathcal{X}).$$

We first consider the finite-horizon case. Fix $(j, X_j) \in \mathcal{S}$ and let $X_{\mathbf{U}_N^*}(\cdot; j, X_j)$ be an optimal trajectory. By the turnpike property with respect to d_2 , there exists a function $\vartheta \in \mathcal{L}$ such that for all $N, L \in \mathbb{N}$ there exists a set

$$\mathcal{Q}(j, X_j, L, N) \subseteq \{0, \dots, N\}, \quad \#\mathcal{Q}(j, X_j, L, N) \leq L,$$

for which

$$d_2(X_{\mathbf{U}_N^*}(k; j, X_j), X^s(j+k)) \leq \vartheta(L)$$

holds for all $k \in \{0, \dots, N\} \setminus \mathcal{Q}(j, X_j, L, N)$.

Using the dominance relation between the pseudometrics, we obtain

$$d_1(X_{\mathbf{U}_N^*}(k; j, X_j), X^s(j+k)) \leq \gamma(\vartheta(L))$$

for all such k . Since $\gamma \circ \vartheta \in \mathcal{L}$, this proves the finite-horizon stochastic turnpike property with respect to d_1 .

The infinite-horizon case follows analogously by replacing the function ϑ with ϑ_∞ and the set $\mathcal{Q}(j, X_j, L, N)$ with $\mathcal{Q}(j, X_j, L, \infty)$.

Finally, if the turnpike property with respect to d_2 is uniform, the function ϑ (respectively ϑ_∞) can be chosen independently of $(j, X_j) \in \mathcal{S}$. Hence, the composed function $\alpha \circ \vartheta$ (respectively $\alpha \circ \vartheta_\infty$) is also independent of (j, X_j) , which proves uniformity for the turnpike property with respect to d_1 . $\square$

Since the concept of dominance between pseudometrics is rather abstract, the following lemma provides an overview of the dominance relations among the pseudometrics introduced in Definition 3.4 and Definition 3.16.

Lemma 3.29. *Consider the metrics from Definition 3.16 and Definition 3.4. Then, for $1 \leq p \leq q \leq \infty$ the following dominance relations hold:*

$$\begin{aligned} (i) \quad d_{L^p} \preceq d_{L^q} & \quad (iii) \quad d_{W^p} \preceq d_{W^q} & (v) \quad d_{LP} \preceq d_{KF} & (vii) \quad d_{M^q} \preceq d_{W^q} \\ (ii) \quad d_{W^q} \preceq d_{L^q} & (iv) \quad d_{KF} \preceq d_{L^q} & (vi) \quad d_{LP} \preceq d_{W^q} \end{aligned}$$

Proof. (i) By Hölder's inequality we can conclude that

$$\mathbb{E}[\|Z\|] = \|Z\|_{L^1} = \|Z \cdot 1\|_{L^1} \leq \|Z\|_{L^r} \|1\|_{L^s} = \|Z\|_{L^r} = \mathbb{E}[\|Z\|^r]^{\frac{1}{r}} \quad (3.27)$$

holds for all $1 \leq r, s \leq \infty$ such that $\frac{1}{r} + \frac{1}{s} = 1$ and $Z \in L^r(\mathcal{X}) \cap L^1(\mathcal{X})$, where $\frac{1}{\infty} := 0$. This particularly implies that $L^r(\mathcal{X}) \subseteq L^1(\mathcal{X})$ and hence,

$$\|Z\|_{L^1} = \infty \quad \Rightarrow \quad \|Z\|_{L^r},$$

showing that the inequality (3.27) also holds for $Z \notin L^1(\mathcal{X})$ with $\infty \leq \infty$. Furthermore, for $Z \in L^1(\mathcal{X})$, $Z \notin L^r(\mathcal{X})$ inequality (3.27) holds trivially since $\|Z\|_{L^1} < \infty$ by definition, showing that the inequality (3.27) holds for all $Z \in \mathcal{R}(\mathcal{X})$.

Hence, setting $Z := \|X - Y\|^p$, $r = \frac{q}{p} \geq 1$, and taking the $(1/p)$ -th power yields

$$\mathbb{E}[\|X - Y\|^{\frac{1}{p}}] \leq \mathbb{E}[\|X - Y\|^{pr}]^{\frac{1}{pr}} = \mathbb{E}[\|X - Y\|^q]^{\frac{1}{q}}, \quad (3.28)$$

which proves $d_{L^p} \preceq d_{L^q}$.

(ii) The dominance $d_{W^q} \preceq d_{L^q}$ follows since

$$d_{W^q}(X, Y) := \inf \{ \|\bar{X} - \bar{Y}\|_{L^q} \mid \bar{X} \sim X, \bar{Y} \sim Y \} \leq \|X - Y\|_{L^q} = d_{L^q}(X, Y)$$

holds for all $X, Y \in \mathcal{R}(\mathcal{X})$.

(iii) Using inequality (3.28), we can directly conclude

$$\begin{aligned} d_{W^p}(X, Y) &:= \inf \{ \|\bar{X} - \bar{Y}\|_{L^p} \mid \bar{X} \sim X, \bar{Y} \sim Y \} \\ &\leq \inf \{ \|\bar{X} - \bar{Y}\|_{L^q} \mid \bar{X} \sim X, \bar{Y} \sim Y \} =: d_{W^q}(X, Y), \end{aligned}$$

for all $X, Y \in \mathcal{R}(\mathcal{X})$, since the infimum is taken over the same set of random variables. This proves the claim.

(iv) Consider $X, Y \in \mathcal{R}(\mathcal{X})$ and define

$$\tilde{\varepsilon} := (\mathbb{E}[\|X - Y\|^q])^{\frac{1}{q+1}}.$$

First assume $d_{L^p}(X, Y) > 0$ implying $\tilde{\varepsilon} > 0$. By Markov's inequality we obtain

$$\mathbb{P}(\|X - Y\| > \tilde{\varepsilon}) \leq \frac{\mathbb{E}[\|X - Y\|^q]}{\tilde{\varepsilon}^q} = \frac{\tilde{\varepsilon}^{q+1}}{\tilde{\varepsilon}^q} = \tilde{\varepsilon}.$$

Hence, $\tilde{\varepsilon}$ is admissible in the definition of the Ky-Fan metric and therefore

$$d_{KF}(X, Y) := \inf \{ \varepsilon > 0 \mid \mathbb{P}(\|X - Y\| > \varepsilon) \leq \varepsilon \} \leq \tilde{\varepsilon}.$$

Since

$$\tilde{\varepsilon} = (\mathbb{E}[\|X - Y\|^q])^{\frac{1}{q+1}} = d_{L^q}(X, Y)^{\frac{q}{q+1}},$$

we obtain

$$d_{KF}(X, Y) \leq \gamma(d_{L^q}(X, Y))$$

with $\gamma(r) := r^{\frac{q}{q+1}} \in \mathcal{K}_\infty$. Now assume $d_{L^q}(X, Y) = 0$. This implies

$$\mathbb{P}(\|X - Y\| > \varepsilon) = 0$$

for all $\varepsilon > 0$. Consequently, $d_{KF}(X, Y) = 0 = \gamma(0)$ holds. Altogether, this proves $d_{KF} \preceq d_{L^q}$.

(v) This claim directly follows due to the definition of the Lévy-Prokhorov as

$$d(X, Y) = d_{LP}(X, Y) := \inf \{ d_{KF}(\bar{X}, \bar{Y}) \mid \bar{X} \sim X, \bar{Y} \sim Y \}.$$

(vi) The claim follows from (iv), since the Lévy-Prokhorov metric is defined as the infimum of the Ky-Fan metric over all random variables with the prescribed distributions, while the Wasserstein distance of order q is defined as the infimum of the L^q -distance over the same class of random variables.

(vii) Since d_{W^q} is a pseudometric we get

$$d_{W^q}(X, 0) \leq d_{W^q}(X, Y) + d_{W^q}(Y, 0), \quad d_{W^q}(Y, 0) \leq d_{W^q}(Y, X) + d_{W^q}(X, 0)$$

by the triangle inequality for all $X, Y \in \mathcal{R}(\mathcal{X})$. Using $d_{W^q}(Y, X) = d_{W^q}(X, Y)$ this yields

$$|d_{W^q}(X, 0) - d_{W^q}(Y, 0)| \leq d_{W^q}(X, Y),$$

which proves the claim since

$$d_{W^q}(Z, 0) = \inf\{\|\bar{Z} - 0\|_{L^q} \mid \bar{Z} \sim Z\} = \|Z\|_{L^q} = \mathbb{E}[\|Z\|^q]^{\frac{1}{q}}$$

holds for all $Z \in \mathcal{R}(\mathcal{X})$. □

As shown in Lemma 3.29, metrics on the space of probability measures are typically dominated by metrics on the space of random variables. The following lemma formalizes this observation by showing that every metric on the space of random variables induces a weaker metric on the distributional level.

Lemma 3.30. *Let d be a metric on the space of random variables $\mathcal{R}(\mathcal{X})$. Then, there exists a metric d_D on the space of probability measures $\mathcal{P}(\mathcal{X})$ such that d_D is dominated by d , i.e., $d_D \preceq d$ holds.*

Proof. Define

$$d_D(X, Y) := \inf\{d(\bar{X}, \bar{Y}) \mid \bar{X} \sim X, \bar{Y} \sim Y\}. \quad (3.29)$$

By construction, d_D inherits symmetry, non-negativity, and the triangle inequality from d . Moreover, $d_D(X, Y) = 0$ whenever $X \sim Y$, and in particular for $X = Y$. Hence, d_D is a pseudometric on $\mathcal{R}(\mathcal{X})$.

To show that d_D induces a metric on $\mathcal{P}(\mathcal{X})$, it remains to verify that $d_D(X, Y) = 0$ implies $X \sim Y$. If $d_D(X, Y) = 0$, then there exist $\bar{X} \sim X$ and $\bar{Y} \sim Y$ such that $d(\bar{X}, \bar{Y}) = 0$. Since d is a metric on $\mathcal{R}(\mathcal{X})$, this implies $\bar{X} = \bar{Y}$, and thus $X \sim Y$. Hence, d_D defines a metric on $\mathcal{P}(\mathcal{X})$.

Finally, d_D is dominated by d , since for all $X, Y \in \mathcal{R}(\mathcal{X})$ we have

$$d_D(X, Y) \leq d(X, Y),$$

which follows directly from the definition of d_D in (3.29). □

A direct consequence of Lemma 3.30 is that dissipativity and turnpike properties on the level of random variables imply the corresponding properties on the distributional level, at least for a suitable metric. The converse, however, does not hold in general, as illustrated by the following example.

Example 3.31. *We consider the stochastic optimal control problem*

$$\begin{aligned} \min_{\mathbf{U} \in \bar{\mathbf{U}}_{\text{ad}}^N(j, X_j)} J_N(X_j, \mathbf{U}) &= \sum_{k=0}^{N-1} d_D(X(k; j, X_j), X_{\text{ref}}) \\ \text{s.t. } X(k+1; j, X_j) &= U_1(k)X(k; j, X_j) + U_2(k)W(k+j), \quad X(0; j, X_j) = X_j, \end{aligned} \quad (3.30)$$

which is a slight modification of [84, Example 3.11]. Here, d_D is an arbitrary metric on the space of probability measures $\mathcal{P}(\mathbb{R}^{n_x})$ and a reference state $X_{\text{ref}} \sim \mathcal{N}(0, s_{\text{ref}}^2)$. Since the costs are stabilizing on $\mathcal{P}(\mathbb{R})$ it is clear that the problem is strictly distributionally dissipative with respect to d_D at $(\mathbf{X}^s, \mathbf{U}^s)$ with $\mathbf{U}^s \equiv (1, 0)$, $\mathbf{X}^s \equiv X_{\text{ref}}$ and storage function $\lambda \equiv 0$.

Now consider an arbitrary metric d on $\mathcal{R}(\mathbb{R})$ and $\tilde{X}_{ref} = -X_{ref} \sim \mathcal{N}(0, s_{ref}^2)$. Then, the pair $(\tilde{\mathbf{U}}^s, \tilde{\mathbf{X}}^s)$ with $\tilde{\mathbf{U}}^s \equiv (1, 0)$, $\tilde{\mathbf{X}}^s \equiv \tilde{X}_{ref}$ is also stationary for problem (3.30). However, since d is a metric on $\mathcal{R}(\mathbb{R})$ and $\tilde{X}_{ref} \neq X_{ref}$ holds almost surely we get

$$d(X^s(k), \tilde{X}^s(k)) = c > 0 \quad \text{for all } k \in \mathbb{N}_0,$$

and hence,

$$d(X^s(k), \tilde{X}^s(k)) \not\rightarrow 0 \quad \text{for } k \rightarrow \infty.$$

Since this contradicts Theorem 3.10, strict dissipativity with respect to a metric d on $\mathcal{R}(\mathbb{R})$ cannot hold.

Now consider the neighborhood $\tilde{\mathcal{S}}$ defined by $\tilde{S}(j) = \{X^s(j)\}$ for all $j \in \mathbb{N}_0$, which is the smallest possible neighborhood of $\mathbf{X}^s$. Then, the problem is trivially cheaply reachable on $\tilde{\mathcal{S}}$, since the stationary control $\mathbf{U}^s$ is optimal for all initial conditions $(j, X^s(j))$ and horizons $N \in \bar{\mathbb{N}}$. Thus, by Theorem 3.19, problem (3.30) has the distributional turnpike property with respect to d_D , which is moreover uniform since $\lambda \equiv 0$ is bounded.

However, since $-X^s(k) \sim X^s(k)$ holds for all $k \in \mathbb{N}_0$, the control

$$\mathbf{U}_{N,j,X^s(j)}^* := ((-1, 0), (1, 0), \dots, (1, 0))$$

is also optimal for all initial conditions $(j, X^s(j))$ and horizons $N \in \bar{\mathbb{N}}$. Nevertheless,

$$d(X^s(k), X_{\mathbf{U}_{N,j,X^s(j)}^*}^s(k; j, X^s(j))) = c > 0 \quad \text{for all } k > 0$$

holds for any metric d on $\mathcal{R}(\mathbb{R})$.

Hence, no local (uniform) turnpike property can hold on the level of random variables for any neighborhood $\mathcal{S}$, since necessarily $\mathcal{S} \supseteq \tilde{\mathcal{S}}$ holds.

This example shows that, without additional assumptions, the layer of random variables does not only imply distributional properties, as established in Lemma 3.30, but is in fact strictly stronger.

In summary, the hierarchy of turnpike and dissipativity notions established in this section is depicted in Figure 3.1.

3.6 Examples

Verifying dissipativity for optimal control problems is usually a hard task even in the deterministic setting. However, in this section, we provide specific examples for which we can establish a form of stochastic dissipativity—and thus also the associated turnpike properties—by explicitly constructing a storage function. In particular, we construct storage functions for a class of generalized stochastic linear-quadratic optimal control problems and for an academic nonlinear stochastic problem.

3.6.1 The generalized stochastic linear-quadratic optimal control problem

Dissipativity and turnpike properties for linear-quadratic optimal control problems have been studied extensively in deterministic settings, cf. [9, 13, 38, 41, 44, 63, 76].

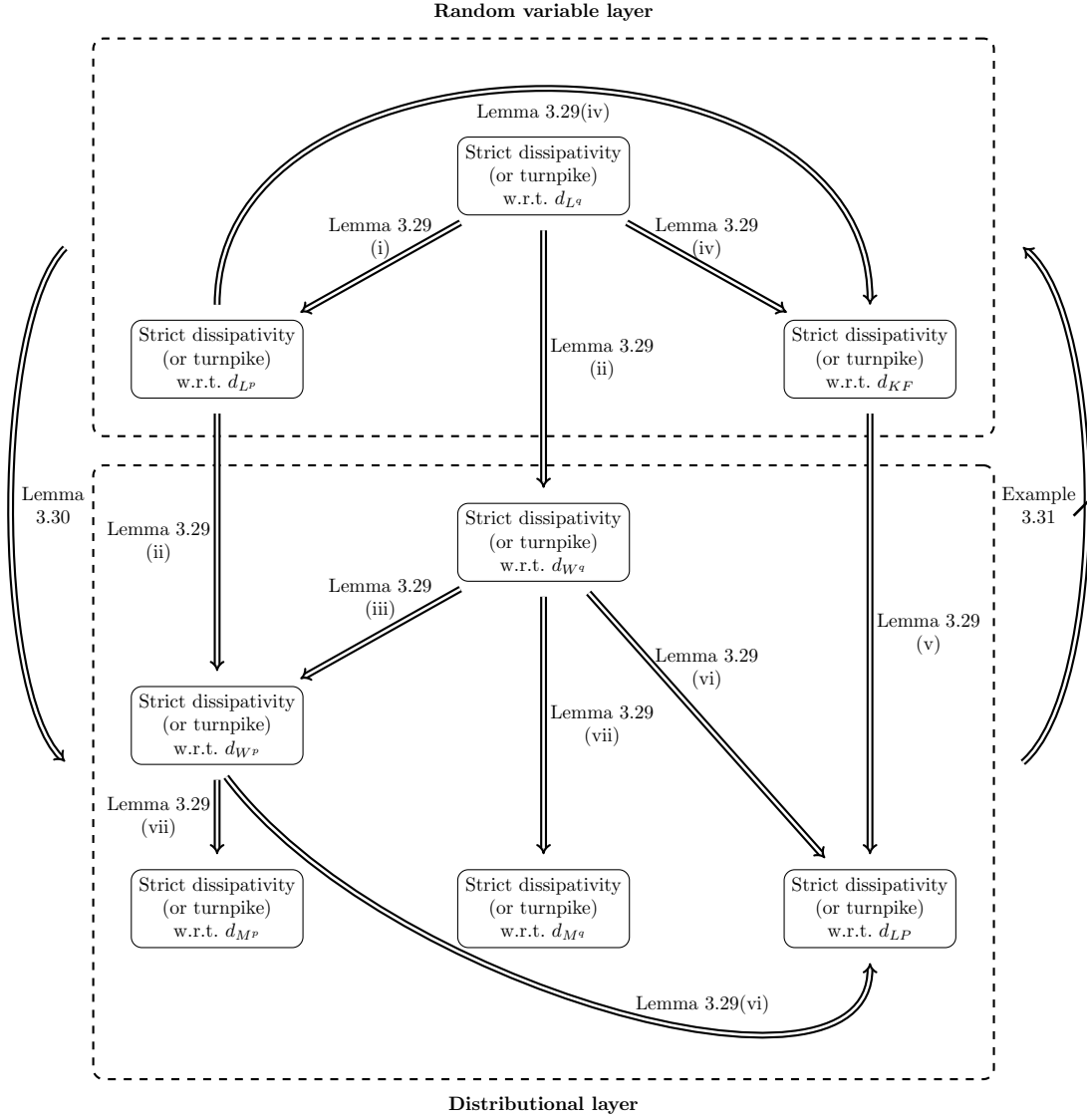


Figure 3.1: Schematic sketch of the strict dissipativity and turnpike hierarchy implied by the dominance relations from Section 3.5

More recently, turnpike results based on optimality conditions have been obtained for stochastic linear-quadratic problems [100, 101] or stochastic linear quadratic mean-field games [15, 69, 102]. In contrast, we establish turnpike properties for stochastic linear-quadratic optimal control problems by means of stochastic dissipativity. In this way, we extend the deterministic results of [41] to the stochastic setting.

Specifically, we consider the following class of generalized discrete-time stochastic linear-quadratic optimal control problems. For matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$ with (A, B) stabilizable, $D \in \mathbb{R}^{n_x \times n_w}$, and a constant shift $z \in \mathbb{R}^{n_x}$, we consider the linear stochastic dynamics

$$X(k+1) = AX(k) + BU(k) + DW(k) + z.$$

Here, we assume that the *i.i.d.* sequence $\mathbf{W}$ has a distribution with finite mean and variance, i.e.,

$$m_W := \mathbb{E}[W(k)] < \infty, \quad \Sigma_W := \text{Cov}(W(k)) < \infty.$$

Furthermore, we consider the stage cost

$$\ell(X, U) = \mathbb{E}[\|X\|_{Q_1}^2 + \|U\|_{R_1}^2 + q^T X + r^T U + c] + \text{Tr}(Q_2 \text{Cov}(X) + R_2 \text{Cov}(U)), \quad (3.31)$$

where $Q_1, Q_2 \in \mathbb{R}^{n_x \times n_x}$, $R_1, R_2 \in \mathbb{R}^{n_u \times n_u}$ with R_1 symmetric and positive definite, and R_2, Q_1, Q_2 symmetric and positive semidefinite, $r \in \mathbb{R}^{n_x}$, $q \in \mathbb{R}^{n_x}$, and $c \in \mathbb{R}$.

Note that, due to equation (2.1), the quadratic terms in the expectation part of the stage cost (3.31) always jointly penalize the quadratic mean and the variance of X and U , whereas the covariance part of the stage cost allows us to model an additional variance penalization that has no impact on the mean.

Then, for a given initial value $X_j \in \mathcal{R}(\mathbb{R}^{n_x})$ at time $j \in \mathbb{N}_0$, the generalized stochastic linear-quadratic optimal control problem on horizon $N \in \mathbb{N}$ reads

$$\begin{aligned} \min_{\mathbf{U} \in \mathbb{U}_{\text{ad}}^N(j, X_j)} J_N(X_j, \mathbf{U}) &= \sum_{k=0}^{N-1} \mathbb{E}[\|X_{\mathbf{U}}(k; j, X_j)\|_{Q_1}^2 + \|U(k)\|_{R_1}^2] \\ &\quad + \mathbb{E}[q^T X_{\mathbf{U}}(k; j, X_j) + r^T U(k) + c] \\ &\quad + \text{Tr}(Q_2 \text{Cov}(X_{\mathbf{U}}(k; j, X_j)) + R_2 \text{Cov}(U(k))) \\ \text{s.t. } X_{\mathbf{U}}(k+1; j, X_j) &= AX_{\mathbf{U}}(k; j, X_j) + BU(k) + DW(k+j) + z, \\ X_{\mathbf{U}}(0; j, X_j) &= X_j. \end{aligned} \quad (3.32)$$

In the following we will show how we can construct a storage function λ for problem (3.32) to show strict dissipativity in the sense of Definition 3.2. This construction is based on reformulations of the stage cost divided into several steps in a series of lemmas, starting with Lemma 3.32.

Lemma 3.32. *Assume that $(x^s, u^s) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ solves the equation*

$$(I - A)x^s - Bu^s = Dm_W + z.$$

Then, the stochastic OCP (3.32) can be reformulated as

$$\begin{aligned}
\min_{\hat{\mathbf{U}} \in \hat{\mathbf{U}}_{\text{ad}}^N(j, \hat{X}_j)} J_N(\hat{X}_j, \hat{\mathbf{U}}) &:= \sum_{k=0}^{N-1} \mathbb{E}[\|\hat{X}_{\hat{\mathbf{U}}}(k; j, \hat{X}_j)\|_{Q_1}^2 + \|\hat{U}(k)\|_{R_1}^2] \\
&\quad + \mathbb{E}[\hat{q}^T \hat{X}_{\hat{\mathbf{U}}}(k; j, \hat{X}_j) + \hat{r}^T \hat{U}(k) + \hat{c}] \\
&\quad + \text{Tr}(Q_2 \text{Cov}(\hat{X}_{\hat{\mathbf{U}}}(k; j, \hat{X}_j)) + R_2 \text{Cov}(\hat{U}(k))) \\
\text{s.t. } \hat{X}_{\hat{\mathbf{U}}}(k+1; j, \hat{X}_j) &= A\hat{X}_{\hat{\mathbf{U}}}(k; j, \hat{X}_j) + B\hat{U}(k) + D\hat{W}(k+j), \\
\hat{X}_{\hat{\mathbf{U}}}(0; j, \hat{X}_j) &= \hat{X}_j := X_j - x^s,
\end{aligned}$$

where $\hat{q} := q + 2Q_1x^s$, $\hat{r} := r + 2R_1u^s$, $\hat{c} := \ell(x^s, u^s)$, $\hat{X}_{\hat{\mathbf{U}}}(k; j, \hat{X}_j) := X_{\mathbf{U}}(k; j, X_j) - x^s$, $\hat{U}(k) := U(k) - u^s$, and $\hat{W}(k) := W(k) - m_W$ for all $k \in \mathbb{N}_0$.

Proof. Assume that $\hat{X}_{\hat{\mathbf{U}}}(k; j, \hat{X}_j) = X_{\mathbf{U}}(k; j, X_j) - x^s$ holds for some $k \in \mathbb{N}_0$. Then, we obtain

$$\begin{aligned}
\hat{X}_{\hat{\mathbf{U}}}(k+1; j, \hat{X}_j) &= A\hat{X}_{\hat{\mathbf{U}}}(k; j, \hat{X}_j) + B\hat{U}(k) + D\hat{W}(k+j) \\
&= A(\hat{X}_{\hat{\mathbf{U}}}(k; j, \hat{X}_j) - x^s) + B(\hat{U}(k) - u^s) + D(W(k+j) - m_W) + z - z \\
&= (AX_{\hat{\mathbf{U}}}(k; j, X_j) + B\hat{U}(k) + DW(k+j) + z) - (Ax^s + Bu^s + Dm_W + z) \\
&= X_{\mathbf{U}}(k+1; j, X_j) - x^s.
\end{aligned}$$

Hence, $\hat{X}_{\hat{\mathbf{U}}}(k; j, \hat{X}_j) = X_{\mathbf{U}}(k; j, X_j) - x^s$ holds for all $k \in \mathbb{N}_0$ by induction, since the equation holds for $k = 0$ due to the definition of the initial condition.

Furthermore, for all $(\hat{X}, \hat{U}) \in \mathcal{R}(\mathbb{R}^{n_x}) \times \mathcal{R}(\mathbb{R}^{n_u})$ it holds that

$$\begin{aligned}
\ell(X, U) &= \mathbb{E}[(\hat{X} + x^s)^T Q_1(\hat{X} + x^s) + (\hat{U} + u^s)^T R_1(\hat{U} + u^s)] \\
&\quad + \mathbb{E}[\hat{q}^T(\hat{X} + x^s) + \hat{r}^T(\hat{U} + u^s) + \hat{c}] \\
&\quad + \text{Tr}(Q_2 \text{Cov}(\hat{X} + x^s) + R_2 \text{Cov}(\hat{U} + u^s)) \\
&= \mathbb{E}[\hat{X}^T Q_1 \hat{X} + \hat{U}^T R_1 \hat{U} + \hat{q}^T \hat{X} + \hat{r}^T \hat{U} + \hat{c}] \\
&\quad + \text{Tr}(Q_2 \text{Cov}(\hat{X}) + R_2 \text{Cov}(\hat{U})),
\end{aligned}$$

which proves the claim. $\square$

Remark 3.33. Note that the assumption of the existence of a solution to the equation $(I - A)x^s - Bu^s = Dm_w + z$ in Lemma 3.32 is needed, since there may be certain choices of $m_w \in \mathbb{R}$ and $z \in \mathbb{R}$ for which there is no solution if we only assume (A, B) to be stabilizable. However, if instead we assume the stronger condition that (A, B) is controllable, then this additional assumption can be omitted.

The contribution of Lemma 3.32 is that we have moved the constant terms $Em_W + z$ from the system dynamics to the stage cost

$$\begin{aligned}
\hat{\ell}(\hat{X}, \hat{U}) &:= \mathbb{E}[\hat{X}^T Q_1 \hat{X} + \hat{U}^T R_1 \hat{U} + \hat{q}^T \hat{X} + \hat{r}^T \hat{U} + \hat{c}] \\
&\quad + \text{Tr}(Q_2 \text{Cov}(\hat{X}) + R_2 \text{Cov}(\hat{U})).
\end{aligned} \tag{3.33}$$

Thus, we only have to handle the constant terms in the stage costs while the system now reads

$$\hat{X}(k+1) = A\hat{X}(k) + B\hat{U}(k) + D\hat{W}(k), \tag{3.34}$$

where $\hat{W}(k)$ has zero mean for all $k \in \mathbb{N}_0$ and is independent of $\hat{X}(k)$ and $\hat{U}(k)$.

For this reason, we first construct a storage function for the stochastic optimal control problem with stage cost $\hat{\ell}(\hat{X}, \hat{U})$ from (3.33) and dynamics (3.34). Then, in Theorem 3.37, we use Lemma 3.32 to conclude that the original problem is strictly L^2 dissipative. The next lemma provides a reformulation of the stage cost $\hat{\ell}(\hat{X}, \hat{U})$, in particular of its quadratic terms.

Note that in the following we can restrict ourselves to random variables $X_k \in L^2(\mathbb{R}^{n_x})$ and $U_k \in L^2(\mathbb{R}^{n_u})$, since the stage cost $\ell(X_k, U_k)$ is infinite otherwise. In this case, the storage function is arbitrary, cf. Remark 3.3. Consequently, it is sufficient to verify the dissipation inequality (3.1) for all $X_k \in L^2(\mathbb{R}^{n_x})$ and $U_k \in L^2(\mathbb{R}^{n_u})$ to conclude stochastic dissipativity.

Lemma 3.34. *Let P^s be the positive semidefinite solution of the discrete-time algebraic Riccati equation*

$$P^s = A^T P A + (Q_1 + Q_2) - A^T P^s B [(R_1 + R_2) + B^T P^s B]^{-1} B^T P^s A, \quad (3.35)$$

and set $K^s := -[(R_1 + R_2) + B^T P^s B]^{-1} B^T P^s A$.

Then, for all $\hat{X}_k \in L^2(\mathbb{R}^{n_x})$, $\hat{U}_k \in L^2(\mathbb{R}^{n_u})$ with $\hat{U}_k \in \check{\mathcal{U}}(\hat{X}_k)$ the identity

$$\begin{aligned} \hat{\ell}(\hat{X}_k, \hat{U}_k) &= \mathbb{E}[\|\hat{U}_k - K \hat{X}_k\|_{\tilde{R}}^2] + \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] + \mathbb{E}[\|\hat{X}_k\|_{P^s}^2] \\ &\quad - \mathbb{E}[\|A\hat{X}_k + B\hat{U}_k + D\hat{W}(k)\|_{P^s}^2] + \mathbb{E}[\hat{q}^T \hat{X}_k + \hat{r}^T \hat{U}_k + \hat{c}] \\ &\quad - \mathbb{E}[\hat{X}_k^T Q_2 \mathbb{E}[\hat{X}_k] - \mathbb{E}[\hat{U}_k]^T R_2 \mathbb{E}[\hat{U}_k]] \end{aligned} \quad (3.36)$$

holds with $\tilde{R} := (R_1 + R_2) + B^T P A$ symmetric and positive definite.

Proof. Using equation (2.1) we can rewrite the stage costs (3.4) as

$$\begin{aligned} \hat{\ell}(\hat{X}_k, \hat{U}_k) &= \mathbb{E}[\hat{X}_k^T Q_1 \hat{X}_k + \hat{U}_k^T R_1 \hat{U}_k + \hat{q}^T \hat{X}_k + \hat{r}^T \hat{U}_k + \hat{c}] + \mathbb{E}[\hat{X}_k^T Q_2 \hat{X}_k] \\ &\quad - \mathbb{E}[\hat{X}_k^T Q_2 \mathbb{E}[\hat{X}_k]] + \mathbb{E}[\hat{U}_k^T R_2 \hat{U}_k] - \mathbb{E}[\hat{U}_k^T R_2 \mathbb{E}[\hat{U}_k]] \\ &= \mathbb{E}[\hat{X}_k^T (Q_1 + Q_2) \hat{X}_k + \hat{U}_k^T (R_1 + R_2) \hat{U}_k] + \mathbb{E}[\hat{q}^T \hat{X}_k + \hat{r}^T \hat{U}_k + \hat{c}] \\ &\quad - \mathbb{E}[\hat{X}_k^T Q_2 \mathbb{E}[\hat{X}_k]] - \mathbb{E}[\hat{U}_k^T R_2 \mathbb{E}[\hat{U}_k]] \end{aligned}$$

for all $\hat{X}_k \in L^2(\mathbb{R}^{n_x})$, $U_k \in L^2(\mathbb{R}^{n_u})$ with $\hat{U}_k \in \check{\mathcal{U}}(\hat{X}_k)$

Furthermore, since P^s is the solution of the algebraic Riccati equation (3.35) it is also a steady-state solution of the Riccati iteration

$$\begin{aligned} P_N(k) &= A^T P_N(k+1) A + (Q_1 + Q_2) - A^T P_N(k+1) B \\ &\quad \cdot [(R_1 + R_2) + B^T P_N(k+1) B]^{-1} B^T P_N(k+1) A \end{aligned}$$

with terminal condition $P_N(N) = P^s$.

Hence, by [6, Chapter 8, Lemma 6.1] we conclude that

$$\begin{aligned} &\mathbb{E}[\hat{X}_k^T (Q_1 + Q_2) \hat{X}_k + \hat{U}_k^T (R_1 + R_2) \hat{U}_k] \\ &= \mathbb{E}[\|\hat{U}_k - K^s \hat{X}_k\|_{\tilde{R}}^2] + \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] \\ &\quad + \mathbb{E}[\|\hat{X}_k\|_{P^s}^2] - \mathbb{E}[\|A\hat{X}_k + B\hat{U}_k + D\hat{W}(k)\|_{P^s}^2], \end{aligned} \quad (3.37)$$

which proves the lemma. $\square$

Equation (3.36) gives us a form of the stage cost that is suitable for the construction of the storage function for our stochastic linear-quadratic problem, starting with the following lemma.

Lemma 3.35. *Assume that $(A, Q_1^{1/2})$ is detectable. Then, there exists an invariant distribution $\hat{\mathbb{P}}_X^s \in \mathcal{P}(\mathbb{R}^{n_x})$ and an initial condition $\hat{X}_0^s \sim \hat{\mathbb{P}}_X^s$ such that the state and control processes defined by*

$$\begin{aligned}\hat{X}^s(k+1) &= A\hat{X}^s(k) + B\hat{U}^s(k) + D\hat{W}(k), \quad \hat{X}^s(0) = \hat{X}_0^s \\ \hat{U}^s(k) &= K^s \hat{X}^s(k)\end{aligned}$$

are a stationary pair according to Definition 2.9 with $\mathbb{E}[\hat{X}^s(k)] = 0$ for all $k \in \mathbb{N}_0$.

Moreover, for all $\hat{X}_k \in L^2(\mathbb{R}^{n_x})$, $\hat{U}_k \in L^2(\mathbb{R}^{n_u})$ with $\hat{U}_k \in \hat{\mathcal{U}}(\hat{X}_k)$ the identity

$$\begin{aligned}\hat{\ell}(\hat{X}_k, \hat{U}_k) - \hat{\ell}(\hat{\mathbf{X}}^s, \hat{\mathbf{U}}^s) + \hat{\lambda}(k, \hat{X}_k) - \hat{\lambda}(k+1, A\hat{X}_k + B\hat{U}_k + D\hat{W}(k)) \\ = \hat{\ell}(\hat{X}_k - \hat{X}^s(k), \hat{U}_k - \hat{U}^s(k)) - \hat{c}\end{aligned}\tag{3.38}$$

holds with

$$\hat{\lambda}(k, \hat{X}_k) := \mathbb{E}[\|\hat{X}_k - \hat{X}^s(k)\|_{P^s}^2 - \|\hat{X}_k\|_{P^s}^2].$$

Proof. Since $(A, Q_1^{1/2})$ is detectable also the pair $(A, (Q_1 + Q_2)^{1/2})$ is detectable since Q_2 is a positive semidefinite matrix. Thus, K^s is a stabilizing feedback and $A + BK^s$ is stable. Hence, since $\mathbb{E}[\hat{W}(k)] = 0$ and $\mathbb{E}[\|\hat{W}(k)\|^2] < \infty$ for all $k \in \mathbb{N}_0$ due to the assumptions on the distribution of $W(k)$, the existence of a stationary distribution $\hat{\mathbb{P}}_X^s$ implying stationarity of $(\hat{\mathbf{X}}^s, \hat{\mathbf{U}}^s)$ follows by [72, Section 10.5.4].

Furthermore, since $\mathbb{E}[\hat{W}(k)] = 0$, it follows directly that $\mathbb{E}[\hat{X}^s(k)] = 0$ for all $k \in \mathbb{N}_0$ since it is the only finite solution of the steady state equation $m_{\hat{X}}^s = (A + BK^s)m_{\hat{X}}^s$ for the expectation $\hat{m}_{\hat{X}}^s$ of $\hat{\mathbf{X}}^s$.

To conclude (3.38), we first observe that

$$\begin{aligned}\mathbb{E}[\|\hat{U}_k - K^s \hat{X}_k\|_{\hat{R}}^2] &= \mathbb{E}[\|\hat{U}_k - K^s \hat{X}^s(k) + K^s \hat{X}^s(k) - K^s \hat{X}_k\|_{\hat{R}}^2] \\ &= \mathbb{E}[\|(\hat{U}_k - \hat{U}^s(k)) - K^s(\hat{X}_k - \hat{X}^s(k))\|_{\hat{R}}^2]\end{aligned}\tag{3.39}$$

holds for all $k \in \mathbb{N}_0$ and $\hat{X}_k \in L^2(\mathbb{R}^{n_x})$, $\hat{U}_k \in L^2(\mathbb{R}^{n_u})$ with $\hat{U}_k \in \hat{\mathcal{U}}(\hat{X}_k)$.

Let us define

$$\begin{aligned}\bar{\ell}(k, \hat{X}_k, \hat{U}_k) &:= \hat{\ell}(\hat{X}_k, \hat{U}_k) - \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] - \mathbb{E}[\|\hat{X}_k\|_{P^s}^2] \\ &\quad + \mathbb{E}[\|A\hat{X}_k + B\hat{U}_k + D\hat{W}(k)\|_{P^s}^2].\end{aligned}$$

Then, since $\hat{X}^s(k)$ has zero mean for all $k \in \mathbb{N}_0$, it follows by Lemma 3.34 and equation (3.39) that

$$\begin{aligned}\bar{\ell}(k, \hat{X}_k, \hat{U}_k) &= \bar{\ell}(k, \hat{X}_k - \hat{X}^s(k), \hat{U}_k - \hat{U}^s(k)) \\ &= \hat{\ell}(\hat{X}_k - \hat{X}^s(k), \hat{U}_k - \hat{U}^s(k)) - \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] - \mathbb{E}[\|\hat{X}_k - \hat{X}^s(k)\|_{P^s}^2] \\ &\quad + \mathbb{E}[\|A(\hat{X}_k - \hat{X}^s(k)) + B(\hat{U}_k - \hat{U}^s(k)) + D\hat{W}(k)\|_{P^s}^2].\end{aligned}\tag{3.40}$$

Moreover, since $\hat{X}^s(k), \hat{U}^s(k)$ as well as $\hat{X}_k, \hat{U}_k$ are independent of $W(k)$, cf. Remark 2.1, we also obtain independence of $\hat{W}(k) = W(k) - m_W$, and thus,

$$\begin{aligned}\mathbb{E}[\|A(\hat{X}_k - \hat{X}^s(k)) + B(\hat{U}_k - \hat{U}^s(k)) + D\hat{W}(k)\|_{P^s}^2] \\ = \mathbb{E}[\|(A\hat{X}_k + B\hat{U}_k) - (A\hat{X}^s(k) + B\hat{U}^s(k))\|_{P^s}^2] + \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] \\ = \mathbb{E}[\|A\hat{X}_k + B\hat{U}_k + D\hat{W}(k) - \hat{X}^s(k+1)\|_{P^s}^2] + \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2].\end{aligned}$$

Hence, after resolving equation (3.40) we get

$$\begin{aligned}
& \hat{\ell}(\hat{X}_k, \hat{U}_k) - \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] + \mathbb{E}[\|\hat{X}_k - \hat{X}^s(k)\|_{P^s}^2] \\
& - \mathbb{E}[\|\hat{X}_k\|_{P^s}^2] - \mathbb{E}[\|A\hat{X}_k + B\hat{U}_k + D\hat{W}(k) - \hat{X}^s(k+1)\|_{P^s}^2] \\
& + \mathbb{E}[\|A\hat{X}_k + B\hat{U}_k + D\hat{W}(k)\|_{P^s}^2] \\
& = \hat{\ell}(\hat{X}_k, \hat{U}_k) - \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] + \hat{\lambda}(k, \hat{X}_k) - \hat{\lambda}(k+1, A\hat{X}_k + B\hat{U}_k + D\hat{W}(k)) \\
& = \hat{\ell}(\hat{X}_k - \hat{X}^s(k), \hat{U}_k - \hat{U}^s(k)).
\end{aligned}$$

Since

$$\begin{aligned}
\hat{\ell}(\hat{X}^s, \hat{U}^s) &= \mathbb{E}[\|\hat{U}^s(k) - K^s \hat{X}^s(k)\|_R^2] + \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] \\
&+ \mathbb{E}[\|\hat{X}^s(k)\|_{P^s}^2] - \mathbb{E}[\|\hat{X}^s(k+1)\|_{P^s}^2] + \hat{c} \\
&= \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] + \hat{c}
\end{aligned} \tag{3.41}$$

holds because $\hat{U}^s(k) = K\hat{X}^s(k)$ and

$$\mathbb{E}[\|\hat{X}^s(k)\|_{P^s}^2] = \mathbb{E}[\|\hat{X}^s(k+1)\|_{P^s}^2]$$

due to the stationarity of $\hat{X}^s$, this shows equation (3.38) and thus, proves the lemma. $\square$

In Lemma 3.35, we showed that the stage cost can be reformulated in terms of shifted arguments by a suitable modification. The next lemma shows how the cost can be modified further to derive a dissipation inequality for the shifted stage cost of the form (3.33).

Lemma 3.36. *Assume that $(A, Q_1^{1/2})$ is detectable. Then, there exists a solution (x_*^s, u_*^s) to $(I - A)x - Bu = 0$, a function $\alpha \in \mathcal{K}_\infty$, a symmetric, positive definite matrix $S \in \mathbb{R}^{n_x \times n_x}$, and a vector $p \in \mathbb{R}^n$, such that for all $\hat{X}_k \in L^2(\mathbb{R}^{n_x})$, $\hat{U}_k \in L^2(\mathbb{R}^{n_u})$ with $\hat{U}_k \in \hat{\mathcal{U}}(\hat{X}_k)$ the inequality*

$$\begin{aligned}
& \hat{\ell}(\hat{X}_k - \hat{X}^s(k), \hat{U}_k - \hat{U}^s(k)) - \hat{\ell}(x_*^s, u_*^s) \\
& + \tilde{\lambda}(k, \hat{X}_k) - \tilde{\lambda}(k+1, A\hat{X}_k + B\hat{U}_k + D\hat{W}(k)) \\
& \geq \alpha \left(\mathbb{E}[\|(\hat{X}_k - \hat{X}^s(k)) - x_*^s\|^2] \right)
\end{aligned} \tag{3.42}$$

holds with

$$\tilde{\lambda}(k, \hat{X}_k) := \mathbb{E}[\|\hat{X}_k - \hat{X}^s(k)\|_S^2 + p^T(\hat{X}_k - \hat{X}^s(k))].$$

Proof. Since $(A, Q_1^{1/2})$ is detectable, we know by [41, Lemma 5.4] that there is $\tilde{S} \in \mathbb{R}^{n_x \times n_x}$ symmetric and positive definite satisfying the matrix inequality

$$Q_1 + \tilde{S} - A^T \tilde{S} A > 0.$$

For a given $\gamma \in (0, 1]$, set $\tilde{S}_\gamma := \gamma \tilde{S}$ and $Q_\gamma := Q_1 + \tilde{S}_\gamma - A^T \tilde{S}_\gamma A$. Then following the calculation of [41, Lemma 4.1] we obtain

$$\begin{aligned}
\hat{\ell}_\gamma(\bar{X}_k, \bar{U}_k) &:= \hat{\ell}(\bar{X}_k, \bar{U}_k) + \mathbb{E}[\|\bar{X}_k\|_{\tilde{S}_\gamma}^2] - \mathbb{E}[\|A\bar{X}_k + B\bar{U}_k\|_{\tilde{S}_\gamma}^2] \\
&= \mathbb{E}[\|(\bar{X}_k, \bar{U}_k)\|_H^2] + \mathbb{E}[\hat{q}^T \bar{X}_k + \hat{r}^T \bar{U}_k + \hat{c}] \\
&\quad + \text{Tr}(Q_2 \text{Cov}(\bar{X}_k) + R_2 \text{Cov}(\bar{U}_k))
\end{aligned} \tag{3.43}$$

with

$$H := \frac{1}{2} \begin{pmatrix} 2Q_\gamma & \gamma G \\ \gamma G & 2R_\gamma \end{pmatrix},$$

$R_\gamma := R_1 - B^T \tilde{S}_\gamma B$ and $G := -A^T \tilde{S}_\gamma B - B^T \tilde{S}_\gamma A$ for all $\bar{X}_k \in L^2(\mathbb{R}^{n_x})$, $\bar{U}_k \in L^2(\mathbb{R}^{n_u})$ with $\bar{U}_k \in \tilde{\mathcal{U}}(\bar{X}_k)$.

Using the Schur complement, we can show that H is positive definite for a sufficient small $\tilde{\gamma} \in (0, 1]$, see [41, Proof of Lemma 4.1]. Thus, we conclude that the function

$$h_{\tilde{\gamma}}(x, u) := (x, u)^T H(x, u) + \hat{q}^T x + \hat{r}^T u + \hat{c}$$

is strictly convex in (x, u) . Then, the optimization problem

$$\min_{(x, u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}} h_{\tilde{\gamma}}(x, u), \quad \text{s.t. } x - Ax - Bu = 0$$

admits a unique global solution (x_*^s, u_*^s) .

Applying [21, Proposition 4.3] we can deduce the existence of a vector $p \in \mathbb{R}^{n_x}$ and a constant $a > 0$ such that

$$h_{\tilde{\gamma}}(x, u) - h_{\tilde{\gamma}}(x_*^s, u_*^s) + p^T x - p^T (Ax + Bu) \geq a \|x - x_*^s\|^2 \quad (3.44)$$

holds for all $(x, u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$.

Additionally, it holds that

$$\begin{aligned} \hat{\ell}_{\tilde{\gamma}}(x_*^s, u_*^s) &= \mathbb{E}[h_{\tilde{\gamma}}(x_*^s, u_*^s)] \\ &= \hat{\ell}(x_*^s, u_*^s) + \mathbb{E}[\|x_*^s\|_{\tilde{S}_{\tilde{\gamma}}}^2] - \mathbb{E}[\|Ax_*^s + Bu_*^s\|_{\tilde{S}_{\tilde{\gamma}}}^2] \\ &= \hat{\ell}(x_*^s, u_*^s) + \mathbb{E}[\|x_*^s\|_{\tilde{S}_{\tilde{\gamma}}}^2] - \mathbb{E}[\|x_*^s\|_{\tilde{S}_{\tilde{\gamma}}}^2] \\ &= \hat{\ell}(x_*^s, u_*^s). \end{aligned} \quad (3.45)$$

Hence, using the equations (3.43), (3.44), and (3.45) and setting $S := \tilde{S}_{\tilde{\gamma}}$ for all $\bar{X}_k \in L^2(\mathbb{R}^{n_x})$, $\bar{U}_k \in L^2(\mathbb{R}^{n_u})$ with $\bar{U}_k \in \tilde{\mathcal{U}}(\bar{X}_k)$ we obtain

$$\begin{aligned} &\hat{\ell}(\bar{X}_k, \bar{U}_k) - \hat{\ell}(x_*^s, u_*^s) + \tilde{\lambda}(\bar{X}_k, \bar{U}_k) - \tilde{\lambda}(A\bar{X}_k + B\bar{U}_k) \\ &= \hat{\ell}_{\tilde{\gamma}}(\bar{X}_k, \bar{U}_k) - \hat{\ell}(x_*^s, u_*^s) + \mathbb{E}[p^T \bar{X}_k] - \mathbb{E}[p^T A\bar{X}_k + B\bar{U}_k] \\ &= \mathbb{E}[h_{\tilde{\gamma}}(\bar{X}_k, \bar{U}_k) - h_{\tilde{\gamma}}(x_*^s, u_*^s) + p^T \bar{X}_k - p^T (A\bar{X}_k + B\bar{U}_k)] \\ &\quad + \text{Tr}(Q_2 \text{Cov}(\bar{X}_k) + R_2 \text{Cov}(\bar{U}_k)) \\ &\geq \mathbb{E}[a \|\bar{X}_k - x_*^s\|^2] + \text{Tr}(Q_2 \text{Cov}(\bar{X}_k) + R_2 \text{Cov}(\bar{U}_k)) \\ &\geq a \mathbb{E}[\|\bar{X}_k - x_*^s\|^2]. \end{aligned}$$

Thus, equation (3.42) with $\alpha(s) := as$ follows by choosing

$$\bar{X}_k = \hat{X}_k - \hat{X}^s(k), \quad \bar{U}_k = \hat{U}_k - \hat{U}^s(k) \quad \hat{X}_k \in L^2(\mathbb{R}^{n_x}), \quad \hat{U}_k \in L^2(\mathbb{R}^{n_u}), \quad \hat{U}_k \in \tilde{\mathcal{U}}(\hat{X}_k)$$

since then

$$\begin{aligned} A\bar{X}_k + B\bar{U}_k &= A(\hat{X}_k - \hat{X}^s(k)) + B(\hat{U}_k - \hat{U}^s(k)) \\ &= A\hat{X}_k + B\hat{U}_k + D\hat{W}(k) - \hat{X}^s(k+1) \end{aligned}$$

holds. □

The following theorem summarizes the results of this section. It shows the exact form of the storage function for the stochastic linear-quadratic OCP and characterizes the corresponding stationary pair.

