

Generalized Hamming weights of additive codes and geometric counterparts

Jozefien D'haeseleer* and Sascha Kurz†

Abstract

We consider the geometric problem of determining the maximum number $n_q(r, h, f; s)$ of $(h-1)$ -spaces in the projective space $\text{PG}(r-1, q)$ such that each subspace of codimension f contains at most s elements. In terms of coding theory, this corresponds to additive codes with a large f th generalized Hamming weight. We also consider the dual problem. Here, we determine the minimum number $b_q(r, h, f; s)$ of $(h-1)$ -spaces in $\text{PG}(r-1, q)$ such that each subspace of codimension f contains at least s elements. We fully determine $b_2(5, 2, 2; s)$ as a function of s . We additionally give bounds and constructions for other parameters. For the computational results we partially use extensive integer linear programming computations.

Keywords: additive codes, Galois geometry, blocking sets, subspace codes

Mathematics Subject Classification: 94B27, 51E22

1 Introduction

It is well known that a linear $[n, k, d]_q$ code C corresponds to a multiset of n points in the projective space $\text{PG}(k-1, q)$ such that each hyperplane contains at most $n-d$ elements. Therefore, instead of asking for linear codes with a large minimum Hamming distance d we can also ask for large multisets of points where not too many elements are contained in a hyperplane. If we replace points by $(h-1)$ -spaces the coding theoretic equivalent is given by additive codes over \mathbb{F}_{q^h} , which are linear over \mathbb{F}_q . Considering multisets of points such that at most s are contained in any subspace of codimension f corresponds to linear codes with a large f th generalized Hamming weight. Here we want to consider the maximum number $n_q(r, h, f; s)$ of $(h-1)$ -subspaces in $\text{PG}(r-1, q)$ with the property each subspace of codimension f contains at most s elements. In coding theory terms we are dealing with additive codes that have a large f th generalized Hamming weight. The special cases where h or f equals 1 have

*Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, 9000 Ghent, Belgium. E-mail: jozefien.dhaeseleer@ugent.be

†Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany.

been extensively studied. Outside of this regime not much seems to be known. By taking the complement we can relate our problem to a dual problem: what is the minimum number $b_q(r, h, f; s)$ of $(h - 1)$ -spaces in $\text{PG}(r - 1, q)$ where each subspace of codimension f contains at least s elements? If $s = 1$ one also speaks of blocking sets of $(h - 1)$ -spaces w.r.t. $(r - f - 1)$ -spaces. The case $h = f = 1$ is a classical problem, and we e.g. have $b_q(3, 1, 1; 1) = q + 1$ attained by all points on a line in $\text{PG}(2, q)$, which is also called a trivial blocking set. For non-trivial blocking sets, which are those not containing a full line in its support, the minimum size rises to $3(p + 1)/2$ for odd primes p [5]. For $h = 2$ and $r - f = 3$ we refer to e.g. [10, 22, 30]. If $s > 1$ one speaks of multiple blocking sets, see e.g. [4]. In [7, Definition 2.1] the authors speak of an s -fold f -blocking set of $\text{PG}(r - 1, q)$ for the special case $h = 1$. In this paper, we use the following definition.

Definition 1.1. A multiset \mathcal{M} of $(h - 1)$ -spaces in $\text{PG}(r - 1, q)$ is called an s -fold blocking set w.r.t. $(v - 1)$ spaces if every $(v - 1)$ -space in $\text{PG}(r - 1, q)$ contains at least s elements from \mathcal{M} .

Whenever the parameters are clear from the context, we just speak of generalized blocking sets.

The remaining part of this paper is structured as follows. In Section 2 we introduce the necessary preliminaries. The relation between the geometric objects and coding theory is outlined in Section 3. In Section 4 we summarize our knowledge on the asymptotic behavior of $n_q(r, h, f; s)$. General constructions are studied in Section 5. In Section 6 we investigate the generalized blocking sets and their minimum possible size $b_q(r, h, f; s)$. In Section 7 we study the maximum number of lines in $\text{PG}(4, q)$ such that each plane contains at most s lines, and we fully determine $b_2(5, 2, 2; s)$ as a function of s . We close with a conclusion and a few open problems in Section 8. Sporadic blocking sets, found by integer linear programming searches, are listed in Appendix A.

2 Preliminaries

The set of all subspaces of \mathbb{F}_q^r , ordered by the incidence relation \subseteq , is called the $(r - 1)$ -dimensional projective geometry over \mathbb{F}_q and denoted by $\text{PG}(r - 1, q)$. Here we use the projective dimension, so that an $(i - 1)$ -space in $\text{PG}(r - 1, q)$ is an i -dimensional space in the vector space setting \mathbb{F}_q^r . We will call 0-, 1-, 2-, 3-, and $(r - 2)$ -spaces points, lines, planes, solids, and hyperplanes, respectively. For two subspaces S and S' we write $S \subseteq S'$ if S is contained in S' . Moreover, we say that S and S' are incident if and only if $S \subseteq S'$ or $S \supseteq S'$. Let $[i]_q := \frac{q^i - 1}{q - 1}$ denote the number of points of an arbitrary $(i - 1)$ -space in $\text{PG}(r - 1, q)$ where $r \geq i$. By convention we set $[0]_q := 0$. More generally, by $\begin{bmatrix} r \\ i \end{bmatrix}_q := \frac{\prod_{j=0}^{i-1} q^{r-j-1}}{\prod_{j=0}^{i-1} q^{i-j-1}} = \frac{\prod_{j=0}^{i-1} [r-j]_q}{\prod_{j=0}^{i-1} [i-j]_q}$ we denote the number of $(i - 1)$ -spaces in $\text{PG}(r - 1, q)$. Duality implies $\begin{bmatrix} r \\ i \end{bmatrix}_q = \begin{bmatrix} r \\ r-i \end{bmatrix}_q$. It is known that the number of j -spaces disjoint from a

fixed m -space in $\text{PG}(n, q)$ equals $q^{(m+1)(j+1)} \binom{n-m}{j+1}_q$, see [32, Section 170].

We can represent an $(i-1)$ -space in $\text{PG}(r-1, q)$ by an $i \times r$ generator matrix over \mathbb{F}_q . A multiset of points \mathcal{M} in $\text{PG}(r-1, q)$ is a mapping from the set of points to \mathbb{N} . For a given point P we call $\mathcal{M}(P)$ its multiplicity. We say that \mathcal{M} is spanning if the points with positive multiplicity span the entire ambient space. The notion of the point multiplicities is extended additively to any subspace S via $\mathcal{M}(S) := \sum_{P \in S} \mathcal{M}(P)$. The relation between multisets of points and linear codes is explained in detail in Section 3. For additive codes we need the following generalization, see e.g. [2].

Definition 2.1. A projective $h - (n, r, s)_q$ system is a multiset \mathcal{S} of n subspaces of $\text{PG}(r-1, q)$ of dimension at most $(h-1)$ such that each hyperplane contains at most s elements of \mathcal{S} , and some hyperplane contains exactly s elements of \mathcal{S} . We say that \mathcal{S} is faithful if all its elements have dimension $(h-1)$. A projective $h - (n, r, s)_q$ system \mathcal{S} is a projective $h - (n, r, s, \mu)_q$ system if each point is contained in at most μ elements from \mathcal{S} , and there is some point that is contained in exactly μ elements from \mathcal{S} .

A faithful projective $1 - (n, r, s)_q$ system \mathcal{S} is just a multiset of points with cardinality n in $\text{PG}(r-1, q)$ with the property that the maximum hyperplane multiplicity $\mathcal{S}(H)$ equals s . Unfaithful projective $h - (n, r, s)_q$ systems also allow the containment of (-1) -dimensional subspaces, which correspond to zero columns in the generator matrix of a corresponding linear code for $h = 1$, see Section 3.

Definition 2.2. By $n_q(r, h; s)$ we denote the maximum number n such that a projective $h - (n, r, s)_q$ system exists.

Note that the elements of \mathcal{S} span the entire ambient space $\text{PG}(r-1, q)$ if and only if $s < n$. If \mathcal{S} is a projective $h - (n, r, s)_q$ system that is not faithful, then we can easily construct a faithful projective $h - (n, r, \leq s)_q$ system \mathcal{S}' by replacing each element $S \in \mathcal{S}$ with dimension smaller than $h-1$ by an arbitrary $(h-1)$ -space containing S . The functions $n_q(r, h; s)$ were e.g. studied in [26], and indirectly in any paper on additive codes with good parameters.

For our situation we need an even more general notion.

Definition 2.3. A projective $(h, f) - (n, r, s)_q$ system, where $h + f \leq r$, is a multiset \mathcal{S} of n subspaces of $\text{PG}(r-1, q)$ of dimension at most $(h-1)$ such that each subspace of codimension f contains at most s elements of \mathcal{S} , and some subspace of codimension f contains exactly s elements of \mathcal{S} . We say that \mathcal{S} is faithful if all elements have dimension $(h-1)$. A projective $(h, f) - (n, r, s)_q$ system \mathcal{S} is a projective $(h, f) - (n, r, s, \mu)_q$ system if each $(f-1)$ -space is contained in at most μ elements from \mathcal{S} , and there is some $(f-1)$ -space that is contained in exactly μ elements from \mathcal{S} .

A projective $(h, 1) - (n, r, s)_q$ system is just a projective $h - (n, r, s)_q$ system, and a general projective $(h, f) - (n, r, s)_q$ system corresponds to an additive

$[n, r/h, d_f]_q^h$ code C with $s = n - d_f$. Here d_f denotes the minimum f th generalized Hamming weight of C , see Section 3. The parameter μ corresponds to the maximum column multiplicity of linear codes over \mathbb{F}_q , if we identify linear dependent non-zero columns of a generator matrix.

Definition 2.4. By $n_q(r, h, f; s)$ we denote the maximum number n such that a projective $(h, f) - (n, r, s)_q$ system exists.

Clearly we can convert any given projective $(h, f) - (n, r, s)_q$ system into a faithful projective $(h, f) - (n, r, \leq s)_q$ system by replacing each element U by an arbitrary $(h - 1)$ -space containing U .

By $\dim(U)$ we denote the projective dimension of a subspace U in \mathbb{F}_q^n , which is one less than the algebraic dimension. With this, the subspace distance is given by $d_S(U, V) = (\dim(U) + 1) + (\dim(V) + 1) - 2(\dim(U \cap V) + 1) = \dim(U) + \dim(V) - 2 \dim(U \cap V)$, which is an even number if $\dim(U) = \dim(V)$.

By $A_q(v, k; 2\delta)$ we denote the maximum number of $(k - 1)$ -spaces in $\text{PG}(v - 1, q)$ with minimum subspace distance 2δ . Here we have $\dim(U \cap V) + 1 \leq k - \delta$ and speak of constant-dimension codes. In the following we give an upper bound and one simple construction for constant-dimension codes. In order to keep the paper self-contained we give a brief proof and description. For more details we refer to the survey [24].

Lemma 2.5. For $v \geq 2k$ we have

$$A_q(v, k; 2\delta) \leq \frac{\begin{bmatrix} v \\ k - \delta + 1 \end{bmatrix}_q}{\begin{bmatrix} k \\ k - \delta + 1 \end{bmatrix}_q}.$$

Proof. Since $\dim(U \cap V) + 1 \leq k - \delta$ for any two different elements U and V of the constant-dimension code, each $(k - \delta)$ -space is contained in at most one $(k - 1)$ -space from the constant-dimension code. \square

A rank metric code M is a subset of $m \times n$ matrices over \mathbb{F}_q equipped with the rank distance $d_r(M, M') = \text{rk}(M - M')$. Assuming $m \leq n$, a Singleton-like upper bound is known and gives $|M| \leq q^{n(m - \delta + 1)}$ for minimum rank distance δ [8]. Codes attaining this bound are called maximum rank distance (MRD) codes. They exist for all parameters, even if one additionally assumes that the matrices form a linear space, i.e. assuming that the code is linearly closed. For a survey on MRD codes we refer to [33]. Given an MRD code M of $m \times n$ matrices over \mathbb{F}_q with minimum rank distance δ , we obtain a lifted MRD (LMRD) code \mathcal{M} by prepending $m \times m$ unit matrices. Interpreted as generator matrices of $(m - 1)$ -spaces in $\text{PG}(n + m - 1, q)$, \mathcal{M} is a set of $q^{n(m - \delta + 1)}$ $(m - 1)$ -spaces in $\text{PG}(n + m - 1, q)$ such that the dimension of the intersection of any two elements is at most $(m - \delta - 1)$ and there exists a special $(n - 1)$ -space S that is disjoint to all elements of \mathcal{M} .

3 Relation to coding theory

A linear $[n, k]_q$ code C is a k -dimensional subspace of the vector space \mathbb{F}_q^n . The elements of C are called codewords and the Hamming weight $\text{wt}(c)$ of a codeword $c \in C$ is the number of non-zero entries. With this, the Hamming distance $d(c_1, c_2)$ between two codewords is given by $\text{wt}(c_1 - c_2)$. The minimum Hamming distance $d(C)$ of a (linear) code is the minimum Hamming distance $d(c_1, c_2)$ between two different codewords. We say that an $[n, k]_q$ code C is an $[n, k, d]_q$ code if its minimum Hamming distance $d(C)$ equals d . A linear code is called Δ -divisible if the weights of all codewords are divisible by Δ . If the non-zero weights of a linear $[n, k]_q$ code are contained in $\{w_1, \dots, w_l\}$ we also speak of an $[n, k, \{w_1, \dots, w_l\}]_q$ code, and an l -weight code if all weights are attained. As a representation for a linear code we use a $k \times n$ generator matrix over \mathbb{F}_q . The dual code C^\perp of an $[n, k]_q$ code C is the $[n, n - k]_q$ code whose codewords are orthogonal to all codewords in C . By d^\perp we denote the corresponding minimum Hamming distance. We say that C has full length if $d^\perp \geq 2$, which is equivalent to the property that there is no zero-column in a given generator matrix for C . It is well known that full length $[n, k]_q$ codes are in one-to-one correspondence to spanning multisets of cardinality n in $\text{PG}(k - 1, q)$, see e.g. [9].¹ The minimum Hamming distance d of a linear code corresponds to the geometric property that the maximum number of elements of the multiset of points that is contained in a hyperplane is given by $n - d$. So, a large minimum Hamming distance corresponds to a small maximum number of points in hyperplanes. Alternatively, minimizing the possible length n of an $[n, k, d]_q$ is equivalent to maximizing the cardinality of a multiset of points in $\text{PG}(k - 1, q)$ with at most s points in each hyperplane, where $s = n - d$.

More generally, a block code C of length n over the alphabet \mathbb{F}_q is just a subset of \mathbb{F}_q^n (equipped with the Hamming metric). If C is linearly closed, i.e. if $c, c' \in C$ and $\alpha, \beta \in \mathbb{F}_q$ implies $\alpha c + \beta c' \in C$, then we have a linear $[n, k]_q$ code, where $k = \log_q |C|$ is called the dimension. An additive code is just a block code that is additively closed, i.e. $c, c' \in C$ implies $c + c' \in C$. Each additive code is linear over some subfield, see e.g. [1]. By an $[n, r/h, d]_q^h$ code we denote an additive code $C \subseteq \mathbb{F}_{q^h}^n$ that is linear over \mathbb{F}_q , has minimum Hamming distance d and cardinality q^r . We call $r/h \in \mathbb{Q}$ its dimension. We can represent an $[n, r/h, d]_q^h$ code as the \mathbb{F}_q row span of an $r \times n$ generator matrix G over \mathbb{F}_{q^h} . Choosing an \mathbb{F}_q basis of \mathbb{F}_{q^h} we can expand this generator matrix to a subfield generator matrix $\tilde{G} \in \mathbb{F}_q^{r \times nh}$. By $\mathcal{X}_G(C)$ we define the multiset of the n subspaces spanned by the n blocks of h columns of \tilde{G} in this way. Note that these subspaces give rise to projective systems.

