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SPECIAL TOPICS

On the existence, linearity and stability of electrostatic hole structures in the Vlasov–Poisson plasma from the perspective of its three velocity-separated evolution equations

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Abstract

An overview of nonlinear electrostatic structures in a collisionless plasma is given, as described by its three Schamel-type evolution equations. Separated in the phase velocity, these equations are related to the three acoustic modes of a two-component plasma, namely ion acoustic, the slow electron acoustic, and the slow ion acoustic mode. In their derivation, a novel coupling method is used that combines the propagation part with the structural part of the coherent wave pattern, with the focus on the exact reproduction of the kinetic equilibrium structures of the Vlasov–Poisson (VP) system. This is where the two central elements of Schamel's equilibrium theory come into play, the nonlinear dispersion relation and the pseudo-potential. Various aspects such as existence, linearity, particle trapping scenario, non-negativity and stability are investigated and the corresponding fundamentals are conveyed. These include the correct understanding of the linear limit as distinct from the linear Vlasov limit and the alleviation of the positivity problem associated with the square root nonlinearity $\sqrt{\phi}\partial_x\phi$ by introducing appropriate pedestals for the electrostatic potential $\phi(x,t)$. A general proof for the existence of solitary ion hole solutions over the entire temperature range is presented: $0 < \theta = \frac{T_e}{T_i} < \infty$, which corrects and extends the more restrictive condition $\theta \leq 3.5$ used in the literature. Ion holes can therefore also exist for hotter electrons. The stability of a solitary electron hole, based on the S-equation, which focuses on a specific macroscopic structural behavior beyond kinetics, and a previous transverse but limited VP instability analysis, exhibits marginal longitudinal stability. The associated linear perturbations are in the form of the asymmetric shift eigenmode of a solvable Schrödinger problem. This finding of the possible dominance of the shift mode perturbation provides a new hint for the anticipated general kinetic proof of marginal stability and transverse instability of electrostatic structures under these conditions including undisclosed potentials.

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1 Introduction

Structure formation in plasma is probably the oldest and most addressed issue in plasma physics with beginnings dating back to the 1920 s with the work of Tonks and Langmuir (1929). Since then, a possibly four-digit number of works have been published. Therefore, it may surprise some readers that, despite its long history, the topic of structure formation has remained interesting and relevant and that there is still much to discover. The crucial point is that the current understanding of electrostatic structures with regard to excitation and description requires revision since it is characterized by the inappropriate application of two fundamental works: the work of Landau and the work of Bernstein, Greene and Kruskal. While the use of both has contributed significantly to the understanding of structures and, not entirely unjustly, enjoy an iconic status, one should not overestimate and apply them where their expertise is no longer valid or where a better, more comprehensive solution exists, respectively.

The first misuse, which refers to Landau's work (Landau 1946), is based on the idea that electrostatic structures, albeit nonlinear, arise from the linear Vlasov approach and owe their excitation to a linear two-stream instability. In such a concept nonlinear particle trapping is taken into account only at larger amplitudes of the electric wave potential $\phi(x)$, i.e. there is a kind of threshold below which linear Vlasov theory can be taken granted. This image still haunts most publications. But it is, of course, only justified if the fundamentals of Landau theory are valid, which requires a sufficiently quiet plasma. However, the latter is not the case. A typical collisionless plasma, whether in space or in fusion devices, is a plasma driven by particle and wave injections and interactions. It is therefore inherently turbulent and exhibits a non-negligible degree of fluctuations (keyword: intermittent turbulence), which are outside the scope of Landau theory.

To avoid misunderstandings, we would like to emphasize that we are not questioning Landau's theory per se, but merely pointing out that it is advisable to know and apply its limitations. In a subcritical, current-carrying plasma, for example, any non-zero perturbation would vanish if one naively applied Landau's scenario. However, numerical and experimental observations show that an excitation generated by a seed in a fluctuating plasma doesn't damp and survives as a hole structure. For this, the assumptions of Landau's theory are clearly not met.

While we don't claim the work of Bernstein, Greene, and Kruskal was flawed, a misconception may arise from their work (Bernstein et al. 1957) and the erroneous assumption that their macroscopic specifications $(\phi(x), v_0)$ are sufficient to capture the observed structural variety.

Since the BGK method has already been extensively discussed in the literature, it suffices to identify and discuss its three main shortcomings.

(1) The BGK method essentially determines the potential profile $\phi(x)$ of the wave structure sought $\phi(x - v_0 t)$ in a self-consistent manner, but not its phase velocity v_0. The latter is imposed from the outside, but without a criterion that allows a distinction between physical and non-physical solutions. In this sense, it defies a self-consistent determination in contrast to Schamel's pseudo-potential method, where, on the microscopic side where the physics takes place, control over the distribution(s) of the trapped particles can be achieved through the concept of trapping scenarios. For this reason, the BGK theory lacks an important component to be a complete theory. (It is therefore not surprising that researchers who relied on the BGK wave picture had little access to velocity related concepts such as slow acoustic modes, nonlinear dispersion relations, negative energy modes, evolution equations involving particle trapping, or trapping scenarios that enable speed control.)
(2) The function space available for BGK modes is already very extensive, since in principle any known, meaningful function for $\phi(x)$ can be assumed. Nevertheless, the so-called "undisclosed potentials" are missing, which, as the pseudo-potential theory shows, constitute the vast majority of potentials. They are characterized by the fact that the inverse function $x(\phi)$, which is obtained by use of the pseudo-potential method, can generally no longer be expressed analytically by known functions, which is reflected in $\phi(x)$.
(3) For slow structures when both electrons and ions experience particle trapping simultaneously, the BGK method offers no solution at all, as it is designed for only one species of particle trapping.

Another difference between the world of Landau and BGK and that of the actual plasma behavior coupled with the pseudo-potential method deserves to be mentioned in advance: the existence of a second linear wave limit, resulting from particle trapping, which implies that its harmonic single wavenumber hole solutions violate Landau's theory. These solutions prove to be marginally stable in a current-carrying plasma, independent of the drift velocity v_D between electrons and ions (see below for further details). Therefore, to capture the actual dynamic behavior of hole structures in plasmas, these two linearities must be recognized and distinguished from one another.

It is hence the pseudo-potential method developed by one of the current authors in 1972, (Schamel 1972), that offers the way out of this dilemma. It focuses on the description of a coherent structure in the advanced nonlinear stage, when the particle trapping process necessary for its existence is more or less completed. For this purpose, the distribution function is given as a solution to the stationary Vlasov equation. This avoids or circumvents, respectively, the intricate excitation process, which cannot be treated mathematically anyway due to the complex (trapped particle) resonant dynamics and the associated intrinsic stochasticity (more on this later).

An abbreviated version of this pseudo-potential method is presented in "Appendix A".

A comprehensive overview of this method can be found in the review article "Pattern formation in Vlasov–Poisson plasmas beyond Landau caused by the continuous spectra of electron and ion holes" published in Rev. Mod. Plasma Phys. 2023 (Schamel 2023).

Our current article can be seen as an extension and in-depth study of this overview.

We present a new proof of existence, reinterpret the linear limit in a concise way, confirm the three evolution equations by a new coupling procedure, and open a new track to treat the still unsolved stability of a solitary wave.

Although we have tried to make the document self-explanatory and have repeated its main ingredients in a considerable detail, the reader may still have some doubts or uncertainties. In this case, in order not to expand the scope of this paper, we would like to ask the reader to consult the above-mentioned review paper for further clarification. And what should not be forgotten in all this fine-tuning is that, from a physical and technical point of view, structure formation in collisionless plasmas is a process that has a significant influence on particle and energy transport, as numerous observations in space, fusion, and laboratory plasmas have shown (Holmes et al. 2018; Ergun et al. 2024; Zhang et al. 2024; Kar and Mukherjee 2013).

