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Research Paper

On rigid varieties isogenous to a product of curves ☆

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ABSTRACT

In this note, we study rigid complex manifolds that are realized as quotients of a product of curves by a free action of a finite group. They serve as higher-dimensional analogues of Beauville surfaces. Using uniformization, we outline the theory to characterize these manifolds through specific combinatorial data associated with the group under the assumption that the action is diagonal and the manifold is of general type. This leads to the notion of a n-fold Beauville structure. We define an action on the set of all n-fold Beauville structures of a given finite group that allows us to distinguish the biholomorphism classes of the underlying rigid manifolds. As an application, we give a classification of these manifolds with group $\mathbb{Z}_5^2$ in the three dimensional case and prove that this is the smallest possible group that allows a rigid, free and diagonal action on a product of three curves. In addition, we provide the classification of rigid 3-folds X given by a group acting faithfully on each factor for any value of the holomorphic Euler number $\chi(\mathcal{O}_X) \geq -5$.

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1. Introduction

A central aspect of any kind of geometric consideration is to study the symmetries of the underlying spaces. In complex algebraic geometry, the symmetries are biholomorphic self-maps of projective manifolds. Given a known manifold X then, in the spirit of Godeaux, we can often construct new and interesting manifolds by taking quotients X/G modulo a free action of a finite group of automorphisms. An important class of such quotients are the varieties isogenous to a product.

Definition 1.1. A complex variety X is isogenous to a product if it is isomorphic to a quotient

$$X \simeq (C_1 \times \ldots \times C_n)/G,$$

where the $C_i's$ are compact Riemann surfaces of genus $g(C_i) \geq 1$ and G is a finite group acting freely on the product $C_1 \times \ldots \times C_n$. We call X *isogenous to a higher product*, if $g(C_i) \geq 2$ for all C_i.

Since Catanese introduced these varieties in [Cat00], they have been studied extensively, especially in dimension two in order to construct and classify surfaces of general type and describe their moduli spaces, cf. [BCG08,CP09,Gle15,Pen11,Fal24]. We point out that a variety X isogenous to a product is of general type if and only if it is isogenous to a higher product. It has been shown by Catanese that a surface S isogenous to a higher product has a unique minimal realization

$$S \simeq (C_1 \times C_2)/G.$$

It is characterized by the property that the diagonal subgroup $G_0 := G \cap (\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2))$ acts faithfully on each curve C_i.

An easy example of a surface isogenous to a higher product is due to Beauville [Bea83b, Exercises X.13 (4) p.118]: consider the product of two Fermat quintics

$$C = \{x_0^5 + x_1^5 + x_2^5 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$$

together with the action of $(a,b) \in \mathbb{Z}_5^2$ defined by

$$(a,b) * \big([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]\big) := \big([\zeta_5^a x_0 : \zeta_5^b x_1 : x_2], [\zeta_5^{a+3b} y_0 : \zeta_5^{2a+4b} y_1 : y_2]\big). \quad (1.1)$$

Since this action is free, the quotient

$$S := (C \times C)/\mathbb{Z}_5^2$$

is a surface isogenous to a product. Its holomorphic Euler number is $\chi(\mathcal{O}_S) = 1$, which is the minimal Euler number for a surface of general type. Remarkably, it turns out that

S is rigid, i.e. it has no non-trivial deformations. Motivated by this example Catanese defined Beauville surfaces as rigid surfaces isogenous to a product [Cat00, Def 3.23]. They are always of general type ([BCG05, cf. Prop 3.2], [BC18, Thm 2.7]) and have an entirely group theoretical description in terms of a so called Beauville structure of the corresponding group. For this reason, Beauville surfaces have been actively studied, not only by algebraic geometers, but also by group theorists, as they provide a rich framework to explore the interplay between these disciplines (cf. [BGV15]). The aim of this paper is to extend the theory of Beauville surfaces to higher dimensions and explore some new phenomena. For this purpose we define:

Definition 1.2. A rigid variety isogenous to a product is called a Beauville manifold.

In contrast to the surface case, the geometry of higher dimensional Beauville manifolds is more involved. First we point out that they are not necessarily of general type. Indeed, according to [BC18, Thm 3.4 and Thm 3.5] we have:

- For all $n \geq 4$ there exists a Beauville n-fold of Kodaira dimension 0.
- For all $n \geq 3$ there exists a Beauville n-fold of Kodaira dimension κ for all $2 \leq \kappa \leq n$.