Theorem 3.1. ([2, Theorem 5]) *If C is an additive $[n, r/h, d]_q^h$ code with generator matrix G , then $\mathcal{X}_G(C)$ is a projective $h - (n, r, n - d)_q$ system \mathcal{S} , and conversely, each projective $h - (n, r, s)_q$ system \mathcal{S} defines an additive $[n, r/h, n - s]_q^h$ code C .*

¹Given a linear $[n, k]_q$ code C with generator matrix G , we can interpret its columns as 1-dimensional vector spaces of \mathbb{F}_q^k or points in $\text{PG}(k - 1, q)$.

The parameters of a linear $[n, k, d]_q$ code C are related by the so-called *Griesmer bound* [14, 34]

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil =: g_q(k, d). \quad (1)$$

From this one can derive the bound

$$\begin{aligned} n &\geq \left\lceil \frac{g_q(r, d \cdot q^{h-1})}{[h]_q} \right\rceil = \left\lceil \frac{\sum_{i=0}^{r-1} \lceil d \cdot q^{h-1-i} \rceil}{[h]_q} \right\rceil \\ &= d + \left\lceil \frac{\sum_{i=1}^{r-h} \left\lceil \frac{d}{q^i} \right\rceil}{[h]_q} \right\rceil = d + \left\lceil \frac{g_q(r-h+1, d) - d}{[h]_q} \right\rceil \end{aligned} \quad (2)$$

for the parameters of an additive $[n, r/h, d]_q^h$ code, see e.g. [2, Theorem 12] or [26, Lemma 15]. Using Theorem 3.1 this gives an upper bound for $n_q(r, h, 1; s)$, which we call the Griesmer upper bound. More precisely, we call the largest integer n that satisfies $[h]_q \cdot n \geq g_q(r, (n-s) \cdot q^{h-1})$ the Griesmer upper bound for $n_q(r, h; s)$, see e.g. [26, Example 5].

The Hamming weight $\text{wt}(c)$ turns \mathbb{F}_q^n into a normed vector space. For $c = (c_1, \dots, c_n) \in \mathbb{F}_q^n$ we call

$$\text{supp}(c) := \{1 \leq i \leq n : c_i \neq 0\} \quad (3)$$

the support of c , so that $\text{wt}(c) = |\text{supp}(c)|$. For some linear subspace C in \mathbb{F}_q^n let

$$\text{supp}(C) := \{1 \leq i \leq n : \exists c = (c_1, \dots, c_n) \in C, c_i \neq 0\} \quad (4)$$

be the support of C and $\dim_{\mathbb{F}_q}(C)$ its (algebraic) \mathbb{F}_q -dimension. For two \mathbb{F}_q vector spaces C, C' in \mathbb{F}_q^n we write $C \subseteq C'$ if C is contained in C' . With this, the f th generalized Hamming weight of a linear code C [16, 20], denoted as $d_f(C)$, is the size of the smallest support of an f -dimensional subcode of C :

$$d_f(C) := \min\{|\text{supp}(C')| : C' \subseteq C, \dim_{\mathbb{F}_q}(C') = f\}. \quad (5)$$

In particular, $d_1(C)$ is the minimum Hamming distance of a linear code C . The sequence $(d_1(C), \dots, d_k(C))$ is called the weight hierarchy of a linear $[n, k]_q$ code C . Clearly, we have $1 \leq d_1(C) \leq \dots \leq d_k(C) \leq n$. The generalized Hamming weights can be used to describe the cryptographic performance of a linear code over the wire-tap channel of type II [36]. Moreover, it can also be used to determine the trellis complexity of the code [6, 11, 12, 19]. The weight hierarchy of a linear code can be obtained from a quadratic form over a finite field [27, 28, 29]. Also the geometric reformulation of the generalized Hamming weights in terms of multisets of points is well known [17, 35]. Let \mathcal{M} be a

multiset of points in $\text{PG}(k-1, q)$ and C its corresponding $[n, k]_q$ code. Then, we have

$$n - d_f(C) = \max \{ \mathcal{M}(U) : U \text{ subspace of codimension } f \} \quad (6)$$

for all $1 \leq f \leq k$. In order to keep the paper self-contained we state a brief argument, c.f. [3]. Given a linear $[n, k, d]_q$ code C , a codeword (a 1-dimensional subcode) of C is obtained by left multiplication of a generator matrix G by a vector $v \in \mathbb{F}_q^k$. Considering v as a point in $\text{PG}(k-1, q)$, the hyperplane v^\perp contains the point x if and only if $\langle v, x \rangle = 0$. Therefore, if we take the set of n points in $\text{PG}(k-1, q)$, corresponding to the columns of G , we have that the codeword vG has weight w if and only if $n - w$ of these points are contained in the hyperplane v^\perp . More generally, for a j -dimensional subspace V of \mathbb{F}_q^n the codimension j subspace V^\perp contains $n - w$ points if the subspace $\{vG : v \in V\}$ has support size w , which proves Equation (6). We can apply the same argument to the subfield generator matrix \tilde{G} of an additive $[n, r/h, d]_q^h$ code C to conclude

$$n - d_f(C) = \max \{ |\{S \in \mathcal{X}_G(C) : S \leq U\}| : U \text{ subspace of codim. } f \} \quad (7)$$

for all $1 \leq f \leq k$. Hence, looking for good additive codes, corresponds to look for large projective systems.

Theorem 3.2. (*Griesmer-type bound*) [15, Theorem 4], [17, Theorem 5]

For each linear $[n, k]_q$ code and each $1 \leq f \leq k$ we have

$$n \geq d_f + \sum_{j=1}^{k-f} \left\lceil \frac{d_f}{[f]_q \cdot q^j} \right\rceil =: g_q^f(k, d_f). \quad (8)$$

Currently we do not know any Griesmer type bound for the f th generalized Griesmer weight of additive codes, which is tight for all sufficiently large minimum distances. This is an important open problem. For $f > 1$ and $h > 1$ we cannot reconstruct the number of $(h-1)$ -spaces in a codimension f space from the number of contained points and the total number of $(h-1)$ -spaces.

4 Asymptotic results

We first state a sum construction and an easy upper bound that can be asymptotically attained.

Lemma 4.1. $n_q(r, h, f; s_1 + s_2) \geq n_q(r, h, f; s_1) + n_q(r, h, f; s_2)$

Proof. Consider the union of a projective $(h, f) - (n_q(r, h, f; s_1), r, s_1)_q$ and a projective $(h, f) - (n_q(r, h, f; s_2), r, s_2)_q$ system. \square

Lemma 4.2. *We have*

$$n_q(r, h, f; s) \leq \frac{\begin{bmatrix} r \\ f \end{bmatrix}_q \cdot s}{\begin{bmatrix} r-h \\ f \end{bmatrix}_q} = \prod_{i=0}^{f-1} \frac{\begin{bmatrix} r-i \\ r-h-i \end{bmatrix}_q}{\begin{bmatrix} r-h-i \\ r-h-i \end{bmatrix}_q} \cdot s. \quad (9)$$

Proof. Let \mathcal{S} be a faithful projective (h, f) - $(n, r, s)_q$ system with $n = n_q(r, h, f; s)$. Since each element $S \in \mathcal{S}$ is contained in $\begin{bmatrix} r-h \\ f \end{bmatrix}_q$ subspaces of codimension f and there are $\begin{bmatrix} r \\ f \end{bmatrix}_q$ subspaces of codimension f in total, we conclude $n \leq \frac{\begin{bmatrix} r \\ f \end{bmatrix}_q \cdot s}{\begin{bmatrix} r-h \\ f \end{bmatrix}_q}$. \square

Considering the set of all $n = \begin{bmatrix} r \\ h \end{bmatrix}_q$ h -spaces in $\text{PG}(r-1, q)$ we see that the upper bound in Lemma 4.2 is tight for $s = \begin{bmatrix} r-f \\ h \end{bmatrix}_q$. Using λ copies of this construction yields

$$\lim_{s \rightarrow \infty} n_q(r, h, f; s) \cdot \frac{\begin{bmatrix} r-h \\ f \end{bmatrix}_q}{\begin{bmatrix} r \\ f \end{bmatrix}_q \cdot s} = 1. \quad (10)$$

For $f = 1$ the Griesmer bound implies that the difference between $n_q(r, h, 1; s)$ and the corresponding Griesmer upper bound tends to zero if s tends to infinity, which is a much tighter statement.² The same stronger result also holds for the cases where $h = 1$ and f is arbitrary. Of course it would be very interesting to have such a result in general. However, the relation to constant-dimension codes in Lemma 4.3 indicates that this might be a hard problem.

Instead of letting s tend to infinity we can also consider $n_q(r, h, f; s)$ as a sequence in the field size q .

Lemma 4.3. *For $1 \leq \delta \leq h$ and $r \geq 2h$ we have*

$$n_q(r, h, r - h - \delta + 1; 1) = A_q(r, h; 2\delta).$$

Proof. Since each $(h + \delta - 2)$ -space contains at most one $(h - 1)$ -space from the projective system, the dimension formula implies that each $(h - \delta)$ -space is contained in at most one $(h - 1)$ -space from the projective system. With this, the distance between the $(h - 1)$ -spaces U and V is $\dim(U) + \dim(V) - 2 \dim(U \cap V) \geq 2h - 2 - 2(h - \delta - 1) = 2\delta$. \square

The special case $\delta = h$ corresponds to partial spreads where many bounds are known, see e.g. [25, Section 9]. For general parameters the following construction using (L)MRD codes is well known.

Proposition 4.4. *For $1 \leq \delta \leq h$ and $r \geq 2h$ we have*

$$n_q(r, h, r - h - \delta + 1; 1) \geq q^{(r-h)(h-\delta+1)}.$$

Proof. Consider an LMRD code \mathcal{M} of $q^{(r-h)(h-\delta+1)}$ $(h-1)$ -spaces in $\text{PG}(r-1, q)$ with minimum rank distance δ , i.e., the projective dimension of the intersection of two different elements of \mathcal{M} is at most $(h - \delta - 1)$. Thus, each subspace of codimension $f = r - h - \delta + 1$ contains at most one element from \mathcal{M} . \square

²While the geometric equivalent of linear or additive codes is very handy for many situations, here the coding theory version looks more nicely. In particular, denoting the minimum length n of an $[n, k, d]_q$ code by $\tilde{n}_q(k, d)$, we have $\lim_{d \rightarrow \infty} \tilde{n}_q(k, d) - g_q(k, d) = 0$. There is a similar formulation for additive codes, see [26].

As an example, for $r = 6$, $h = 3$, and $\delta = 2$ we obtain a set of q^6 planes in $\text{PG}(5, q)$ with the property each solid contains at most one plane (and each 4-space contains at most q^3 planes). Via Lemma 4.3 we can replace the used LMRD codes by any other constant-dimension code with the same minimum subspace distance, see e.g. [24] for a survey on some constructions from the literature. For the aforementioned parameters we remark that this construction is not optimal, since there is a construction known with $q^6 + 2q^2 + q + 1$ planes, see [18].

Corollary 4.5. *For $1 \leq \delta \leq h$ and $r \geq 2h$ we have*

$$n_q(r, h, r - h - \delta + 1; s) \geq s \cdot q^{(r-h)(h-\delta+1)}.$$

For the special cases where either $r = 2h$ or $s = 1$ the construction with the LMRD codes is asymptotically tight if q tends to infinity.

To prove this, we use the so-called q -Pochhammer symbol

$$(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i).$$

In particular, we will use that

$$1 \leq q^{-b(a-b)} \cdot \begin{bmatrix} a \\ b \end{bmatrix}_q \leq \frac{1}{(1/q; 1/q)_b}, \quad (11)$$

see e.g. [21].

Proposition 4.6. *For $1 \leq \delta \leq h$ we have*

$$\lim_{q \rightarrow \infty} \frac{n_q(2h, h, h - \delta + 1; s)}{s \cdot q^{h(h-\delta+1)}} = 1.$$

Proof. Lemma 4.2 yields

$$n_q(2h, h, h - \delta + 1; s) \leq \frac{\begin{bmatrix} 2h \\ h - \delta + 1 \end{bmatrix}_q \cdot s}{\begin{bmatrix} h \\ h - \delta + 1 \end{bmatrix}_q}.$$

Applying the q -Pochhammer symbols to both numerator and denominator, and using (11), we have

$$\begin{aligned} n_q(2h, h, h - \delta + 1; s) &\leq \frac{q^{(h+\delta-1)(h-\delta+1)}}{q^{(\delta-1)(h-\delta+1)}} \cdot \frac{1}{(1/q; 1/q)_{h-\delta+1}} \cdot s \\ &= q^{h(h-\delta+1)} \cdot \frac{1}{(1/q; 1/q)_{h-\delta+1}} \cdot s, \end{aligned}$$

where the second factor tends to 1 as q approaches infinity. \square

As in the proof of Proposition 4.6, Lemmas 2.5 and 4.3, together with Inequality (11) yield the upper bound, while Proposition 4.4 provides a matching lower bound.

Proposition 4.7. For $1 \leq \delta \leq h$ and $r \geq 2h$ we have

$$\lim_{q \rightarrow \infty} \frac{n_q(r, h, r - h - \delta + 1; 1)}{q^{(r-h)(h-\delta+1)}} = 1.$$

We remark that Lemma 2.5 is known as the anticodeword bound in the context of subspace codes and that tighter bounds are known, see e.g. [24].

5 General constructions

In this section we want to study known constructions for linear codes from the literature and generalize them to our context.

In coding theory it is well known that the problem of determining the minimum possible length of an $[n, k, d]_q$ code as a function of d is a finite problem for given parameters k and q . More precisely, if the minimum distance d is sufficiently large, then the Griesmer bound can always be attained with equality. A corresponding construction was given by Solomon and Stiffler [34]. In geometric terms this means that the determination of the function $n_q(r, 1, 1; \cdot)$ in terms of s is a finite, but still rather hard, problem for each given pair of parameters r and q . In [26] this result was generalized to additive codes, i.e. also applies to $n_q(r, h, 1; \cdot)$ for arbitrary h . In order to describe the Solomon–Stiffler construction and its generalization, we have to introduce further notation. For each subspace S in $\text{PG}(r-1, q)$ we denote its characteristic function by χ_S , i.e. we have, for a point $P \in \text{PG}(r-1, q)$ that $\chi_S(P) = 1$ if $P \in S$ and $\chi_S(P) = 0$ otherwise.

Definition 5.1. We say that a multiset of points \mathcal{M} in $\text{PG}(r-1, q)$ is h -partitionable if there exist $(h-1)$ -spaces S_1, \dots, S_l , for some integer l , such that $\mathcal{M} = \sum_{i=1}^l \chi_{S_i}$, i.e. \mathcal{M} can be partitioned into $(h-1)$ -spaces.

To ease the notation and to avoid technical difficulties, we choose a chain of subspaces $S_1 \subseteq S_2 \subseteq \dots \subseteq S_r$ in $\text{PG}(r-1, q)$, where S_i has projective dimension $(i-1)$.

Definition 5.2. Given a chain of subspaces $S_1 \subsetneq S_2 \subsetneq \dots \subsetneq S_r$ in $\text{PG}(r-1, q)$, we say that $\sum_{i=1}^r a_i S_i$ is h -partitionable over \mathbb{F}_q if the multiset of points $\sum_{i=1}^r a_i \chi_{S_i}$ in $\text{PG}(r-1, q)$ is h -partitionable, where $a_i \in \mathbb{Z}$ for all $1 \leq i \leq r$.