Our paper is organized as follows. In Sect. 2 we mention some innovations which have been made so far by the Schamel method (S-method), and Sect. 3 refers to some achievements obtained numerically with the Schamel equation (S-equation). Section 4 describes some features of pattern formation, its descriptions, foundations, and history. Section 5 describes how to obtain a dark soliton via a negative pedestal, while Sect. 6 derives the formulas for the three acoustic phase velocities in the long-wavelength limit. In Sect. 7, we present a new proof for the existence of solitary ion holes, which has not been found in the literature so far. Using a new coupling method, we generally rederive the S-equation in Sect. 8 and apply it to determine the S-equation for the three acoustic modes in Sects. 9–11. Section 12 addresses the stability of structures within the S-equation with an outlook on their microscopic stability behavior. With a Summary and Conclusions in Sect. 13 and two Appendices we conclude the paper.

2 Innovations obtained by the Schamel method

The progress made since then with this theory has been documented in several review articles (Schamel 1979, 1982c, 1986, 2000; Korn and Schamel 1996b; Luque and Schamel 2005; Eliasson and Shukla 2006) and some of its innovations are listed here again for the memory and information of the readership, respectively.

1. Foundation of the Schamel method—1972 (Schamel 1972).
2. Modified Korteweg de Vries (Schamel) equation—1973 (Schamel 1973).
3. Small amplitude solitary electron holes—1979 (Schamel 1979b) and ion holes—1980 (Schamel and Bujarbarua 1980).
4. Its generalization for finite amplitudes—1981 (Bujarbarua and Schamel 1981).
5. Analytical evidence of a strong double layer with missing small-amplitude limit—1983 (Schamel and Bujarbarua 1983).
6. Analysis of the stability of solitary electron and ion holes (Schamel 1982a, b) (more on this later).

7. Inclusion of a drift v_D for the treatment of current-carrying plasmas (Maslov and Schamel 1993; Schamel and Maslov 1994; Korn and Schamel 1996a, b).
8. Theory of current-free double layers (Goswami et al. 2008).
9. Consistent derivations of the energy of a structured plasma (Schamel 2000; Griessmeier et al. 2002a; Griessmeier and Schamel 2002; Luque and Schamel 2005; Das et al. 2018; Schamel 2023).
10. Demonstration of increased resistivity in the presence of hole structures and dissipation (Korn and Schamel 1996b; Luque and Schamel 2005).
11. Ion mobility as a key to dissipative electron–hole equilibria in driven plasmas (Korn and Schamel 1996b; Luque and Schamel 2005).
12. Spectral Fourier-Hermite code for optimal numerical hole treatment (Korn and Schamel 1996a, b; Luque and Schamel 2005).
13. Proof of the failure of the linear Landau theory for harmonic hole trains in the infinitesimal amplitude limit (Schamel 2018a).
14. Consistent description of coherent structures in circular particle accelerators such as synchrotrons and storage rings (Schamel 1997, 1998; Schamel and Fedele 2000; Griessmeier et al. 2002b; Schamel and Luque 2004; Blaskiewicz et al. 2004; Luque and Schamel 2005).
15. Quantum tunneling of trapped hole electrons via the Wigner–Moyal formalism (Luque et al. 2004; Luque and Schamel 2005).
16. Holes for different background distributions: non-extensive kappa, Fermi–Dirac and arbitrary distributions (Tribeche et al. 2012; Schamel and Eliasson 2016; Schamel 1975).
17. Electron and ion holes of opposite polarity (Schamel 2023; Schamel and Chakrabarti 2023b)
18. Multiple trapping scenarios to account for ergodic particle trajectories in the vicinity of separatrices (Schamel 2023).
19. Undisclosed potentials, unlimited diversity and loss of identifiability of hole structures (Schamel 2020a, b; Schamel et al. 2020a, b).
20. Imperfect electron and ion trapping scenarios (Das et al. 2018; Schamel 2015; Schamel et al. 2018)
21. Existence of a second-order electron hole of the Gaussian type: $e^{-sinh(x)^2}$ (Schamel 2020a, b).
22. Nonlinear proof of the "Resonance Broadening" by the Γ_s—trapping scenario, s = e,i, (Schamel 2020a; Schamel et al. 2020a).

3 Numerical achievements made with the Schamel equation

At about the same time as the establishment of the pseudo-potential method, HS discovered the corresponding evolution equation for ion-acoustic structures using reductive perturbation theory, which he called the modified KdV equation (Schamel 1973). This was then called the Schamel equation by Verheest and Hereman (1994) to make it more precise and to distinguish it from other modified KdV equations. Since then it has been known as the Schamel equation (S-equation) and has become an essential framework for exploring acoustic wave dynamics in plasma systems (Schamel 1973; Ali et al. 2017; Chowdhury et al. 2018; Mushtaq and Shah 2006; Williams et al. 2014; Saha and Chatterjee 2015a, b) and recently in metamaterials (Mogilevich and Popova 2023; Zemlyanukhin et al. 2019), damping systems (Shan 2019; Sultana and Kourakis 2022) and electrical circuits (Aziz et al. 2020; Kengne et al. 2020).

The S-equation is frequently compared to the well-known Korteweg–de Vries (KdV) equation (Zabusky and Kruskal 1965), as both share the same dispersion relation, and reads in a canonical form as

$$\phi_t + \sqrt{\phi}\phi_x + \phi_{xxx} = 0, \tag{1}$$

where $\phi(x,t)$ is the electrostatic potential of the structure and $\phi_{x,t}$ stands for $\frac{\partial \phi}{\partial x,t}$. However, to address practical considerations, particularly in numerical studies, a modified version known as the modular Schamel equation has been suggested (Schamel 1973; xxx yyy),

$$\phi_t + \sqrt{|\phi|}\phi_x + \phi_{xxx} = 0. \tag{2}$$

From a mathematical standpoint, the S-equation presents unique challenges due to its square-rooted nonlinear term and the modulus nonlinear term. These features complicate both analytical and numerical treatments. Additionally, unlike integrable equations such as the Gardner equation or the modified Korteweg–de Vries (mKdV) equation, the S-equation lacks integrability. This non-integrability introduces further complexity when attempting to derive exact solutions or apply standard analytical techniques commonly used in plasma physics studies (Ruderman et al. 2008, 2023; Schamel 1972, 1973). Recent studies by Flamarion et al. (2023) and Didenkulova et al. (2023) have explored the interaction of solitary wave solutions in the context of the S-equation. Although the S-equation is nonintegrable, the collisions between solitary waves are nearly elastic, meaning that after interaction, the solitary waves largely retain their original form. However, Didenkulova et al. (2023) noted that in bipolar interactions, energy tends to transfer from the smaller wave (in modulus) to the larger one. This energy redistribution, combined with the dispersive tails generated during the collision, contributes to the formation of freak waves. A more detailed explanation of this freak wave formation mechanism was provided in a subsequent work (Flamarion et al. 2024, 2025).

The response of solitary waves to external forces has also been a focus of study within the framework of the S-equation. For instance, Chowdhury et al. (2018) derived a forced version of the S-equation, incorporating the effects of an external time-dependent force.

A damped/growing S-equation, in canonical form

$$\phi_t + \sqrt{\phi}\phi_x + \phi_{xxx} + C\phi = 0. \tag{3}$$

where C is a constant that has been introduced in the literature (Shan 2019; Schamel and Maslov 1994).

Sultana and Kourakis (2022) later employed this equation in their analysis of electrostatic potential, examining the nonlinear characteristics of dissipative ion-acoustic solitary waves in the presence of trapped electrons.

4 Some characteristic features of pattern formation in VP plasmas

In recent decades, it has been shown that the equilibrium theory presented in Schamel (2023), Schamel and Chakrabarti (2023a) is the appropriate, indeed the only method that can completely describe the spectrum of long-lived electrostatic structures in VP plasmas. The reason for this is that both aspects, shape and speed of a structure, are taken into account equally and only mathematically sound distribution functions are used; moreover, a veritable treasure trove of free parameters is available for structural adaptation. In this theory, the t-independent Vlasov equations for electrons and ions are solved exactly using the two concepts of constants of motion and trapping scenarios (TS), respectively. In this method self-consistency is achieved by using the pseudo-potential.

Under certain circumstances, however, a simpler description is available, namely when the structures move at one of the three acoustic speeds, which are the slow ion acoustic (SIA), the ion acoustic (IA) and the slow electron acoustic (SEA) speed. One can then resort to S-equations that are formulated in macroscopic space-time (x, t) instead of microscopic phase space (x, v, t) for electrons (or (x, u, t) for ions) and are thus predestined to describe the structural dynamics on a much simpler basis. This reduction reflects the mathematical realization of a strong convergence in macroscopic dynamics, whereas in microscopic dynamics only a weak convergence is found (Mouhot and Villani 2011).