Theorem 3.37. *Assume that (x^s, u^s) solves the equation $(I - A)x^s - Bx^s = Dm_W + z$ and $(A, Q_1^{1/2})$ is detectable. Then, there exist an invariant distribution $\mathbb{P}_X^s \in \mathcal{P}(\mathbb{R}^{n_x})$ and an initial condition $X_0^s \sim P_X^s X$ such that the state and control processes*

$$\begin{aligned} X^s(k+1) &= AX^s(k) + BU^s(k) + DW(k), \quad X^s(0) = X_0^s \\ U^s(k) &= K^s(X^s(k) - x_*^s - x^s) + u_*^s + u^s \end{aligned}$$

with K^s from Lemma 3.34 define a stationary pair according to Definition 2.9 and the stochastic OCP (3.32) is strictly L^2 dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$.

Moreover, for $\alpha \in \mathcal{K}_\infty$ from Lemma 3.36 the dissipation inequality (3.1) is satisfied with storage function

$$\lambda(k, X_k) := \mathbb{E}[\|X_k - (X^s(k) - x_*^s)\|_{P+S}^2 - \|X_k - x^s\|_P^2 + p^T(X_k - (X^s(k) - x_*^s))] \quad (3.46)$$

for all $X_k \in L^2(\mathbb{R}^{n_x})$, $U_k \in L^2(\mathbb{R}^{n_u})$ with $U_k \in \check{\mathcal{U}}(X_k)$. Here, P is the solution of the algebraic Riccati equation (3.35) and S , q , and (x_*^s, u_*^s) are from Lemma 3.36.

Proof. By Lemma 3.35, we know that the state and control processes

$$\begin{aligned} \hat{X}^s(k+1) &= A\hat{X}^s(k) + B\hat{U}^s(k) + D\hat{W}(k), \\ \hat{U}^s(k) &= K\hat{X}^s(k) \end{aligned}$$

define a stationary pair for the system (3.34) with $\hat{X}^s(k) \sim \hat{\mathbb{P}}_X^s$.

Now let us assume that $X^s(k) = \hat{X}^s(k) + x^s + x_*^s$ for some $k \in \mathbb{N}_0$. Then, we get

$$\begin{aligned} X^s(k+1) &= AX^s(k) + BU^s(k) + DW(k) + z \\ &= A(\hat{X}^s(k) + x^s + x_*^s) + B(K^s(X^s(k) - x_*^s - x^s) + u_*^s + u^s) \\ &\quad + DW(k) + z \\ &= (A + BK^s)\hat{X}^s(k) + (Ax_*^s + Bu_*^s) + DW(k) + Ax^s + Bu^s + z \quad (3.47) \\ &= (A + BK^s)\hat{X}^s(k) + x_*^s + DW(k) + (x^s - Dm_W) \\ &= (A + BK^s)\hat{X}^s(k) + D\hat{W}(k) + x^s + x_*^s \\ &= \hat{X}^s(k+1) + x^s + x_*^s. \end{aligned}$$

Thus, if we set $X_0^s := \hat{X}_0^s + x^s + x_*^s$ as the initial condition, we get $X^s(k) = \hat{X}^s(k) + x^s + x_*^s$ for all $k \in \mathbb{N}_0$ by induction. Since $\hat{X}^s(k) \sim \hat{\mathbb{P}}_X^s$ for all $k \in \mathbb{N}_0$ we can directly conclude that $X^s(k) \sim \mathbb{P}_X^s$ for all $k \in \mathbb{N}_0$ where the distribution $\mathbb{P}_X^s$ (in the sense of a measure) is given by

$$\mathbb{P}_X^s(B) = \hat{\mathbb{P}}_X^s(B - (x^s + x_*^s)), \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^n),$$

with $B + c := \{b + c \mid b \in B\}$ for $c \in \mathbb{R}^n$. Hence, $(\mathbf{X}^s, \mathbf{U}^s)$ defines a stationary pair according to Definition 2.9.

To infer the second statement of the proof, we use Lemma 3.35 and 3.36 to conclude that

$$\begin{aligned}
& \hat{\ell}(\hat{X}_k, \hat{U}_k) - \hat{\ell}(\hat{X}^s(k), \hat{U}^s(k)) - \hat{\ell}(x_*^s, u_*^s) + \hat{c} \\
& + \hat{\lambda}(k, \hat{X}_k) - \hat{\lambda}(k+1, A\hat{X}_k + B\hat{U}_k + D\hat{W}(k)) \\
& + \tilde{\lambda}(k, \hat{X}_k) - \tilde{\lambda}(k+1, A\hat{X}_k + B\hat{U}_k + D\hat{W}(k)) \\
& \geq \alpha \left(\mathbb{E}[\|(\hat{X}_k - \hat{X}^s(k)) - x_*^s\|^2] \right)
\end{aligned} \tag{3.48}$$

holds for all $\hat{X}_k \in \mathcal{R}(\mathbb{R}^{n_x})$ and $\hat{U}_k \in \check{\mathcal{U}}(\hat{X}_k)$.

Furthermore, by Lemma 3.32, the transformations $\hat{X}(k) = X - x^s$, and equation (3.47) we get

$$\begin{aligned}
\ell(X^s(k), U^s(k)) &= \ell(\hat{X}^s(k) + x^s + x_*^s, \hat{U}^s(k) + u^s + u_*^s) \\
&= \hat{\ell}(\hat{X}^s(k) + x_*^s, \hat{U}^s(k) + u_*^s) \\
&= \mathbb{E}[(\hat{X}^s(k) + x_*^s)^T Q_1 (\hat{X}^s(k) + x_*^s)] + \mathbb{E}[(\hat{U}^s(k) + u_*^s)^T R_1 (\hat{U}^s(k) + u_*^s)] \\
&\quad + \mathbb{E}[\hat{q}^T (\hat{X}^s(k) + x_*^s) + \hat{r}^T (\hat{U}^s(k) + u_*^s) + \hat{c}] \\
&\quad + \text{Tr}(Q_2 \text{Cov}(\hat{X}^s(k) + x_*^s) + R_2 \text{Cov}(\hat{U}^s(k) + u_*^s)) \\
&= \hat{\ell}(\hat{X}^s(k), \hat{U}^s(k)) + \hat{\ell}(x_*^s, u_*^s) - \hat{c} + 2\mathbb{E}[\hat{X}^s(k)^T Q_1 x_*^s] + 2\mathbb{E}[\hat{U}^s(k)^T R_1 u_*^s] \\
&= \hat{\ell}(\hat{X}^s(k), \hat{U}^s(k)) + \hat{\ell}(x_*^s, u_*^s) - \hat{c}
\end{aligned} \tag{3.49}$$

since $\hat{X}^s(k)$ and $\hat{U}^s(k)$ have zero mean and x_*^s, u_*^s as deterministic values are stochastically independent of $\hat{X}^s(k)$ and $\hat{U}^s(k)$.

Hence, using the transformation from Lemma 3.32 and setting

$$\lambda(k, X_k) := \hat{\lambda}(k, X_k - x^s) + \tilde{\lambda}(k, X_k - x^s)$$

we obtain by (3.48) and (3.49) that

$$\begin{aligned}
& \ell(X_k, U_k) - \ell(X^s(k), U^s(k)) + \lambda(k, X_k) - \lambda(k+1, AX_k + BU_k + W(k)) \\
& = \hat{\ell}(X_k - x^s, U_k - u^s) - \hat{\ell}(\hat{X}^s(k), \hat{U}^s(k)) - \hat{\ell}(x_*^s, u_*^s) + \hat{c} \\
& \quad + \lambda(k, X_k) - \lambda(k+1, AX_k + BU_k + W(k)) \\
& \geq \alpha \left(\mathbb{E}[\|(X_k(k) - x^s) - \hat{X}^s(k) - x_*^s\|^2] \right) \\
& = \alpha \left(\mathbb{E}\|X_k(k) - X^s(k)\|^2 \right)
\end{aligned}$$

holds for all $X_k \in \mathcal{R}(\mathbb{R}^{n_x})$ and $U_k \in \check{\mathcal{U}}(X_k)$. This proves the theorem, since $\lambda(k, X_k)$ is uniformly bounded from below because of the positive definiteness of S , and thus is a storage function in the sense of Definition 3.2. $\square$

Before illustrating these theoretical findings by numerical simulations, we give two remarks discussing the restrictiveness of the assumptions and the connection to dissipativity for the deterministic counterpart of problem (3.32).

Remark 3.38. *The assumptions of Theorem 3.37 are not restrictive. The non-existence of (x^s, u^s) implies that the expectation of $X(k)$ cannot remain constant in k . Hence, this condition is necessary for the existence of a stationary pair. Moreover, the detectability condition is known to be necessary for strict dissipativity in the deterministic case [41], which is a special case of our stochastic setting, as shown in Remark 3.39.*

Remark 3.39. Consider the stochastic optimal control problem

$$\begin{aligned} \min_{\mathbf{U} \in \mathbb{U}_{\text{ad}}^N(0, X_0)} J_N(X_0, \mathbf{U}) &:= \sum_{k=0}^{N-1} \mathbb{E}[\|X(k)\|_Q^2 + \|U(k)\|_R^2 + r^T X(k) + v^T U(k) + c] \\ \text{s.t. } X(k+1) &= AX(k) + BU(k) + Dm_W + z, \quad X(0) = X_0 \end{aligned}$$

with constant additive disturbance $\mathbf{W} \equiv m_W$. Then, the stochastic OCP is strictly dissipative in L^2 at $(\mathbf{X}^s, \mathbf{U}^s) \equiv (x^s + x_*^s, u^s + u_*^s)$ with storage function

$$\lambda(k, X) = \mathbb{E}[\|X - x^s\|_{P^s+S}^2 - \|X - x^s\|_{P^s}^2 + q^T(X - x^s)] = \mathbb{E}[\|X - x^s\|_S^2 + q^T(X - x^s)]$$

according to Theorem 3.37. Furthermore, if we additionally restrict ourselves to Dirac distributions for $X(k)$ and $U(k)$, i.e., if $X(k) \in \mathbb{R}^{n_x}$ and $U(k) \in \mathbb{R}^{n_u}$ hold for all $k \in \mathbb{N}_0$, then the storage function coincides with that of the deterministic generalized linear-quadratic optimal control problem, see [41] for the special case $m_W = z = 0$.

Remark 3.39 in particular shows that our stochastic dissipativity results are consistent with the deterministic theory. Furthermore, this observation implies that the expectations of the stationary processes, $\mathbb{E}[X^s(k)] = x^s + x_*^s$ and $\mathbb{E}[U^s(k)] = u^s + u_*^s$, form an optimal equilibrium of the deterministic optimal control problem, i.e., they constitute a solution of

$$\min_{x, u} \ell(x, u) \quad \text{s.t. } x = Ax + Bu + Dm_W + z \quad (3.50)$$

by dissipativity arguments.

Until now, we have shown that the generalized linear-quadratic stochastic optimal control problem is L^2 -dissipative. Using the results from Section 3.4 and Section 3.5, we can conclude that this problem exhibits several stochastic turnpike properties, both on the level of random variables and of distributions, if the problem is cheaply reachable. This is shown by the following lemma.

Lemma 3.40. Consider the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ from Theorem 3.37 and assume that $(A, Q_1^{1/2})$ is detectable. Then, for every constant

$$C > \mathbb{E}[\|X^s(k)\|^2],$$

the stationary process $\mathbf{X}^s$ is cheaply reachable for problem (3.32) on the neighborhood $\mathcal{S}$ defined by

$$\mathcal{S}(j) := \{X_j \in \mathcal{R}(\mathbb{R}^{n_x}) \mid \mathbb{E}[\|X_j\|^2] < C\}, \quad j \in \mathbb{N}_0.$$

Proof. Let $C > \mathbb{E}[\|X^s(k)\|^2]$ and consider an arbitrary initial value

$$X_j \in \mathcal{S}(j) = \{X \in \mathcal{R}(\mathbb{R}^{n_x}) \mid \mathbb{E}[\|X\|^2] < C\} \subseteq L^2(\mathbb{R}^{n_x}).$$

Furthermore, consider the linear feedback law

$$U(k) = K^s(X_U(k; j, X_j) - x^s - x_*^s) + u^s + u_*^s, \quad (3.51)$$

where K^s , x^s , u^s , x_*^s , and u_*^s are taken from Theorem 3.37.

Define the shifted variables

$$\begin{aligned}\tilde{X}(k) &:= X_{\mathbf{U}}(k; j, X_j) - (x^s + x_*^s), \\ \tilde{U}(k) &:= U(k) - (u^s + u_*^s), \\ \hat{W}(k) &:= W(k) - m_W.\end{aligned}$$

Then, arguing as in Lemma 3.32, the shifted dynamics satisfy

$$\tilde{X}(k+1) = (A + BK^s)\tilde{X}(k) + D\hat{W}(k), \quad \tilde{X}(0) = X_j - (x^s + x_*^s).$$

Using the shifted variables and the definitions

$$\begin{aligned}\tilde{q} &:= q + 2Q(x^s + x_*^s), \\ \tilde{r} &:= r + 2R(u^s + u_*^s), \\ \tilde{c} &:= \ell(x^s + x_*^s, u^s + u_*^s),\end{aligned}$$

we obtain

$$\begin{aligned}\ell(X_{\mathbf{U}}(k; j, X_j), U(k)) &= \mathbb{E}[\|\tilde{X}(k)\|_{Q+K^s R K^s}^2] \\ &\quad + (\tilde{q}^T + \tilde{r}^T K^s) \mathbb{E}[\tilde{X}(k)] + \tilde{c} - \|\mathbb{E}[\tilde{X}(k)]\|_{Q_2+K^s R_2 K^s}^2.\end{aligned}\tag{3.52}$$

Define

$$F(X) := \mathbb{E}[\|X\|_{P^s}^2 + p^{sT} X] + \|\mathbb{E}[X]\|_{P_2^s}^2,$$

where P^s is the solution of the algebraic Riccati equation (3.35), P_2^s solves the Lyapunov equation

$$(A + BK^s)^T P_2^s (A + BK^s) - P_2^s = (Q_2 + K^{sT} R_2 K^s),\tag{3.53}$$

and p^s solves

$$\begin{aligned}((A + BK^s) - I)^T p^s + 2((A + BK^s)^T P^s - P^s)(x^s + x_*^s) \\ = -(\tilde{q} + K^{sT} \tilde{r}).\end{aligned}$$

Since P^s solves the Riccati equation, we obtain

$$(A + BK^s)^T P^s (A + BK^s) - P^s = -(Q + K^{sT} R K^s).\tag{3.54}$$

Moreover, $\hat{W}(k)$ has zero mean and is independent of $\tilde{X}(k)$. Hence,

$$\begin{aligned}F(X_{\mathbf{U}}(k+1; j, X_j)) - F(X_{\mathbf{U}}(k; j, X_j)) \\ = -\mathbb{E}[\|\tilde{X}(k)\|_{Q+K^s R K^s}^2] - (\tilde{q} + K^{sT} \tilde{r})^T \mathbb{E}[\tilde{X}(k)] \\ + \|\mathbb{E}[\tilde{X}(k)]\|_{Q_2+K^s R_2 K^s}^2 + \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2].\end{aligned}$$

Combining this identity with (3.52) yields

$$\begin{aligned}F(X_{\mathbf{U}}(k+1; j, X_j)) - F(X_{\mathbf{U}}(k; j, X_j)) \\ = -\ell(X_{\mathbf{U}}(k; j, X_j), U(k)) + \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] + \tilde{c}.\end{aligned}$$

Since $\tilde{c} = \ell(x^s + x_*^s, u^s + u_*^s) = \hat{\ell}(x^s, u^s)$ holds for $\hat{\ell}$ from (3.33) by (3.41) and (3.49), we obtain

$$\ell(\mathbf{X}^s, \mathbf{U}^s) = \mathbb{E}[\|D\hat{W}(k)\|_{P^s}^2] + \tilde{c}.$$

Therefore,

$$\ell(X_{\mathbf{U}}(k; j, X_j), U(k)) - \ell(\mathbf{X}^s, \mathbf{U}^s) = F(X_{\mathbf{U}}(k; j, X_j)) - F(X_{\mathbf{U}}(k+1; j, X_j)).$$

Summation from $k = 0$ to $N - 1$ gives the telescoping estimate

$$\sum_{k=0}^{N-1} (\ell(X_{\mathbf{U}}(k; j, X_j), U(k)) - \ell(\mathbf{X}^s, \mathbf{U}^s)) = F(X_j) - F(X_{\mathbf{U}}(N; j, X_j)).$$

Combining equation (3.53) and equation (3.54) and using $Q = Q_1 + Q_2$, $R = R_1 + R_2$, we obtain that $\tilde{P}^s := P^s + P_2^s$ solves the Lyapunov equation

$$\begin{aligned} (A + BK^s)^T \tilde{P}^s (A + BK^s) - \tilde{P}^s &= - (Q + K^{sT} R K^s) + (Q_2 + K^{sT} R_2 K^s) \\ &= - (Q_1 + K^{sT} R_1 K^s). \end{aligned}$$

Since $(A, Q_1^{1/2})$ is detectable the matrix $A + BK^s$ is stable and thus, the matrix $\tilde{P}^s$ is positive definite due to the positive definiteness of R_1 .

Consequently, there exists a constant $C_F \geq 0$ such that

$$F(X) \geq \|\mathbb{E}[X]\|_{\tilde{P}^s}^2 + p^{sT} \mathbb{E}[X] \geq -C_F \quad \text{for all } X \in L^2(\mathbb{R}^{n_x}).$$

Moreover, since $X_j \in L^2(\mathbb{R}^{n_x})$ and $W(k)$ has finite second moments, we obtain

$$X_{\mathbf{U}}(k; j, X_j) \in L^2(\mathbb{R}^{n_x}) \quad \text{for all } k \in \mathbb{N}_0.$$

Using $\mathbb{E}[\|X_j\|^2] < C$ and Cauchy–Schwarz, we estimate

$$\begin{aligned} F(X_j) &\leq \varrho(P^s) \mathbb{E}[\|X_j\|^2] + \|p^s\| \sqrt{\mathbb{E}[\|X_j\|^2]} \\ &\leq \varrho(P^s) C + \|p^s\| \sqrt{C}, \end{aligned}$$

where $\varrho(P^s)$ denotes the spectral radius of P^s .

Therefore,

$$\begin{aligned} V_N(X_j) - N\ell(\mathbf{X}^s, \mathbf{U}^s) &\leq \sum_{k=0}^{N-1} (\ell(X_{\mathbf{U}}(k; j, X_j), U(k)) - \ell(\mathbf{X}^s, \mathbf{U}^s)) \\ &\leq \varrho(P^s) C + \|p^s\| \sqrt{C} + C_F, \end{aligned}$$

which proves cheap reachability on $\mathcal{S}$. $\square$

To illustrate the different turnpike properties from Definition 3.16, we use the following concrete example, which is a modified version of [77, Section 3.3].

Example 3.41. *We consider the stochastic optimal control problem*

$$\begin{aligned} \min_{\mathbf{U}} \sum_{k=0}^{N-1} &\mathbb{E}[\|X_{\mathbf{U}}(k; 0, X_0)\|_{Q_1} + U(k)^2 + q^T X_{\mathbf{U}}(k; 0, X_0) - 0.5U(k)] \\ &+ \text{Tr}(Q_2 \text{Cov}(X_{\mathbf{U}}(k; j, X_j))) \\ \text{s.t. } X_{\mathbf{U}}(k+1; 0, X_0) &= \begin{pmatrix} 1.12 & 0 \\ 0.26 & 0.88 \end{pmatrix} X_{\mathbf{U}}(k; j, X_j) + \begin{pmatrix} 0.05 \\ -0.05 \end{pmatrix} U(k) + W(k), \\ X_{\mathbf{U}}(0; 0, X_0) &= X_0 \sim \mathcal{N}((0.5, 0.8)^T, \text{diag}(0.5^2, 0.8^2)) \end{aligned} \tag{3.55}$$

with $Q_1 = \text{diag}(1, 5)$, $Q_2 = \text{diag}(5, 0)$, and $q = (1, 0)^T$.

In the following, we consider two settings: Gaussian distributed noise

$$W(k) \sim \mathcal{N}((0.2, 0.2)^T, \text{diag}(0.03, 0.03)) \quad (3.56)$$

and gamma distributed noise

$$W(k) = (W_1(k), W_2(k)) \quad W_i(k) \sim \Gamma(1.33, 0.15), \quad i = 1, 2, \quad (3.57)$$

with $\text{Cov}(W_1(k), W_2(k)) = 0$.

We assume the Gaussian distributed setting from (3.56). Then, the corresponding stationary pair from Theorem 3.37 is given by

$$\begin{aligned} X^s(k+1) &= \begin{pmatrix} 1.12 & 0 \\ 0.26 & 0.88 \end{pmatrix} X^s(k) + \begin{pmatrix} 0.05 \\ -0.05 \end{pmatrix} U^s(k) + W(k) \\ U^s(k) &= K^s(-(-1.116, -0.199)^T) - 1.323 \end{aligned}$$

with $K^s = -(7.679, -0.388)$ and initial distribution $X^s(0) \sim \mathcal{N}(m^s, \Sigma^s)$ where

$$m^s = (-1.116, -0.199)^T, \text{ and } \Sigma^s = \begin{pmatrix} 0.063 & 0.054 \\ 0.054 & 0.619 \end{pmatrix} \quad (3.58)$$

(all numbers are rounded to four digits).

Here we have obtained K^s by solving the Riccati equation (3.35), and the expectation m^s as well as the linear shift in $U^s(k)$ are determined by solving the deterministic steady state problem (3.50). The covariance Σ^s is then computed by solving the Lyapunov equation $\Sigma^s = (A+BK)\Sigma^s(A+BK)^T + \Sigma_W$, cf. [87, Lemma 3]. Note that, in general, this approach is not sufficient to fully characterize the stationary distribution. However, since we consider a purely Gaussian setting here, the stationary distribution is Gaussian due to the linearity of the system and is thus fully characterized by its first two moments.

To visualize the different turnpike phenomena, we solve the optimal control problem from Example 3.41 over different optimization horizons $N = 20, 30, \dots, 80$. To this end, we use Polynomial Chaos Expansions (PCE) and its *Julia* implementation *PolyChaos.jl* [74] to obtain numerical solutions. For an introduction to PCE, we refer to [99], and for more details on its use in our simulations to [77].

Figure 3.2 shows the L^2 distance between the stationary process $\mathbf{X}^s$ and the solutions of the optimal control problem (3.55), where we can clearly observe the L^2 turnpike property implied by L^2 dissipativity. Here, we approximated the L^2 distance via Monte-Carlo simulations

$$\mathbb{E}[\|X_{\mathbf{U}^*}(k; 0, X_0) - X^s(k)\|^2] \approx \frac{1}{n} \sum_{j=1}^n \|X_U(k; 0, X_0)(\omega_j) - X^s(k)(\omega_j)\|^2,$$

where the PCE solutions were sampled for $n = 10^4$ noise realizations

$$(W(0)(\omega_j), \dots, W(N-1)(\omega_j)) = (w_j(0), \dots, w_j(N-1)), \quad j = 1, \dots, n$$

and with stochastically independent initial values X_0 and $X^s(0)$.

To illustrate the pathwise-in-probability turnpike property of the solutions implied by L^2 dissipativity, we consider two fixed but different realization paths of

$$(W(0)(\omega_i), \dots, W(N-1)(\omega_i)) = (w_i(0), \dots, w_i(N-1)), \quad i = 1, 2.$$

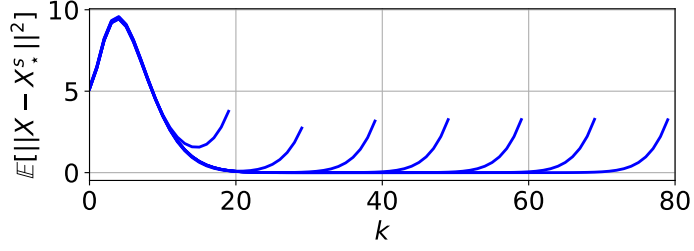


Figure 3.2: L^2 distance between the stationary pair and the optimal solutions of problem (3.55) on different horizons.

We simulate the states and controls as well as the stationary pair according to the dynamics of problem (3.55) subject to these fixed disturbance realizations and random initial values following the desired distributions. Figure 3.3 shows the resulting turnpike behavior of the realizations. Notice that Figure 3.3 is not a direct visualization of the pathwise-in-probability turnpike property, since a single realization is a null set in the probabilistic sense. However, it illustrates the pathwise behavior and thus supports that the pathwise turnpike property holds.

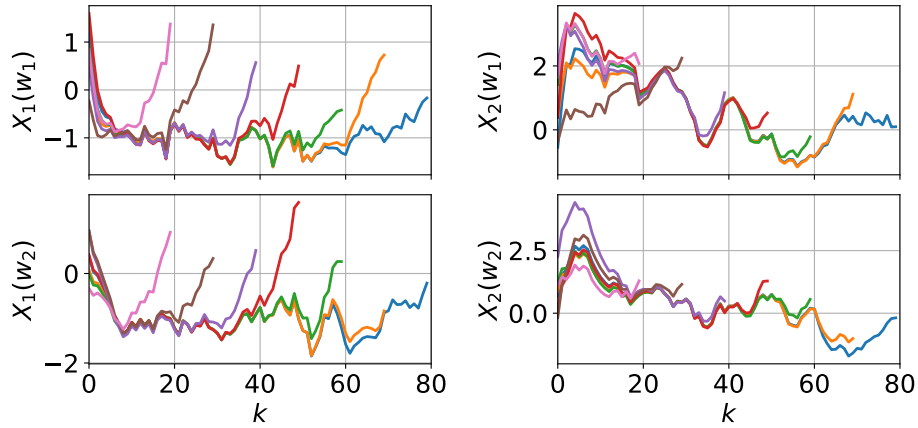


Figure 3.3: Realizations of optimal state trajectories of problem (3.55) on different horizons with identical Gaussian disturbance realization in each row.

Since, for Gaussian distributions, the exact calculation of the Wasserstein distance of order 2 is possible, we also plot this distance between the solutions of the stochastic optimal control problem (3.55) and the stationary process $\mathbf{X}^s$. The results are shown in Figure 3.4. Again, we can observe that the system exhibits the distributional turnpike property with respect to the Wasserstein distance, as implied by the L^2 dissipativity.

As we have shown in Lemma 3.29, the distributional turnpike property with respect to the Wasserstein distance allows us to make additional statements concerning the first two moments of the solutions via the absolute moments metric from (3.6). Figure 3.5 depicts the evolution of the expectation and the variance of the solutions for different horizon lengths together with the corresponding moments of the stationary distribution. Once more, we can observe that the trajectories spend most of their time near the stationary solution, which visualizes the turnpike property for moments.

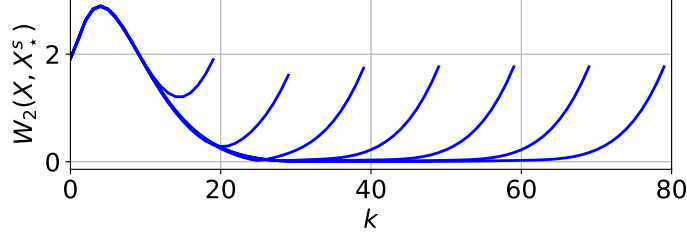


Figure 3.4: Wasserstein distance of order 2 between the stationary distribution and the distribution of the optimal trajectories of problem (3.55) on different horizons N .

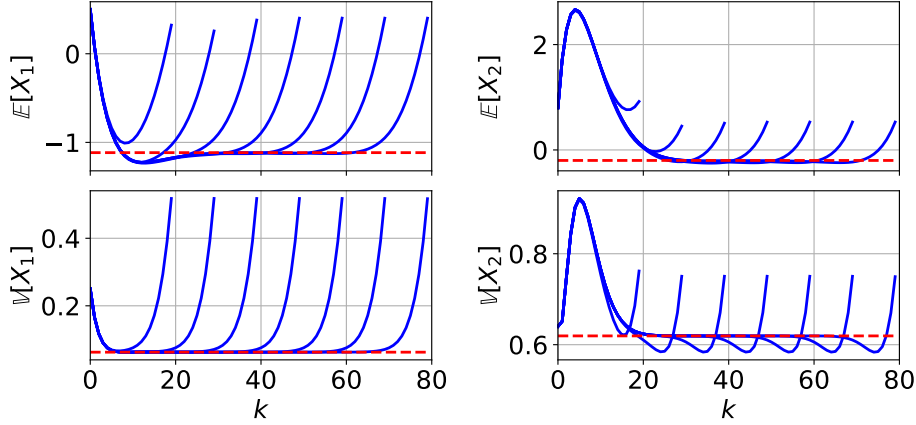


Figure 3.5: Evolution of the expectations and variances of the optimal trajectories of problem (3.55) on different horizons N (solid blue) together with the corresponding moment of the stationary distribution (dashed red).

Next, we turn to the case of non-Gaussian disturbances to demonstrate that our theoretical results remain applicable. Specifically, we consider the Gamma-distributed disturbance from (3.57).

First, we plot solutions for two fixed realizations of the gamma-distributed noise, see Figure 3.6. Once again, we observe the pathwise turnpike property analogous to Figure 3.3, which illustrates that the pathwise-in-probability turnpike property also holds for non-Gaussian distributions.

Unfortunately, in this example, the numerical validation of the L^2 turnpike property and the distributional behavior is quite challenging. This is due to the difficulty of computing the distributions of the stationary pair, since these distributions are no longer fully characterized by their first two moments, as in the Gaussian setting of the previous example. Thus, we would have to solve the problem (3.12) explicitly, which is in general very hard, since we have to perform a minimization over the infinite-dimensional spaces of probability measures and feedbacks.

Hence, we do not attempt to plot the counterparts of the quantities shown in Figure 3.2 and Figure 3.4 in the non-Gaussian setting. However, via PCE we can still calculate the first two moments of the stationary distribution.

Note that these stationary moments are the same as in equation (3.58), since our gamma-distributed noise has the same expectation and variance as the Gaussian noise before, and the evolution of the moments depends only on the moments of the

noise and not on its exact distribution. Thus, we obtain the same results as shown in Figure 3.5.

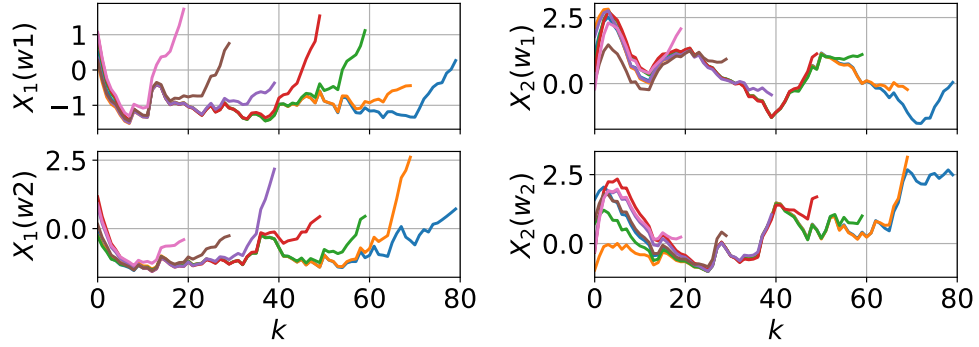


Figure 3.6: Realizations of optimal state trajectories of problem (3.55) on different horizons with identical gamma-distributed disturbance realization in each row.

In order to provide an alternative illustration of the distributional turnpike property without knowledge of the stationary distribution, Figure 3.7 shows the evolution of the probability density functions (PDFs) defining the marginal distributions of the components of solution on horizon $N = 80$. We observe that the PDFs are almost constant in the middle of the time horizon, which is a clear indicator of a turnpike property in distribution. Moreover, we observe the asymmetric nature of the PDF of the first component.

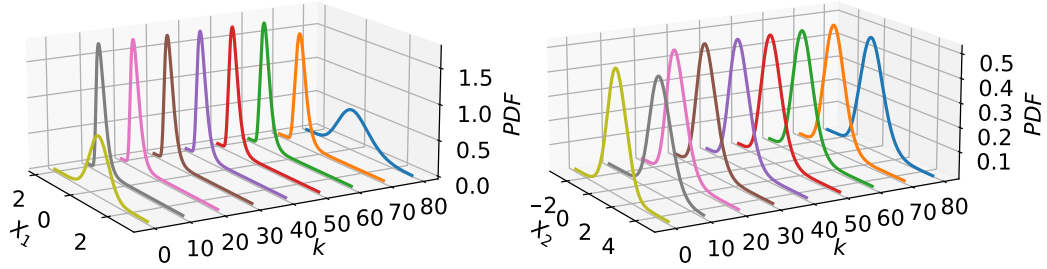


Figure 3.7: Evolution of the PDFs defining the marginal distributions of the optimal trajectory of problem (3.55) on horizon $N = 80$ for gamma-distributed disturbances.

Additionally, Figure 3.8 compares the PDFs in the Gaussian and gamma-distributed setting for the solutions on horizon $N = 80$ at time $k = 40$. Given that this is the middle of the time horizon, it can be reasonably assumed that these PDFs are a good approximation of the stationary distributions according to the definition of the turnpike property. As we can see, in the gamma-distributed setting the stationary distribution is asymmetric and differs significantly from the Gaussian one. Thus, we observe that the explicit behavior of the solutions depends on the specific distribution of the noise, although the proofs of this section do not take it directly into account.

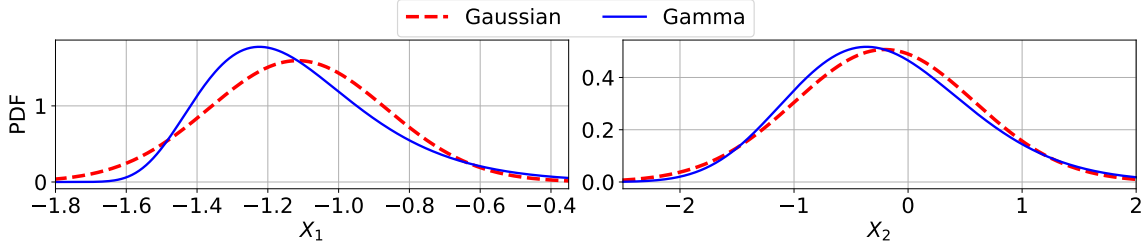


Figure 3.8: Comparison of the PDFs defining the marginal distributions of the optimal trajectory of problem (3.55) on horizon $N = 80$ at time $k = 40$ for Gaussian and gamma-distributed disturbances.

3.6.2 A nonlinear example

To show that the stochastic dissipativity theory introduced in this chapter is not limited to linear dynamics, we consider the nonlinear dynamics

$$X(k+1) = (U(k) - X(k))^2 + W(k)$$

and the stage cost

$$\ell(X, U) = \mathbb{E}[(U - X)^4] + \gamma \mathbb{E}[U]^2,$$

where $\gamma \geq 0$ is a regularization parameter. Furthermore, we assume that the *i.i.d.* sequence $\mathbf{W}$ has zero mean and finite fourth moment.

Then, the nonlinear stochastic optimal control problem on horizon $N \in \bar{\mathbb{N}}$ with initial value X_j at time $j \in \mathbb{N}_0$ reads

$$\begin{aligned} \min_{\mathbf{U}} J_N(X_j, \mathbf{U}) &= \sum_{k=0}^{N-1} \mathbb{E}[(U(k) - X_{\mathbf{U}}(k; j, X_j))^4] + \gamma \mathbb{E}[U(k)]^2 \\ \text{s.t. } X_{\mathbf{U}}(k+1; j, X_j) &= (U(k) - X_{\mathbf{U}}(k; j, X_j))^2 + W(k+j), \\ X_{\mathbf{U}}(0; j, X_j) &= X_j. \end{aligned}$$

It is easy to observe that the pair $(\mathbf{X}^s, \mathbf{U}^s)$ satisfying

$$\begin{aligned} X^s(k+1) &= (U^s(k) - X^s(k))^2 + W(k), \\ U^s(k) &= X^s(k), \quad X^s(0) = X_0^s \sim W(0) \end{aligned} \tag{3.59}$$

defines a stationary pair with $X^s(k+1) = W(k)$ and hence $X^s(k) \sim W(k)$ for all $k \in \mathbb{N}_0$.

Furthermore, for the stationary pair (3.59) we obtain

$$\ell(\mathbf{X}^s, \mathbf{U}^s) = \mathbb{E}[(U^s(k) - X^s(k))^4] + \gamma \mathbb{E}[U^s(k)]^2 = \gamma \mathbb{E}[W(k)]^2 = 0.$$

Thus, for

$$\lambda(k, X_k) := \mathbb{E}[(X_k - X^s(k))^2]$$

we get

$$\begin{aligned} &\ell(X_k, U_k) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(k, X_k) - \lambda(k+1, (U_k - X_k)^2 + W(k)) \\ &= \mathbb{E}[(U_k - X_k)^4] + \gamma \mathbb{E}[U_k]^2 + \mathbb{E}[(X_k - X^s(k))^2] \\ &\quad - \mathbb{E}[(U_k - X_k)^2 + W(k) - X^s(k+1)]^2 \\ &= \mathbb{E}[(U_k - X_k)^4] + \gamma \mathbb{E}[U_k]^2 + \mathbb{E}[(X_k - X^s(k))^2] - \mathbb{E}[(U_k - X_k)^2]^2 \\ &= \gamma \mathbb{E}[U_k]^2 + \mathbb{E}[(X_k - X^s(k))^2] \\ &\geq \mathbb{E}[(X_k - X^s(k))^2] \end{aligned}$$

for all $(X_k, U_k) \in \mathcal{R}(\mathbb{R}) \times \mathcal{R}(\mathbb{R})$, showing that the problem is stochastically L^2 dissipative.

Furthermore, for all $X_j \in \mathcal{R}(\mathbb{R})$ with $\mathbb{E}[|X_j|] \leq C$ we get

$$V_N(j, X_j) \leq \ell(X_j, X_j) + \sum_{k=0}^{N-1} \ell(\mathbf{X}^s, \mathbf{U}^s) = \gamma \mathbb{E}[X_j]^2 \leq \gamma C^2.$$

Hence, for all

$$C > 0 = \mathbb{E}[|X^s(k)|] = \mathbb{E}[|W(k)|]$$

the problem is cheaply reachable on the neighborhood $\mathcal{S}$, where

$$\mathcal{S}(j) := \{X \in \mathcal{R}(\mathbb{R}) \mid \mathbb{E}[|X|] \leq C\}, \quad \text{for all } j \in \mathbb{N}_0.$$

It should be noted that, in the case $\gamma = 0$, strict L^2 -dissipativity also holds for arbitrary disturbance distributions. However, in this situation the optimal solutions approach the stationary process after a single time step, resulting in rather trivial system behavior.

To illustrate the stochastic turnpike phenomena as in the previous section, we consider the following concrete example.

Example 3.42. *We consider the stochastic optimal control problem*

$$\begin{aligned} \min_{\mathbf{U}} J_N(X_j, \mathbf{U}) &= \sum_{k=0}^{N-1} \mathbb{E}[(U(k) - X_{\mathbf{U}}(k; 0, X_0))^4] + 50\mathbb{E}[U(k)]^2 \\ \text{s.t. } X_{\mathbf{U}}(k+1; 0, X_0) &= (U(k) - X_{\mathbf{U}}(k; 0, X_0))^2 + W(k), \\ X_{\mathbf{U}}(0; 0, X_0) &= 5.5 \end{aligned} \tag{3.60}$$

with disturbance

$$W(k)(\omega) = \begin{cases} 0.5 & \text{with probability } 0.2, \\ -0.125 & \text{with probability } 0.8 \end{cases}, \quad \text{for all } k \in \mathbb{N}_0.$$

To obtain the optimal solutions for Example 3.42, we solved the optimal control problem (3.60) on horizons $N = 3, 5, 7, 9, 11$ by direct minimization over all realization paths, which is feasible due to the finite support of the disturbance, specifically a two-point distribution. The implementation was carried out in *Python* using *CasADi* [4] and the nonlinear solver *IPOPT* [104].

Figure 3.6.2 shows three realizations of the optimal trajectories for the different prediction horizons together with the corresponding realization of the optimal stationary process. We observe that increasing the horizon length leads to trajectories that are increasingly close to the stationary process. In particular, for $N = 9$ and $N = 11$, the stationary process is closely approximated at the end of the horizon. This suggests a pathwise-in-probability turnpike property. In contrast to Section 3.6.1, no leaving arc is observed, which can be attributed to the stabilizing nature of the stage cost.

As already mentioned, due to the finite support of the disturbance, it is not only possible to compute individual realization paths, but also the full set of all possible realization paths. This yields a binomial tree-like structure for each optimal solution as well as for the optimal stationary process, as shown in Figure 3.6.2. As

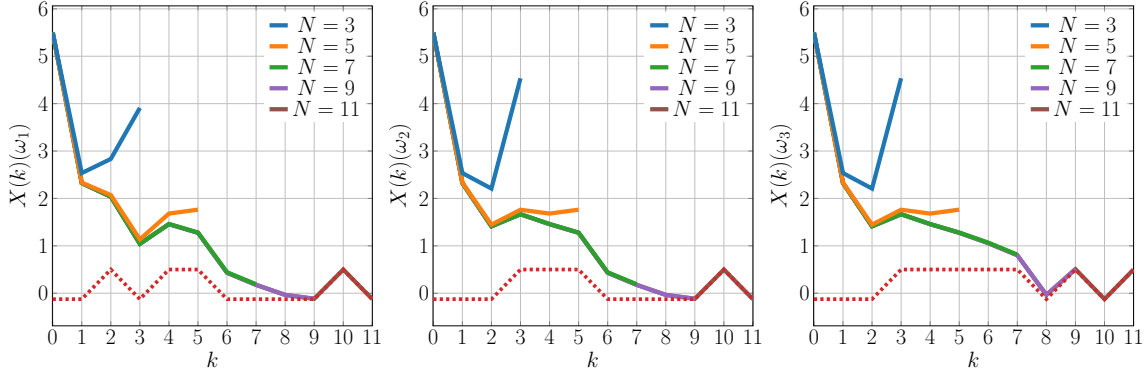


Figure 3.9: Realizations of optimal state trajectories of problem (3.60) on different horizons, together with the corresponding realization of the stationary process (dashed red) with identical disturbance realization in each sub-plot.

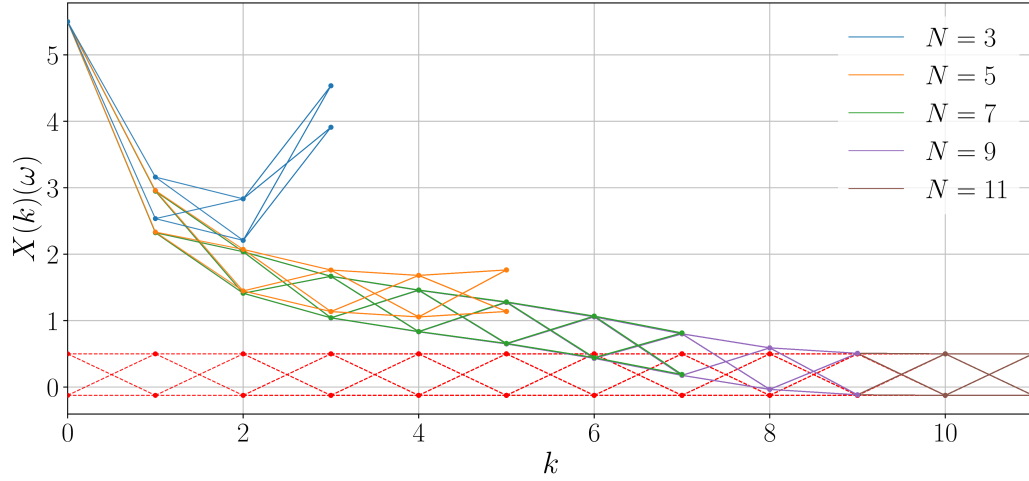


Figure 3.10: Evolution of all realization paths of the optimal solutions of problem (3.60) on different horizons, together with the realizations of the optimal stationary process (dashed red).

can be observed, these trees exhibit behavior consistent with that of the individual realization paths, thereby indicating an L^2 turnpike property.

Since we regard this representation as the most informative visualization for optimal trajectories under finite-support disturbances, we will use it throughout this thesis as the primary tool to numerically indicate the presence of stochastic turnpike properties in the case of two-point disturbances.

To conclude this section, Figure 3.6.2 shows the distributional behavior of the optimal solutions. We observe that the expectation of the optimal trajectories converge to the stationary expectation as the prediction horizon increases, and the same holds for the variance. In particular, the stationary variance is reached within one step.

In addition, we observe that the distributional distance, measured in terms of the first-order Wasserstein distance, approaches zero as the horizon length increases, thereby illustrating distributional turnpike property.

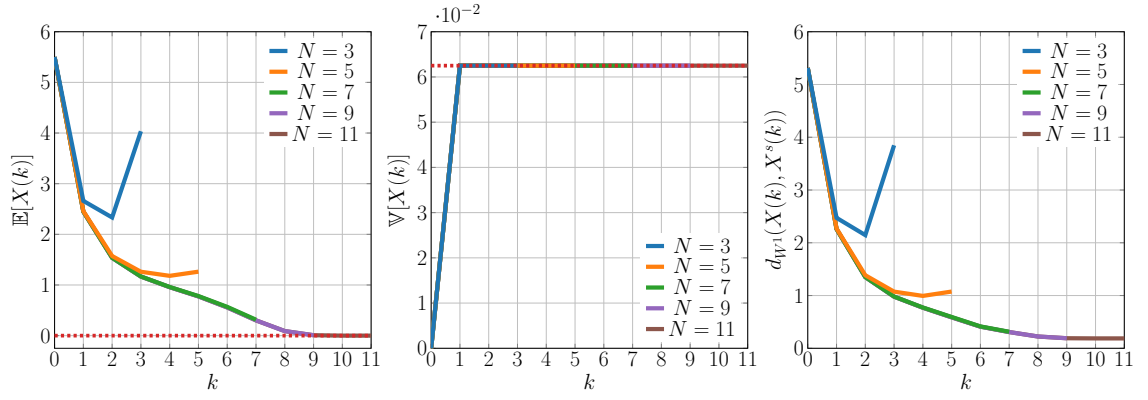


Figure 3.11: Evolution of the mean and variance of the optimal solutions of problem (3.60) on different horizons together with the stationary moment (dashed red), as well as the Wasserstein distance of order 1 to the optimal stationary process.

4 Closed-loop analysis of stochastic economic MPC with an expected cost criterion

Model predictive control (MPC) is one of the most prominent methodologies for the long-time control of dynamical systems. Its success is based on its ability to systematically incorporate system dynamics, constraints, and performance objectives into an online optimization framework. While the deterministic theory of MPC is by now well established, including rigorous results on feasibility, stability, and performance, extending these guarantees to stochastic systems remains a challenging problem due to the presence of uncertainty in both system evolution and cost accumulation; see, e.g., [70].