For example, we trivially have that S_3 is 3-partitionable over \mathbb{F}_q . The existence of plane spreads in $\text{PG}(5, q)$ implies that S_6 is 3-partitionable over \mathbb{F}_q . Since $[7]_q$ is not divisible by $[3]_q$, we have that S_7 is not 3-partitionable over \mathbb{F}_q , while $(q^2 + q + 1) \cdot S_7$ is 3-partitionable. For the details on the underlying constructions we refer to [26].

Using a specific parameterization of the minimum distance d the Griesmer bound in Inequality (1) can be written more explicitly as follows. Let k and d be positive integers. Write d as

$$d = \sigma q^{k-1} - \sum_{i=1}^{k-1} \varepsilon_i q^{i-1}, \tag{12}$$

where $\sigma \in \mathbb{N}_{>0}$, and the $0 \leq \varepsilon_i < q$ are integers for all $1 \leq i \leq k-1$. Then, Inequality (1) is satisfied with equality if and only if

$$n = \sigma[k]_q - \sum_{i=1}^{k-1} \varepsilon_i [i]_q, \quad (13)$$

which is equivalent to

$$n - d = \sigma[k-1]_q - \sum_{i=1}^{k-1} \varepsilon_i [i-1]_q. \quad (14)$$

Remark 5.3. Given k and d , Equation (12) always determines σ and the ε_i uniquely. This is different for Equation (14) given k and $n-d=s$.

By relaxing to $0 \leq \varepsilon_i \leq q$ we can ensure existence and uniqueness are enforced by additionally requiring $\varepsilon_j = 0$ for all $j < i$ where $\varepsilon_i = q$ for some i . The same is true for Equation (13) given k and n . For more details, we refer to [13, Chapter 2], which also gives pointers to Hamada's work on minihypers. We will mostly state our corresponding results referring to Equation (12) and using the coding theoretic formulation.

Given arbitrary $\varepsilon_1, \dots, \varepsilon_{k-1} \in \mathbb{Z}$, we have that $\mathcal{M} = \sigma\chi_{S_k} - \sum_{i=1}^{k-1} \varepsilon_i \chi_{S_i}$ is a multiset of points in $\text{PG}(k-1, q)$, which is a projective $1 - (n, k, s)_q$ system for all sufficiently large $\sigma \in \mathbb{N}$. Here $n = \sigma[k]_q - \sum_{i=1}^{k-1} \varepsilon_i [i]_q$ and $s = \sigma[k-1]_q - \sum_{i=1}^{k-1} \varepsilon_i [i-1]_q$, see e.g. [26, Lemma 23]. While $\sigma S_k - \sum_{i=1}^{k-1} \varepsilon_i S_i$ is obviously 1-partitionable over \mathbb{F}_q if σ is sufficiently large, there are further conditions for being h -partitionable when $h > 1$ as well as more sophisticated constructions for the partition, see [26].

Recall that a multiset \mathcal{M} of $(h-1)$ -spaces in $\text{PG}(r-1, q)$ is an s -fold blocking set w.r.t. $(v-1)$ spaces if every $(v-1)$ -space in $\text{PG}(r-1, q)$ contains at least s elements from \mathcal{M} . Note that the smallest size of an s -fold blocking set w.r.t. $(v-1)$ -spaces in $\text{PG}(r-1, q)$ is denoted by $b_q(r, h, r-v; s)$. Whenever the parameters are clear from the context, we just speak of generalized blocking sets.

A first attempt to generalize the Solomon–Stiffler construction is given by the following lemma.

Lemma 5.4. *Let $h, f, r \in \mathbb{N}$ with $h+f \leq r$, $\varepsilon_i \in \mathbb{N}$ for $h+f \leq i \leq r-1$ and $\sigma \in \mathbb{N}$ sufficiently large, e.g. $\sigma \geq \sum_{i=h+f}^{r-1} \varepsilon_i$. Then, we have $n_q(r, h, f; s) \geq n$, where $n := \sigma \cdot \begin{bmatrix} r \\ h \end{bmatrix}_q - \sum_{i=h+f}^{r-1} \varepsilon_i \cdot \begin{bmatrix} i \\ h \end{bmatrix}_q$ and $s := \sigma \cdot \begin{bmatrix} r-f \\ h \end{bmatrix}_q - \sum_{i=h+f}^{r-1} \varepsilon_i \cdot \begin{bmatrix} i-f \\ h \end{bmatrix}_q$.*

Proof. Consider the following multiset \mathcal{M} of $(h-1)$ -spaces in $\text{PG}(r-1, q)$. Starting from σ copies of every $(h-1)$ -space in $\text{PG}(r-1, q)$ we remove the $(h-1)$ -spaces contained in ε_i $(i-1)$ -spaces for all $h+f \leq i \leq r-1$, so that $|\mathcal{M}| = n$. Since the set of all $(h-1)$ -spaces contained in an arbitrary $(i-1)$ -space is an $\begin{bmatrix} i-f \\ h \end{bmatrix}_q$ -fold blocking set w.r.t. $(r-f-1)$ -spaces in $\text{PG}(r-1, q)$, every codimension f space in $\text{PG}(r-1, q)$ contains at most s elements from \mathcal{M} . \square

Choosing $\sigma = \varepsilon_4 = 1$ we e.g. obtain $n_2(5, 2, 2; 6) \geq 120$. Similarly, $\sigma = \varepsilon_4 = 2$ yields $n_2(5, 2, 2; 12) \geq 240$.

The essential idea in the proof of Lemma 5.4 is the blocking property of subspaces, so that we state the following alternative.

Lemma 5.5. *Assume $h, f, r \in \mathbb{N}$ with $h + f \leq r$, $l \in \mathbb{N}$, $\varepsilon_i \in \mathbb{N}$ for $1 \leq i \leq l$. Let \mathcal{B}_i be an s_i -fold blocking set of $(h - 1)$ -spaces with respect to $(r - f - 1)$ -spaces for every $1 \leq i \leq l$. Moreover, let $|\mathcal{B}_i| = n_i$, and let $\sigma \in \mathbb{N}$ be sufficiently large. Then, we have $n_q(r, h, f; s) \geq n$, where $n := \sigma \cdot \begin{bmatrix} r \\ h \end{bmatrix}_q - \sum_{i=1}^l \varepsilon_i \cdot n_i$ and $s := \sigma \cdot \begin{bmatrix} r-f \\ h \end{bmatrix}_q - \sum_{i=1}^l \varepsilon_i \cdot s_i$.*

Proof. Form a multiset \mathcal{M} of $(h - 1)$ -spaces in $\text{PG}(r - 1, q)$, starting from σ copies of every $(h - 1)$ -space in $\text{PG}(r - 1, q)$, and removing the $(h - 1)$ -spaces contained in ε_i copies of \mathcal{B}_i . The value for s follows from the definition of s_i -fold blocking set: every codimension f subspace contains at least s_i elements from \mathcal{B}_i , so removing ε_i copies reduces the count by $\varepsilon_i \cdot s_i$. \square

In Section 6 we will consider blocking sets that have a smaller cardinality than the one consisting of all $(h - 1)$ -spaces in a fixed subspace.

A simple but very effective variant of Lemma 5.5 is given by the removal of a single blocking set.

Lemma 5.6. *If \mathcal{B} is a s -fold blocking set of $(h - 1)$ -spaces in $\text{PG}(r - 1, q)$ with respect to subspaces of codimension f that has maximum multiplicity at most m , then we have*

$$n_q\left(r, h, f; m \cdot \begin{bmatrix} r-f \\ h \end{bmatrix}_q - s\right) \geq m \cdot \begin{bmatrix} r \\ h \end{bmatrix}_q - |\mathcal{B}|. \quad (15)$$

6 Blocking sets

In Section 1 we have introduced the notion $b_q(r, h, f; s)$ for the minimum size of an s -fold blocking set of $(h - 1)$ -spaces with respect to subspaces of codimension f . Recall that this is the minimum number of $(h - 1)$ -spaces in $\text{PG}(r - 1, q)$ such that each subspace of codimension f contains at least s members. If we additionally assume that the maximum multiplicity of an $(h - 1)$ -space is m , then we use the notation $b_q(r, h, f; s, m)$ for the minimum possible cardinality. So, we obviously have $b_q(r, h, f; s, m) \geq b_q(r, h, f; s, m')$ and $b_q(r, h, f; s, m) \geq b_q(r, h, f; s)$ for all $m, m' \in \mathbb{N}$ with $m \leq m'$. We note that allowing subspaces of projective dimension smaller than $h - 1$ would not decrease those numbers. A straightforward counting argument gives a first lower bound.

Lemma 6.1. *We have*

$$b_q(r, h, f; s) \geq \frac{\begin{bmatrix} r \\ f \end{bmatrix}_q \cdot s}{\begin{bmatrix} r-h \\ f \end{bmatrix}_q} = \prod_{i=0}^{f-1} \frac{\begin{bmatrix} r-i \\ r-h-i \end{bmatrix}_q}{\begin{bmatrix} r-h-i \\ r-h-i \end{bmatrix}_q} \cdot s. \quad (16)$$

Proof. Each $(h - 1)$ -dimensional element is contained in $\begin{bmatrix} r-h \\ f \end{bmatrix}_q$ subspaces of codimension f and there are $\begin{bmatrix} r \\ f \end{bmatrix}_q$ subspaces of codimension f in total. This implies $b_q(r, h, f; s) \geq \frac{\begin{bmatrix} r \\ f \end{bmatrix}_q \cdot s}{\begin{bmatrix} r-h \\ f \end{bmatrix}_q}$. \square

A well-known construction is to use all $(h - 1)$ -spaces contained in a fixed $(h + f - 1)$ -space. Furthermore, note that the union of two blocking sets again gives a blocking set. Using this, we get the following proposition and lemma.

Proposition 6.2. *For $r, h, f \in \mathbb{N}$ with $h + f \leq r$ we have $b_q(r, h, f; 1) \leq \begin{bmatrix} h+f \\ h \end{bmatrix}_q$.*

Lemma 6.3. $b_q(r, h, f; s_1 + s_2) \leq b_q(r, h, f; s_1) + b_q(r, h, f; s_2)$.

From Proposition 6.2 we can e.g. conclude $b_q(5, 2, 2; 1) \leq \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = q^4 + q^3 + 2q^2 + q + 1$. Next we describe an improved construction from [10]. For a given integer $l \geq 3$ consider $\text{PG}(2l - 2, q)$ and an arbitrary point P . With this, let \mathcal{S} be a set of $\frac{q^{2l-2}-1}{q^2-1}$ planes through P that form a geometric line spread in the quotient geometry through P . Fix an l -space U through P . Let \mathcal{B} consist of the $\frac{q^l-1}{q-1}$ lines in U through P together with the $q^2 \frac{q^{2l-2}-1}{q^2-1}$ lines that lie in a plane of \mathcal{S} but do not contain P . Then every $(l - 1)$ -space in $\text{PG}(2l - 2, q)$ contains at least one line from \mathcal{B} .

This construction is shown to be optimal in the theorem below.

Theorem 6.4. *([10, Theorem 1.2]) For each integer $l \geq 3$ we have $b_q(2l - 1, 2, l - 1; 1) \geq \frac{q^{2l}-q^2}{q^2-1} + \frac{q^l-1}{q-1}$ and the above example is the only one in which equality holds.*

We remark that in [10] sets of lines were considered. However, the statement remains obviously true for multisets of lines. For $l = 3$ we obtain $b_q(5, 2, 2; 1) = q^4 + 2q^2 + q + 1$, i.e. the subspace construction is improved by q^3 lines.

6.1 The minimum number of lines in $\text{PG}(4, q)$ such that every plane contains at least s elements

In this subsection we want to focus on the values $b_q(5, 2, 2; s)$. From Theorem 6.4, Proposition 6.2, and Lemma 6.3 we directly conclude:

Lemma 6.5. *For each $0 \leq s' \leq q^2 + q$ and $t \geq 0$ with $s = t[3]_q + s'$ we have $b_q(5, 2, 2; s' + t[3]_q) \leq (q^4 + 2q^2 + q + 1) \cdot s' + \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \cdot t$.*

In the following we present some constructions that are better for specific choices for s and we look into lower bounds improving upon Lemma 6.1.

Lemma 6.6. *In $\text{PG}(4, q)$ there exists a q -fold blocking set w.r.t. planes consisting of $q^2(q + 1) + q^3(q^2 + 1)$ (pairwise different) lines.*

Proof. For a point P , let $\mathcal{P}_1, \dots, \mathcal{P}_q$ be q disjoint sets of $q^2 + 1$ planes, all containing the point P and each forming a line spread in the factor geometry through P . Hence in the factor geometry, these line spreads are contained in a parallelism. With this, consider the set \mathcal{L}_1 of lines consisting of all $q^2 \cdot q \cdot (q^2 + 1)$ lines contained in one of the planes of the \mathcal{P}_i that do not contain P . Let \mathcal{L}_2 be the $q^2(q + 1)$ lines that do contain P but are disjoint to a fixed line L , which is contained in one of the planes in one of the \mathcal{P}_i . Then we denote \mathcal{L} by $\mathcal{L}_1 \cup \mathcal{L}_2$. Let π be a plane with $P \in \pi$. If $\pi = \langle P, L \rangle$, then π contains all q^2 lines of \mathcal{L} not through P . If $P \in \pi$, but $\pi \neq \langle P, L \rangle$, then π intersects $\langle P, L \rangle$ in a point or in one line and hence π contains at least q lines from \mathcal{L} that contain point P .

Let π be a plane not containing P . The image of π in the factor geometry through P contains one line in each of the lines spreads, so that π contains q lines from \mathcal{L} . \square

For $q = 2$ the corresponding blocking set has size 52, which is indeed the minimum size for a double blocking set as verified by a small ILP computation, see Lemma 6.10. In Lemma 6.14 we give a lower bound, which shows that $|B| \geq q^5 + q^3 + q^2 + q$, so that Lemma 6.6 can be improved by at most $q^3 - q$.

Lemma 6.7. *In $\text{PG}(4, q)$ there exists a $(q + 1)$ -fold blocking set w.r.t. planes consisting of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q + (q + 1)q^2(q^2 + 1)$ (pairwise different) lines.*

Proof. Let P be an arbitrary point in $\text{PG}(4, q)$ and $\mathcal{P}_1, \dots, \mathcal{P}_{q+1}$ be sets of $q^2 + 1$ planes containing P , with the extra property that each forms a partial line parallelism in the factor geometry through P . With this, consider the set of lines \mathcal{L} consisting of all $q^2(q + 1)(q^2 + 1)$ lines contained in one of the planes of the \mathcal{P}_i that do not contain P and the $\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q$ lines that contain P . Denote the corresponding set of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q + (q + 1)q^2(q^2 + 1)$ lines by \mathcal{L} .

Let π be an arbitrary plane that contains P . The $q + 1$ lines in π that contain point P are all contained in \mathcal{L} .

Let π' be a plane not containing P . The image of π' in the factor geometry through P contains one line in each of the lines spreads, so that π' contains $q + 1$ lines from \mathcal{L} . \square

For $q = 2$ this gives a 3-fold blocking set of cardinality 75. By a sequence of ILP computations, see Lemma 6.11, we can verify that this is the minimum possible cardinality (even for multisets of lines).

Lemma 6.8. *In $\text{PG}(4, q)$ there exist a q^2 -fold blocking set w.r.t. planes consisting of $q^6 + q^4 + q^3 + q^2$ lines (maximum line multiplicity q^2).*

Proof. For an arbitrary plane E let the multiset of lines \mathcal{L} consist of q^2 copies of each line in E and a single copy of each of the q^6 lines outside of E , so that $|\mathcal{L}| = q^6 + q^4 + q^3 + q^2$.

