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Albanese map for Kähler manifolds with nef anticanonical bundle

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Abstract
We study the structure of the Albanese map for Kähler manifolds with nef anticanonical bundle. First, we give a result for fourfolds whose Albanese torus is an elliptic curve. In the general case, for manifolds of arbitrary dimension, we consider two cases: the general fiber of the Albanese map is either a Calabi-Yau manifold or a projective space. In the first case, we show that the manifold itself must be Calabi-Yau. In the second case, we provide a more topological proof of a result by Cao and Höring, which states that the manifold must be the projectivization of a numerically flat vector bundle.

Keywords Kähler manifolds · Nef anti-canonical bundles · Albanese maps · Structure theorems

Mathematics Subject Classification Primary 32J25; Secondary 32J27 · 14E30

1 Introduction

In this work, we study the Albanese map of compact Kähler manifolds with a nef anticanonical line bundle. The conjectural picture is the following:

Conjecture 1 *Let X be a compact Kähler manifold with nef anticanonical line bundle. Then the Albanese morphism of X is a locally trivial fibration with connected fibers.*

The conjecture is fully proven when X is projective, in [17]. However, in the compact Kähler case, it is still widely open, and it was only proven recently in dimension three [53]. The main difficulty in the Kähler case arises from the fact that the Albanese morphism is no longer projective, since the fibers are not necessarily projective. Therefore, the theory of positivity of direct images - a key technique in the projective case - does not apply directly. Although it is conjectured by Campana that the MRC quotient stated in Conjecture 2 below is a projective morphism with the base being Calabi-Yau, the Albanese map can only be projective if the Hyperkähler components in the Beauville-Bogomolov decomposition of the

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MRC quotient of X are all projective. In the whole paper, a simply connected compact Kähler manifold X with $c_1(X) = 0 \in H^2(X, \mathbb{R})$ will be called a Calabi-Yau manifold.

After revealing the structure of the Albanese map for Kähler threefolds, a next step is to consider dimension 4. Using the recent developments in four-dimensional Kähler MMP from [22], we first present a result about the Albanese morphism for fourfolds:

Theorem A *Let X be a non-projective compact Kähler fourfold with $-K_X$ nef such that $\tilde{q}(X) = q(X) = 1$. Then, up to a finite étale cover, X is either the product of a K3 surface and the projectivization of a rank two numerically flat vector bundle over an elliptic curve, or the product of a simply connected Calabi-Yau threefold with an elliptic curve.*

In the general case of arbitrary dimension, we mainly deal with two cases. In the first case, we assume that the fibers of the Albanese map are simply connected Calabi-Yau manifolds. In this setting, we have the following result:

Theorem B *Let X be a compact Kähler manifold with $-K_X$ nef and $\tilde{q}(X) = q(X)$. If the general fiber of the Albanese map is a Calabi-Yau manifold, then $c_1(X) = 0$.*

Here, the key idea is to use the close relation between the curvature of the relative canonical bundle and deformation theory in order to replace [7], which is not known in the Kähler case. The second result covers the case where the fibers of the Albanese map are projective spaces:

Theorem C *Let X be a compact Kähler manifold such that $-K_X$ is nef. Assume that the general fiber of the Albanese map is $\mathbb{P}^{r-1}$. Then $X \simeq \mathbb{P}(E)$ for some numerically flat vector bundle E of rank r over $A(X)$, up to a finite étale cover.*

To fix our notation, a $\mathbb{P}^r$−bundle over a manifold Z (for some $r \geq 1$) is a proper morphism between complex manifolds $p : Y_Z \to Z$ such that every fiber $f^{-1}(z)$ is a projective space. The above result was known previously by [18]. Here, we give a completely different and more topological proof. The key observation is that, using the results of [15] and [1], the Albanese morphism is smooth in codimension 2, and the manifold is the projectivization of some vector bundle in codimension 1. Using the result of [2], we can extend the vector bundle to a numerically flat vector bundle over the Albanese torus by carefully controlling its first and second Chern classes via the codimension condition.

We note that Example 4.1 gives a compact Kähler manifold admitting a surjective, non-projective Albanese map whose general fiber is projective. This example was indicated to the authors by Andreas Höring. In particular, our result cannot be obtained by the argument of [52, Theorem 4.1].

Finally, we discuss the algebraic approximation of compact Kähler manifolds with nef anticanonical line bundle. Recall that X is said to admit an algebraic approximation if there exists a deformation $\mathfrak{X} \to \Delta$ over the unit disk such that the central fiber X_0 is isomorphic to X, and there exists a sequence $t_i \to 0$ in Δ such that all the fibers X_{t_i} are projective. We have the following structural conjecture, which is proven up to dimension 3 in [53, Theorem 1.3]. It is completely settled in the projective case in [19].

Conjecture 2 *Let X be a compact Kähler manifold with a nef anti-canonical bundle. There exists a fibration $\varphi : X \to Y$ with the following properties:*

- *$\varphi : X \to Y$ is a locally constant fibration (i.e. $\varphi : X \simeq (\tilde{Y} \times F)/\pi_1(Y) \to Y = \tilde{Y}/\pi_1(Y)$ is induced by the natural projection, where $\tilde{Y}$ is the universal cover of Y, F the fiber of φ, together with a representation $\rho : \pi_1(X) \to \mathrm{Aut}(F)$);*

- *Y is a compact Kähler manifold with $c_1(Y) = 0$;*
- *F (the fiber of $\varphi : X \to Y$) is rationally connected.*

The concept of locally trivial morphism is introduced in [52]. Assuming this conjecture, we can show the following result:

Theorem D *If Conjecture 2 holds true, then any compact Kähler manifold X with nef anti-canonical divisor can be algebraically approximated.*

2 An isotriviaity result in the Calabi-Yau case

We start by recalling the definition of the coarse moduli space of polarized Kähler manifolds with $c_1 = 0$ according to [34, Section 5] which is denoted by $\mathfrak{M}$ and which will be used in the proof of the next proposition. For this prupose, we fix an underlying smooth structure (X, α) consisting of a compact connected smooth manifold X and a class $\alpha \in H^2(X, \mathbb{R})$. As a set, $\mathfrak{M}$ is the set of isomorphism classes of polarized Kahler manifold (X', γ) which are diffeomorphic to (X, α), i.e., there exists a dlffeomorphism $\varphi : X' \to X$ such that $\varphi^*\alpha = \gamma$. Since the underlying smooth structure is invariant under (polarized) deformations, for any point of $\mathfrak{M}$ represented by a polarized Kahler manifold (X', γ) we have the natural map

$$\pi : S(X', \gamma) \to \mathfrak{M}$$

where $S(X', \gamma)$ is the Kuranishi space of (X', γ) which is smooth. By [34, (5.2)], we have a bijection

$$S(X', \gamma)/G(X', \gamma) \to \pi(S(X', \gamma)) \subset \mathfrak{M}$$

where $G(X', \gamma) = \mathrm{Aut}(X', \gamma)/\mathrm{Aut}^0(X')$ is a finite group. Here $\mathrm{Aut}(X', \gamma)$ means the automorphisms of X' perserving γ and $\mathrm{Aut}^0(X')$ is the connected component of the identity in $\mathrm{Aut}(X')$. With these natural maps π, $\mathfrak{M}$ can be endowed with a complex structure as a complex orbifold.

We first give an alternative proof of a special case of the isotriviality theorem in [64, Remark 3.2] using the moduli of Kähler polarized Calabi-Yau manifolds:

Proposition 2.1 *If $f : X \to T$ is a holomorphic submersion between compact Kähler manifolds with Calabi-Yau fibers X_y and a torus base T, then f is a holomorphic fiber bundle.*

Proof First we want to show that the fibration is isotrivial. In the projective case, this follows from [26, Theorem A]. The same is true in our situation: By the work of Fujiki and Schumacher, see [34, Thm. 5.4] and also [33, 58], there exists a coarse moduli space of polarized Kähler manifolds with $c_1 = 0$, denoted by $\mathfrak{M}$, which has the structure of a complex orbifold. In analogy to the canonically polarized case, by the work of [8] or [63], this moduli space carries an orbifold Finsler metric whose sectional curvature can be bounded above by a negative constant. Hence, the moduli of Kähler polarized Calabi-Yau manifolds is Kobayashi hyperbolic in the orbifold sense, cf. [26, Thm. C].

Now we can prove the isotriviality as in [26, pf. of Thm. C]: Suppose that f is not isotrivial. The Kähler form ω_X induces a polarization $\lambda_{X/T} \in R^1 f_*\Omega_{X/T}(T)$, cf. [34]. Hence we obtain a moduli map $\varphi : T \to \mathfrak{M}$ which is not constant. By construction, $\mathfrak{M}$ is locally the quotient of Kuranishi spaces, so the map φ is locally liftable and thus an orbifold morphism. For a pair of points $p, q \in T$ such that $\varphi(p) \neq \varphi(q)$, we obtain by the distance decreasing property of the Kobayashi pseudo-metric, see [26, Lemma 3.4], that $d_Y(p,q) \geq d_{\mathfrak{M}}(\varphi(p), \varphi(q)) > 0$, which contradicts the fact that the complex torus T is hyperbolically special, i.e. the Kobayashi pseudo distance vanishes identically on T. Note that the Kobayashi pseudo distance vanishes identically on Euclidean space and every holomorphic map between A_1 and A_2 of complex spaces is distance-decreasing with respect to the Kobayashi pseudo distance of A_1 and A_2. □

As a corollary, we have the following application:

Proposition 2.2 *Let X be a compact Kähler manifold and $f : X \to T$ be a smooth morphism onto a complex torus. Assume that each fiber of f is a simply connected Calabi-Yau manifold. Then $c_1(X) = 0$.*

Proof By Proposition 2.1, we have no deformation of complex structure among the fibers of f which in turn implies that the Weil-Petersson metric on T vanishes identically. To conclude, we use the fact that for a smooth fibration of Calabi-Yau manifolds, there is a close relation between the variation of complex structure and the positivity of the relative canonical bundle. To this end, we consider the fiberwise unique Ricci-flat metrics ω_s in the corresponding Kähler classes, which are given by the polarization obtained from restricting the given Kähler class to each fiber. The relative volume form $\omega^n_{X/T} = g dV$, where $\dim X_s = n$, induces a hermitian metric g^{-1} on the relative canonical bundle $K_{X/T} = K_X$ (see [34, Proposition 3.6]). Its curvature is given, up to a numerical constant, by the pullback of the Weil-Petersson form downstairs, i.e. the L^2 inner product of harmonic representatives of the Kodaira-Spencer classes, see e.g. [9, Eqn. 1.1]:

$$2\pi c_1(K_X, g^{-1}) = \sqrt{-1}\partial\overline{\partial} \log g = \frac{1}{\mathrm{vol}\, X_s} f^* \omega^{WP}.$$

But by the isotriviality, the Weil-Petersson form ω^{WP} vanishes identically, hence $c_1(K_X, g^{-1}) = 0$. □

3 Structure of the Albanese map for 4-folds when the base is a curve

Proposition 3.1 *(= Theorem A) Let X be a non-projective compact Kähler fourfold with $-K_X$ nef such that $\tilde{q}(X) = q(X) = 1$. Then up to a finite étale cover, X is either the product of a K3 surface and the projectivization of a rank two numerically flat vector bundle over an elliptic curve or the product of a simply connected Calabi-Yau threefold with an elliptic curve.*

Proof Let $\alpha : X \to A(X)$ be the Albanese morphism which is smooth by [15], since the base is of dimension one. In particular, by [25, Proposition 3.12], $\tilde{q}(F) = 0$ where F is any fiber. Note that if K_X is $\mathbb{Q}-$effective, $c_1(X) = 0$ and X is the product of a simply connected Calabi-Yau threefold with an elliptic curve up to a finite étale cover. In the following, we always assume that K_X is not $\mathbb{Q}-$effective. Up to a finite étale cover, we can assume that $\pi_1(X)$ is free abelian, see [56].