As explained in more detail in the review article (Schamel 2023), the S-equation is an asymptotic evolution equation. In the case of a collisionless plasma, it describes the spatio-temporal behavior of electrostatic structures $\phi(x,t)$ after the violent particle trapping processes have already taken place. This means it only takes into account plasma states when they have entered into the "calmer waters", just before reaching the stationary equilibrium state. It is therefore not suited as a model that includes linear waves from fluid or linearized Vlasov descriptions because trapping effects are neglected in these approaches. It also fails of course to solve Cauchy's initial value problem of the Vlasov–Poisson system (VP system) in which $\{f_e(x,v,0), f_i(x,u,0), \phi(x,0)\}$ are prescribed as initial values. The latter is a mathematically intractable problem because stochasticity and non-integrability play a role during the trapping process; these are processes such as folding, trapping, detrapping, filamentation, etc., which occur in resonant wave-particle interaction in phase space resulting into a non-treatability. This means that all references to linear wave solutions, whether from the fluid description or the linear Vlasov description (Landau, van Kampen), can be forgotten. Our solutions have already passed through and left this early stage of evolution. However, linearization within the S-equation is still possible as explained later.

We return to the theory of equilibrium structures in VP systems (Schamel 2023) and consider a positive potential structure, which propagates in the laboratory frame with the velocity v_0, the phase velocity. It is based on the positive pedestal:

$$0 \leq \phi(x - v_0 t) \leq \psi \ll 1,$$

where ψ is the amplitude of the structure. The governing equations, whose derivation is repeated in a compact form for the interested reader in the "Appendix A" inclusively its normalizations, are given by

$$k_{0+}^2 - \frac{1}{2} Z_r' \left(\frac{v_0}{\sqrt{2}} \right) - \frac{\theta}{2} Z_r' \left(\frac{u_0}{\sqrt{2}} \right) = B_e + \frac{3}{2} B_i \theta^{3/2} \tag{4}$$

and

$$\begin{aligned} -\mathcal{V}(\phi) = \frac{k_{0+}^2}{2} \phi(\psi - \phi) + \frac{B_e}{2\sqrt{\psi}} \phi^2 \left(\sqrt{\psi} - \sqrt{\phi} \right) + \\ \frac{B_i \theta^{3/2}}{4\sqrt{\psi}} \left(\phi \sqrt{\psi} (3\phi - 5\psi) + 2 \left[\psi^{5/2} - (\psi - \phi)^{5/2} \right] \right), \end{aligned} \tag{5}$$

valid for a current-free $v_D = 0$ plasma. The first part refers to the nonlinear dispersion relation (NDR) that is the determining equation for the phase velocity v_0, the second part is the pseudo-potential that governs the structure $\phi(x)$ itself.

The system (4),(5) corresponds exactly to (24),(25) of Schamel (2000). The corresponding set of equations, (36),(37), formulated for a negative pedestal (see later), is presented in "Appendix B".

A detailed discussion of the existence and energy dependence of structures with respect to the three variables (k_{0+}^2, B_e, B_i) was presented by Luque and Schamel (2005) and revealed a variety of new patterns.

Figure 1 shows $-\frac{1}{2} Z_r'(x)$ as a function of x. Its zero point at $x = 0.924$ defines the slow acoustic phase velocities via the nonlinear dispersion relation, namely the electronic with $v_0 = 0.924\sqrt{2} = 1.307$ and the ionic with $u_0 := \sqrt{\frac{\theta}{\delta}} v_0 = 1.307$. Both corresponding mode structures can be macroscopically described by a Schamel equation.

Figure 2 shows the high-frequency electronic part of the NDR (4) with B:=B_e, indicating a multitude of new solutions besides theLangmuir and the slow electron-acoustic branch. See also Trivedi and Ganesh (2018), Schamel (2012, 2019).

The pseudo-potential (5) in its canonical form has the noteworthy property that it is independent of the phase velocity v_0 and thus cannot be used for its determination. This system represents periodic and cnoidal waves and has been thoroughly discussed as a two parametric system (k_{0+}, B_e) by Korn and Schamel (1996a). The potential structure, with the exception of the two solitary parts, is periodic in space and is characterized by the parameter $S = 4B_e/k_{0+}^2$ which lies in the interval $-8 \leq S \leq \infty$, i.e. between the bounds $S = -8$ and $S = \infty$. It is hence embedded

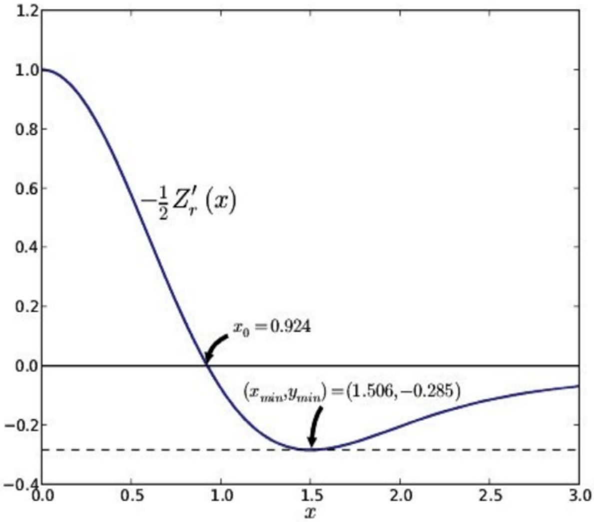

Fig. 1 The quantity -$\frac{1}{2}Z_r'$ as a function of x, where $Z(z)$ is the complex plasma dispersion function

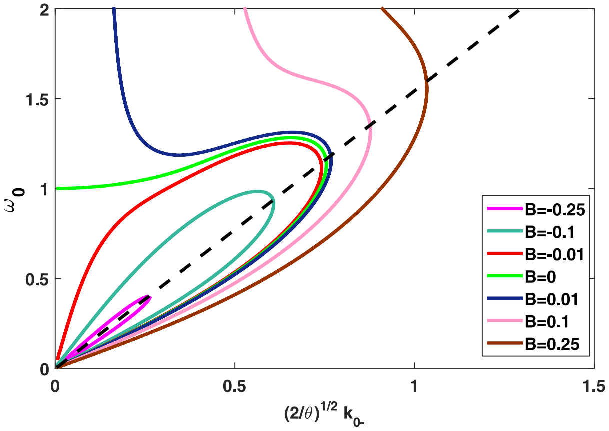

Fig. 2 The high-frequency electronic part of the NDR (4) with $B := B_e, \theta = 0 = B_i, \omega_0 := k_{0+}v_0$

between a solitary hump ("bright soliton"), when $k_{0+} = 0$, and a solitary dip ("dark soliton"), when $k_{0+}^2 = -B_e/2 > 0$. The $k_{0+} = 0$ solitary wave is the well-known:

$$\phi(x) = \psi \operatorname{sech}^4\left(\frac{\sqrt{B_e}}{4}x\right) \tag{6}$$

whereas the solitary dip solution, shown in Fig. 3 of Schamel (2023) or Fig. 1 of Schamel and Chakrabarti (2023a), is more complicated.

Of special interest thereby is the single harmonic wave solution which lies in between at S = 0 and is given by:

$$\phi(x) = \frac{\psi}{2}(1 + \cos k_{0+}x) \geq 0 \tag{7}$$

It refers to $B_e = 0$ and is the linearized solution within Schamel's theory, since the trapping nonlinearity, represented by the B_e term, vanishes.

This linear wave solution, which is lifted by $\psi/2$ in comparison to an ordinary sinusoidal wave, is therefore strictly non-negative like all structures are. It has, as said, nothing to do with the ordinary linear waves stemming from linearized Vlasov–Poisson system (or fluid system). The reason is that the electron distribuition $f_e(x - v_0t, v)$ is still nonlinearly distorted within the resonant trapped electron range even when $B_e = 0$.