Now let π be an arbitrary plane. If π intersects E in a line, then this line is contained q^2 times in \mathcal{L} . For $\pi = E$ we have $q^2 + q + 1$ lines in \mathcal{L} , each with multiplicity q^2 . If π intersects E in a point, then π contains q^2 lines disjoint to E , which are all contained in \mathcal{L} . \square

For $q = 2$ this gives a 4-fold blocking set of cardinality 92, whose minimality can be concluded from Lemma 6.14, see Theorem 6.16. However, there are lines that are taken four times. In Appendix A we list examples showing $b_2(5, 2, 2; 4, 1) \leq 102$ and $b_2(5, 2, 2; 4, 2) \leq 98$. The best known and indeed optimal construction for $b_2(5, 2, 2; 5)$ is given by $b_2(5, 2, 2; 5) \leq b_2(5, 2, 2; 1) + b_2(5, 2, 2; 4) = 27 + 92 = 119$, see Lemma 6.3. Again, this construction comes with a large maximum line multiplicity. Moreover, using ILP, we found examples to prove $b_2(5, 2, 2; 5, 1) \leq 123$, $b_2(5, 2, 2; 5, 3) \leq 121$, and $b_2(5, 2, 2; 5, 4) \leq 120$, see Appendix A.

Lemma 6.9. *In $\text{PG}(4, q)$ there exist a $(q^2 + q)$ -fold blocking set w.r.t. planes consisting of $q^6 + q^5 + q^4 + 2q^3 + 2q^2 + q = [6]_q + q^3 + q^2 - 1$ lines (with maximum line multiplicity $q^2 + q$).*

Proof. Let L be an arbitrary line and $S \supseteq L$ be an arbitrary solid. With this, let \mathcal{L}_1 be the set of all $(q+1)^2q$ lines that intersect L in a point and are contained in S . Moreover, let \mathcal{L}_2 be the set of lines that intersect S in a point and are disjoint to L . As blocking set \mathcal{B} we choose $q^2 + q$ times the line L , q times the elements of \mathcal{L}_1 , and once the elements of \mathcal{L}_2 , so that $|B| = (q^2 + q) + (q+1)^2q^2 + (q+1)q^5 = q^6 + q^5 + q^4 + 2q^3 + 2q^2 + q$.

Now we check the possible cases for a plane π . If $L \subseteq \pi$, then L is contained $q^2 + q$ times in \mathcal{B} . If π is disjoint to L , then $q^2 + q$ elements of \mathcal{L}_2 are contained in π . If π intersects L in a point and is contained in S , then π contains $q + 1$ elements from \mathcal{L}_1 . If π intersects L in a point and is not contained in S , then π contains one element from \mathcal{L}_1 and q^2 elements from \mathcal{L}_2 . \square

For $q = 2$ Lemma 6.9 gives a 6-fold blocking set of cardinality 138, whose optimality is implied by Lemma 6.14, see Theorem 6.17. However, there exists a line that is taken six times, so that we give examples showing $b_2(5, 2, 2; 6, 1) \leq 146$, $b_2(5, 2, 2; 6, 2) \leq 142$, $b_2(5, 2, 2; 6, 3) \leq 142$, and $b_2(5, 2, 2; 6, 5) \leq 141$ in Appendix A.

s	$b_2(5, 2, 2; s)$	construction	lower bound	$b_2(5, 2, 2; s, 1) \leq$
1	27	Theorem 6.4	Theorem 6.4	27
2	52	Lemma 6.6	Lemma 6.10	52
3	75	Lemma 6.7	Lemma 6.11	75
4	92	Lemma 6.8	Lemma 6.14	98
5	119	Lemma 6.3	Lemma 6.13	123
6	138	Lemma 6.9	Lemma 6.14	146
7	155	Proposition 6.2	Lemma 6.1	155

Table 1: Exact values for $b_2(5, 2, 2; s)$ and upper bounds for $b_2(5, 2, 2; s, 1)$.

In Table 1 we have summarized the upper bounds for $b_2(5, 2, 2; s)$ based on the constructions described so far, where $1 \leq s \leq 7$. For future reference we have added the currently best known upper bound for $b_2(5, 2, 2; s, 1)$ in the last

column. Either the mentioned construction in the unrestricted cases automatically satisfies a maximum line multiplicity of one or the example was found by ILP computations. In the remaining part of this subsection we will present matching lower bounds. The lower bounds for $s \in \{2, 3, 5\}$ were obtained by tailored ILP computations, see Lemma 6.10, Lemma 6.11, and Lemma 6.13, using the ILOG CPLEX solver.

Lemma 6.10. $b_2(5, 2, 2; 2) \geq 52$.

Proof. Direct ILP computation. \square

Lemma 6.11. $b_2(5, 2, 2; 3) \geq 75$.

Proof. We utilize several ILP computations. If the maximum line multiplicity is 3, then the minimum possible cardinality is 75. If the maximum line multiplicity is 2 and there are two lines L, L' with multiplicity 2, then the minimum possible cardinality is 75, independent of $\dim(L \cap L')$. If the maximum line multiplicity of 2 is attained at a unique line, then the minimum possible cardinality is at least 75. If the maximum line multiplicity is one and there exists a plane with three contained lines through a point, then the minimum possible cardinality is 75. If the maximum line multiplicity is one then there has to be a configuration as described before see Lemma 6.7. \square

Lemma 6.12. *The unique example attaining $b_2(5, 2, 2; 4) = 92$ is given by the construction in the proof of Lemma 6.8.*

Proof. We utilize several ILP computations. If the maximum line multiplicity is at most three, then the cardinality is larger than 92. If there is a unique line with multiplicity 4, then the cardinality is larger than 94. If there are two disjoint lines with multiplicity four, then the minimum possible cardinality is 95. So, we prescribe two intersecting lines L, L' with multiplicity four and minimize the chosen number of lines in $E := \langle L, L' \rangle$ given a cardinality of 92. It turns out that all seven lines in E need to have multiplicity 4 each. Prescribing such a configuration and cardinality 92 results in a unique ILP solution. \square

Lemma 6.13. $b_2(5, 2, 2; 5) \geq 119$.

Proof. We utilize several ILP computations. If there is a line L with multiplicity at least 6, then the minimum possible cardinality is 120. For maximum line multiplicity five the minimum possible cardinality is 119. For maximum line multiplicity at most four we considered a pair of lines L, L' intersecting in a point, where L attains the maximum multiplicity and L' has the largest possible multiplicity of all lines intersecting L . For each choice of these two multiplicities we have checked by an ILP computation that cardinality 118 is infeasible. \square

Lemma 6.14. *Let \mathcal{B} be an s -fold blocking set of lines w.r.t. planes in $\text{PG}(4, q)$. Then, we have*

$$|\mathcal{B}| \geq (q^4 + q^2 + q + 1) \cdot s - q(q + 1) \cdot \mathcal{B}(L)$$

for each line L , where $\mathcal{B}(L)$ denotes its multiplicity in \mathcal{B} .

Proof. Fix a line L and let \mathcal{P}_1 be the set of planes that contain L and \mathcal{P}_2 the set of planes that are disjoint to L , so that $|\mathcal{P}_1| = q^2 + q + 1$ and $|\mathcal{P}_2| = q^6$. Consider the multiset $\mathcal{P} := q^2 \cdot \mathcal{P}_1 + \mathcal{P}_2$ of $q^6 + q^4 + q^3 + q^2$ planes. Note that L is contained in all elements of \mathcal{P}_1 and therefore in $q^2 \cdot (q^2 + q + 1)$ elements of \mathcal{P} . Furthermore, any line L' that is disjoint to L is contained in q^2 elements of \mathcal{P}_2 and \mathcal{P} . And all other lines (i.e. those that intersect L in a point) are contained in a unique element from \mathcal{P}_1 and therefore in q^2 elements from \mathcal{P} . Consider an s -fold blocking set \mathcal{B} of lines with respect to planes. We double count the set $S = \{(l, \pi) : l \in \mathcal{B}, l \subset \pi, \pi \in \mathcal{P}\}$; which gives that

$$\mathcal{B}(L)q^2(q^2 + q + 1) + \sum_{l' \neq L, l' \cap L \neq \emptyset} \mathcal{B}(l')q^2 + \sum_{l' \cap L = \emptyset} \mathcal{B}(l')q^2 \geq (q^6 + q^4 + q^3 + q^2)s,$$

which is equivalent to

$$q^2|\mathcal{B}| + \mathcal{B}(L)q^2(q^2 + q) \geq (q^6 + q^4 + q^3 + q^2)s,$$

and hence, proves the lemma. \square

We note that Lemma 7.3 is a complementary bound.

Lemma 6.15. *If $b_q(5, 2, 2; s) \leq (q^4 + q^2 + q + 1) \cdot s$, then we have*

$$b_q(5, 2, 2; s + t[3]_q) = b_q(5, 2, 2; s) + t \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q,$$

for all $t \in \mathbb{N}$.

Proof. Since $b_q(5, 2, 2; [3]_q) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$, Lemma 6.3 yields $b_q(5, 2, 2; s + t[3]_q) \leq b_q(5, 2, 2; s) + t \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$ for all $t \in \mathbb{N}$. Now assume that \mathcal{B} is a $(s + t[3]_q)$ -fold blocking set of n lines w.r.t. planes where $n < b_q(5, 2, 2; s) + t \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$. W.l.o.g. we assume that t is minimal with this property, which implies the existence of a line L with multiplicity $\mathcal{B}(L) = 0$. Lemma 6.14 gives

$$\begin{aligned} |\mathcal{B}| &\geq (q^4 + q^2 + q + 1) \cdot (s + t[3]_q) \\ &= (q^4 + q^2 + q + 1) \cdot s + t \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q + tq(q + 1), \end{aligned}$$

which is a contradiction. \square

Theorem 6.16. *For all $t \in \mathbb{N}$ we have*

$$b_q(5, 2, 2; t[3]_q + q^2) = t \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q + q^2 \cdot (q^4 + q^2 + q + 1).$$

If equality is attained, then every line has multiplicity at least t .

Proof. Lemma 6.8 gives a matching construction for $t = 0$, so that Proposition 6.2 and Lemma 6.3 imply the corresponding upper bound for all $t \in \mathbb{N}$. Lemma 6.14 gives a matching lower bound for $t = 0$, so that the statement follows from Lemma 6.15. \square

Since the construction in the proof of Lemma 6.8 gives the unique 4-fold blocking set of 92 lines in $\text{PG}(4, 2)$ w.r.t. planes, see Lemma 6.12, there is a unique example attaining $b_2(5, 2, 2; 4 + 7t)$ for all $t \in \mathbb{N}$.

Theorem 6.17. *For all $t \in \mathbb{N}$ we have*

$$b_q(5, 2, 2; t[3]_q + q^2 + q) = t \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q + (q^2 + q)(q^4 + q^2 + q + 1).$$

If equality is attained, then every line has multiplicity at least t .

Proof. Lemma 6.9 gives a matching construction for $t = 0$, so that Proposition 6.2 and Lemma 6.3 imply the corresponding upper bound for all $t \in \mathbb{N}$. Lemma 6.14 gives a matching lower bound for $t = 0$, so that the statement follows from Lemma 6.15. \square

Theorem 6.18. *For each $t \in \mathbb{N}$ we have*

- $b_2(5, 2, 2; 1 + 7t) = 27 + 155t$,
- $b_2(5, 2, 2; 2 + 7t) = 52 + 155t$,
- $b_2(5, 2, 2; 3 + 7t) = 75 + 155t$,
- $b_2(5, 2, 2; 4 + 7t) = 92 + 155t$,
- $b_2(5, 2, 2; 5 + 7t) = 119 + 155t$,
- $b_2(5, 2, 2; 6 + 7t) = 138 + 155t$,
- $b_2(5, 2, 2; 7 + 7t) = 155(t + 1)$.

Proof. For the constructions and upper bounds for $b_2(5, 2, 2; s)$ for $1 \leq s \leq 7$ we refer to Table 1. Since $n_2(5, 2, 2; 7) = 155$, Lemma 6.3 extends the upper bounds to all $t \in \mathbb{N}$. Therefore, it remains to give the lower bounds. Lemma 6.14 with $\mathcal{B}(L) = 0$ shows that these upper bounds for $b_2(5, 2, 2; s)$ are tight for $s \in \{4, 6\}$, so that we can apply Lemma 6.15. Applying Lemma 6.14 with $s = 8$ and $\mathcal{B}(L) = 0$ would give a lower bound of $184 > 182$, so that we may suppose that $\mathcal{B}(L) \geq 1$ for each L . Hence, it suffices to determine $b_2(5, 2, 2; 1)$, which is done in [10]. Applying Lemma 6.14 with $\mathcal{B}(L) = 0$ gives a lower bound that matches the size of the stated constructions for $s \in \{9, 10\}$, see Table 1 and Lemma 6.3, and is strictly larger for $s = 12$. So, it suffices to determine $b_2(5, 2, 2; s)$ for $s \in \{2, 3, 5\}$, see Lemma 6.10, Lemma 6.11, and Lemma 6.13 for the corresponding lower bounds. \square

In Table 2 we fix $q = 3$ and summarize our knowledge on $b_3(5, 2, 2; s)$ for $1 \leq s \leq 13$.

s	$b_3(5, 2, 2; s)$	construction	lower bound
1	103	Theorem 6.4	Theorem 6.4
2	188–206	Lemma 6.3	Lemma 6.14
3	282–306	Lemma 6.6	Lemma 6.14
4	376–400	Lemma 6.7	Lemma 6.14
5	470–502	ILP	Lemma 6.14
6	564–600	ILP	Lemma 6.14
7	658–690	ILP	Lemma 6.14
8	752–784	ILP	Lemma 6.14
9	846	Lemma 6.8	Lemma 6.14
10	940–949	Lemma 6.3	Lemma 6.14
11	1034–1050	ILP	Lemma 6.14
12	1128	Lemma 6.9	Lemma 6.14
13	1210	Proposition 6.2	Lemma 6.1

Table 2: Bounds for $b_3(5, 2, 2; s)$.

6.2 Generalizations to other parameters

The constructions from Lemma 6.8 and Lemma 6.9 can be described from a more general point of view. In $\text{PG}(r-1, q)$ let S_1, \dots, S_{r-1} be a chain of subspaces with $\dim(S_i) = i-1$ for $1 \leq i \leq r-1$. The set of $(h-1)$ -spaces is partitioned into classes $\mathcal{H}_1, \dots, \mathcal{H}_u$ according to the intersection dimensions with those S_i . The set of subspaces of codimension f is partitioned into classes $\mathcal{F}_1, \dots, \mathcal{F}_v$ according to the intersection dimensions with the S_i . By $\beta_{i,j}$ we denote the number of elements from \mathcal{H}_i that are contained in an arbitrary element $\pi \in \mathcal{F}_j$. As blocking set we choose $\mathcal{B} = \sum_{i=1}^u \alpha_i \mathcal{H}_i$, where $\alpha_i \in \mathbb{N}$ for $1 \leq i \leq u$. Given this framework, we obtain a simple optimization problem: choose $\alpha_i \in \mathbb{N}$ minimizing $\sum_{i=1}^u \alpha_i \cdot |\mathcal{H}_i|$ such that $\sum_i \alpha_i \beta_{i,j} \geq s$ for all $1 \leq j \leq v$. Of course also the easy construction from Proposition 6.2 can be described in this way.

Example 6.19. Let K be a chamber in $\text{PG}(4, q)$, which is a maximal flag $\{\pi_0, \pi_1, \pi_2, \pi_3\}$, where $\pi_0 \subset \pi_1 \subset \pi_2 \subset \pi_3$ and $\dim(\pi_i) = i$. For $\text{PG}(4, q)$ and $(h, f) = (2, 2)$ we obtain the ten line classes and ten plane classes listed in Table 3, according to their intersection dimensions K . Therefore, as an example, \mathcal{H}_3 consists of all lines, contained in π_2 (and hence also in π_3) and meeting the line π_1 precisely in the point π_0 . On the other hand, \mathcal{F}_8 consists of all planes in π_3 , that meet π_2 precisely in the point P .