By [22, Theorem 1.2], we can consider the relative K_X-MMP with respect to α,

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A(X) \\ {\scriptstyle\beta}\,\vdots & \nearrow_{\gamma} & \\ Y & & \end{array}$$

where the last step of β is a Mori fiber space. Notice that the restriction of the relative K_X-MMP with respect to α to any fiber F is the K_F-MMP since $K_X|_F = (K_X + F)|_F = K_F$ by the adjunction formula. But since any fiber F of α is a compact Kähler threefold with nef anticanonical divisor such that $\tilde{q}(F) = 0$, we claim that its MMP consists of at most one step of a Mori fiber space (i.e. the MMP contains only one step that is a Mori fiber space or the MMP is the identity map). Thus β is a morphism and β consists of at most one step of a Mori fiber space. In fact, since $\tilde{q}(F) = 0$, $\pi_1(F)$ is finite by the structural theorem for compact Kähler threefolds with nef anticanonical divisor proven in [53]. Since $\pi_2(A(X)) = 0$, the homotopy exact sequence implies that $\pi_1(F)$ is a subgroup of $\pi_1(X)$ which is free abelian. Thus F is simply connected. Therefore, F is either a Calabi-Yau threefold, a product of $\mathbb{P}^1$ with a K3 surface or a rational connected threefold. In the third case, $H^{2,0}(F) = H^{1,0}(F) = 0$ and thus $H^{2,0}(X) = 0$, contradicting the assumption that X is non-projective. Thus the third possibility can be ruled out. Note that the dimension of Y is 4 or 3.

If $\dim(Y) = 4$, any fiber F of α is a simply connected Calabi-Yau threefold and β is an isomorphism. By Proposition 2.2 we have $c_1(X) = 0$, contradicting the assumption that K_X is not $\mathbb{Q}-$effective.

If $\dim(Y) = 3$, the map γ is a K3-fibration between compact Kähler manifolds. Because $\dim(Y) = 3$, F is a product of $\mathbb{P}^1$ with a K3 surface and the restriction of β to F is the projection onto the K3 surface. In particular, the set-theoretic fiber of γ is alway a K3 surface. Since β is a $\mathbb{Q}-$conic bundle (see [53]) and X is smooth, β is a conic bundle outside a closed analytic subset of codimension at least 2 in Y. The proof of [44, Lemma 4.1] shows that Y is klt which is in particular Cohen-Macaulay. Since the fibers of γ are K3 sufaces, γ is flat by miracle flatness. We claim that Y is smooth. In the following, we give two different proofs of this claim. The first one uses the singular Beauville-Bogomolov decomposition theorem. Since $c_1(Y) = 0$, up to passing to a quasi-étale cover, Y is a three dimensional torus or a product of K3 surface S with an elliptic curve E. In the first case, a normalisation of the base change of X (by a quasi-étale and hence étale cover over X as X is smooth) will have three linearly independend global holomorphic one forms which is impossible. In the second case, the map $S \times E \to Y \to A(X)$ is constant over the slices $S \times \{p\}$ for any $p \in E$. Then the induced map $E \to A(X)$ is of degree d for some $d \in \mathbb{N}$. Since the set-theoretic fiber of γ is alway a connected K3 surface, the fibers of $S \times E \to Y$ has always d points. The map $S \times E \to Y$ is thus étale and Y is smooth.

The second proof is more elementary but lengthy. By the proof of [36, 9.C.4] on the local structure of klt singularities, there exists a finite set of points Z such that locally near any point of $Y \setminus Z$, the singularity is given by an open subset of $(\mathbb{C}^2/G) \times \mathbb{C}$ containing the origin where G is a finite subgroup of $GL(2, \mathbb{C})$. We claim that $Y \setminus Z$ is in fact smooth. Consider the normalisation of the (local) base change of $X \to Y$ which gives a germ of a $\mathbb{Q}-$conic bundle $\tilde{\beta} : \tilde{X} \to \mathbb{C}^3$ near the origin with $\tilde{X} \to X$ quasi-étale. The action of G lifts on the base change which induces a map from $G \times \tilde{X}$ to the base change. By the universal property of normalisation, this gives an action on $\tilde{X}$. Since X is smooth, $\tilde{X}$ is also smooth. (In other words, $\tilde{\beta}$ is a conic bundle.) In particular, the action of G on $\tilde{X}$ is free and the map $\tilde{X} \to X$ is in fact étale. Note that thus the action of G on $\tilde{\beta}^{-1}(0)$ is also free. Since $\tilde{X} \to \mathbb{C}^3$ is a conic bundle outside $(0, 0) \times \mathbb{C}$ whose set-theoretic fibers are rational curves, by [53, Proposition 3.4(3)], $\tilde{X} \to \mathbb{C}^3$ is a trivial conic bundle over $\mathbb{C}^3$. Note that $\beta^{-1}(0) = \tilde{\beta}^{-1}(0)/G$. Hurwitz forumula implies that G is trivial. This finishes the proof of the claim that $Y \setminus Z$ is in fact smooth. In other words, Y is smooth in codimension 2.

We continue to show that Y is in fact smooth. Note that γ is thus generically smooth. For general $t \in A(X)$, $\chi(\mathcal{O}_{Y_t})$ is 2 as Y_t is a smooth K3 surface. The special fibers of γ

are the union of a K3 surface with some possible embeded points as α is a submersion and Y is smooth in codimension 2. In particular, there is no embedded curve. But the possible embeded points will only decrease the arithmetic genus $\chi(\mathcal{O}_{Y_t})$. Since the arithmetic genus is constant in the flat family γ, the morphism is smooth. Since γ and $A(X)$ are smooth also Y is smooth.

Again by Proposition 2.2, up to some finite étale cover, Y is a product of $A(X)$ with a K3 surface. Alternatively, we may argue as follows: Since $-K_X$ is nef and β is a conic bundle (see [53, Definition 3.1, Definition 3.3]) by (the proof of) [22, Theorem 8.12], $-K_Y$ is psef by [53, Proposition 3.5]. On the other hand, Y is not uniruled which implies that K_Y is psef by [10, Corollary 1.2]. More precisely, the rational curves of Y are contained in the fibers of γ since $A(X)$ contains no rational curves. Thus, if Y were uniruled the fibers of γ would be uniruled, too, contradicting the fact that the fibers of γ are K3 surfaces. Thus $c_1(Y) = \frac{1}{4}\beta_*(c_1(-K_X)^2) = 0$ by conic bundle formula [53, Proposition 3.5]. More precisely, by conic bundle formula [53, Proposition 3.5] and the assumption that $-K_X$ is nef, $-4c_1(K_Y) - c_1(\Delta)$ is pseudoeffective where Δ is the discriminant divisor. Since K_Y is pseudoeffective, $c_1(Y) = 0$ and there is no discriminant divisor.

By [53, Proposition 3.4(3)], there is no singular fiber as there is no discriminant divisor. By the Fischer-Grauert theorem [29], β is locally trivial since $\mathbb{P}^1$ is rigid and thus $\beta_*(-K_X)$ is locally free. By [53, Claim 2.8], $\beta_*(-K_X)$ is weakly positively curved. On the other hand, $c_1(\beta_*(-K_X)) = 0$ by [53, Proposition 3.4 (6)]. By [68, Main theorem], $\beta_*(-K_X)$ is numerically flat. Recall that Y is a product of a K3 surface S and an elliptic curve (i.e. the Albanese torus $A(X)$) (by choosing a suitable finite étale cover). By the next Lemma, $\beta_*(-K_X)$ is the pull back of a numerically flat vector bundle V over the elliptic curve $A(X)$. The pullpack of $\beta_*(-K_X)$ over the universal cover of Y (isomorphic to $S \times \mathbb{C}$) is trivial. In particular, we can identify the universal cover of $S \times \mathbb{P}(V)$ as $S \times \mathbb{C} \times \mathbb{P}^2$. Consider the base change

$$\begin{array}{ccc}
(S \times \mathbb{C} \times \mathbb{P}^2) \times_{\mathbb{P}(\beta_*(-K_X))} X & \longrightarrow & X \\
\downarrow & & \downarrow \\
S \times \mathbb{C} \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}(\beta_*(-K_X)) \\
\downarrow & & \downarrow \\
S \times \mathbb{C} & \longrightarrow & Y.
\end{array}$$

Being an étale base change of a conical bundle, the composition $(S \times \mathbb{C} \times \mathbb{P}^2) \times_{\mathbb{P}(\beta_*(-K_X))} X \to S \times \mathbb{C}$ is a conical bundle, too. By [53, Proposition 3.4 (2)], X can be embedded in $S \times \mathbb{P}(V)$. Under this identification, X is defined by some section of $H^0(S \times \mathbb{C} \times \mathbb{P}^2, \mathcal{O}_{S\times\mathbb{C}} \boxtimes \mathcal{O}_{\mathbb{P}^2}(2))$ invariant under the action of the fundamental group. Since S is compact and simply connected, the section is constant in S. By Künneth formula, X is isomorphic to a product of S with the zero set of a section of $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(2))$. In othe words, X is a product of S and a ruled surface over an elliptic curve. Then by [5, Proposition 4.1, Chap. V], the ruled surface over the elliptic curve is the projectivization of some vector bundle. Up to a finite étale cover, [13, Lemma 7.4] implies that we may assume its determinant line bundle to be trivial. The nefness of the anticanonical line bundle then implies that the vector bundle is actually numerically flat. Without passing to a finite étale cover, the possibilities are given in [13, Theorem 3.1 (5)]. □

Lemma 3.2 *Let X, Y be two compact Kähler manifolds. Assume that $\pi_1(Y) = 0$. Let E be a numerically flat vector bundle over $X \times Y$. Then there exists a numerically flat vector bundle F on X such that $E = p_1^* F$ where $p_1 : X \times Y \to X$ is the natural projection.*

Proof By [25, Theorem 1.18], E is a successive extension of hermitian flat vector bundles on $X \times Y$. Since any hermitian flat vector bundle corresponds to a representation of $\pi_1(X \times Y) \simeq \pi_1(X)$, these hermitian flat vector bundles are pullbacks of some hermitian flat vector bundles on X.

Now consider an extension of pullbacks of hermitian flat vector bundles E_1, E_2 on X which corresponds to an element of

$$\begin{aligned}&H^1(X \times Y, p_1^* \mathcal{H}om(E_2, E_1))\\ &\simeq H^1(X, \mathcal{H}om(E_2, E_1)) \otimes H^0(Y, \mathcal{O}_Y) \oplus H^0(X, \mathcal{H}om(E_2, E_1)) \otimes H^1(Y, \mathcal{O}_Y)\\ &\simeq H^1(X, \mathcal{H}om(E_2, E_1))\end{aligned}$$

by the Künneth formula. In particular, the extension is the pullback of some extension of the hermitian flat vector bundles on X. Induction on the rank of E finishes the proof. □

4 First case: fibers are Calabi-Yau

4.1 Preliminaries

In this part, we recall the real polarized Hodge structures following [64, Section 3].

Let $f : X \to Y$ be a family of Calabi-Yau manifolds, i.e. a submersion between compact Kähler manifolds such that any fiber is a Calabi-Yau manifold. Let $P \subset H^{n-m}(X_y, \mathbb{C}) =: H$ be the primitive cohomology induced by the Kähler class $\{\omega_X|_{X_y}\}$. Let $H_{\mathbb{R}} = P \cap H^{n-m}(X_y, \mathbb{R})$, $h^{p,q} = \dim_{\mathbb{C}} P \cap H^q(X_y, \Omega^p_{X_y})$ for $p+q = n-m$, and call Q the quadratic form on H given by

$$Q(\alpha, \beta) = (-1)^{(n-m)(n-m-1)/2} \int_{X_y} \alpha \wedge \beta,$$

which is called a polarization (even though no integrality assumption is made).

Then the construction by Griffiths [38, Section 8] gives a classifying space $\mathcal{D}$ for real polarized Hodge structure of type $\{H_{\mathbb{R}}, h^{p,q}, Q\}$, and a well-defined holomorphic period map $p : \tilde{Y} \to \mathcal{D}$, where $\tilde{Y}$ is the universal cover of Y. If $\pi : \tilde{Y} \to Y$ is the universal covering map, and $y = \pi(z)$, then we have $p(z) = \{P^{p,q} := P \cap H^q(X_y, \Omega^p_{X_y})\}$. Furthermore, p is horizontal in the sense that $p_*(T_{\tilde{Y}}) \subset T^h_{\mathcal{D}}$, where $T^h_{\mathcal{D}}$ is the horizontal subbundle of $T_{\mathcal{D}} \subset \oplus_{r>0} \mathrm{Hom}(P^{p,q}, P^{p-r,q+r})$.