Our second remark is that even in the case of a higher product, we cannot assume that the diagonal subgroup

$$G_0 := G \cap (\mathrm{Aut}(C_1) \times \ldots \times \mathrm{Aut}(C_n))$$

acts faithfully on each curve C_i. Indeed, in [FG16] the authors provide a classification of all 3-folds X isogenous to a higher product of curves with $\chi(\mathcal{O}_X) = -1$ under the assumption that the action is diagonal, i.e. $G = G_0$ and faithful on each curve C_i. Among the 54 families there are no rigid examples. However, dropping the faithfulness on the factors, a rigid example with $\chi(\mathcal{O}_X) = -1$ is easy to construct as a modification of the original Beauville surface (1.1): we take the hyperelliptic curve D of genus two given by the affine equation $y^2 = x^5 - 1$ and define X as the quotient of $C^2 \times D$ modulo the following free $\mathbb{Z}_5^2$-action:

$$(a, b) * \big([x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [x, y]\big) := \big([\zeta_5^a x_0 : \zeta_5^b x_1 : x_2], [\zeta_5^{a+3b} y_0 : \zeta_5^{2a+4b} y_1 : y_2], [\zeta_5^a x, y]\big).$$

Clearly, the action is not faithful on the third factor. However, the quotient X is rigid and $\chi(\mathcal{O}_X) = -1$. This observation serves as a motivation for the following questions about Beauville 3-folds of general type:

- Is $\mathbb{Z}_5^2$ the smallest group attached to a Beauville 3-fold?
- What is the number of biholomorphism classes of Beauville 3-folds with group $\mathbb{Z}_5^2$ and Euler number $\chi(\mathcal{O}_X) = -1$?

- What is the largest integer $\chi \leq -2$ such that there exists a Beauville threefold X with $\chi(\mathcal{O}_X) = \chi$ under the assumption that the group acts faithfully on each C_i? Is it possible to classify these 3-folds up to biholomorphism?

For simplicity, we restrict our focus to unmixed Beauville 3-folds, i.e. to the case where G acts diagonally on the (higher) product of three curves. Our results are the following:

Theorem 1.3. *The smallest group attached to an unmixed Beauville 3-fold X is $\mathbb{Z}_5^2$. These 3-folds have either $\chi(\mathcal{O}_X) = -1$ or -5. There are 8 biholomorphism classes of such 3-folds with $\chi(\mathcal{O}_X) = -1$ and 77 with $\chi(\mathcal{O}_X) = -5$.*

Theorem 1.4. *The number $\mathcal{N}$ of biholomorphism classes of unmixed Beauville 3-folds X with holomorphic Euler number $\chi(\mathcal{O}_X) = \chi \in \{-5, -4, -3, -2\}$, such that the corresponding group acts faithfully on each factor of the product is:*

χ	-5	-4	-3	-2
$\mathcal{N}$	78	8	0	1

We will now explain how the paper is organized. In Section 2, we recall the basic theory of varieties isogenous to a product of curves and introduce Beauville manifolds. Crucial for our analysis is the existence and uniqueness of a minimal realization of a Beauville manifold or more generally of a variety isogenous to a higher product of curves of unmixed type. Even though this result is well-known in the surface case [Cat00, Cor. 3.12 and Prop. 3.13] and folklore in higher dimensions, we decided to include a proof, since we could not find a reference for dimension $n \geq 3$. In Section 3, we use Riemann's existence theorem to give a purely group theoretical description of unmixed Beauville manifolds of general type. More precisely, we explain how to attach to the minimal realization of a given Beauville n-fold an *n-fold Beauville structure* of the corresponding group G. Then we provide a natural action of the group $\mathrm{Aut}(G) \times (\mathcal{B}_3 \wr \mathfrak{S}_n)$ on the set $\mathcal{UB}_n(G)$ of all (unmixed) Beauville structures of G, where $\mathcal{B}_3$ is the Artin-Braid group on three strands. We show that the biholomorphism classes of unmixed Beauville n-folds are in $1:1$ correspondence with the orbits of this action. Unfortunately, for certain groups, it can be very difficult and computationally expensive to determine these orbits. This difficulty is resolved in Section 4, where we present an effective method to compute the orbits, that extends the *automorphism of Braid type* approach of [Fal24, Section 1.2] from surfaces to arbitrary dimensions. These results allow us to employ the *Database of topological types of actions on curves* of [CGP23] for explicit computations. In Section 5, we discuss the notion of the Beauville dimension of a finite group G, which is the minimum dimension of a Beauville manifold with group G. This concept was introduced by Carta and Fairbairn in [CF22] under the assumption that the group acts faithfully on each factor. We drop this assumption and show that the group $\mathbb{Z}_n^3$ has Beauville dimension three if and only if $\gcd(n, 6) = 1$. In Section 6, we prove our main theorems using a MAGMA [BCP97] implementation of the algorithm from Section 4. The reader can find a MAGMA implementation on the webpage https://www.komplexe-analysis.uni-bayreuth.de/de/team/gleissner-christian/index.php.