The corresponding intersection numbers $\beta_{i,j}$ are given in Table 4. Note that we have $\beta_{i,j} \in \{0, 1, q, q^2\}$.

We can generalize Lemma 6.14 as follows.

Lemma 6.20. *Let $h = 2$, $f \geq h$, and $r > h + f$. Then, for any s -fold blocking*

i	\mathcal{H}_i	$\#$	j	\mathcal{F}_j	$\#$
1	(0, 1, 1, 1)	1	1	(0, 1, 2, 2)	1
2	(0, 0, 1, 1)	q	2	(0, 1, 1, 2)	q
3	(-1, 0, 1, 1)	q^2	3	(0, 0, 1, 2)	q^2
4	(0, 0, 0, 1)	q^3	4	(-1, 0, 1, 2)	q^3
5	(-1, 0, 0, 1)	q^3	5	(0, 1, 1, 1)	q^2
6	(-1, -1, 0, 1)	q^4	6	(0, 0, 1, 1)	q^3
7	(0, 0, 0, 0)	q^3	7	(-1, 0, 1, 1)	q^4
8	(-1, 0, 0, 0)	q^4	8	(0, 0, 0, 1)	q^4
9	(-1, -1, 0, 0)	q^5	9	(-1, 0, 0, 1)	q^5
10	(-1, -1, -1, 0)	q^6	10	(-1, -1, 0, 1)	q^6

Table 3: Line and plane classes in $\text{PG}(4, q)$ according to the intersection dimensions with a chamber, i.e. a maximal flag.

j/i	1	2	3	4	5	6	7	8	9	10
1	1	q	q^2	0	0	0	0	0	0	0
2	1	0	0	q	q^2	0	0	0	0	0
3	0	1	0	q	0	q^2	0	0	0	0
4	0	0	1	0	q	q^2	0	0	0	0
5	1	0	0	0	0	0	q	q^2	0	0
6	0	1	0	0	0	0	q	0	q^2	0
7	0	0	1	0	0	0	0	q	q^2	0
8	0	0	0	1	0	0	q	0	0	q^2
9	0	0	0	0	1	0	0	q	0	q^2
10	0	0	0	0	0	1	0	0	q	q^2

Table 4: Intersection numbers $\beta_{i,j}$ in $\text{PG}(4, q)$ w.r.t. lines and planes.

set of lines in $\text{PG}(r-1, q)$ w.r.t. to subspace of codimension f we have

$$|\mathcal{B}| \geq \frac{1}{\beta_{2,1}} \cdot \left(\alpha_1 + \frac{\beta_{2,1} - \beta_{3,1}}{\beta_{3,2}} \cdot \alpha_2 \right) \cdot s - \frac{\beta_{1,1} - \beta_{2,1}}{\beta_{2,1}} \cdot \mathcal{B}(L)$$

for each line L , where $\mathcal{B}(L)$ denotes its multiplicity in \mathcal{B} ,

$$\begin{aligned} \alpha_1 &= \begin{bmatrix} r-2 \\ f \end{bmatrix}_q, & \alpha_2 &= q^{2(r-f)} \cdot \begin{bmatrix} r-2 \\ r-f \end{bmatrix}_q, \\ \beta_{i,1} &= \begin{bmatrix} r-i-1 \\ f \end{bmatrix}_q & \text{for } i &\in \{1, 2, 3\} \\ \beta_{3,2} &= q^{2(r-f-2)} \begin{bmatrix} r-4 \\ f-2 \end{bmatrix}_q. \end{aligned}$$

Proof. Let \mathcal{L}_2 be the set of lines in $\text{PG}(r-1, q)$ that intersect L in a point and \mathcal{L}_3 be the set of lines that are disjoint to L . Set $\mathcal{L}_1 := \{L\}$ and $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$, i.e. the set of all lines in $\text{PG}(r-1, q)$. By \mathcal{P}_1 we denote the set of subspaces of codimension f that contain L and by \mathcal{P}_2 we denote the set of subspaces of codimension f that are disjoint to L .

Recall that the number of j -spaces disjoint from a fixed m -space in $\text{PG}(n, q)$ equals $q^{(m+1)(j+1)} \begin{bmatrix} n-m \\ j+1 \end{bmatrix}_q$. With this we have

$$\alpha_1 := |\mathcal{P}_1| = \begin{bmatrix} r-2 \\ f \end{bmatrix}_q \quad (17)$$

and

$$\alpha_2 := |\mathcal{P}_2| = q^{2(r-f)} \begin{bmatrix} r-2 \\ r-f \end{bmatrix}_q. \quad (18)$$

A line $l' \in \mathcal{L}_i$ is contained in an element $\pi \in \mathcal{P}_1$ if the i -space $\langle l', L \rangle$ is contained in π . Hence, $\beta_{i,1} = \begin{bmatrix} r-i-1 \\ f \end{bmatrix}_q$. A line $l' \in \mathcal{L}_1 \cup \mathcal{L}_2$ meets the line L and hence, cannot be contained in an element of \mathcal{P}_2 ; which implies $\beta_{1,2} = \beta_{2,2} = 0$. For a line $l' \in \mathcal{L}_3$, we need to define the number $\beta_{3,2}$ of $(r-f-1)$ -spaces in $\text{PG}(r-1, q)$ through l' and disjoint from L . This equals the number of $(r-f-3)$ -spaces in $\text{PG}(r-3, q)$, disjoint from a line, which equals $q^{2(r-f-2)} \begin{bmatrix} r-4 \\ f-2 \end{bmatrix}_q$.

Let \mathcal{B} be an s -fold blocking set of lines in $\text{PG}(r-1, q)$ w.r.t. a subspace of codimension f . Choose $t \in \mathbb{R}_{\geq 0}$ such that $\beta_{3,1} + t \cdot \beta_{3,2} = \beta_{2,1}$, i.e.

$$t := \frac{\beta_{2,1} - \beta_{3,1}}{\beta_{3,2}}. \quad (19)$$

With this, we double count the set $S = \{(l, \alpha) : l \in \mathcal{B}, l \subset \alpha, \dim(\alpha) = r-f-1\}$.

$$\sum_{E \in \mathcal{P}_1} \sum_{U \subseteq E: U \in \mathcal{L}} \mathcal{B}(U) + t \cdot \sum_{E \in \mathcal{P}_2} \sum_{U \subseteq E: U \in \mathcal{L}} \mathcal{B}(U) \geq (\alpha_1 + t \cdot \alpha_2) \cdot s. \quad (20)$$

Note that $\mathcal{B}(U)$ is counted $\beta_{2,1}$ times in the sum on the left hand side for all $U \in \mathcal{L}_2 \cup \mathcal{L}_3$ while $\mathcal{B}(L)$ is counted $\beta_{1,1}$ times. \square

It seems tempting to generalize Theorem 6.16 (or Theorem 6.17). However, there are some issues that we cannot resolve. Motivated by Lemma 6.8 we state the following generalized construction:

Lemma 6.21. *For $r \geq 4$ there exist a q^2 -fold blocking set in $\text{PG}(r, q)$ w.r.t. planes consisting of $q^{2(r-1)} + q^2 \cdot \binom{r-1}{2}_q$ lines (maximum line multiplicity q^2).*

Proof. For an arbitrary but fixed $(r-2)$ -space S , let \mathcal{L}_1 be the set of $\binom{r-1}{2}_q$ lines contained in S and let \mathcal{L}_2 the set of $q^{2(r-1)}$ lines disjoint to S . With this we set $\mathcal{B} = q^2 \cdot \mathcal{L}_1 + \mathcal{L}_2$ and check that \mathcal{B} is indeed a q^2 -fold blocking set w.r.t. planes. \square

So, we especially have $b_q(6, 2, 3; q^2) \leq q^8 + q^6 + q^5 + 2q^4 + q^3 + q^2$. Applying Lemma 6.20 for these parameters with $\mathcal{B}(L) = 0$ gives $b_q(6, 2, 3; q^2) \geq q^8 + q^6 + q^5 + q^4 + q^3 + q^2$, i.e. there remains a gap of q^4 . For $q = 2$ those bounds give $380 \leq b_2(6, 2, 3; 4) \leq 396$. Solving our standard ILP model with the additional constraint $x_L = 0$, i.e. $\mathcal{B}(L) = 0$, gives $b_2(6, 2, 3; 4) = 396$, so that Lemma 6.21 is optimal for $(r, q) = (5, 2)$. We remark that the corresponding LP relaxation yields the lower bound $b_2(6, 2, 3; 4) \geq 380$ only, and hence, the bound in Lemma 6.20 is optimal for these parameters if we only rely on counting arguments and the extra information $\mathcal{B}(L) = 0$. Using $\mathcal{B}(L) = 0$ and $\mathcal{B}(L') = 0$ for two disjoint lines the corresponding LP relaxation gives $b_2(6, 2, 3; 4) \geq 385.3333$. For $b_q(6, 2, 3; q^2 + q)$ similar computations can be performed.

7 The maximum number of lines in $\text{PG}(4, q)$ such that each plane contains at most s lines

Here we want to determine bounds for $n_q(5, 2, 2; s)$. Recall the correspondence, by duality, between $n_q(5, 2, 2; s)$ and the generalized blocking sets. More precisely, if we have a configuration with maximum line multiplicity at most s , then $n_q(5, 2, 2; s) = s \cdot \binom{5}{2}_q - b_q(5, 2, 2; s \cdot [3]_q - s, s)$.

Theorem 7.1. *We have $n_q(5, 2, 2; 1) = q^3 + 1$.*

Proof. From Lemma 4.3 we conclude $n_q(5, 2, 2; 1) = A_q(5, 2; 4)$. Here $A_q(5, 2; 4)$ is the maximum cardinality of a partial line spread in $\text{PG}(4, q)$, which is well known, see e.g. [31, Theorem 5]. \square

Corollary 7.2. *We have $b_q(5, 2, 2; q^2 + q, 1) = \binom{5}{2}_q - (q^3 + 1)$.*

From Lemma 6.17 we know that $b_q(5, 2, 2; q^2 + q) = (q^2 + q)(q^4 + q^2 + q + 1)$, which shows that here again, the last parameter m plays an important role.

We can easily formulate the problem of the determination of $n_q(r, h, f; s)$ as an integer linear programming (ILP) problem. To this end let \mathcal{H} denote the set of all $(h-1)$ -spaces and \mathcal{F} denote the set of all $(r-f-1)$ -spaces in $\text{PG}(r-1, q)$. As variables we choose $x_H \in \mathbb{N}$ for all $H \in \mathcal{H}$ to model the multiplicities of

the chosen $(h - 1)$ -spaces. The condition that each subspace of codimension f contains at most s elements can be modeled as $\sum_{H \in \mathcal{H}: H \subseteq F} x_H \leq s$ for all $F \in \mathcal{F}$. As target function we choose $\sum_{H \in \mathcal{H}} x_H$, i.e. the number of selected $(h - 1)$ -spaces. Typically this ILP can be solved directly for rather small values of r, h, f , and q only. In order to obtain lower bounds we can e.g. prescribe some automorphisms. For upper bounds we can add tailored extra constraints or prescribe a few $(h - 1)$ -spaces to reduce the symmetry of the formulation.

From Theorem 7.1 and Lemma 4.1 we have $n_q(5, 2, 2; 2) \geq 2(q^3 + 1) \in \Theta(q^3)$. From Lemma 4.2 we conclude $n_q(5, 2, 2; 2) \leq 2 \cdot \frac{(q^4 + q^3 + q^2 + q + 1) \cdot (q^2 + 1)}{q^2 + q + 1} \in \Theta(q^4)$, so that the question for the right order of magnitude, in terms of q , arises. By ILP computations we found examples showing $n_2(5, 2, 2; 2) \geq 32$, $n_3(5, 2, 2; 2) \geq 97$, and $n_5(5, 2, 2; 2) \geq 493$. We remark that improved constructions for $n_q(5, 2, 1; 2)$ have been recently obtained in [23].

Lemma 7.3. *Let \mathcal{L} be a multiset of lines in $\text{PG}(4, q)$ such that each plane contains at most s lines. Then, we have*

$$|\mathcal{L}| \leq (q^4 + q^2 + q + 1) \cdot s - q(q + 1) \cdot \mathcal{L}(L), \quad (21)$$

for each line L , where $\mathcal{L}(L)$ denotes the multiplicity of L in \mathcal{L} .

Proof. Fix a line L and let \mathcal{P}_1 be the set of planes that contain L and \mathcal{P}_2 the set of planes that are disjoint to L . Then we have $|\mathcal{P}_1| = q^2 + q + 1$ and $|\mathcal{P}_2| = q^6$. Consider the multiset $\mathcal{P} := q^2 \cdot \mathcal{P}_1 + \mathcal{P}_2$ of $q^6 + q^4 + q^3 + q^2$ planes. Note that L is contained in all elements of \mathcal{P}_1 and so in $q^2 \cdot (q^2 + q + 1)$ elements of \mathcal{P} . Any line L' that is disjoint to L is contained in q^2 elements of \mathcal{P}_2 and \mathcal{P} . Moreover, all other lines (i.e. those that intersect L in a point) are contained in a unique element from \mathcal{P}_1 and so q^2 elements from \mathcal{P} . Consider a projective $(2, 2) - (|\mathcal{L}|, 5, s)$ system \mathcal{L} .

We double count the set $S = \{(l, \pi) : l \in \mathcal{L}, l \subset \pi, \pi \in \mathcal{P}\}$; which gives that

$$\mathcal{L}(L)q^2(q^2 + q + 1) + \sum_{l' \neq L, l' \cap L \neq \emptyset} \mathcal{L}(l')q^2 + \sum_{l' \cap L = \emptyset} \mathcal{L}(l')q^2 \leq (q^6 + q^4 + q^3 + q^2)s,$$

which is equivalent to

$$q^2|\mathcal{L}| + \mathcal{L}(L)q^2(q^2 + q) \leq (q^6 + q^4 + q^3 + q^2)s,$$

and hence, proves the lemma. \square

Lemma 7.4. *Let B be the smallest s -fold blocking set of lines in $\text{PG}(4, q)$ with respect to planes and maximum multiplicity m . Then*

$$155m - |B| = 155m - b_2(5, 2, 2; s, m) \leq n_2(5, 2, 2; 7m - s) \leq 23(7m - s) - 6m.$$

Proof. Follows immediately from Lemmas 5.6 and 7.3. \square

Lemma 7.5. *We have $166 \leq n_2(5, 2, 2; 8) \leq 172$, $323 \leq n_2(5, 2, 2; 15) \leq 327$, and $478 \leq n_2(5, 2, 2; 22) \leq 482$.*

Proof. The upper bound follows immediately from Lemma 7.4 with $s = 6$ and $m \in \{2, 3, 4\}$. For the lower bound, we also use the bounds $b_2(5, 2, 2; 6, 2) \leq 144$, $b_2(5, 2, 2; 6, 3) \leq 142$ and $b_2(5, 2, 2; 6, 4) \leq 142$. \square

Lemma 7.6. *We have $32 \leq n_2(5, 2, 2; 2) \leq 34$, $187 \leq n_2(5, 2, 2; 9) \leq 195$, $344 \leq n_2(5, 2, 2; 16) \leq 350$, and $500 \leq n_2(5, 2, 2; 23) \leq 505$.*

Proof. The upper bound follows immediately from Lemma 7.4 for $s = 5$ and $m \in \{1, 2, 3, 4\}$. For the lower bound, we also use the bounds $b_2(5, 2, 2; 5, 2) \leq b_2(5, 2, 2; 5, 1) \leq 123$, $b_2(5, 2, 2; 5, 3) \leq 121$ and $b_2(5, 2, 2; 5, 4) \leq 120$. In the case of $m = 1$ we can find a better upper bound: let \mathcal{P} be a faithful $(2, 2) - (n, 5, 2)_2$ projective system. If there exists a line L in \mathcal{P} with multiplicity $\mathcal{P}(L)$ at least 2, then Lemma 7.3 implies $n \leq 34$. For maximum line multiplicity one, we utilize an ILP computation to verify $n \leq 34$. $b_2(5, 2, 2; 5, 4) \leq 120$. \square