This applies to the three S-equations, as well, since they represent these equilibria in the special case of the three acoustic phase velocities, namely when $v_0 \approx \sqrt{\delta}$ (the ion acoustic wave limit, IAW), when $v_0 \approx 1.307$ (the slow electron acoustic wave limit, SEAW), and when $u_0 \approx 1.307$ (the slow ion acoustic wave limit, SIAW), where δ is the mass ratio m_e/m_i.

5 The electronic dark soliton and the negative pedestal

In a short insert, we point out that sometimes a negative pedestal is useful for non-positive structures

$$-1 \ll -\psi \leq \hat{\phi}(x - v_0t) \leq 0.$$

An example is the ion hole, where ion trapping plays the crucial role. This mode rests on the slow ion acoustic velocity, $u_0 = 1.307$, and the solitary wave potential assumes the form $\hat{\phi}(x) = -\psi\,\mathrm{sech}^4\left(\frac{\sqrt{B_i}}{4}x\right)$, as first developed in Schamel and Bujarbarua (1980). In the present case, it is the electronic dark soliton that could benefit from this shift of $\phi(x)$ to $\hat{\phi}(x) := \phi(x) - \psi$, provided that the pseudo-potential and hence the S-equation take a simpler form in this new variable. To check this we first note that the pseudo-potential in ϕ becomes for the dark soliton:

$$-\mathcal{V}(\phi) = \frac{B_e}{4}\phi\left(3\phi - \psi - \frac{2}{\sqrt{\psi}}\phi^{3/2}\right)$$

which translates into

$$-\mathcal{V}(\hat{\phi}) = \frac{B_e}{2}(\hat{\phi} + \psi)\left[\frac{3}{2}\hat{\phi} + \psi - \frac{1}{\sqrt{\psi}}(\hat{\phi} + \psi)^{3/2}\right]$$

written in the new dependent variable $\hat{\phi}$. However, as it turns out, our hope is not fulfilled. The complexity of $\phi(x)$ in the case of electronic dark solitons also carries over to the new expression, which has not become much simpler. The complexity remains and is thus confirmed.

Note that the equivalence of both pedestals was proven in Schamel (2000). Of course, one can also use an intermediate pedestal, such as $-\psi/2 \leq \tilde{\phi} \leq \psi/2 << 1$ with $\tilde{\phi}$ as the new dependent variable. However, this does not increase the attractiveness of the equations and formulas.

6 The three acoustic solutions of the NDR

As mentioned above, the NDR has three acoustic solutions: the ion acoustic wave solution (IAW), the slow electron acoustic wave solution (SEAW) and the slow ion acoustic wave solution (SIAW).

The IAW is obtained by using the approximation

$$-\frac{1}{2}Z_r'\Big(\frac{v_0}{\sqrt{2}}\Big) \approx 1 - v_0^2 + \cdots \tag{8}$$

$$-\frac{1}{2}Z_r'\Big(\frac{u_0}{\sqrt{2}}\Big) \approx -\frac{1}{u_0^2}\Big(1 + \frac{3}{u_0^2} + \cdots\Big) \tag{9}$$

and results in

$$v_0 = \sqrt{\delta}(1 - K_1),\ \text{where}\ K_1 = \frac{k_{0+}^2}{2} - \frac{3}{2\theta} - \frac{\delta}{2} - \frac{B_e}{2} - \frac{3}{4}\theta^{3/2}B_i, \tag{10}$$

v_0 and K_1 are assumed to be small.

The SEAW uses, assuming $v_0 \sim 1.307$,

$$-\frac{1}{2}Z_r'\Big(\frac{v_0}{\sqrt{2}}\Big) \approx \frac{1.307 - v_0}{1.307};\ \ -\frac{1}{2}Z_r'\Big(\frac{u_0}{\sqrt{2}}\Big) \approx \frac{-\delta}{\theta v_0^2};\ \ u_0 = \sqrt{\frac{\theta}{\delta}}v_0 \text{ and } \theta = T_e/T_i, \tag{11}$$

and yields

$$v_0 = 1.307(1 + K_2),\ \text{where}\ K_2 = k_{0+}^2 - \frac{\delta}{1.307^2} - B_e. \tag{12}$$

Since $v_0 \sim 1.307$ and thus $u_0 \sim \sqrt{\frac{\theta}{\delta}}1.307$, the ion trapping term in (4) is negligible from the outset, since $\theta^{3/2}B_i \sim \theta^{3/2}e^{-u_0^2/2} \sim \theta^{3/2}exp(-\frac{1.71\theta}{2\delta}) \ll 1$.

The SIAW, which is the slowest, uses (8) with $v_0 = 0$ and the expansion of $-\frac{1}{2}Z_r'\Big(\frac{u_0}{\sqrt{2}}\Big)$ around $u_0 = 1.307$ and becomes

$$u_0 = 1.307(1 + K_3),\ \text{where}\ K_3 = \frac{1 + k_{0+}^2 - B_e}{\theta} - \frac{3}{2}\sqrt{\theta}B_i. \tag{13}$$

We note that using the Taylor expansion requires $|K_3| << 1$, which in turn implies $\theta >> 1$. Our analytical expression for u_0 as a solution of the NDR, as we will use it later, therefore requires $\theta >> 1$.

7 Existence of solitary ion holes propagating at SIAW speed for arbitrary $\theta = \frac{T_e}{T_i}$

However, this does not mean that $\theta >> 1$ must always hold for structures propagating at the SIAW speed. The point is that the NDR also admits solutions that cannot be obtained via a Taylor expansion and therefore require a deviation from $|K_3| << 1$, allowing $\theta \approx O(1)$. The proof which is exemplarily performed for solitary ion holes at isothermal electrons, $k_{0-} = 0 = B_e$, see (36), is as follows. (We prefer the description for a negative pedestal in "Appendix B" and use $v_0 = 0$.)

The NDR to be solved is given by: $1 - \frac{\theta}{2} Z_r'\left(\frac{u_0}{\sqrt{2}}\right) = B_i \theta^{3/2}$.

Since the right hand side has to be positive it holds $-\frac{1}{2} Z_r'\left(\frac{u_0}{\sqrt{2}}\right) > -\frac{1}{\theta}$. Taking the minimum of the left hand side, which is -0.285, we get $\theta \leq 3.5$ as the preferential existence region. This does however not imply that higher θ values, being above 3.5 or even $\theta >> 1$ are excluded, as can be seen by the following more detailed investigation.

We take a point at the negative SIAW branch of $-\frac{1}{2} Z_r'\Big(x\Big)$ as for example the minimum point: $(1.506, -0.285)$, see Fig. 1. If we define $\mu := B_i \theta^{3/2}$ then we get $0.285\theta = 1 - \mu$ or $\theta = 3.5(1 - \mu)$. By insertion into the definition of μ we find a relation between μ and B_i given by $B_i(\mu) = \mu\Big(3.5(1-\mu)\Big)^{-3/2} > 0$. A solution exists for $0 < \mu < 1$, and given μ, then θ follows from $\theta = (\mu/B_i)^{2/3} = 3.5(1 - \mu)$. This corresponds to the previous, too rough approximation.

But if we instead chose a value of $-\frac{1}{2} Z_r'\Big(x\Big) = -\epsilon$, with $0 < \epsilon \leq 0.285$, which is the negative part of the SIAW branch, then $B_i(\mu) = \epsilon^{3/2} \frac{\mu}{(1-\mu)^{3/2}}$, which again exists between $0 < \mu < 1$ with an asymptote at $\mu = 1$. It then holds $\theta = \frac{1-\mu}{\epsilon}$, which, for given μ, can become very large as $\epsilon \to 0$, the approximation we used in the paper.

We conclude that ion holes, as a solution of the NDR, can be found at arbitrary values of $\theta = \frac{T_e}{T_i}$ which include regions of hot ions as well as hot electrons. The restriction $\theta \leq 3.5$ for ion holes predicted in Chen et al. (2004) is thus lifted and invalidates the issue in the debate between (Hutchinson 2023) and Schamel and Chakrabarti (2023b).