By [37, Theorem 1.27], [20, Theorem 5.3.4], the differential $p_* : T_{\tilde{Y},z} \to T^h_{\mathcal{D},p(z)}$ is a composition of the Kodaira-Spencer map $T_{\tilde{Y},z} \to H^1(X_y, T_{X_y})$, $(y = \pi(z))$, and the cup product map $w : H^1(X_y, T_{X_y}) \to T^h_{\mathcal{D},p(z)}$. For a detailed reference, we refer to [65, Theorem 10.4]. By [37, Proposition 3.6], w is injective. This result is known as the infinitesimal Torelli theorem for Calabi-Yau manifolds.

Here we give an example of a compact Kähler manifold admitting a surjective non-projective Albanese map of which the general fibre is projective.

Example 4.1 Let $\alpha : S \to \mathbb{P}^1$ be the elliptic fibration of a non-projective K3 surface. Such an example can be constructed for example as follows. Take a $2-$dimensional torus T_1 of algebraic dimension 1. Let $a : T_1 \to T_2$ be its algebraic reduction which is a smooth non-projective elliptic fibration. Consider the $\mathbb{Z}_2-$action on T_1 by changing the sign of each coordinates as the Kummer surface. Consider a similar action on T_2. Then a is an equivariant morphism which induces a map between a Kummer surface $T_1/\mathbb{Z}_2 \to T_2/\mathbb{Z}_2 \simeq \mathbb{P}^1$. Composition with the minimal resolution of $T_1/\mathbb{Z}_2$ (which is a K3 surface S) gives $\alpha : S \to \mathbb{P}^1$. Let E be an elliptic curve and $E \to \mathbb{P}^1$ a non-trivial meromorphic function. Consider the base change and the minimal resolution of singularities of the base change

$$\begin{array}{ccccc} X & \longrightarrow & E \times_{\mathbb{P}^1} S & \longrightarrow & S \\ & \searrow^{\beta} & \downarrow & & \downarrow{\scriptstyle \alpha} \\ & & E & \longrightarrow & \mathbb{P}^1 \end{array}$$

By construction, β is an elliptic fibration. Note that X is non-projective and compact Kähler as S. In particular, β is not a projective morphism. Since X is non-uniruled, K_X is nef. We claim that β is the Albanese map of X. By classificiation of compact Kähler surface [5, Table 10 on Page 244], the possibilities of X are a K3 surface, a torus, or a minimal properly elliptic surface. By the universal property of the Albanese map and the surjectivity of β, the Albanese torus of X is of dimension at least 1. Thus we may rule out the case of a K3 surface. If X is a torus, consider the Stein factorisation $X \to X_1 \to S$ of the map $X \to S$ where X_1 is normal, $X \to X_1$ is a fibration and $X_1 \to S$ is a finite morphism. Note that X and X_1 are bimeromorphic. By [5, (2.5), Chap. III], the exceptional divisor of $X \to X_1$ is a union of rational curves. Since X is a torus which has no rational curve, X is isomorphic to X_1. In other words, $X \to S$ is a finite morphism (i.e. a ramified covering). By [5, formula (20), Chap. I on page 53], the difference between K_X and the pull back of K_S is the ramification divisor. Since X is a torus and S is a K3 surface, the ramification divisor is 0. In other words, X is an unramified cover of S which is impossible since S is simply connected. In conclusion, X is not a torus and X is a minimal properly elliptic surface.

From the proof of [5, Proposition (5.1), Chap. VII] (which shows the uniqueness of the elliptic fibration for a minimal properly elliptic surface) and all the possibilities listed on [5, p. 256], $\chi(X, \mathcal{O}_X) \geq 1$. By [5, Corollary (11.2), Chap. III], $R^1\beta_*\mathcal{O}_X$ is locally free. Since X is non-projective and compact Kähler, $h^2(X, \mathcal{O}_X) \neq 0$. By the Leray spectral sequence for β and $\mathcal{O}$, we have an isomorphism $H^2(X, \mathcal{O}_X) \simeq H^1(E, R^1\beta_*\mathcal{O}_X)$ and an exact sequence

$$0 \to H^1(E, \beta_*\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(E, R^1\beta_*\mathcal{O}_X) \to 0.$$

Recall that by the positivity result of [5, Theorem (18.2), Chap. III], the degree of the line bundle $\beta_*(K_{X/E})$ is positive. By relatively duality (cf. proof of [5, Theorem (12.1), Chap. V], $\beta_*(K_{X/E})$ is dual to $R^1\beta_*\mathcal{O}_X$. In particular, by Serre duality and the fact the E is an elliptic curve, the degree of line bundle $R^1\beta_*\mathcal{O}_X$ is strictly negative or otherwise $R^1\beta_*\mathcal{O}_X$ is trivial. If $R^1\beta_*\mathcal{O}_X$ is trivial, β_*K_X is trivial, too. Thus $H^0(X, K_X) = H^0(E, \beta_*K_X)$ has a global non-vanishing section. In other words, K_X is trivial which contradicts the condition that X is a minimal properly elliptic surface. Thus by degree reasons, $H^0(E, R^1\beta_*\mathcal{O}_X) = 0$. By the short exact sequence $h^1(X, \mathcal{O}_X) = 1$. Finally, by the universal property of the Albanese map and the fact that β is an elliptic fibration, β is the Albanese map of X.

4.2 Main results

Following the arguments of [64, Theorem 3.1], we have the following result:

Proposition 4.2 *(= Theorem B) Let X be a compact Kähler manifold of dimension n such that $\tilde{q}(X) = q(X)$, the general fiber is a Calabi-Yau manifold and $-K_X$ is nef. Then we have $c_1(X) = 0$.*

Proof Let $\alpha : X \to A(X)$ be the Albanese morphism which is smooth in codimension 1 by [15]. If $\dim(A(X)) = n$, by [15], X is a modification of $A(X)$. In particular, we have equality of Kodaira dimensions $\kappa(X) = \kappa(A(X)) = 0$ and thus K_X is psef. Since $-K_X$ is nef, $c_1(X) = 0$. Beauville-Bogomolov decomposition implies that in fact $X \simeq A(X)$. Therefore, we only consider the case where $\dim(A(X)) < n$. Without loss of generality, we may assume also that $q(X) \geq 1$.

Let Z be a closed analytic subset of $A(X)$ of codimension at least two such that $\alpha : \alpha^{-1}(A(X) \setminus Z) \to A(X) \setminus Z$ is smooth.

Consider the universal cover $\pi : \mathbb{C}^{2\dim(A(X))} \to A(X)$. By the codimension condition of Z, $\pi|_{\pi^{-1}(A(X)\setminus Z)} : \pi^{-1}(A(X) \setminus Z) \to A(X) \setminus Z$ is the universal cover of $A(X) \setminus Z$. (By the stratification of a closed analytic subset (cf. e.g. [24, Chapter II, Proposition 5.6]), we may assume that Z is smooth. But this case is well-known, see for example [35, Théorème 2.3].)

Endow the local system $R^k\alpha_*\mathbb{R}$ associated to $\alpha : \alpha^{-1}(A(X) \setminus Z) \to A(X) \setminus Z$ with the geometric real variation of polarized Hodge structures, where $k = n - \dim(A(X))$, cf. [64]. The construction of period domains and period mappings for real polarized Hodge structures by Griffiths gives a well-defined holomorphic period map $p : \pi^{-1}(A(X)\setminus Z) \to \mathfrak{D}$ where $\mathfrak{D}$ is the associated period domain. For example, the period domain is the upper half plane $\mathbb{H}$ (biholomorphic to the unit disc) if the fiber F is an elliptic curve. In the case $\dim(A(X)) = n - 2$, as observed on page 305 of [46], the period domain of real polarized K3 surfaces is the non-compact Hermitian symmetric domain $O(2, 19)/SO(2) \times O(1, 19)$. By Harish-Chandra's embedding theorem, the period domain can be realized as a bounded symmetric domain in a complex vector space. By Hartogs theorem, the period map p extends to a holomorphic function on $\mathbb{C}^{2\dim(A(X))}$. In the general case, the extension follows from [39, Corollary 9.5] based on the geodesic completeness of the period domain and the distance decreasing property of morphisms between manifolds whose curvature have strict negative upper bounds (which implies the continuous hence holomorphic extension).

The period map is thus constant by Liouville's theorem in the case $\dim(A(X)) = n - 1$ or $\dim(A(X)) = n - 2$. In the general case, it follows from Schwarz lemma [64, Lemma 3.3] and the existence of a hermitian metric on $\mathfrak{D}$ whose holomorphic sectional curvature is bounded from above by a strictly negative constant along the horizontal subbundle of $T\mathfrak{D}$ defined by Griffiths [38].

But now the differential of the period map p at z is a composition of the Kodaira-Spencer map $\rho_z : T_z(\pi^{-1}(A(X) \setminus Z)) \to H^1(\alpha^{-1}(\pi(z)), T_{\alpha^{-1}(\pi(z))})$, and the cup product map $H^1(\alpha^{-1}(\pi(z)), T_{\alpha^{-1}(\pi(z))}) \to T_{p(z)}\mathfrak{D}$. Since the second map is injective by infinitesimal Torelli, the Kodaira-Spencer map of the pullback family over $\pi^{-1}(A(X) \setminus Z)$ is identically zero.

By [47, Theorem 4.6], since $h^1(F, T_F) = h^{1,n-1}(F)$ which is constant for the Kähler fibration (see e.g. [65, Section 9.3.2]), α is locally trivial over $A(X) \setminus Z$. No deformation of complex structure among the fibers of α implies that the Weil-Petersson metric on $A(X) \setminus Z$ is identically zero. Again by the result of [9], the pullback of the Weil-Petersson metric represents the first Chern class of relative canonical divisor $K_{(X\setminus\alpha^{-1}(Z))/(A(X)\setminus Z)}$ modulo a constant factor, which implies that the first Chern class of $X \setminus \alpha^{-1}(Z)$ is trivial.

The preimage of the non-flat locus W of α is codimension at least 2 by Lemma 4.4. Since α is equidimensional over its flat locus, the preimage of the non-smooth locus of α is also codimension at least 2. Thus, by Lemma 4.3, $H^2(X, \mathbb{C}) \simeq H^2(X \setminus \alpha^{-1}(Z), \mathbb{C})$ and hence

$c_1(X) = 0$. Note that the isomorphism is given by the pullback of the restriction map, thus the restriction of $c_1(X)$ is $c_1(X \setminus \alpha^{-1}(Z))$. □

Lemma 4.3 *Let X be complex manifold of dimension n containing a closed analytic subset W of codimension at least k. Then we have for $i \leq 2(k-1)$,*

$$H^i(X, \mathbb{C}) \simeq H^i(X \setminus W, \mathbb{C}).$$

Proof By Poincaré duality, it is equivalent to show that for $i \leq 2(k-1)$,

$$H_c^{2n-i}(X, \mathbb{C}) \simeq H_c^{2n-i}(X \setminus W, \mathbb{C})$$

which follows from long exact sequence

$$H_c^{2n-i-1}(W, \mathbb{C}) = 0 \to H_c^{2n-i}(X \setminus W, \mathbb{C}) \to H_c^{2n-i}(X, \mathbb{C}) \to H_c^{2n-i}(W, \mathbb{C}) = 0.$$

Here, the vanishing of cohomologies follows from dimension reasons. □

Lemma 4.4 *Let X be a compact Kähler manifold of dimension n such that $-K_X$ is nef. Let Y be a compact Kähler manifold such that $c_1(Y) = 0$ with a holomorphic map $\alpha : X \to Y$. Assume that the non-flat locus $W \subset Y$ is of codimension at least 2. Then $\alpha^{-1}(W)$ is of codimension at least 2.*