2. Generalities on Beauville manifolds

In this section, we present the basic theory of varieties isogenous to a product of curves via uniformization. More precisely, we extend the results [Cat00, Cor. 3.9, Prop. 3.11, Cor. 3.12, Prop. 3.13] of Catanese on surfaces, which were established using different methods.

In particular, we focus on Beauville manifolds, which are the rigid varieties isogenous to a product of curves. In higher dimensions they serve as natural generalizations of Beauville surfaces.

Recall that a variety X isogenous to a product is smooth and projective. The Kodaira dimension $\kappa(X)$ is equal to the number of curves of genus $g(C_i) \geq 2$ in the product $C_1 \times \ldots \times C_n$. The n-fold self-intersection of the canonical class K_X^n, the topological Euler number $e(X)$ and the holomorphic Euler number $\chi(\mathcal{O}_X)$ are given in terms of the genera $g(C_i)$ and the order of the group.

Proposition 2.1. *Let $X \simeq (C_1 \times \ldots \times C_n)/G$ be a variety isogenous to a product, then*

$$\chi(\mathcal{O}_X) = \frac{(-1)^n}{|G|} \prod_{i=1}^{n} \big(g(C_i) - 1\big), \quad K_X^n = (-1)^n n! 2^n \chi(\mathcal{O}_X) \quad \textit{and} \quad e(X) = 2^n \chi(\mathcal{O}_X).$$

Definition 2.2. A rigid variety isogenous to a product is called a Beauville manifold.

Remark 2.3. A Beauville surface is always isogenous to a higher product, i.e. of general type ([BCG05, cf. Prop. 3.2] and [BC18, Thm. 2.7]). This is not true anymore for Beauville manifolds of higher dimension:

1. For all $n \geq 4$ there exists a Beauville n-fold of Kodaira dimension 0 (see [BC18, Thm. 3.4]).
2. For all $n \geq 3$ there exists a Beauville n-fold of Kodaira dimension κ for all $2 \leq \kappa \leq n$ (see [BC18, Thm. 3.5]).
3. There is no rigid and free action on a 3-dimensional complex torus. In particular, there are no Beauville 3-folds of Kodaira dimension 0. (see [DG23, Thm. 1.1(a)]).
4. The existence of Beauville manifolds of Kodaira dimension 1 is still an open question. However, there are no such manifolds if the action is diagonal and faithful on each factor, cf. Definition 2.7. For this result, see [BGK25, Cor. 3.11].
5. The existence of rigid 3-folds of Kodaira dimension 0 and rigid n-folds of Kodaira dimension 1 for all $n \geq 3$ is known (see [Bea83a] and [BG20]). They are obtained as certain resolutions of quotients of a product of curves by a non-free action.