If we are not mistaken, it is these ion hole-like structures that propagate at the nonlinear slow ion acoustic speed (and not at the usual ion acoustic speed) that have been detected in large quantities in the solar wind (Ergun et al. 2024). These refer to regions with essentially hot ions and include regions with Boltzmann electrons ($\beta = 1$) or electrons with a flattened distribution ($\beta = 0$), but also regions in which a positive slope of the ion distribution function is no longer necessarily present. We point out, however, that the explanations given by the authors (Ergun et al. 2024) require at least an update, since they are essentially based on the linear Vlasov theory adapted to a changed background, and thus ignore the fact that saturated holes require a completely different function space, modified exclusively by particle trapping (Schamel 2023). In more or less turbulent real plasmas, holes are triggered by seed fluctuations and predominantly persist over a longer period of time only in linearly stable plasmas. The emission of linear waves by holes, for example during acceleration, and their subsequent growth in linearly unstable plasmas prevent the formation of holes as equilibrium states, see Schamel (2023) and unpublished work of Mandal et al. (2018).

8 An alternative, generalized access to the Schamel equation

The new concept for the derivation of the S-equation, which represents the entire spectrum of wave structures in the three acoustic limits, has been introduced in Schamel (2020a). It consists in the Ansatz

$$\Big(\phi_t + v_0\phi_x\Big) + c\Big[-\mathcal{V}''(\phi)\phi_x - \phi_{xxx}\Big] = 0 \tag{14}$$

and arises from the fact that in equilibrium $\phi(x - v_0 t)$ both expressions $(\cdot)$ and $[\cdot]$ vanish identically. The first is obvious and relates to the propagational part of the coherent wave pattern, the second arises from the x-derivative of Poisson's equation and involves its structural part, nonlinearity and dispersion. We hence have added two terms which vanish in equilibrium. Appropriate space-time dependent solutions should therefore not deviate too much from equilibrium; to what extent, however, remains an open question.

To justify this derivation, it can be said that it reproduces the usual results of the reductive perturbation method when the latter is feasible, but has a much wider range of applications when, for example, suitable scaling properties are no longer available. This is the case for the majority of structures namely when further trapping scenarios (not treated in the present article) are in action leading to the so-called undisclosed potentials. Therefore, this derivation here is much broader applicable from the outset.

In Eq. (14), v_0 has to be taken up to first order, as derived in (10), (12) and (13), whereas the coupling constant c depends on the chosen case. In principle, c can be any constant. However, to represent a dynamic evolution and not just the equilibrium, c must be chosen appropriately. A proper choice for the correct coupling constant c is that in the harmonic limit ($B_e = 0 = B_i$) the dispersion effect is exclusively represented by the term ϕ_{xxx}. The correct implementation can then be recognized by the fact that k_{0+}^2 no longer appears in the equation.

9 The ion acoustic case ($T_e >> 3.5T_i$)

In the present ion acoustic case this results in $c = -\frac{\sqrt{\delta}}{2}$.

We note furthermore that by applying (37), it generally holds that

$$-\mathcal{V}''(\phi) = -k_{0+}^2 + B_e\left(1 - \frac{15}{8}\sqrt{\frac{\phi}{\psi}}\right) + B_i\theta^{3/2}\left(\frac{3}{2} - \frac{15}{8}\sqrt{1 - \frac{\phi}{\psi}}\right) \quad (15)$$

such that (14) becomes with $c = -\frac{\sqrt{\delta}}{2}$

$$\frac{1}{\sqrt{\delta}}\phi_t + \left(1 + \frac{15}{16}\frac{B_e}{\sqrt{\psi}}\sqrt{\phi}\right)\phi_x + \frac{1}{2}\phi_{xxx} = 0, \quad (16)$$

which has the form of (1) in the frame traveling with ion acoustic velocity $(x - \sqrt{\delta}t)$. Since B_i is extremely small in the case $\theta >> 1$, namely $B_i \propto \theta e^{-\theta/2}$, we have neglected its contribution $(\frac{15B_i\theta^{3/2}}{16}\sqrt{1 - \frac{\phi}{\psi}})\phi_x$ to (16). This evolution equation corresponds exactly to the original equation (15) of Schamel (1973), its derivation is therefore equivalent to the reductive perturbation method used there.

It is an extended version of the S-equation and applies for the whole spectrum of waves $-8 \leq S := \frac{64B_e\sqrt{\psi}}{15k_{0+}^2} \leq \infty$. It thus includes the bright soliton, when $k_{0+}^2 = 0$ or $S = \infty$, and the dark soliton, when $S = -8$, as well as the single harmonic wave, when $S = 0$ or $B_e = 0$. It coincides with the S-equation (15) of Schamel (2023) for the bright soliton and with (39) of Schamel (2023) for the dark soliton.

There is a difference to (9) in Schamel and Chakrabarti (2023a) since the k_{0+}^2 term is now absent. The reason that in Schamel and Chakrabarti (2023a) only $v_0 = \sqrt{\delta}$ was used rather than the full expression including K_1. In the corrected version, single harmonic waves show dispersion in the ion acoustic limit.

10 The slow electron acoustic case ($B_i = 0$)

Using v_0 from (12), applying the same selection procedure for c as before, which yields $c = 1.307$, and neglecting the small term proportional to δ, we obtain from (14)

$$\frac{1}{1.307}\phi_t + \left(1 - \frac{15}{8}B_e\sqrt{\frac{\phi}{\psi}}\right)\phi_x - \phi_{xxx} = 0. \quad (17)$$

This slightly modified *S*-equation is again identical to the previous results, such as (5) and (14) of Schamel and Chakrabarti (2023a) for the bright soliton and (29) of Schamel and Chakrabarti (2023a) for the dark soliton. In the harmonic case, we get back (12). We emphasize that the trapping nonlinearity and the dispersion effect in (16), (17) occur in the same way.

A generalization of (17) was presented in Schamel (2020a), given by (19) and (A4), respectively, for which two further electron trapping scenarios are responsible.

11 The slow ion acoustic case ($T_e >> 3.5T_i$)

Finally, for this slowest hole structure, we get by applying (14) with $c = \frac{\sqrt{\delta}}{\theta^{3/2}}1.307$ and using u_0 from (13) the following S-type evolution equation

$$\frac{\sqrt{\theta}}{1.307\sqrt{\delta}}\phi_t + \Big(1 + \frac{1}{\theta} - \frac{15}{8\theta}B_e\sqrt{\frac{\phi}{\psi}} - \frac{15}{8}B_i\theta^{1/2}\sqrt{1 - \frac{\phi}{\psi}}\Big)\phi_x - \frac{1}{\theta}\phi_{xxx} = 0. \quad (18)$$

Its general equilibrium solution is again typically periodic fitted to a positive pedestal and is determined by the two nonlinearities, the electronic $\sqrt{\frac{\phi}{\psi}}$ and the ionic $\sqrt{1 - \frac{\phi}{\psi}}$. It depends in general on (θ, B_e, B_i) and belongs to the class of undisclosed potentials, since the integral (8) of Schamel (2023), which leads to $x(\phi)$, cannot be solved analytically.

As a limiting subset, a two-parametric bright soliton exists when $k_{0+}^2 = 0$ and $B_e > \theta^{3/2}B_i$, which extends the ordinary sech^4 solitary electron hole by the α -ion trapping effect. A two-parametric dark soliton, on the other hand, exist as well. It requires $2k_{0+}^2 + B_e - B_i\theta^{3/2} = 0$ and $B_e < \theta^{3/2}B_i$ and extends the ordinary $-\operatorname{sech}^4$ ion hole by the β-electron trapping effect. In the special case of Boltzmann electrons, when $B_e = 0$ or $(\beta = 1, v_0 = 0)$, we find $u_0 = 1.307(1 + \frac{1}{\theta} - B_i\sqrt{\theta})$ and $-\mathcal{V}(\hat{\varphi}) = \frac{\theta^{3/2}B_i\psi^2}{2}\hat{\varphi}^2[1 - \sqrt{-\hat{\varphi}}]$ with $\hat{\varphi} := \frac{\phi-\psi}{\psi}$ and hence get back the normal $-\operatorname{sech}^4$ solitary ion hole (Schamel and Chakrabarti 2023a).

We emphasize that the restriction to $\theta >> 3.5$ does not mean that there are no hole solutions with $\theta < 3.5$. However, an S-equation is only derivable for hot electrons.