This challenge is particularly pronounced in the context of economic MPC, where the stage cost is not required to be positive definite with respect to a prescribed reference point. In the deterministic setting, dissipativity and turnpike arguments have led to a comprehensive understanding of stability and performance properties for economic MPC without requiring terminal ingredients; see [5, 18, 22, 27, 34, 39]. For stochastic MPC, however, corresponding results are significantly less developed.

Existing approaches to stochastic MPC stability analysis often rely on concepts originating from robust control; see, for example, [64], or require suitable terminal ingredients. Representative examples include practical stability results for multistage MPC schemes [65], expectation-based stability estimates [68], stability guarantees for linear systems with terminal constraints [58], and drift-based approaches [16]. Recent developments based on probabilistic reachable sets and constraint tightening have further enabled the treatment of chance-constrained formulations; see [48, 49, 55].

Despite these advances, many available stochastic stability results provide asymptotic bounds only for averaged quantities, such as expectation-based input-to-state stability estimates, and do not explicitly characterize the limiting stochastic behavior of the closed-loop process.

Performance analysis is even less developed. Existing results typically rely on upper bounds for the expected average closed-loop performance that are generally not tight; see, for example, [16]. Another line of research studies dynamic regret formulations for stochastic linear-quadratic control problems with additive and multiplicative uncertainties under finite-support assumptions [95]. However, performance characterizations directly linked to stationary stochastic optimal control problems, analogous to the deterministic economic MPC theory, are largely missing in the current literature.

The aim of this chapter is to address these limitations for discrete-time stochastic economic MPC with an expected cost criterion and without probabilistic constraints. Building on the stochastic dissipativity and turnpike framework developed in the

previous chapter, we establish a closed-loop analysis that extends central results from deterministic economic MPC to the stochastic setting.

The first main contribution is a stability analysis based on stochastic dissipativity. The dissipativity properties established for stochastic optimal control problems are exploited to construct Lyapunov functions for the MPC closed loop, which allows us to derive practical asymptotic stability results without the use of terminal ingredients. In contrast to most existing approaches, the asymptotic behavior of the closed-loop system is characterized explicitly by a stationary stochastic process. This yields stability results with respect to several stochastic notions of convergence, including convergence in the p -th mean, convergence in probability, convergence in distribution, and convergence of moments.

The second main contribution concerns performance analysis. We show that the closed-loop trajectories satisfy both averaged and non-averaged near-optimality estimates. These results provide stochastic counterparts of classical performance guarantees from deterministic economic MPC and establish tight long-term performance bounds for the expected cost criterion considered here.

A further contribution concerns the practical implementation of the MPC scheme. The theoretical developments are formulated for an abstract stochastic MPC framework that assumes knowledge of the state distribution. We prove, however, that this formulation is statistically equivalent to an implementable MPC algorithm that requires only state measurements generated along a single realization of the closed-loop trajectory. This establishes the practical relevance of the proposed approach and allows the theoretical results to be transferred directly to implementable MPC schemes.

The chapter is organized as follows. Section 4.1 introduces the stochastic MPC problem formulation and the abstract and implementable MPC schemes that form the basis of the subsequent analysis. The main stability results are presented in Section 4.2, while Section 4.3 is devoted to the corresponding performance analysis. Numerical illustrations of the theoretical results are provided in Section 4.4.

The majority of the results in this chapter have been published in [88, 90].

4.1 Setting and preliminaries

Throughout this chapter, we assume that Assumption 2.21 holds. More precisely, the mapping $\mathbb{L}$ in the stage costs is given by the expectation, i.e.,

$$\ell(X, U) = \mathbb{E}[g(X, U)],$$

where $g : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$ is a lower semicontinuous function that is bounded from below. Moreover, probabilistic constraints of the form (2.3) are not considered, and we only impose pathwise constraints of the form

$$h(X(\omega), U(\omega)) \leq 0 \quad \text{for almost all } \omega \in \Omega.$$

Here, $h : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^{n_c}$ is a Borel-measurable function satisfying the viability condition that, for every $x \in \mathcal{X}$, there exists some $u \in \mathcal{U}$ such that

$$h(x, u) \leq 0.$$

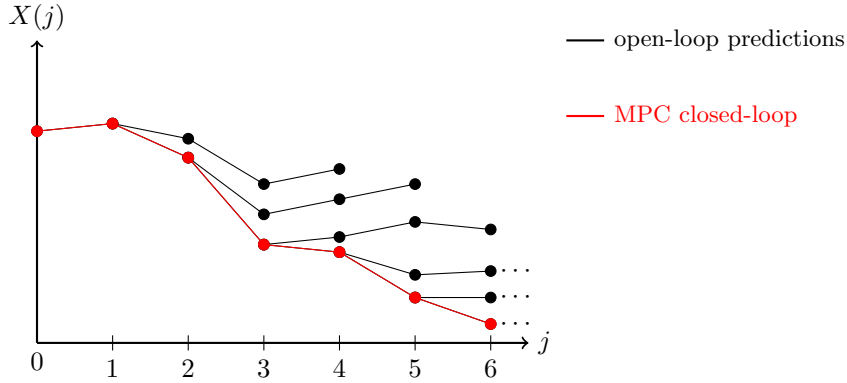


Figure 4.1: Illustration of the conceptual idea of MPC.

Hence, the stochastic optimal control problem considered in this chapter reads

$$\begin{aligned}
& \underset{\mathbf{U} \in \mathcal{U}_{\text{ad}}^N(j, X_j)}{\text{minimize}} && J_N(X_j, \mathbf{U}) := \sum_{k=0}^{N-1} \ell(X_{\mathbf{U}}(k; j, X_j), U(k)) \\
& \text{s.t.} && X_{\mathbf{U}}(k+1; j, X_j) = f(X_{\mathbf{U}}(k; j, X_j), U(k), W(k+j)), \\
& && X_{\mathbf{U}}(0; j, X_j) = X_j, \\
& && h(X_{\mathbf{U}}(k; j, X_j)(\omega), U(k)(\omega)) \leq 0, \quad \text{almost surely.}
\end{aligned} \tag{4.1}$$

The idea of model predictive control (MPC) is to approximate a solution of the infinite-horizon stochastic optimal control problem (4.1) by iteratively solving finite-horizon problems over shorter horizons. At each iteration, MPC predicts the system's future behavior over the finite horizon and computes an optimal control sequence. Only the first control action is applied to the system, after which the horizon is shifted forward. This approach is illustrated in Figure 4.1.

If we adapt the classical MPC approach directly to our stochastic setting this leads to Algorithm 1 where we consider full knowledge of the random variable in every MPC iteration since our states are random variables. Furthermore, we recall that every optimal control sequence obtained in Step 2 of the algorithm is generated by a deterministic sequence of state-dependent feedback policies.

Algorithm 1 Abstract stochastic MPC algorithm

- for** $j = 0, 1, \dots$
- 1.) Set $X_j = X(j)$.
 - 2.) Solve the stochastic optimal control problem (4.1) and obtain the optimal control sequence

$$\begin{aligned}
\mathbf{U}_{N,j,X_j}^* &= (U_{N,j,X_j}^*(0), \dots, U_{N,j,X_j}^*(N-1)) \\
&= (\pi_0^*(X(0; j, X_j)), \dots, \pi_{N-1}^*(X(N-1; j, X_j))).
\end{aligned}$$

- 3.) Apply the MPC feedback $\mu_N^a(X(j)) := U_{N,j,X_j}^*(0)$ to system (2.3) and get the next state $X(j+1)$.
-

While the abstract MPC Algorithm 1 on the space of random variables is useful for our theoretical analysis, its practical relevance is limited. This is because, in

a real plant, one can typically only measure the current realization of a random variable and not its distribution.

Hence, instead of using the full random variable as the initial condition when solving the open-loop problem in each MPC iteration, one uses its current realization in order to obtain an implementable algorithm. Such an approach is described in Algorithm 2.

Algorithm 2 Implementable stochastic MPC algorithm

for $j = 0, 1, \dots$

1.) Measure the state $x_j := X(j, \omega)$ and set $X_0 \equiv x_j$.

2.) Solve the stochastic optimal control problem (4.1) and obtain the optimal control sequence

$$\begin{aligned} \mathbf{U}_{N,j,x_j}^* &= (U_{N,j,x_j}^*(0), \dots, U_{N,j,x_j}^*(N-1)) \\ &= (\pi_0^*(X(0; j, x_j)), \dots, \pi_{N-1}^*(X(N-1; j, x_j))). \end{aligned}$$

3.) Apply the MPC feedback $\mu_N^i(x_j) := U_{N,x_j}^*(0)$ to system (2.3).

Since Algorithm 2 computes only a single solution path corresponding to the observed realizations, the full feedback law on the space of random variables can be recovered by running Algorithm 2 for all possible realizations. Note that, in contrast to Algorithm 1, the MPC feedback obtained in each iteration is not only induced by a deterministic feedback policy, but is in fact a deterministic value, since the initial condition $x_j \in \mathcal{X}$ is deterministic.

However, it is not immediately clear whether the two feedback laws obtained from the abstract Algorithm 1 and the implementable Algorithm 2 coincide, i.e., whether

$$\mu_N^a(X(\omega)) = \mu_N^i(x) \quad \text{whenever } X(\omega) = x.$$

If this were the case, it would in particular imply that for $\mu_N \in \{\mu_N^a, \mu_N^i\}$ the corresponding closed-loop trajectories

$$\begin{aligned} X_{\mu_N}(k+1; j, X_j) &= f(X_{\mu_N}(k; j, X_j), \mu_N(X_{\mu_N}(k; j, X_j)), W(j+k)), \\ X_{\mu_N}(0; j, X_j) &= X_j, \end{aligned} \tag{4.2}$$

as well as the closed-loop performance

$$J_K^{cl}(X_0, \mu) := \sum_{j=0}^{K-1} \ell(X_\mu(j; 0, X_0), \mu(X_\mu(j; 0, X_0))), \tag{4.3}$$

coincide for all $j \in \mathbb{N}_0$ and all $X_j, X_0 \in \mathcal{R}(\mathcal{X})$.

In general, such a relation need not hold for arbitrary cost and constraint structures, cf. Example 5.6 in Section 5.2. However, under Assumption 2.21, the pathwise dynamic programming principle from Theorem 2.22 applies. This guarantees that the two algorithms coincide, as shown in the following corollary.

Corollary 4.1. *Consider a horizon N , an initial time $j \in \mathbb{N}_0$, and an initial state $X_j \in \mathcal{R}(\mathcal{X})$. Let μ_N^a be the feedback law generated by the abstract MPC Algorithm 1.*

Then, there exists a measurable feedback law μ_N^i generated pathwise by the implementable Algorithm 2 such that

$$\mu_N^a(X_{\mu_N^a}(k; j, X_j)) = \mu_N^i(X_{\mu_N^i}(k; j, X_j)), \quad X_{\mu_N^a}(k; j, X_j) = X_{\mu_N^i}(k; j, X_j) \quad (4.4)$$

holds almost surely for all $k \in \mathbb{N}_0$. Moreover, the closed-loop costs coincide, i.e.,

$$J_K^{cl}(X_0, \mu_N^a) = J_K^{cl}(X_0, \mu_N^i)$$

holds for all $X_0 \in \mathcal{R}(\mathcal{X})$.

Proof. Consider that $X_{\mu_N^a}(k; j, X_j) = X_{\mu_N^i}(k; j, X_j)$ for some $k \in \mathbb{N}_0$. Then, using Theorem 2.22 in combination with Lemma 2.23 we obtain that

$$\mu_N^a(X_{\mu_N^a}(k; j, X_j)) = \mu_N^i(X_{\mu_N^i}(k; j, X_j))$$

holds almost surely, which also yields

$$X_{\mu_N^a}(k+1; j, X_j) = X_{\mu_N^i}(k+1; j, X_j).$$

Hence, equation (4.4) follows by induction since the initial condition

$$X_{\mu_N^a}(0; j, X_j) = X_{\mu_N^i}(0; j, X_j) = X_j$$

is fixed.

Finally, since the stage cost satisfies $\ell(X, U) = \mathbb{E}[g(X, U)]$ and thus depends only on the joint distribution of (X, U) , the identity of the closed-loop costs follows directly from (4.4). $\square$

Remark 4.2. (i) As we do not assume uniqueness of the optimal controls, Algorithm 1 may not yield a unique MPC closed-loop trajectory. However, the estimates we derive below are valid for all possible closed-loop trajectories.

(ii) Since Algorithm 2 computes a universally measurable feedback law, a measurable selection from the pointwise computed control values is required to guarantee measurability of the resulting feedback policy used in Corollary 4.1. However, in practice, Algorithm 2 only finitely many evaluations of $\mu_N = \mu_N^i$ will be performed. Since a finite selection of values is always measurable, the requirement of μ_N^i being measurable has no practical relevance for the implementation of Algorithm 2.

Since our proof techniques directly apply only to Algorithm 1, Corollary 4.1 is pivotal for our analysis. It guarantees that the results derived in the remainder of this paper for the MPC closed-loop trajectory generated by the abstract MPC Algorithm 1 also hold for the MPC closed-loop trajectories generated by the practically implementable MPC Algorithm 2, thereby establishing the practical relevance of our results. Hence, in the following we simply write μ_N instead of μ_N^a or μ_N^i , since the two feedback laws can be used interchangeably.

However, we emphasize that if full distributional information were available at every iteration, so that the abstract Algorithm 1 could be applied directly, then the results of this chapter would also hold for $\mu_N = \mu_N^a$ without the restrictions on the cost and constraint structure imposed by Assumption 2.21.

Before studying the properties of the closed-loop trajectory (4.2) in terms of stability and performance, we make the following additional dissipativity and continuity assumptions necessary for our analysis.

Assumption 4.3. We assume that the stochastic optimal control problem (4.1) is strictly stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to some pseudometric d in the sense of Definition 3.2 and that there exists $r > 0$ such that

(i) the storage function $\lambda : \mathbb{N}_0 \times \mathcal{R}(\mathcal{X}) \rightarrow \mathbb{R}$ is uniformly bounded on $\bar{B}_r^d(\mathbf{X}^s)$, cf. (3.18);

(ii) the stationary process $\mathbf{X}^s$ is cheaply reachable on $\bar{B}_r^d(\mathbf{X}^s)$, cf. Definition 3.17.

Assumption 4.4. Consider the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ and the pseudometric d from Assumption 4.3. Then, we assume that the optimal value function is approximately continuous at $\mathbf{X}^s$, i.e. there exist $\varepsilon > 0$, $\gamma_V : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $\gamma_V(N, r) \rightarrow 0$ if $N \rightarrow \infty$ and $r \rightarrow 0$, and $\gamma_V(\cdot, r), \gamma_V(N, \cdot)$ monotonous for fixed r and N such that

$$|V_N(j, X_j) - V_N(j, X^s(j))| \leq \gamma_V(N, d(X_j, X^s(j)))$$

holds for all $N \in \mathbb{N}$ and $(j, X_j) \in \bar{B}_\varepsilon^d(\mathbf{X}^s)$.

Note that the dissipativity assumption from Assumption 4.3 typically becomes less restrictive for weaker metrics in the sense of Definition 3.26. In contrast, the continuity property from Assumption 4.4 becomes more restrictive for weaker metrics. Therefore, it is not immediate that the assumptions of this chapter remain satisfied when replacing a metric d_1 by a weaker metric $d_2 \preceq d_1$. However, as explained in Remark 4.30, the metric d has no qualitative influence on the performance results derived in Section 4.3. Instead, the choice of the metric mainly affects the stability and convergence properties. In Remark 4.20 we therefore discuss which stability and convergence implications can still be obtained for weaker metrics.

Remark 4.5. (i) Note that Assumption 4.3, in particular, guarantees that the uniform finite-horizon turnpike property on $\bar{B}_r^d(\mathbf{X}^s)$ holds. In order to conclude the turnpike property on the infinite horizon as well, one additionally has to impose the continuity property from Definition 3.21, as shown in Theorem 3.25.

(ii) The term “approximately” in Assumption 4.4 reflects the fact that small discontinuities of the optimal value function are permitted at points away from the stationary process. Moreover, for any finite horizon N , bounded discontinuities of V_N are allowed even at points on the stationary process. Additionally, note that this continuity property directly carries over to the shifted optimal value function on finite horizon defined in (3.23), since the identity $\hat{V}_N(j, X_j) - \hat{V}_N(j, X^s(j)) = V_N(j, X_j) - V_N(j, X^s(j))$ holds for all $N \in \mathbb{N}$.

4.2 Practical asymptotic stability

After introducing the stochastic MPC algorithm under consideration, we now turn to the analysis of the stability properties of the resulting closed-loop trajectory (4.2). To this end, we first introduce a suitable notion of stability, starting with the following definition of forward invariance.

Definition 4.6 (Forward invariant neighbourhood). A neighborhood $\mathcal{S}$ of a stationary process $\mathbf{X}^s$ is called forward-invariant for the feedback controlled system (4.2) if

$$f(X_j, \mu_N(X_j), W(j)) \in \mathcal{S}(j+1) \quad \text{for all } (j, X_j) \in \mathcal{S}.$$

Next we introduce the notion of P-practical asymptotic stability, which will serve as the stability notion considered in the following. Note that the specific type of stability is again determined by the choice of the pseudometric d , as in the case of dissipativity and the turnpike property, cf. Definition 3.4 and Definition 3.16.

Moreover, if d is the L^p metric, the resulting stability notion is associated with convergence in the p -th mean. If d is the Ky–Fan metric, it corresponds to convergence in probability, and if d is the Lévy–Prokhorov metric, it corresponds to convergence in distribution. Hence, our notion of stability encompasses the common types of stochastic convergence.

Definition 4.7 (Uniform P-practical asymptotic stability). *Consider a forward invariant neighborhood $\mathcal{S}$ of $\mathbf{X}^s$ and subsets $P(j) \subset S(j)$ with $X^s(j) \in P(j)$ for all $j \in \mathbb{N}_0$.*

Then, the stationary process $\mathbf{X}^s$ is called P-practically uniformly asymptotically stable on $\mathcal{S}$ with respect to the pseudometric d if there exists a comparison function $\beta \in \mathcal{KL}$ such that

$$d(X_{\mu_N}(j; j_0, X_{j_0}), X^s(j + j_0)) \leq \beta(d(X, X^s(j_0)), j) \quad (4.5)$$

holds for all $X_{j_0} \in S(j_0)$ and all $j_0, j \in \mathbb{N}_0$ with $j \geq j_0$ and $X_{\mu_N}(j; j_0, X_{j_0}) \notin P(j)$.

Since the decay condition (4.5) is only required to hold outside the set $P(j)$, P-practical asymptotic stability does not enforce convergence of the system state exactly to the stationary process as time tends to infinity. In contrast to (exact) asymptotic stability, it merely guarantees convergence to a neighborhood of the stationary process. Typically, this neighborhood is chosen to be small (cf. Corollary 4.19 below).

To establish this form of stability, one usually does not verify the definition directly, but instead employs the concept of Lyapunov functions. A Lyapunov function in our time-varying setting is defined as follows.

Definition 4.8 (Uniform time-varying Lyapunov function). *Consider a neighborhood $\mathcal{Y}$ of $\mathbf{X}^s$. A function $V : \mathcal{Y} \rightarrow \mathbb{R}_0^+$ is called a uniform time-varying Lyapunov function on $Y(j)$ if there are $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_V \in \mathcal{K}$ such that*

$$(i) \quad \alpha_1(d(X_j, X^s(j))) \leq V(j, X_j) \leq \alpha_2(d(X_j, X^s(j)));$$

$$(ii) \quad V(j + 1, f(X_j, \mu_N(X_j), W(j))) \leq V(j, X_j) - \alpha_V(d(X_j, X^s(j)))$$

holds for all $(j, X_j) \in \mathcal{S}$ with $f(X_j, \mu_N(X_j), W(j)) \in Y(j + 1)$.

The following theorem shows that the existence of a uniform time-varying Lyapunov function in the sense of Definition 4.8 indeed implies uniform P-practical asymptotic stability in the sense of Definition 4.7. Thus, it provides a convenient tool for verifying stability.

Theorem 4.9 (cf. [37, Theorem 2.23]). *Consider a forward invariant neighborhood $\mathcal{S}$ of $\mathbf{X}^s$ and subsets $P(j) \subset S(j)$ with $X^s(j) \in P(j)$ for all $j \in \mathbb{N}_0$. If there exists a uniform time-varying Lyapunov function V on $Y(j) = S(j) \setminus P(j)$, then $\mathbf{X}^s$ is uniformly P-practically asymptotically stable on $\mathcal{S}$.*

Our goal in this section is to construct a Lyapunov function for the feedback-controlled system (4.2). In model predictive control, a natural candidate for a Lyapunov function is typically the optimal value function, cf. [37].

However, with respect to stability without terminal ingredients, the optimal value function (2.14) of the original problem is not suitable. Instead, we consider the following modified problem.

For a strictly stochastically dissipative optimal control problem (2.13) and $N \in \mathbb{N}$, we define the modified costs

$$\tilde{J}_N(j, X_j, \mathbf{U}) := \sum_{k=0}^{N-1} \tilde{\ell}(j, X_{\mathbf{U}}(k; j, X_j), U(k)), \quad (4.6)$$

where the modified stage cost are given by

$$\tilde{\ell}(j, X, U) := \ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(X) - \lambda(f(X, U, W(j))). \quad (4.7)$$

The corresponding optimal value function of the modified problem is then defined as

$$\tilde{V}_N(j, X_j) := \inf_{\mathbf{U} \in \tilde{\mathbb{U}}_{\text{ad}}^N(j, X_j)} \tilde{J}_N(j, X_j, \mathbf{U}) \quad (4.8)$$

and a minimizer of (4.8) is denoted by $\tilde{U}_N^*$ or $\tilde{U}_{N,j,X_j}^*$ if we want to emphasize the dependency on the initial condition.

In contrast to the original problem, the modified problem introduced above employs stage costs that are adjusted by the storage function λ from Definition 3.2. Since λ may depend on time—unless the assumptions of Lemma 3.6 hold—the modified stage costs, and consequently the cost functional $\tilde{J}_N(j, X_j, \mathbf{U})$, become explicitly time-dependent, rather than depending on time only indirectly through the dynamics as before.

Moreover, due to the dissipation inequality (3.1), we obtain

$$\tilde{\ell}(j, X, U) \geq \alpha(d(X, X^s(j)))$$

for all $X \in \mathcal{R}(\mathcal{X})$, $U \in \mathbb{U}(X)$, and $j \in \mathbb{N}_0$, without any further modification.

However, since optimizing the modified problem generally leads to different optimal trajectories—i.e., in general $\tilde{\mathbf{U}}_N^* \neq \mathbf{U}_N^*$ —it is not immediate that the turnpike property of the original problem carries over to the modified one. This issue is addressed in the following lemma.

Lemma 4.10. *Let Assumption 4.3 hold and consider the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ and the pseudometric d from Assumption 4.3. Then, the modified stochastic optimal control problem (4.8) has the uniform stochastic finite-horizon turnpike property on $\bar{B}_r^d(\mathbf{X}^s)$ with respect to the pseudometric d .*

Proof. Consider $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$ and let $\mathbf{U}_N^*$ be the optimal solution of the stochastic optimal control problem (4.1) with initial time j and state X_j . Then, by optimality it follows that

$$\begin{aligned} \tilde{V}_N(j, X_j) &\leq \tilde{J}_N(j, X_j, \mathbf{U}_N^*) \\ &= \lambda(j, X_j) + J_N(X_j, \mathbf{U}_N^*) - N\ell(\mathbf{X}^s, \mathbf{U}^s) - \lambda(j + N, X_{\mathbf{U}_N^*}(N; j, X_j)). \end{aligned}$$

Using the cheap reachability from Assumption 4.3(ii) we get

$$J_N(X_j, \mathbf{U}_N^*) - N\ell(\mathbf{X}^s, \mathbf{U}^s) = V_N(j, X_j) - N\ell(\mathbf{X}^s, \mathbf{U}^s) \leq C_V.$$

Furthermore, due to the uniform boundedness of λ from Assumption 4.3(i) we know that $\lambda(j, X_j) < C_\lambda$ for some $C_\lambda > 0$ and due to the definition of dissipativity we obtain that there is a $C_\lambda^l > 0$ such that $\lambda(j, X) > -C_\lambda^l$ for all $j \in \mathbb{N}_0$ and $X \in \mathcal{R}(\mathcal{X})$. Hence, we get

$$\tilde{V}_N(j, X_j) \leq \tilde{C}_V := C_\lambda + C_\lambda^l + C_V \quad (4.9)$$

for all $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$.

Now consider $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$, the corresponding optimal control sequence $\tilde{\mathbf{U}}_N^*$ to the modified problem (4.8), and suppose that equation (3.13) from the finite-horizon turnpike property from Definition 3.12 does not hold with

$$\vartheta(L) := \alpha^{-1} \left(\frac{\tilde{C}_V}{L} \right) = \alpha^{-1} \left(\frac{C_\lambda + C_\lambda^l + C_V}{L} \right), \quad \vartheta \in \mathcal{L}. \quad (4.10)$$

Then, there exist $N \in \mathbb{N}$, a number $L \in \mathbb{N}$, and a set $\mathcal{M} \subseteq \{0, \dots, N-1\}$ with $\#\mathcal{M} \geq L+1$ such that

$$d(X_{\tilde{\mathbf{U}}_N^*}(k; j, X_j), X^s(j+k)) > \vartheta(L) \quad \text{for all } k \in \mathcal{M},$$

which implies

$$\tilde{V}_N(j, X_j) \geq \sum_{k \in \mathcal{M}} \alpha(\vartheta(L)) > L\alpha(\vartheta(L)) = \tilde{C}_V.$$

Hence, the uniform stochastic finite-horizon turnpike property holds since this contradicts (4.9) and ϑ does not depend on $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$. $\square$

We emphasize that the function $\vartheta \in \mathcal{L}$ characterizing the turnpike property of the original stochastic optimal control problem (2.13) and the corresponding function for the modified problem can be chosen to be the same, cf. (3.20) in Theorem 3.19 and (4.10) in Lemma 4.10. Hence, in the following we do not need to distinguish between different functions ϑ when invoking the turnpike property for either the original or the modified problem.

Remark 4.11. *Analogously to Theorem 3.19, a non-uniform version of the turnpike property for the modified problem could also be established without the additional assumption of a bounded storage function. However, since we require the uniform version in this chapter and assume uniform boundedness of the storage function anyway, we omit the proof of the non-uniform version.*

Before finally showing P-practical asymptotic stability of the closed-loop system, we establish a series of preparatory lemmas relating the costs of optimal solutions corresponding to different horizons, based on dissipativity and the turnpike property.

To this end, we impose the following continuity assumptions on the storage function and on the modified optimal value function. Note that, although the continuity assumption on the modified optimal value function is similar to Assumption 4.4 for the original problem, we do not allow the upper bound to depend on the horizon N . This means that for the modified optimal value function, no discontinuities along the stationary process are permitted.

Assumption 4.12. Consider the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$, the pseudometric d , and the storage function λ from Assumption 4.3. We assume that there exist $\tilde{r} \geq \tilde{\varepsilon} > 0$ and $\gamma_\lambda, \gamma_{\tilde{V}} \in \mathcal{K}_\infty$ such that

- (i) $|\lambda(j, X_j) - \lambda(j, X^s(j))| \leq \gamma_\lambda(d(X_j, X^s(j)))$ for all $(j, X_j) \in \bar{B}_\varepsilon^d(\mathbf{X}^s)$;
- (ii) $|\tilde{V}_N(j, X_j) - \tilde{V}_N(j, X^s(j))| \leq \gamma_{\tilde{V}}(d(X_j, X^s(j)))$ for all $(j, X_j) \in \bar{B}_{\tilde{r}}^d(\mathbf{X}^s)$, $N \in \mathbb{N}$.

The following lemma compares the performance of optimal trajectories on horizons N and $N + 1$ for the modified problem.

Lemma 4.13. Let Assumptions 4.3 and 4.12(ii) hold. Consider $\vartheta \in \mathcal{L}$ from (4.10) and $\tilde{r}$ from Assumption 4.12. Then, for all $(j, X_j) \in \bar{B}_{\tilde{r}}^d(\mathbf{X}^s)$ and all $N, L \in \mathbb{N}$ with $L \geq \vartheta^{-1}(\tilde{r})$, there exists a set $\bar{\mathcal{Q}}(j, X_j, L, N)$ with $\#\bar{\mathcal{Q}}(j, X_j, L, N) \leq 2L$ such that for all $M \in \{0, \dots, N\} \setminus \bar{\mathcal{Q}}(j, X_j, L, N)$ the estimate

$$\tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_N^*) = \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_{N+1}^*) + R_1(j, X_j, M, N) \quad (4.11)$$

holds with $|R_1(j, X_j, M, N)| \leq 2\gamma_{\tilde{V}}(\vartheta(L))$.

Proof. Let $\tilde{\mathbf{U}}_N^*$ and $\tilde{\mathbf{U}}_{N+1}^*$ be the optimal control sequences of the modified problem (4.8) on horizon N and $N + 1$. Then, by optimality and applying the stochastic dynamic programming principle we obtain

$$\begin{aligned} \tilde{V}_N(j, X_j) &= \tilde{J}_N(j, X_j, \tilde{\mathbf{U}}_N^*) \\ &= \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_N^*) + \tilde{V}_{N-M}(j + M, X_{\tilde{\mathbf{U}}_N^*}(M; j, X_j)) \\ &\leq \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_{N+1}^*) + \tilde{V}_{N-M}(j + M, X_{\tilde{\mathbf{U}}_{N+1}^*}(M; j, X_j)). \end{aligned} \quad (4.12)$$

Defining the remainder terms

$$\begin{aligned} \tilde{R}_1(j, X_j, M, N) &:= \tilde{V}_{N-M}(j + M, X_{\tilde{\mathbf{U}}_N^*}(M; j, X_j)) \\ &\quad - \tilde{V}_{N-M}(j + M, X^s(j + M)) \\ \tilde{R}_2(j, X_j, M, N) &:= \tilde{V}_{N-M}(j + M, X_{\tilde{\mathbf{U}}_{N+1}^*}(M; j, X_j)) \\ &\quad - \tilde{V}_{N-M}(j + M, X^s(j + M)) \end{aligned} \quad (4.13)$$

we derive from equation (4.12) that

$$\tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_N^*) \leq \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_{N+1}^*) - \tilde{R}_1(j, X_j, M, N) + \tilde{R}_2(j, X_j, M, N).$$

Due to Lemma 4.10 we know that there are sets $\tilde{\mathcal{Q}}(j, X_j, L, N) \subseteq \{0, \dots, N\}$ and $\tilde{\mathcal{Q}}(j, X_j, L, N + 1) \subseteq \{0, \dots, N + 1\}$ such that

$$\begin{aligned} d(X_{\tilde{\mathbf{U}}_N^*}(M; j, X_j), X^s(j + M)) &\leq \vartheta(L), \\ d(X_{\tilde{\mathbf{U}}_{N+1}^*}(M; j, X_j), X^s(j + M)) &\leq \vartheta(L) \end{aligned}$$

holds for all $M \in \{0, \dots, N\} \setminus \tilde{\mathcal{Q}}(j, X_j, L, N)$ with

$$\bar{\mathcal{Q}}(j, X_j, L, N) := \tilde{\mathcal{Q}}(j, X_j, L, N) \cup \tilde{\mathcal{Q}}(j, X_j, L, N + 1)$$

and $\#\bar{\mathcal{Q}}(j, X_j, L, N) \leq 2L$ since each of the sets $\tilde{\mathcal{Q}}$ contains at most L elements.

Now consider $L \geq \vartheta^{-1}(\tilde{\varepsilon})$. Then, we get $\vartheta(L) \leq \tilde{\varepsilon}$ and thus

$$\begin{aligned} |\tilde{R}_1(j, X_j, M, N)| &\leq \gamma_{\tilde{V}}(d(X_{\tilde{U}_N^*}(M; j, X_j), X^s(j+M))) \leq \gamma_{\tilde{V}}(\vartheta(L)) \\ |\tilde{R}_2(j, X_j, M, N)| &\leq \gamma_{\tilde{V}}(d(X_{\tilde{U}_{N+1}^*}(M; j, X_j), X^s(j+M))) \leq \gamma_{\tilde{V}}(\vartheta(L)) \end{aligned}$$

for all $M \in \{0, \dots, N\} \setminus \bar{\mathcal{Q}}(j, X_j, L, N)$.

We continue with repeating the above calculations for $\tilde{J}_M(j, X_j, \tilde{U}_{N+1}^*)$ which yields

$$\tilde{J}_M(j, X_j, \tilde{U}_{N+1}^*) \leq \tilde{J}_M(j, X_j, \tilde{U}_N^*) - R_3(j, X_j, M, N) + R_4(j, X_j, M, N)$$

with

$$\begin{aligned} \tilde{R}_3(j, X_j, M, N) &:= \tilde{V}_{N+1-M}(j+M, X_{\tilde{U}_{N+1}^*}(M; j, X_j)) \\ &\quad - \tilde{V}_{N+1-M}(j+M, X^s(j+M)) \\ \tilde{R}_4(j, X_j, M, N) &:= \tilde{V}_{N+1-M}(j+M, X_{\tilde{U}_N^*}(M; j, X_j)) \\ &\quad - \tilde{V}_{N+1-M}(j+M, X^s(j+M)). \end{aligned} \tag{4.14}$$

These residuals can be estimated in the same way as $\tilde{R}_1$ and $\tilde{R}_2$, i.e. the inequalities $|\tilde{R}_3(j, X_j, M, N)| \leq \gamma_{\tilde{V}}(\vartheta(L))$ and $|\tilde{R}_4(j, X_j, M, N)| \leq \gamma_{\tilde{V}}(\vartheta(L))$ hold for all $M \in \bar{\mathcal{Q}}(j, X_j, L, N)$. Thus, by defining

$$R_1(j, X_j, M, N) := \tilde{J}_M(j, X_j, \tilde{U}_N^*) - \tilde{J}_M(j, X_j, \tilde{U}_{N+1}^*)$$

we obtain

$$\begin{aligned} |R_1(j, X_j, M, N)| &= |\tilde{J}_M(j, X_j, \tilde{U}_N^*) - \tilde{J}_M(j, X_j, \tilde{U}_{N+1}^*)| \\ &\leq \max\{|-R_1(j, X_j, M, N) + R_2(j, X_j, M, N)|, \\ &\quad |-R_3(j, X_j, M, N) + R_4(j, X_j, M, N)|\} \\ &\leq \max\{|R_1(j, X_j, M, N)| + |R_2(j, X_j, M, N)|, \\ &\quad |R_3(j, X_j, M, N)| + |R_4(j, X_j, M, N)|\} \\ &\leq \max\{2\gamma_{\tilde{V}}(\vartheta(L)), 2\gamma_{\tilde{V}}(\vartheta(L))\} = 2\gamma_{\tilde{V}}(\vartheta(L)) \end{aligned}$$

for all $M \in \{0, \dots, N\} \setminus \bar{\mathcal{Q}}(j, X_j, L, N)$, which proves the claim. $\square$

Assumption 4.14. *For the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ from Assumption 4.3 it holds that $\tilde{\ell}(j, X^s(j), U^s(j)) = 0$ for all $j \in \mathbb{N}_0$.*

Assumption 4.14 states that the modified stage costs evaluated at the stationary pair are not only bounded from below by zero, but are in fact equal to zero. This is equivalent to requiring that the storage function is constant along the optimal trajectory, i.e.,

$$\lambda(j, X^s(j)) = \lambda(j+1, X^s(j+1)) \quad \text{for all } j \in \mathbb{N}_0,$$

since the stage costs $\ell(\mathbf{X}^s, \mathbf{U}^s)$ remain constant along the stationary pair. For time-invariant problems, this condition always holds, as then $X^s(j) = X^s(j+1)$ for all $j \in \mathbb{N}_0$. For time-dependent problems, this equality does not necessarily hold by definition. However, for all problems we are aware of for which strict dissipativity has

been theoretically guaranteed, cf. [84, 89], this property holds, and hence we do not consider this assumption to be restrictive. Moreover, on the layer of distributions, the storage function can be chosen to be law- and time-invariant, cf. Lemma 3.6. Consequently, for strict distributional dissipativity (or strict dissipativity with respect to a law-invariant pseudometric), Assumption 4.14 can be imposed without loss of generality.

Under Assumption 4.14 and by means of Lemma 4.13, we can now show that the optimal value function of the modified problem attains nearly identical values for the horizons N and $N + 1$.

Lemma 4.15. *Let Assumptions 4.3, 4.12(ii), and 4.14 hold. Consider $\vartheta \in \mathcal{L}$ from (4.10), $\tilde{r}$ from Assumption 4.12, and $\tilde{\mathcal{Q}}(j, X_j, L, N)$ from Lemma 4.13. Then, for all $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$, all $N, L \in \mathbb{N}$ with $L \geq \vartheta^{-1}(\tilde{r})$, and all $M \in \{0, \dots, N\} \setminus \tilde{\mathcal{Q}}(j, X_j, L, N)$ the estimate*

$$\tilde{V}_{N+1}(j, X_j) = \tilde{V}_N(j, X_j) + R_2(j, X_j, M, N) \quad (4.15)$$

holds with $|R_2(j, X_j, M, N)| \leq 4\gamma_{\tilde{V}}(\vartheta(L))$ and $\mathcal{M}(j, X_j, L, N)$ from Lemma 4.13.

Proof. Using the remainder term $\tilde{R}_1(j, X_j, M, N)$ from equation (4.13) we obtain

$$\begin{aligned} \tilde{V}_N(j, X_j) &= \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_N^*) + \tilde{V}_{N-M}(j + M, X^s(j + M)) \\ &\quad + \tilde{R}_1(j, X_j, M, N) \\ &= \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_N^*) + \tilde{R}_1(j, X_j, M, N). \end{aligned} \quad (4.16)$$

Here, the last equality follows since Assumption 4.14 implies that $\tilde{V}_K(j, X^s(j)) = 0$ for all $j \in \mathbb{N}_0$ and $K \in \mathbb{N}$. Moreover, with an analogous computation we get

$$\tilde{V}_{N+1}(j, X_j) = \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_{N+1}^*) + \tilde{R}_3(j, X_j, M, N) \quad (4.17)$$

with $R_3(j, X_j, M, N)$ from equation (4.14).

By using Lemma 4.13 and the identities (4.16) and (4.17) we thus obtain

$$\begin{aligned} \tilde{V}_{N+1}(j, X_j) &= \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_{N+1}^*) + R_3(j, X_j, M, N) \\ &= \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_N^*) + R_3(j, X_j, M, N) - R_5(j, X_j, M, N) \\ &= \tilde{V}_N(j, X_j) - \tilde{R}_1(j, X_j, M, N) + \tilde{R}_3(j, X_j, M, N) - R_1(j, X_j, M, N) \end{aligned}$$

with

$$\begin{aligned} |\tilde{R}_1(j, X_j, M, N)| &\leq \gamma_{\tilde{V}}(\vartheta(L)), \\ |\tilde{R}_3(j, X_j, M, N)| &\leq \gamma_{\tilde{V}}(\vartheta(L)), \\ |R_1(j, X_j, M, N)| &\leq 2\gamma_{\tilde{V}}(\vartheta(L)) \end{aligned}$$

for all $M \in \{0, \dots, N\} \setminus \tilde{\mathcal{Q}}(j, X_j, L, N)$, which proves the claim with

$$R_2(j, X_j, M, N) := -\tilde{R}_1(j, X_j, M, N) + \tilde{R}_3(j, X_j, M, N) - R_1(j, X_j, M, N).$$

□

Lemma 4.16. *Let Assumptions 4.3 and 4.4 hold. Consider $\vartheta \in \mathcal{L}$ from (3.20) and ε from Assumption 4.4. Then, for all $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$ and all $N, L \in \mathbb{N}$ with $L \geq \vartheta^{-1}(\varepsilon)$ there exists a set $\mathcal{Q}(j, X_j, L, N)$ with $\#\mathcal{Q}(j, X_j, L, N) \leq L$ such that for all $M \in \{0, \dots, N\} \setminus \mathcal{Q}(j, X_j, L, N)$ and all $\mathbf{U} \in \check{\mathbf{U}}_{\text{ad}}^M(j, X_j)$ with $d(X_{\mathbf{U}}(M; j, X_j), X^s(j+M)) \leq \vartheta(L)$, the estimate*

$$J_M(j, X_j, \mathbf{U}_N^*) \leq J_M(j, X_j, \mathbf{U}) + R_3(j, X_j, M, N) \quad (4.18)$$

holds with $|R_3(j, X_j, M, N)| \leq 2\gamma_V(N - M, \vartheta(L))$.

Proof. The proof of inequality (4.18) is performed by contradiction. To this end, consider an arbitrary $M \leq N$ and assume that there is a control sequence $\mathbf{U} \in \check{\mathbf{U}}_{\text{ad}}^M(j, X_j)$ such that

$$J_M(j, X_j, \mathbf{U}) + \bar{R}_1(j, X_j, M, N) + \bar{R}_2(j, X_j, M, N) < J_M(j, X_j, \mathbf{U}_N^*) \quad (4.19)$$

with

$$\begin{aligned} \bar{R}_1(j, X_j, M, N) &:= V_{N-M}(j+M, X_{\mathbf{U}}(M; j, X_j)) \\ &\quad - V_{N-M}(j+M, X^s(j+M)) \\ \bar{R}_2(j, X_j, M, N) &:= V_{N-M}(j+M, X^s(j+M)) \\ &\quad - V_{N-M}(j+M, X_{\mathbf{U}_N^*}(M; j, X_j)). \end{aligned}$$

Then, it follows that

$$\begin{aligned} &\hat{J}_M(j, X_j, \mathbf{U}) + V_{N-M}(j+M, X_{\mathbf{U}}(M; j, X_j)) \\ &= J_M(j, X_j, \mathbf{U}) + V_{N-M}(j+M, X^s(j+M)) + R_1(j, X_j, M, N) \\ &= J_M(j, X_j, \mathbf{U}) + V_{N-M}(j+M, X_{\mathbf{U}_N^*}(M; j, X_j)) \\ &\quad + \bar{R}_1(j, X_j, M, N) + \bar{R}_2(j, X_j, M, N) \\ &< J_M(j, X_j, \mathbf{U}_N^*) + V_{N-M}(j+M, X_{\mathbf{U}_N^*}(M; j, X_j)) = V_N(j, X_j) \end{aligned}$$

which contradicts the optimality of $\mathbf{U}_N^*$. Thus, we can conclude that

$$J_M(j, X_j, \mathbf{U}_N^*) \leq J_M(j, X_j, \mathbf{U}) + R_3(j, X_j, M, N) \quad (4.20)$$

holds with

$$R_3(j, X_j, M, N) := \bar{R}_1(j, X_j, M, N) + \bar{R}_2(j, X_j, M, N).$$

It remains to show that the bound on $|R_3(j, X_j, M, N)|$ holds. By Lemma 3.19 we know that there is a set $\mathcal{Q}(j, X_j, L, N)$ such that $d(\bar{X}_{\mathbf{U}^*}, X^s(j+M)) \leq \vartheta(L)$ for all $M \in \{0, \dots, N\} \setminus \mathcal{Q}(j, X_j, L, N)$.

Now consider $L \geq \vartheta^{-1}(\varepsilon)$, and $\mathbf{U} \in \check{\mathbf{U}}_{\text{ad}}^M(j, X_j)$ with $d(X_{\mathbf{U}}(M; j, X_j), X^s(j+M)) \leq \vartheta(L)$. Then, according to the continuity assumption of the optimal value function, see Assumption 4.4, it follows that

$$\begin{aligned} |\bar{R}_1(j, X_j, M, N)| &\leq \gamma_V(N - M, \vartheta(L)) \\ |\bar{R}_2(j, X_j, M, N)| &\leq \gamma_V(N - M, \vartheta(L)) \end{aligned}$$

which implies $|R_3(j, X_j, M, N)| \leq 2\gamma_V(N - M, \vartheta(L))$ and thus concludes the proof. $\square$

In the final preparatory lemma of the stability analysis, we consider a control sequence $\hat{\mathbf{U}}$ constructed as follows. On an initial segment, it coincides with the optimal control sequence $\mathbf{U}_N^*$ of the original (unmodified) problem, until the corresponding trajectory approaches the stationary state $\mathbf{X}^s$. From that point onward, we apply the optimal control sequence of the modified problem.

The lemma shows that this composite control sequence yields a cost that is nearly identical to the cost obtained by applying the optimal control sequence of the modified problem over the entire horizon.

Lemma 4.17. *Let Assumptions 4.3, 4.4, 4.12, and 4.14 hold. Consider $\vartheta \in \mathcal{L}$ from (4.10), ε from Assumption 4.4, $\tilde{\varepsilon}$ from Assumption 4.12, and $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$. Let $N, L \in \mathbb{N}$ with $L \geq \vartheta^{-1}(\min\{\varepsilon, \tilde{\varepsilon}\})$, and $\bar{\mathbf{U}}$ be the solution of*

$$\min_{\mathbf{U} \in \check{\mathbf{U}}_{\text{ad}}^{N-M}(j+M, \bar{X}_M)} \tilde{J}_{N-M}(j+M, \bar{X}_M, \mathbf{U})$$

with $\bar{X}_M := X_{\mathbf{U}_N^*}(M; j, X_j)$.

Then, there exists a set $\hat{\mathcal{Q}}(j, X_j, L, N)$ with $\#\hat{\mathcal{Q}}(j, X_j, L, N) \leq 2L$ such that for all $M \in \{0, \dots, N\} \setminus \hat{\mathcal{Q}}(j, X_j, L, N)$ the composite control sequence $\hat{\mathbf{U}} \in \check{\mathbf{U}}_{\text{ad}}^N(j, X_j)$ defined by $\hat{U}(j) := U_N^*(j)$ for $j = 0, \dots, M-1$ and $\hat{U}(j+M) := \bar{U}(j)$ for $j = 0, \dots, N-M$ satisfies

$$\tilde{J}_N(j, X_j, \hat{\mathbf{U}}) = \tilde{V}_N(j, X_j) + R_4(j, X_j, M, N)$$

with $|R_4(j, X_j, M, N)| \leq 2\gamma_{\bar{V}}(\vartheta(L)) + 2\gamma_{\lambda}(\vartheta(L)) + 2\gamma_V(N-M, \vartheta(L))$.