Lemma 7.7. *We have $53 \leq n_2(5, 2, 2; 3) \leq 59$, $212 \leq n_2(5, 2, 2; 10) \leq 218$, $367 \leq n_2(5, 2, 2; 17) \leq 373$, and $n_2(5, 2, 2; 24) = 528$.*

Proof. The upper bound follows from Lemma 7.4 for $s = 4$ and $m \in \{1, 2, 3, 4\}$. For the lower bound, we also use the bounds $b_2(5, 2, 2; 4, 1) \leq 102$, $b_2(5, 2, 2; 4, 2) \leq 98$, and $b_2(5, 2, 2; 4, 4) \leq 92$. In the case of $m = 1$ we can find a better upper bound: let \mathcal{P} be a faithful $(2, 2) - (n, 5, 3)_2$ projective system. If there exists a line L in \mathcal{P} with multiplicity $\mathcal{P}(L)$ at least 2, then Lemma 7.3 implies $n \leq 57$. For maximum line multiplicity one, we utilize an ILP computation to verify $n \leq 59$. \square

Proposition 7.8. *For $t \in \mathbb{N}$ we have $n_2(5, 2, 2; 7t + 4) = 155t + 80$.*

Proof. From Lemma 7.4 with $s = 3$ and $m = t + 1$, we have that $155(t + 1) - b_2(5, 2, 2; 3, 1) \leq 155(t + 1) - b_2(5, 2, 2; 3, t + 1) \leq n_2(5, 2, 2; 7t + 4) \leq 23(7t + 4) - 6\mathcal{L}(l)$. Using $b_2(5, 2, 2; 3, 1) = b_2(5, 2, 2; 3) = 75$ we get the right lower bound. Now, let \mathcal{P} be a faithful $(2, 2) - (n, 5, 7t + 4)_2$ projective system. If there exists a line L in \mathcal{P} with multiplicity $\mathcal{L}(L)$ at least $t + 2$, then we find the right lower bound $n \leq 155t + 80$. For maximum line multiplicity $t + 1$ we conclude $n \leq (t + 1) \cdot 155 - b_2(5, 2, 2; 3, 1) = 155t + 80$, which proves the statement. \square

Actually, Lemma 5.6 and the construction of a blocking set in Lemma 6.7 imply the following lemma.

Lemma 7.9. *We have $n_q(5, 2, 2; q^2) \geq q^4 \cdot (q^2 + 1)$.*

Proposition 7.10. *For $t \in \mathbb{N}$, we have $n_2(5, 2, 2; 7t + 5) = 155t + 103$.*

Proof. From Lemma 7.4 with $s = 2$ and $m = t + 1$, we find the right lower bound using $b_2(5, 2, 2; 2, 1) = 52$. For the upper bound, let \mathcal{P} be a faithful $(2, 2) - (n, 5, 7t + 5)_2$ projective system. If there exists a line L in \mathcal{P} with multiplicity $\mathcal{P}(L)$ at least $t + 2$, then Lemma 7.3 implies $n \leq 155t + 103$. For maximum line multiplicity $t + 1$, we conclude $n \leq (t + 1) \cdot 155 - b_2(5, 2, 2; 2, 1) = 155t + 103$. \square

Theorem 7.11. *For each $t \geq 0$, we have*

$$n_q(5, 2, 2; t \cdot [3]_q + q^2 + q) = t \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q + q^6 + q^5 + q^4 + 2q^3.$$

Proof. Consider the set of all lines in $\text{PG}(4, q)$ with multiplicity $(t + 1)$ and subtract those from a blocking set \mathcal{B} as in Theorem 6.17. Since the total number of lines is given by $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$, we have $n_q(5, 2, 2; t[3]_q + q^2 + q) \geq (t + 1) \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q - (q^4 + 2q^2 + q + 1)$.

Now consider a multiset \mathcal{L} of lines in $\text{PG}(4, q)$ such that each plane contains at most $t \cdot [3]_q + q^2 + q$ lines and that $|\mathcal{L}| > t \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q + q^6 + q^5 + q^4 + 2q^3$. If there exists a line L with $\mathcal{L}(L) \geq t + 2$, then Lemma 7.3 yields

$$\begin{aligned} |\mathcal{L}| &\leq (q^4 + q^2 + q + 1) \cdot (t \cdot [3]_q + q^2 + q) - (t + 2)q(q + 1) \\ &= t \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q + q^6 + q^5 + q^4 + 2q^3 - q, \end{aligned}$$

which is a contradiction. Thus, the maximum line multiplicity $\mathcal{L}(L)$ is at most t and we denote the complementary multiset of lines by \mathcal{B} . Since each plane contains at most $q^2 + q = [3]_q - 1$ elements from \mathcal{S} , the elements of \mathcal{B} block every plane at least once. From Theorem 6.17 we conclude

$$\begin{aligned} |\mathcal{L}| &= (t + 1) \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q - |\mathcal{B}| \leq (t + 1) \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q - (q^4 + 2q^2 + q + 1) \\ &= t \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q + q^6 + q^5 + q^4 + 2q^3, \end{aligned}$$

which is a contradiction, and hence, proves the theorem. \square

We have summarized our information on $n_2(5, 2, 2; s)$ in Table 5.

s	$n_2(5, 2, 2; s)$	s	$n_2(5, 2, 2; s)$	s	$n_2(5, 2, 2; s)$	s	$n_2(5, 2, 2; s)$
1	9	8	166–172	15	323–327	22	478–482
2	32–34	9	187–195	16	344–350	23	500–505
3	53–59	10	212–218	17	367–373	24	528
4	80	11	235	18	390	25	545
5	103	12	258	19	413	26	568
6	128	13	283	20	438	27	593
7	155	14	310	21	465	28	620

Table 5: Bounds for $n_2(5, 2, 2; s)$.

We can easily generalize Lemma 7.3 to $\text{PG}(n, q)$. For an even more general version, formulated in terms of blocking sets, we refer to Lemma 6.20.

Lemma 7.12. *Let \mathcal{L} be a multiset of lines in $\text{PG}(n, q)$ such that each plane contains at most s lines. Then, we have*

$$|\mathcal{L}| \leq \left(q^4 \frac{(q^{n-1} - 1)(q^{n-2} - 1)}{(q^3 - 1)(q^2 - 1)} + [n - 1]_q \right) \cdot s - q[n - 2]_q \cdot \mathcal{L}(L) \quad (22)$$

for each line L , where $\mathcal{L}(L)$ denotes the multiplicity of L in \mathcal{L} .

Proof. Fix a line L and let \mathcal{P}_1 be the set of planes that contain L , and \mathcal{P}_2 be the set of planes that are disjoint to L . Hence, $|\mathcal{P}_1| = [n - 1]_q$ and $|\mathcal{P}_2| = q^6 \begin{bmatrix} n-1 \\ 3 \end{bmatrix}_q$. Consider the multiset $\mathcal{P} := q^2[n - 3]_q \cdot \mathcal{P}_1 + \mathcal{P}_2$ of $q^2[n - 3]_q[n - 1]_q + q^6 \begin{bmatrix} n-1 \\ 3 \end{bmatrix}_q$ planes. Note that L is contained in all elements of \mathcal{P}_1 and so in $q^2[n - 3]_q[n - 1]_q$ elements of \mathcal{P} . Any line L' that is disjoint to L is contained in $q^2[n - 3]_q$ elements of \mathcal{P}_2 and \mathcal{P} . All other lines (i.e. those that intersect L in a point) are contained in a unique element from \mathcal{P}_1 and so $q^2[n - 3]_q$ elements from \mathcal{P} . Consider a projective $(2, n - 2) - (|\mathcal{L}|, n + 1, s)$ system \mathcal{L} .

We double count the set $S = \{(l', \pi) : l' \in \mathcal{L}, l' \subset \pi, \pi \in \mathcal{P}\}$; which gives that

$$\begin{aligned} \mathcal{L}(L)q^2[n - 3]_q[n - 1]_q + \sum_{l' \neq L, l' \cap L \neq \emptyset} \mathcal{L}(l')q^2[n - 3]_q + \sum_{l' \cap L = \emptyset} \mathcal{L}(l')q^2[n - 3]_q \\ \leq \left([n - 1]_q q^2[n - 3]_q + q^6 \begin{bmatrix} n-1 \\ 3 \end{bmatrix}_q \right) s. \end{aligned}$$

This is equivalent to

$$q^2[n - 3]_q |\mathcal{L}| + \mathcal{L}(L)q^3[n - 3]_q[n - 2]_q \leq \left([n - 1]_q q^2[n - 3]_q + q^6 \begin{bmatrix} n-1 \\ 3 \end{bmatrix}_q \right) s.$$

Hence,

$$|\mathcal{L}| \leq s \left([n - 1]_q + q^4 \frac{(q^{n-1} - 1)(q^{n-2} - 1)}{(q^3 - 1)(q^2 - 1)} \right) - \mathcal{L}(L)q([n - 2]_q),$$

which proves the lemma. \square

s	$n_3(5, 2, 2; s)$	s	$n_3(5, 2, 2; s)$	s	$n_3(5, 2, 2; s)$
1	28	6	465–558	11	1004–1023
2	105–186	7	562–651	12	1107
3	190–279	8	660–744	13	1210
4	275–372	9	810–837		
5	366–465	10	904–930		

Table 6: Bounds for $n_3(5, 2, 2; s)$.

From Theorem 6.4, Lemma 6.6, Theorem 7.1, Lemma 6.14, and Proposition 6.2 we conclude $b_3(5, 2, 2; 1, 1) = 103$, $b_3(5, 2, 2; 3, 1) \leq 306$, $b_3(5, 2, 2; 12, 1) = 1182$, and $b_3(5, 2, 2; 13, 1) = 1210$, respectively. For $b_3(5, 2, 2; 2, 1) \leq 206$, $b_3(5, 2, 2; 5, 1) \leq 550$, $b_3(5, 2, 2; 6, 1) \leq 648$, $b_3(5, 2, 2; 7, 1) \leq 745$, $b_3(5, 2, 2; 8, 1) \leq 844$, $b_3(5, 2, 2; 9, 1) \leq 935$, $b_3(5, 2, 2; 10, 1) \leq 1020$, and $b_3(5, 2, 2; 11, 1) \leq 1105$ we refer to Appendix A. Using Lemma 5.6 we obtain the lower bounds for $n_3(5, 2, 2; s)$ for $1 \leq s \leq 13$, as summarized in Table 6.

8 Conclusion and open problems

We have introduced the maximum number $n_q(r, h, f; s)$ of $(h - 1)$ -spaces in $\text{PG}(r - 1, q)$ such that each subspace of codimension f contains at most s elements. These numbers are complemented by the minimum number $b_q(r, h, f; s)$ of $(h - 1)$ -spaces in $\text{PG}(r - 1, q)$ such that each subspace of codimension f contains at least s elements. Both notions are rather general. As an example, the case $(h, f) = (1, 1)$ corresponds to linear codes with their geometric reformulation as multisets of points. If we keep $f = 1$ but consider $h > 1$, then we are dealing with additive codes. For $h = 1$ and $f > 1$ we are confronted with linear codes w.r.t. to the f th generalized Hamming weight. So, in this paper, we generalize both concepts to one more general structure. Due to this general setting, one cannot expect to determine these number in full generality. While we have some results on the asymptotic behavior, even the question for the right order of magnitude remains open in most cases. Besides a few general insights we mostly focused on $n_q(5, 2, 2; s)$ and $b_q(5, 2, 2; s)$, where we mostly assume $q \in \{2, 3\}$. As a first specific open problem we ask for the right order of magnitude of $n_q(5, 2, 2; 2)$ in terms of q .

In Theorem 6.18 we have fully determined the minimum number $b_2(5, 2, 2; s)$ of lines in $\text{PG}(4, 2)$ such that each plane contains at least s elements as a function of s . However, this result is still based on integer linear programming computations and we propose it as an open problem to replace some of these by theoretical lower bounds. The techniques used in [10, 30] may serve as a blueprint. If we restrict the maximum multiplicity of the lines, then in most cases we only presented upper bounds by listing explicit examples found by ILP searches. It would be interesting to determine the exact values. For $b_3(5, 2, 2; s)$ we have presented partial results, see Table 2.

In Theorem 7.11 we have fully determined $n_q(5, 2, 2; t \cdot [3]_q + q^2 + q)$. The underlying construction fits into the framework of Lemma 5.5: starting from the set of all lines in $\text{PG}(4, q)$ we can remove any set of lines that blocks all planes to obtain a lower bound for $n_q(5, 2, 2; q^2 + q)$. Choosing the trivial blocking set consisting of all $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = q^4 + q^3 + q^2 + q + 1$ lines in a solid yields $n_q(5, 2, 2; q^2 + q) \geq q^6 + q^5 + q^4 + q^3$, i.e. $n_2(5, 2, 2; 6) \geq 120$. Choosing the blocking set obtained from the $q^4 + q^3 + q^2 + q + 1$ lines in the orbit of a Singer-cycle of $\text{PG}(4, q)$ yields $n_q(5, 2, 2; q^2 + q) \geq q^6 + q^5 + q^4 + q^3 + q^2$, i.e. $n_2(5, 2, 2; 6) \geq 124$. The best choice of the blocking set yields the lower bound from Theorem 7.11, i.e. $n_2(5, 2, 2; 6) \geq 128$, which is tight. So far, all of our lower bounds for $n_2(5, 2, 2; s)$

are of this type. Finding a good lower bound for $n_q(5, 2, 2; 2)$ seems to be a challenging problem.

While there is a Griesmer type bound for linear and additive codes that determines $n_q(r, h, 1; s)$ for all sufficiently large values of s , we currently do not know such a bound for the cases $h, f \geq 2$.

In order to turn the determination of $n_q(5, 2, 2; s)$ and $b_q(5, 2, 2; s)$ as a function of s , given some fixed field size q , into a finite computational problem, we have presented Lemma 7.3 and Lemma 6.14. Both bounds are generalized to some extent, but still do not cover the whole parameter space of (r, h, f) . We can conclude that in this paper, we give a new, rather general research direction, in which many things can still be investigated.

Acknowledgements

The authors would like to thank Timothy Alderson, Simeon Ball, and Tabriz Popatia for the discussions on additive codes during the seventh Irsee conference. There, we uncovered the relation between the geometric objects we study in this paper, and additive codes with respect to the generalized Hamming weight. Both authors discussed the initial ideas for this paper at that conference.

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 $(11010), (00010), (00001)$. There are 21 lines of multiplicity 2 and 56 lines of multiplicity 1.

$b_2(5, 2, 2; 5, 1) \leq 123$: $(00010), (00100), (00100), (00101), (00101), (01000),$
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 $(11011), (11010), (11011), (11011), (11011), (11000), (11100), (11100), (11100),$
 $(00101), (00110), (00111), (00110), (00111), (00001), (00010), (00011), (00011)$.

$b_2(5, 2, 2; 5, 3) \leq 121$: $(00010), (00010), (00010), (00100), (00100), (01000),$
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 $(11101), (11101), (11101), (11101), (11110), (11110), (11110), (11100), (11101),$
 $(00010), (00011), (00011), (00011), (00001), (00001), (00001), (00001)$. The seven lines of multiplicity 3 are the lines contained in a plane π . The 48 lines of multiplicity 0 intersect π in a point and there are no lines of multiplicity 2.