In this paper, we are mainly interested in the special case where X is isogenous to a higher product of curves. In order to study group actions on a product $C_1 \times \ldots \times C_n$ of compact Riemann surfaces with $g(C_i) \geq 2$, it is important to understand the structure of the automorphism group of the product. This group has a simple description in terms of the automorphism groups $\mathrm{Aut}(C_i)$ of the factors, thanks to the lemma below:

Lemma 2.4. *Let $D_1, \ldots, D_k$ be pairwise non-isomorphic compact Riemann surfaces with $g(D_i) \geq 2$. Then for all positive integers $n_1, \ldots, n_k$ it holds:*

$$\mathrm{Aut}(D_1^{n_1} \times \ldots \times D_k^{n_k}) = (\mathrm{Aut}(D_1) \wr \mathfrak{S}_{n_1}) \times \ldots \times (\mathrm{Aut}(D_k) \wr \mathfrak{S}_{n_k}).$$

Here $\mathrm{Aut}(D_i) \wr \mathfrak{S}_{n_i} = \mathrm{Aut}(D_i)^{n_i} \rtimes \mathfrak{S}_{n_i}$ *denotes the* wreath product*.*

Proof. Any automorphism $\varphi \in \mathrm{Aut}(D_1^{n_1} \times \ldots \times D_k^{n_k})$ lifts to an automorphism $\hat{\varphi}$ of the universal cover, which is a product of unit discs Δ^n. The claim follows from the well known fact that

$$\mathrm{Aut}(\Delta^n) = \mathrm{Aut}(\Delta) \wr \mathfrak{S}_n.$$

See, [Nar95, Proposition 3, p.68]. □

The above lemma motivates the following definition:

Definition 2.5. A n-dimensional variety X isogenous to a higher product is said to be of unmixed type, if there is a realization $X \simeq (C_1 \times \ldots \times C_n)/G$, such that G acts diagonally on the product: $G \leq \mathrm{Aut}(C_1) \times \ldots \times \mathrm{Aut}(C_n)$. Otherwise, we say that X is of mixed type.

Proposition 2.6. *A variety X is isogenous to a higher product of curves if and only if there exists an unramified cover*

$$f : C_1 \times \ldots \times C_n \to X, \qquad \textit{where} \quad g(C_i) \geq 2.$$

Proof. Assume that f exists, then the universal cover of X is the polydisc Δ^n and the cover f is induced by a finite index subgroup

$$\Gamma_1 \times \ldots \times \Gamma_n < \Gamma,$$

where $\Gamma < \mathrm{Aut}(\Delta^n)$ is the fundamental group of X and Γ_i the fundamental group of C_i. An element of Γ is of the form

$$\gamma_\tau(z_1, \ldots, z_n) = (\gamma_1(z_{\tau(1)}), \ldots, \gamma_n(z_{\tau(n)})),$$

for some permutation $\tau \in \mathfrak{S}_n$ and automorphisms $\gamma_i \in \mathrm{Aut}(\Delta)$. This implies

$$\gamma_\tau(\Gamma_1 \times \cdots \times \Gamma_n)\gamma_\tau^{-1} = \gamma_1\Gamma_{\tau(1)}\gamma_1^{-1} \times \cdots \times \gamma_n\Gamma_{\tau(n)}\gamma_n^{-1}.$$

The normal core i.e. the largest subgroup of $\Gamma_1 \times \cdots \times \Gamma_n$ that is normal in Γ is therefore given by

$$\mathrm{core}_\Gamma(\Gamma_1 \times \cdots \times \Gamma_n) = \bigcap_{\gamma_\tau \in \Gamma} \gamma_\tau(\Gamma_1 \times \cdots \times \Gamma_n)\gamma_\tau^{-1} = \prod_{i=1}^{n} \bigcap_{\gamma_\tau \in \Gamma} \gamma_i\Gamma_{\tau(i)}\gamma_i^{-1}.$$

Since the core of a finite index subgroup has finite index, we obtain a finite Galois cover

$$\Delta/\Gamma_1' \times \cdots \times \Delta/\Gamma_n' \to X, \quad \text{where} \quad \Gamma_i' := \bigcap_{\gamma_\tau \in \Gamma} \gamma_i\Gamma_{\tau(i)}\gamma_i^{-1}. \quad \square$$

We want to point out that a realization of a variety X isogenous to a higher product as a quotient $(C_1 \times \ldots \times C_n)/G$ is in general not unique. However, we can always find a so called *minimal realization* which is unique up to isomorphism. In this paper, we want to stick to the case where X is of unmixed type. Then, given a realization, we obtain G-actions $\psi_i \colon G \to \mathrm{Aut}(C_i)$ on the factors which are not necessarily faithful. We denote by K_i the kernel of ψ_i and define the quotient G/K_i, which then acts faithfully on C_i as $\overline{G}_i$.