Proof. Due to the definition of the composite control sequence $\hat{\mathbf{U}}$ it holds that

$$\begin{aligned} \tilde{J}_N(j, X_j, \hat{\mathbf{U}}) &= \tilde{J}_M(j, X_j, \mathbf{U}_N^*) + \tilde{J}_{N-M}(j+M, \bar{X}_M, \bar{\mathbf{U}}) \\ &= \tilde{J}_M(j, X_j, \mathbf{U}_N^*) + \tilde{V}_{N-M}(j+M, \bar{X}_M) \\ &= J_M(j, X_j, \mathbf{U}_N^*) - M\ell(\mathbf{X}^s, \mathbf{U}^s) \\ &\quad + \lambda(j, X_j) - \lambda(j+M, \bar{X}_M) + \tilde{V}_{N-M}(j+M, \bar{X}_M). \end{aligned} \quad (4.21)$$

Now consider $\tilde{R}_1(j, X_j, M, N)$ from equation (4.13) and

$$\begin{aligned} \hat{R}_1(j, X_j, M, N) &:= \tilde{V}_{N-M}(j+M, \bar{X}_M) - \tilde{V}_{N-M}(j+M, X^s(j+M)) \\ \hat{R}_2(j, X_j, M, N) &:= \lambda(j+M, X^s(j+M)) - \lambda(j+M, \bar{X}_M) \\ \hat{R}_3(j, X_j, M, N) &:= \lambda(j+M, \tilde{X}_M) - \lambda(j+M, X^s(j+M)) \end{aligned}$$

with $\tilde{X}_M := X_{\bar{\mathbf{U}}_N^*}(M; j, X_j)$. Then, by Theorem 4.16 it holds that

$$\begin{aligned} &J_M(j, X_j, \mathbf{U}_N^*) + \lambda(j, X_j) - \lambda(j+M, \bar{X}_M) + \tilde{V}_{N-M}(j+M, \bar{X}_M) \\ &= J_M(j, X_j, \mathbf{U}_N^*) + \lambda(j, X_j) - \lambda(j+M, X^s(j+M)) \\ &\quad + \tilde{V}_{N-M}(j+M, X^s(j+M)) + \hat{R}_1(j, X_j, M, N) + \hat{R}_2(j, X_j, M, N) \\ &= J_M(j, X_j, \mathbf{U}_N^*) + \lambda(j, X_j) - \lambda(j+M, \tilde{X}_M) + \tilde{V}_{N-M}(j+M, \tilde{X}_M) \\ &\quad + \hat{R}_1(j, X_j, M, N) + \hat{R}_2(j, X_j, M, N) + \hat{R}_3(j, X_j, M, N) - \tilde{R}_1(j, X_j, M, N) \\ &\leq J_M(j, X_j, \tilde{\mathbf{U}}_N^*) + \lambda(j, X_j) - \lambda(j+M, \tilde{X}_M) + \tilde{V}_{N-M}(j+M, \tilde{X}_M) \\ &\quad + \hat{R}_1(j, X_j, M, N) + \hat{R}_2(j, X_j, M, N) + \hat{R}_3(j, X_j, M, N) \\ &\quad - \tilde{R}_1(j, X_j, M, N) + R_3(j, X_j, M, N). \end{aligned}$$

Hence, using the dynamic programming principle we obtain

$$\begin{aligned}\tilde{J}_N(j, X_j, \hat{\mathbf{U}}) &\leq \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_N^*) + \tilde{V}_{N-M}(j+M, \tilde{X}_M) + \hat{R}_4(j, X_j, M, N) \\ &= \tilde{V}_N(j, X_j) + \hat{R}_4(j, X_j, M, N)\end{aligned}$$

with

$$\begin{aligned}\hat{R}_4(j, X_j, M, N) &:= \hat{R}_1(j, X_j, M, N) + \hat{R}_2(j, X_j, M, N) + \hat{R}_3(j, X_j, M, N) \\ &\quad - \tilde{R}_1(j, X_j, M, N) + R_3(j, X_j, M, N)\end{aligned}$$

since

$$J_M(j, X_j, \tilde{\mathbf{U}}_N^*) - M\ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda(j, X_j) - \lambda(j+M, \tilde{X}_M) = \tilde{J}_M(j, X_j, \tilde{\mathbf{U}}_N^*).$$

Moreover, by the definition of the optimal value function we get $\tilde{V}_N(j, X_j) \leq \tilde{J}_N(j, X_j, \hat{\mathbf{U}})$ implying that $\hat{R}_4(j, X_j, M, N) \geq 0$. Hence, we can conclude that there exists a residual $R_4(j, X_j, M, N)$ with

$$0 \leq R_4(j, X_j, M, N) \leq \hat{R}_4(j, X_j, M, N)$$

such that

$$\tilde{J}_N(j, X_j, \hat{\mathbf{U}}) = \tilde{V}_N(j, X_j) + R_4(j, X_j, M, N).$$

Thus, it remains to derive the upper bound on $|R_4(j, X_j, M, N)|$. For this purpose, consider $L \in \mathbb{N}$ large enough such that $\vartheta(L) \leq \min\{\varepsilon, \tilde{\varepsilon}\}$ holds with ε from Assumption 4.4 and $\tilde{\varepsilon}$ from Assumption 4.12.

Then, using the turnpike properties from Theorem 3.19 and Lemma 4.10, and the continuity Assumptions 4.4 and 4.12 we can conclude that there exist sets $\mathcal{Q}(j, X_j, L, N)$ and $\tilde{\mathcal{Q}}(j, X_j, L, N)$ such that

$$|\hat{R}_1(j, X_j, M, N)| \leq \gamma_{\tilde{V}}(\vartheta(L)), \quad |\hat{R}_2(j, X_j, M, N)| \leq \gamma_{\lambda}(\vartheta(L))$$

for all $M \in \{0, \dots, N\} \setminus \mathcal{Q}(j, X_j, L, N)$ and

$$|\hat{R}_3(j, X_j, M, N)| \leq \gamma_{\tilde{V}}(\vartheta(L)), \quad |\tilde{R}_1(j, X_j, M, N)| \leq \gamma_{\lambda}(\vartheta(L))$$

for all $M \in \{0, \dots, N\} \setminus \tilde{\mathcal{Q}}(j, X_j, L, N)$.

Thus, setting $\hat{\mathcal{Q}}(j, X_j, L, N) = \mathcal{Q}(j, X_j, L, N) \cup \tilde{\mathcal{Q}}(j, X_j, L, N)$ and using these estimates and Lemma 4.16 we can conclude that

$$\begin{aligned}|R_4(j, X_j, M, N)| &\leq |\hat{R}_4(j, X_j, M, N)| \\ &\leq |\hat{R}_1(j, X_j, M, N)| + |\hat{R}_2(j, X_j, M, N)| + |\hat{R}_3(j, X_j, M, N)| \\ &\quad + |\tilde{R}_1(j, X_j, M, N)| + |R_3(j, X_j, M, N)| \\ &\leq 2\gamma_{\tilde{V}}(\vartheta(L)) + 2\gamma_{\lambda}(\vartheta(L)) + 2\gamma_V(N-M, \vartheta(L))\end{aligned}$$

holds for all $M \in \hat{\mathcal{Q}}(j, X_j, L, N)$, which shows the claim, since each of the sets $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ contains at most L elements. $\square$

Using the preparatory results established above, we can now prove that the modified optimal value function $\tilde{V}_N$ serves as a Lyapunov function for the closed-loop system under the MPC feedback law μ_N . Consequently, the closed-loop system is P-practically asymptotically stable according to Theorem 4.9.

Note that the “practical” neighborhood $P(j)$ of the stationary process can be made arbitrarily small by increasing the optimization horizon N . This observation will be used in Corollary 4.19.

Theorem 4.18. *Let Assumptions 4.3, 4.4, 4.14, and 4.12 hold. Then, there exists $\theta_0 > 0$ and $N_0 \in \mathbb{N}$ such that for all $0 < \theta \leq \theta_0$ the following holds:*

There exists a comparison function $\delta \in \mathcal{L}$ such that for all $N \geq N_0$ the optimal value function $\tilde{V}_N$ is a uniform time-varying Lyapunov function for the closed-loop system (4.2) on $Y(j) = S(j) \setminus P(j)$ for the families of forward invariant sets $S(j) = \tilde{V}_N^{-1}(j, [0, \theta])$ and $P(j) = \tilde{V}_N^{-1}(j, [0, \delta(N)])$.

Proof. Due to Assumption 4.3 it follows that

$$\begin{aligned} \tilde{V}_N(j, X_j) &= \inf_{\mathbf{U} \in \tilde{\mathcal{U}}_{\text{ad}}^N(j, X_j)} \sum_{k=0}^{N-1} \tilde{\ell}(j+k, X_{\mathbf{U}}(k; j, X_j), U(k)) \\ &\geq \inf_{\mathbf{U} \in \tilde{\mathcal{U}}_{\text{ad}}^N(j, X_j)} \sum_{k=0}^{N-1} \alpha(d(X_{\mathbf{U}}(k; j, X_j), X^s(j+k))) \\ &\geq \alpha(d(X_j, X^s(j))) \end{aligned} \quad (4.22)$$

holds for all $X_j \in \mathcal{R}(\mathcal{X})$ and $j \in \mathbb{N}_0$. Thus, the comparison function $\alpha_1 = \alpha \in \mathcal{K}_\infty$ yields a valid lower bound of the optimal value function.

In the next step we establish an upper bound. Since Assumption 4.14 implies that $\tilde{V}_N(j, X^s(j)) = 0$ for all $j \in \mathbb{N}_0$ we obtain by Assumption 4.12(ii) that

$$|\tilde{V}(j, X_j)| \leq \gamma_{\tilde{V}}(d(X_j, X^s(j)))$$

for all $X_j \in \bar{B}_{\tilde{r}}^d(X^s(j))$, $j \in \mathbb{N}_0$. Furthermore, by dissipativity we can conclude that for $\theta_1 := \alpha^{-1}(\tilde{r})$ the inequality

$$\theta_1 \geq \tilde{V}_N(X_j) \geq \alpha(d(X_j, X^s(j))) \quad (4.23)$$

holds, which yields $\tilde{V}_N^{-1}(j, [0, \theta_1]) \subseteq \bar{B}_{\tilde{r}}^d(X^s(j))$ for all $j \in \mathbb{N}_0$. Thus, the comparison function $\alpha_2 = \gamma_{\tilde{V}} \in \mathcal{K}_\infty$ yields an upper bound of $\tilde{V}_N(j, X_j)$ for all $X_j \in \tilde{V}_N^{-1}(j, [0, \theta_1])$, $j \in \mathbb{N}_0$.

The next part of the proof shows the decrease of the modified optimal value function along the MPC closed-loop trajectory. To this end, consider the composite control sequence $\hat{\mathbf{U}}$ from Lemma 4.17 and define $X_j^+ := X_{\hat{\mathbf{U}}}(1; j, X_j)$. Then, by Lemma 4.17 we obtain

$$\begin{aligned} \tilde{V}_N(j, X_j) + R_4(j, X_j, M, N) &= \tilde{J}_N(j, X_j, \hat{\mathbf{U}}) \\ &= \tilde{\ell}(j, X_j, \mu_N(X_j)) + \tilde{J}_{N-1}(j+1, X_j^+, \hat{\mathbf{U}}(\cdot+1)). \end{aligned}$$

Moreover, since $\hat{\mathbf{U}}(\cdot+1)$ corresponds to the optimal control sequence $\mathbf{U}_{N-1, X_j^+}^*$ up to horizon $M-1$, according to the stochastic dynamic programming principle we can apply Lemma 4.17 again to $\tilde{J}_{N-1}(j+1, X_j^+, \hat{\mathbf{U}}(\cdot+1))$ and get

$$\begin{aligned} \tilde{V}_N(j, X_j) + R_4(j, X_j, M, N) \\ = \tilde{\ell}(j, X_j, \mu_N(X_j)) + \tilde{V}_{N-1}(j+1, X_j^+) + R_4(j+1, X_j^+, M-1, N-1). \end{aligned} \quad (4.24)$$

Furthermore, applying Lemma 4.15 to equation (4.24) in order to estimate the optimal value function with the optimization horizon $N-1$ by the optimal value function with the optimization horizon N leads to

$$\begin{aligned} \tilde{V}_N(j+1, X_j^+) &= \tilde{V}_N(j, X_j) - \tilde{\ell}(j, X_j, \mu_N(X_j)) - R_2(j+1, X_j^+, M-1, N-1) \\ &\quad - R_4(j+1, X_j^+, M-1, N-1) + R_4(j, X_j, M, N). \end{aligned} \quad (4.25)$$

Now, consider $\theta_2 := \alpha^{-1}(r)$ with r from Assumption 4.3. Then, analogous to equation (4.23) for all $X_j \in \tilde{V}_N^{-1}(j, [0, \theta_2])$ we obtain $X_j \in \bar{B}_r^d(X^s(j))$ and

$$\theta_2 \geq \tilde{V}_N(X_j) \geq \tilde{\ell}(j+1, X_j^+, U_N^*(1)) \geq \alpha(d(X_j^+, X^s(j+1)))$$

which implies $X_j^+ \in \bar{B}_r^d(X^s(j+1))$. Thus, choosing $L \geq \lceil \vartheta^{-1}(\min\{\varepsilon, \tilde{\varepsilon}\}) \rceil$ and using the estimates on the residual terms from Lemma 4.15 and Lemma 4.17 we obtain that

$$\begin{aligned} & -R_2(j+1, X_j^+, M-1, N-1) - R_4(j+1, X_j^+, M-1, N-1) + R_4(j, X_j, M, N) \\ & \leq |R_2(j+1, X_j^+, M-1, N-1)| + |R_4(j+1, X_j^+, M-1, N-1)| \\ & \quad + |R_4(j, X_j, M, N)| \\ & \leq 8\gamma_{\tilde{V}}(\vartheta(L)) + 4\gamma_{\lambda}(\vartheta(L)) + 4\gamma_V(N-M, \vartheta(L)) \end{aligned} \tag{4.26}$$

holds for all

$$M \in \{0, \dots, N\} \setminus (\bar{\mathcal{Q}}(j+1, X_j^+, L, N-1) \cup \hat{\mathcal{Q}}(j+1, X_j^+, L, N-1) \cup \hat{\mathcal{Q}}(j, X_j, L, N)).$$

Because each of the sets $\bar{\mathcal{Q}}(j+1, X_j^+, L, N-1)$, $\hat{\mathcal{Q}}(j+1, X_j^+, L, N-1)$, $\hat{\mathcal{Q}}(j, X_j, L, N)$ contains at most $2L$ elements, we can guarantee for $N \geq 12L$ that there exists at least one such M satisfying $M \leq \frac{N}{2}$ and $N-M \geq \frac{N}{2}$.

Hence, for all $N \geq N_0 := 12\lceil \vartheta^{-1}(\min\{\varepsilon, \tilde{\varepsilon}\}) \rceil$ an upper bound $\nu(N)$ of (4.26) only depending on N is given by

$$\nu(N) := 8\gamma_{\tilde{V}}\left(\vartheta\left(\frac{N}{12}\right)\right) + 4\gamma_{\lambda}\left(\vartheta\left(\frac{N}{12}\right)\right) + 4\gamma_V\left(\frac{N}{2}, \vartheta\left(\frac{N}{12}\right)\right).$$

Using this estimate, the upper bound on $\tilde{V}_N$, and dissipativity in equation (4.25) we obtain the inequality

$$\tilde{V}_N(j+1, X_j^+) \leq \tilde{V}_N(j, X_j) - \eta(\tilde{V}_N(j, X_j)) + \nu(N)$$

with $\eta := \alpha \circ \alpha_2^{-1} \in \mathcal{K}_{\infty}$. Now define $P(j) := \tilde{V}_N^{-1}(j, [0, \delta(N)])$ with

$$\delta(N) := \max\{\eta^{-1}(2\nu(N)), \eta^{-1}(\nu(N)) + \nu(N)\}.$$

Then, for all $X_j \in \tilde{V}_N^{-1}(j, [0, \theta_2]) \setminus P(j)$ the inequality

$$\tilde{V}_N(j, X_j) \geq \delta(N) \geq \eta^{-1}(2\nu(N))$$

holds, which implies

$$\nu(N) \leq \frac{\eta(\tilde{V}_N(j, X_j))}{2}.$$

Thus, using the lower bound on $\tilde{V}_N$ we obtain

$$\tilde{V}_N(j+1, X_j^+) \leq \tilde{V}_N(j, X_j) - \eta(\tilde{V}_N(j, X_j)) + \nu(N) \tag{4.27}$$

$$\begin{aligned} & \leq \tilde{V}_N(j, X_j) - \frac{\eta(\tilde{V}_N(j, X_j))}{2} \\ & \leq \tilde{V}_N(j, X_j) - \frac{\eta(\alpha_1(d(X_j, X^s(j))))}{2}, \end{aligned} \tag{4.28}$$

which shows the decrease property of $\hat{V}_N$ for all $X_j \in \tilde{V}_N^{-1}(j, [0, \theta_2]) \setminus P(j)$ with $\alpha_V(r) = \nu(\alpha_1(r))/2$. Thus, for $P(j) = \tilde{V}_N^{-1}(j, [0, \delta(N)])$ and $S(j) = \tilde{V}_N^{-1}(j, [0, \theta])$ with $\theta \leq \theta_0 := \min\{\alpha^{-1}(r), \alpha^{-1}(\tilde{r})\}$ all conditions from Definition 4.8 are satisfied on $Y(j) = S(j) \setminus P(j)$.

It remains to show that the sets $S(j)$ and $P(j)$ are forward invariant. We start with the set $P(j)$. Let $X_j \in P(j)$, and hence $\tilde{V}_N(j, X_j) \leq \delta(N)$. In order to show that $X_j^+ \in P(j+1)$, we distinguish two different cases.

1.) Assume that $\eta(\tilde{V}_N(j, X_j)) \geq \nu(N)$ holds. Then, by (4.27) we obtain

$$\tilde{V}_N(j+1, X_j^+) \leq \tilde{V}_N(j, X_j) - \eta(\tilde{V}_N(j, X_j)) + \nu(N) \leq \tilde{V}_N(j, X_j) \leq \delta(N).$$

2.) Assume that $\eta(\tilde{V}_N(j, X_j)) < \nu(N)$ holds. Then, using (4.27) we get

$$\begin{aligned} \tilde{V}_N(j+1, X_j^+) &\leq \tilde{V}_N(j, X_j) - \eta(\tilde{V}_N(j, X_j)) + \nu(N) \\ &\leq \tilde{V}_N(j, X_j) + \nu(N) \\ &\leq \eta^{-1}(\nu(N)) + \nu(N) \leq \delta(N). \end{aligned}$$

Hence, in both cases we receive $\tilde{V}_N(j+1, X_j^+) \leq \delta(N)$ which implies $X_j^+ \in P(j+1)$, and thus, shows the forward invariance of $P(j)$. Now consider $X_j \in S(j) \setminus P(j)$. Then, by equation (4.28) we obtain

$$\tilde{V}_N(j+1, X_j^+) \leq \tilde{V}_N(j, X_j) - \frac{\eta(\alpha_1(d(X, X^s(j))))}{2} \leq \tilde{V}_N(j, X_j) \leq \theta,$$

which implies $X_j^+ \in S(j+1)$, and thus, together with the invariance of $P(j)$ proves the forward invariance of $S(j)$. $\square$

Note that Theorem 4.18 establishes only local P-practical asymptotic stability, since the underlying assumptions are imposed locally. However, by suitably strengthening Assumption 4.3 and Assumption 4.12(ii), this result can be extended in the following semi-global sense.

Corollary 4.19. *Let Assumptions 4.3, 4.4, 4.12, and 4.14 hold. Furthermore, let the Assumptions 4.3 and 4.12 hold for all $r, \tilde{r} > 0$.*

Then, the stationary process $\mathbf{X}^s$ is semi-globally practically asymptotically stable with respect to the optimization horizon N , i.e., there exists $\beta \in \mathcal{KL}$ such that the following property holds: For each $\Delta_1 > 0$ and $\Delta_2 > \Delta_1$ there exists $N_{\Delta_1, 2} \in \mathbb{N}$ such that for all $N \geq N_{\Delta_1, 2}$, all $j \in \mathbb{N}_0$, and all $(j_0, X_{j_0}) \in \bar{B}_{\Delta_2}^d(\mathbf{X}^s)$ the inequality

$$d(X_{\mu_N}(j; j_0, X_{j_0}), X^s(j + j_0)) \leq \max\{\beta(d(X_{j_0}, X^s(j_0)), j), \Delta_1\}$$

holds.

Proof. Since the Assumptions 4.3 and 4.12 hold for all $r, \tilde{r} > 0$, we can set $r = \tilde{r} = \alpha^{-1}(\gamma_{\tilde{V}}(\Delta_2))$ implying that we can choose $\theta = \theta_0 = \gamma_{\tilde{V}}(\Delta_2)$ in Theorem 4.18. Further, consider $N_{\Delta_1, 2} \geq N_0$ large enough such that $\alpha(\delta(N_{\Delta_1, 2})) \leq \Delta_1$ holds with $N_0 > 0$ and $\delta \in \mathcal{L}$ from Theorem 4.18. Then, the statement follows directly by Theorem 4.18, since these choices imply that $\bar{B}_{\Delta_2}^d(X^s(j)) \subseteq S(j)$ and $P(j) \subseteq \bar{B}_{\Delta_1}^d(X^s(j))$ hold for all $j \in \mathbb{N}_0$ and $S(j), P(j)$ from Theorem 4.18. $\square$

Remark 4.20. *Although it is not clear whether the assumptions of this chapter being satisfied for a metric d_1 implies that they are also satisfied for every weaker metric $d_2 \preceq d_1$, we can still derive estimates with respect to d_2 . In particular, from Theorem 4.18 we obtain that*

$$d_2(X_{\mu_N}(j; j_0, X_{j_0}), X^s(j + j_0)) \leq \gamma^{-1}(\beta(d_1(X_{j_0}, X^s(j_0)), j)) \quad (4.29)$$

holds for all $X_{j_0} \in \tilde{V}_N^{-1}(j, [0, \theta])$ and all $j_0, j \in \mathbb{N}_0$ with $j \geq j_0$ and $X_{\mu_N}(j; j_0, X_{j_0}) \notin \tilde{V}_N^{-1}(j_0, [0, \delta(N)])$. Moreover, Corollary 4.19 yields that

$$d_2(X_{\mu_N}(j; j_0, X_{j_0}), X^s(j + j_0)) \leq \max\{\gamma^{-1}(\beta(d_1(X_{j_0}, X^s(j_0)), j)), \gamma^{-1}(\Delta_1)\} \quad (4.30)$$

for all $\Delta_1 > 0$, $\Delta_2 > \Delta_1$, $N \geq N_{\Delta_1, 2}$, all $j \in \mathbb{N}_0$, and all $(j_0, X_{j_0}) \in \bar{B}_{\Delta_2}^d(\mathbf{X}^s)$, where $\gamma \in \mathcal{K}_\infty$ is given by the dominance relation from Definition 3.26. Hence, a stability-type inequality also holds with respect to the weaker metric d_2 , even though the assumptions are not necessarily satisfied for this metric. However, the admissible set of initial conditions (j_0, X_{j_0}) and the corresponding convergence rate are still determined by the original metric d_1 . Consequently, (4.29) and (4.30) guarantee convergence of the closed-loop trajectories to the stationary stochastic process with respect to the weaker metric d_2 for all initial values that are sufficiently close to the stationary process measured in the metric d_1 .

4.3 Near-optimal performance results

After analyzing the stability properties of the closed-loop solution, we now turn to the performance of the trajectories generated by the proposed MPC algorithm. Our objective is to show that these trajectories are near-optimal.

In the following, we derive estimates for both the non-averaged and the averaged performance. In particular, we show that the MPC closed-loop trajectory is approximately overtaking optimal and that its averaged performance approximately attains the optimal value characterized in Lemma 3.8.

The term “approximately” hereby means that the performance gap between the MPC solution and the optimal value can be bounded by error terms that depend solely on either the optimization horizon N or the number of closed-loop iterations K . In particular, as $N \rightarrow \infty$ and $K \rightarrow \infty$, these error terms vanish, and the performance of the closed-loop and the optimal trajectory coincide.

4.3.1 Non-averaged performance estimates

We begin by deriving non-averaged performance estimates. As in the previous section, we first prepare the main results by establishing a series of auxiliary lemmas. Since the arguments in this chapter rely crucially on the infinite-horizon turnpike property, we impose the following assumption.

Assumption 4.21. *The continuity property of the infinite-horizon shifted optimal value function from Definition 3.21 with respect to the pseudometric d from Assumption 4.3 holds, i.e., there exist $\varepsilon > 0$ and a function $\gamma_{V_\infty} \in \mathcal{K}_\infty$ such that*

$$\left| \hat{V}_\infty(j, X_j) - \hat{V}_\infty(j, X^s(j)) \right| \leq \gamma_{V_\infty}(d(X_j, X^s(j)))$$

holds for all $(j, X_j) \in \bar{B}_\varepsilon^d(\mathbf{X}^s)$.

Note that this assumption, together with Assumption 4.3, implies the uniform infinite-horizon turnpike property on $\bar{B}_r^d(\mathbf{X}^s)$.

Moreover, whenever we refer to the functions ϑ from (3.20) or ϑ_∞ from (3.25), we always mean their uniform versions. In particular, we consider $C_\lambda^j = C_\lambda$ in (3.20) and (3.25).

The following lemma relates the cost of infinite-horizon optimal trajectories to the cost evaluated on finite horizons.

Lemma 4.22. *Let Assumptions 4.3 and 4.21 hold. Consider $\vartheta_\infty \in \mathcal{L}$ from equation (3.25) and $\hat{\varepsilon}$ from Assumption 4.21.*

Then, for all $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$ and $N, L \in \mathbb{N}$ with $L \geq \vartheta_\infty^{-1}(\hat{\varepsilon})$ there exists a set $\hat{\mathcal{Q}}(j, X_j, L, \infty)$ with $\#\hat{\mathcal{Q}}(j, X_j, L, \infty) \leq L$ such that for all $M \in \mathbb{N} \setminus \hat{\mathcal{Q}}(j, X_j, L, \infty)$ the estimate

$$\hat{V}_\infty(j, X_j) = \hat{J}_M(X_j, \mathbf{U}_\infty^*) + \hat{V}_\infty(j + M, X^s(j + M)) + \hat{R}_1(j, X_j, M) \quad (4.31)$$

holds with $|\hat{R}_1(j, X_j, M)| \leq \gamma_{V_\infty}(\vartheta_\infty(L))$.

Proof. The infinite horizon dynamic programming principle from Theorem 2.18 yields

$$\hat{V}_\infty(j, X_j) = \hat{J}_M(X_j, \mathbf{U}_\infty^*) + \hat{V}_\infty(j + M, X_{\mathbf{U}_\infty^*}(M; j, X_j))$$

for all $M \in \mathbb{N}$. Hence, equation (4.31) holds with

$$\hat{R}_1(j, X_j, M) = \hat{V}_\infty(j + M, X_{\mathbf{U}_\infty^*}(M; j, X_j)) - \hat{V}_\infty(j + M, X^s(j + M)).$$

By the uniform infinite-horizon turnpike property from Theorem 3.25 we know that there exists a set $\hat{\mathcal{Q}}(j, X_j, L, \infty)$ with $\#\hat{\mathcal{Q}}(j, X_j, L, \infty) \leq L$ such that

$$d(X_{\mathbf{U}_\infty^*}(M; j, X_j), X^s(j + M)) \leq \vartheta_\infty(L) \leq \hat{\varepsilon}$$

for all $L \geq \vartheta_\infty^{-1}(\hat{\varepsilon})$ and $M \in \mathbb{N} \setminus \hat{\mathcal{Q}}(j, X_j, L, \infty)$.

Thus, using the continuity Assumption 4.21 we obtain

$$\begin{aligned} |R_1(j, X_j, M)| &= |\hat{V}_\infty(j + M, X_{\mathbf{U}_\infty^*}(M; j, X_j)) - \hat{V}_\infty(j + M, X^s(j + M))| \\ &\leq \gamma_{V_\infty}(\vartheta_\infty(L)) \end{aligned}$$

for all $M \in \mathbb{N} \setminus \hat{\mathcal{Q}}(j, X_j, L, \infty)$. □

Lemma 4.22 states that the optimal cost on infinite horizon is approximately equal to the cost of the optimal trajectory evaluated only up to some appropriately chosen time index M plus the optimal cost in the stationary process. The next lemma shows that this statement also holds for the optimal cost on finite horizons $N \in \mathbb{N}$.

Lemma 4.23. *Let Assumptions 4.3 and 4.4 hold. Consider $\vartheta \in \mathcal{L}$ from equation (3.20) and ε from Assumption 4.4.*

Then, for all $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$ and all $N, L \in \mathbb{N}$ with $L \geq \vartheta^{-1}(\varepsilon)$ there exists a set $\mathcal{Q}(j, X_j, L, N)$ with $\#\mathcal{Q}(j, X_j, L, N) \leq L$ such that for all $M \in \{0, \dots, N\} \setminus \mathcal{Q}(j, X_j, L, N)$ the estimate

$$\hat{V}_N(j, X_j) = \hat{J}_M(X_j, \mathbf{U}_N^*) + \hat{V}_{N-M}(j + M, X^s(j + M)) + \hat{R}_2(j, X_j, M, N) \quad (4.32)$$

holds with $|\hat{R}_2(j, X_j, M, N)| \leq \gamma_V(N - M, \vartheta(L))$.

Proof. The finite horizon dynamic programming principle from Theorem 2.17 yields

$$\hat{V}_N(j, X_j) = \hat{J}_M(X_j, \mathbf{U}_N^*) + \hat{V}_{N-M}(j+M, X_{\mathbf{U}_N^*}(M; j, X_j))$$

for all $M \in \{0, \dots, N\}$. Hence, equation (4.32) holds with

$$\hat{R}_2(j, X_j, M, N) = \hat{V}_{N-M}(j+M, X_{\mathbf{U}_N^*}(M, X_j)) - \hat{V}_{N-M}(j+M, X^s(j+M)).$$

By the uniform finite-horizon turnpike property from Theorem 3.25 we know that there exists a set $\mathcal{Q}(j, X_j, L, N)$ with $\#\mathcal{Q}(j, X_j, L, N) \leq L$ such that

$$d(X_{\mathbf{U}_\infty^*}(j; M, X), X^s(j+M)) \leq \vartheta(L) \leq \varepsilon$$

for all $L \geq \vartheta^{-1}(\varepsilon)$ and $M \in \{0, \dots, N\} \setminus \mathcal{Q}(j, X_j, L, N)$. Thus, we get

$$\begin{aligned} |\hat{R}_2(j, X_j, M, N)| &= |\hat{V}_{N-M}(j, X_{\mathbf{U}_N^*}(M; j, X_j)) - \hat{V}_{N-M}(j+M, X^s(j+M)^s)| \\ &\leq \gamma_V(N-M, \vartheta(L)) \end{aligned}$$

for all $M \in \{0, \dots, N\} \setminus \mathcal{Q}(j, X_j, L, N)$. □

The final lemma estimates the difference between the costs of the finite and infinite horizon optimal control sequences.

Lemma 4.24. *Let Assumptions 4.3, 4.4, and 4.21 hold. Consider $\vartheta \in \mathcal{L}$ from equation (3.20), $\vartheta_\infty \in \mathcal{L}$ from equation (3.25), ε from Assumption 4.4, and $\hat{\varepsilon}$ from Assumption 4.21.*

Then, for all $(j, X_j) \in \bar{B}_r^d(\mathbf{X}^s)$, all $N \in \mathbb{N}$, all $L \in \mathbb{N}$ with

$$L \geq \max\{\vartheta^{-1}(\hat{\varepsilon}), \vartheta_\infty^{-1}(\hat{\varepsilon}), \vartheta_\infty^{-1}(\varepsilon), \vartheta^{-1}(\varepsilon)\},$$

and all $M \in \{0, \dots, N\} \setminus \left(\mathcal{Q}(j, X_j, L, N) \cup \hat{\mathcal{Q}}(j, X_j, L, \infty)\right)$ the estimate

$$\hat{J}_M(X_j, \mathbf{U}_\infty^*) = \hat{J}_M(X_j, \mathbf{U}_N^*) + \hat{R}_3(j, X_j, M, N)$$

holds with

$$|\hat{R}_3(j, X_j, M, N)| \leq \gamma_V(N-M, \vartheta_\infty(L)) + \gamma_V(N-M, \vartheta(L)) + \gamma_{V_\infty}(\vartheta(L)) + \gamma_{V_\infty}(\vartheta_\infty(L)).$$

Proof. Consider $R_1(j, X_j, M)$ from Lemma 4.22 and define

$$\tilde{R}_1(j, X_j, M, N) := \hat{V}_\infty(j+M, X_{\mathbf{U}_N^*}(M; j, X_j)) - \hat{V}_\infty(j+M, X^s(j+M)).$$

Then, analogous to the proof Lemma 4.22 using the infinite-horizon continuity property from Assumption 4.21 and the uniform finite-horizon turnpike property from Theorem 3.19 one can show that $|\tilde{R}_1(j, X_j, M, N)| \leq \gamma_{V_\infty}(\vartheta(L))$ for all $L \geq \vartheta^{-1}(\hat{\varepsilon})$ and $M \in \{0, \dots, N\} \setminus \mathcal{Q}(j, X_j, L, N)$.

Furthermore, by the infinite horizon dynamic programming principle from Theorem 2.18 it holds that

$$\begin{aligned} &\hat{J}_M(X_j, \mathbf{U}_\infty^*) + \hat{V}_\infty(j+M, X_{\mathbf{U}_\infty^*}(M; j, X_j)) \\ &\leq \hat{J}_M(X_j, \mathbf{U}_N^*) + \hat{V}_\infty(j+M, X_{\mathbf{U}_N^*}(M; j, X_j)), \end{aligned}$$

which implies

$$\hat{J}_M(X_j, \mathbf{U}_\infty^*) \leq \hat{J}_M(X_j, \mathbf{U}_N^*) + \tilde{R}_1(j, X_j, M, N) - \hat{R}_1(j, X_j, M)$$

with $|\hat{R}_1(j, X_j, M)| \leq \gamma_{V_\infty}(\vartheta_\infty(L))$ for all $L \geq \vartheta_\infty^{-1}(\hat{\varepsilon})$, and all $M \in \mathbb{N} \setminus \hat{\mathcal{Q}}(j, X_j, L, \infty)$ due to Lemma 4.22.

To show the converse inequality, we consider $R_2(j, X_j, M, N)$ from Lemma 4.23 and define

$$\tilde{R}_2(j, X_j, M, N) := \hat{V}_{N-M}(j+M, X_{\mathbf{U}_\infty^*}(M; j, X_j)) - \hat{V}_{N-M}(j+M, X^s(j+M)).$$

Again, using the finite-horizon continuity property from Assumption 4.4 and the infinite-horizon turnpike property from Theorem 3.25 one can show analogous to Lemma 4.23 that $|\tilde{R}_2(j, X_j, M, N)| \leq \gamma_V(N-M, \vartheta_\infty(L))$ for all $L \geq \vartheta_\infty^{-1}(\varepsilon)$ and all $M \in \mathbb{N} \setminus \hat{\mathcal{Q}}(j, X_j, L, \infty)$.

By the finite horizon dynamic programming principle from Theorem 2.17 we get

$$\begin{aligned} & \hat{J}_M(X_j, \mathbf{U}_N^*) + \hat{V}_{N-M}(j+M, X_{\mathbf{U}_N^*}(M; j, X_j)) \\ & \leq \hat{J}_M(X_j, \mathbf{U}_\infty^*) + \hat{V}_{N-M}(j+M, X_{\mathbf{U}_\infty^*}(M; j, X_j)), \end{aligned}$$

which yields

$$\hat{J}_M(X_j, \mathbf{U}_N^*) \leq \hat{J}_M(X_j, \mathbf{U}_\infty^*) + \tilde{R}_2(j, X_j, M, N) - \hat{R}_2(j, X_j, M, N),$$

where $|\hat{R}_2(j, X_j, M, N)| \leq \gamma_V(N-M, \vartheta(L))$ holds for all $L \geq \vartheta^{-1}(\varepsilon)$ and all $M \notin \mathcal{Q}(j, X_j, L, N)$, cf. Lemma 4.23.

Hence, we conclude for all $L \geq \max\{\vartheta^{-1}(\hat{\varepsilon}), \vartheta_\infty^{-1}(\hat{\varepsilon}), \vartheta_\infty^{-1}(\varepsilon), \vartheta^{-1}(\varepsilon)\}$ and all $M \in \{0, \dots, N\} \setminus (\mathcal{Q}(X, L, N) \cup \mathcal{Q}(X, L, \infty))$ that

$$\begin{aligned} |R_3(j, X_j, M, N)| &:= |\hat{J}_M(X_j, \mathbf{U}_N^*) - \hat{J}_M(X_j, \mathbf{U}_\infty^*)| \\ &\leq \max\{|\tilde{R}_1(j, X_j, M, N)| + |R_1(j, X_j, M)|, \\ &\quad |\tilde{R}_2(j, X_j, M, N)| + |R_2(j, X_j, M, N)|\} \\ &\leq \max\{\gamma_{V_\infty}(\vartheta(L)) + \gamma_{V_\infty}(\vartheta_\infty(L)), \gamma_V(N-M, \vartheta_\infty(L)) + \gamma_V(N-M, \vartheta(L))\} \\ &\leq \gamma_{V_\infty}(\vartheta(L)) + \gamma_{V_\infty}(\vartheta_\infty(L)) + \gamma_V(N-M, \vartheta_\infty(L)) + \gamma_V(N-M, \vartheta(L)), \end{aligned}$$

which proves the claim. $\square$

Now we can state our first main result for the closed-loop performance of the stochastic MPC Algorithm. Here we denote by $\hat{J}_K^{cl}(X_0, \mu_N)$ the closed-loop cost (4.3) with respect to the shifted stage cost from (3.4), i.e.,

$$\hat{J}_K^{cl}(X_0, \mu) := \sum_{j=0}^{K-1} \hat{\ell}(X_\mu(j; 0, X_0), \mu(X_\mu(j; 0, X_0))).$$

Theorem 4.25. *Let Assumptions 4.3, 4.4, 4.12, 4.14, and 4.21 hold. Then, there exist $\theta_0 > 0$ and $\delta_1 \in \mathcal{L}$ such that for all $K \in \mathbb{N}$, each sufficient large $N \in \mathbb{N}$, and all $X_0 \in \tilde{V}_N^{-1}([0, \theta_0])$ the closed-loop cost satisfies*

$$\hat{J}_K^{cl}(X_0, \mu_N) + \hat{V}_\infty(K, X_{\mu_N}(K; 0, X_0)) \leq \hat{V}_\infty(0, X_0) + K\delta_1(N).$$

Proof. Consider θ_0 from Theorem 4.18, $X_0 \in \tilde{V}_N^{-1}(0, [0, \theta_0])$, and the corresponding closed-loop trajectory $X_{\mu_N}(\cdot; 0, X_0)$. Using the finite horizon dynamic programming principle, cf. Theorem 2.17, and $\mu_N(X_{\mu_N}(j; 0, X_0)) := U_{N, X_{\mu_N}(j; 0, X_0)}^*(0)$ we know that

$$\begin{aligned} & \hat{\ell}(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(j; 0, X_0))) \\ &= \hat{V}_N(j, X_{\mu_N}(j; 0, X_0)) - \hat{V}_{N-1}(j+1, X_{\mu_N}(j+1; 0, X_0)) \end{aligned}$$

holds for all $j \in \mathbb{N}$. By the definition of the optimal value function and the fact that

$$U_{N, X_{\mu_N}(j; 0, X_0)}^*(s+1) = U_{N-1, X_{\mu_N}(j+1; 0, X_0)}^*(s)$$

for all $s = 0, \dots, N-2$, we get

$$\begin{aligned} & \hat{V}_N(j, X_{\mu_N}(j; 0, X_0)) - \hat{V}_{N-1}(j+1, X_{\mu_N}(j+1; 0, X_0)) \\ &= \hat{J}_N(X_{\mu_N}(j; 0, X_0), \mathbf{U}_{N, X_{\mu_N}(j; 0, X_0)}^*) - \hat{J}_{N-1}(X_{\mu_N}(j+1; 0, X_0), \mathbf{U}_{N-1, X_{\mu_N}(j+1; 0, X_0)}^*) \\ &= \hat{J}_K(X_{\mu_N}(j; 0, X_0), \mathbf{U}_{N, X_{\mu_N}(j; 0, X_0)}^*) - \hat{J}_{K-1}(X_{\mu_N}(j+1; 0, X_0), \mathbf{U}_{N-1, X_{\mu_N}(j+1; 0, X_0)}^*) \end{aligned}$$

for all $K \in \{1, \dots, N\}$.

Now consider $L \geq \max\{\vartheta^{-1}(\hat{\varepsilon}), \vartheta_{\infty}^{-1}(\hat{\varepsilon}), \vartheta_{\infty}^{-1}(\varepsilon), \vartheta^{-1}(\varepsilon)\}$ and $N > 8L$. Then, there exists at least one $K \in \{1, \dots, N\}$ with

$$K \notin \mathcal{Q}(j, X_{\mu_N}(j; 0, X_0), L, N) \cup \mathcal{Q}(j, X_{\mu_N}(j; 0, X_0), L, \infty)$$

and

$$K-1 \notin \mathcal{Q}(j+1, X_{\mu_N}(j+1; 0, X_0), L, N-1) \cup \mathcal{Q}(j+1, X_{\mu_N}(j+1; 0, X_0), L, \infty)$$

such that $K \leq \frac{N}{2}$.

Furthermore, since the choice of N and L imply that

$$N \geq 8\lceil \vartheta^{-1}(\min\{\varepsilon, \hat{\varepsilon}\}) \rceil,$$

by Theorem 4.18 we know that

$$X_{\mu_N}(j; 0, X_0) \in \tilde{V}_N^{-1}(j, [0, \theta_0]) \subseteq \bar{B}_r^d(X^s(j))$$

holds for all $j \in \mathbb{N}_0$.

Hence, we can apply Lemma 4.24 twice with $M = K$, $N = N$ and $M = K-1$, $N = N-1$, respectively, to conclude that

$$\begin{aligned} & \hat{J}_K(X_{\mu_N}(j; 0, X_0), \mathbf{U}_{N, X_{\mu_N}(j; 0, X_0)}^*) - \hat{J}_K(X_{\mu_N}(j+1; 0, X_0), \mathbf{U}_{N-1, X_{\mu_N}(j+1; 0, X_0)}^*) \\ &= \hat{J}_K(X_{\mu_N}(j; 0, X_0), \mathbf{U}_{\infty, X_{\mu_N}(j; 0, X_0)}^*) - \hat{J}_K(X_{\mu_N}(j+1; 0, X_0), \mathbf{U}_{\infty, X_{\mu_N}(j+1; 0, X_0)}^*) \\ & \quad - \hat{R}_3(j, X_{\mu_N}(j; 0, X_0), K, N) + \hat{R}_3(j+1, X_{\mu_N}(j+1; 0, X_0), K-1, N-1). \end{aligned}$$

Moreover, by applying Lemma 4.22 for $M = K$ and $M = K-1$ we get

$$\begin{aligned} & \hat{J}_K(X_{\mu_N}(j; 0, X_0), \mathbf{U}_{\infty, X_{\mu_N}(j; 0, X_0)}^*) - \hat{J}_K(X_{\mu_N}(j+1; 0, X_0), \mathbf{U}_{\infty, X_{\mu_N}(j+1; 0, X_0)}^*) \\ &= \hat{V}_{\infty}(j, X_{\mu_N}(j; 0, X_0)) - \hat{V}_{\infty}(j+1, X_{\mu_N}(j+1; 0, X_0)) \\ & \quad - \hat{R}_1(j, X_{\mu_N}(j; 0, X_0), K) + \hat{R}_1(j+1, X_{\mu_N}(j+1; 0, X_0), K-1). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \hat{\ell}(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(j; 0, X_0))) \\ &= \hat{V}_\infty(j, X_{\mu_N}(j; 0, X_0)) - \hat{V}_\infty(j+1, X_{\mu_N}(j+1; 0, X_0)) \\ & \quad + \hat{R}_4(j, X_{\mu_N}(j; 0, X_0), K, N) \end{aligned} \quad (4.33)$$

with

$$\begin{aligned} & \hat{R}_4(j, X_{\mu_N}(j; 0, X_0), K, N) \\ &= -\hat{R}_1(j, X_{\mu_N}(j; 0, X_0), K) + \hat{R}_1(j+1, X_{\mu_N}(j+1; 0, X_0), K-1) \\ & \quad - \hat{R}_3(j, X_{\mu_N}(j; 0, X_0), K, N) + \hat{R}_3(j+1, X_{\mu_N}(j+1; 0, X_0), K-1, N-1). \end{aligned}$$

In addition, from the monotonicity of γ_V together with Lemmas 4.22 and 4.24 we obtain the bound

$$\begin{aligned} |\hat{R}_4(j, X_{\mu_N}(j; 0, X_0), K, N)| &\leq 4\gamma_{V_\infty}(\vartheta_\infty(L)) + 2\gamma_V(N-K, \vartheta_\infty(L)) \\ &\quad + 2\gamma_V(N-K, \vartheta(L)) + 2\gamma_{V_\infty}(\vartheta(L)). \end{aligned}$$

Using again the monotonicity of the γ_V and $L < \frac{N}{8}$, $N-K \geq \frac{N}{2}$ we get

$$|\hat{R}_4(j, X_{\mu_N}(j; 0, X_0), K, N)| \leq \delta_1(N)$$

with

$$\delta_1(N) := 4\gamma_{V_\infty}\left(\vartheta_\infty\left(\frac{N}{8}\right)\right) + 2\gamma_V\left(\frac{N}{2}, \vartheta_\infty\left(\frac{N}{8}\right)\right) + 2\gamma_V\left(\frac{N}{2}, \vartheta\left(\frac{N}{8}\right)\right) + 2\gamma_{V_\infty}\left(\vartheta\left(\frac{N}{8}\right)\right).$$

Summing along the closed-loop trajectory and applying equation 4.33 thus yields

$$\begin{aligned} \hat{J}_K^{cl}(X_0, \mu_N) &= \sum_{j=0}^{K-1} \hat{\ell}(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(j; 0, X_0))) \\ &\leq \sum_{j=0}^{K-1} \left(\hat{V}_\infty(j, X_{\mu_N}(j; 0, X_0)) - \hat{V}_\infty(j+1, X_{\mu_N}(j+1; 0, X_0)) + \delta_1(N) \right) \\ &= \hat{V}_\infty(0, X_{\mu_N}(0)) - \hat{V}_\infty(K, X_{\mu_N}(K)) + K\delta_1(N) \end{aligned}$$

which proves the claim. $\square$

The identity established in Theorem 4.25 admits the following interpretation: if the closed-loop trajectory is followed for K steps and subsequently continued by an infinite-horizon optimal trajectory, then the resulting performance is near-optimal. More precisely, the gap to optimality is bounded by $K\delta(N)$. Note that the formulation of Theorem 4.25 necessarily relies on the shifted cost functional, since otherwise the infinite-horizon optimal value function would fail to attain a finite value, even at the stationary solution.

An alternative way to characterize (near) infinite-horizon optimality without introducing a shifted cost is to revert to the concept of overtaking optimality, cf. Definition 2.15. Before presenting this characterization in Corollary 4.27, however, we show that the closed-loop trajectory is also finite-horizon near-optimal when compared to all other trajectories that exhibit the same stability properties.

To this end, for $K \in \mathbb{N}$ and $\kappa > 0$ we define the set of admissible control sequences on the horizon K whose corresponding state trajectory reaches the κ -tube around $\mathbf{X}^s$ at time K by

$$\tilde{\mathbb{U}}_{\text{ad}}^{K,\kappa}(0, X_0) := \{\mathbf{U} \in \tilde{\mathbb{U}}_{\text{ad}}^K(0, X_0) \mid X_{\mathbf{U}}(K; 0, X_0) \in \bar{B}_{\kappa}^d(X^s(K))\}.$$

Theorem 4.26. *Let Assumptions 4.3, 4.4, 4.14, 4.12, and 4.21 hold. Consider $\alpha \in \mathcal{K}_{\infty}$ from Assumption 4.3, $\delta \in \mathcal{L}$, $N_0 \in \mathbb{N}$ and $\theta_0 > 0$ from Theorem 4.18, $\beta \in \mathcal{KL}$ from (4.5), and $\nu \in \mathcal{L}$ from equation (4.2). Then, there exist $\delta_2, \delta_3 \in \mathcal{L}$ such that for all sufficiently large $N, K \in \mathbb{N}$ and all $X_0 \in \tilde{V}_N^{-1}([0, \theta_0])$ the inequality*

$$J_K^{\text{cl}}(X_0, \mu_N) \leq \inf_{\mathbf{U} \in \tilde{\mathbb{U}}_{\text{ad}}^{K,\kappa}((0, X_0))} J_K^{\text{cl}}(X_0, \mathbf{U}) + \delta_2(N) + K\nu(N) + \delta_3(K)$$

holds with $\kappa = \max\{\beta(d(X_0, X^s(0)), K), \alpha^{-1}(\delta(N))\}$.

Proof. Consider $X_j \in \tilde{V}_N^{-1}(j, [0, \theta_0])$ and set $X_j^+ := f(X_j, \mu_N(X_j), W(j))$. Then, from equation (4.25) we obtain for all $N \geq N_0$ the identity

$$\begin{aligned} \tilde{\ell}(j, X_j, \mu_N(X_j)) &= \tilde{V}_N(j, X_j) - \tilde{V}_N(j+1, X_j^+) - R_2(j+1, X_j^+, M-1, N-1) \\ &\quad - R_4(j+1, X_j^+, M-1, N-1) + R_4(j, X_j, M, N) \end{aligned}$$

with

$$|R_4(j, X_j, M, N) - R_4(j+1, X_j^+, M-1, N-1) - R_2(j+1, X_j^+, M-1, N-1)| \leq \nu(N)$$

and ν from equation (4.2).