Proof We follow the arguments of [19, Lemma 3.2]. Let ω_X be a Kähler form on X. By Hironaka's flattening theorem [42], we have a diagram

$$\begin{array}{ccccc} \tilde{X} & \longrightarrow & X_1 & \longrightarrow & X \\ & \searrow^{\beta} & \downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\alpha} \\ & & \tilde{Y} & \xrightarrow{\ \tau\ } & Y \end{array}$$

such that π which is the composition $\tilde{X} \to X_1 \to X$, $X_1 \to X$, τ are modifications, $\tilde{X}, \tilde{Y}$ are smooth and $\tilde{\alpha}$ is flat. Denote

$$K_{\tilde{X}} = \pi^* K_X + E, \, K_{\tilde{Y}} = F$$

where E, F effective divisors. Let $\omega_{\tilde{X}}$ be a Kähler form on $\tilde{X}$. For any $\epsilon > 0$, $c_1(\pi^*(-K_X)) + \epsilon\omega_{\tilde{X}}$ is Kähler. Since $\beta(E) \neq \tilde{Y}$ (as $\beta(E)$ is contained in the exceptional divisor of τ), $c_1(K_{\tilde{X}} + \pi^*(-K_X)) + \epsilon\omega_{\tilde{X}}$ is Kähler when restricting to a general fiber of β. By [55, Theorem 1.1], $c_1(K_{\tilde{X}/\tilde{Y}} + \pi^*(-K_X)) + \epsilon\omega_{\tilde{X}}$ is psef. Thus after passing to the limit, $E - \beta^* F$ is psef and we get

$$(E - \beta^* F, \pi^*(\omega_X)^{n-1}) = -(\beta^* F, \pi^*(\omega_X)^{n-1}) \geq 0.$$

Therefore, $\beta^*(F)$ must be $\pi-$exceptional and, in particular, $\alpha^{-1}(W)$ is of codimension at least 2. □

As a direct application, we have the following corollary:

Corollary 4.5 *Let X be a compact Kähler manifold such that $\tilde{q}(X) = q(X) = 2$ and $-K_X$ is nef. Then the Albanese map of X is flat.*

Remark 4.6 As conjectured in [7], if X is not uniruled, K_X should be psef. If $-K_X$ is nef, we should have $c_1(X) = 0$. The interest of the above arguments is a substitute of [7] to prove the vanishing of Chern class.

For example, let X be a compact Kähler non-uniruled $n-$fold with nef anticanonical divisor. If $q(X) = \tilde{q}(X) \geq n-3$, by adjunction formula, the anticanonical divisor of a general fiber F is nef. We claim that the general fiber is not uniruled. If F is uniruled (which is meaningful by [31]), there exists a rational curve passing through a general point of X which implies that X is uniruled which is a contradiction. By the condition on the dimension, K_F is also psef which implies that the fibers are Calabi-Yau. Proposition 4.2 then implies that $c_1(X) = 0$.

Combined with Proposition 3.1, the only unknown cases of the conjecture of [7] for compact Kähler fourfold with nef anticanonical divisor are simply connected ones. Note that it is not enough to apply [10, Theorem 1.1] to conclude in this case. In fact, in the three-dimensional case, the non-projectivity implies the existence of a non-trivial holomorphic two-form. We can then apply [10, Theorem 1.1] to conclude that the non-pseudoeffectivity of the canonical line bundle implies uniruledness. However, in dimension four, to apply [10, Theorem 1.1], we would need a non-trivial holomorphic three-form. But this is not always the case, even if the anticanonical line bunde is nef as shown in the following example. Conjecturally, a compact Kähler non-uniruled fourfold with nef anticanonical divisor should have vanishing first Chern class. If such a manifold X is not projective and simply connected, $h^{2,0}(X) \neq 0$ and then X should be hyperkähler or a product of K3 surfaces. But there exists a hyperkähler fourfold with trivial $H^{3,0}(X)$.

5 Second case: fibers are projective spaces

Lemma 5.1 *Let X be the complement of a closed analytic subset of codimension at least 2 in a complex torus. Then any $\mathbb{P}^{r-1}-$bundle over X is isomorphic to $\mathbb{P}(E)$ for some holomorphic vector bundle E over X with* $\det(E) = \mathcal{O}_X$ *up to some finite étale cover.*

Proof Denote $X = A \setminus Z$ where A is a torus and Z is of codimension at least 2. The obstruction of a $\mathbb{P}^{r-1}-$bundle over X being isomorphic to $\mathbb{P}(E)$ for some holomorphic vector bundle E over X with $\det(E) = \mathcal{O}_X$ lies in $H^2(A \setminus Z, \mathbb{Z}_r)$ by the theory of Brauer groups. We refer to [28, Section 1,2] for more information about Brauer groups.

Here, we follow the proof of [13, Lemma 7.4]. From the exact sequence

$$0 \to \mathbb{Z}_r \to \mathrm{SL}(r, \mathcal{O}_X) \to \mathrm{PSL}(r, \mathcal{O}_X) \to 0,$$

we see that the obstruction lies in

$$H^2(A \setminus Z, \mathbb{Z}_r) = H^2(A, \mathbb{Z}_r) = H^2(A, \mathbb{Z})/H^2(A, r\mathbb{Z})$$

by the following Lemma 5.2. Now to show that $H^2(A, \mathbb{Z}_r) = H^2(A, \mathbb{Z})/H^2(A, r\mathbb{Z})$, we argue as follows: Since the torus A is an Eilenberg-MacLane space, the singular cohomology of the torus can be calculated by the group cohomology of its fundamental group. Consider the long exact sequence of group cohomology associated to the short exact sequence of groups

$$0 \to r\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_r \to 0.$$

We have an exact sequence

$$0 \to H^2(A, \mathbb{Z})/H^2(A, r\mathbb{Z}) \to H^2(A, \mathbb{Z}_r) \to H^3(A, r\mathbb{Z}) \to H^3(A, \mathbb{Z})$$

which implies the desired isomorphism as the map $H^3(A, r\mathbb{Z}) \to H^3(A, \mathbb{Z})$ is injective.

Thus the obstruction vanishes after passing to some finite étale cover. Notice that any finite étale cover of $A \setminus Z$ extends uniquely to a finite étale cover of A. This finishes the proof of the lemma. □

Lemma 5.2 *Let X be the complement of a closed analytic subset of codimension at least k in a complex manifold Y. Then for any $i \leq 2k-1$, $H^i(X, \mathbb{Z}) \simeq H^i(Y, \mathbb{Z})$. Similar results hold if we change the coefficient ring to $\mathbb{Z}_r$ for any $r \geq 2$.*

Proof By stratification of closed analytic subset (cf. e.g. [24, Chap II, Proposition 5.6]), without loss of generality, we may assume that X is the complement of a smooth complex submanifold of codimension at least k in a complex manifold Y. Consider long exact sequence

$$H^{i-1}(Y, X, \mathbb{Z}) \to H^i(Y, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(Y, X, \mathbb{Z}).$$

It is enough to show that for $i \leq 2k-1$, by considering a tubular neighborhood of $Y \setminus X$ and the excision theorem,

$$H^i(Y, X, \mathbb{Z}) \simeq H^i(Y \setminus X \times \mathbb{D}, Y \setminus X \times \mathbb{D}^*, \mathbb{Z}) = 0$$

where $\mathbb{D}$ is the disk of dimension of the real codimension of $Y \setminus X$ and $\mathbb{D}^*$ is the punctured disk. Since $H^\bullet(\mathbb{D}, \mathbb{D}^*, \mathbb{Z})$ is finite generated free group, Künneth theorem implies that

$$H^i(Y \setminus X \times \mathbb{D}, Y \setminus X \times \mathbb{D}^*, \mathbb{Z}) \simeq \oplus_k H^{i-k}(Y \setminus X, \mathbb{Z}) \otimes H^k(\mathbb{D}, \mathbb{D}^*, \mathbb{Z}) = 0$$

by the codimension condition. □

Remark 5.3 In general, the obstruction of a $\mathbb{P}^{r-1}$-bundle being the projectivization of some holomorphic vector bundle lies in the torsion part of $H^2(X, \mathcal{O}_X^*)$. Here, we have improved the method of [57] and our argument works in the more general setting. Let us compare our argument with theirs. In [57], the authors show the vanishing of the torsion part of $H^2(X, \mathcal{O}_X^*)$ in the setting of Lemma 5.1 when the dimension of X is two. More precisely, they show that $H^2(X, \mathcal{O}_X) = 0$ by using a result of Malgrange, the remark following [51, Thm. 8]. This implies that the torsion part of $H^2(X, \mathcal{O}_X^*)$ is the same as the torsion part of $H^3(X, \mathbb{Z})$. Note that the latter is trivial by Lemma 5.2. However, in general, their arguments fail in arbitrary dimension at the point that $H^2(X, \mathcal{O}_X)$ can be non-trivial.

To give an easy and concrete example, let X be the complement of $A_1 \times \{p\}$ in $A_1 \times A_2$ where A_1, A_2 are complex tori with A_1 of dimension at least 2. Consider the restriction maps

$$H^2(A_1 \times A_2, \mathcal{O}_{A_1 \times A_2}) \to H^2(X, \mathcal{O}_X) \to H^2(A_1 \times \{q\}, \mathcal{O}_{A_1 \times \{q\}})$$

where $q \neq p \in A_2$. By the Künneth formula, the composition map is surjective with non-trivial image. In particular, $H^2(X, \mathcal{O}_X) \neq 0$.

We give a more topological proof of the following result in [18].

Theorem 5.4 *(= Theorem C) Let X be a compact Kähler manifold such that $-K_X$ is nef. Assume that the general fiber of the Albanese map is $\mathbb{P}^{r-1}$. Then $X \simeq \mathbb{P}(E)$ for some numerical flat vector bundle of rank r over $A(X)$, up to some finite étale cover.*

Proof Let Z be the non-flat locus of α. By Lemma 4.4, Z is of codimension at least 3. By [1, Theorem 3], since the Albanese map is smooth in codimension one with $\mathbb{P}^{r-1}$ as the general fiber, $\alpha_{\alpha^{-1}(A(X)\setminus Z)}$ is a $\mathbb{P}^{r-1}$−bundle over $A(X) \setminus Z$. By Lemma 5.1, there exists a holomorphic vector bundle E over $A(X) \setminus Z$ of rank r with $\det(E) = \mathcal{O}_X$ such that

$$X|_{\alpha^{-1}(A(X)\setminus Z)} \simeq \mathbb{P}(E)$$

up to passing to some finite étale cover of X.

In particular, $c_1(E) = 0$ and $-K_X = \mathcal{O}_X(r)$ on $A(X) \setminus Z$. By the Leray-Hirsch theorem, we have

$$H^2(\alpha^{-1}(A(X) \setminus Z), \mathbb{R}) \simeq \mathbb{R}c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \oplus \alpha^* H^2(A(X) \setminus Z, \mathbb{R}).$$

Let ω be a normalised Kähler form on X (i.e. such that the restriction of $\omega + c_1(K_X)$ to the general fiber of α is trivial, hence for all fibers over $A(X) \setminus Z$ by the Leray-Hirsch theorem). We claim that there exists $\beta \in H^{1,1}(A(X), \mathbb{R})$ such that we have

$$c_1(K_X) + \omega = \alpha^* \beta$$

in $H^{1,1}(X, \mathbb{R})$. By Lemma 4.3, it is enough to show the existence of $\beta \in H^{1,1}(A(X), \mathbb{R})$ satisfying the equality on $\alpha^{-1}(A(X) \setminus Z)$. Define

$$\beta := \frac{1}{r^{r-1}} \alpha_*(c_1(-K_X)^{r-1} \cdot \omega).$$

The claim follows by construction. More precisely, over $\alpha^{-1}(A(X) \setminus Z)$, $\omega = c_1(\mathcal{O}_X(r)) + \alpha^* \beta$ for some β. Since $\alpha_*(c_1(\mathcal{O}_X(1))^r) = c_1(E) = 0$ and by the projection formula,

$$\beta = \frac{1}{r^{r-1}} (\alpha_*(c_1(\mathcal{O}_X(r))^r) + \alpha_*(\alpha^* \beta \cdot c_1(\mathcal{O}(r))^{r-1})) = \frac{1}{r^{r-1}} \alpha_*(c_1(-K_X)^{r-1} \cdot \omega).$$

(The same equality holds in finer cohomology as integral Bott-Chern cohomology as shown in [69].)