Definition 2.7. A diagonal G-action on $C_1 \times \ldots \times C_n$ is called

1. minimal, if $K_1 \cap \ldots \cap \widehat{K}_i \cap \ldots \cap K_n = \{1_G\}$ for all i.
2. absolutely faithful, if all kernels K_i are trivial.

Clearly, an absolutely faithful action is also minimal and in dimension two the notions coincide.

Theorem 2.8. *Every variety isogenous to a higher product of curves of unmixed type has a unique minimal realization, i.e. a realization obtained by a minimal action.*

To prepare for the proof of this theorem, we need to recall the structure of the fundamental group of a variety isogenous to a higher product of unmixed type, cf. [DP12].

Remark 2.9. Let $(C_1 \times \ldots \times C_n)/G$ be a not necessarily minimal realization of a variety X isogenous to a higher product of curves of unmixed type. Considering the universal cover $\pi_i \colon \Delta \to C_i$, we obtain the short exact sequence

$$1 \to \Gamma_i \to \mathbb{T}_i \xrightarrow{\rho_i} \overline{G}_i \to 1.$$

As above Γ_i is isomorphic to the fundamental group of C_i and $\mathbb{T}_i$ is the group of all possible lifts of the elements in $\overline{G}_i$, i.e. the orbifold fundamental group:

$$\mathbb{T}_i = \{\gamma \in \mathrm{Aut}(\Delta) \mid \text{ exists } \ \overline{g} \in \overline{G}_i \ \text{ such that } \ \pi_i \circ \gamma = \psi_i(g) \circ \pi_i\},$$

see [Cat15, Chapter 6] for an in depth discussion of orbifold fundamental groups. Similarly, we take the universal cover $\Delta^n \to C_1 \times \ldots \times C_n$ and get the short exact sequence

$$1 \to \Gamma_1 \times \ldots \times \Gamma_n \to \Gamma \to G \to 1.$$

The group Γ consists of the lifts of the elements in G to $\mathrm{Aut}(\Delta^n)$. It is isomorphic to the fundamental group of X, because the action of G is free. Since G acts diagonally on the product of curves, an automorphism $g \in G$ lifts to an automorphism $(\gamma_1, \ldots, \gamma_n) \in \mathrm{Aut}(\Delta)^n$, if and only if each $\psi_i(g)$ lifts to $\gamma_i \in \mathrm{Aut}(\Delta)$. Therefore, we can write Γ in the following way:

$$\Gamma = \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{T}_1 \times \ldots \times \mathbb{T}_n \mid \overline{g} = \rho_i(\gamma_i) \in \overline{G}_i \text{ for all } 1 \leq i \leq n \text{ and some } g \in G\}.$$

Proof of Theorem 2.8. (I) To show the existence of a minimal realization, we start with an arbitrary realization

$$X \simeq \frac{C_1 \times \ldots \times C_n}{G}.$$

For each i, we consider the normal subgroups $H_i := K_1 \cap \ldots \cap \widehat{K}_i \cap \ldots \cap K_n \trianglelefteq G$. Note that H_i acts trivially on C_j for all $i \neq j$ and freely on C_i. In particular, the genus of the quotient curve C_i/H_i is at least 2. We take the product $H := H_1 \cdot \ldots \cdot H_n \trianglelefteq G$ of our normal subgroups and form the double quotient

$$X \simeq \frac{(C_1 \times \ldots \times C_n)/H}{G/H} \simeq \frac{C_1/H_1 \times \ldots \times C_n/H_n}{G/H}.$$

By construction, the induced G/H-action on the product $C_1/H_1 \times \ldots \times C_n/H_n$ is minimal.