Hence, using that $\tilde{V}_N^{-1}(j, [0, \theta_0])$ is a forward invariant set for the closed-loop system and $X_0 \in \tilde{V}_N^{-1}(0, [0, \theta_0])$ holds, summing the cost along the closed-loop trajectory yields

$$\begin{aligned} &\sum_{j=0}^{K-1} \tilde{\ell}(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(j; 0, X_0))) \\ &\leq \tilde{V}_N(0, X_0) - \tilde{V}_N(K, X_{\mu_N}(K)) + K\nu(N). \end{aligned} \tag{4.34}$$

Now consider $N, K \in \mathbb{N}$ such that $\max\{\beta(d(X_0, X^s(0)), K), \alpha^{-1}(\delta(N))\} \leq \tilde{\varepsilon}$ holds with $\tilde{\varepsilon}$ from Assumption 4.12(ii). Thus, if $K \leq N$ holds, we can conclude by Assumption 4.12 that

$$\begin{aligned} \tilde{J}_K(X_0, \mathbf{U}) &= \tilde{J}_K(X_0, \mathbf{U}) + \tilde{V}_{N-K}(K, X_{\mathbf{U}}(K)) - \tilde{V}_{N-K}(K, X_{\mathbf{U}}(K)) \\ &\geq \tilde{V}_N(0, X_0) - \gamma_{\tilde{V}}(\kappa) \end{aligned}$$

holds for all $\mathbf{U} \in \tilde{\mathbb{U}}_{\text{ad}}^{K,\kappa}((0, X_0))$. On the other hand, if $K \geq N$ the non-negativity of $\tilde{\ell}$ implies that $\tilde{J}_K(X_0, \mathbf{U}) \geq \tilde{V}_N(0, X_0)$.

Hence, using Assumption 4.12(i) we obtain

$$\begin{aligned}
& J_K^{cl}(X_0, \mu_N) \\
&= \sum_{j=0}^{K-1} \left(\tilde{\ell}(j, X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(j; 0, X_0))) \right. \\
&\quad \left. - \lambda(0, X_0) + \lambda(K, X_{\mu_N}(K)) + \ell(\mathbf{X}^s, \mathbf{U}^s) \right) \\
&\leq \tilde{V}_N(0, X_0) - \tilde{V}_N(K, X_{\mu_N}(K)) + K\nu(N) - \lambda(0, X_0) + \lambda(K, X_{\mu_N}(K)) + \ell(\mathbf{X}^s, \mathbf{U}^s) \\
&\leq \tilde{J}_K(X_0, \mathbf{U}) + \gamma_{\tilde{V}}(\kappa) - \tilde{V}_N(K, X_{\mu_N}(K)) + K\nu(N) \\
&\quad - \lambda(0, X_0) + \lambda(K, X_{\mu_N}(K)) + \ell(\mathbf{X}^s, \mathbf{U}^s) \\
&\leq J_K(X_0, \mathbf{U}) + \gamma_{\tilde{V}}(\kappa) - \tilde{V}_N(K, X_{\mu_N}(K)) + K\nu(N) - \lambda(X_{\mathbf{U}}(K)) + \lambda(K, X_{\mu_N}(K)) \\
&\leq J_K(X_0, \mathbf{U}) + \gamma_{\tilde{V}}(\kappa) + K\nu(N) + 2\gamma_{\lambda}(\kappa).
\end{aligned}$$

Using the definition of κ and $X_{\mu_N}(j; 0, X_0) \in \tilde{V}_N^{-1}([0, \theta_0]) \subseteq \bar{B}_r^d(X^s(j))$ we can estimate

$$\gamma_{\tilde{V}}(\kappa) + 2\gamma_{\lambda}(\kappa) \leq \delta_2(N) + \delta_3(K)$$

with

$$\delta_2(N) := \gamma_{\tilde{V}}(\alpha^{-1}(\delta(N))) + 2\gamma_{\lambda}(\alpha^{-1}(\delta(N)))$$

and

$$\delta_3(K) := \gamma_{\tilde{V}}(\beta(r, K)) + 2\gamma_{\lambda}(\beta(r, K)),$$

which proves the theorem. $\square$

Using Theorem 4.26 and the uniform infinite-horizon turnpike property from Theorem 3.25 we can additionally show that the closed-loop solution is approximately overtaking optimal, which characterizes the performance of our stochastic MPC scheme on the infinite-horizon.

Corollary 4.27. *Let Assumptions 4.3, 4.4, 4.12, 4.14, and 4.21 hold. Consider $\nu \in \mathcal{L}$ from equation (4.2) and $\delta_2 \in \mathcal{L}$ from Theorem 4.26. Then, for all sufficiently large N and all $X_0 \in \tilde{V}_N^{-1}(0, [0, \theta_0])$ the inequality*

$$\liminf_{K \rightarrow \infty} \left(\sum_{j=0}^{K-1} \left(\ell(X_{\mathbf{U}}(j; 0, X_0), U(j)) - \ell(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(j; 0, X_0))) \right) \right. \\
\left. + K\nu(N) + \delta_2(N) \right) \geq 0$$

holds for all $\mathbf{U} \in \tilde{\mathcal{U}}_{\text{ad}}^{\infty}(0, X_0)$.

Proof. Choose $N, K \in \mathbb{N}$ large enough such that Theorem 4.26 holds and consider the infinite-horizon optimal control sequence $\mathbf{U}_{\infty}^*$. Due to the infinite-horizon turnpike property, cf. Theorem 3.25, we can conclude that there exist a $K_0 \geq K$ such that

$$d(X_{\mathbf{U}_{\infty}^*}(\tilde{K}; 0, X_0), X^s(\tilde{K})) \leq \alpha^{-1}(\delta(N)), \quad \beta(d(X_0, X^s(0)), \tilde{K}) \leq \alpha^{-1}(\delta(N))$$

holds for all $\tilde{K} \geq K_0$.

Hence, we obtain $\mathbf{U}_\infty^* \in \check{\mathcal{U}}_{\text{ad}}^{\tilde{K}, \kappa}(0, X_0)$ and by Theorem 4.26 that

$$J_K^{\text{cl}}(X_0, \mu_N) \leq J_K^{\text{cl}}(X_0, \mathbf{U}_\infty^*) + \delta_2(N) + \tilde{K}\nu(N) + \delta_3(\tilde{K})$$

holds for all $\tilde{K} \geq K_0$.

Since $\delta_3(\tilde{K}) \rightarrow 0$ for $\tilde{K} \rightarrow \infty$, taking the limit yields

$$\begin{aligned} \liminf_{K \rightarrow \infty} \left(\sum_{j=0}^{K-1} \left(\ell(X_{\mathbf{U}_\infty^*}(j; 0, X_0), U_\infty^*(j)) \right. \right. \\ \left. \left. - \ell(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(j; 0, X_0))) \right) \right. \\ \left. + K\nu(N) + \delta_2(N) \right) \geq 0. \end{aligned}$$

Because of the optimality of $\mathbf{U}_\infty^*$ we can further conclude that

$$\liminf_{K \rightarrow \infty} \left(\sum_{j=0}^{K-1} \left(\ell(X_{\mathbf{U}}(j; 0, X_0), U(j)) - \ell(X_{\mathbf{U}_\infty^*}(j; 0, X_0), U_\infty^*(j)) \right) \right) \geq 0$$

holds for all $\mathbf{U} \in \check{\mathcal{U}}_{\text{ad}}^\infty(0, X_0)$, which proves the claim. $\square$

Remark 4.28. One might be worried by the K -dependence of the error term $K\nu(N)$ in Corollary 4.27, because this implies that the deviation from the optimal cost grows linearly in K . However, as the accumulated non-shifted cost itself grows linearly in K (except in the very particular case when the stationary cost $\ell(\mathbf{X}^s, \mathbf{U}^s)$ equals 0), the relative deviation to the non-shifted optimal cost is constant in K .

4.3.2 Averaged performance estimates

After deriving an estimate for the non-averaged performance in Corollary 4.27, we can combine this result with the stability properties established in Theorem 4.9 to obtain bounds on the averaged closed-loop performance, defined by

$$\bar{J}_K^{\text{cl}}(X_0, \mu_N) := \frac{1}{K} J_K^{\text{cl}}(X_0, \mu_N).$$

Since we assume strict stochastic dissipativity, cf. Assumption 4.3, by Lemma 3.8 we can directly conclude that the stag cost $\ell(\mathbf{X}^s, \mathbf{U}^s)$ along the stationary pair is a lower bound on $\bar{J}_K^{\text{cl}}(X_0, \mu_N)$ for $K \rightarrow \infty$, i.e.,

$$\liminf_{K \rightarrow \infty} \bar{J}_K^{\text{cl}}(X_0, \mu_N) \geq \ell(\mathbf{X}^s, \mathbf{U}^s)$$

holds for all $X_0 \in \mathcal{R}(\mathcal{X})$.

The following theorem shows that the asymptotic average performance of the closed-loop trajectory can be made arbitrarily close to this lower bound as the optimization horizon N increases.

Theorem 4.29. *Let Assumptions 4.3, 4.4, 4.12, 4.14, and 4.21 hold. Then, for all $N \in \mathbb{N}$ sufficiently large and all $X_0 \in \tilde{V}_N^{-1}(0, [0, \theta_0])$ the averaged closed-loop cost satisfies*

$$\ell(\mathbf{X}^s, \mathbf{U}^s) \leq \limsup_{K \rightarrow \infty} \bar{J}_K^{\text{cl}}(X_0, \mu_N) \leq \ell(\mathbf{X}^s, \mathbf{U}^s) + \nu(N)$$

with $\nu(N)$ from equation (4.2) and $\theta_0 > 0$ from Theorem 4.18.

Proof. Using equation (4.34) we can conclude that

$$\begin{aligned}
& \hat{J}_K^{cl}(X_0, \mu_N) \\
&= \sum_{j=0}^{K-1} \tilde{\ell}(j, X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(j; 0, X_0))) - \lambda(0, X_0) + \lambda(K, X_{\mu_N}(K)) \quad (4.35) \\
&\leq \tilde{V}_N(0, X_0) - \tilde{V}_N(K, X_{\mu_N}(K)) + K\nu(N) - \lambda(0, X_0) + \lambda(K, X_{\mu_N}(K))
\end{aligned}$$

holds.

Now consider $N \in \mathbb{N}$ large enough such that $\alpha^{-1}(\delta(N)) \leq \tilde{\varepsilon}$ holds with $\tilde{\varepsilon}$ from Assumption 4.12 (ii) and choose K_0 such that $\beta(d(X_0, X^s(0)), K_0) \leq \alpha^{-1}(\delta(N))$ holds. Then, by the stability property from Theorem 4.9 for all $K \geq K_0$ we can conclude that $X_{\mu_N}(K) \in \bar{B}_{\tilde{\varepsilon}}^d(X^s(j))$.

Using equation (4.35), Assumption 4.3 (i) and 4.12 (ii), and the non-negativity of $\tilde{\ell}$ this implies

$$\begin{aligned}
\hat{J}_K^{cl}(X_0, \mu_N) &\leq \tilde{V}_N(0, X_0) - \tilde{V}_N(K, X_{\mu_N}(K)) + K\nu(N) - \lambda(0, X_0) + \lambda(K, X_{\mu_N}(K)) \\
&\leq \theta_0 + K\nu(N) - \lambda(0, X_0) + \lambda(K, X^s(K)) + \gamma_\lambda(\tilde{\varepsilon}) \\
&\leq \theta_0 + C_\lambda^l + C_\lambda + \gamma_\lambda(\tilde{\varepsilon}) + K\nu(N),
\end{aligned}$$

where $-C_\lambda^l$ is a uniform lower bound on the storage function λ .

Dividing by K and taking the $\limsup_{K \rightarrow \infty}$ then yields

$$\begin{aligned}
\limsup_{K \rightarrow \infty} \bar{J}_K^{cl}(X_0, \mu_N) &= \limsup_{K \rightarrow \infty} \frac{1}{K} \hat{J}_K^{cl}(X_0, \mu_N) - \ell(\mathbf{X}^s, \mathbf{U}^s) \\
&\leq \limsup_{K \rightarrow \infty} \frac{\theta_0 + C_\lambda^l + C_\lambda + \gamma_\lambda(\tilde{\varepsilon})}{K} + \nu(N) \\
&= \nu(N),
\end{aligned}$$

which proves the claim. $\square$

Remark 4.30. *In contrast to the stability results from Theorem 4.18 and Corollary 4.19, the performance bounds obtained in Theorem 4.25, Corollary 4.27, and Theorem 4.29 do not depend qualitatively on the metric d for which the assumptions of this chapter are satisfied. However, the set of admissible initial conditions for which these results hold may still depend on the metric d , and the precise form of the error term $\nu(N)$ is determined by the chosen metric. Hence, the metric d may still influence the results quantitatively, while the qualitative behavior of the performance bounds is independent of the metric. In particular, the error term $\nu(N)$ vanishes as $N \rightarrow \infty$, independently of the metric.*

4.4 Numerical results

To illustrate our theoretical findings numerically, we implemented Algorithm 2 in *Python*. Since the examples considered in the following are driven by disturbances with finite support, we solve the resulting stochastic optimal control problems by direct optimization over all possible realization paths using *IPOPT* [104] via *CasADi* [4].

In particular, we consider an example motivated by the nonlinear academic example from Section 3.6.2 and an example modeling the motion of a vehicle subject to external forces.

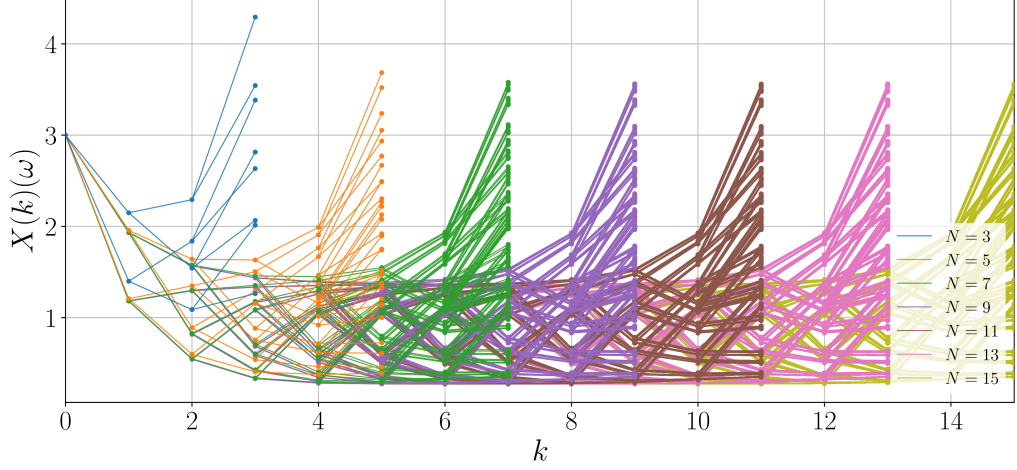


Figure 4.2: Evolution of all realization paths of the optimal solutions of problem (4.36) on different horizons.

Example 1

We consider the one-dimensional nonlinear optimal control problem

$$\begin{aligned}
 \min_{\mathbf{U} \in \tilde{\mathbf{U}}_{\text{ad}}^N(j, X_j)} J_N(X_j, \mathbf{U}) &:= \sum_{k=0}^{N-1} \mathbb{E}[(U(k) - X_{\mathbf{U}}(k; j, X_j))^4 + \gamma U(k)^2] \\
 \text{s.t. } X_{\mathbf{U}}(k+1; j, X_j) &= (U(k) - X_{\mathbf{U}}(k; j, X_j))^2 + W(k+j), \\
 X_{\mathbf{U}}(0; j, X_j) &= X_j
 \end{aligned} \tag{4.36}$$

where $\gamma \geq 0$ is a regularization parameter and $W(k)$ follows a two-point distribution, i.e.,

$$W(k) = \begin{cases} 1 & \text{with probability 0.7} \\ 0.25 & \text{with probability 0.3} \end{cases}$$

for all $k \in \mathbb{N}_0$

For $\gamma = 0$, we have shown analytically in Section 3.6.2 that the problem is L^2 -dissipative. However, in this case the optimal solutions reach the turnpike exactly after one step. To obtain a more complex behavior of the solutions, we therefore set $\gamma = 25$ leading to a slight modification of problem 3.60.

Although in this case stochastic dissipativity can no longer be guaranteed theoretically, stochastic turnpikes can still be observed numerically, cf. Figure 4.4. There, we see that all possible realization paths are close to each other in the middle of the optimization horizon, indicating a pathwise—and thus also a distributional—turnpike property.

Hence, based on our theoretical findings, we expect to observe stability both in a pathwise sense and in distribution. To this end, we generated 5 000 sample paths of the closed-loop solution up to time $k = 150$ using Algorithm 2 with initial condition $X_0 = 3$.

Furthermore, to estimate the characteristics of the stationary stochastic process, we employed the values attained at the middle of the horizon by the solution on horizon $N = 15$ —shown in red in Figure 4.4—as suggested by the turnpike property.

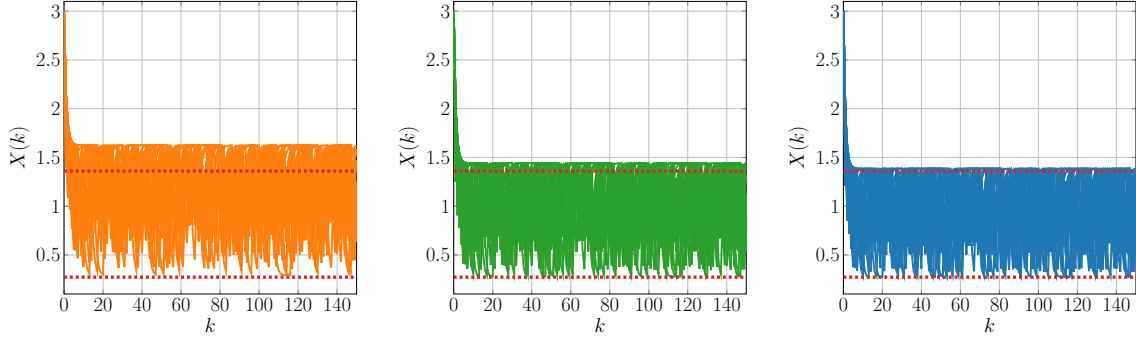


Figure 4.3: 15 sample paths of the closed-loop trajectories of problem (4.36) for different prediction horizons $N = 3$ (left), $N = 4$ (middle), and $N = 5$ (right) together with the upper and lower bound of the support of the stationary process (dashed red).

Figure 4.4 depicts the sample paths of the closed-loop solutions together with the upper and lower bounds of the stationary process realizations. As the horizon increases, the paths converge toward the region of the stationary process realizations, highlighting the practical asymptotic stability established in Theorem 4.18.

In Figure 4.4, we show the empirical cumulative distribution function (CDF) of the closed-loop state $X_{\mu_N}(150; 0, X_0)$ for different prediction horizons N . It can be observed that, as the horizon N increases, the distribution of the closed-loop state converges towards the distribution of the stationary process, illustrating stability in distribution.

This behavior is further supported by the evolution of the moments: Figure 4.4 displays the mean and variance of the closed-loop trajectory for different prediction horizons, both of which approach the corresponding values of the stationary process as N increases.

Together, these results indicate that the closed-loop system not only exhibits pathwise practical asymptotic stability, but also achieves distributional stability in accordance with the theoretical predictions of Section 2.2.

In addition to the stability results, the performance estimates derived in Section 4.3 provide both averaged and non-averaged bounds for the MPC closed loop. These results are illustrated in Figure 4.4.

It can be observed that, for sufficiently long horizons K , the cumulative closed-loop costs increase approximately linearly, with different slopes corresponding to different prediction horizons N . These differences in slope are a direct consequence of the $K\nu(N)$ terms in Corollary 4.27.

Similarly, the effect of the horizon length N is visible in the averaged performance: as N increases, the averaged costs converge more closely to a neighborhood of the stationary stage cost $\ell(\mathbf{X}^s, \mathbf{U}^s)$, as illustrated in the zoomed view. This is in line with the predictions of Theorem 4.29.

Overall, these results confirm that increasing the prediction horizon improves both the long-term and average closed-loop performance, reflecting the theoretical performance guarantees of our MPC scheme

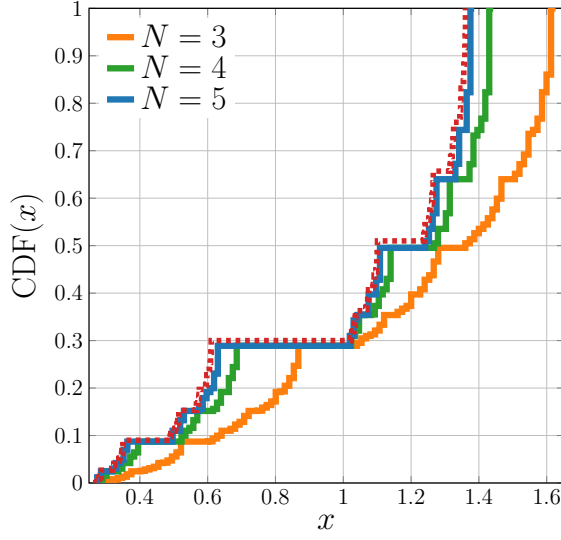


Figure 4.4: Empirical cumulative distribution function of the distributions of the closed-loop trajectory of problem (4.36) at time $k = 150$ for different horizons, together with the approximate stationary distribution (dashed red).

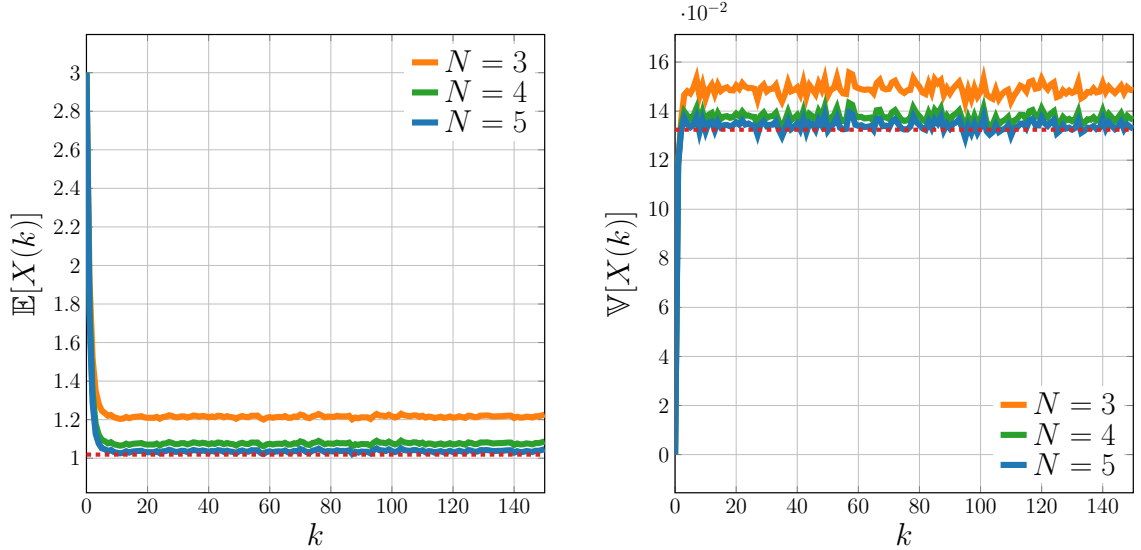


Figure 4.5: Empirical mean and variance of the components of the closed-loop trajectory for problem (4.36) for different prediction horizons, together with the approximate stationary values (red).

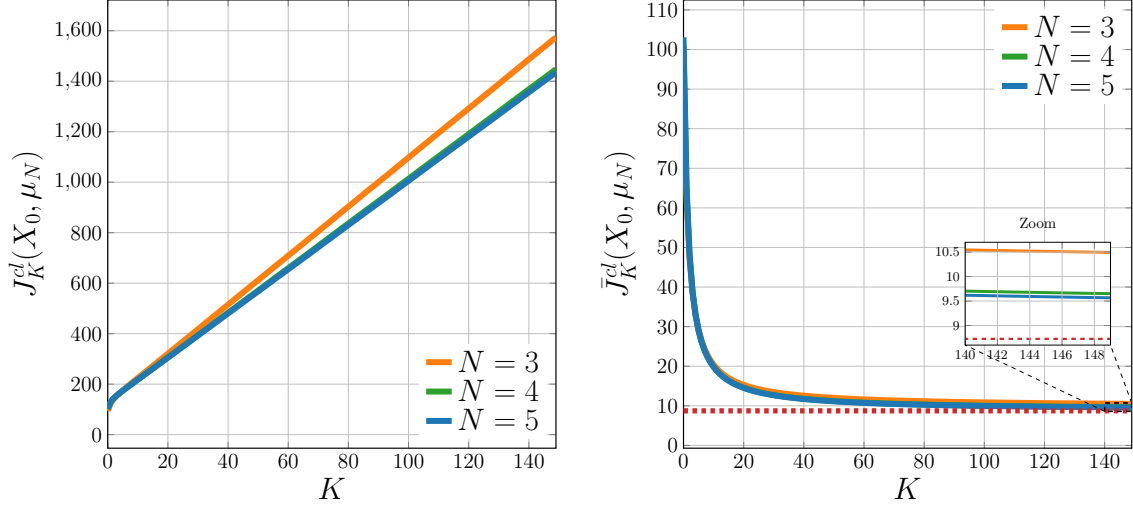


Figure 4.6: Sample-based approximations of the cumulative costs and averaged cost of the closed-loop solutions of problem (4.36) for different prediction horizons, together with the approximation of the stationary stage cost (red).

Example 2

As a second example, we consider the motion of a vehicle modeled by the system of differential equations

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= u(t) - F_2(x_2(t)) + F_1.\end{aligned}$$

Here, $x_1(t) \in \mathbb{R}$ denotes the position of the vehicle, $x_2(t) \in \mathbb{R}$ its velocity, and the control input $u(t) \in \mathbb{R}$ its acceleration. The term

$$F_2(x_2) = \kappa x_2^2, \quad \kappa \in \mathbb{R},$$

represents a velocity-dependent damping force opposing the acceleration, while F_1 denotes an additional external force acting on the system.

To obtain a discrete-time stochastic model, we apply an Euler discretization with step size $h = 1$ and model the term F_2 as a stochastic disturbance $W(k)$. This can, for example, be interpreted as a wind force influencing the motion of the vehicle. This yields the discrete-time dynamics

$$\begin{aligned}X_1(k+1) &= X_1(k) + X_2(k), \\ X_2(k+1) &= X_2(k) + U(k) - \kappa X_2(k)^2 + W(k).\end{aligned}$$

In the stage cost, we penalize the distance between the vehicle position and a given setpoint x_{ref} as well as the control effort, leading to the stochastic stage cost

$$\ell(X, U) = \mathbb{E}[(X_1 - x_{\text{ref}})^2 + \gamma U^2].$$

Hence, in summary, we consider the following two-dimensional stochastic optimal

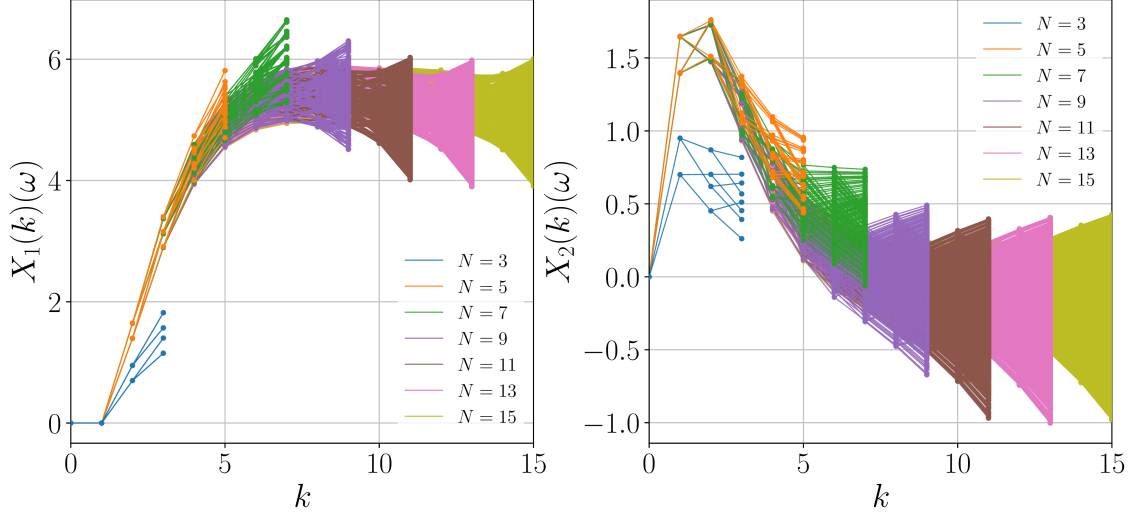


Figure 4.7: Evolution of all realization paths of the optimal solutions of problem (4.37) on different horizons.

control problem

$$\begin{aligned}
& \min_{\mathbf{U} \in \tilde{\mathcal{U}}_{\text{ad}}^N(j, X_j)} J_N(X_j, \mathbf{U}) := \sum_{k=0}^{N-1} \mathbb{E} \left[(X_{1,\mathbf{U}}(k; j, X_j) - x_{\text{ref}})^2 + \gamma U(k)^2 \right] \\
& \text{s.t.} \quad X_{1,\mathbf{U}}(k+1; j, X_j) = X_{1,\mathbf{U}}(k; j, X_j) + X_{2,\mathbf{U}}(k; j, X_j), \\
& \quad \quad X_{2,\mathbf{U}}(k+1; j, X_j) = X_{2,\mathbf{U}}(k; j, X_j) + U(k) \\
& \quad \quad \quad - \kappa X_{2,\mathbf{U}}(k; j, X_j)^2 + W(k+j), \\
& \quad \quad X_{\mathbf{U}}(0; j, X_j) = X_j.
\end{aligned} \tag{4.37}$$

For the numerical simulations, we further set the friction parameter $\kappa = 0.2$, the control weight $\gamma = 5$, the reference position $x_{\text{ref}} = 5$, and consider the disturbance

$$W(k) = \begin{cases} 0.1 & \text{with probability 0.2,} \\ -0.15 & \text{with probability 0.8,} \end{cases} \quad k \in \mathbb{N}_0.$$

Unfortunately, a theoretical verification of stochastic dissipativity for this example is rather difficult. Hence, we again solve the stochastic optimal control problem (4.37) numerically for different horizon lengths and the initial condition $(j, X_j) = (0, (0, 0)^T)$. The results are shown in Figure 4.4, where we observe that all realizations remain close to each other in a stationary sense for most of the horizon, thereby providing numerical evidence for a turnpike property at the level of realizations.

Furthermore, for the closed-loop illustrations, we generated 20 000 samples of the closed-loop trajectories up to time $k = 150$ using Algorithm 2 with initial condition $X_0 = (0, 0)^T$ for each prediction horizon $N = 3, 4, 5$.

To illustrate the theoretical findings regarding stability and performance, we generated figures analogous to those presented in the previous example. The stationary quantities appearing in the figures were approximated by the values attained in the middle of the optimal trajectory corresponding to a prediction horizon of $N = 15$, as in the previous example.

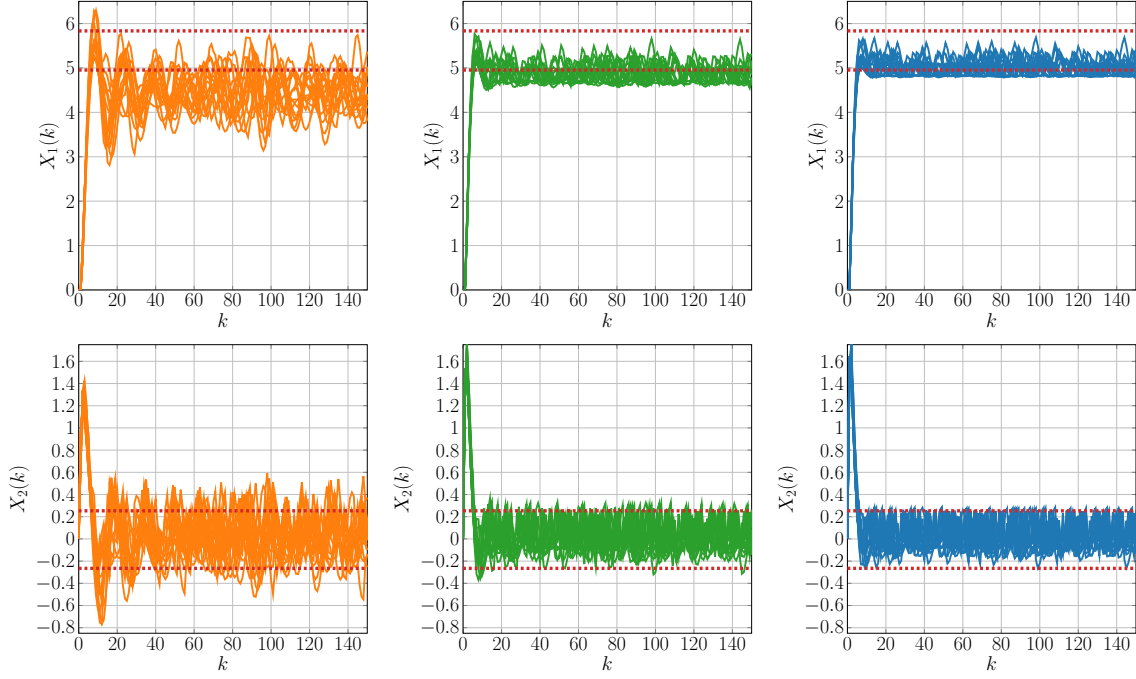


Figure 4.8: 15 sample paths of the closed-loop trajectories of problem (4.37) for different prediction horizons $N = 3$ (left), $N = 4$ (middle), and $N = 5$ (right) together with the upper and lower bound of the support of the stationary process (dashed red).

Figure 4.4 shows 15 individual realization paths of the closed-loop trajectories for different prediction horizons. We observe that, as the horizon length increases, the closed-loop trajectories remain closer to the support of the optimal stationary process in both components. This provides numerical evidence supporting the pathwise stability properties predicted by the theory.

In Figure 4.4, the empirical cumulative distribution functions (CDFs) of the marginal distributions of the closed-loop state components at time $k = 150$ are shown. We observe that, as the prediction horizon increases, the empirical CDFs become increasingly close to the corresponding stationary CDFs, which illustrates the stability in distribution of the closed-loop trajectories.

A similar behavior can also be observed for the expectation and variance of the closed-loop state components. Figure 4.4 shows that, as the prediction horizon increases, both the expectation and the variance converge towards values that become increasingly close to the corresponding stationary moments. This provides numerical evidence for stability of the closed-loop trajectories at the level of moments.

Finally, Figure 4.4 compares the closed-loop costs for the prediction horizons $N = 3, 4, 5$. It can be observed that, for sufficiently large K , the cumulative costs grow approximately linearly, with slopes depending on the prediction horizon N . This is again consistent with the predictions of Corollary 4.27, as discussed in the previous example.

Moreover, as the prediction horizon increases, the averaged closed-loop performance approaches increasingly small neighborhoods of the stationary cost, in agreement with Theorem 4.29. While the improvement from $N = 3$ to $N = 4$ is particularly pronounced, a further performance gain can still be observed for $N = 5$, as

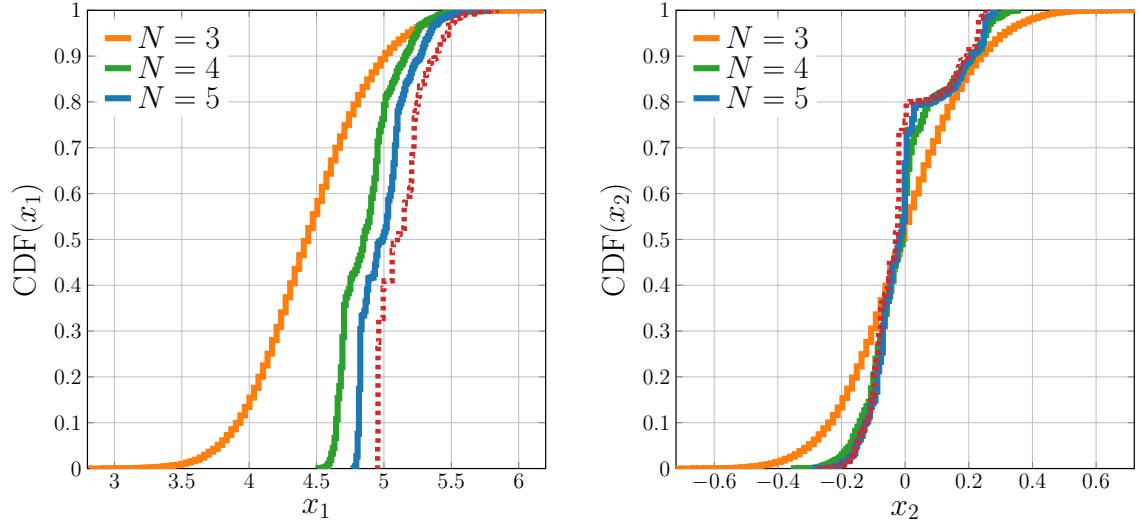


Figure 4.9: Empirical cumulative distribution function of the distributions of the closed-loop trajectory of problem (4.37) at time $k = 150$ for different horizons, together with the approximate stationary distribution (dashed red).

illustrated by the zoomed view in Figure 4.4. Overall, these results confirm that increasing the prediction horizon improves both the long-term and average closed-loop performance, reflecting the theoretical performance guarantees of our MPC scheme.

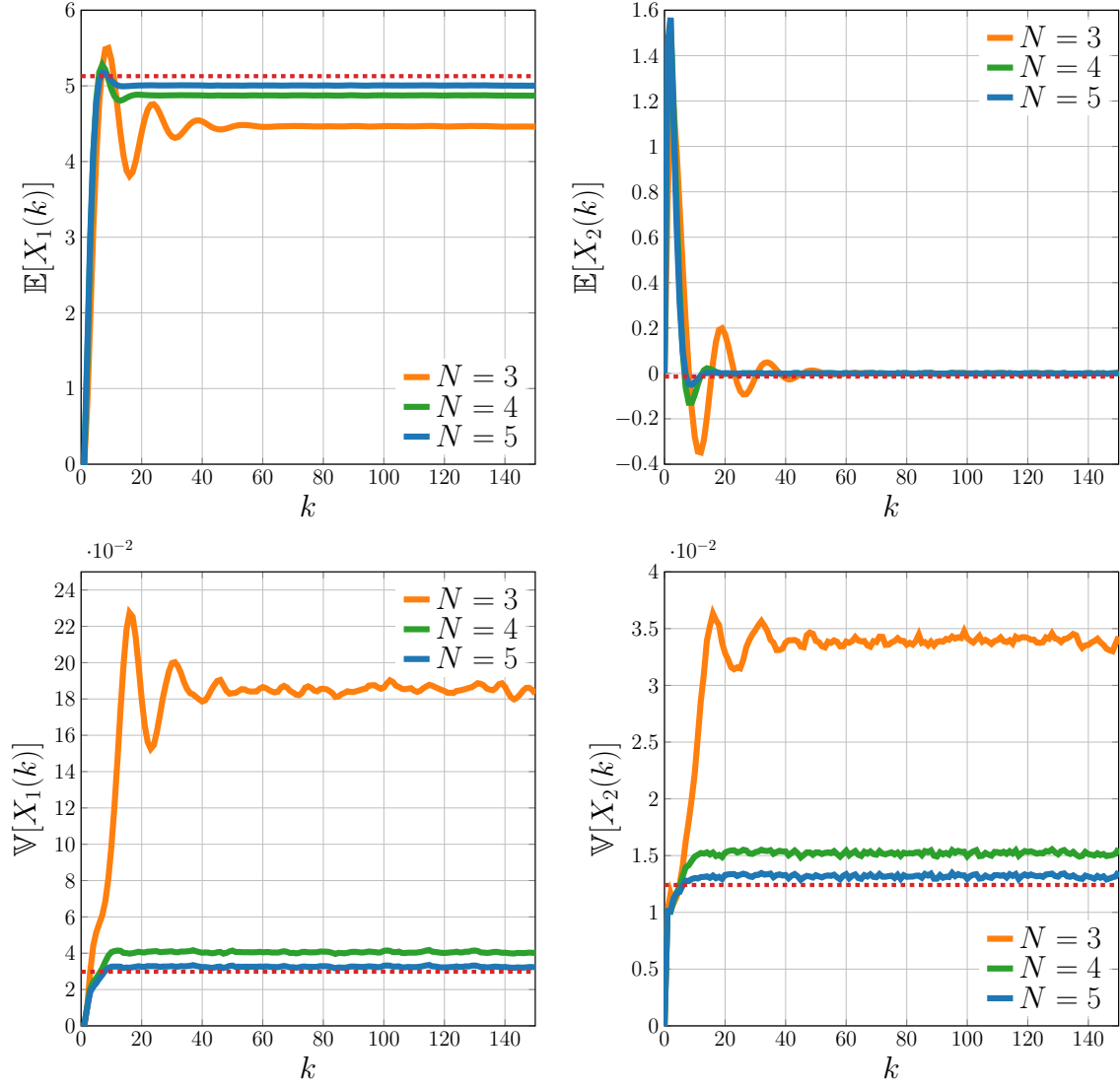


Figure 4.10: Empirical mean and variance of the components of the closed-loop trajectory for problem (4.37) for different prediction horizons, together with the approximate stationary values (red).

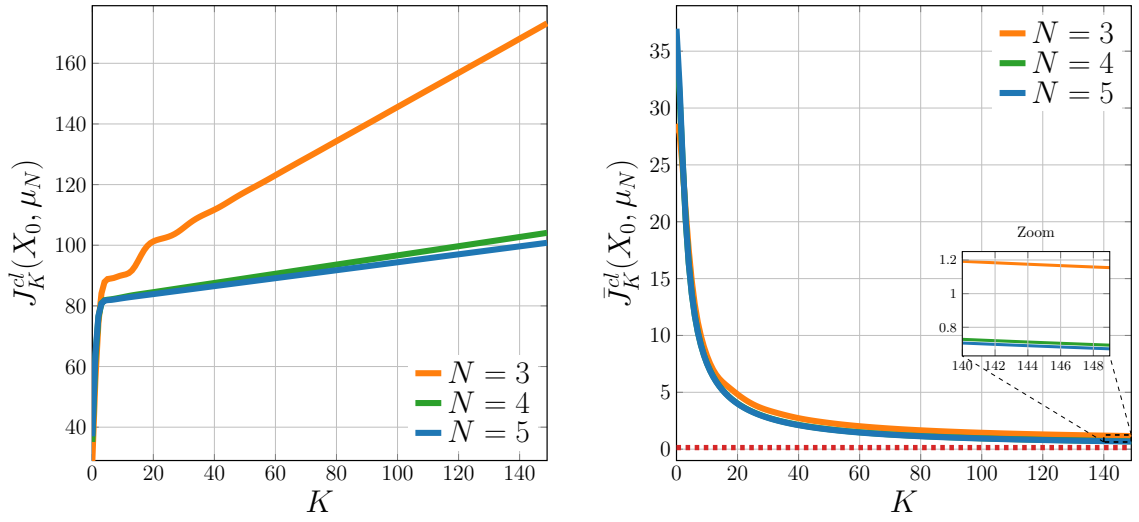


Figure 4.11: Sample-based approximations of the cumulative costs and averaged cost of the closed-loop solutions of problem (4.37) for different prediction horizons, together with the approximation of the stationary stage cost (red).

5 First results on closed-loop guarantees for stochastic MPC with risk-averse formulations

The previous chapter considered stochastic model predictive control (MPC) with expected value objectives and established closed-loop guarantees based on stochastic dissipativity and turnpike concepts. While expectation-based formulations provide a natural stochastic extension of deterministic MPC, they do not explicitly account for rare but potentially severe events. In many applications, such events must be addressed either by penalizing unfavorable outcomes in the objective or by enforcing constraints in a risk-averse manner.

This motivates the study of risk-averse formulations in stochastic MPC. Broadly speaking, risk can be incorporated into stochastic predictive control in two fundamentally different ways. The first approach modifies the objective function by replacing the expectation operator with a suitable risk measure. The second introduces risk-averse constraint formulations, thereby extending classical chance-constrained approaches. While both approaches aim to mitigate the impact of low-probability adverse events, they lead to conceptually different problem classes and require distinct analytical tools.

Risk-averse objective formulations have received increasing attention in stochastic optimal control and MPC. Common examples include conditional value-at-risk, averaged value-at-risk, and more general coherent risk measures; see, e.g., [83, 94]. Such formulations allow the explicit penalization of unfavorable tail events and have therefore been studied extensively in stochastic optimal control; see, for example, [2, 45, 50, 79]. In the context of MPC, [96] investigates linear systems subject to stochastic disturbances using coherent risk measures, while [97] considers a related framework for nonlinear systems.

In parallel, risk-averse constraint handling has emerged as an important research direction in stochastic MPC, where chance constraints have become a standard tool for incorporating uncertainty into constraint formulations; see, e.g., [7, 25, 49, 49, 55, 78, 91, 92]. However, while chance constraints provide probabilistic feasibility guarantees, they typically do not account for the severity of constraint violations when these occur. Risk-averse constraint formulations based on suitable risk measures address this limitation by quantifying not only the likelihood but also the magnitude of unfavorable violations; cf. [23, 108].

Despite these developments, rigorous closed-loop guarantees for stochastic MPC with risk-averse formulations remain limited. In particular, a systematic dissipativity-based framework for the analysis of stability and performance in risk-averse stochastic MPC is still lacking.

This chapter develops such a framework for discrete-time stochastic MPC with risk-averse formulations. Building on the stochastic dissipativity concepts developed

in the preceding chapters, we investigate two complementary classes of problems.

The first part considers stochastic MPC with risk-averse objective functions and without probabilistic constraints. Using parameterized risk measures, we derive performance guarantees that relate the MPC closed-loop performance to the optimal stationary cost. This extends the expected-value analysis of the previous chapter to a broader class of objective functionals.

The second part addresses stochastic MPC with expectation-based objectives and risk-averse constraints. For this setting, we extend the indirect feedback approach from chance-constrained stochastic MPC [48,49] to general risk-measure-based constraints. Based on stochastic dissipativity arguments, we establish averaged closed-loop performance guarantees for linear systems subject to possibly non-Gaussian disturbances and for general stage costs.

Taken together, the results of this chapter provide closed-loop guarantees for stochastic MPC under risk-averse formulations. They show that dissipativity-based analysis extends beyond expectation-based settings and provide a basis for a unified treatment of risk-averse stochastic economic MPC.

The chapter is organized as follows. Section 5.1 introduces the required notions of risk measures. Section 5.2 analyzes stochastic MPC with risk-averse objective functions and derives averaged performance guarantees and closed-loop convergence results. Section 5.3 then considers risk-averse constraint formulations and establishes constraint satisfaction and averaged performance bounds based on an indirect feedback approach.

This chapter comprises the results of [85,86], while extending the moment-based reformulations from [86] to generalized linear-quadratic problems in Section 5.3.3. Section 5.2.2 presents previously unpublished results on closed-loop trajectory convergence under risk-averse cost formulations.

5.1 Fundamentals of risk measures

Before investigating stochastic MPC schemes involving risk measures, we briefly recall some basic concepts from the theory of risk measures. We begin with the following definition, which introduces risk measures in their most general form.

Definition 5.1 (Risk measure). *A function $\rho : \mathcal{R}(\mathcal{X}) \rightarrow \bar{\mathbb{R}}$ is called risk measure if it is*

- (i) *monotone, i.e. for all $Z_1, Z_2 \in \mathcal{R}(\mathbb{R})$ it holds that $Z_1 \geq Z_2$ a.s. implies $\rho(Z_1) \geq \rho(Z_2)$;*
- (ii) *translation invariant, i.e. for all $Z \in \mathcal{R}(\mathbb{R})$ and $m \in \mathbb{R}$ it holds that $\rho(Z+m) = \rho(Z) + m$.*

Remark 5.2. (i) *In our definition of a risk measure ρ , large values of the random variable Z correspond to high risk $\rho(Z)$. Hence, Z is interpreted as a loss. This convention is standard in insurance mathematics. In contrast, in financial mathematics it is common to associate low values of Z with high risk $\tilde{\rho}(Z)$, since Z is typically interpreted as the payoff of a portfolio. However, the two formulations are easily converted into one another via the relation $\rho(Z) = \tilde{\rho}(-Z)$.*

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- (ii) We allow the risk measure ρ to take values in $[-\infty, +\infty]$, whereas in much of the literature risk measures are typically defined as mappings into $\mathbb{R}$ or $(-\infty, \infty]$. The reason for this more general setting is that we do not want to restrict the domain of ρ to specific spaces such as L^p . In nonlinear systems, it cannot be guaranteed a priori that the resulting state and control processes remain in a fixed L^p space. Allowing ρ to be defined on general random variables therefore ensures that the framework remains applicable without imposing artificial integrability assumptions.