In particular, $-K_X$ is $\alpha-$ample and α is projective. Note that E can be extended over $A(X)$ as a reflexive sheaf $\mathcal{F}$ by [62, Theorem 2] since the codimension of Z is at least 3. We claim that the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ can be extended over X. The line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ can also be endowed with a smooth metric whose curvature has finite L^2 norm and whose local potentials of $\mathcal{O}_{\mathbb{P}(E)}(1)$ are the local potentials of $-K_X$ divided by r. By the removable singularity theorem (see [2, Lemma 1]), the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ has a reflexive extension. Its direct image is a reflexive extension of E by [41, Cor.1.7] (which is thus $\mathcal{F}$). (Note that the proof of [41] goes through as long as the preimage of the non-flat locus is of codimension at least two.) Note that the extended line bundle of $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef and $\alpha-$ample as $-K_X$. By the positivity of direct images [53, Proposition 2.6], $\mathcal{F}$ is weakly positively curved. Since $c_1(E) = 0$, $c_1(\mathcal{F}) = 0$ on $A(X)$.

Next, we show that $\mathcal{F}$ is locally free by [2]. By nefness of the extended line bundle of $\mathcal{O}_{\mathbb{P}(E)}(1)$ and the fact that Z is of codimension at least 3 (such that $H^4(A(X) \setminus Z, \mathbb{R}) \simeq H^4(A(X), \mathbb{R})$), $c_1^2(\mathcal{F}) - c_2(\mathcal{F}) = -c_2(\mathcal{F})$ contains a closed positive current whose restriction on $A(X) \setminus Z$ represents $\alpha_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{r+1})$. We refer to [66, Remark 1] for the construction of Chern classes of a coherent sheaf over $A(X) \setminus Z$ in the de Rham cohomologies. If $\mathcal{F}$ is stable, by [2, Corollary 3], the equality case of the Bogomolov-Gieseker inequality implies that $\mathcal{F}$ is a projectively flat vector bundle. Since $c_1(\mathcal{F}) = 0$, $\mathcal{F}$ is hermitian flat.

We show in the following that $\mathcal{F}$ is a numerically flat vector bundle following the arguments of [14, Proposition 2.7], if $\mathcal{F}$ is not stable. If $\mathcal{F}$ is not stable, consider the Harder-Narasimhan filtration of $\mathcal{F}$ respect to ω, say

$$\mathcal{F}_0 = 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_m := \mathcal{F}$$

where $\mathcal{F}_i / \mathcal{F}_{i-1}$ is ω-stable for every i and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$, and where $\mu_j = \mu_\omega(\mathcal{F}_j / \mathcal{F}_{j-1})$ is the slope of $\mathcal{F}_j / \mathcal{F}_{j-1}$ with respect to ω. Now, consider the coherent subsheaf $\mathcal{S} = \mathcal{F}_{m-1}$. Without loss of generality, we can assume that $\mathcal{S}$ is reflexive by taking the double dual if necessary, as this preserves the rank, first Chern class and slope. Then we get a short exact sequence

$$0 \to \mathcal{S} \to \mathcal{F} \to \mathbb{Q} \to 0.$$

As a quotient of $\mathcal{F}$, $\mathbb{Q}$ moduo its torsion part is also weakly positively curved. In particular, $c_1(\mathbb{Q})$ is pseudoeffective. Together with the stability condition of the filtration and the fact that

$$0 = c_1(\mathcal{F}) = c_1(\mathcal{S}) + c_1(\mathbb{Q}),$$

we can easily check that $c_1(\mathcal{S}) = c_1(\mathbb{Q}) = 0$.

By [25, Corollary 1.19], the inclusion $\mathcal{S} \to \mathcal{F}$ is a bundle morphism on the locally free locus of $\mathcal{F}$ (i.e. $A(X) \setminus Z$). In particular, $\mathbb{Q}$ is locally free on $A(X) \setminus Z$. Compared to the case of an arbitrary weakly positive reflexive sheaf with vanishing first Chern class over a projective manifold as studied in [14], there exist smooth metrics with lower bounds on the curvature of $\mathcal{O}_{\mathbb{P}(E)}(1)$ (induced from $-K_X$ and the nefness condition of $-K_X$). Now we deviate from [14] and give a more simplified argument based on these metrics. Restriction of these metrics on $\mathcal{O}_{\mathbb{P}(\mathbb{Q})}(1)$ to $A(X) \setminus Z$ shows that over $A(X) \setminus Z$, the cohomology class

$$\alpha_*(c_1(\mathcal{O}_{\mathbb{P}(\mathbb{Q})}(1))^{\mathrm{rank}\mathbb{Q}+1}) = c_1^2(\mathbb{Q}) - c_2(\mathbb{Q}) = -c_2(\mathbb{Q})$$

contains a closed positive $(2,2)-$current. Since Z is of dimension at least 3, this current extends across Z. In particular, $-c_2(\mathbb{Q}^{**})$ contains a closed postive current. On the other hand, $\mathbb{Q}^{**}$ is stable. The equality case of the Bogomolov-Gieseker inequality is attained and thus $\mathbb{Q}^{**}$ is also a hermitian flat vector bundle by [2].

(If the rank $r = 2$, $\mathbb{Q}$ can only be of rank 1. We have that $c_1(\mathbb{Q}^{**}) = c_1(\mathbb{Q}) = 0$ and $\mathbb{Q}^{**}$ is a hermitian flat line bundle.)

The natural morphism on the locally free locus of $\mathcal{F}$

$$\wedge^{\mathrm{rank}\mathcal{F}-\mathrm{rank}\mathbb{Q}-1}\mathcal{F} \otimes \det(\mathbb{Q})^{-1} \to \mathcal{S}$$

implies that $\mathcal{S}$ is weakly positively curved after extension to $A(X)$ and it admits a sequence of smooth metrics with increasing lower bound on $A(X) \setminus Z$. Since

$$c_2(\mathcal{F}) = c_2(\mathbb{Q}^{**}) + c_2(\mathcal{S}) + c_1(\mathcal{S}) \cdot c_1(\mathbb{Q}) = c_2(\mathcal{S}),$$

if $\mathcal{S}$ is not stable, we can continue to think about its Harder-Narasimhan filtration and by induction on the rank conclude that $\mathcal{S}$ is a numerically flat vector bundle.

By [68, Lemma 4], the restriction map

$$H^1(A(X), \mathcal{H}om(\mathbb{Q}^{**}, \mathcal{S})) \to H^1(A(X) \setminus Z, \mathcal{H}om(\mathbb{Q}^{**}, \mathcal{S}))$$

is surjective. Hence the extension class in $H^1(A(X) \setminus Z, \mathcal{H}om(\mathbb{Q}^{**}, \mathcal{S}))$ obtained from the Harder-Narasimhan filtration on $A(X) \setminus Z$ can be extended to an extension class in $H^1(A(X) \setminus Z, \mathcal{H}om(\mathbb{Q}^{**}, \mathcal{S}))$. The extended class determines a vector bundle whose restriction to $A(X) \setminus Z$ is isomorphic to $\mathcal{F}$ by construction. It follows that this vector bundle coincides with $\mathcal{F}$ on $A(X)$ since $\mathcal{F}$ is reflexive. Note that $\mathcal{F}$ is a numerically flat vector bundle in this case.

In conclusion, we have shown that X is bimeromorphic to $\mathbb{P}(\mathcal{F})$ over $A(X)$ which is isomorphic in codimension one. Following [49, Lemma 6.39], we show that it is in fact an isomorphism. Let $p : \mathbb{P}(\mathcal{F}) \to A(X)$ be the natural projection. Note that $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(r)$ is $p-$ample such that its strict transform $-K_X$ is nef. It is enough to show that for any $\alpha-$ample Cartier divisor D on X, the induced Cartier divisor D' on $\mathbb{P}(\mathcal{F})$ is $p-$nef. Since p is a projective morphism, it is enough to show for any (compact) curve C contained in a fiber of p, the intersection number $(c_1(D') \cdot C) \geq 0$. On the other hand, since p is locally trivial, C deforms to a curve over $A(X) \setminus Z$ which is thus isomorphic to a curve in X. Thus $(c_1(D') \cdot C) \geq 0$ since D is $\alpha-$ample. □

Remark 5.5 Notice that manifolds fulfilling the requirements of Theorem 5.4 (or compact Kähler manifolds whose finite étale cover is such a manifold) can be algebraically approximated. Indeed, the tangent bundle of $\mathbb{P}(E)$ of a numerically flat vector bundle over a torus is nef by [13, Lemma 7.4] (the proof works identically in the analytic setting). Then the algebraic approximation exists by [16, Theorem 6.3]. In fact, by Simpson's correspondance [61], E is flat which corresponds to a representation of $\pi_1(X)$ in $\mathrm{GL}(r, \mathbb{C})$ where r is the rank of E. Since the fundamental group is preserved under deformation, we can deform $\mathbb{P}(E)$ when deforming the torus to get the algebraic approximation.

Remark 5.6 As shown in [21, Example 1.2], Hirzebruch surfaces can admit non-trivial (flat) deformations over a polydisc of dimension at least 2 which are locally trivial outside the origin. In particular, the analogous result of [1] does not hold for Hirzebruch surfaces. By [6, Theorem 1.3], the only globally rigid rational surface is $\mathbb{P}^2$. (Here $\mathbb{P}^2$ being globally rigid means that any surface deformation equivalent to $\mathbb{P}^2$ is $\mathbb{P}^2$.) It seems that the above topological approach (by extension from a big Zariski open set, i.e. the complement in the Albanese torus is of codimension at least 2) is hard to generalise to a Del Pezzo fibration. In particular, it seems unclear how to classify uniruled compact Kähler fourfolds with nef anticanonical line bundle and two-dimensional Albanese torus. (Notice that unlike to the pseudoeffectivity of the anticanonical line bundle, the example in [21] also shows that the nefness of the anticanonical line bundle is not a closed condition.)

The above phenomenon in remark 5.6 can be ruled out if each fiber of the deformation has nef anticanonical line bundle. In particular, assuming that the Albanese morphism is smooth and fibers are surfaces, we get a structure theorem, cf. [57, Cor. 3.4], as we show in the next section.

6 The general uniruled case when fibers are surfaces

We start with a Kähler version of [57, Proposition 0.4]:

Proposition 6.1 *Let X, Y be compact Kähler manifolds, $\pi : X \to Y$ a smooth morphism with $-K_F$ is nef for all fibers F which are moreover non-minimal surfaces. Assume that there exists a (-1)-curve C in some fiber F and a Kähler class ω on X such that*

$$(K_F + C) \cdot \omega < 0.$$

Then there exists an étale base change $\sigma : X' = X \times_Y Y' \to Y'$ induced by an étale map $Y' \to Y$, and a smooth effective divisor $S \subset X'$ such that the restriction $\sigma|_S : S \to Y'$ yields a $\mathbb{P}^1$-bundle structure on S, and $S \cap F$ is a (-1)-curve in F for all fibers F. Hence X' can be blown down over Y' along S.

Proof For convenience of the reader, we reproduce the arguments from [57, Proposition 0.4]. First, note that all non-minimal surfaces F with $-K_F$ nef are isomorphic to the plane $\mathbb{P}^2$ blown up in at most 9 points in sufficiently general position.

We fix the Kähler class ω on X. Consider the fiber F of π in the assumption and take a (-1)-curve $E \subset F$ such that $\omega \cdot E$ is minimal among all (-1)-curves in F. By assumption,

$$(K_F + E) \cdot \omega < 0.$$

In the following, we study deformations of this $(-1)-$curve in X that satisfy this additional condition on the intersection number and we will pay special attention to their degenerations.

Step 1: Deformation of $(-1)-$*curves in* X.

The normal bundles of E satisfy

$$0 \to N_{E/F} \simeq \mathcal{O}(-1) \to N_{E/X} \to N_{F/X}|_E \simeq \mathcal{O}_E^{\dim Y} \to 0.$$

Since $H^1(E, \mathcal{H}om(N_{F/X}|_E, N_{E/F})) = H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, the short exact sequence splits and the normal bundle is of the form

$$N_{E|X} = \mathcal{O}^{\dim Y} \oplus \mathcal{O}(-1).$$

Since $H^1(E, \mathcal{H}om(N_{F/X}|_E, N_{E/F})) = 0$, the germ of the (reduced) Douady space is locally isomorphic to the Kuranishi space (see e.g. [54, proof of Theorem 2]). In particular, E corresponds to a regular point of the Douady space (more precisely the deformation constructed by Kodaira's theory on complete families of compact complex submanifolds.) Note that the irreducible component of the Douady space containing E is of dimension dim Y (as $H^0(E, N_{E/X})) = \mathbb{C}^{\dim Y}$) which is moreover compact. The compactness follows from [32]. We will show in the following that it is in fact smooth.