(II) To prove the uniqueness, we consider a biholomorphism between two minimal realizations of a variety X isogenous to a higher product:

$$f \colon \frac{C_1 \times \ldots \times C_n}{G} \to \frac{D_1 \times \ldots \times D_n}{G'}.$$

The map f lifts to an automorphism $\hat{f} \in \mathrm{Aut}(\Delta^n) = \mathrm{Aut}(\Delta) \wr \mathfrak{S}_n$. Up to permutation of the curves D_i, we may assume that

$$\hat{f}(z_1, \ldots, z_n) = \big(\hat{f}_1(z_1), \ldots, \hat{f}_n(z_n)\big).$$

Conjugation with $\hat{f}$ induces an isomorphism between the deck transformation groups of the universal covers

$$f_*\colon \Gamma \to \Gamma', \qquad (\gamma_1, \ldots, \gamma_n) \mapsto \big(\hat{f}_1 \gamma_1 \hat{f}_1^{-1}, \ldots, \hat{f}_n \gamma_n \hat{f}_n^{-1}\big).$$

Our goal is to show that f_* restricts to an isomorphism between the fundamental groups of the product of curves

$$f_*(\Gamma_1 \times \ldots \times \Gamma_n) = \Gamma_1' \times \ldots \times \Gamma_n'. \tag{2.1}$$

Under this condition the uniqueness follows, since then f_* induces an isomorphism of Galois groups

$$G \simeq \Gamma/(\Gamma_1 \times \ldots \times \Gamma_n) \to \Gamma'/(\Gamma_1' \times \ldots \times \Gamma_n') \simeq G'$$

and a biholomorphic lift of f to the product of curves:

$$\begin{array}{ccc} (C_1 \times \ldots \times C_n)/G & \xrightarrow{f} & (D_1 \times \ldots \times D_n)/G' \\ \uparrow & & \uparrow \\ C_1 \times \ldots \times C_n & \xrightarrow{\tilde{f}} & D_1 \times \ldots \times D_n. \end{array}$$

To verify equation (2.1), it suffices to show that

$$f_*(1, \ldots, 1, \gamma_i, 1, \ldots, 1) \in \Gamma_1' \times \ldots \times \Gamma_n',$$

for all $1 \leq i \leq n$ and $\gamma_i \in \Gamma_i$. Here we use the structure of Γ' given in Remark 2.9 and assume for simplicity that $i = 1$. Take $\gamma_1 \in \Gamma_1$, then

$$f_*(\gamma_1, 1, \ldots, 1) = \big(\hat{f}_1 \gamma_1 \hat{f}_1^{-1}, 1, \ldots, 1\big) \in \Gamma'.$$

Hence there exists $g \in G'$ such that

$$\overline{g} = \rho_1'\big(\hat{f}_1 \gamma_1 \hat{f}_1^{-1}\big) \in G'/K_1 \qquad \text{and} \qquad \overline{g} = \rho_j'(1) = 1 \in G'/K_j'$$

for all $j \geq 2$. This implies $g \in K_2' \cap \ldots \cap K_n' = \{1_{G'}\}$ and therefore also $\hat{f}_1 \gamma_1 \hat{f}_1^{-1} \in \ker(\rho_1') = \Gamma_1'$. $\square$

Remark 2.10. Theorem 2.8 allows us to attach, to any variety isogenous to a higher product of unmixed type, a product of curves $C_1 \times \ldots \times C_n$ together with a finite group G acting minimally on the product. We will use this fact in the next section to give a purely group theoretical description of unmixed Beauville manifolds.

Proposition 2.11. *A diagonal and free G-action on a product of compact Riemann surfaces*

$$Y = C_1 \times \ldots \times C_n \qquad \textit{with} \qquad g(C_i) \geq 2$$

yields a Beauville manifold, if and only if each C_i *is a* triangle curve*:*

1. $C_i/\overline{G}_i \simeq \mathbb{P}^1$ *and*
2. $C_i \to C_i/\overline{G}_i$ *is branched in three points, where* $\overline{G}_i = G/K_i$*.*

Proof. The tangent bundle of Y decomposes as

$$\Theta_Y = p_1^*\Theta_{C_1} \oplus \cdots \oplus p_n^*\Theta_{C_n},$$

where $p_i\colon Y \to C_i$ is the projection onto the i-th factor. Consequently,

$$H^1(Y, \Theta_Y) = H^1(Y, p_1^*\Theta_{C_1}) \oplus \cdots \oplus H^1(Y, p_n^*\Theta_{C_n}).$$