An important consequence of (18) is that both solitons, the bright and the dark, cannot exist simultaneously, since they have a mutually exclusive range of validity. An interaction between the two does not occur and therefore cannot be analyzed numerically by a single equation. What might interest us, however, is the influence of the ions B_i on the bright soliton and the electrons B_e on the dark soliton. For this reason, we generalize the equation and rewrite it as follows

$$\phi_\tau + \Big(1 + \frac{1}{\theta} - \mathtt{b}_e\sqrt{|\phi|} - \mathtt{b}_i\sqrt{|\psi - \phi|}\Big)\phi_x - \frac{1}{\theta}\phi_{xxx} = 0, \quad (19)$$

where we defined $\tau := \frac{1.307\sqrt{\delta}}{\theta^{3/2}}t$, used the modular extension and introduced the ψ -independent constants $\mathtt{b}_e := \frac{15B_e}{8\theta\sqrt{\psi}}$ and $\mathtt{b}_i := \frac{15B_i\theta^{1/2}}{8\sqrt{\psi}}$. While in (16),(17) ψ effectively disappeared, in (19) it now appears explicitly, which presents a new numerical challenge.

In a potentially sophisticated numerical simulation that could be groundbreaking by considering both trapping scenarios simultaneously, in the case of a single maximum of $\phi(x)$, one could allow ψ to be weakly time-dependent and replace it at each time step by the current maximum value of the evolving perturbation to account for the time-dependent changes in phase velocity and trapping. Physically, this could be explained by the fact that evolution occurs extremely slowly such that the system goes through a series of quasi-equilibria. One possible application is the splitting of an initially bell-shaped potential profile into a sequence of successively increasing solitons in order to observe the effect of the ion trapping (Schamel 1973). Another application is the time-limited acceleration of a bright soliton, or at least its initial development, discovered in Schamel et al. (2017, 2020a, 2020b),Mandal et al. (2018, 2020). These numerical Vlasov simulations show a transition of a bright SIA soliton into a bright IA soliton, where the final velocity is independent of the initial velocity of the seed-initiated SIA soliton. The latter indicates a universal character of evolution and can be interpreted as an indication that the soliton tends to a state of lower energy through acceleration, acting like an attractor (Schamel 2023). This potentially releases energy for the emission of linear Vlasov waves, which lead to increased plasma destabilization and intensification of intermittent turbulence, especially in a linear unstable two-stream plasma. In these simulations, the importance of the ionic species was manifest, and to identify the structure, a second trapping scenario, the Γ_e scenario, was necessary to explain the observed central dip in the perturbed electron density and the shoulders of the trapped electron distribution near the separatrix (Schamel et al. 2020a). During acceleration, the structure hardly changed its shape and strength.

It may hence be necessary to add in this expected new simulation, besides a non-zero drift velocity, this second trapping scenario Γ_e that affects v_0 but not $\mathcal{V}(\phi)$ (Schamel 2000, 2023; Schamel et al. 2020a and perhaps a third non-perturbative scenario, the D_e scenario, that accounts for single Gaussian holes (Schamel et al. 2020b).

This type of single-hole acceleration was also the subject of a recent numerical study of Guillevic et al. (2025). Using a Schamel-type electron distribution function and a sech^4 profile, an acceleration of the structure was observed in the linearly stable regime of a $\delta = \frac{1}{30}$, current-carrying, $\theta = 1$ "plasma". However, the observed acceleration was accompanied by an increase of the phasetrophy (Diamond et al. 2010, 2011) and of the amplitude and width of the structure, which differs from the previously described simulations. It must be noted that the two cases are not fully comparable due to the different parameter regimes. Nevertheless, uncertainties regarding stability remain and need to be investigated.

Ultimately, however, only a direct comparison with kinetic simulations at the microscopic level will show to what extent these approaches with possible multiple trapping scenarios can represent the real structural behavior controlled by chaotic particle trapping.

12 Stability

The stability of trapped particle equilibria is within the microscopic VP system of equations a mathematically delicate problem, since it remains unsolved even as a linear stability theory. An exception are single harmonic structures, which turn out to be linearly marginally stable and independent of the drift velocity v_D in a current-carrying plasma (Schamel 2018a, 2023), a fact that, incidentally, contradicts the Landau analysis of linear Vlasov–Poisson plasmas. There are two main reasons for this failure. First, most authors did not have a clean, internally consistent equilibrium solution $(\phi(x), v_0)$ at their disposal and second, the resulting eigenvalue problem for the linear perturbation $\phi_1(x,t)$ remained unsolved. Only by a mathematically unproven truncation of the Taylor series of the nonlocal spectral operator of infinite order after the second term could further progress be made. This approach, called fluid limit (Lewis and Symon 1979), was exploited by the present author of this study to derive a linear stability theory for solitary structures (Schamel 1982a, 1986, 2023), which is exact within its limitation.

It is therefore interesting to see how the achieved marginal stability is also reflected in the S-equation and how its solution can be classified in relation to the overall unknown solution of the problem. If we consider the bright soliton solution (6) as the simplest, non-trivial form of the unperturbed structure $\phi_0(x)$ together with v_0 from (10) and the S-equation for the ion acoustic case (16), we obtain for the first-order perturbation $\phi_1(x,t)$ in linear approximation

$$\frac{1}{\sqrt{\delta}}\phi_{1t} + \phi_{1x} + \frac{15B_e}{16\sqrt{\psi}}\Big(\sqrt{\phi_0}\phi_{1x} + \frac{\phi_{0x}}{2\sqrt{\phi_0}}\phi_1\Big) + \frac{1}{2}\phi_{1xxx} = 0. \tag{20}$$

The zeroth order solution for $\phi_0(x - v_0 t)$ is satisfied by the soliton solution (6) itself with v_0 from (10). Using the new variables $\tau := \sqrt{\delta}t$, $\xi := \alpha x$ with $\alpha := \frac{\sqrt{B_e}}{4}$ we get the somewhat simpler pde

$$\phi_{1\tau} + \alpha\phi_{1\xi} + 15\alpha^3\Big(\sqrt{\frac{\phi_0}{\psi}}\phi_1\Big)_\xi + \frac{1}{2}\alpha^3\phi_{1\xi\xi\xi} = 0. \tag{21}$$

Next we show that a steady-state, co-propagating $\phi_1(\xi,\tau) = \varphi\Big(\alpha(x - v_0 t)\Big) = \varphi\Big(\xi - \alpha(1+8\alpha^2)\tau\Big)$ is a solution of this equation where we used the simplest nontrivial form of v_0 namely $v_0 = \sqrt{\delta}(1 + B_e/2)$. By insertion we find

$$-8\varphi_\xi + 15\Big(\sqrt{\frac{\phi_0}{\psi}}\varphi\Big)_\xi + \frac{1}{2}\varphi_{\xi\xi\xi} = 0, \tag{22}$$

which can immediately be integrated, assuming $(\varphi, \varphi_{\xi\xi})$ vanish as $|\xi| \to \infty$, and get

$$\varphi_{\xi\xi} - 16\varphi + 30\,\mathrm{sech}^2(\xi)\varphi = 0. \tag{23}$$

We hence arrived at a Schrödinger eigenvalue problem (Schamel 1982a, b)

$$\Lambda\eta_n := \alpha^2\Big(\partial_\xi^2 - 16 + 30\,\mathrm{sech}^2(\xi)\Big)\eta_n = -\lambda_n\eta_n, \tag{24}$$

which has five discrete eigenstates. The lowest order two are given by the symmetric ground state $\eta_0(\xi) = \operatorname{sech}^5(\xi), \lambda_0 = -9\alpha^2$ and the asymmetric first excited state $\eta_1(\xi) = \operatorname{sech}^4(\xi)\tanh(\xi), \lambda_1 = 0$. The latter is created by a simple shift of the original structure and is also known under Goldstone mode. Its is easily seen that our solution corresponds to the shift or Goldstone mode.