To fix notations, there exists a compact complex space Y' and a closed analytic subset $M \subset X \times Y'$, flat over Y' such that $M \cap (X \times 0) = E$, identifying $X \times 0$ with X. We let

$$E_t = M \cap (X \times t)$$

and shall identify X with $X \times t$.

Claim 1: Every E_t is contained in some fiber F' of π.

Otherwise, $\pi(E_t)$ will be a non-trivial effective curve. This contradicts the fact that the E_t are in the same cohomology class and thus they are the $\pi(E_t)$. Note that $[\pi(E)] = 0 \in H^{2\dim(Y)-2}(Y, \mathbb{Z})$. In particular, there exists a commutative diagram by rigidity lemma (see e.g. [23, Lemma 1.15])

$$\begin{array}{ccc} M & \longrightarrow & X \\ \downarrow & & \downarrow \pi \\ Y' & \xrightarrow{\varphi} & Y. \end{array}$$

By Remmert's proper mapping theorem, the holomorphic map $Y' \to Y$ is surjective. More precisely, otherwise, the image of Y' is a proper analytic subset of Y. In particular, by semicontinuity of dimension of fiber, the preimage of $\pi(E)$ has strictly positive dimension. In other words, F contains a family of rational curves containing E of strictly positive dimension. This contradicts the fact that E is a (-1)-curve, thus it does not deform in F. This finishes the proof of Claim 1.

Let us continue to study (local) deformations of E. To show that the morphism $Y' \to Y$ is in fact locally isomorphic near the point coresponding to E, we need a local description of the Douady space using Kodaira's theory (see e.g. [48, Main Theorem]). Locally, one can cover X by coordinate charts $(z_i^\lambda, w_j^\lambda)$, $i = 1, 2$; $1 \leq j \leq \dim(Y)$ where λ is the index of the coordinate chart such that $\pi(z_i^\lambda, w_j^\lambda) = w_j^\lambda$ and E is locally given by $\{z_2^\lambda = w_j^\lambda = 0, \forall j\}$. By Kodaira's theory, the deformation of E is locally given by $(z_1^\lambda, z_2^\lambda = \psi^\lambda(z_1^\lambda, t), w_j^\lambda = \varphi_j^\lambda(z_1^\lambda, t))$ for some local holomorphic functions $\psi^\lambda(z_1^\lambda, t), \varphi_j^\lambda(z_1^\lambda, t)$ where t is the coordinate of the parameter space. The differential of the map $t \mapsto (\psi^\lambda(z_1^\lambda, t), \varphi_j^\lambda(z_1^\lambda, t))$ gives an identification of the tangent space of the parameter space at 0 with $H^0(E, N_{E/X})$. In particular, the differential of the map $t \mapsto (\varphi_j^\lambda(z_1^\lambda, t))$ is isomorphic. (And $\psi^\lambda(z_1^\lambda, t)$ is independent of t.) On the other hand, in local coordinates, $\varphi(t) = (\varphi_j^\lambda(z_1^\lambda, t))$ which implies that φ is locally isomorphic.

Note that the above calculation is valid for any smooth (-1)-rational curve. Observe next that

$$-K_X \cdot E_t = -K_X \cdot E = 1.$$

Step 2: Degeneration of (-1)-curves in X under the condition on the intersection number. Consider some E_{t_0}. Write

$$E_{t_0} = \sum_{i\geq 0} a_i C_i$$

with irreducible curves C_i (as a cycle, neglecting the possible embedded point). Since $-K_{F'}$ is nef, we conclude (after renumbering possibly) that

$$a_0 = 1, -K_{F'} \cdot C_0 = 1$$

and that

$$-K_{F'} \cdot C_i = 0, i \geq 1.$$

Since E_{t_0} is a deformation of E and the intersection number is invariant under deformation, we have that

$$(K_{F'} + E_{t_0}) \cdot \omega < 0.$$

Claim 2: C_0 is a (-1)-curve in F' and the $C_i, i \geq 1$, are (-2)-curves.
Define

$$\mathcal{I} := \mathcal{O}_{F'}(-E_{t_0}) \subset \mathcal{O}_{F'}.$$

Consider the map

$$H^1(F', \mathcal{O}_{F'}) \to H^1(F', \mathcal{O}_{F'}/\mathcal{I}) \to H^2(F', \mathcal{I}) = H^0(F', K_{F'} + \sum a_i C_i) = 0$$

where the isomorphism follows from the Serre duality and the vanishing follows from the inequality $(K_{F'} + C) \cdot \omega < 0$. Since F' is rational, $H^1(F', \mathcal{O}_{F'}/\mathcal{I}) = H^1(F', \mathcal{O}_{F'}) = 0$.

Since $H^1(E_{t_0}, \mathcal{O}_{E_{t_0}}) = 0$, any irreducible component is a smooth rational curve. More precisely, let $E_{t_0}^\nu$ be the normalisation of E_{t_0} with $\nu : E_{t_0}^\nu \to E_{t_0}$ a finite morphism. The cokernel of $\mathcal{O}_{E_{t_0}} \to \nu_*(\mathcal{O}_{E_{t_0}}^\nu)$ is supported on finitely many closed points and hence has vanishing higher cohomology. On the other hand, the second cohomology of the kernel of $\mathcal{O}_{E_{t_0}} \to \nu_*(\mathcal{O}_{E_{t_0}}^\nu)$ is trivial since E_{t_0} is of dimension one. By considering the associated long exact sequence, we find that

$$H^1(E_{t_0}, \mathcal{O}_{E_{t_0}}) \to H^1(E_{t_0}, \nu_*\mathcal{O}_{E_{t_0}^\nu}) = H^1(E_{t_0}^\nu, \mathcal{O}_{E_{t_0}^\nu})$$

is surjective. In particular, any irreducible component is a smooth rational curve.

The adjunction formula shows that C_0 is a $(-1)-$curve in F' and the $C_i, i \geq 1$, are $(-2)-$curves. This finishes the proof of Claim 2.

Claim 3: E_{t_0} has only one irreducible component.
Now we look at the deformations of C_0 and obtain a family $(C_s)_{s\in A}$. By the discussion on the deformation of (-1)-curves in X (as above), we get a morphism $f : A \to Y$ such that

$f(0) = \varphi(t_0)$ which is an isomorphism near $0 \in A$ for some small disc A. Therefore we can consider the (non-effective) family of cycles $(E_t - C_t)_{t \in A}$ so that

$$E_{t_0} - C_0 = \sum_{i \geq 1} a_i C_i.$$

(Strictly speaking, we would need to consider the fiber product of A and Y' over Y to get a common parameter space for both families of deformations (E_t) and (C_s), but we surpress this here by abuse of notation.) By the choice of E, we claim that $\omega \cdot E_t$ is minimal among the (-1)-curves in $\pi^{-1}(t)$ for general t, therefore $\omega \cdot E_t \leq \omega \cdot C_t$ and we conclude

$$\omega \cdot \sum_{i \geq 1} a_i C_i = 0$$

and therefore $a_i = 0$ for $i \geq 1$ so that E_{t_0} is irreducible and reduced. This would finish our proof of claim 3.

Let us prove our claim on the minimality. As above, let F' be the fiber that contains E_{t_0}. In fact, we have that

$$\omega \cdot (K_{F'} + C_0) < 0,$$

since otherwise we would have

$$C_0 \cdot \omega \geq -K_{F'} \cdot \omega > E \cdot \omega = E_{t_0} \cdot \omega,$$

which gives a contradiction. Here, the second inequality follows from the assumption

$$(K_F + E) \cdot \omega < 0.$$

Recall that a general deformation of the (-1)-curve C_0 is still a smooth (-1)-curve in the corresponding fiber. By the above discussion, a special deformation of the (-1)-curve C_0 still has a (-1)-curve (as an irreducible component). In any case, there is - possibly a component of - some deformation C_s that is a smooth (-1)-curve sitting in the initial fiber F that we startet with. This allows for comparing the intersecton numbers for E_t and C_s with ω with the value given at the fiber F. The minimality of $\omega \cdot E_t$ now follows from the minimality of $E \cdot \omega$.

Claim 4: Y' is smooth at t_0.
Since E_{t_0} is a smooth (-1)-curve, we have

$$h^1(E_{t_0}, N_{E_{t_0}/X}) = 0$$

and Y' is smooth at t_0 (as discussed in Step 1).

Step 3: Construction of base change and blow down.
In conclusion, Y' is smooth and $M \to Y'$ is a $\mathbb{P}^1$-bundle. The fact that any point of Y' represents some (-1)-curve in some fiber of π and the fact that φ is a local isomorphism at a point corresponding to a (-1)-curve (as discussion on deformation of (-1)-curves in Step 1) implies that $\varphi : Y' \to Y$ is étale. Define $S = \mathrm{pr}_1(M) \times_Y Y'$.

The existence of the blow down follows from the Grauert-Fujiki contraction theorem. An english reference is [30, Theorem 2]. In loc.cit. the assumption is that the conormal bundle is σ-ample such that all relative higher cohomologies of its multiple vanish. Here, the restriction of S to any fiber of σ is a disjoint union of (-1)-curves in a smooth rational surface with nef anticanonical line bundle which thus satisfies the required assumptions. □

Remark 6.2 Note that the assumption on the negativity of the intersection number $(K_F+C)\cdot\omega$ is necessary for the existence of a blow-down as in Proposition 6.1 for a well-chosen Kähler form ω on X if the manifold is still Kähler with $-K_F$ big after blowing down. More precisely if, after a blow down $p: X \to X_1$, the anticanonical line bundle of the Albanese map of X_1 is relatively big and the total space X_1 is still Kähler, we can choose a Kähler metric on X that satifies the condition on the intersection number in Proposition 6.1. This is the case which we consider in the rest of this section. Since the assumption is invariant under finite étale covers, we may assume that there exists a blow down $p: X \to X_1$ of X to a compact Kähler manifold X_1 which is obtained by blowing down (-1)-curves in the fibers of π. Let ω_{X_1} be a Kähler form on X_1 and let E be the exceptional divisor of p. For $\epsilon > 0$ small enough, $p^*\omega_{X_1} - \epsilon c_1(E)$ is still a Kähler class on X. Let F be a fiber of π and C a (-1)-curve in F such that $p(C)$ is a point. Let C_i, $(1 \leq i \leq k)$ be the exceptional divisors of $p|_F$. Note that the C_i are disjoint. Then we have for ϵ small enough

$$\begin{aligned}
&(K_F + C) \cdot (p^*\omega_{X_1} - \epsilon c_1(E))|_F \\
&= \left(p|_F^* K_{p(F)} + c_1\left(\sum_i C_i + C\right)\right) \cdot \left(p|_F^*\omega_{X_1}|_{p(F)} - \epsilon c_1\left(\sum_i C_i\right)\right) \\
&= (k+1)\epsilon + K_{p(F)} \cdot \omega_{X_1}|_{p(F)} < 0.
\end{aligned}$$

The assumption is satisfied if $(K_F)^2 \geq 2$ and $-K_X$ is nef. In fact, assume on the contrary that for any Kähler form ω and any $(-1)-$curve C in any fiber F we have

$$(K_F + C) \cdot \omega \geq 0.$$

In particular, the intersection number of $(K_F + C)$ with any nef class is non-negative. Thus

$$(K_F + C) \cdot (-K_X)|_F = (K_F + C) \cdot (-K_F) = -(K_F)^2 + 1 \geq 0$$

which is impossible.

Note that the proof of Proposition 6.1 holds if we replace the Kähler form ω by some big class α such that the restriction of α to any fiber F is modified Kähler (cf. e.g. [67]). In particular, by Demailly's theory of Monge-Ampère operators (see e.g. [24, Section 4, Chap III]), for any effective curve C in F, the intersection number $C \cdot \alpha|_F$ is strictly positive.