$b_2(5, 2, 2; 5, 4) \leq 120$: $(00010), (00010), (00010), (00010), (00101), (01001),$
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 $(01100), (01101), (01101), (01110), (10000), (10000), (10000), (10000), (10000),$
 $(00011), (00010), (00011), (00001), (00001), (00001), (00001), (00001), (00010),$

$(10000), (10001), (10010), (10011), (10100), (10101), (10110), (10111), (11000), (11001), (11010), (11011), (11100), (11101), (11110), (11111), (10000), (10001), (10010), (10011), (10100), (10101), (10110), (10111), (11000), (11001), (11010), (11011), (11100), (11101), (11110), (11111)$. The seven lines of multiplicity 4 are the lines of a plane π . The eight lines of multiplicity 2 are disjoint to π and the 64 lines of multiplicity 0 intersect π in a point.

The complement of a partial line spread of size 9 gives $b_2(5, 2, 2; 6, 1) \leq 146$:

$(00010), (00011), (00100), (00101), (00110), (00111), (01000), (01001), (01010), (01011), (01100), (01101), (01110), (01111), (10000), (10001), (10010), (10011), (10100), (10101), (10110), (10111), (11000), (11001), (11010), (11011), (11100), (11101), (11110), (11111)$.

$b_2(5, 2, 2; 6, 2) \leq 144$:

$(00100), (00101), (00110), (00111), (01000), (01001), (01010), (01011), (01100), (01101), (01110), (01111), (10000), (10001), (10010), (10011), (10100), (10101), (10110), (10111), (11000), (11001), (11010), (11011), (11100), (11101), (11110), (11111)$.

(10211), (10211), (10211), (10211), (10222), (10222), (10222), (10222), (10211),
 (01101), (01101), (01101), (01101), (01002), (01002), (01002), (01002), (01102),
 (10211), (10211), (10211), (10220), (10212), (10212), (10220), (10220), (10212),
 (01102), (01102), (01102), (01010), (01111), (01011), (01112), (01012), (01110),
 (10221), (10210), (10221), (10210), (10221), (10210), (10222), (10211), (10222),
 (01010), (01111), (01011), (01112), (01012), (01110), (01010), (01111), (01011),
 (10211), (10222), (10211), (10220), (10212), (10212), (10220), (10212), (10212),
 (01112), (01012), (01110), (01020), (01122), (01021), (01120), (01022), (01121),
 (10221), (10210), (10221), (10210), (10221), (10210), (10222), (10211), (10222),
 (01020), (01122), (01021), (01120), (01022), (01121), (01020), (01122), (01021),
 (10211), (10222), (10211), (10201), (10201), (10201), (10201), (10201), (10201),
 (01120), (01022), (01121), (01210), (01210), (01210), (01210), (01211), (01211),
 (10201), (10201), (10201), (10201), (10201), (10201), (10202), (10202), (10202),
 (01211), (01211), (01212), (01212), (01212), (01212), (01210), (01210), (01210),
 (10202), (10202), (10202), (10202), (10202), (10202), (10202), (10202), (10202),
 (01210), (01211), (01211), (01211), (01211), (01212), (01212), (01212), (01212),
 (10200), (10200), (10200), (10200), (10200), (10200), (10200), (10200), (10200),
 (01210), (01210), (01210), (01210), (01211), (01211), (01211), (01211), (01212),
 (10200), (10200), (10200), (10201), (10201), (10201), (10202), (10202), (10202),
 (01212), (01212), (01212), (01221), (01222), (01220), (01221), (01222), (01220),
 (10200), (10200), (10200), (10201), (10201), (10201), (10202), (10202), (10202),
 (01221), (01222), (01220), (01202), (01200), (01201), (01202), (01200), (01201),
 (10200), (10200), (10200), (11000), (11000), (11000), (11010), (11010), (11010),
 (01202), (01200), (01201), (00001), (00001), (00001), (00001), (00001), (00001),
 (11020), (11120), (11000), (11020), (11000), (11022), (11000), (11001), (11001),
 (00001), (00001), (00100), (00110), (00101), (00111), (00102), (00112), (00100),
 (11021), (11001), (11020), (11001), (11022), (11002), (11022), (11002), (11021),
 (00110), (00101), (00111), (00102), (00112), (00100), (00110), (00101), (00111),
 (11002), (11020), (11000), (11000), (11000), (11001), (11001), (11001), (11002),
 (00102), (00112), (00120), (00121), (00122), (00120), (00121), (00122), (00120),
 (11002), (11002), (11010), (11010), (11022), (11010), (11021), (11011), (11011),
 (00121), (00122), (00110), (00121), (00111), (00122), (00112), (00120), (00110),
 (11021), (11011), (11020), (11011), (11022), (11012), (11022), (11012), (11021),
 (00121), (00111), (00122), (00112), (00120), (00110), (00121), (00111), (00122),
 (11012), (11020), (11020), (11020), (11021), (11021), (11021), (11021), (11022),
 (00112), (00120), (00100), (00101), (00102), (00100), (00101), (00102), (00100),
 (11022), (11022), (11210), (11220), (11220), (11200), (12000), (12000), (12010),
 (00101), (00102), (00001), (00001), (00001), (00001), (00001), (00001), (00001),
 (12210), (12020), (12020), (12220), (12220), (12000), (12010), (12000), (12011),
 (00001), (00001), (00001), (00001), (00001), (00100), (00110), (00101), (00111),
 (12000), (12012), (12001), (12011), (12012), (12001), (12012), (12001), (12002),
 (00102), (00112), (00100), (00110), (00101), (00111), (00102), (00112), (00100),
 (12012), (12002), (12010), (12011), (12002), (12010), (12000), (12000), (12001),
 (00110), (00101), (00111), (00102), (00112), (00120), (00121), (00122), (00120),
 (12001), (12001), (12002), (12002), (12002), (12010), (12021), (12021), (12022),
 (00121), (00122), (00120), (00121), (00122), (00100), (00110), (00101), (00111),
 (12010), (12020), (12011), (12022), (12011), (12020), (12011), (12021), (12012),
 (00102), (00112), (00100), (00110), (00101), (00111), (00102), (00112), (00100),
 (12020), (12012), (12021), (12012), (12022), (12010), (12010), (12010), (12010),
 (00110), (00101), (00111), (00102), (00112), (00120), (00121), (00122), (00120),
 (12011), (12011), (12012), (12012), (12012), (12012), (12100), (12110), (12120),
 (00121), (00122), (00120), (00121), (00122), (00001), (00001), (00001), (00001),
 (00010), (00010), (00010), (10000), (10010), (10020),
 (00001), (00001), (00001), (00001), (00001), (00001).

A blocking set showing $b_3(5, 2, 2; 2, 1) \leq 206$ is given by:

(01010), (01011), (01010), (01012), (01012), (01012), (01011), (01011),
 (00111), (00112), (00112), (00110), (00111), (00112), (00110), (00100),
 (01020), (01022), (01020), (01022), (01021), (01021), (01021), (00100),
 (00101), (00101), (00102), (00102), (00100), (00101), (00102), (00120),
 (10010), (10011), (10010), (10012), (10012), (10012), (00120), (10021),
 (00121), (00120), (00122), (00121), (00122), (00120), (00121), (01000),
 (10020), (10022), (10020), (10022), (10021), (10021), (10021), (10020),
 (01001), (01001), (01002), (01002), (01000), (01001), (01002), (01110),
 (10020), (10022), (10020), (10022), (10021), (10021), (10000), (10000),
 (01111), (01112), (01112), (01110), (01110), (01111), (01112), (01210),
 (10000), (10001), (10001), (10001), (10002), (10002), (10002), (10011),
 (01212), (01210), (01211), (01212), (01210), (01211), (01212), (01220),
 (10010), (10011), (10010), (10012), (10012), (10012), (10012), (10111),
 (01221), (01220), (01222), (01221), (01222), (01220), (01221), (01010),
 (10110), (10111), (10110), (10112), (10112), (10112), (01010), (10111),
 (01011), (01012), (01012), (01010), (01011), (01012), (01010), (01110),
 (10110), (10111), (10110), (10112), (10112), (10112), (10112), (10120),
 (01111), (01112), (01112), (01110), (01111), (01112), (01110), (01100),
 (10120), (10122), (10120), (10122), (10121), (10121), (10121), (10120),
 (01101), (01101), (01102), (01102), (01100), (01101), (01102), (01210),
 (10120), (10122), (10120), (10122), (10121), (10121), (10220), (10222),
 (01211), (01212), (01212), (01210), (01210), (01211), (01212), (01020),
 (10220), (10222), (10220), (10222), (10221), (10221), (10221), (10211),
 (01021), (01020), (01022), (01021), (01020), (01021), (01022), (01100),
 (10210), (10211), (10210), (10212), (10212), (10212), (10210), (10210),
 (01101), (01101), (01102), (01102), (01100), (01101), (01102), (01200),
 (10210), (10210), (10210), (10212), (10212), (10212), (11000), (11000),
 (01201), (01201), (01202), (01202), (01200), (01201), (01202), (00010),
 (11000), (11001), (11001), (11002), (11002), (11002), (11000), (11000),
 (00012), (00010), (00011), (00012), (00010), (00011), (00012), (00010),
 (11100), (11101), (11101), (11102), (11102), (11102), (12000), (12000),
 (00012), (00010), (00011), (00012), (00010), (00011), (00012), (00110),
 (12000), (12001), (12001), (12001), (12001), (12002), (12002), (12011),
 (00112), (00110), (00111), (00112), (00110), (00111), (00112), (00120),

(10202), (10222), (10202), (10222), (10211), (10202), (10211), (10202), (10211),
 (01021), (01120), (01022), (01121), (01000), (01100), (01001), (01101), (01002),
 (10202), (10210), (10201), (10210), (10201), (10210), (10201), (10211), (10202),
 (01102), (01010), (01111), (01011), (01112), (01012), (01110), (01010), (01111),
 (10211), (10202), (10211), (10202), (10210), (10201), (10210), (10201), (10210),
 (01011), (01112), (01012), (01110), (01020), (01122), (01021), (01120), (01022),
 (10201), (10211), (10202), (10211), (10202), (10211), (10202), (10211), (10202),
 (01121), (01020), (01122), (01021), (01120), (01022), (01121), (01020), (01122),
 (10212), (10200), (10212), (10200), (10220), (10212), (10220), (10212), (10220),
 (01021), (01120), (01022), (01121), (01000), (01100), (01001), (01101), (01002),
 (10212), (10210), (10221), (10210), (10210), (10221), (10210), (10220), (10212),
 (01102), (01000), (01100), (01001), (01101), (01002), (01102), (01010), (01111),
 (10220), (10212), (10220), (10212), (10220), (10212), (10220), (10212), (10220),
 (01011), (01112), (01012), (01110), (01020), (01122), (01021), (01120), (01022),
 (10212), (10221), (10210), (10221), (10210), (10221), (10210), (10222), (10211),
 (01121), (01020), (01122), (10222), (10222), (10222), (10222), (10221), (10221),
 (10222), (10211), (10222), (10211), (10211), (10210), (10211), (10212), (10221),
 (10221), (10222), (10222), (10222), (10220), (10220), (10200), (10201), (10200),
 (01201), (01202), (01200), (01201), (01210), (01211), (01212), (01210), (10200),
 (10200), (10200), (10200), (10200), (10200), (10201), (10201), (10202), (10202),
 (01212), (01221), (01222), (01220), (01202), (01200), (01201), (01202), (01200),
 (10202), (10200), (10200), (10200), (10211), (10211), (10211), (10210), (10210),
 (01201), (01202), (01200), (01201), (01210), (01211), (01212), (01221), (10210),
 (10210), (10210), (10210), (10211), (10211), (10211), (10210), (10210), (10210),
 (01220), (01202), (01200), (01201), (00001), (00001), (00010), (00011), (00010),
 (11101), (11002), (11102), (11020), (11120), (11000), (11020), (11001), (11021),
 (00011), (00010), (00011), (00001), (00001), (00100), (00110), (00100), (00110),
 (11002), (11022), (11000), (11011), (11001), (11012), (11002), (11010), (11000),
 (00100), (00110), (00111), (00122), (00111), (00122), (00111), (00122), (00121),
 (11001), (11002), (11010), (11011), (11012), (11010), (11021), (11011), (11022),
 (00121), (00121), (00101), (00101), (00101), (00112), (00120), (00112), (00120),
 (11012), (11020), (11020), (11021), (11022), (11210), (11201), (11202), (11200),
 (00112), (00120), (00102), (00102), (00102), (00001), (00012), (00012), (00012),
 (11200), (12000), (12200), (12000), (12000), (12200), (12001), (12201), (12002),
 (00001), (00001), (00001), (00011), (00012), (00011), (00012), (00011), (00012),
 (12010), (12210), (12000), (12010), (12001), (12011), (12002), (12012), (12000),
 (00001), (00001), (00100), (00110), (00100), (00110), (00100), (00110), (00111),
 (12022), (12001), (12002), (12002), (12021), (12000), (12001), (12002), (12002),
 (00122), (00111), (00122), (00111), (00122), (00121), (00121), (00121), (00102),
 (12020), (12011), (12021), (12012), (12022), (12010), (12011), (12012), (12020),
 (00112), (00102), (00112), (00102), (00112), (00120), (00120), (00120), (00101),
 (12021), (12022), (12100), (12101), (12100), (12102), (12120), (10020),
 (00101), (00101), (00001), (00010), (00010), (00010), (00001), (00001).

A blocking set showing $b_3(5, 2, 2; 6, 1) \leq 648$ is given by: (00100), (00102),
 (00011), (00012), (00012), (00010), (00001), (00001), (00012), (00010), (00011),
 (01101), (01002), (01102), (01010), (01110), (01020), (01120), (01000), (01021),
 (00010), (00012), (00010), (00001), (00001), (00001), (00001), (00102), (00112),
 (01001), (01022), (01002), (01000), (01000), (01012), (01000), (01011), (01000),
 (00102), (00112), (00102), (00112), (00110), (00121), (00111), (00122), (00112),
 (01010), (01010), (01010), (01001), (01012), (01001), (01001), (01011), (01002),
 (00120), (00110), (00121), (00111), (00122), (00112), (00120), (00110), (00121),
 (01002), (01010), (01002), (01012), (01000), (01001), (01002), (01010), (01010),
 (00111), (00122), (00112), (00120), (00120), (00120), (00120), (00100), (00101),
 (01010), (01011), (01011), (01011), (01012), (01012), (01012), (01010), (01020),
 (00102), (00100), (00101), (00102), (00100), (00101), (00102), (00110), (00121),
 (01011), (01021), (01012), (01022), (01020), (01021), (01022), (01210), (01202),
 (00110), (00121), (00110), (00121), (00100), (00100), (00100), (00001), (00011),
 (01200), (01201), (01220), (01200), (10000), (10000), (10000), (10002), (10002),
 (00011), (00011), (00001), (00001), (00010), (00010), (00011), (00012), (00010),
 (10002), (10000), (10000), (10001), (10001), (10002), (10002), (10002), (10002),
 (00012), (00100), (00110), (00101), (00111), (00100), (00110), (00101), (00111),
 (10002), (10002), (10000), (10001), (10002), (10002), (10002), (10010), (10011),
 (00102), (00112), (00121), (00122), (00121), (00122), (00120), (00102), (00112),
 (10011), (10012), (10012), (10012), (10010), (10010), (10012), (10010), (10012),
 (00100), (00110), (00102), (00112), (00100), (00110), (00101), (00111), (00102),
 (10010), (10011), (10011), (10011), (10012), (10010), (10010), (10010), (10022),
 (00112), (00121), (00122), (00120), (00120), (00121), (00120), (00100), (00110),
 (10020), (10022), (10021), (10020), (10021), (10020), (10021), (10022), (10021),
 (00101), (00111), (00100), (00110), (00102), (00112), (00100), (00110), (00121),
 (10022), (10020), (10021), (10021), (10021), (10100), (10100), (10101), (10102),
 (00120), (00121), (00121), (00122), (00001), (00001), (00001), (00012), (00010),
 (10120), (10200), (10202), (10201), (10202), (10201), (10202), (10000), (10000),
 (00001), (00011), (00012), (00010), (00011), (00012), (00010), (01000), (01100),
 (10000), (10000), (10000), (10000), (10000), (10001), (10001), (10001), (10001),
 (01001), (01101), (01002), (01102), (01000), (01100), (01001), (01101), (01002),
 (10001), (10001), (10002), (10002), (10002), (10002), (10000), (10000), (10000),
 (01102), (01000), (01100), (01001), (01101), (01002), (01102), (01010), (01111),
 (10000), (10000), (10000), (10000), (10000), (10000), (10000), (10000), (10000),
 (01011), (01112), (01012), (01110), (01020), (01122), (01021), (01120), (01022),
 (10000), (10001), (10001), (10001), (10001), (10001), (10001), (10012), (10010),
 (01121), (01020), (01122), (01021), (01120), (01022), (01121), (01000), (01100),
 (10012), (10010), (10012), (10010), (10012), (10010), (10012), (10010), (10012),
 (01001), (01101), (01002), (01102), (01010), (01111), (01011), (01112), (01012),
 (10010), (10010), (10010), (10012), (10010), (10012), (10010), (10020), (10022),
 (01110), (01020), (01122), (01021), (01120), (01022), (01121), (01000), (01100),

(12000), (12012), (12001), (12012), (12001), (12010), (12002), (12010), (12002),
(00102), (00112), (00101), (00111), (00102), (00112), (00101), (00111), (00102),
(12011), (12000), (12022), (12001), (12020), (12002), (12021), (12000), (12000),
(00112), (00111), (00122), (00111), (00122), (00111), (00122), (00120), (00122),
(12001), (12001), (12002), (12002), (12010), (12021), (12011), (12022), (12012),
(00120), (00122), (00120), (00122), (00100), (00110), (00100), (00110), (00100),
(12020), (12010), (12011), (12012), (12020), (12021), (12022), (12100), (12100),
(00110), (00121), (00121), (00121), (00101), (00101), (00101), (00001), (00010),
(12100), (12101), (12101), (12102), (12110), (12120), (12100), (12120),
(00011), (00010), (00011), (00010), (00011), (00001), (00001).