We can therefore conclude that a solitary electron hole (or bright soliton) exhibits marginal stability when derived from the S-equation. The undamped perturbation in this approach corresponds exactly to the first excited state, the shift mode, in Schamel's kinetic theory of restricted transverse instability (Schamel 1982a), where, by applying the ground state, instability was found to be present only transversely. The use of this shift mode η_1 instead of η_0 gives a new relevance to this analysis, although it is actually based on an unproven approximation, the truncation or fluid approach (Lewis and Symon 1979), in the generally still unsolved kinetic VP-linear stability problem (Schamel 1982a). We mention that such an asymmetric shift mode was also recently observed in a PIC simulation (Hutchinson 2018, 2019). However, in contrast to the author, we see neither a justification nor a need for a new stability analysis dealing with the "kinetic jetting of marginally passing electrons", since regularly passing particles are naturally taken into account in our analysis of the transverse instability.

We note that our result is in agreement with the previous macroscopic marginal stability analyses of solitons by Kuznetsov (1984) and of periodic structures by Bronski et al. (2016) in KdV-like equations.

Finally, we would like to point out, admittedly somewhat speculatively, the universal character of marginal stability of electrostatic structures in general or at least for this particular shift mode. The reason for this is that marginal stability could also be seen microscopically, namely for single harmonic wave structures (Schamel 2018a, 2023). However, to prove marginal stability for the entire spectrum of structures, including solitary and cnoidal waves, the solution of a non-local eigenvalue problem $\Big((5), (6)$ of Schamel (1982a); (18), (24) of Schamel (2018a) or (55) of Schamel (2023) $\Big)$ is required, which is still pending except for harmonic structures, where the marginally stable perturbation can be considered as a shift mode (non-validity of Landau approach!).

But maybe someone will come along who can prove the marginal stability for a general $\phi_0(x)$ including undisclosed potentials by using the shift mode in this eigenvalue problem as the eigenmode perturbation $\phi_1(x) = \phi_0'(x)$ and thus free the Schamel's restricted transversal instability theory from its limitation? A big advantage would be that the x-dependence in all subsequent formulas could be replaced by a ϕ_0 -dependence. This holds for the non-local integral of (6) of Schamel (1982a), where dx could be replaced by $\frac{d\phi_0}{\pm\sqrt{-2\mathcal{V}(\phi_0)}}$, but also for the x-derivative in (7) of Schamel (1982a) where ∂_x could be replaced by $\pm\sqrt{-2\mathcal{V}(\phi_0)}\frac{d}{d\phi_0}$. This would therefore also open the door for undisclosed potentials $\phi_0(x)$, which represent by far the majority of potentials.

However, the final word on the stability of solitary hole structures has not yet been spoken. One should not forget that a short, transient acceleration of a seed-generated soliton from the slow ion acoustic branch to the ion acoustic branch has been observed numerically (Schamel et al. 2017, 2020a, b; Mandal et al. 2018; Schamel 2023), thus skipping a forbidden region of the NDR. This suggests that a soliton is either unstable through the ground state of linear stability analysis or that a nonlinear destabilization process is actually at work and the whole game of "linearity versus nonlinearity" is repeated.

13 Summary and conclusions

In summary, we have rededuced the S-equation for the entire spectrum of periodic electron and ion holes, including bright and dark solitons, focusing on the softest trapping scenario and on small structures propagating at the three possible acoustic phase velocities in a two-component plasma. The single harmonic wave as a solution of the linearized S-equation did thereby not pose a problem with non-negativity because $(1 + \cos x)/2$ was used rather than functions like $\sin x$, $\cos x$, or other linear wave solutions such as Airy functions.

One important point is that all solutions are nonlinear solutions of the underlying phase space dynamics. This is easily seen in the harmonic SEA wave case where $B_e \approx (1 - \beta - v_0^2) = 0$ requires an electron trapping parameter $\beta < -0.71$, which represents a depressed region around the phase velocity in $f_e(x, v)$. The parameter β is thereby related to the first order perturbative trapping scenario, see (1) of Schamel (2023). This harmonic wave as a nonlinear solution of the VP system is not subject to Landau damping or growth, for example in a current-carrying plasma, as it is microscopically stable, independent of the strength of the current or the drift velocity (Schamel 2023). However, the limits and extensions of the S-equation have not yet been thoroughly explored and certainly require further investigation.

One example of a kinetic soliton behavior was explicitly mentioned, namely the numerically observed, transient acceleration of a bright soliton during its propagation in a subcritical plasma (Mandal et al. 2018) in which the final state was independent of the initial seed velocity and the soliton amplitude remained nearly constant. It is known from fundamental theory (Schamel 2023) that a soliton with a higher velocity can achieve a lower energy state, which can even become negative. In a linearly stable situation, a soliton can therefore act as an attractor and settle at a higher speed with or without growth. In the opposite, supercritical regime, however, as seen in unpublished simulations of Mandal et al., fluctuation-seeded solitary holes are destabilized by the emission of growing linear Vlasov waves and dissolve. Therefore, we anticipate that phase space holes occur predominantly in subcritical plasmas contrary to the currently prevailing opinion.

Since there are a variety of trapping scenarios, as discussed in Schamel (2023), the *S* equation is open to further nonlinear extensions, although nonnegativity remains a problem.

An important aspect of multiple simultaneously active trapping scenarios is that $\phi(x)$ becomes a mathematically unknown function representing an almost unlimited variety of structures, a point that still requires further intensive treatment. An attractive point of our derivation of the S-equation is, that in contrast to the BGK theory, the explicit form of $\phi(x)$ is not required, since the potential is already uniquely characterized by the pseudo-potential.

We emphasize that this work could not have been carried out using the BGK method, as it lacks the mechanism for determining the phase velocity v_0 and cannot handle potentials $\phi(x)$ whose functional behavior is analytically unknown.

Finally, the S equation as a macroscopic representation of deeper microscopic dynamics cannot, of course, reflect all aspects of phase space dynamics, so it is up to future generations to explore its scope. The coalescence of holes will clearly be outside their scope of application.

Appendices

The Schamel method in a compact form

Our main goal is to derive equations (4),(5) and search for stationary solutions of the Vlasov–Poisson (VP) system using the method developed by Schamel half a century ago in Schamel (1972). This VP system consists of the Vlasov equation for electrons and ions (we assume protons) as well as Poisson's equation and is for normalized quantities in the rest frame of the wave given by

$$[v\partial_x + \phi'(x)\partial_v]f_e(x,v) = 0 \qquad [u\partial_x - \theta\phi'(x)\partial_u]f_i(x,u) = 0 \qquad \phi''(x) = \int \mathrm{d}v f_e(x,v) - \int \mathrm{d}u f_i(x,u). \tag{25}$$

The normalization is standard, electron (ion) quantities are normalized by their thermal expressions and space and time by the Debye length and the inverse plasma frequency, respectively.

A solution of the Vlasov equations is provided by the two sets of constants of motion: $\epsilon_e = \frac{v^2}{2} - \phi$, $\sigma_e = \frac{v}{|v|}$ and $\epsilon_i = \frac{u^2}{2} - \theta(\psi - \phi)$, $\sigma_i = \frac{u}{|u|}$, respectively, in which the sign constants refer to untrapped particles only. For simplicity, we choose a Maxwellian currentless background plasma, assume a positive pedestal $0 \le \phi(x) \le \psi$, and select the smoothest trapping scenario where the distribution of trapped particles is analytic.

It should be noted that distributions of trapped particles with a singular slope at the separatrixes are also permissible, but are not considered here mainly for the sake of simplicity. They considerably expand the wave spectrum, but at the same time impede the search for structures; see point 20 in Sect. 2 and Schamel (2023).

The solution is given by the Ansatz

$$f_e(x,v) = \frac{1+k_{0+}^2\psi}{\sqrt{2\pi}}\Big[\theta(\epsilon_e)\exp\Big(-\frac{1}{2}(\sigma_e\sqrt{2\epsilon_e} - v_0)^2\Big) + \theta(-\epsilon_e)\exp(-v_0^2/2)\exp(-\beta\epsilon_e)\Big] \tag{26}$$

$$f_i(x,u) = \frac{1+K_i}{\sqrt{2\pi}}\Big[\theta(\epsilon_i)\exp\Big(-\frac{1}{2}(\sigma_i\sqrt{2\epsilon_i} - u_0)^2\Big) + \theta(-\epsilon_i)\exp(-u_0^2/2)\exp(-\alpha\epsilon_i)\Big], \tag{27}$$

which are known as Schamel distributions. The first part $\theta(\epsilon_{e,i})$, where $\theta(x)$ is the Heavyside step function, refers to free or passing particles, the second part $\theta(-\epsilon_{e,i})$ to trapped particles. It holds $u_0 = \sqrt{\frac{\theta}{\delta}}v_0$, and the phase velocity v_0 in $f_e(x,v)$ has to be replaced by $\tilde{v}_D := |v_D - v_0|$ in case of a current-carrying plasma with a finite drift velocity v_D between electrons and ions. A negative trapping parameter $\beta(\alpha)$ represents a dip in the trapped electron (ion) distribution.