In particular, if $(K_F)^2 \geq 2$, $-K_X$ is nef and the other assumptions of Proposition 6.1 are satisifed, up to some finite étale cover, we can blow down $p: X \to X'$ to some compact manifold X' of class $\mathscr{C}$ which admits a smooth Del Pezzo fibration of degree at least 3 onto another compact manifold of class $\mathscr{C}$. Note that $p_*(c_1(-K_X) + \epsilon\omega)$ is a big class for any $\epsilon > 0$ whose restriction to any Del Pezzo surface fiber is modified Kähler.

Thus, if the Del Pezzo surface fiber is not $\mathbb{P}^2$ or a product $\mathbb{P}^1 \times \mathbb{P}^1$, one can apply again Proposition 6.1 to get a further blown-down (after passing to a suitable finite étale cover).

The following proposition is a special case of [18]. However, we find that its proof has its own interest using only deformation theory under the additional assumption that the Albanese morphism is smooth.

Proposition 6.3 *Let X be a uniruled compact Kähler manifold of dimension n such that $\tilde{q}(X) = q(X) = n-2$ and $-K_X$ is nef. Assume that the Albanese morphism $\alpha: X \to A(X)$ is smooth. Moreover, assume that for any fiber F of α,*

$$(K_F)^2 \geq 2.$$

Then, up to some finite étale cover, the possibilities of X are one of the following:

(1) $X \simeq \mathbb{P}(E)$ for some numerically flat vector bundle (of rank 3) over $A(X)$;
(2) $X \simeq \mathbb{P}(E_1) \times_{A(X)} \mathbb{P}(E_2)$ for some numerically flat vector bundles E_1 and E_2 (of rank 2) over $A(X)$;
(3) we have a factorisation

$$X \xrightarrow{\gamma_1} X_1 \cdots \xrightarrow{\gamma_k} X_k \xrightarrow{\rho} A(X)$$

with γ_j blow-ups of étale multi-sections, $X_k \simeq \mathbb{P}(E_1) \times_{A(X)} \mathbb{P}(E_2)$ for some numerically flat vector bundles E_1, E_2 over $A(X)$ or $X_k \simeq \mathbb{P}(E)$ for some numerically flat vector bundle over $A(X)$.

Proof First note that all non-minimal surfaces F with $-K_F$ nef are isomorphic to the plane $\mathbb{P}^2$ blown up in at most 9 points in sufficiently general position (by the Kodaira-Enriques classification). All minimal rational ones are either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. The case of $\mathbb{P}^2$ is treated as follows: By [13, Lemma 7.4], up to some finite étale cover, we have $X \simeq \mathbb{P}(E)$ for some vector bundles E over $A(X)$ with trivial determinant. Since $-K_X$ is nef, E is nef, hence numerically flat.

The case of $\mathbb{P}^1 \times \mathbb{P}^1$ can be treated similarly: Notice that the proof of [13, Lemma 9.4] works also in the compact Kähler setting. Up to some finite étale cover, we have $X \simeq \mathbb{P}(E_1) \times_{A(X)} \mathbb{P}(E_2)$ for some vector bundles E_1, E_2 over $A(X)$ with trivial determinant. By positivity of direct images applied to the submersions $X \to \mathbb{P}(E_i)$ $(i = 1, 2)$ with the condition that $-K_X$ is nef, $\mathbb{P}(E_i)$ $(i = 1, 2)$ has nef anticanonical line bundle which implies that E_1 and E_2 are numerically flat.

The remaining case can be treated as follows: By the preceding remark 6.2, the factorisation as stated in case (3) exists. Thus, X_k is a compact manifold of class $\mathscr{C}$ whose fibers of the Albanese morphism are either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. The conclusion follows from [13, Lemma 9.4] and [28]. Note that proof of [13, Lemma 9.4] also works for compact manifolds of class $\mathscr{C}$ by replacing the Chow variety by the corresponding Barlet space whose irreducible components are known to be compact (see e.g. [50, corollary on page 145]). In particular, all X_i's are in fact Kähler. Note that the Albanese morphism is always projective in this case. (In fact in this case, the Albanese morphism can be chosen to be a MRC quotient map.)

We claim that the Albanese morphism is a locally constant fibration (cf. [52, Definition 2.3]) by applying [52, Proposition 2.5]. Since all spaces are smooth, it is enough to show the existence of a relatively ample line bundle whose direct image is numerically flat. By [53, Theorem 2.6], it is enough to show the existence of a relatively ample and pseudoeffective line bundle. Denote by $\pi : X \to X_k$ the composition with exceptional divisor E. Since $-K_{X_k}$ is relatively ample with respect to the natural projection $X_k \to A(X)$, for $m \gg 0$, $m\pi^*(-K_{X_k}) - E$ is $\alpha-$ample. On the other hand, $-K_X = \pi^*(-K_{X_k}) - E$ which implies that $m\pi^*(-K_{X_k}) - E = -mK_X + (m-1)E$ is pseudoeffective.

Consider the last statement about the numerical flatness of the bundles E_1, E_2 and E. Let us first consider the case of E. Note that since the Albanese map is locally constant, the Albanese map is induced from a representation $\rho_1 : \pi_1(\mathrm{A}(X)) \to \mathrm{Aut}(F)$ where F is a fiber of the Albanese map. Let $\{g_k\}_{k \in I}$ be the image of a set of generators of $\pi_1(\mathrm{A}(X))$ under ρ_1. Since γ_1 is a blow-up of étale multi-sections, for any $k \in I$, g_k defines an automorphism of F which maps the exceptional divisor of $\gamma_1|_F$ to itself. By the following Lemma 6.4, we get a representation $\rho_2 : \pi_1(\mathrm{A}(X)) \to \mathrm{Aut}(\gamma_1(F))$ by composition. By the rigidity lemma [23, Lemma 1.15], X_1 is isomorphic to the complex manifold induced from the representation $\rho_2 : \pi_1(\mathrm{A}(X)) \to \mathrm{Aut}(\gamma_1(F))$. That is, the map $X_1 \to \mathrm{A}(X)$ is also locally constant. By induction, we can conclude that ρ corresponds to a representation $\pi_1(\mathrm{A}(X)) \to PGL(3, \mathbb{C})$ which is a $\mathbb{P}^2-$bundle over $\mathrm{A}(X)$. By abuse of notation we denote the corresponding representation also by ρ. By considering $PGL(3,\mathbb{C})$ as a $\pi_1(\mathrm{A}(X))-$module with the trivial action, we see that the group cohomology

$$H^1(\pi_1(\mathrm{A}(X)), PGL(3,\mathbb{C})) = Hom(\pi_1(\mathrm{A}(X)), PGL(3;\mathbb{C}))$$

contains the above representation $\rho : \pi_1(Alb(X)) \to PGL(3,\mathbb{C})$. Since $\mathrm{A}(X)$ is the Eilenberg-MacLane space associated to $\pi_1(\mathrm{A}(X))$, we have

$$H^1(\pi_1(\mathrm{A}(X)), PGL(3,\mathbb{C})) \simeq H^1(\mathrm{A}(X), PGL(3,\mathbb{C})).$$

Note that the $\mathbb{P}^2-$bundle ρ is not only an element in $H^1(\mathrm{A}(X), PGL(3,\mathcal{O}_{\mathrm{A}(X)}))$ but also the image of some element under the natural morphism

$$H^1(\mathrm{A}(X), PGL(3,\mathbb{C})) \to H^1(\mathrm{A}(X), PGL(3,\mathcal{O}_{\mathrm{A}(X)})).$$

Since E is defined as a preimage of ρ under the natural morphism

$$H^1(\mathrm{A}(X), SL(3,\mathcal{O}_{\mathrm{A}(X)})) \to H^1(\mathrm{A}(X), PGL(3,\mathcal{O}_{\mathrm{A}(X)}))$$

and by considering the commuting diagram

$$\begin{array}{ccccc} H^1(\mathrm{A}(X), SL(3,\mathbb{C})) & \longrightarrow & H^1(\mathrm{A}(X), PSL(3,\mathbb{C})) & \longrightarrow & H^2(\mathrm{A}(X),\mathbb{Z}_3) \\ \downarrow & & \downarrow & & \downarrow = \\ H^1(\mathrm{A}(X), SL(3,\mathcal{O}_{\mathrm{A}(X)})) & \longrightarrow & H^1(\mathrm{A}(X), PSL(3,\mathcal{O}_{\mathrm{A}(X)})) & \longrightarrow & H^2(\mathrm{A}(X),\mathbb{Z}_3) \end{array}$$

we can conclude that E is a flat vector bundle. By Simpson's correspondance [61, Corollary 3.10], E is a successive extension of Hermitian flat vector bundles which are thus numerically flat. More precisely, Simpson's correspondance implies that E is a semistable Higgs vector bundle and [61, Theorem 2] implies that E is a successive extension of stable Higgs vector bundles which correspond to irreducible representations of the fundamental group. Since the fundamental group of a torus is finitely generated and abelian, the irreducible representations are one dimensional. In other words, those stable Higgs vector bundles are flat line bundles. Moreover, since a torus is compact Kähler, $\partial\overline{\partial}-$lemma implies that the flat line bundles are in fact Hermitian flat. The case of E_1, E_2 can be treated similarly. □

Lemma 6.4 *Let $f : X \to Y$ be a modification of projective manifolds with exceptional divisor E. Assume that f has connected fibers and $f(E)$ is of dimension 0. Then there exists a group morphism*

$$\{g \in \mathrm{Aut}(X) : g(E) = E\} \to \mathrm{Aut}(Y).$$

Proof Let g be an automorphism of X such that $g(E) = E$. Since the fibers of f are connected, by the induced topology on E, the connected components of E are distinct fibers of f. The automorphism g permutes different connected components of E. Since by our assumption, any fiber contracted under the morphism $X \xrightarrow{g} X \xrightarrow{f} Y$ is also contracted by f, the rigidity lemma [23, Lemma 1.15] implies that we have a holomorphic morphism $g' : Y \to Y$ such that the above composition is given by $X \xrightarrow{f} Y \xrightarrow{g'} Y$. Because $f(E)$ is of dimension 0 by assumption, the map g' is unique. By the uniqueness it is easy to conclude that the above defined morphism $\{g \in \mathrm{Aut}(X) : g(E) = E\} \to \mathrm{Aut}(Y)$ is a group morphism. □

Note that the assumption about $f(E)$ being of dimension 0 is necessary as shown by the example of the Atiyah flop:

Example 6.5 Consider the threefold singularity in $\mathbb{C}^4$ given by the hypersurface

$$X_0 = \{(x, y, z, w) \in \mathbb{C}^4 \mid xy - zw = 0\}.$$

Note that the permutation of coordinates $\sigma_0(x, y, z, w) = (x, y, w, z)$ defines a non-trivial automorphism of X_0. The variety X_0 defines an ordinary double point singularity at the origin. Blowing up the origin gives a resolution of singularities X with exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The above permutation induces an automorphism of X which we will denote by σ. The variety X_0 admits two distinct small resolutions by blowing down X, obtained by introducing two different $\mathbb{P}^1$-bundle structures over $\mathbb{P}^1$. Specifically, consider the two small resolutions X_+ and X_-, given as the total spaces of the projections

$$X_+ = \{(x, y, z, w, [\lambda : \mu]) \mid xy = zw, x\mu = z\lambda\} \subset \mathbb{C}^4 \times \mathbb{P}^1$$

and

$$X_- = \{(x, y, z, w, [\lambda : \mu]) \mid xy = zw, x\mu = w\lambda\} \subset \mathbb{C}^4 \times \mathbb{P}^1.$$

Each small resolution replaces the singular point with an exceptional divisor isomorphic to $\mathbb{P}^1$. The birational transformation that replaces X_+ with X_- is called Atiyah's flop. The spaces X, X_+, X_- are smooth. By the rigidity lemma [23, Lemma 1.15], we have a commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ X_+ & \longrightarrow & X_-. \end{array}$$

However, there is no commuting diagram replacing X_+ by X_- in the above diagram.

Remark 6.6 We find that there is a gap in the proof of [57, Proposition 0.4]. The following arguments are given in the proof: Since F' is realised as blow-up of $\mathbb{P}^2$ in 9 points, we may take a general line l in F'. This general line l can be deformed in X to a general line in a neighbouring F_s. For general s we have

$$(K_{F_s} + E_s) \cdot l_s < 0,$$

where E_s is one of the $(-1)-$curves in our family sitting inside F_s. The following example shows that the intersection number of $(K_{F_s} + E_s)$ with l_s can be 0 for some $(-1)-$curve. This comes from the fact that F' may have different contractions to $\mathbb{P}^2$. Note that the result itself holds as a consequence of structure theorem of projective manifolds with nef anticanonical line bundle as shown in [19].