Each summand can be computed using the Künneth formula. Since

$$p_i^*\Theta_{C_i} = \left(\bigotimes_{j\neq i} p_j^*\mathcal{O}_{C_j}\right) \otimes p_i^*\Theta_{C_i},$$

we obtain

$$H^1(Y, p_i^*\Theta_{C_i}) = \bigoplus_{l_1+\cdots+l_n=1} \left(\bigotimes_{j\neq i} H^{l_j}(C_j, \mathcal{O}_{C_j})\right) \otimes H^{l_i}(C_i, \Theta_{C_i}) = H^1(C_i, \Theta_{C_i}),$$

where the final equality follows from the fact that $H^0(C_j, \mathcal{O}_{C_j}) = \mathbb{C}$ and $H^0(C_i, \Theta_{C_i}) = 0$ for $g(C_i) \geq 2$. Summing over all i, we find:

$$H^1(Y, \Theta_Y) = H^1(C_1, \Theta_{C_1}) \oplus \cdots \oplus H^1(C_n, \Theta_{C_n}).$$

Since the G-action on Y is diagonal, we obtain

$$H^1(Y, \Theta_Y)^G = 0 \quad \text{if and only if} \quad H^1(C_i, \Theta_{C_i})^{\overline{G}_i} = 0 \quad \text{for all } i.$$

By using [Bea83b, Examples VI.12 (2)], the condition $H^1(C_i, \Theta_{C_i})^{\overline{G}_i} = 0$ is easily seen to be equivalent to (1) and (2). □

Corollary 2.12. *An unmixed Beauville manifold is always regular, i.e. it has no non-zero global holomorphic* 1*-forms.*

Proof. The irregularity of $X = (C_1 \times \ldots \times C_n)/G$ is given by $q(X) = \sum_{i=1}^n g(C_i/\overline{G}_i) = 0$. □

3. Group theoretical description of Beauville n-folds

In this section we briefly recall the theory of *triangle curves* from the group theoretical point of view. This allows us to give a group theoretical description of unmixed Beauville n-folds that provides a way to classify them up to biholomorphism.

As we mentioned in the previous section, triangle curves are finite Galois covers of the projective line branched on three points $\mathcal{B} := \{-1, 0, 1\} \subset \mathbb{P}^1$. Note that the fundamental group of the complement $\mathbb{P}^1 \setminus \mathcal{B}$ is generated by three simple loops γ, δ and ϵ around the points $-1, 0$ and 1, respectively. These loops satisfy a single relation and we get:

$$\pi_1(\mathbb{P}^1 \setminus \mathcal{B}, \infty) = \langle \gamma, \delta, \epsilon \mid \gamma \cdot \delta \cdot \epsilon = 1 \rangle. \text{ (Fig. 1)}$$

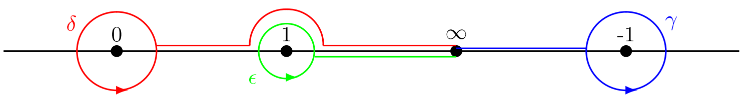

Fig. 1. The three generators γ, δ, ϵ of the fundamental group of $\mathbb{P}^1 \setminus \mathcal{B}$.

Definition 3.1. Let G be a finite group. A triple $S = [a, b, c]$ of non-trivial group elements is called a *spherical triple of generators* or shortly a *generating triple* of G if

$$G = \langle a, b, c \rangle \qquad \text{and} \qquad a \cdot b \cdot c = 1_G$$

The *type of* S is defined as

$$T(S) := [\mathrm{ord}(a), \mathrm{ord}(b), \mathrm{ord}(c)].$$

Observe that a generating triple $S = [a, b, c]$ of a finite group G induces a surjective homomorphism

$$\eta_S \colon \pi_1(\mathbb{P}^1 \setminus \mathcal{B}, \infty) \to G \qquad \text{by} \qquad \gamma \mapsto a, \quad \delta \mapsto b, \quad \text{and} \quad \epsilon \mapsto c.$$

Using *Riemann's existence theorem*, the homomorphism η_S yields a Galois triangle cover

$$f_S \colon (C_S, q_0) \to (\mathbb{P}^1, \infty)$$

with branch locus $\mathcal{B}$ together with a unique isomorphism $\psi \colon G \to \mathrm{Deck}(f_S)$ such that the composition

$$(\psi \circ \eta_S) \colon \pi_1(\mathbb{P}^1 \setminus \mathcal{B}, \infty) \to G \to \mathrm{Deck}(f_S)$$

is the monodromy map of the associated unramified cover. All triangle covers arise in this way.