To obtain the densities, we have to integrate over the entire velocity space.

For small amplitudes, $\psi << 1$, we get, with $Z_r(x) := \frac{1}{\sqrt{\pi}} P \int dt \frac{exp(-t^2)}{t-x}$ being the real part of the complex plasma dispersion function,

$$n_e = 1 + \frac{k_{0+}^2}{2}\psi - \frac{1}{2}Z_r'(\frac{v_0}{\sqrt{2}})\phi - \frac{5B_e}{4\sqrt{\psi}}\phi^{3/2} + ... \tag{28}$$

$$n_i = 1 + K_i - \frac{\theta}{2}Z_r'(\frac{u_0}{\sqrt{2}})(\psi - \phi) - \frac{5B_i}{4\sqrt{\psi}}[\theta(\psi - \phi)]^{3/2} + ... \tag{29}$$

In these equations we have defined:

$$B_e := \frac{16(1-\beta-v_0^2)}{15\sqrt{\pi}}\exp(-v_0^2/2)\sqrt{\psi} \qquad B_i := \frac{16(1-\alpha-u_0^2)}{15\sqrt{\pi}}\exp(-u_0^2/2)\sqrt{\psi} \tag{30}$$

Both expressions coincide e.g. with (20b) of Schamel (2000). In the solitary wave limit, $k_{0+}^2 \to 0$, both densities have to be equal at infinity, i.e. at $\phi = 0$. This demand provides K_i which becomes $K_i = \left(\frac{1}{2}Z_r'(\frac{u_0}{\sqrt{2}}) + \frac{5B_i\sqrt{\theta}}{4}\right)\theta\psi$. After inserting K_i into (29) the ion density simplifies to

$$n_i = 1 + \frac{\theta}{2}Z_r'(\frac{u_0}{\sqrt{2}})\phi + \frac{5B_i\theta^{3/2}\psi}{4}[1 - (1 - \frac{\phi}{\psi})^{3/2}] + ... \tag{31}$$

To solve Poisson's equation, we introduce the (provisional) pseudo-potential in formal analogy to classical mechanics: $\mathcal{V}_0(\phi; v_0)$ by $\phi''(x) = n_e(\phi) - n_i(\phi) =: -\mathcal{V}_0'(\phi; v_0)$, where the derivative refers to ϕ. We get $-\mathcal{V}_0'(\phi; v_0) = \frac{k_{0+}^2}{2}\psi - [\frac{1}{2}Z_r'(\frac{v_0}{\sqrt{2}}) + \frac{\theta}{2}Z_r'(\frac{u_0}{\sqrt{2}})]\phi + \frac{5}{4\sqrt{\psi}}\left(B_e\phi^{3/2} + B_i(\theta\psi)^{3/2}[1 - (1 - \frac{\phi}{\psi})^{3/2}]\right)$. By ϕ integration, assuming that $\mathcal{V}_0(\phi; v_0)$ vanishes at $\phi = 0$ we get

$$-\mathcal{V}_0(\phi; v_0) = \frac{k_{0+}^2}{2}\psi\phi - [\frac{1}{2}Z_r'(\frac{v_0}{\sqrt{2}}) + \frac{\theta}{2}Z_r'(\frac{u_0}{\sqrt{2}})]\frac{\phi^2}{2} - \frac{B_e}{2\sqrt{\psi}}\phi^{5/2} - \frac{5B_i\theta^{3/2}\psi}{4}\left(\phi - \frac{2\psi}{5} + \frac{2\psi}{5}(1 - \frac{\phi}{\psi})^{5/2}\right) \tag{32}$$

By x-integration of the Poisson equation we obtain the pseudo-energy:

$$\frac{\phi'(x)^2}{2} + \mathcal{V}_0(\phi; v_0) = 0.$$

Since at the potential maximum $\phi=\psi$ the slope of $\phi(x)$ (or the first derivative $\phi'(x)$) vanishes, we arrive directly at: $\mathcal{V}_0(\psi;v_0)=0$, which is a determining equation for v_0.

This equation is commonly referred to as the nonlinear dispersion relation (NDR) and is:

$$k_{0+}^2-\frac{1}{2}Z_r'\left(\frac{v_0}{\sqrt{2}}\right)-\frac{\theta}{2}Z_r'\left(\frac{u_0}{\sqrt{2}}\right)=B_e+\frac{3}{2}B_i\theta^{3/2} \tag{33}$$

Its solution v_0 provides the first part of our problem of finding a suitable $\phi(x-v_0t)$. The second part, the determination of the shape of $\phi(x)$, follows directly from the canonical form of the pseudo-energy:

$$\frac{\phi'(x)^2}{2}+\mathcal{V}(\phi)=0 \tag{34}$$

The canonical pseudo-potential $\mathcal{V}(\phi)$ is thereby obtained by replacing the v_0-dependent part in $\mathcal{V}_0(\phi;v_0)$ by the NDR and is:

$$-\mathcal{V}(\phi)=\frac{k_{0+}^2}{2}\phi(\psi-\phi)+\frac{B_e}{2\sqrt{\psi}}\phi^2\left(\sqrt{\psi}-\sqrt{\phi}\right)+\frac{B_i\theta^{3/2}}{4\sqrt{\psi}}\left(\phi\sqrt{\psi}(3\phi-5\psi)+2[\psi^{5/2}-(\psi-\phi)^{5/2}]\right) \tag{35}$$

Equations (33)–(35) provide the general solution of our problem with two trapping scenarios B_e, B_i. The NDR and the pseudo-potential are identical to earlier expressions such as (24),(25) or (44),(45) of Schamel (2000) or (51),(52) of Schamel (2023), respectively.

An alternative formulation that better illustrates the symmetry between electron and ion holes is given by the following expression:

$$-\mathcal{V}(\phi)/\psi^2=\left(\frac{k_{0+}^2}{2}-\frac{B_i\theta^{3/2}}{4}\right)\varphi(1-\varphi)+\frac{B_e}{2}\varphi^2(1-\sqrt{\varphi})+\frac{B_i\theta^{3/2}}{4}(1-\varphi)^2\left(1-\sqrt{1-\varphi}\right),$$

where $\varphi:=\phi/\psi$.

The basic equations for a negative pedestal

To obtain the equations for a negative pedestal corresponding to (4),(5), we have to change from k_{0+}^2 to $k_{0-}^2:=k_{0+}^2+\frac{1}{2}(B_e-B_i\theta^{3/2})$ and from $\phi(x)$ to $\hat{\phi}(x)=\phi(x)-\psi$ according to Schamel (2000). The result is

$$k_{0-}^2-\frac{1}{2}Z_r'(\frac{v_0}{\sqrt{2}})-\frac{\theta}{2}Z_r'(\frac{u_0}{\sqrt{2}})=\frac{3}{2}B_e+B_i\theta^{3/2} \tag{36}$$

and

$$-\mathcal{V}(\hat{\phi}) = -\frac{k_{0-}^2}{2}\hat{\phi}(\psi+\hat{\phi}) + \frac{B_i\theta^{3/2}}{2\sqrt{\psi}}\hat{\phi}^2\left(\sqrt{\psi}-\sqrt{-\hat{\phi}}\right) + \frac{B_e}{4\sqrt{\psi}}\left(\hat{\phi}\sqrt{\psi}(3\hat{\phi}+5\psi) + 2[\psi^{5/2}-(\psi+\hat{\phi})^{5/2}]\right). \quad (37)$$

The difference in k_{0s}, s = +, − is that for a bright soliton the asymptotic condition is different for each pedestal, namely $\phi(x) = 0$ for the positive and $\hat{\phi}(x) = -\psi$ for the negative pedestal.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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