Example 6.7 Let F_s be a smooth cubic surface. It can be realised as a blow up of $\mathbb{P}^2$ in 6 points in general position. It is classical that there exist 27 rational curves on F (which are $(-1)-$curves). These rational curves correspond to the 6 exceptional curves created by blowing up, the birational transforms of the 15 lines through pairs of the 6 points in $\mathbb{P}^2$ and the birational transforms of the 6 conics containing all but one of the 6 points. Let E_s be one of the birational transforms of the 6 conics containing all but one of the 6 points and l_s be the pull back of general line of $\mathbb{P}^2$. Then we have

$$(K_{F_s} + E_s) \cdot l_s = 0.$$

Example 6.8 Let S_8 be a Del Pezzo surface of degree 1. Let $\pi : S_8 \to \mathbb{P}^2$ be the blow down with exceptional curves $E_i (1 \le i \le 8)$. By [27, Proposition 8.2.19] there exist $(-1)-$curves $C_i (1 \le i \le 8)$ on S_8 such that $C_i + E_i = -2K_{S_8}$. Then

$$\pi_* C_i = -2K_{\mathbb{P}^2} = \mathcal{O}(6)$$

With the choice $F_s = S_8$, $E_s = C_i$, l_s the strict transforms of $\mathcal{O}(1)$,

$$(K_{F_s} + E_s) \cdot l_s \ge 0.$$

(In fact it is enough to choose a Del Pezzo surface of degree 2.)

Remark 6.9 The above proof of Proposition 6.3 works if X is a compact Kähler 4-fold with nef anticanonical bundle that admits a smooth fibration π onto a K3 surface Y under the assumption on the intersection number. In this case, there are three possiblities:

(1) $Y \times \mathbb{P}^2$;
(2) $Y \times \mathbb{P}^1 \times \mathbb{P}^1$;
(3) we have a factorisation

$$X \xrightarrow{\gamma_1} X_1 \cdots \xrightarrow{\gamma_k} X_k \xrightarrow{\rho} Y$$

with γ_j blow-ups of étale multi-sections, $X_k \simeq \mathbb{P}(E_1) \times_Y \mathbb{P}(E_2)$ for some numerically flat vector bundles E_1, E_2 over Y or $X_k \simeq \mathbb{P}(E)$ for some numerically flat vector bundle over Y.

More precisely, for the first and second case, the anticanonical line bundle of the fiber $-K_F$ is very ample. By the Kodaira vanishing theorem, $H^1(F, -mK_F) = 0$, so $-K_X$ is $\pi-$ample by [4, Theorem 4.2]. Then we can argue as in [19] (cf. e.g. [53, Theorem 2.6]), to see that $\pi_*(-K_X)$ is numerically flat. By [52, Proposition 2.5], the conclusion follows. (Note that the Brauer group theory implies that there exists a $\mathbb{P}^r-$bundle over a K3 surface which cannot be written as the projectivization of some vector bundle.) The remaining case follows similarly as the end of Proposition 6.3.

Conjecturally, all compact Kähler non-projective fourfolds X with nef anticanonical line bundle are, up to a finite étale cover, one of the following types:

(1) $c_1(X) = 0$;
(2) there is a locally trivial fibration $\pi : X \to Y$ from a compact Kähler fourfold X to a K3 surface or a two dimensional torus Y with rational fibers;
(3) $X = \mathbb{P}(E)$ for some rank 2 numerically flat vector bundle over a torus of dimension three;
(4) the product of a K3 surface and a projectivization of a rank two numerically flat vector bundle over an elliptic curve.

The present unknown cases are the simply connected ones with trivial canonical line bundle and case (2) without assuming that the fibration is smooth and having the assumption on the intersection number.

Remark 6.10 Let X be a compact Kähler manfiold. Assume that, up to some finite étale cover, X is the projectivization $\mathbb{P}(E)$ of some numerically flat vector bundle E over a torus T. Applying [43, Corollary 2.11] to the regular foliation $T_{X/T}$ induced from the projective bundle structure, X itself admits a submersion $\varphi : X \to Y$ to a compact Kähler manifold Y with $c_1(Y) = 0$. (For more details, see the proof of [53, Theorem 4.8].) By the Fischer-Grauert theorem, φ is locally trivial. The fibration $\varphi : X \to Y$ is the Mori fiber space of X;

hence $-K_X$ is a φ-ample line bundle. By [53, Theorem 2.6], since Y is smooth, we can find a φ-ample line bundle B such that $\varphi_*(pB)$ is numerically flat for $1 \ll p \in \mathbb{Z}_+$. Then we see that $\varphi : X \to Y$ is a locally constant fibration by [52, Proposition 2.5]. Proposition 7.3 in the next section implies that X can be algebraically approximated.

7 Algebraic approximation

The following lemma is well-known to experts. For convenience of the reader, we give a proof here which was communicated to us by F. Campana.

Lemma 7.1 *Let X be a compact Kähler manifold. Assume that the rational quotient $R(X)$ of X is projective. Then X is projective.*

Proof By [12, Théorème 6] (see also [11, Theorem 3.4]), to show that X is projective, it is enough to show that any two general points of X can be connected by a chain of (compact) curves. Without loss of generality, we can assume that the rational quotient is a holomorphic map since as a compact Kähler manifold, the projectivity of any modification of X would imply that X is Moishezon (and hence projective).

Take a very ample line bundle A over $R(X)$ which exists by the projectivity assumption. By Bertini's theorem, any two general points of $R(X)$ can be connected by a smooth curve C defined as a complete intersection of sections in the linear system of A such that the general fiber over C is smooth and rationally connected. By Sard's theorem, we can furthermore assume that the base change X_C is smooth. It is enough to show that X_C is projective which implies that any two points of X_C can be connected by a chain of (compact) curves.

By Kodaira's criterion (see e.g. [24, Chap. VII, (14.4)]), as X_C is a compact Kähler manifold, it is enough to show that $H^0(X_C, \Omega^2_{X_C}) = 0$. Let $C_0 \subset C$ be the smooth locus such that each fiber is a rationally connected manifold. Any section vanishing over a non-empty Zariski open set of X_C has to vanish identically. Therefore, it is enough to show that $H^0(X_{C_0}, \Omega^2_{X_{C_0}}) = 0$. We denote by $f : X_{C_0} \to C_0$ the restricted map which gives a short exact sequence of vector bundles

$$0 \to f^*\Omega^1_{C_0} \to \Omega^1_{X_{C_0}} \to \Omega^1_{X_{C_0}/C_0} \to 0$$

that induces a filtration on $\Omega^2_{X_{C_0}}$ (cf. [40, Ch.II, Ex. 5.16 (d)]):

$$0 \subset f^*\Omega^2_{C_0} \subset f^*\Omega^1_{C_0} \otimes \Omega^1_{X_{C_0}} \subset \Omega^2_{X_{C_0}}.$$

The graded pieces are

$$R_0 := f^*\Omega^2_{C_0},\ R_1 := \Omega^1_{X_{C_0}/C_0} \otimes f^*\Omega^1_{C_0},\ R_2 := \Omega^2_{X_{C_0}/C_0}.$$

It is enough to show that the graded pieces have no global sections. $R_0 = 0$ since C is of dimension 1. $H^0(X_{C_0}, \Omega^2_{X_{C_0}/C_0}) = H^0(C_0, f_*\Omega^2_{X_{C_0}/C_0}) = 0$ since by Grauert's theorem, for any $z \in C_0$, $H^0(X_z, \Omega^2_{X_{C_0}/C_0}|_{X_z}) = 0$. Note that a rationally connected manifold has no non-trivial holomorphic forms. $H^0(X_{C_0}, \Omega^1_{X_{C_0}/C_0} \otimes f^*\Omega^1_{C_0}) = H^0(C_0, f_*(\Omega^1_{X_{C_0}/C_0} \otimes f^*\Omega^1_{C_0})) = H^0(C_0, f_*(\Omega^1_{X_{C_0}/C_0}) \otimes \Omega^1_{C_0})) = 0$ where the last equality follows from Grauert's theorem. This finishes the proof of the lemma. □

Remark 7.2 Similar arguments in fact show the following statement: Let $f : X \to B$ be a proper holomorphic submersion with connected fibers between two connected complex manifolds. If the fibers are rationally connected we have

$$H^0(X, \Omega_X^p) = f^*(H^0(B, \Omega_B^p))$$

for any $p \geq 0$.

Proposition 7.3 *(= Theorem D) Assume that Conjecture 2 holds. Then any compact Kähler manifold X with nef anticanonical divisor can be algebraically approximated.*

Proof Consider the fibration $\varphi : X \to Y$ predicted in Conjecture 2. By the well-known fact that a Calabi-Yau manifold can be algebraically approximated, there exists a deformation $\pi : \mathfrak{Y} \to \Delta$ of Y over a small disk Δ such that for Euclidean dense $t \in \Delta$, $Y_t := \pi^{-1}(t)$ is a projective Calabi-Yau manifold. Let $p : \tilde{\mathfrak{Y}} \to \mathfrak{Y}$ be the (holomorphic) universal cover of $\mathfrak{Y}$.

We claim that $\tilde{Y}_t := (p \circ \pi)^{-1}(t)$ is the universal cover of Y_t for any t. Since a base change of étale morphism is still étale, the projection $\tilde{Y}_t \to Y_t$ is étale (i.e. a unbranched covering map). Since Δ is contractible, by Ehrensmann's theorem, there exist 2 pointwise linear independent smooth vector fields over $\mathfrak{Y}$. These vector fields define a diffeomorphism between $\mathfrak{Y}$ and $Y_0 \times \Delta$. Since p is étale, the pullback of these vector fields are well defined on $\tilde{\mathfrak{Y}}$. Up to a possible shrinking of Δ, the pullback fields define a diffeomorphism between $\tilde{\mathfrak{Y}}$ and $\tilde{Y}_0 \times \Delta$. In particular, for any t, $\pi_1(\tilde{Y}_t) = \pi_1(\tilde{\mathfrak{Y}}) = 0$ since Δ is contractible. This finishes the proof of the claim.

Consider $\mathfrak{X} := (\tilde{\mathfrak{Y}} \times F)/\pi_1(Y) \xrightarrow{\tilde{\varphi}} \mathfrak{Y} \to \Delta$ which defines a deformation of X. Since the group action is free, $\mathfrak{X}$ is smooth. For any $t \in \Delta$, the morphism $X_t := (\pi \circ \tilde{\varphi})^{-1}(t) \to Y_t$ has rational connected fiber whose base is non-uniruled. Thus Y_t is a MRC quotient of X_t. For dense t, Lemma 7.1 implies that X_t is projective which finishes the proof. Note that for such dense t, $-K_{X_t}$ is nef. Indeed, since $-K_X$ is nef, by definition, for any $\epsilon > 0$, there exists a smooth metric h_ϵ such that its Chern curvature satisfies $c_1(-K_X, h_\epsilon) \geq -\epsilon\omega$. We pull back h_ϵ to the universal cover $q : \tilde{Y} \times F \to X$ of X whose restriction to $\{\tilde{y}\} \times F$ for any $\tilde{y} \in \tilde{Y}$ defines a $\pi_1(Y)-$invariant metric on $-K_F$. The product metric defined by the restricted h_ϵ and the Ricci flat metric on $\tilde{Y}_t$ is $\pi_1(Y)-$invariant and thus descends to X_t which implies that $-K_{X_t}$ is nef. □

Remark 7.4 The above proposition shows that the nefness of the anticanonical line bundle is a deformation invariant property. However, the strong pseudoeffectivity of the tangent bundle is not a deformation invariant property in general. For example, the blow-up of $\mathbb{P}^2$ in at most 4 points (in general position) has a strongly pseudoeffective tangent bundle as shown in [45]. However, any toric variety has strongly pseudoeffective tangent bundle which can be taken as blow-up of $\mathbb{P}^2$ at arbitrary many points (in special position).

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