Having introduced risk measures in their general form, we now refine this concept by specifying structural properties that are commonly imposed in the literature. The following definition collects standard properties such as normalization, convexity, positive homogeneity, sub-additivity, and law invariance.

Definition 5.3. A risk measure $\rho : \mathcal{R}(\mathbb{R}) \rightarrow \bar{\mathbb{R}}$ is called

- (i) normalized if $\rho(0) = 0$;
- (ii) convex if $\rho(\beta Z_1 + (1 - \beta)Z_2) \leq \beta\rho(Z_1) + (1 - \beta)\rho(Z_2)$ for all $Z_1, Z_2 \in \mathcal{R}(\mathbb{R})$, $\beta \in [0, 1]$;
- (iii) positive homogeneous if $\rho(\beta Z) = \beta\rho(Z)$ for all $Z \in \mathcal{R}(\mathbb{R})$ and $\beta \geq 0$;
- (iv) sub-additive if $\rho(Z_1 + Z_2) \leq \rho(Z_1) + \rho(Z_2)$ for all $Z_1, Z_2 \in \mathcal{R}(\mathbb{R})$;
- (v) coherent if it has the properties from (i)-(iv);
- (vi) law-invariant if $\rho(Z_1) = \rho(Z_2)$ holds for all $Z_1, Z_2 \in \mathcal{R}(\mathbb{R})$ with $Z_1 \sim Z_2$.

We remark that every positively homogeneous risk measure is necessarily normalized. Moreover, it can be shown that if a normalized risk measure satisfies any two of the three properties (ii), (iii), and (iv) from Definition 5.3, then the third property follows automatically. For more details on risk measures and their properties we also refer to [29].

Some examples of risk measures and their properties are given below.

- The expected value $\mathbb{E}[X]$ is a law-invariant coherent risk measure.
- The value at risk with confidence level $1 - \alpha \in (0, 1)$ is defined as

$$\text{VaR}_{1-\alpha}(Z) := \inf_{t \in \mathbb{R}} \{t : \mathbb{P}(Z \leq t) \geq 1 - \alpha\}. \quad (5.1)$$

This law-invariant risk measure is normalized and positively homogeneous, but it is neither sub-additive nor convex and therefore not coherent.

- The conditional value at risk (also known as averaged value at risk or expected shortfall) with confidence level $1 - \alpha \in (0, 1)$ is a coherent risk measure defined as

$$\text{CVaR}_{1-\alpha}(Z) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_{1-\gamma}(Z) d\gamma. \quad (5.2)$$

-
- Let $M_Z(t) = \mathbb{E}[e^{tZ}]$ be the moment generating function of $Z \in \mathcal{R}(\mathbb{R})$ at $t > 0$. Then, the entropic value at risk with confidence level $1 - \alpha \in (0, 1)$ is a law-invariant coherent risk measure defined as

$$\text{EVaR}_{1-\alpha}(Z) := \inf_{t>0} \{t^{-1} \ln(M_Z(t)/\alpha)\} \quad (5.3)$$

where we consider $\ln(M_Z(t)/\alpha) = \infty$ if $M_Z(t) = \infty$.

- Consider $\mathcal{D}(\Omega) := \{\xi : \Omega \rightarrow [0, \infty) \mid \int_{\Omega} \xi d\mathbb{P} = 1\}$ and a ϕ -divergence, i.e., a convex lower semicontinuous function $\phi : \mathbb{R} \rightarrow (-\infty, \infty]$ with $\phi(1) = 0$ and $\phi(z) = +\infty$ for $z < 0$. Then, the ϕ -divergence risk measure with constraint-level $c \geq 0$ is defined as

$$\text{Div}_c^\phi(Z) := \sup_{\xi \in \mathcal{A}} \int_{\Omega} \xi Z(\omega) d\mathbb{P}(\omega) \quad (5.4)$$

with

$$\mathcal{A} := \left\{ \xi \in \mathcal{D}(\Omega) : \int_{\Omega} \phi(\xi(\omega)) d\mathbb{P}(\omega) \leq c \right\}. \quad (5.5)$$

This risk measure is law-invariant and coherent.

- If ϕ is the Kullback-Leibler-divergence

$$\phi_{KL}(z) := \begin{cases} z \ln(z) - z + 1 & \text{for } z \geq 0 \\ \infty & \text{for } z < 0 \end{cases} \quad (5.6)$$

the risk measure from (5.4) is called Kullback-Leibler-divergence risk measure. Since this risk measure belongs to the class of ϕ -divergence risk measures, it is also law-invariant and coherent.

Remark 5.4. *The names conditional value at risk, averaged value at risk, and expected shortfall are usually used interchangeable in the literature. However, depending on the reference, the name conditional value at risk is sometimes also used as another name for the tail conditional expectation (or tail value at risk) given by*

$$\text{TCE}_\alpha := \mathbb{E}[Z \mid Z \geq \text{VaR}_\alpha(Z)].$$

While these two concepts coincide for continuous random variables, this is in general not the case for discrete ones, cf. [1].

All the risk measures listed above are widely used in the literature, cf. [8, 29, 83, 94], and will serve as concrete examples throughout this chapter.

We emphasize that, among these examples, only the expectation may attain the value $-\infty$ for any $Z \in \mathcal{R}(\mathbb{R})$, although in our general framework a risk measure is allowed to take values in $\bar{\mathbb{R}}$. Thus, the deviation from the standard formulations in the literature, as discussed in Remark 5.2 (ii), is primarily motivated by mathematical rigor rather than practical relevance. In contrast, the value $+\infty$ may also arise for other risk measures. For instance, the entropic value at risk defined in (5.3) attains the value $+\infty$ if the corresponding moment generating function is infinite for all $t > 0$.

In general, evaluating and optimizing risk measures is challenging, particularly when the underlying probability distributions are arbitrary. To address this issue, we focus on a class of risk measures that admit a *parametric representation*, thereby providing additional structure that can be exploited analytically and computationally. Risk measures of this type have also been considered in [8, 45, 93]. In particular, if the stage costs are defined through mappings of this type, the resulting stochastic optimal control problem can be reformulated as a two-stage optimization problem, where the minimization is performed jointly over the control sequence $\mathbf{U} \in \check{\mathcal{U}}_{\text{ad}}^N(j, X_j)$ and an additional deterministic real-valued parameter.

Definition 5.5 (Parameterized risk measures). *A risk measure $\rho : \mathcal{R}(\mathcal{X}) \rightarrow \mathbb{R} \cup \{\infty\}$ is called parameterized if there exists a set $\Xi \subset \mathbb{R}^{n_\xi}$, $n_\xi \in \mathbb{N}$ and a function $\Psi : \mathbb{R} \times \Xi \rightarrow \mathbb{R}$ such that*

$$\rho(Z) = \inf_{\xi \in \Xi} \mathbb{E}[\Psi(Z, \xi)]. \quad (5.7)$$

Note that for every proper, closed and monotone non-decreasing function $\psi : \mathbb{R} \rightarrow (-\infty, \infty]$, the mapping

$$\rho(Z) := \inf_{\xi \in \mathbb{R}} \mathbb{E}[\xi + \psi(Z - \xi)] \quad (5.8)$$

defines a risk measure according to Definition 5.1 given in parametric form (5.7). Moreover, if ψ is additionally convex the risk measure (5.8) is also convex. However, it is in general not positive homogeneous, and thus, not coherent. Several risk measures that are used in practice can be represented in the parametric form (5.7):

- The expected value can be parameterized by setting $\Psi(z, \xi) = z$ in (5.7).
- The conditional value-at-risk can be parameterized as

$$\text{CVaR}_{1-\alpha}(Z) = \inf_{\xi \in \mathbb{R}} \mathbb{E}[\xi + \alpha^{-1} \max\{0, Z - \xi\}]. \quad (5.9)$$

- Every ϕ -divergence risk measure can be parameterized as

$$\text{Div}_c^\phi(Z) = \inf_{\xi \in \mathbb{R}^+ \times \mathbb{R}} \mathbb{E} \left[\xi_1 c + \xi_2 + \xi_1 \phi^* \left(\frac{Z - \xi_2}{\xi_1} \right) \right], \quad (5.10)$$

where ϕ^* is the Legendre-Fenchel conjugate of ϕ .

- The Kullback-Leibler-divergence risk measure can be parameterized as

$$\text{Div}_c^{\phi_{KL}} = \inf_{\xi \in \mathbb{R}^+ \times \mathbb{R}} \mathbb{E} \left[\xi_1 c + \xi_2 + \xi_1 e^{\left(\frac{Z - \xi_2}{\xi_1} \right)} - \xi_1 \right]. \quad (5.11)$$

Note that, due to their definition as the expectation of a deterministic function, cf. (5.7), every parameterized risk measure is law-invariant. In particular, this property holds for all risk measures listed above.

5.2 Asymptotic estimates for problems with risk-averse costs

Now, our aim is to adapt the results from Chapter 4 to settings where risk measures appear in the cost structure. In particular, we consider stage costs of the form

$$\ell(X, U) = \mathbb{L}[g(X, U)] = \rho(g(X, U)), \quad (5.12)$$

where $\rho : \mathcal{R}(\mathbb{R}) \rightarrow \bar{\mathbb{R}}$ is a law-invariant risk measure and $g : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$ is a deterministic function bounded from below.

As in Chapter 4, we still assume that only pathwise constraints of the form (2.3) are imposed, and probabilistic constraints of the form (2.3) are not considered. First results on extending the framework to include probabilistic constraints formulated via risk measures will be discussed in Section 5.3.

Hence, the risk-averse stochastic optimal control problem under consideration is given by

$$\begin{aligned} \underset{\mathbf{U} \in \mathcal{U}_{\text{ad}}^N(j, X_j)}{\text{minimize}} \quad & J_N(X_j, \mathbf{U}) := \sum_{k=0}^{N-1} \ell(X_{\mathbf{U}}(k; j, X_j), U(k)) \\ \text{s.t.} \quad & X_{\mathbf{U}}(k+1; j, X_j) = f(X_{\mathbf{U}}(k; j, X_j), U(k), W(k+j)), \\ & X_{\mathbf{U}}(0; j, X_j) = X_j, \\ & h(X_{\mathbf{U}}(k; j, X_j)(\omega), U(k)(\omega)) \leq 0, \quad \text{almost surely.} \end{aligned} \quad (5.13)$$

The crucial difference when considering the risk-aware stochastic optimal control problem (5.13), instead of the standard stochastic optimal control problem (4.1) with costs expressed as expectations, is that, in general, the pointwise dynamic programming principle from Theorem 2.22 no longer holds. This phenomenon is illustrated by the following numerical example.

Example 5.6. *Consider the stochastic optimal control problem*

$$\begin{aligned} \underset{\mathbf{U} \in \mathcal{U}_{\text{ad}}^N(j, X_j)}{\text{minimize}} \quad & J_N(X_j, \mathbf{U}) := \sum_{k=0}^{N-1} \mathbb{L}[X_{\mathbf{U}}(k; j, X_j)^2 + U(k)^2] \\ \text{s.t.} \quad & X_{\mathbf{U}}(k+1; j, X_j) = X_{\mathbf{U}}(k; j, X_j) + U(k) + W(k+j) \\ & X_{\mathbf{U}}(0; j, X_j) = X_j \end{aligned} \quad (5.14)$$

with disturbance

$$W(j)(\omega) = \begin{cases} 0.5 & \text{with probability } 0.5 \\ -0.5 & \text{with probability } 0.5 \end{cases}, \quad \text{for } j \in \mathbb{N}_0.$$

In Figure 5.6 the solutions obtained for $\mathbb{L} = \mathbb{E}$ and $\mathbb{L} = \text{CVaR}_{0.99}$ with initial condition $(j, X_j) = (0, 1.5)$ on horizon $N = 5$ are shown. Furthermore, it illustrates the solutions on the shifted and shortened horizon after one step when optimizing separately for each possible realization of the optimal solution. More precisely, it shows the solutions on horizon 4 with initial conditions $(1, X_{\mathbf{U}}(1; 0, 1.5)(\omega_1))$ and $(1, X_{\mathbf{U}}(1; 0, 1.5)(\omega_2))$. For all numerical simulations concerning $\mathbb{L} = \text{CVaR}_{0.99}$ we used the same smooth approximation as explained in Section 5.2.3.

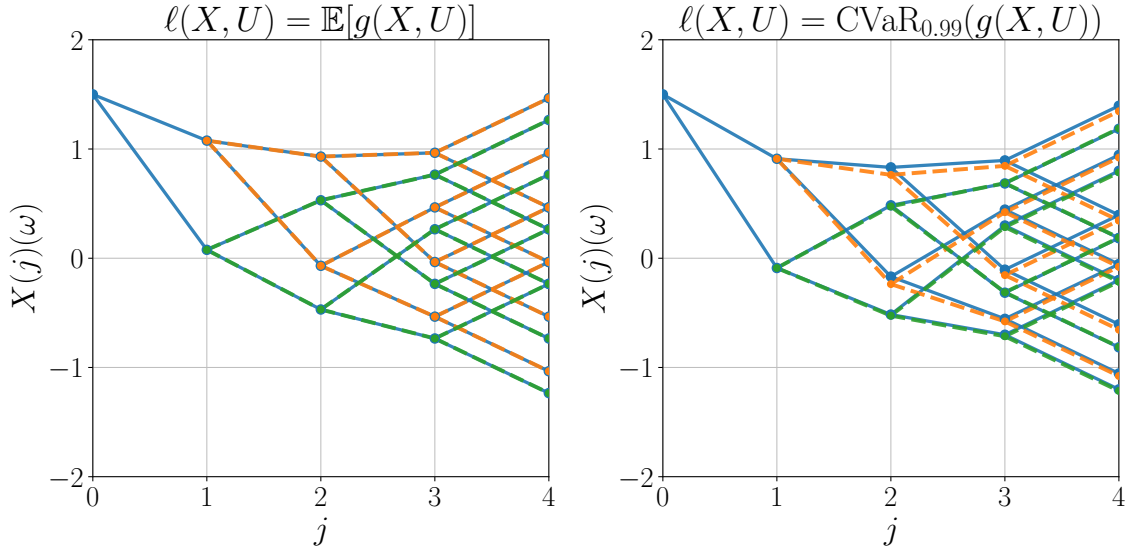


Figure 5.1: Optimal solutions of problem (5.14) (solid orange) for $\mathbb{L} = \mathbb{E}$ (left) and $\mathbb{L} = \text{CVaR}_{0.99}$ (right), together with the optimal solutions on the shifted and shortened horizon after one step when optimizing separately for each possible realization of the optimal solution (dashed blue and dashed green).

As the pointwise dynamic programming principle from Theorem 2.22 indicates the solutions for $\mathbb{L} = \mathbb{E}$ are the same, while the solutions for $\mathbb{L} = \text{CVaR}_{0.99}$ deviate. Hence, the pointwise dynamic programming principle does not hold for problem (5.14) with $\mathbb{L} = \text{CVaR}_{0.99}$.

To address this issue, we restrict ourselves to the class of parameterized risk measures. Accordingly, we impose the following assumption.

Assumption 5.7. *The mapping $\mathbb{L}$ from (5.12) is a parameterized risk measure given by*

$$\rho(Z) = \inf_{\xi \in \Xi} \mathbb{E}[\Psi(Z, \xi)]. \quad (5.15)$$

However, restricting ourselves to parameterized risk measures alone does not yet allow us to directly apply the results from Chapter 4 to an implementable stochastic MPC scheme. This limitation also arises from the fact that the infimum in equation (5.15) is taken over the random variable Z . Consequently, knowledge of a single path is generally insufficient to evaluate the risk measure correctly.

However, by fixing a parameter $\bar{\xi} \in \Xi$ in (5.15), instead of taking the infimum over all $\xi \in \Xi$, we obtain

$$\ell_{\bar{\xi}}(X, U) := \mathbb{E}[\Psi(g(X, U), \bar{\xi})] \geq \inf_{\xi \in \Xi} \mathbb{E}[\Psi(g(X, U), \xi)] = \ell(X, U).$$

Hence, by using the stage costs $\ell_{\bar{\xi}}(X, U)$ in place of the original stage costs, we optimize an upper bound of the costs of the original stochastic optimal control problem, which is in expectation form as considered in Chapter 4.

This naturally leads to the following risk-averse stochastic optimal control problem:

$$\begin{aligned}
& \underset{\mathbf{U} \in \mathcal{U}_{\text{ad}}^N(j, X_j)}{\text{minimize}} & J_{N, \bar{\xi}}(X_j, \mathbf{U}) &:= \sum_{k=0}^{N-1} \ell_{\bar{\xi}}(X_{\mathbf{U}}(k; j, X_j), U(k)) \\
& \text{s.t.} & X_{\mathbf{U}}(k+1; j, X_j) &= f(X_{\mathbf{U}}(k; j, X_j), U(k), W(k+j)), \\
& & X_{\mathbf{U}}(0; j, X_j) &= X_j, \\
& & h(X_{\mathbf{U}}(k; j, X_j)(\omega), U(k)(\omega)) &\leq 0, \quad \text{almost surely.}
\end{aligned} \tag{5.16}$$

To ensure that the stochastic optimal control problem satisfies Assumption 2.21, and thus falls within the setting where the pointwise dynamic programming principle from Theorem 2.22 applies, we impose the following assumption.

Assumption 5.8. *For the fixed parameter $\bar{\xi} \in \Xi$ in problem (5.16), the composition $\Psi(\cdot, \bar{\xi}) \circ g$ is bounded from below.*

The corresponding risk-averse stochastic MPC algorithm is given by Algorithm 3. Note that, under Assumption 5.8, the pointwise dynamic programming principle applies. Consequently, the pointwise implementation of the algorithm coincides with the abstract MPC algorithm defined on the space of random variables, as explained in Section 4.1 and established in Theorem 4.1.

Algorithm 3 Implementable risk-averse stochastic MPC algorithm

Fix a parameter $\bar{\xi} \in \Xi$ and consider the resulting stochastic optimal control problem (5.16).

for $j = 0, 1, \dots$

1.) Measure the state $x_j := X(j, \omega)$ and set $X_0 \equiv x_j$.

2.) Solve the stochastic optimal control problem (5.16) with fixed parameter $\bar{\xi}$ and obtain the optimal control sequence

$$\begin{aligned}
\mathbf{U}_{N,j,x_j}^* &= (U_{N,j,x_j}^*(0), \dots, U_{N,j,x_j}^*(N-1)) \\
&= (\pi_0^*(X(0; j, x_j)), \dots, \pi_{N-1}^*(X(N-1; j, x_j))).
\end{aligned}$$

3.) Apply the MPC feedback $\mu_{N,\bar{\xi}}(x_j) := U_{N,x_j}^*(0)$ to system (2.3).

5.2.1 Averaged performance estimates

Proceeding as in Chapter 4 would allow us to derive results for the stochastic optimal control problem with fixed parameter from (5.16). However, such estimates would be formulated with respect to the modified stage costs $\ell_{\bar{\xi}}(X, U)$ and the associated stationary pair $(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s)$, rather than the original stage costs $\ell(X, U)$ and the corresponding stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$.

In order to relate these quantities to the original problem, we impose the following continuity assumption on the function Ψ defining the parameterized risk measure from Assumption 5.7.

Assumption 5.9. *The function Ψ satisfies the following uniform continuity assumption: for all $\xi_1 \in \Xi$ there exists $\alpha_\Psi \in \mathcal{K}_\infty$ such that*

$$|\Psi(z, \xi_1) - \Psi(z, \xi_2)| \leq \alpha_\Psi(\|\xi_1 - \xi_2\|)$$

holds for all $z \in \mathbb{R}$ and $\xi_2 \in \Xi$.

Furthermore, we impose the following two dissipativity assumptions: one for the risk-averse stochastic optimal control problem with fixed parameter from (5.16) and one for the original problem from (5.13).

Assumption 5.10. *The risk-averse stochastic optimal control problem with fixed parameter $\bar{\xi} \in \Xi$ from (5.16) is stochastically dissipative at a stationary pair $(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s)$.*

Assumption 5.11. *The original stochastic optimal control problem (5.13) is stochastically dissipative at a stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ and there exists*

$$\xi^s \in \arg \min_{\xi \in \Xi} \ell_\xi(\mathbf{X}^s, \mathbf{U}^s). \quad (5.17)$$

Remark 5.12. *Note that if the optimal stationary parameter ξ^s from (5.17) exists, then (strict) stochastic dissipativity of the original stochastic optimal control problem (5.13) implies (strict) stochastic dissipativity of problem (5.16) with fixed parameter $\bar{\xi} = \xi^s$. Consequently, Assumption 5.10 follows from Assumption 5.11 for $\bar{\xi} = \xi^s$, but in general not for other choices of $\bar{\xi}$.*

Based on these assumptions, we can now relate the optimal stationary cost of the optimal control problem with fixed parameter to that of the original problem, as shown in the following theorem.

Theorem 5.13. *Let Assumption 5.9, Assumption 5.10, and Assumption 5.11 hold. Then, there exists $\alpha_\Psi \in \mathcal{K}_\infty$ such that*

$$\ell_{\bar{\xi}}(\mathbf{U}_{\bar{\xi}}^s, \mathbf{X}_{\bar{\xi}}^s) - \ell(\mathbf{X}^s, \mathbf{U}^s) \leq \alpha_\Psi(\|\bar{\xi} - \xi^s\|).$$

Proof. By Assumption 5.10 we can conclude that

$$\ell_{\bar{\xi}}(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s) - \ell_{\bar{\xi}}(\mathbf{X}^s, \mathbf{U}^s) \leq 0$$

holds, cf. Theorem 3.9. Thus, by the uniform continuity property of Ψ from Assumption 5.9 we get that there is an $\alpha_\Psi \in \mathcal{K}_\infty$ such that

$$\begin{aligned} & \ell_{\bar{\xi}}(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s) - \ell_{\xi^s}(\mathbf{X}^s, \mathbf{U}^s) \\ &= \ell_{\bar{\xi}}(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s) - \ell_{\bar{\xi}}(\mathbf{X}^s, \mathbf{U}^s) + \ell_{\bar{\xi}}(\mathbf{X}^s, \mathbf{U}^s) - \ell_{\xi^s}(\mathbf{X}^s, \mathbf{U}^s) \\ &\leq \ell_{\bar{\xi}}(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s) - \ell_{\bar{\xi}}(\mathbf{X}^s, \mathbf{U}^s) + |\ell_{\bar{\xi}}(\mathbf{X}^s, \mathbf{U}^s) - \ell_{\xi^s}(\mathbf{X}^s, \mathbf{U}^s)| \\ &\leq \mathbb{E}[\|\Psi(g(\mathbf{X}^s, \mathbf{U}^s), \bar{\xi}) - \Psi(g(\mathbf{X}^s, \mathbf{U}^s), \xi^s)\|] \\ &\leq \mathbb{E}[\alpha_\Psi(\|(g(\mathbf{X}^s, \mathbf{U}^s), \bar{\xi}) - (g(\mathbf{X}^s, \mathbf{U}^s), \xi^s)\|)] \\ &\leq \alpha_\Psi(\|\bar{\xi} - \xi^s\|). \end{aligned}$$

This shows the claim since

$$\ell(\mathbf{X}^s, \mathbf{U}^s) = \ell_{\xi^s}(\mathbf{X}^s, \mathbf{U}^s)$$

holds due to the choice of ξ^s from (5.17). □

Remark 5.14. In Assumption 5.11 we assume uniform continuity for all $z \in \mathbb{R}$. However, if we take a look at the proof of Corollary 5.15, we can see that it is sufficient that this condition holds for all $z \in \text{Im}(g(\mathbf{X}^s, \mathbf{U}^s)) \subseteq \mathbb{R}$. Hence, it is particularly fulfilled if Ψ is continuous and the realizations of the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ lie in a compact set.

Using this theorem, we are now able to derive a performance estimate for the closed-loop trajectory generated by Algorithm 3 in terms of the original stage cost, as stated in the following corollary. Note that the averaged performance of the closed-loop trajectory $X_{\mu_{N,\bar{\xi}}}(\cdot; 0, X_0)$ with respect to the original stage cost is given by

$$\bar{J}_K^{cl}(X_0, \mu_{N,\bar{\xi}}) = \frac{1}{K} \sum_{j=0}^{K-1} \ell(X_{\mu_{N,\bar{\xi}}}(j; 0, X_0), \mu_{N,\bar{\xi}}(X_{\mu_{N,\bar{\xi}}}(j; 0, X_0))). \quad (5.18)$$

Corollary 5.15. Let Assumption 5.8, Assumption 5.9, Assumption 5.10, and Assumption 5.11 hold. Furthermore, let the assumptions of Theorem 4.29 hold for the risk-averse stochastic optimal control problem (5.16) with fixed parameter $\bar{\xi} \in \Xi$.

Then, there exist functions $\delta \in \mathcal{L}$, $\alpha \in \mathcal{K}_\infty$ and a neighborhood $\mathcal{S}_{\bar{\xi}}$ of the stationary process $\mathbf{X}_{\bar{\xi}}^s$ such that the averaged closed-loop cost (5.18) satisfies

$$\ell(\mathbf{X}^s, \mathbf{U}^s) \leq \limsup_{K \rightarrow \infty} \bar{J}_K^{cl}(X_0, \mu_{N,\bar{\xi}}) \leq \ell(\mathbf{X}^s, \mathbf{U}^s) + \delta(N) + \alpha(\|\bar{\xi} - \xi^s\|) \quad (5.19)$$

for all $X_0 \in S_{\bar{\xi}}(0)$ and sufficiently large $N \in \mathbb{N}$.

Proof. Consider the averaged closed-loop cost

$$\bar{J}_{K,\bar{\xi}}^{cl}(X_0, \mu_{N,\bar{\xi}}) := \frac{1}{K} \sum_{j=0}^{K-1} \ell_{\bar{\xi}}(X_{\mu_{N,\bar{\xi}}}(j; 0, X_0), \mu_{N,\bar{\xi}}(X_{\mu_{N,\bar{\xi}}}(j; 0, X_0)))$$

measured with respect to the stage costs with fixed parameter $\bar{\xi} \in \Xi$ from problem (5.16).

Since the Assumptions of Theorem 4.29 hold for the risk-averse stochastic optimal control problem (5.16) with fixed parameter $\bar{\xi} \in \Xi$, we can conclude that there exists $\delta \in \mathcal{L}$ and a neighborhood $\mathcal{S}_{\bar{\xi}}$ of the stationary process $\mathbf{X}_{\bar{\xi}}^s$ such that

$$\begin{aligned} \limsup_{K \rightarrow \infty} \bar{J}_{K,\bar{\xi}}^{cl}(X_0, \mu_{N,\bar{\xi}}) &\leq \ell_{\bar{\xi}}(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s) + \delta(N) \\ &= \ell_{\bar{\xi}}(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \ell(\mathbf{X}^s, \mathbf{U}^s) + \delta(N) \end{aligned}$$

for all $X_0 \in S_{\bar{\xi}}(0)$ and sufficiently large $N \in \mathbb{N}$.

Using Theorem 5.13 yields that there exists $\alpha_\Psi \in \mathcal{K}_\infty$ such that

$$\limsup_{K \rightarrow \infty} \bar{J}_{K,\bar{\xi}}^{cl}(X_0, \mu_{N,\bar{\xi}}) = \ell(\mathbf{X}^s, \mathbf{U}^s) + \delta(N) + \alpha_\Psi(\|\bar{\xi} - \xi^s\|).$$

Now consider the averaged closed-loop cost (5.18) measured with respect to the original risk-averse stage costs. Since the original risk-averse stochastic optimal control problem (5.13) is stochastically dissipative due to assumption 5.11 it is also optimally operated, cf. Lemma 3.8, which implies

$$\ell(\mathbf{X}^s, \mathbf{U}^s) \leq \liminf_{K \rightarrow \infty} \bar{J}_K^{cl}(X_0, \mu_{N,\bar{\xi}}) \leq \limsup_{K \rightarrow \infty} \bar{J}_K^{cl}(X_0, \mu_{N,\bar{\xi}}).$$

Furthermore, we know by definition that

$$\ell_{\bar{\xi}}(X, U) \leq \ell(X, U)$$

holds, implying

$$\bar{J}_{K, \bar{\xi}}^{cl}(X_0, \mu_{N, \bar{\xi}}) \leq \bar{J}_K^{cl}(X_0, \mu_{N, \bar{\xi}}) \quad (5.20)$$

for all $K \in \mathbb{N}$.

Hence, taking the $\liminf_{K \rightarrow \infty}$ in equation (5.20) and using the estimates from above we obtain the desired estimate from equation (5.19) for $\alpha = \alpha_{\Psi}$. $\square$

Corollary 5.15 shows that the asymptotic averaged performance depends not only on the length of the optimization horizon N , but also on the deviation of the fixed parameter $\bar{\xi}$ from the stationary parameter ξ^s defined in (3.10).

5.2.2 Trajectory convergence results

Next, we aim to derive statement for the convergence of the closed-loop trajectory. To this end, we have to make the following assumption, replacing the dissipativity from Assumption 5.11 with strict stochastic dissipativity on the distributional layer.

Assumption 5.16. *Assumption 5.11 holds and the original stochastic optimal control problem (5.13) is strictly stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to a law-invariant pseudometric d_D .*

Based on this assumption, we can now derive a bound on the distance between the optimal stationary pair of the original problem and that of the problem with fixed parameter, depending on how well the fixed parameter approximates the stationary parameter.

Theorem 5.17. *Let Assumption 5.9, Assumption 5.10, and Assumption 5.16 hold. Then, there exists $\alpha_s \in \mathcal{K}_{\infty}$ such that*

$$d(X_{\bar{\xi}}^s(k), X^s(k)) \leq \alpha_s(\|\bar{\xi} - \xi^s\|) \quad \text{for all } k \in \mathbb{N}_0,$$

where d is the law-invariant pseudometric d_D from Assumption 5.16.

Proof. Since the stochastic optimal control problem (5.13) is strictly stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$ with respect to a law-invariant pseudometric d_D according to Assumption 5.16 we can conclude by Lemma 3.6 that there exists a law-invariant storage function $\lambda_D : \mathcal{R}(\mathcal{X}) \rightarrow \mathbb{R}$ and $\alpha \in \mathcal{K}_{\infty}$ such that

$$\ell(X, U) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda_D(X) - \lambda_D(f(X, U, W(k))) \geq \alpha(d_D(X, X^s(k))) \quad (5.21)$$

holds for $k \in \mathbb{N}_0$ and all Markovian controls $U \in \mathbb{U}(X)$.

Thus, inserting the stationary process $(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s)$ in (5.21) yields

$$\begin{aligned} \alpha(d_D(X_{\bar{\xi}}^s(k), X^s(k))) &\leq \ell(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s) - \ell(\mathbf{X}^s, \mathbf{U}^s) + \lambda_D(X_{\bar{\xi}}^s(k)) - \lambda_D(X_{\bar{\xi}}^s(k+1)) \\ &= \ell(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s) - \ell(\mathbf{X}^s, \mathbf{U}^s) \end{aligned} \quad (5.22)$$

for all $k \in \mathbb{N}_0$ due to the stationarity of $(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s)$ and the law-invariance of λ_D .

Using the estimate

$$\ell(\mathbf{X}_{\bar{\xi}}^s, \mathbf{U}_{\bar{\xi}}^s) - \ell(\mathbf{X}^s, \mathbf{U}^s) \leq \alpha_{\Psi}(\|\bar{\xi} - \xi^s\|)$$

from Theorem 5.13 the inequality (5.22) yields

$$d_D(X_{\bar{\xi}}^s(k), X^s(k)) \leq \alpha^{-1}(\alpha_{\Psi}(\|\bar{\xi} - \xi^s\|))$$

which proves the claim with $\alpha_s := \alpha^{-1} \circ \alpha_{\Psi}$. $\square$

Remark 5.18. In Theorem 5.17, strict stochastic dissipativity with respect to a law-invariant pseudometric d_D is crucial, since otherwise we cannot conclude that there exists a storage function λ_D such that

$$\lambda_D(X_{\bar{\xi}}^s(k)) - \lambda_D(X_{\bar{\xi}}^s(k+1)) = 0$$

holds in (5.22). In general, strict stochastic dissipativity only guarantees the existence of a storage function

$$\lambda : \mathbb{N}_0 \times \mathcal{R}(\mathcal{X}) \rightarrow \mathbb{R},$$

for which no analogous estimate

$$\lambda(k, X_{\bar{\xi}}^s(k)) - \lambda(k+1, X_{\bar{\xi}}^s(k+1)) = 0$$

can be derived, since $(k, X_{\bar{\xi}}^s(k)) \neq (k+1, X_{\bar{\xi}}^s(k+1))$ and the storage function may depend on the specific representation of $X_{\bar{\xi}}^s(k)$ as a random variable.

Using Theorem 5.17, we derive convergence of the closed-loop trajectory to the optimal stationary process $\mathbf{X}^s$ of the original problem based on the stability properties established in Section 4.2. The following corollary shows that the stability results obtained there with respect to the stationary process $\mathbf{X}_{\bar{\xi}}^s$ can be transferred to convergence toward $\mathbf{X}^s$ for the closed-loop trajectory generated by Algorithm 3. To this end, we employ the semi-global version of P -practical asymptotic stability from Corollary 4.19. Nevertheless, we expect that a corresponding result also holds in the non-semi-global setting if the size of the sets $P(j)$ is characterized appropriately.

Corollary 5.19. Let Assumption 5.9, Assumption 5.10, and Assumption 5.16 hold. Furthermore, let the assumptions of Corollary 4.19 with respect to a pseudometric $d \succeq d_D$ hold for the risk-averse stochastic optimal control problem (5.16) with fixed parameter $\bar{\xi} \in \Xi$.

Then, there exists $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ such that the following property holds: For each $\Delta_1 > 0$ and $\Delta_2 > \Delta_1$, there exists $N_{\Delta} \in \mathbb{N}$ such that for all $N \geq N_{\Delta}$, all $j \in \mathbb{N}_0$, and all $(j_0, X_{j_0}) \in \bar{B}_{\Delta_2}^d(\mathbf{X}_{\bar{\xi}}^s)$ the inequality

$$\lim_{j \rightarrow \infty} d_D(X_{\mu_{N,\bar{\xi}}}(j; j_0, X_{j_0}), X^s(j)) \leq \alpha_1(\Delta_1) + \alpha_2(\|\bar{\xi} - \xi^s\|) \quad (5.23)$$

holds.

Proof. From Corollary 4.19 we can conclude that there exists $\beta_{\bar{\xi}} \in \mathcal{KL}$ such that the following property holds: For each $\Delta_1 > 0$ and $\Delta_2 > \Delta_1$ there exists $N_{\Delta} \in \mathbb{N}$ such that for all $N \geq N_{\Delta}$, all $j \in \mathbb{N}_0$, and all $(j_0, X_{j_0}) \in \bar{B}_{\Delta_2}^d(\mathbf{X}_{\bar{\xi}}^s)$ the inequality

$$d(X_{\mu_N}(j; j_0, X_{j_0}), X_{\bar{\xi}}^s(j + j_0)) \leq \max\{\beta_{\bar{\xi}}(d(X_{j_0}, X_{\bar{\xi}}^s(j_0)), j), \Delta_1\} \quad (5.24)$$

holds.

Furthermore, using the triangle inequality and that $d_D \prec d$ we obtain that there exists $\gamma \in \mathcal{K}_\infty$ such that

$$\begin{aligned} d_D(X_{\mu_N}(j; j_0, X_{j_0}), X^s(j + j_0)) \\ \leq d_D(X_{\mu_N}(j; j_0, X_{j_0}), X_\xi^s(j + j_0)) + d_D(X_\xi^s(j + j_0), X^s(j + j_0)) \\ \leq \gamma(d(X_{\mu_N}(j; j_0, X_{j_0}), X_\xi^s(j + j_0))) + d_D(X_\xi^s(j + j_0), X^s(j + j_0)) \end{aligned}$$

Hence, by the estimate (5.24) and Theorem 5.17 we get

$$d_D(X_{\mu_N}(j; j_0, X_{j_0}), X^s(j + j_0)) \leq \gamma(\max\{\beta_{\bar{\xi}}(d(X_{j_0}, X_\xi^s(j_0)), j), \Delta_1\}) + \alpha_s(\|\bar{\xi} - \xi^s\|),$$

which yields

$$\lim_{j \rightarrow \infty} d_D(X_{\mu_N}(j; j_0, X_{j_0}), X^s(j + j_0)) \leq \gamma(\Delta_1) + \alpha_s(\|\bar{\xi} - \xi^s\|)$$

and thus proves the claim with $\alpha_1 = \gamma$, $\alpha_2 = \alpha_s$. $\square$

Corollary 5.19 again shows that fixing the parameter $\bar{\xi}$ in Algorithm 3 introduces an additional error term determined by the deviation of $\bar{\xi}$ from the stationary parameter ξ^s , similar to Corollary 5.15. Moreover, the corollary only yields convergence on the distributional layer, even if the assumptions hold with respect to a stronger pseudometric on the space of random variables. The reason is that otherwise the term

$$\lambda(k, X_\xi^s(k)) - \lambda(k + 1, X_\xi^s(k + 1))$$

would not vanish at $\mathbf{X}_\xi^s$, as discussed in Remark 5.18. Thus, to obtain convergence with respect to the stronger norm, one would need to estimate this term or, at least, show that

$$\lim_{k \rightarrow \infty} (\lambda(k, X_\xi^s(k)) - \lambda(k + 1, X_\xi^s(k + 1))) = 0,$$

which we leave as an open question for future research.

5.2.3 Numerical simulations

To conclude this section, we provide numerical simulations illustrating the results derived above. For this purpose, we first consider the following nonlinear stochastic optimal control problem

$$\begin{aligned} \underset{\mathbf{U} \in \mathcal{U}_{\text{ad}}^N(j, X_j)}{\text{minimize}} \quad & J_N(X_j, \mathbf{U}) := \sum_{k=0}^{N-1} \mathbb{L}[X_{\mathbf{U}}(k; j, X_j)^2 + \gamma U(k)^2] \\ \text{s.t.} \quad & X_{\mathbf{U}}(k + 1; j, X_j) = (X_{\mathbf{U}}(k; j, X_j) - U(k))^2 + W(k + j) \\ & X_{\mathbf{U}}(0; 0, X_0) = X_0 \end{aligned} \tag{5.25}$$

where $\gamma \geq 0$ is a regularization parameter and $W(k)$ follows a two-point distribution such that

$$W(j)(\omega) := \begin{cases} 1 & \text{with probability } 0.7 \\ 0.25 & \text{with probability } 0.3 \end{cases} \quad \text{for } j \in \mathbb{N}_0.$$

For $\mathbb{L}[Z] = \mathbb{E}[Z]$, this problem constitutes a slight modification of the first example from Section 4.4, where it was shown numerically that the problem exhibits stochastic turnpike phenomena for $\gamma > 0$. Moreover, for $\gamma = 0$ it was shown analytically in [84] that the problem satisfies all stochastic turnpike properties from Definition 3.16.

For $\mathbb{L}[Z] = \mathbb{E}[Z]$ and $\mathbb{L}[Z] = \text{Div}_{0.5}^{\phi_{KL}}(Z)$, where $\text{Div}_{0.5}^{\phi_{KL}}(Z)$ is the Kullback-Leibler-divergence risk measure from equation (5.6) with constraint level $c = 0.5$, the corresponding state and control trajectories on horizons $N = 3, 5, 7, 9, 11$ with $\gamma = 15$ and initial condition $(j, X_j) = (0, 1.5)$ are shown in Figure 5.2.

We can again observe stochastic turnpike properties when a risk measure is used as the mapping $\mathbb{L}[Z]$ in (5.25). Moreover, the qualitative behavior of the solutions remains similar for $\mathbb{L}[Z] = \mathbb{E}[Z]$ and $\mathbb{L}[Z] = \text{Div}_{0.5}^{\phi_{KL}}(Z)$. However, the use of the risk measure in the cost appears to lead to a consolidation of the trajectories, making large state values less likely.

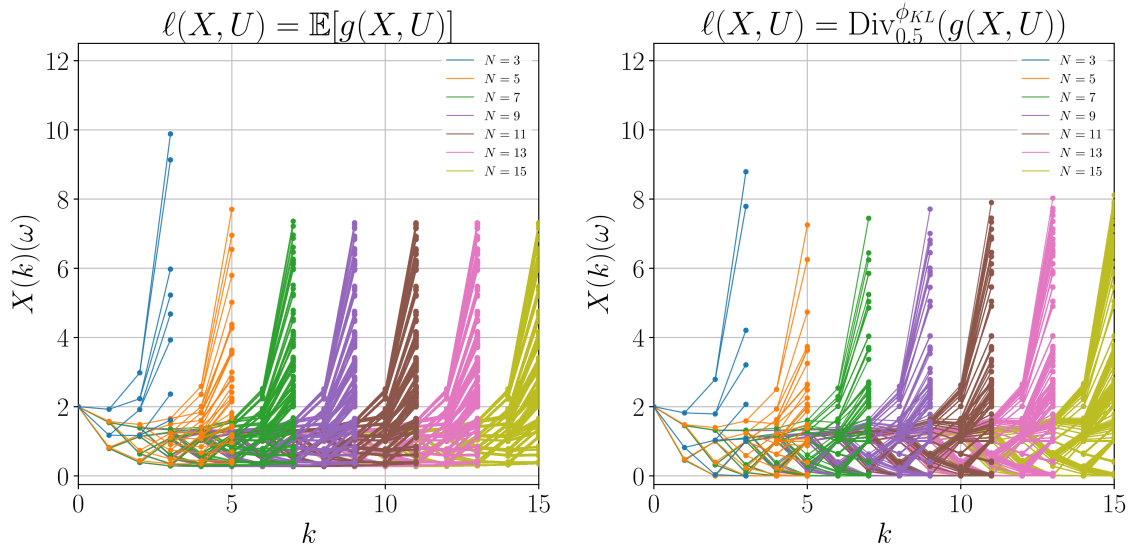


Figure 5.2: Evolution of all realization paths of the optimal solutions of problem (5.25) on different horizons for $\mathbb{L}[Z] = \mathbb{E}[Z]$ or $\mathbb{L}[Z] = \text{Div}_{0.5}^{\phi_{KL}}(Z)$.

In addition, we computed the parameters realizing the optimal stage costs at each step along the optimal trajectory, i.e.,

$$\xi_k^* \in \arg \min_{\xi \in \Xi} \mathbb{E}[\Psi(\xi, X_{\mathbf{U}_N^*}(k; 0, X_0))] \quad (5.26)$$

for horizons $N = 3, 5, 7, 9, 13$. The results are also shown in Figure 5.3, where we observe that these parameters exhibit a turnpike property analogous to that of the optimal states and controls.

This observation suggests that the quantities evaluated at the middle of the horizon provide a good approximation of the stationary quantities $(\mathbf{X}^s, \mathbf{U}^s)$ and ξ^s . Due to the results of this section the convergence to the optimal stationary process $\mathbf{X}^s$ and the distance between the asymptotic averaged cost and the optimal stationary cost for the closed-loop trajectory computed by Algorithm 3 is determined by the optimization horizon and how well the fixed parameter $\bar{\xi}$ fits the stationary one ξ^s .

To compute the closed-loop trajectories numerically, we ran Algorithm 3 with the deterministic initial value $X_0 = 1.5$ and generated 1 000 sample paths.

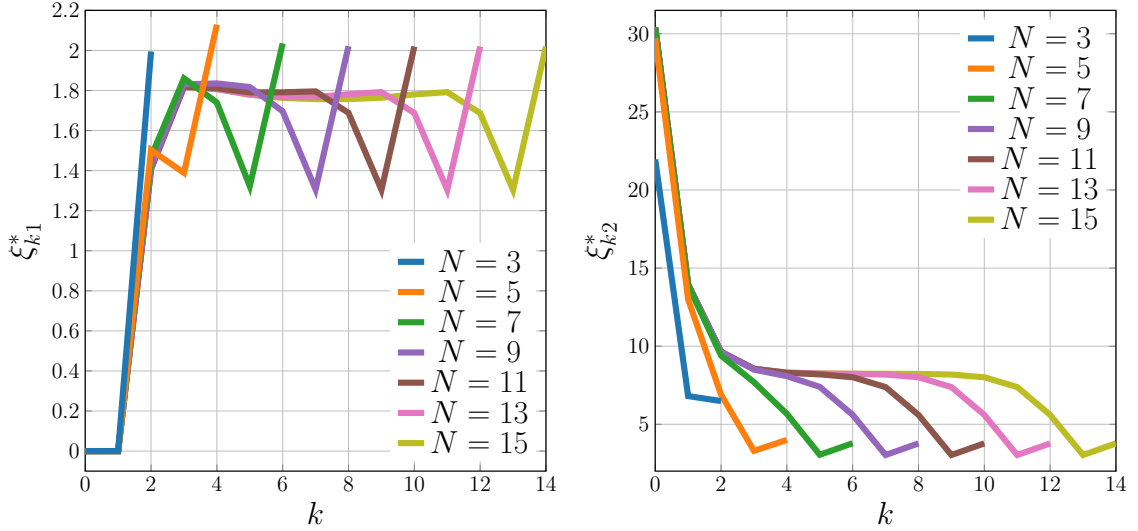


Figure 5.3: Evolution of the parameters realizing the optimal stage cost of problem (5.25) on different horizons for $\mathbb{L}[Z] = \text{Div}_{0.5}^{\phi_{KL}}(Z)$.

Figure 5.4 shows the averaged performance results for horizon lengths $N = 6, 7, 8, 9$ with parameter $\bar{\xi} = \xi^s$ in Algorithm 3. Furthermore, the figure also illustrates the performance for a fixed horizon $N = 9$ and different parameter choices $\bar{\xi} \in \{\xi^s, \xi_1, \xi_2\}$, where $\xi_1 = \xi^s + (1.5, 1.5)$ and $\xi_2 = \xi^s + (2.5, 2.5)$.

Increasing the horizon length N clearly improves the averaged performance, while deviations from the optimal stationary parameter ξ^s lead to a deterioration in performance, in agreement with Theorem 5.15. Furthermore, the horizon length appears to have a considerably stronger influence on the performance than the choice of $\bar{\xi}$, suggesting that the proposed approach is relatively robust with respect to errors in the parameter estimation in terms of performance, at least for this example.

Furthermore, based on the generated samples for the different parameters $\bar{\xi}$ and horizons N considered above, we computed the empirical cumulative distribution functions (CDFs) of the closed-loop state $X_{\mu_{N,\bar{\xi}}}(250; 0, X_0)$. As suggested by Corollary 5.19, we observe that the distribution gets closer to the stationary distribution as the horizon length increases or as the parameter $\bar{\xi}$ provides a better approximation of the stationary parameter. These results are shown in Figure 5.5. This behavior may be explained by the fact that risk measures typically place greater emphasis on states with higher probability, and the asymmetry in the system nonlinear system dynamics from equation (5.25), cf. Figure 5.2.