A blocking set showing $b_3(5, 2; 7, 1) \leq 745$ is given by:

(00101), (00102), (00102), (00120), (01000), (01100), (01000), (00110), (00110),
(00011), (00012), (00010), (00001), (00001), (00001), (00010), (00011), (00001),
(01100), (01000), (01100), (01001), (01101), (01001), (01101), (01001), (01101),
(00012), (00012), (00010), (00010), (00010), (00011), (00011), (00012), (00010),
(01002), (01102), (01002), (01002), (01002), (01102), (01102), (01110), (01020),
(00010), (00011), (00011), (00012), (00012), (00010), (00010), (00001), (00001),
01120, 01000, 01022, 01001, 01020, 01002, 01021, 01000, 01012, 01012,
00001, 00101, 01111, 01001, 01011, 00111, 00111, 00111, 01000, 01012,
01000, 01010, 01001, 01001, 01010, 01001, 01001, 01002, 01011, 01002,
(00112), (00120), (00110), (00121), (00112), (00120), (00110), (00121), (00121),
01012, 01000, 01001, 01002, 01010, 01011, 01010, 01011, 01011, 01012,
(00120), (00122), (00122), (00122), (00100), (00102), (00100), (00102), (00100),
01012, 01010, 01020, 01011, 01021, 01012, 01022, 01020, 01021,
(00102), (00110), (00121), (00110), (00121), (00110), (00121), (00100), (00100),
01022, 01210, 01202, 01201, 01200, 01200, 01202, 01201, 01201,
(00100), (00001), (00011), (00012), (00010), (00011), (00012), (00010), (00011),
01200, 01202, 01220, 01200, 01200, 01200, 01200, 01200, 01200, 01200,
(00012), (00010), (00001), (00001), (00010), (00011), (00012), (00100), (00100),
10000, 10000, 10000, 10001, 10001, 10000, 10000, 10001, 10001,
(00101), (00111), (00101), (00111), (00121), (00122), (00122), (00100), (00110),
10010, 10011, 10011, 10012, 10011, 10012, 10011, 10012, 10012, 10012,
(00101), (00111), (00100), (00110), (00101), (00111), (00102), (00112), (00100),
10010, 10012, 10010, 10010, 10010, 10010, 10011, 10011, 10011, 10012,
(00110), (00101), (00111), (00102), (00112), (00121), (00122), (00120), (00121),
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(00122), (00121), (00122), (00120), (00100), (00110), (00101), (00111), (00102),
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A blocking set showing $b_3(5, 2, 2; 8, 1) \leq 844$ is given by: $(\begin{smallmatrix} 00100 \\ 00001 \end{smallmatrix})$, $(\begin{smallmatrix} 00110 \\ 00001 \end{smallmatrix})$,

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 (12011), (10212), (10212), (10210), (10210), (10210), (10211), (11012), (10211),
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 (01222), (01110), (00111), (01112), (01120), (01121), (01122), (01120), (01120),
 (11011), (10211), (10211), (10200), (10200), (10212), (12010), (10212), (10220),
 (00112), (01121), (01122), (01122), (01122), (01122), (01122), (01100), (01100),
 (10220), (10220), (10221), (10112), (10221), (10221), (10201), (10222), (10220),
 (01101), (01102), (01100), (00110), (01101), (01102), (01221), (01102), (01110),
 (10220), (10221), (11011), (10221), (10221), (10200), (10222), (12011), (10222),
 (01111), (01110), (00111), (01112), (10220), (10222), (10222), (10220), (10220),
 (01120), (01121), (01121), (01122), (01220), (01121), (01122), (01201), (00120),
 (11000), (10222), (12020), (11002), (10221), (12021), (11000), (10222), (12021),
 (00121), (01202), (00121), (00122), (01200), (00122), (00120), (01201), (00120),
 (11002), (10221), (10220), (10220), (10220), (11002), (10221), (10222), (12020),
 (00121), (01202), (01200), (00122), (01202), (00120), (01201), (01200), (00120),
 (11001), (12021), (11000), (12022), (11000), (11000), (11000), (11000), (11001),
 (00121), (00121), (00122), (00122), (00001), (00010), (00011), (00012), (00010),
 (11010), (11001), (11002), (12120), (11002), (11020), (12102), (12100), (12101),
 (00001), (00012), (00010), (00001), (00011), (00001), (00010), (00011), (00012),
 (12101), (12102), (12100), (12100), (12101), (12102), (11200), (11200), (11200),
 (00010), (00011), (00012), (00010), (00011), (00012), (00001), (00010), (00011),
 (11200), (11201), (11201), (11201), (11201), (11202), (12100), (11202), (11201),
 (00012), (00010), (00011), (00012), (00010), (00001), (00011), (00012), (00001),
 (12000), (12000), (12000), (12000), (12001), (12001), (12002), (12002), (12002),
 (00001), (00010), (00011), (00012), (00010), (00010), (00011), (00001), (00012),
 (12010), (00001).

A blocking set showing $b_3(5, 2, 2; 11, 1) \leq 1105$ is given by: (00010), (01110),
 (00001), (00001), (00001), (00001), (00001), (00001), (00001), (00001),
 (00100), (01101), (00100), (00100), (00011), (00010), (00012), (00010), (00010),
 (00101), (01101), (00102), (00102), (00001), (00011), (00011), (00012), (00012),
 (00011), (00012), (00010), (00001), (00011), (00011), (00012), (00012), (00001),
 (01100), (01000), (01011), (01000), (01000), (01010), (01000), (01012), (01001),
 (00011), (00001), (00100), (00010), (00011), (00100), (00012), (00100), (00010),
 (01010), (01001), (01011), (01001), (01011), (01002), (01002), (01210), (01012),
 (00001), (00011), (00102), (00012), (00101), (00010), (00001), (00011), (00101),

(01002), (01010), (01010), (01020), (01012), (01000), (01000), (01001), (01000),
(00012), (00102), (00101), (00001), (00102), (00100), (00101), (00100), (00102),
(01021), (01001), (01011), (01001), (01011), (01002), (01201), (01002), (01021),
(00100), (00101), (00120), (00102), (00110), (00100), (00010), (00101), (00110),
(01002), (01001), (01000), (01000), (01001), (01001), (01011), (01002), (01200),
(00102), (00120), (00110), (00111), (00110), (00111), (00122), (00110), (00011),
(01002), (01022), (01002), (01000), (01000), (01020), (01001), (01011), (01011),
(00111), (00111), (00112), (00122), (00120), (00121), (00100), (00121), (00121),
(01001), (01011), (01002), (01020), (01002), (01020), (01010), (01010), (01202),
(00122), (00112), (00121), (00112), (00122), (00101), (00121), (00111), (00101),
(01020), (01010), (01010), (01021), (01012), (01201), (01012), (01021), (01010),
(00111), (00110), (00112), (00101), (00110), (00011), (00111), (00112), (00122),
(01022), (01012), (01200), (01012), (01022), (01012), (01020), (01022), (01022),
(00101), (00120), (00012), (00121), (00110), (00122), (00102), (00102), (00102),
(01020), (01021), (01020), (01020), (01020), (01021), (01021), (01021), (01021),
(00110), (00111), (00011), (00120), (00121), (00120), (00121), (00122), (00120),
(01201), (01022), (01022), (01220), (01200), (01200), (10000), (10000), (10000),
(00012), (00121), (00122), (00001), (00001), (00001), (01111), (00010), (00011),
(10000), (10000), (10000), (10001), (11110), (10001), (10001), (10001), (10002),
(01110), (00012), (01112), (00010), (00001), (00011), (01110), (00012), (01112),
(10002), (12220), (10002), (10002), (10002), (10001), (10000), (10000), (10000),
(00010), (00001), (00011), (01110), (00012), (01112), (00100), (00101), (01101),
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(10001), (10021), (10002), (10021), (10002), (10021), (10002), (10012), (00100),
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(10010), (10002), (01011), (10011), (10012), (10011), (10012), (10022), (10012),
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(01122), (00110), (00011), (00112), (01122), (00110), (00011), (00111), (01100),
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(10000), (00100), (01001), (01011), (01002), (01211), (01010), (01010), (01012), (01212),
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(10210), (12000), (11022), (11021), (10210), (12001), (10220), (10220), (10220),
(01210), (00102), (00100), (00101), (01212), (00101), (01020), (01021), (01022),
(10221), (10221), (10221), (10221), (10212), (10222), (12002), (10222), (10200),
(01020), (00102), (01021), (01022), (01210), (01020), (00102), (01022), (01100),
(10200), (10200), (10201), (11011), (10201), (10201), (10200), (10202), (10200),
(01101), (01102), (01100), (00110), (01101), (01102), (01221), (01102), (01110),
(10200), (10201), (10201), (10202), (12010), (10202), (10202), (10200), (10200),
(01111), (01112), (01111), (01110), (00111), (01111), (01112), (01120), (01121),
(10200), (10201), (11012), (10201), (10201), (10202), (10202), (12011), (10202),
(01122), (01120), (00112), (01121), (01122), (01220), (01120), (00112), (01121),
(10202), (10210), (10210), (10210), (11010), (10211), (10211), (10202), (10212),
(01122), (01100), (01101), (01100), (01100), (00110), (01101), (01102), (01221),
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(00110), (01101), (01102), (01110), (01110), (01111), (01112), (01110), (00111),
(10211), (10212), (10212), (10212), (10212), (10212), (10210), (10210), (10210),
(01112), (01222), (01110), (00111), (01111), (01112), (01120), (01121), (01122),
(10211), (11011), (10211), (10211), (10200), (10212), (12010), (10212),
(01120), (00112), (01121), (01122), (01220), (01120), (00112), (01121), (01122),
(10220), (10220), (10220), (10221), (11012), (10221), (10221), (10201), (10222),
(01100), (01101), (01102), (01100), (00110), (01101), (01102), (01221), (01100),
(12010), (10222), (10222), (10220), (10220), (10221), (11011), (10221), (10200),
(00110), (01101), (01102), (01112), (01110), (00111), (01111), (01112), (01222),
(10222), (12011), (10222), (10222), (10222), (10220), (10220), (10221), (11010),
(01110), (00111), (01111), (01112), (01120), (01121), (01122), (01120), (00112),
(10221), (10221), (10221), (10222), (12012), (10222), (10222), (10220), (12022),
(01121), (01122), (01220), (01120), (00112), (01121), (01122), (01201), (00120),
(11000), (10222), (12020), (10221), (10221), (10221), (11000), (10222), (12021),
(00121), (01202), (00121), (00122), (01200), (00122), (00120), (01201), (00120),
(11002), (10221), (10221), (10220), (10220), (10220), (10220), (11002), (10221),
(00121), (01202), (00121), (00122), (01200), (00122), (01202), (00120), (01201),
(10222), (12020), (11001), (12021), (11000), (12022), (11000), (11000), (11000),
(01200), (00120), (00121), (00121), (00122), (00122), (00001), (00010), (00011),
(11000), (11001), (11010), (11001), (11002), (12120), (11002), (11020), (11000),
(00012), (00010), (00001), (00012), (00010), (00001), (00011), (00001), (00010),
(12100), (12101), (12101), (12102), (12100), (12100), (12101), (12102), (12110),
(00011), (00012), (00010), (00011), (00012), (00010), (00011), (00012), (00001),
(11200), (11200), (11200), (11200), (11200), (11201), (11201), (11201), (12100),
(00001), (00010), (00011), (00012), (00010), (00011), (00012), (00010), (00001),
(11202), (11202), (11210), (11220), (11220), (12000), (12000), (12000), (12001),
(00011), (00012), (00001), (00001), (00001), (00010), (00011), (00012), (00010),
(12001), (12002), (12020), (12002), (12010),
(00011), (00010), (00001), (00012), (00001).

A blocking set with line multiplicity 4 showing $b_3(5, 2, 2; 7) \leq 690$ is given

by: (00100), (00102), (00100), (00101), (00101), (00102), (00102), (00102),
(00010), (00011), (00011), (00012), (00012), (00010), (00011), (00012), (00010),
(01010), (01010), (01010), (01110), (01110), (01110), (01110), (01000), (01000),
(00001), (00001), (00001), (00001), (00001), (00001), (00001), (00001), (00100),
(01020), (01000), (01022), (01000), (01021), (01001), (01021), (01001), (01020),
(00110), (00101), (00111), (00102), (00112), (00100), (00110), (00101), (00111),
(01001), (01022), (01002), (01022), (01002), (01021), (01002), (01020), (01000),
(00102), (00112), (00100), (00110), (00101), (00111), (00102), (00112), (00120),
(01000), (01000), (01001), (01001), (01001), (01002), (01002), (01010), (01010),
(00121), (00122), (00120), (00121), (00122), (00120), (00121), (00122), (00110),
(01020), (01010), (01022), (01010), (01021), (01011), (01021), (01011), (01020),
(00121), (00111), (00122), (00112), (00120), (00110), (00121), (00111), (00122),
(01011), (01022), (01012), (01022), (01012), (01021), (01012), (01020), (01020),
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(01020), (01020), (01021), (01021), (01021), (01022), (01022), (01220), (01220),
(00101), (00102), (00100), (00101), (00102), (00100), (00101), (00102), (00001),
(01220), (01220), (01220), (10000), (10000), (10000), (10000), (10001), (10001),
(00001), (00001), (00001), (00010), (00011), (00012), (00010), (00011), (00012),
(10002), (10002), (10002), (10100), (10100), (10100), (10100), (10100), (10110),
(00010), (00011), (00012), (00001), (00001), (00001), (00001), (00001), (00001),
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(00012), (00012), (00012), (00012), (00010), (00010), (00010), (00010), (00011),
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