As a second example we consider the linear-quadratic stochastic optimal control problem

$$\begin{aligned}
 & \underset{\mathbf{U} \in \mathcal{U}_{\text{ad}}^N(j, X_j)}{\text{minimize}} & J_N(X_j, \mathbf{U}) &:= \sum_{k=0}^{N-1} \mathbb{L}[X_{\mathbf{U}}(k; j, X_j)^2 + 5U(k)^2] \\
 & \text{s.t.} & & X_{\mathbf{U}}(k+1; j, X_j) = 1.5X_{\mathbf{U}}(k; j, X_j) + U(k) + W(k+j) \\
 & & & X_{\mathbf{U}}(0; j, X_j) = X_j
 \end{aligned} \tag{5.27}$$

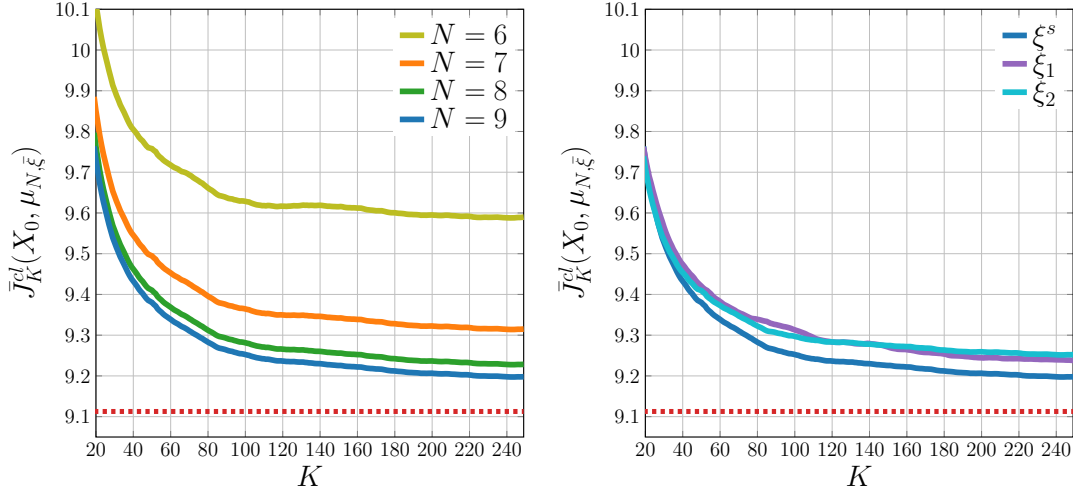


Figure 5.4: Optimal stationary cost $\ell(\mathbf{X}^s, \mathbf{U}^s)$ (dashed red) and averaged cost of the closed-loop trajectory of problem (5.25) with $\mathbb{L}[Z] = \text{Div}_{0.5}^{\phi_{KL}}(Z)$ for ξ^s and varying horizons $N = 6, 7, 8, 9$ (left) and for horizon $N = 9$ and varying parameters ξ^s, ξ_1, ξ_2 (right).

where $W(k)$ follows a two-point distribution such that

$$W(j)(\omega) := \begin{cases} 0.6 & \text{with probability } 0.5 \\ -0.6 & \text{with probability } 0.5 \end{cases} \quad \text{for } j \in \mathbb{N}_0.$$

For $\mathbb{L}[Z] = \mathbb{E}[Z]$ this corresponds to a standard stochastic linear-quadratic problem. For this setting we have already shown analytically that the problem exhibits stochastic turnpike properties in various forms, cf. Section 3.6.1. The optimal trajectories of problem (5.27) with $\mathbb{L}[Z] = \mathbb{E}[Z]$ for horizons $N = 3, 5, 7, 9, 13$ and $X_0 = 1.5$ are shown in the left column of Figure 5.6.

If, instead, a risk measure is used as the mapping $\mathbb{L}[Z]$, we can still observe stochastic turnpike properties. This is illustrated in the right column of Figure 5.6 for the conditional value-at-risk from (5.2) with confidence level $1 - \alpha = 0.95$. For numerical purposes we used the softplus function $\log(1 + e^z)$ in the simulations instead of the function $\max\{0, z\}$ appearing in the parameterized form (5.9). Note that using the softplus function yields an approximation of the averaged value-at-risk. However, it can also be interpreted as a risk measure in its own right due to (5.8) with $\psi(z) = \log(1 + e^z)$.

The results shown in Figure 5.6 indicate that although the qualitative behavior is similar for both the expectation and the risk measure used in the cost, the overall width of the possible state trajectories is smaller when the risk-averse formulation is used instead of the expectation.

Additionally, as in the previous example, we computed the parameters realizing the optimal stage costs at each step along the optimal trajectory, cf. (5.26). These are shown in Figure 5.7, where the parameters also exhibit a clear turnpike property. Hence, we can again approximate the stationary qualities by the turnpike.

As in the previous example, Algorithm 3 was applied with deterministic initial value $X_0 = 1.5$, and 1000 sample paths were generated to compute the closed-loop trajectories numerically.

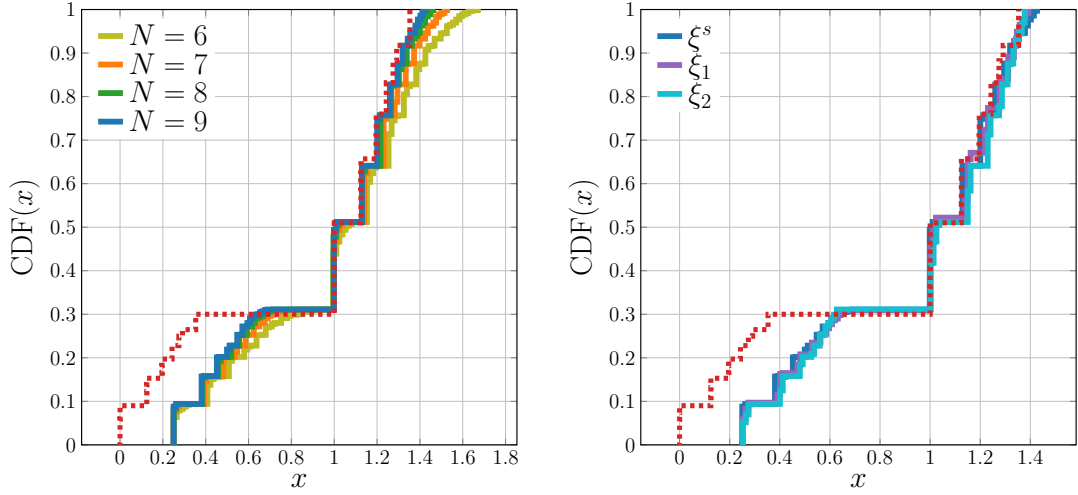


Figure 5.5: CDF of the stationary distribution (dashed red) and empirical CDFs of the closed-loop trajectory of problem (5.25) with $\mathbb{L}[Z] = \text{Div}_{0.5}^{\phi_{KL}}(Z)$ at time $k = 150$ on different horizons $N = 6, 7, 8, 9$ (left) and for horizon $N = 9$ and different parameters ξ^s, ξ_1, ξ_2 (right).

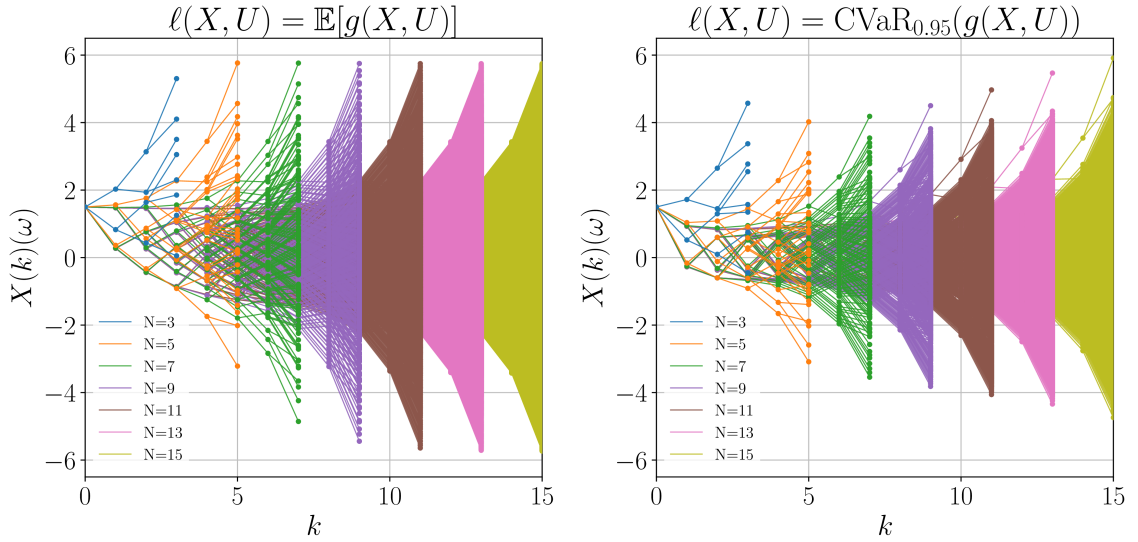


Figure 5.6: Evolution of all realization paths of the optimal solutions of problem (5.27) on different horizons for $\mathbb{L}[Z] = \mathbb{E}[Z]$ or $\mathbb{L}[Z] = \text{CVaR}_{0.95}(Z)$.

The averaged performance results for varying horizons $N = 6, 7, 8, 9$ with fixed parameter $\xi = \xi^s$, as well as for varying parameters $\xi = \xi^s, \xi_1, \xi_2$ with $\xi_1 = 0.75\xi^s$ and $\xi_2 = 0.5\xi^s$ for fixed horizon $N = 9$, are shown in Figure 5.8. We observe that increasing the horizon length N leads to improved performance, while deviations from the parameter ξ^s deteriorate the performance. However, in contrast to Figure 5.4, a sufficiently large deviation from the optimal parameter ξ^s leads to a significant deterioration in performance. This indicates that an accurate parameter estimation is important, although the overall sensitivity with respect to the parameter remains moderate.

Again, we computed the empirical cumulative distribution functions (CDFs) of the closed-loop state $X_{\mu_{N,\bar{\xi}}}(250; 0, X_0)$ based on the generated samples for the different

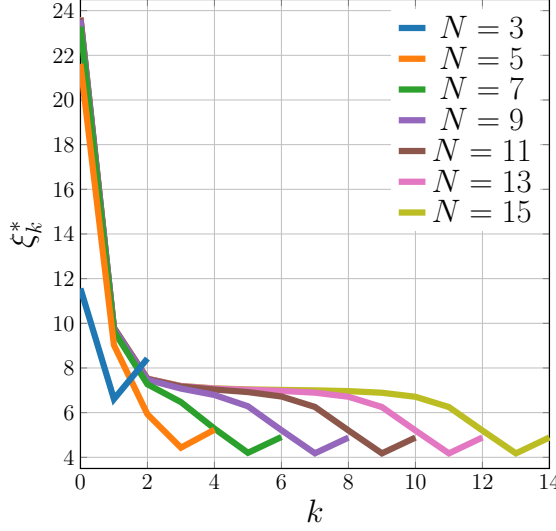


Figure 5.7: Evolution of the parameters realizing the optimal stage cost of problem (5.25) on different horizons for $\mathbb{L}[Z] = \text{CVaR}_{0.95}(Z)$.

parameters $\bar{\xi}$ and horizons N considered above. Similarly, as before, as suggested by Corollary 5.19, we observe that the distribution gets closer to the stationary distribution as the horizon length increases or as the parameter $\bar{\xi}$ provides a better approximation of the stationary parameter. However, in contrast to Figure 5.9, the deviation from the stationary distribution appears to be more evenly distributed across the domain. This behavior may be explained by the symmetric structure of the linear dynamics in (5.27).

5.3 Stochastic linear MPC with risk-averse constraints

In this section we will present first MPC results for problems with probabilistic constraints. More precisely, in terms of costs, we adapt the setting from Chapter 4, i.e., we consider stage costs of the form

$$\ell(X, U) = \mathbb{E}[g(X, U)]$$

for some lower-semicontinuous function $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}$.

However, in contrast to Chapter 4, for $G \in \mathbb{R}^{n_c \times n_x}$, $H \in \mathbb{R}^{n_c \times n_u}$, and $d \in \mathbb{R}^{n_c}$, we consider linear risk-averse constraints of the form

$$\mathbb{T}[GX + HU] \leq d. \quad (5.28)$$

Here, the mapping $\mathbb{T} : \mathcal{R}(\mathbb{R}^{n_c}) \rightarrow \bar{\mathbb{R}}^{n_c}$ is defined componentwise by risk measures ρ_i , $i = 1, \dots, n_c$, i.e.,

$$\mathbb{T}[Z] = (\rho_1(Z_1), \dots, \rho_{n_c}(Z_{n_c})) \quad \text{for } Z \in \mathcal{R}(\mathbb{R}^{n_c}). \quad (5.29)$$

Note that monotonicity and translation invariance directly transfer from the risk measures ρ_i to the mapping $\mathbb{T}$. Hence, it holds that

$$\mathbb{T}[Z_1] \geq \mathbb{T}[Z_2] \quad \text{for all } Z_1, Z_2 \in \mathcal{R}(\mathbb{R}^{n_c}) \text{ with } Z_1 \geq Z_2 \text{ a.s.}$$

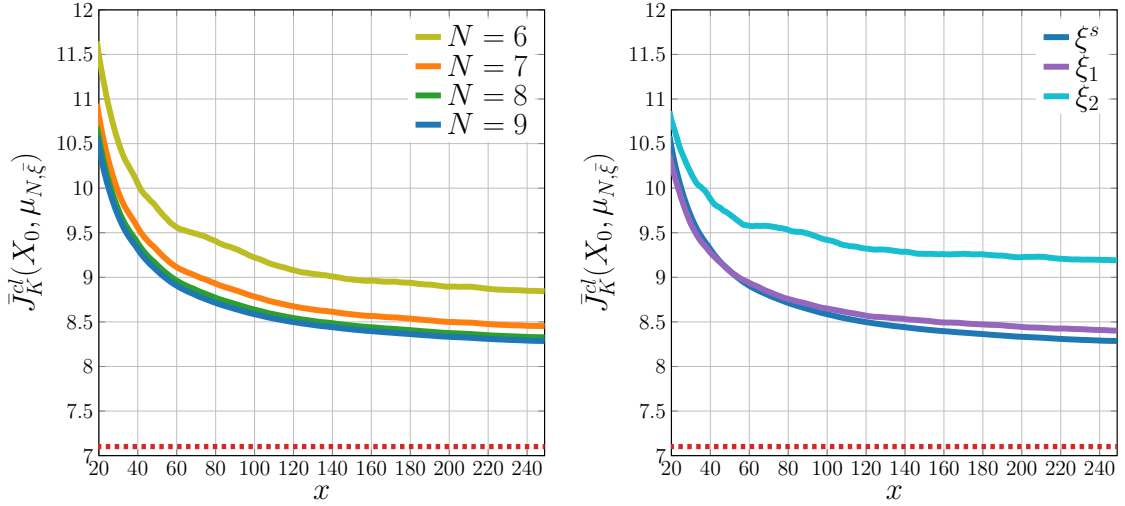


Figure 5.8: Optimal stationary cost $\ell(\mathbf{X}^s, \mathbf{U}^s)$ (red dashed) and averaged cost for ξ^s on different horizons $N = 6, 7, 8, 9$ (left) and for horizon $N = 9$ and different parameters ξ^s, ξ_1, ξ_2 (right) for the problem (5.27) with $\mathbb{L}[Z] = \text{CVaR}_{0.95}(Z)$.

and

$$\mathbb{T}[Z + m] = \mathbb{T}[Z] + m \quad \text{for all } Z \in \mathcal{R}(\mathbb{R}^{n_c}) \text{ and } m \in \mathbb{R}^{n_c}.$$

Furthermore, we restrict ourselves to linear stochastic systems

$$X(k+1) = AX(k) + BU(k) + DW(k) \quad (5.30)$$

with $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, such that (A, B) is stabilizable, and $D \in \mathbb{R}^{n_x \times n_w}$. We assume that the *i.i.d.* sequence $\mathbf{W}$ has finite first moment and variance denoted by

$$m_W := \mathbb{E}[W(k)] \in \mathbb{R}^{n_w}, \quad \Sigma_W := \text{Cov}(W(k)) \in \mathbb{R}^{n_w \times n_w}, \quad \text{for all } k \in \mathbb{N}_0.$$

Then, the optimal control problem considered in this section reads

$$\begin{aligned} \min_{\mathbf{U} \in \mathbb{U}_{\text{ad}}^N(j, X_j)} J_N(X_j, \mathbf{U}) &:= \sum_{k=0}^{N-1} \ell(X_{\mathbf{U}}(k; j, X_j), U(k)) \\ \text{s.t. } X_{\mathbf{U}}(k+1; j, X_j) &= AX_{\mathbf{U}}(k; j, X_j) + BU(k) + DW(k+j), \\ X_{\mathbf{U}}(0; j, X_j) &= X_j, \\ \mathbb{T}[GX_{\mathbf{U}}(k; j, X_j) + HU(k)] &\leq d. \end{aligned} \quad (5.31)$$

Remark 5.20. Note that in problem (5.31) we optimize over the set $\mathbb{U}_{\text{ad}}^N(j, X_j)$ and hence also consider non-Markovian control sequences. While this does not provide any advantage for the exact optimal solution compared to a restriction to Markovian controls, cf. Theorem 2.13, we will later see that the indirect feedback scheme proposed in this section yields a non-Markovian control sequence.

Our goal is to approximately solve this problem on the infinite horizon using model predictive control. However, similar to the previous section, the risk-averse constraints (5.28) require full distributional information of the state and control in

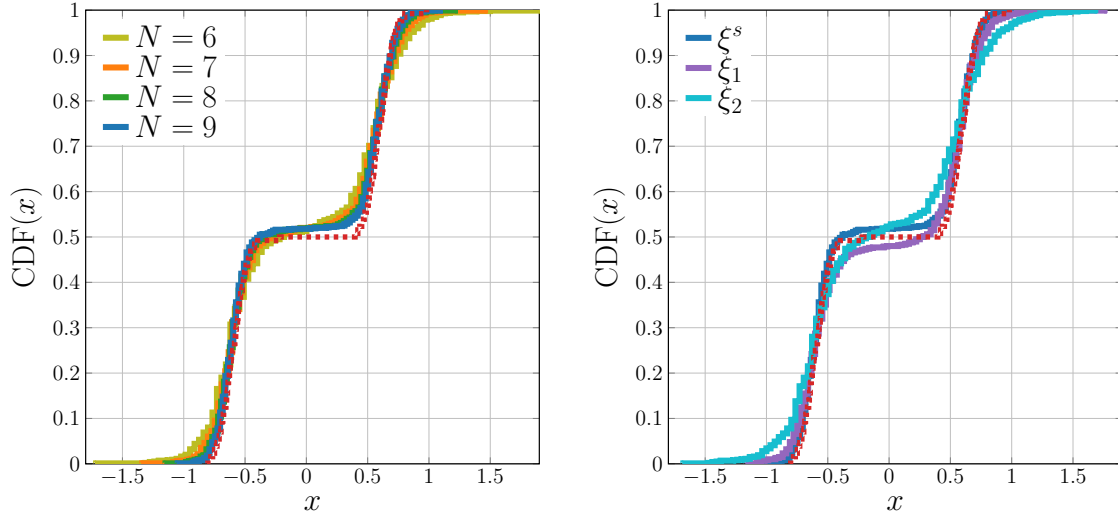


Figure 5.9: CDF of the stationary distribution (dashed red) and empirical CDFs of the closed-loop trajectory of problem (5.27) with $\mathbb{L}[Z] = \text{CVaR}_{0.95}(Z)$ at time $k = 150$ on different horizons $N = 6, 7, 8, 9$ (left) and for horizon $N = 9$ and different parameters ξ^s, ξ_1, ξ_2 (right).

order to evaluate the constraints exactly, and pathwise information alone is not sufficient to obtain an optimal solution. Moreover, it is possible that certain realizations of a random variable are infeasible, although the random variable itself is feasible, since the constraints are not enforced almost surely. This implies that recursive feasibility is also an issue if we use current measurements of the closed-loop solution as initial values. Hence, the implementable MPC Algorithm 2 from Chapter 4 is not suitable for such problems.

In the following, we now aim to derive a recursively feasible and implementable algorithm to solve risk-constrained problems via MPC, ensure closed-loop constraint satisfaction, and provide performance estimates.

5.3.1 The indirect feedback approach

To compute an approximate solution to the infinite horizon version of problem (5.31), we employ an indirect-feedback stochastic MPC scheme similar to [48, 49]. The basic idea of the indirect-feedback approach is to use a deterministic prediction $z(k) \in \mathbb{R}^{n_x}$ for the evaluation of the constraints in open loop, while the cost is optimized based on the most recent state measurement. The term *indirect* refers to the fact that the state measurement enters the optimization problem only indirectly via the cost evaluation.

To obtain a prediction for constraint evaluation in the MPC scheme, we employ a linear-affine feedback parametrization of the form

$$U(k) = KX(k) + \bar{u}(k),$$

where $K \in \mathbb{R}^{n_u \times n_x}$ is a fixed feedback gain that stabilizes the pair (A, B) , and $\bar{u}(k) \in \mathbb{V} \subseteq \mathbb{R}^{n_u}$ is the free control input. Substituting this parametrization into the

stochastic dynamics yields

$$\begin{aligned} X(k+1; j, X_j) &= AX(k; j, X_j) + BU(k) + DW(k+j) \\ &= (A + BK)X(k; j, X_j) + B\bar{u}(k) + DW(k+j), \\ X(0; j, X_j) &= X_j. \end{aligned} \quad (5.32)$$

We decompose the resulting dynamics into a nominal component $z(k; j, X_j) \in \mathbb{R}^{n_x}$ and a stochastic error component $E(k; j, X_j) \in \mathcal{R}(\mathbb{R}^{n_x})$ such that

$$X(k; j, X_j) = z(k; j, X_j) + E(k; j, X_j) \quad \text{for all } k \in \mathbb{N}_0. \quad (5.33)$$

The associated dynamics for the nominal prediction z and the error E are given by

$$\begin{aligned} z(k+1; j, X_j) &= (A + BK)z(k; j, X_j) + B\bar{u}(k) + D\mathbb{E}[W(k+j)], \\ z(0; j, X_j) &= \mathbb{E}[X_j], \end{aligned} \quad (5.34)$$

and

$$\begin{aligned} E(k+1; j, X_j) &= (A + BK)E(k; j, X_j) + D(W(k+j) - \mathbb{E}[W(k+j)]), \\ E(0; j, X_j) &= X_j - \mathbb{E}[X_j]. \end{aligned} \quad (5.35)$$

For a given initial value $X_j \in \mathcal{R}(\mathcal{X})$, the dynamics of z in (5.34) are purely deterministic, since $\bar{u}(k) \in \mathbb{R}^{n_u}$ for all $k \in \mathbb{N}_0$. Analogously to (2.4), for a given control sequence $\bar{\mathbf{u}} = (\bar{u}(0), \dots, \bar{u}(N-1))$, we denote the solution of (5.34) by $z_{\bar{\mathbf{u}}}(k; j, X_j)$.

In contrast, the error E is a random variable and thus a stochastic quantity. It does not depend on the control sequence $\bar{\mathbf{u}}$, but only on the fixed linear feedback K , which is used to pre-stabilize the error system (5.35). Accordingly, we do not use a subscript $\bar{\mathbf{u}}$ for the solution of (5.35), and denote it by $E(k; j, X_j)$.

Hence, the proposed decomposition of the dynamics induces a separation of the state into a deterministic, controlled component and a stochastic, uncontrolled component, which can be exploited to reformulate the constraints. Using the translation invariance of $\mathbb{T}$, the parametrization $U(k) = KX(k) + \bar{u}(k)$, and the decomposition from (5.33), we obtain for a fixed initial condition (j, X_j)

$$\begin{aligned} &\mathbb{T}[GX_{\mathbf{U}}(k; j, X_j) + HU(k)] \\ &= \mathbb{T}[G(z_{\bar{\mathbf{u}}}(k; j, X_j) + E(k; j, X_j)) + H(K(z_{\bar{\mathbf{u}}}(k; j, X_j) + E(k; j, X_j)) + \bar{u}(k))] \\ &= \mathbb{T}[(G + HK)z_{\bar{\mathbf{u}}}(k; j, X_j) + (G + HK)E(k; j, X_j) + H\bar{u}(k)] \\ &= (G + HK)z_{\bar{\mathbf{u}}}(k; j, X_j) + H\bar{u}(k) + \mathbb{T}[(G + HK)E(k; j, X_j)]. \end{aligned}$$

Thus, we can rewrite the risk-averse constraints as

$$(G + HK)z_{\bar{\mathbf{u}}}(k; j, X_j) + H\bar{u}(k) \leq d(k; j, X_j)$$

with

$$d(k; j, X_j) = d - \mathbb{T}[(G + HK)E(k; j, X_j)].$$

In addition to this constraint reformulation, we introduce terminal ingredients to formulate the open-loop problem for the indirect stochastic MPC scheme, namely a terminal set $\mathbb{Z}_f$ for the nominal prediction and a terminal penalty $F(X) = \mathbb{E}[g_f(X)]$ with $g_f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$. Such terminal ingredients are standard in MPC to ensure

recursive feasibility and stability, cf. [37, 67], and are also used to derive closed-loop guarantees in Section 5.3.2.

Summarizing the above, the corresponding open-loop optimal control problem on a horizon $N \in \mathbb{N}$ for the indirect feedback approach is given by

$$\begin{aligned}
& \underset{\bar{\mathbf{u}}}{\text{minimize}} && J_N(\bar{\mathbf{u}}, x_j, X_j) := \sum_{k=0}^{N-1} \ell(X(k), U(k)) + F(X(N)) \\
& \text{s.t.} && X_{\mathbf{U}}(k+1; j, x_j) = AX_{\mathbf{U}}(k; j, x_j) + BU(k) + DW(k+j), \\
& && U(k) = KX_{\mathbf{U}}(k; j, x_j) + \bar{u}(k), \quad X(0; j, x_j) = x_j, \\
& && z_{\bar{\mathbf{u}}}(k+1; j, X_j) = (A+BK)z_{\bar{\mathbf{u}}}(k; j, X_j) + B\bar{u}(k) + D\mathbb{E}[W(k+j)], \\
& && E(k+1; j, X_j) = (A+BK)E(k; j, X_j) + D(W(k+j) - \mathbb{E}[W(k+j)]), \\
& && z_{\bar{\mathbf{u}}}(0; j, X_j) = \mathbb{E}[X_j], \quad E(0; j, X_j) = X_j - \mathbb{E}[X_j], \\
& && \bar{u}(k) \in \mathbb{V}, \quad z_{\bar{\mathbf{u}}}(N; j, X_j) \in \mathbb{Z}_f, \\
& && (G+HK)z_{\bar{\mathbf{u}}}(k; j, X_j) + H\bar{u}(k) \leq d(k; j, X_j), \\
& && d(k; j, X_j) = d - \mathbb{T}[(G+HK)E(k; j, X_j)].
\end{aligned} \tag{5.36}$$

The corresponding MPC scheme is given in Algorithm 4. As described above, we use the current measurement $x_j \in \mathbb{R}^{n_x}$ as the initial value for the state and control variables entering the cost, while the prediction $X_j \in \mathcal{R}(\mathbb{R}^{n_x})$ initializes the nominal state z and the error E used for constraint evaluation.

The optimal value function of problem (5.36) is defined as

$$V_N(j, x_j, X_j) := \inf_{\bar{\mathbf{u}} \in \mathbb{V}_{\text{ad}}^N(j, x_j, X_j)} J_N(\bar{\mathbf{u}}, x_j, X_j), \tag{5.37}$$

where $\mathbb{V}_{\text{ad}}^N(j, x_j, X_j)$ denotes the set of all admissible control sequences for (5.36), i.e.,

$$\begin{aligned}
\mathbb{V}_{\text{ad}}^N(j, x_j, X_j) := \{ & \bar{\mathbf{u}} \in (\mathbb{R}^{n_u})^N : \text{the constraints in (5.36) are satisfied} \\
& \text{for the initial condition } (j, x_j, X_j) \}.
\end{aligned}$$

It should be noted that, due to the initialization at time $j = 0$, we obtain

$$z_0 = \mathbb{E}[X_0] = \mathbb{E}[X_{\mu_N}(0; 0, X_0)],$$

and thus

$$X_{\mu_N}(0; 0, X_0) = z_0 + E_0.$$

However, since the current measurement x_j influences the optimal solution $\bar{\mathbf{u}}_N^*$ of (5.36), and the nominal state z_j depends on $u_j = \bar{u}_N^*(0)$ for all times $j \geq 1$, it generally no longer holds that

$$z_j = \mathbb{E}[X_{\mu_N}(j; 0, X_0)].$$

Indeed, z_j is affected indirectly by randomness through the feedback of the measured state. Moreover, since the optimization itself depends on z_j via the constraints, it follows recursively that u_j , and hence z_{j+1} , depend on the entire history of measurements $x_0, \dots, x_j$. In particular, we obtain

$$z_j = \mathbb{E}[X_{\mu_N}(j; 0, X_0) \mid \mathcal{F}_{j-1}](\omega),$$

Algorithm 4 Indirect feedback stochastic MPC algorithm

Input: Fixed stabilizing feedback $K \in \mathbb{R}^{n_u \times n_x}$, feasible initial state $X_{\mu_N}(0; 0, X_0) = X_0$.

for $j = 0, 1, \dots$

1.) Measure the state $x_j = X_{\mu_N}(j; 0, X_0)(\omega)$.

2.) Solve the stochastic optimal control problem (5.36) and obtain the solution

$$\bar{\mathbf{u}}_N^* := (\bar{u}_N^*(0), \dots, \bar{u}_N^*(N-1)) \in (\mathbb{R}^{n_u})^N.$$

3.) Compute $X_{j+1} = z_{j+1} + E_{j+1}$ via

$$u_j = \bar{u}_N^*(0)$$

$$z_{j+1} = (A + BK)z_j + Bu_j + D\mathbb{E}[W(j)] = z_{\bar{\mathbf{u}}_N^*}(1; j, X_j) = \mathbb{E}[X_{j+1}]$$

$$E_{j+1} = (A + BK)E_j + D(W(j) - \mathbb{E}[W(j)]) = E(1; j, X_j) = X_{j+1} - \mathbb{E}[X_{j+1}].$$

4.) Apply the MPC feedback

$$\mu_N(x_0, \dots, x_j) := Kx_j + u_j$$

to system (5.32).

where $\mathcal{F}_j$ denotes the natural filtration of the closed-loop trajectory $X_{\mu_N}(\cdot; 0, X_0)$, cf. Definition 2.2. On the other hand, since the evolution of the error E does not depend on the control u_j , it is independent of the measurements $x_0, \dots, x_j$. Consequently, for all $j \in \mathbb{N}_0$,

$$X_{\mu_N}(j; 0, X_0) = \mathbb{E}[X_{\mu_N}(j; 0, X_0) \mid \mathcal{F}_{j-1}] + E_j \quad (5.38)$$

holds. This highlights that Algorithm 4 induces a *non-Markovian* feedback law and that z_j must be treated as a stochastic quantity in the closed-loop analysis. This is also the reason why we write $\mu_N(x_0, \dots, x_j)$ in Algorithm 4. Analogously to the implementable Algorithm 2 from Chapter 4, the feedback law μ_N is computed pointwise. The corresponding feedback on the space of random variables is then given by

$$\mu_N(X(0), \dots, X(k))(\omega) := \mu_N(X(0)(\omega), \dots, X(k)(\omega)).$$

Furthermore, it should be noted that, since the dynamics of E do not depend on the solution of the open-loop optimization problem, we obtain

$$E(j; 0, X_0) = E(k; j, X_j) =: E_{j+k}, \quad \text{for all } j, k \in \mathbb{N}_0.$$

Hence, the values E_j and the sequence $d(k; j, X_j)$ defining the upper bound on the nominal state z in problem (5.36) can be computed offline in advance via

$$\begin{aligned} E_{j+1} &= (A + BK)E_j + D(W(j) - \mathbb{E}[W(j)]), \quad E_0 = X_0 - \mathbb{E}[X_0], \\ d_{j+k} &:= d - \mathbb{T}[(G + HK)E_{j+k}] = d(k; j, X_j). \end{aligned}$$

5.3.2 Closed-loop guarantees

Next, we establish closed-loop guarantees for Algorithm 4, in particular, closed-loop constraint satisfaction and averaged (near-)optimality. In this section, we do not

impose any of the simplifying assumptions introduced later in Section 5.3.3. Consequently, the results derived here hold for general initial conditions, distributions, and stage costs. Moreover, as will be shown in Section 5.3.3, all assumptions required in this section are satisfied in the linear-quadratic setting, which allows us to directly transfer the guarantees derived in this section to the computationally tractable Algorithm 5.

Recursive feasibility

Before addressing constraint satisfaction and optimality estimates, we first establish recursive feasibility of Algorithm 4. That is, if the open-loop problem (5.36) is feasible for an initial condition X_0 , then feasibility is preserved along the closed-loop trajectory.

To this end, we introduce the following assumption, which is in line with [48, Assumption 1], and ensures that there exists a subset of admissible nominal states z which is forward invariant under a control law $u_f \in \mathbb{V}$.

Assumption 5.21. *There exists a set $\mathbb{Z}_f \neq \emptyset$ and a control $u_f \in \mathbb{V}$ such that*

$$(G + HK)z + Hu_f \leq d - \sup_{j \in \mathbb{N}_0} \mathbb{T}[(G + HK)E(j; 0, X_0)] \quad (5.39)$$

and

$$(A + BK)z + Bu_f + Dm_W \in \mathbb{Z}_f$$

holds for all $z \in \mathbb{Z}_f$.

Note that, since the feedback gain $K \in \mathbb{R}^{n_u \times n_x}$ stabilizes the pair (A, B) , the system (5.35) with $W(j) \in L^p(\mathbb{R}^{n_w})$ and $X_0 \in L^p(\mathbb{R}^{n_x})$ for some $1 \leq p \leq \infty$ satisfies

$$\sup_{j \in \mathbb{N}_0} \|E(j; 0, X_0)\|_{L^p} < \infty.$$

Hence, the supremum in (5.39) is finite, provided that the mapping $\mathbb{T}$ is locally bounded on L^p -bounded sets, i.e., for every $C > 0$,

$$\sup_{\|Y\|_{L^p} \leq C} \|\mathbb{T}[Y]\| < \infty.$$

For sufficiently large p , this property is satisfied for mappings $\mathbb{T}$ induced by common risk measures such as the expectation, the value at risk (5.1), the conditional value at risk (5.2), or the entropic value at risk (5.3), which will be of particular interest in Section 5.3.3.

Moreover, for any normalized, monotone and translative risk measure ρ it holds that

$$|\rho(X)| \leq \|X\|_{L^\infty}, \quad X \in L^\infty(\mathbb{R}_x^n).$$

Hence, for $p = \infty$, this property is satisfied for all such risk measures and induced mappings $\mathbb{T}$, and therefore the assumption is not restrictive when considering regular enough disturbances and initial values.

Consequently, Assumption 5.21 can always be satisfied by choosing $\mathbb{Z}_f := \{z^s\}$ and $u_f = \bar{u}^s$, provided that there exists an equilibrium $(z^s, \bar{u}^s) \in \mathbb{R}^{n_x} \times \mathbb{V}$ of the dynamics (5.34) satisfying (5.39).

Under Assumption 5.21, recursive feasibility of Algorithm 4 follows analogously to [48, Theorem 1].

Theorem 5.22. *If problem (5.36) is feasible for the initial condition $z_0 = \mathbb{E}[X_0]$, then Algorithm 4 is recursive feasible, i.e., feasible for all times $j \in \mathbb{N}_0$.*

Proof. Let $\bar{\mathbf{u}}_N^* = (\bar{u}_N^*(0), \dots, \bar{u}_N^*(N-1))$ be an optimal solution of problem (5.36) at time $j \in \mathbb{N}_0$. Then, by Assumption 5.21 the control sequence

$$\mathbf{v} := (\bar{u}_N^*(1), \dots, \bar{u}_N^*(N-1), u_f)$$

satisfies the constraints in (5.36) for

$$z_{j+1} = (A + BK)z_j + B\bar{u}_N^*(0) + Dm_W$$

and all measurements $x_j \in \mathbb{R}^{n_x}$. Hence, $\mathbf{v}$ is an admissible control sequence for time $j+1$, which proves the claim since problem (5.36) is feasible at time $j=0$ by assumption. $\square$

Remark that, although recursive feasibility itself does not directly depend on the horizon N , initial feasibility typically does, since reachability of the terminal set $\mathbb{Z}_f$ must be guaranteed.

Constraint satisfaction

Since, in closed loop, the value z_j represents $\mathbb{E}[X_{\mu_N}(j; 0, X_0) \mid \mathcal{F}_{j-1}]$ rather than the unconditional expectation $\mathbb{E}[X_{\mu_N}(j; 0, X_0)]$, it is not immediately clear that the proposed risk-averse constraints are satisfied in closed loop. The following theorem shows that the constraints imposed in open loop are indeed sufficient to guarantee closed-loop constraint satisfaction.

Theorem 5.23. *The risk-averse constraints are satisfied in closed loop, i.e.,*

$$\mathbb{T}[GX_{\mu_N}(j; 0, X_0) + H\mu_N(X_{\mu_N}(0; 0, X_0), \dots, X_{\mu_N}(j; 0, X_0))] \leq d$$

holds for all times $j \in \mathbb{N}_0$, where the feedback μ_N is computed pointwise according to Algorithm 4.

Proof. According to (5.38), the closed-loop trajectory can be written as

$$X_{\mu_N}(j; 0, X_0) = \mathbb{E}[X_{\mu_N}(j; 0, X_0) \mid \mathcal{F}_{j-1}] + E_j.$$

Furthermore, the feedback law generated by Algorithm 4 is given by

$$\mu_N(X_{\mu_N}(0; 0, X_0), \dots, X_{\mu_N}(j; 0, X_0)) = KX_{\mu_N}(j; 0, X_0) + \bar{U}(j),$$

where $\bar{U}(j)$ is defined pointwise via $\bar{U}(j)(\omega) := u_j$.

Using this representation, the closed-loop constraints can be written as

$$\mathbb{T}[(G + HK)(\mathbb{E}[X_{\mu_N}(j; 0, X_0) \mid \mathcal{F}_{j-1}] + E_j) + H\bar{U}(j)].$$

By the open-loop constraints, we have

$$\begin{aligned} (G + HK)z_j + Hu_j &= (G + HK)\mathbb{E}[X_{\mu_N}(j; 0, X_0) \mid \mathcal{F}_{j-1}](\omega) + H\bar{U}(j)(\omega) \\ &\leq d_j = d - \mathbb{T}[(G + HK)E_j] \end{aligned}$$

for all $\omega \in \Omega$ and $j \in \mathbb{N}_0$.

Note that $\mathbb{T}[(G + HK)E_j] \in \mathbb{R}^{n_x}$ is a constant for all $j \in \mathbb{N}_0$. Hence, using the monotonicity and translation invariance of $\mathbb{T}$, we obtain

$$\begin{aligned} & \mathbb{T}[(G + HK)(\mathbb{E}[X_{\mu_N}(j; 0, X_0) \mid \mathcal{F}_{j-1}] + E_j) + H\bar{U}(j)] \\ &= \mathbb{T}[(G + HK)\mathbb{E}[X_{\mu_N}(j; 0, X_0) \mid \mathcal{F}_{j-1}] + H\bar{U}(j) + (G + HK)E_j] \\ &\leq \mathbb{T}[d - \mathbb{T}[(G + HK)E_j] + (G + HK)E_j] \\ &= d - \mathbb{T}[(G + HK)E_j] + \mathbb{T}[(G + HK)E_j] \\ &= d, \end{aligned}$$

for all $j \in \mathbb{N}_0$, which proves the claim. $\square$

Note that the proof of constraint satisfaction relies on the decomposition of the closed-loop state into a stochastic component $E(j)$, which is independent of the control, and a nominal component $z(j)$, which can be constrained almost surely in a suitable manner. Hence, we conjecture that similar results can also be obtained for alternative decompositions and control parametrizations that preserve these structural properties.

Remark 5.24. *In general, it is difficult to evaluate*

$$d_j = d - \mathbb{T}[(G + HK)E_j] \quad (5.40)$$

exactly, except for special cases such as those considered in Section 5.3.3. One possible approach to approximate these terms is to use Monte Carlo sampling. However, since such methods may underestimate the risk $\mathbb{T}[(G + HK)E_j]$, the guarantees of Theorem 5.23 may no longer hold.

Alternatively, one can compute upper bounds T_j such that

$$\mathbb{T}[(G + HK)E_j] \leq T_j, \quad \text{for all } j \in \mathbb{N}_0,$$

and replace d_j by

$$\tilde{d}_j := d - T_j$$

in problem (5.36). While this tightening leads to a more conservative formulation, the results of this section remain valid.

Averaged performance optimality

As a final step in our closed-loop analysis, we establish optimality estimates for the averaged performance. These results are based on the notion of stochastic dissipativity introduced in Chapter 3. Furthermore, we impose the following structural assumption on the terminal cost.

Assumption 5.25. *The Assumption 5.21 holds and there exists a stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ and a constant $C_f \geq 0$ such that for all $X_k \in \mathcal{R}(\mathbb{R}^{n_x})$ and the terminal control u_f from Assumption 5.21 the inequality*

$$\begin{aligned} & F(AX_k + B(KX_k + u_f) + DW(k)) \\ & \leq F(X_k) - \ell(X_k, KX_k + u_f) + \ell(\mathbf{X}^s, \mathbf{U}^s) + C_f \end{aligned} \quad (5.41)$$

holds for all $k \in \mathbb{N}_0$.

In Section 5.3.3, we provide a construction of such terminal costs for the linear-quadratic case. Note that, due to the law-invariance of the terminal cost F , it suffices to verify that (5.41) holds for some $k \in \mathbb{N}_0$, as this then implies that it holds for all $k \in \mathbb{N}_0$, since $\mathbf{W}$ is an *i.i.d.* sequence.

The following theorem establishes an asymptotic upper bound on the averaged performance

$$\bar{J}_L^{cl}(X_0, \mu_N) = \frac{1}{L} \sum_{k=0}^{L-1} \ell(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(0; 0, X_0), \dots, X_{\mu_N}(j; 0, X_0))) \quad (5.42)$$

under Assumption 5.25. The proof follows similar arguments as in the proof of [48, Theorem 3], while extending the result beyond quadratic costs and yielding a characterization in terms of a stationary pair.

Theorem 5.26. *Let Assumption 5.25 hold. Then, it holds that*

$$\limsup_{L \rightarrow \infty} \bar{J}_L^{cl}(X_0, \mu_N) \leq \ell(\mathbf{X}^s, \mathbf{U}^s) + C_f.$$

Proof. Consider $j \in \mathbb{N}_0$, a given measurement $x_j = X_{\mu_N}(j; 0, X_0)(\omega) \in \mathbb{R}^{n_x}$, and the prediction

$$X_j = z_j + E_j = \mathbb{E}[X_{\mu_N}(j; 0, X_0) \mid \mathcal{F}_{j-1}](\omega) + E_j$$

from Algorithm 4. Furthermore, assume that

$$\mathbf{u}_N^* \in \arg \min V_N(j, x_j, X_j)$$

holds, where V_N is the optimal value function from (5.37).

Then, we obtain

$$V_N(j, x_j, X_j) = J_N(\mathbf{U}_N^*, x_j, X_j) = \sum_{k=0}^{N-1} \ell(X_{\mathbf{U}^*}(k; j, x_j), U_N^*(k)),$$

with

$$\begin{aligned} X_{\mathbf{U}_N^*}(k+1; j, x_j) &= AX_{\mathbf{U}^*}(k; j, x_j) + BU^*(k) + DW(k+j), & X_{\mathbf{U}_N^*}(0; j, x_j) &= x_j \\ U_N^*(k) &= KX_{\mathbf{U}_N^*}(k; j, x_j) + u_N^*(k). \end{aligned}$$

As shown in the proof of Theorem 5.22 the control sequence

$$\mathbf{v} = (u_N^*(1), \dots, u_N^*(N-1), u_f)$$

with $u_f \in \mathbb{V}$ from Assumption 5.21 is admissible for time $j+1$, i.e., admissible for the prediction

$$X_{j+1} = z_{j+1} + E_{j+1} = \mathbb{E}[X_{\mu_N}(j+1; 0, X_0) \mid \mathcal{F}_j](\omega) + E_{j+1} =: \tilde{X}_{j+1}$$

and any measurement $x_{j+1} = X_{\mu_N}(j+1; 0, X_0)(\omega)$.

Hence, using the law-invariance of the costs and that $\mathbf{W}$ is *i.i.d.* we get

$$\begin{aligned} & \mathbb{E} \left[V_N(j+1, X_{\mu_N}(j+1; 0, X_0), \tilde{X}_{j+1}) \mid \mathcal{F}_j \right] (\omega) \\ & \leq \sum_{k=1}^{N-1} \ell(X_{\mathbf{U}^*}(k; j, x_j), U_N^*(k)) \\ & \quad + F \left(AX_{\mathbf{U}_N^*}(N; j, x_j) + B(KX_{\mathbf{U}_N^*}(N; j, x_j) + u_f) + DW(N+1+j) \right). \end{aligned}$$

Using Assumption 5.25 this leads to the estimate

$$\begin{aligned} & V_N(j, x_j, X_j) - \mathbb{E} \left[V_N(j+1, X_{\mu_N}(j+1; 0, X_0), \tilde{X}_{j+1}) \mid \mathcal{F}_j \right] (\omega) \\ & \leq \ell(X_{\mu_N}(j; 0, X_0)(\omega), \mu_N(X_{\mu_N}(0; 0, X_0)(\omega), \dots, X_{\mu_N}(j; 0, X_0)(\omega))) \\ & \quad - \ell(\mathbf{X}^s, \mathbf{U}^s) - C_f \end{aligned}$$

Taking the expectation then yields

$$\begin{aligned} & \ell(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(0; 0, X_0), \dots, X_{\mu_N}(j; 0, X_0))) \\ & \leq \ell(\mathbf{X}^s, \mathbf{U}^s) + C_f + \mathbb{E}[V_N(j, X_{\mu_N}(j; 0, X_0), \tilde{X}_j)] \\ & \quad - \mathbb{E}[V_N(j+1, X_{\mu_N}(j+1; 0, X_0), \tilde{X}_{j+1})] \end{aligned}$$

and thus, we get

$$\begin{aligned} & \limsup_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \ell(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(0; 0, X_0), \dots, X_{\mu_N}(j; 0, X_0))) \\ & \leq \limsup_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(\ell(\mathbf{X}^s, \mathbf{U}^s) + C_f \right. \\ & \quad \left. + \mathbb{E}[V_N(j, X_{\mu_N}(j; 0, X_0), \tilde{X}_j)] - \mathbb{E}[V_N(j+1, X_{\mu_N}(j+1; 0, X_0), \tilde{X}_{j+1})] \right) \\ & \leq \ell(\mathbf{X}^s, \mathbf{U}^s) + C_f \\ & \quad + \limsup_{L \rightarrow \infty} \frac{1}{L} \left(\mathbb{E}[V_N(0, X_{\mu_N}(0; 0, X_0), \tilde{X}_0)] - \mathbb{E}[V_N(L, X_{\mu_N}(L; 0, X_0), \tilde{X}_L)] \right) \\ & \leq \limsup_{L \rightarrow \infty} \frac{1}{L} (\mathbb{E}[V_N(X_0, z_0, E_0)] - NC_g) + \ell(\mathbf{X}^s, \mathbf{U}^s) + C_f \\ & = \ell(\mathbf{X}^s, \mathbf{U}^s) + C_f, \end{aligned}$$

where $C_g \in \mathbb{R}$ is a lower bound on the deterministic stage costs g . $\square$

Theorem 5.26 shows that the asymptotic averaged performance can be bounded from above depending on how well the terminal cost is designed with respect to the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$, that is, how small the constant C_f can be chosen. Moreover, using the optimal operation property from Lemma 3.8, we can further establish near-optimality and show that the averaged performance (5.42) asymptotically approaches the optimal value if C_f is zero.

Corollary 5.27. *Let Assumptions 5.25 hold and assume that the stochastic optimal control problem (5.31) is stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$. Then, the closed-loop solution computed pointwisely by Algorithm 4 has near-optimal averaged performance, i.e.,*

$$\ell(\mathbf{X}^s, \mathbf{U}^s) \leq \liminf_{L \rightarrow \infty} \bar{J}_L^{cl}(X_0, \mu_N) \leq \limsup_{L \rightarrow \infty} \bar{J}_L^{cl}(X_0, \mu_N) \leq \ell(\mathbf{X}^s, \mathbf{U}^s) + C_f.$$

Moreover, if $C_f = 0$ holds in Assumption 5.25, then the closed-loop solution has optimal averaged performance, i.e.,

$$\lim_{L \rightarrow \infty} \bar{J}_L^{cl}(X_0, \mu_N) = \ell(\mathbf{X}^s, \mathbf{U}^s).$$

Proof. Since the closed-loop solution generated by Algorithm 4 satisfies the constraints, cf. Theorem 5.23, we can apply Lemma 3.8 to conclude that

$$\ell(\mathbf{X}^s, \mathbf{U}^s) \leq \liminf_{L \rightarrow \infty} \frac{1}{L} \sum_{k=0}^{L-1} \ell(X_{\mu_N}(j; 0, X_0), \mu_N(X_{\mu_N}(0; 0, X_0), \dots, X_{\mu_N}(j; 0, X_0))).$$

Note that this is possible, since Lemma 3.8 holds explicitly also for non-Markovian controls. The upper bound on the averaged performance follows by applying Theorem 5.26. $\square$

5.3.3 Moment-based reformulation for linear-quadratic problems

Although our theory applies to general cost functions and disturbance distributions, constructing suitable terminal penalties and solving the open-loop problem (5.36) is, in general, challenging. In this section, we introduce simplifying assumptions that yield an implementable version of Algorithm 4, relying only on the expectation and covariance of the relevant quantities.

We consider linear-quadratic stage costs of the form

$$\ell(X, U) = \mathbb{E}[\|X\|_Q^2 + \|U\|_R^2 + q^T X + r^T U + c], \quad (5.43)$$

with $Q \in \mathbb{R}^{n_x \times n_x}$ symmetric and positive semidefinite, $R \in \mathbb{R}^{n_u \times n_u}$ symmetric and positive definite, $(A, Q^{1/2})$ detectable, $q \in \mathbb{R}^{n_x}$, $r \in \mathbb{R}^{n_u}$, and $c \in \mathbb{R}$.

Furthermore, we consider a terminal penalty of the form

$$F(X) = \mathbb{E}[\|X\|_{P_f}^2 + p_f^T X], \quad (5.44)$$

where $P \in \mathbb{R}^{n_x \times n_x}$ is symmetric and positive semidefinite and $p \in \mathbb{R}^{n_x}$.

Then, for a given measurement $x_j \in \mathbb{R}^{n_x}$, we can evaluate the cost in (5.36) as

$$\begin{aligned} & \sum_{k=0}^{N-1} \left(\|m_X(k)\|_Q^2 + \|K m_X(k) + u(k)\|_R^2 + q^T m_X(k) + r^T (K m_X(k) + u(k)) \right. \\ & \quad \left. + \text{Tr}((Q + K^T R K) \Sigma_X(k)) \right) \\ & + \|m_X(N)\|_{P_f}^2 + p_f^T m_X(N) + \text{Tr}(P_f \Sigma_X(N)) \end{aligned}$$

with

$$\begin{aligned} m_X(k+1) &= \mathbb{E}[X(k)] = (A + BK)m_X(k) + Bu(k) + Dm_W, \quad m_X(0) = x_j, \\ \Sigma_X(k+1) &= \text{Cov}(X(k)) = (A + BK)\Sigma_X(k)(A + BK)^T + D\Sigma_W D^T, \quad \Sigma_X(0) = 0, \end{aligned}$$

where $m_W := \mathbb{E}[W(k)]$ denotes the mean and $\Sigma_W := \text{Cov}(W(k))$ the covariance of the disturbance.

Moreover, we assume that the disturbance follows a Gaussian distribution, i.e.,

$$W(j) \sim \mathcal{N}(m_W, \Sigma_W) \quad \text{for all } j \in \mathbb{N}_0,$$

and that, for every $i = 1, \dots, n_c$, the risk measures ρ_i defining the risk-averse constraints in (5.29) are given by either the expectation, the value-at-risk (5.1), the conditional value-at-risk (5.2), or the entropic value-at-risk (5.3).

Since $W(k)$ is Gaussian, we conclude that for an initial value

$$X_0 \sim \mathcal{N}(m_{X_0}, \Sigma_{X_0}),$$

the error E is Gaussian with zero mean for all times, i.e.,

$$E_j \sim \mathcal{N}(0, \Sigma_E(j)) \quad \text{for all } j \in \mathbb{N}_0,$$

with covariance given by

$$\Sigma_E(j+1) = (A + BK)\Sigma_E(j)(A + BK)^T + D\Sigma_W D^T, \quad \Sigma_E(0) = \Sigma_{X_0}. \quad (5.45)$$

Using this observation, we can evaluate d_j from (5.40) explicitly. Indeed, for any random variable $Y \sim \mathcal{N}(m_Y, s_Y^2)$ with mean $m_Y \in \mathbb{R}$ and variance $s_Y^2 \in \mathbb{R}_{\geq 0}$, the risk measures ρ_i defining the constraints can be written as

$$\rho_i(Y) = m_Y + s_Y R_i(\alpha), \quad \alpha \in (0, 1), \quad i = 1, \dots, n_c, \quad (5.46)$$

with

$$R_i(\alpha) = \begin{cases} 0 & \text{if } \rho_i(Y) = \mathbb{E}[Y], \\ \Phi^{-1}(1 - \alpha) & \text{if } \rho_i(Y) = \text{VaR}_{1-\alpha}(Y), \\ \frac{\varphi(\Phi^{-1}(1-\alpha))}{\alpha} & \text{if } \rho_i(Y) = \text{CVaR}_{1-\alpha}(Y), \\ \sqrt{-2 \ln(\alpha)} & \text{if } \rho_i(Y) = \text{EVaR}_{1-\alpha}(Y), \end{cases} \quad (5.47)$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ denotes the standard normal density and $\Phi(x)$ the standard normal distribution function.

Applying this to the constraint expression $Y_i = (G + HK)_i E_j$, we obtain

$$d_j = d - \begin{pmatrix} s_1(j) R_1(\alpha) \\ \vdots \\ s_{n_c}(j) R_{n_c}(\alpha) \end{pmatrix},$$

where

$$s_i(j) = \sqrt{(G + HK)_i \Sigma_E(j) (G + HK)_i^T}$$

denotes the standard deviation of the i -th constraint component.

Remark 5.28. The constraint $\text{VaR}_{1-\alpha}(Y) \leq d$ is equivalent to the chance constraint formulation $\mathbb{P}(Y \leq d) \geq 1 - \alpha$. Hence, our framework includes chance constraints as a special case and can therefore be viewed as an extension of [48] with respect to the class of admissible constraints.

Moreover, since K stabilizes the pair (A, B) , the covariance dynamics of the error E from (5.45) converge to a steady-state covariance Σ_E^s , which is the unique solution of the Lyapunov equation

$$\Sigma_E^s = (A + BK)\Sigma_E^s(A + BK)^T + D\Sigma_W D^T.$$

Now assume that $\Sigma_{X_0} \leq \Sigma_E^s$. Then, by monotonicity of the Lyapunov recursion, it follows that

$$\Sigma_E(j) \leq \Sigma_E^s \quad \text{for all } j \in \mathbb{N}_0.$$

Consequently, the risk offsets satisfy

$$d_j \geq d^s := d - \begin{pmatrix} s_1^s R_1(\alpha) \\ \vdots \\ s_{n_c}^s R_{n_c}(\alpha) \end{pmatrix},$$

where

$$s_i^s := \sqrt{(G + HK)_i \Sigma_E^s (G + HK)_i^T}.$$

Hence, as discussed after Assumption 5.21, any admissible steady state $(z^s, \bar{u}^s) \in \mathbb{R}^{n_x} \times \mathbb{V}$ of the dynamics (5.34), i.e., any pair satisfying

$$z^s = (A + BK)z^s + B\bar{u}^s + Dm_W = Az^s + B(Kz^s + \bar{u}^s) + Dm_W, \quad (5.48)$$

and

$$(G + HK)z^s + H\bar{u}^s \leq r^s, \quad (5.49)$$

can be used to define a terminal set $\mathbb{Z}_f = \{z^s\}$ and a terminal control $u_f = \bar{u}^s$ fulfilling Assumption 5.21.

Using such a terminal set together with the simplifications introduced in this section, the moment-based open-loop problem reduces to

$$\begin{aligned} \underset{\bar{\mathbf{u}}}{\text{minimize}} \quad & J_N(\bar{\mathbf{u}}, x_j, z_j, E_j) := \sum_{k=0}^{N-1} \left(\|m_X(k)\|_Q^2 + \|Km_X(k) + \bar{u}(k)\|_R^2 + q^T m_X(k) \right. \\ & \quad \left. + r^T (Km_X(k) + \bar{u}(k)) + c + \text{Tr}((Q + K^T RK)\Sigma_X(k)) \right) \\ & \quad \left. + \|m_X(N)\|_{P_f}^2 + p_f^T m_X(N) + \text{Tr}(P_f \Sigma_X(N)) \right. \\ & m_X(k+1) = (A + BK)m_X(k) + B\bar{u}(k) + Dm_W, \quad m_X(0) = x_j, \\ & \Sigma_X(k+1) = (A + BK)\Sigma_X(k)(A + BK)^T + D\Sigma_W D^T, \quad \Sigma_X(0) = 0, \\ & z(k+1) = (A + BK)z(k) + B\bar{u}(k) + Dm_W, \quad z(0) = z_j \\ & \Sigma_E(j+k+1) = (A + BK)\Sigma_E(j+k)(A + BK)^T + D\Sigma_W D^T, \quad \Sigma_E(j) = \Sigma_{E_j} \\ & \bar{u}(k) \in \mathbb{V}, \quad z(N) \in \mathbb{Z}_f, \quad (G + HK)z(k) + H\bar{u}(k) \leq d_{j+k} \\ & d_{j+k} = d - (s_1(j+k)R_1(\alpha), \dots, s_{n_c}(j+k)R_{n_c}(\alpha))^T, \\ & s_i(j+k) = \sqrt{(G + HK)_i \Sigma_E(j+k)(G + HK)_i^T}, \quad i = 1, \dots, n_c. \end{aligned} \quad (5.50)$$

The corresponding moment-based indirect-feedback SMPC scheme is given in Algorithm 5. Note that this algorithm only requires the propagation of deterministic quantities, namely means and covariance matrices. In particular, no random variables need to be propagated online. Hence, the resulting optimization problem is computationally tractable.

Since the terminal set $\mathbb{Z}_f$ was designed via (5.48) and (5.49) such that Assumption 5.21 is satisfied, recursive feasibility from Theorem 5.22 and closed-loop constraint satisfaction from Theorem 5.23 also hold for the moment-based Algorithm 5.

To ensure that the performance bounds from Corollary 5.27 also hold, we have to show that the stochastic optimal control problem (5.31) with stage cost (5.43) is stochastically dissipative and that suitable terminal costs satisfying Assumption 5.25 can be designed. The first point is addressed by the following corollary.

Algorithm 5 Moment-based indirect feedback stochastic MPC algorithm

Input: Fixed stabilizing feedback $K \in \mathbb{R}^{n_u \times n_x}$, feasible initial state $X_{\mu_N}(0; 0, X_0) = X_0$ with $Cov(X_0) \leq \Sigma_E^s$.

for $j = 0, 1, \dots$

1.) Measure the state $x_j = X_{\mu_N}(j; 0, X_0)(\omega)$.

2.) Solve the stochastic optimal control problem (5.50) and obtain the solution

$$\bar{\mathbf{u}}_N^* := (\bar{u}_N^*(0), \dots, \bar{u}_N^*(N-1)) \in (\mathbb{R}^{n_u})^N.$$

3.) Compute the predictions

$$\begin{aligned} z_{j+1} &= (A + BK)z_j + Bu_j + Dm_W = z(1), \quad u_j = \bar{u}_N^*(0) \\ \Sigma_{E_{j+1}} &= (A + BK)\Sigma_{E_j}(A + BK)^T + D\Sigma_W D^T = \Sigma(j+1) \end{aligned}$$

4.) Apply the MPC feedback

$$\mu_N(x_0, \dots, x_j) := Kx_j + u_j$$

to system (5.32).

Corollary 5.29. Assume that the stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ from Theorem 3.37 is admissible for problem (5.31), i.e.,

$$\mathbb{T}[GX^s(k) + HU^s(k)] \leq d, \quad \text{for all } k \in \mathbb{N}_0. \quad (5.51)$$

Then, the stochastic optimal control problem (5.31) is stochastically dissipative at $(\mathbf{X}^s, \mathbf{U}^s)$.

Proof. The claim directly follows by Theorem 3.37 since dissipativity of the unconstrained problem at $(\mathbf{X}^s, \mathbf{U}^s)$ implies dissipativity for the constrained problem at the same stationary pair $(\mathbf{X}^s, \mathbf{U}^s)$ if $(\mathbf{X}^s, \mathbf{U}^s)$ is admissible. $\square$

Note that, due to the stationarity of $(\mathbf{X}^s, \mathbf{U}^s)$ and the law invariance of $\mathcal{T}$, it suffices that (5.51) holds for some $k \in \mathbb{N}_0$ in order to conclude that it holds for all $k \in \mathbb{N}_0$.

To also obtain an upper bound on the asymptotic averaged performance, we next provide suitable choices for P_f and p_f in Theorem 5.31 such that Assumption 5.25 is satisfied. Before doing so, however, we prove the following lemma, which provides a clearer interpretation of the constant C_f in Theorem 5.31.

Lemma 5.30. Assume that P solves the Lyapunov equation

$$(A + BK)^T P (A + BK) - P = -(Q + K^T R K) \quad (5.52)$$

and $(z^s, \bar{u}^s)$ satisfies the conditions (5.48) and (5.49).

Then, there exists a stationary distribution $\tilde{\mathbb{P}}_X^s$ such that the pair $(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s)$ with

$$\begin{aligned} \tilde{X}^s(k+1) &= A\tilde{X}^s(k) + B\tilde{U}^s(k) + DW(k), \quad \tilde{X}^s(0) \sim \tilde{\mathbb{P}}_X^s, \\ \tilde{U}^s(k) &= K\tilde{X}^s(k) + \bar{u}^s = K(\tilde{X}^s(k) - z^s) + u^s \end{aligned} \quad (5.53)$$

defines a stationary pair with cost

$$\ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s) = \mathbb{E}[\|D(W(k) - m_W)\|_{P_f}^2] + \ell(z^s, u^s), \quad u^s := \bar{u}^s + Kz^s.$$

Proof. Consider the shifted stationary process $\hat{X}^s(k) := \tilde{X}^s(k) - z^s$ for all $k \in \mathbb{N}_0$. Then, the dynamics (5.53) imply that

$$\begin{aligned}\hat{X}^s(k+1) &= A\hat{X}^s(k) + B\hat{U}^s(k) + D\hat{W}(k) = (A + BK)\hat{X}^s(k) + D\hat{W}(k) \\ \hat{U}^s(k) &:= \tilde{U}^s(k) - u^s = K\hat{X}^s(k), \quad \hat{W}(k) := W(k) - m_W.\end{aligned}\tag{5.54}$$

Since K stabilizes the pair (A, B) and $\mathbb{E}[\hat{W}(k)] = 0$, $\mathbb{E}[\|W(k)\|^2] < \infty$ by [72, Section 10.5.4] we can conclude that there exists a stationary distribution $\hat{P}_X^s$ for system (5.54) such that $\hat{\mathbf{X}}^s$ is a stationary process with initial condition $\hat{X}^s(0) \sim \hat{P}_X^s$ and $\mathbb{E}[\hat{X}^s(k)] = 0$ for all $k \in \mathbb{N}_0$.

Hence, system (5.53) defines a stationary pair with initial condition

$$\tilde{X}^s(0) = \hat{X}^s(0) + z^s \sim \tilde{\mathbb{P}}_X^s$$

since $z^s \in \mathbb{R}^{n_x}$ is only a constant shift.

Furthermore, it holds that

$$\begin{aligned}\ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s) &= \mathbb{E}[\|\hat{X}^s(k)\|_Q^2 + \|\hat{U}^s(k)\|_R^2 + \hat{q}^T \hat{X} + \hat{r}^T \hat{U} + \hat{c}] \\ &= \mathbb{E}[\|\hat{X}^s(k)\|_{Q+K^T RK}^2] + (\hat{q}^T + \hat{r}^T K) \mathbb{E}[\hat{X}^s(k)] + \hat{c} \\ &= \mathbb{E}[\|\hat{X}^s(k)\|_{Q+K^T RK}^2] + \hat{c}\end{aligned}$$

with $\hat{q} := q + 2Qz^s$, $\hat{r} := r + 2Ru^s$, and $\hat{c} := \ell(z^s, u^s)$.

Thus, by the Lyapunov equation (5.52) and the stationarity of $\hat{\mathbf{X}}^s$ we obtain

$$\begin{aligned}\ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s) &= \mathbb{E}[\|\hat{X}^s(k)\|_{Q+K^T RK}^2] + \hat{c} \\ &= \mathbb{E}[\|\tilde{X}^s(k) - x^s\|_{P_f - (A+BK)^T P_f (A+BK)}^2] + \hat{c} \\ &= \mathbb{E}[\|\hat{X}^s(k)\|_P^2] - (\mathbb{E}[\|(A+BK)\hat{X}^s(k)\|_P^2] + \mathbb{E}[\|D\hat{W}(k)\|_P^2] - \mathbb{E}[\|D\hat{W}(k)\|_P^2]) \\ &= \mathbb{E}[\|\hat{X}^s(k)\|_P^2] - \mathbb{E}[\|\hat{X}^s(k+1)\|_{P_f}^2] + \mathbb{E}[\|D\hat{W}(k)\|_P^2] + \hat{c} \\ &= \mathbb{E}[\|D\hat{W}(k)\|_P^2] + \hat{c}\end{aligned}$$

since $\hat{W}(k)$ is independent of $\hat{X}^s(k)$ for all $k \in \mathbb{N}_0$. □

Next, we finally show how to construct appropriate weights in the terminal cost to satisfy Assumption 5.25 in our linear-quadratic setting, where the constant C_f can be calculated as the difference between two stationary costs using Lemma 5.30.

Lemma 5.31. *Let the Assumption of Corollary 5.29 hold. Consider the terminal costs (5.44) and the equilibrium $(z^s, \bar{u}^s)$ defining the terminal set $\mathbb{Z}_f$ and satisfying (5.48) and (5.49). Assume that P_f solves the Lyapunov equation (5.52) and p_f solves the linear equation*

$$((A + BK) - I)p_f + 2((A + BK)^T P_f - P_f)z^s = -(q + 2Qz^s + K^T(r + 2Ru^s)).$$

Then, Assumption 5.25 is satisfied with

$$C_f := \ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s) - \ell(\mathbf{X}^s, \mathbf{U}^s) \geq 0,$$

where $(\mathbf{X}^s, \mathbf{U}^s)$ is the stationary pair from Theorem 5.29 and $(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s)$ is the stationary pair induced by $(z^s, \bar{u}^s)$ from Lemma 5.30.

Proof. Consider $k \in \mathbb{N}_0$, $X_k \in \mathcal{R}(\mathbb{R}^{n_x})$, and set

$$X_k^+ = AX_k + BU_k + DW(k), \quad U_k = KX_k + \bar{u}^s, \quad u^s := \bar{u}^s + Kz^s.$$

Switching to the shifted variables

$$\hat{X}_k^+ := X_k^+ - z^s, \quad \hat{X} := X_k - z^s, \quad \hat{U} := U_k - u^s, \quad \hat{W}(k) := W(k) - m_W,$$

yields

$$\hat{X}_k^+ = A\hat{X}_k + B\hat{U}_k + D\hat{W}(k) = (A + BK)\hat{X}_k + D\hat{W}(k)$$

and

$$\begin{aligned} \ell(X_k, U_k) &= \mathbb{E}[\|\hat{X}_k\|_Q^2 + \|\hat{U}_k\|_R^2 + \hat{q}^T \hat{X} + \hat{r}^T \hat{U} + \hat{c}] \\ &= \mathbb{E}[\|\hat{X}_k\|_{Q+K^T RK}^2] + (\hat{q}^T + \hat{r}^T K) \mathbb{E}[\hat{X}_k] + \hat{c} \end{aligned}$$

with $\hat{q} := q + 2Qz^s$, $\hat{r} := r + 2Ru^s$, and $\hat{c} := \ell(z^s, u^s)$.

Using that $\hat{W}(k)$ has zero mean and is independent of $\hat{X}_k$ it follows that

$$\begin{aligned} F(X_k^+) - F(X_k) &= F(\hat{X}_k^+ + z^s) - F(\hat{X}_k + z^s) \\ &= \mathbb{E}[\|\hat{X}_k\|_{(A+BK)^T P_f (A+BK) - P_f}^2 + \|\hat{W}(k)\|_{P_f}^2] \\ &\quad + \mathbb{E}[2\hat{X}_k^T (A+BK)^T P_f z^s - 2\hat{X}_k^T P_f z^s + p^T ((A+BK) - I)\hat{X}_k + D\hat{W}(k)] \\ &= \mathbb{E}[-\|\hat{X}_k\|_{Q+K^T RK}^2] + \mathbb{E}[\|D\hat{W}(k)\|_{P_f}^2] \\ &\quad + (((A+BK) - I)p_f + 2(A+BK)^T P_f z^s - 2P_f z^s)^T \mathbb{E}[\hat{X}_k] \\ &= \mathbb{E}[-\|\hat{X}_k\|_{Q+K^T RK}^2] - (\hat{q} + K^T \hat{r})^T \mathbb{E}[\hat{X}_k] - \hat{c} + \mathbb{E}[\|D\hat{W}(k)\|_{P_f}^2] + \hat{c} \\ &= -\ell(X_k, U_k) + \mathbb{E}[\|D\hat{W}(k)\|_{P_f}^2] + \hat{c}. \end{aligned}$$

This shows the claim, since $\ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s) = \mathbb{E}[\|D\hat{W}(k)\|_{P_f}^2] + \hat{c}$ holds by Lemma 5.30 and $\ell(\mathbf{X}^s, \mathbf{U}^s) \leq \ell(\tilde{\mathbf{X}}^s, \tilde{\mathbf{U}}^s)$ holds by Theorem 3.9 due to dissipativity. $\square$

Theorem 5.31 provides a constructive way for the computation of terminal costs in the linear-quadratic setting considered in this section. With this choice of the terminal cost, Corollary 5.27 ensures that the indirect feedback Algorithm 5 achieves near-optimal asymptotic performance with respect to the averaged cost criterion.

Moreover, Lemma 5.30 characterizes the resulting performance gap in terms of the deviation between the optimal stationary process and the stationary process induced by the fixed feedback gain K and the steady state $(z^s, \bar{u}^s)$. This representation shows that the asymptotic performance is fully determined by the choice of $(K, (z^s, \bar{u}^s))$ relative to the optimal stationary solution.

In particular, averaged optimality is obtained by selecting K and $(z^s, \bar{u}^s)$ such that the induced stationary process coincides with the optimal one. This is summarized in the following corollary.

Corollary 5.32. *Let the Assumptions of Theorem 5.31 hold with*

$$K = K^s, \quad z^s = x^s + x_*, \quad \bar{u}^s = -K(x^s + x_*) + (u^s + u_*^s),$$

where x^s, x_*^s, u^s, u_*^s and K^s are from Theorem 3.37.

Then, the closed-loop solution generated by Algorithm 5 is averaged cost optimal, i.e.,

$$\lim_{L \rightarrow \infty} \bar{J}_L^{\text{cl}}(X_0, \mu_N) = \ell(\mathbf{X}^s, \mathbf{U}^s)$$

holds, where $(\mathbf{X}^s, \mathbf{U}^s)$ is the stationary pair from Theorem 3.37.

Proof. Follows by Corollary 5.15 using Corollary 5.29 and Theorem 5.31. $\square$

Remark 5.33. *In this section, we have exploited the Gaussian assumption on $\mathbf{W}$ in order to obtain exact reformulations of the considered risk measures in terms of the first two moments, cf. (5.46). However, as discussed in Remark 5.24, these exact representations can be replaced by suitable upper bounds, leading to more conservative constraint formulations. Consequently, the presented results can be extended to non-Gaussian disturbances, provided that the employed risk measures admit distribution-independent bounds depending only on mean and variance.*

For instance, for value-at-risk constraints and equivalently for chance constraints, such an upper bound can be derived via Markov's inequality. This yields

$$\text{VaR}_{1-\alpha}(Z) \leq \mathbb{E}[Z] + \sqrt{\frac{\mathbb{V}[Z]}{1-\alpha}} \quad \text{for all } Z \in \mathcal{R}(\mathbb{R}).$$

5.3.4 Numerical simulations

In this section, we illustrate the theoretical results by means of a DC-DC converter regulation problem, which has previously been used as a benchmark in stochastic MPC studies [14, 91, 92].

The system dynamics are of the form (5.30), with

$$A = \begin{pmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{pmatrix}, \quad B = \begin{pmatrix} 4.798 \\ 0.115 \end{pmatrix},$$

$D = I_2$, and additive noise $W(k) \sim \mathcal{N}(m_w, \Sigma_W)$ with expectation $m_w = 0.05$ and covariance $\Sigma_W = 0.1I_2$.

We consider quadratic stage costs of the form (5.43) with parameters

$$Q = \text{diag}(1, 10), \quad R = 5, \quad q = 0, \quad r = 0, \quad c = 0.$$

Note that, although there are no linear terms in the stage cost, it is still necessary to compute the linear weight p_f in the terminal cost since the noise has non-zero mean.

Additionally, we impose a scalar risk constraint on the first state component given by

$$\mathbb{T}[CX] = \rho(X_1(k)) \leq 2, \quad \rho \in \{\mathbb{E}, \text{VaR}_{0.6}, \text{CVaR}_{0.6}, \text{EVaR}_{0.6}\}, \quad (5.55)$$

with $C = (1, 0)$.

To construct the terminal constraint set, we use the optimal steady state of the deterministic problem, i.e., the pair $(x^s + x_*^s, u^s + u_*^s)$ from Corollary 5.32.

Since the problem has the linear-quadratic structure discussed in Section 5.3.3, we apply Algorithm 5 to compute a feedback-based closed-loop trajectory. All simulations are performed using a prediction horizon of $N = 12$ and 15 000 Monte Carlo samples.

The feedback gain is chosen as $K = K^s$, where K^s is the stabilizing Riccati-based solution from Theorem 3.37. The initial condition is deterministic and given by $X_0 = (1.8, 1.5)^T$.

For each time step k and constraint mapping $\mathbb{T}$, we evaluate the quantity

$$\rho(X_{\mu_N,1}(k; 0, X_0)), \quad \rho \in \{\mathbb{E}, \text{VaR}_{0.6}, \text{CVaR}_{0.6}, \text{EVaR}_{0.6}\}, \quad (5.56)$$

where $X_{\mu_N,1}(k; 0, X_0)$ denotes the first component of the closed-loop state generated by Algorithm 5. Figure 5.10 shows the resulting trajectories for a different constraint mapping $\mathbb{T}$ in each row.

Furthermore, for all $\alpha \in (0, 1)$ and $Z \in \mathcal{R}(\mathbb{R})$, the well-known ordering

$$\text{VaR}_{1-\alpha}(Z) \leq \text{CVaR}_{1-\alpha}(Z) \leq \text{EVaR}_{1-\alpha}(Z)$$

holds under the respective definitions used here. Consequently, the relative conservativeness of the constraints follows the same ordering, which is also reflected in Figure 5.10.

To compare the performance for different choices of K in Algorithm 5, we construct a second stabilizing feedback

$$\tilde{K} := -(\tilde{R} + B^T \tilde{P}_f B)^{-1} B^T \tilde{P}_f A, \quad (5.57)$$

where $\tilde{P}_f$ is the solution of the modified Riccati equation

$$\tilde{P}_f = A^T \tilde{P}_f A + \tilde{Q} - A^T \tilde{P}_f B (\tilde{R} + B^T \tilde{P}_f B)^{-1} B^T \tilde{P}_f A,$$

with weighting matrices $\tilde{Q} := 0.01Q$ and $\tilde{R} := 200R$.

We then rerun Algorithm 5 for these different feedback choices, where the constraints in (5.55) are defined using $\text{CVaR}_{0.6}$.

As Corollary 5.32 indicates, for $K = K^s$ the averaged closed-loop cost (5.42) converges to the optimal stationary cost, as also illustrated in Figure 5.11. For $K = \tilde{K}$, Corollary 5.27 guarantees only a suboptimal performance bound, and convergence to the optimal stationary cost is not ensured in general. Figure 5.11 shows that the averaged closed-loop cost approaches a different asymptotic value in this case, illustrating that the choice of feedback gain influences the long-term performance, in accordance with the theoretical results.

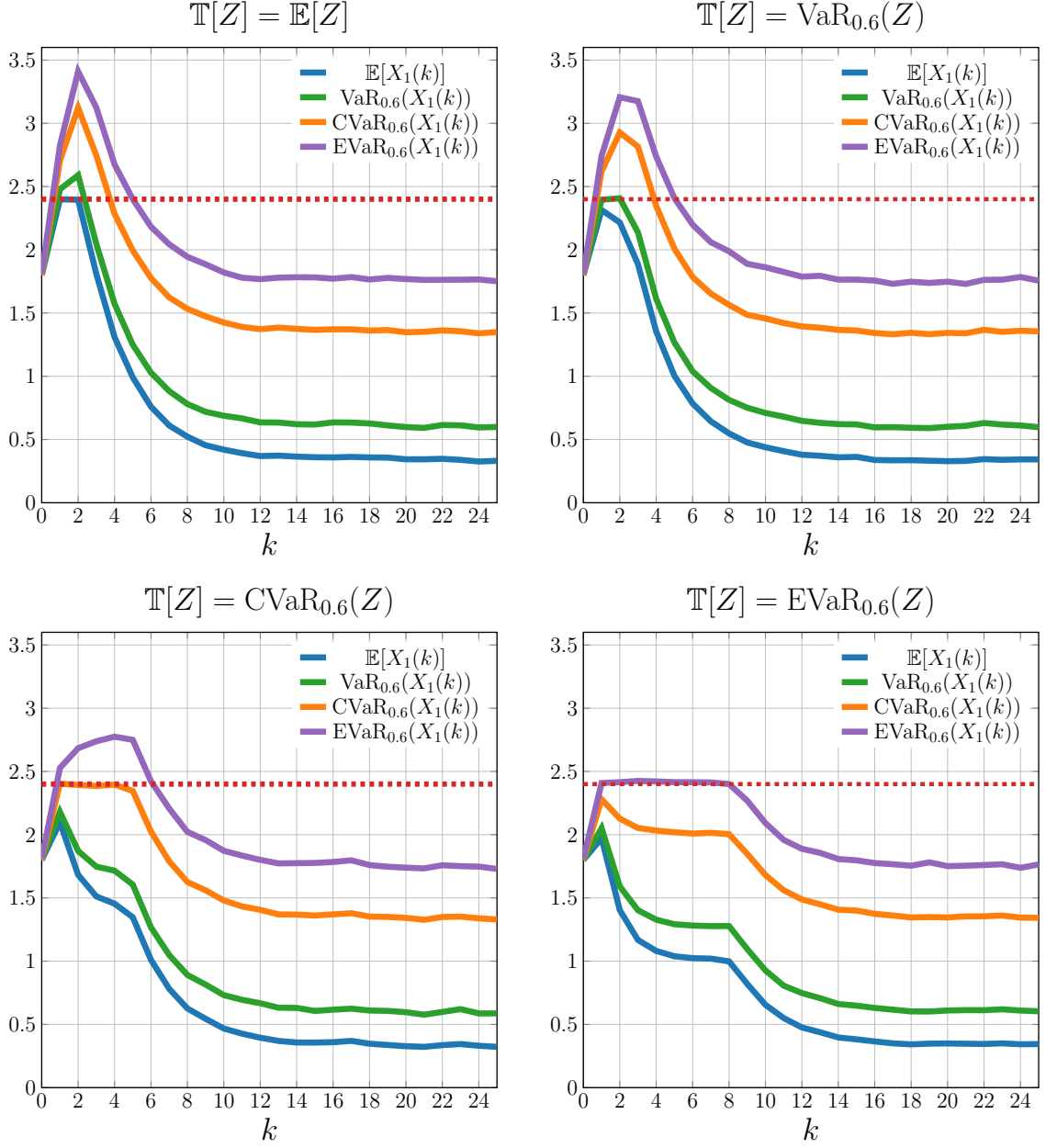


Figure 5.10: Constraint bound (dashed red) together with the evolution of all risk measures from (5.56) applied to the first component of the closed-loop trajectory calculated by the moment-based indirect feedback stochastic MPC Algorithm 5 when one of them is used to define the constraint.

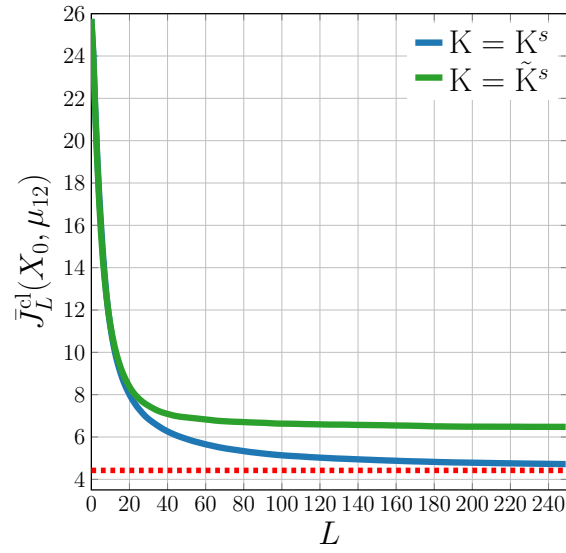


Figure 5.11: Averaged cost of the closed-loop trajectory calculated by the moment-based indirect feedback stochastic MPC Algorithm 5 for the Riccati feedback $K = K^s$ from (5.57) and the feedback $K = \tilde{K}^s$ from (5.57), together with the optimal stationary cost (dashed red).

6 Outlook and future research

In this thesis, we have formalized dissipativity and turnpike properties in stochastic settings and demonstrated how these notions can be used to analyze closed-loop properties of stochastic MPC schemes.

To conclude this thesis, we outline several directions for future research.

Sufficient and necessary conditions for stochastic dissipativity. As discussed in Sections 3.6 and 4.4, verifying stochastic dissipativity is, in general, a challenging task. It would therefore be desirable to derive tractable sufficient and necessary conditions for stochastic (strict) dissipativity.

In particular, a deeper investigation of the connections between stochastic dissipativity and notions such as *irreducibility*, *aperiodicity*, *positive recurrence*, and *ergodicity*, which play a central role in stochastic control theory, cf. [72], appears promising. These concepts are closely related to the existence and uniqueness of stationary distributions; see, for instance, [71]. Moreover, recurrence properties have been employed in the stability analysis of stochastic control systems [32] and discounted stochastic optimal control problems [73]. Establishing systematic links between such results and the dissipativity framework developed in this thesis therefore constitutes an interesting direction for future research.

Another important objective is the derivation of converse results to Theorem 3.19. More precisely, it would be desirable to identify conditions under which stochastic turnpike properties imply stochastic strict dissipativity, thereby obtaining a stochastic counterpart of the deterministic converse results in [36]. In the deterministic setting, such converse implications typically rely on suitable controllability assumptions. Consequently, the development and analysis of appropriate notions of stochastic controllability appear to be an essential prerequisite for extending these results to stochastic systems. Such a converse result would be particularly useful from a practical perspective. Indeed, turnpike behavior can often be identified numerically, as demonstrated in several examples throughout this thesis. Establishing an implication from turnpike properties to strict dissipativity could therefore provide a means of obtaining numerical evidence for strict dissipativity without requiring the explicit construction of a storage function.

Performance and stability estimates for suboptimal solutions. In the closed-loop analysis of Chapters 4 and 5, we have assumed that the underlying stochastic optimal control problems can be solved exactly at each optimization step. In practice, however, this assumption is often unrealistic, since stochastic optimal control problems are typically too complex to be solved online within reasonable computation times, especially in the presence of unbounded disturbances, probabilistic constraints, or long prediction horizons.

To reduce the computational burden, various approximation techniques have been proposed, including fixed feedback parametrizations or restrictions to particular con-

trol classes and scenario-tree reduction methods, cf. [46, 59, 61, 62]. Since such approaches generally yield only suboptimal solutions, it remains an open question to what extent stochastic dissipativity and turnpike properties are preserved under suboptimal control laws and how these concepts can be exploited to derive stability and performance guarantees for the resulting stochastic MPC closed-loop systems.

Dissipativity of stochastic linear-quadratic optimal control problems with risk-averse costs. In Section 3.6.1, we have established stochastic dissipativity results for generalized stochastic linear-quadratic optimal control problems with expected-value cost functionals. Since risk-averse cost formulations have also been considered in Section 5.2, it is natural to ask whether the dissipativity results can be extended to linear-quadratic problems with risk-averse stage costs.

A possible approach is to reformulate risk-averse cost functionals in terms of expectations, similar to the strategy used in Section 5.2. However, for parametric risk measures, this reformulation typically destroys the quadratic structure of the stage cost, which is crucial for the dissipativity analysis in the linear-quadratic setting.

An alternative approach is based on the dual representation of coherent risk measures. In particular, for a coherent risk measure ρ , one has

$$\rho(Z) = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[Z] = \sup_{\mathbb{Q} \in \mathcal{M}} \int_{\Omega} Z d\mathbb{Q},$$

where $\mathcal{M}$ is a set of probability measures that are absolutely continuous with respect to the nominal probability measure $\mathbb{P}$, see [29].

While this representation is typically not advantageous from a numerical perspective, since it introduces an additional optimization over probability measures, it may nevertheless be useful for analytical purposes. In particular, it could provide a suitable starting point for constructing storage functions in risk-averse linear-quadratic problems, as it preserves the quadratic structure of the underlying stage cost within each measure $\mathbb{Q} \in \mathcal{M}$.

Dissipativity-based analysis of nonlinear problems with risk-averse constraints.

In Section 5.3, we extended the framework of [48, 49] from linear stochastic systems with chance constraints to a setting with risk-averse constraints. For this class of problems, we developed a dissipativity-based closed-loop analysis and established performance guarantees in an averaged sense.

In parallel, nonlinear extensions of chance-constrained stochastic MPC have been developed in [55], where closed-loop stability and feasibility results are obtained using contraction metric techniques. These results demonstrate that nonlinear stochastic MPC problems with probabilistic constraints can be treated in a rigorous manner beyond the linear setting.

A natural question is therefore whether these nonlinear techniques can be combined with the risk-averse framework introduced in Section 5.3. In particular, it is of interest to investigate whether dissipativity-based arguments can be extended to nonlinear systems with risk-measure-based constraints while still guaranteeing closed-loop feasibility and performance bounds. Moreover, it remains an open question whether dissipativity can also be exploited to derive stability properties of the resulting closed-loop trajectories in this setting.

Such an extension would provide a unified framework for nonlinear stochastic MPC under general risk-averse constraint formulations.

Stochastic dissipativity and turnpikes in continuous-time problems. In this thesis, we have analyzed stochastic optimal control problems in discrete time. A natural question is whether our notions of stochastic dissipativity and turnpike properties can also be formulated for controlled stochastic differential equations (SDEs) and used to analyze the corresponding continuous-time optimal control problems.

This extension is particularly relevant in applications where continuous-time models are standard, such as in mean-field control and mean-field game theory. In these settings, turnpike phenomena have already been observed, cf. [12]. This suggests that a corresponding dissipativity-based framework could also exist in continuous time.

Such an extension would be especially interesting because, in mean-field problems, the distribution of the state is typically assumed to be available at each time instant. This structural property could potentially simplify parts of the analysis developed in this thesis, since several results in Chapter 4 rely on distributional information that is explicitly accessible in this setting.

However, extending the framework to continuous time also introduces substantial technical challenges. In particular, stochastic dissipativity and turnpike notions would need to be formulated for SDEs, where solutions are defined in an Itô sense and the analysis of energy-like inequalities requires stochastic calculus tools. This likely necessitates the use of Itô calculus and related techniques to establish analogues of the discrete-time dissipation inequalities and long-term optimality properties.

Developing such a theory therefore appears to be a promising but technically demanding direction for future research.

Stochastic optimal control and MPC for problems with time-varying disturbances. So far, the analysis in this thesis has been restricted to *i.i.d.* disturbance sequences, implying that the distribution of the noise process is constant over time. This assumption is central to the characterization of stationary solutions and simplifies both dissipativity and turnpike analysis. However, many practical applications involve non-stationary uncertainty, where the disturbance distribution evolves over time. Typical examples include Gaussian noise with time-varying mean or covariance, or more generally exogenous stochastic processes with slowly varying statistics.

In the deterministic setting, time-varying optimal control problems and their dissipativity-based analysis have been studied in [40], and corresponding MPC frameworks have been developed in [39]. These results show that turnpike and dissipativity concepts can be extended to non-autonomous systems by appropriately adapting the notion of stationarity.

The present stochastic framework already incorporates a form of time-dependence at the level of sample paths. This suggests that a natural extension would be to combine time-varying stochastic dynamics with the existing dissipativity-based approach. A central difficulty in this direction is that time-varying disturbances induce a genuinely non-stationary evolution at the level of distributions, so that stationary measures and stationary optimality conditions are no longer directly applicable. In addition, the characterization of “steady-state” solutions becomes more involved, as

it may require replacing stationary distributions by time-dependent or asymptotically periodic reference processes.

Developing a unified dissipativity and turnpike theory for such non-stationary stochastic systems therefore remains an open research direction.

List of Figures

3.1	Schematic sketch of the strict dissipativity and turnpike hierarchy implied by the dominance relations from Section 3.5	44
3.2	L^2 distance between the stationary pair and the optimal solutions of problem (3.55) on different horizons.	57
3.3	Realizations of optimal state trajectories of problem (3.55) on different horizons subject to Gaussian disturbances	57
3.4	Wasserstein distance between the stationary distribution and the distribution of the optimal trajectories of problem (3.55) on different horizons	58
3.5	Evolution of the expectations and variances of the optimal trajectories of problem (3.55) on different horizons	58
3.6	Realizations of optimal state trajectories of problem (3.55) on different horizons subject to gamma-distributed disturbances	59
3.7	Evolution of the probability density functions defining the marginal distributions of the optimal trajectory of problem (3.55) for gamma-distributed disturbances	59
3.8	Comparison of the probability density functions defining the marginal distributions of the optimal trajectory of problem (3.55) for Gaussian and gamma-distributed disturbances	60
3.9	Realizations of optimal state trajectories of problem (3.60) on different horizons	62
3.10	Evolution of all realization paths of the optimal solutions of problem (3.60) on different horizons	62
3.11	Evolution of the mean and variance of the optimal solutions of problem (3.60) on different horizons together as well as the Wasserstein distance to the optimal stationary process	63
4.1	Illustration of the conceptual idea of MPC	67
4.2	Evolution of all realization paths of the optimal solutions of problem (4.36) on different horizons	93
4.3	Sample paths of the closed-loop trajectories of problem (4.36) for different horizons	94
4.4	Empirical cumulative distribution function of the distributions of the closed-loop trajectory of problem (4.36) for different horizons	95
4.5	Empirical mean and variance of the components of the closed-loop trajectory of problem (4.36) for different horizons	95
4.6	Sample-based approximations of the cumulative and averaged cost of the closed-loop solutions of problem (4.36) for different horizons	96
4.7	Evolution of all realization paths of the optimal solutions of problem (4.37) on different horizons	97
4.8	Sample paths of the closed-loop trajectories of problem (4.37) for different horizons	98

4.9	Empirical cumulative distribution function of the distributions of the closed-loop trajectory of problem (4.37) for different horizons	99
4.10	Empirical mean and variance of the components of the closed-loop trajectory of problem (4.37) for different horizons	100
4.11	Sample-based approximations of the cumulative and averaged costs of the closed-loop solutions of problem (4.37) for different horizons . .	101
5.1	Optimal solutions of problem (5.14) for different cost mappings, together with the optimal solutions on the shifted and shortened horizon after one step	109
5.2	Evolution of all realization paths of the optimal solutions of problem (5.25) on different horizons for different cost mappings	116
5.3	Evolution of the parameters realizing the optimal stage cost of problem (5.25) on different horizons for risk-averse costs	117
5.4	Averaged cost of the closed-loop trajectory of problem (5.25) with risk-averse costs for varying parameters and varying horizons	118
5.5	Empirical commutative distribution functions of the closed-loop trajectory of problem (5.25) with risk-averse costs for varying parameters and varying horizons	119
5.6	Evolution of all realization paths of the optimal solutions of problem (5.27) on different horizons for different cost mappings	119
5.7	Evolution of the parameters realizing the optimal stage cost of problem (5.27) on different horizons for risk-averse costs	120
5.8	Averaged cost for varying parameters and varying horizons for problem (5.27) with risk-averse costs	121
5.9	Empirical commutative distribution functions of the closed-loop trajectory of problem (5.27) with risk-averse costs for varying parameters and varying horizons	122
5.10	Evolution of all risk measures from (5.56) applied to the first component of the closed-loop trajectory calculated by the moment-based indirect feedback stochastic MPC Algorithm 5	139
5.11	Averaged cost of the closed-loop trajectory calculated by the moment-based indirect feedback stochastic MPC Algorithm 5 for different linear feedbacks	140

List of Algorithms

1	Abstract stochastic MPC algorithm	67
2	Implementable stochastic MPC algorithm	68
3	Implementable risk-averse stochastic MPC algorithm	110
4	Indirect feedback stochastic MPC algorithm	125
5	Moment-based indirect feedback stochastic MPC algorithm	134

Bibliography

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Publications

The following list contains all publications authored or co-authored by the author of this thesis.

- [P1] G. Eichfelder, L. Grüne, L. Krügel, and J. Schießl. Relaxed dissipativity assumptions and a simplified algorithm for multiobjective MPC. *Computational Optimization and Applications*, 86:1081–1116, 2023.
- [P2] R. Ou, L. Januzi, J. Schießl, M. H. Baumann, L. Grüne, and T. Faulwasser. PolyOCP.jl : A Julia Package for Stochastic OCPs and MPC. arXiv preprint arXiv:2511.19084, 2025.
- [P3] R. Ou, J. Schießl, M. H. Baumann, L. Grüne, and T. Faulwasser. A polynomial chaos approach to stochastic LQ optimal control : Error bounds and infinite-horizon results. *Automatica*, 174, 2025.
- [P4] J. Schießl, M. H. Baumann, T. Faulwasser, and L. Grüne. On the relationship between stochastic turnpike and dissipativity notions. *IEEE Transactions on Automatic Control*, 70(6):3527–3539, 2025.
- [P5] J. Schießl and L. Krügel. nMPyC A Python Package for Solving Optimal Control Problems via Model Predictive Control. In *Model Predictive Control : Engineering Methods for Economists*, volume 31 of *Dynamic Modeling and Econometrics in Economics and Finance*, pages 111–133. Springer, Cham, Switzerland, 2025.
- [P6] J. Schießl, R. Ou, M. H. Baumann, T. Faulwasser, and L. Grüne. Towards turnpike-based performance analysis of risk-averse stochastic predictive control. In *2025 IEEE 64th Conference on Decision and Control (CDC) : Proceedings*, pages 329–335. Rio de Janeiro, Brazil, 2025.
- [P7] J. Schießl, R. Ou, M. H. Baumann, T. Faulwasser, and L. Grüne. Closed-loop analysis of linear stochastic MPC with risk-averse constraints. arXiv preprint arXiv:2604.11183, April 2026.
- [P8] J. Schießl, R. Ou, T. Faulwasser, M. H. Baumann, and L. Grüne. Pathwise turnpike and dissipativity results for discrete-time stochastic linear-quadratic optimal control problems. In *2023 IEEE 62nd Conference on Decision and Control (CDC): Proceedings*, pages 2790–2795. 2023.
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- [P10] J. Schießl, R. Ou, T. Faulwasser, M. H. Baumann, and L. Grüne. Turnpike and Dissipativity in Generalized Discrete-Time Stochastic Linear-Quadratic Optimal Control. *SIAM Journal on Control and Optimization*, 63(2):1432–1457, 2025.
- [P11] J. Schießl, H. Selder, R. Ou, M. H. Baumann, T. Faulwasser, and L. Grüne. Stability and performance of stochastic economic MPC : Stochastic characterization of the closed-loop asymptotics. arXiv preprint arXiv:2510.19416, 2025.

The content of the publications [P4, P8, P10] is included Chapter 3, while the results of the publication [P9] and the submitted preprint [P11] are presented in Chapter 4. Chapter 5 contains results from the publication [P6] and the submitted preprint [P7].

Results from all other publications listed above are not included in this thesis.

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