Remark 3.2.

1. The genus of C_S, the order of G and the orders of the generators a, b and c are related by *Hurwitz's formula*:

$$2g(C_S) - 2 = |G| \left(1 - \frac{1}{\mathrm{ord}(a)} - \frac{1}{\mathrm{ord}(b)} - \frac{1}{\mathrm{ord}(c)}\right).$$

   In particular we observe that $g(C_S) \geq 2$, if and only if

$$\frac{1}{\mathrm{ord}(a)} + \frac{1}{\mathrm{ord}(b)} + \frac{1}{\mathrm{ord}(c)} < 1.$$

   In this case the generating triple $S = [a, b, c]$ is said to be *hyperbolic*, which we will assume throughout this section.
2. The stabilizer set of S is defined as

$$\Sigma_S := \bigcup_{g \in G} (g\langle a\rangle g^{-1} \cup g\langle b\rangle g^{-1} \cup g\langle c\rangle g^{-1})$$

   It consists of the elements in G which have at least one fixed point on the curve C_S.

Remark 3.3. An unmixed Beauville n-fold $X = (C_1 \times \ldots \times C_n)/G$ of general type yields an n-tuple $[S_1, \ldots, S_n]$, such that

1. $S_i = [a_i, b_i, c_i]$ is a hyperbolic generating triple of $\overline{G}_i = G/K_i$, where $K_i \trianglelefteq G$ is the kernel of

$$\psi_i \colon G \to \mathrm{Aut}(C_i).$$

2. The intersection $K_1 \cap \ldots \cap \widehat{K}_i \cap \ldots \cap K_n$ is trivial for all i.
3. $\bigcap_{i=1}^{n} \Sigma_{S_i} \cdot K_i = \{1_G\}$.

The condition that S_i is hyperbolic tells us $g(C_{S_i}) \geq 2$. The second condition reflects the *minimality of the realization* and the third the *freeness of the action*. Conversely, any such tuple gives rise to an unmixed Beauville n-fold of general type.

Definition 3.4. Let G be a finite group. An n-tuple $[S_1, \ldots, S_n]$ of hyperbolic generating triples such that the conditions (1), (2) and (3) from Remark 3.3 hold is called an *n-fold unmixed Beauville structure* for G. The set of n-fold unmixed Beauville structures for G is denoted by $\mathcal{UB}_n(G)$.

The group G is called an *n-fold unmixed Beauville group* if G admits an *n-fold unmixed Beauville structure.*

In the following we derive a criterion which allows us to decide whether two unmixed Beauville structures yield biholomorphic Beauville manifolds (cf. [BCG05, Proposition 4.2] for the surface case). For this purpose it is important to understand when two generating triples lead to the same triangle cover.

Definition 3.5. A *twisted covering isomorphism* of two triangle G-covers $f_i \colon C_i \to \mathbb{P}^1$, branched on $\mathcal{B} = \{-1, 0, 1\}$, is a pair (u, v) of biholomorphic maps

$$u \colon C_1 \to C_2 \qquad \text{and} \qquad v \colon \mathbb{P}^1 \to \mathbb{P}^1$$

such that $v(\mathcal{B}) = \mathcal{B}$ and $v \circ f_1 = f_2 \circ u$.

Remark 3.6. Let $\psi_i \colon G \to \mathrm{Deck}(f_i)$ be the corresponding G-actions, then the existence of a twisted covering isomorphism is equivalent to the existence of an automorphism $\alpha \in \mathrm{Aut}(G)$ and a biholomorphism $u \colon C_1 \to C_2$ such that

$$\psi_2(\alpha(g)) \circ u = u \circ \psi_1(g) \quad \text{for all} \quad g \in G.$$

As we shall see, this holds if and only if the corresponding generating triples belong to the same orbit of a certain group action on the set $\mathcal{S}(G)$ of all hyperbolic generating triples of G. First of all, there is a natural action of the *Artin-Braid* group

$$\mathcal{B}_3 := \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

on $\mathcal{S}(G)$ defined by:

$$\sigma_1 * [a, b, c] := [aba^{-1}, a, c] \qquad \text{and} \qquad \sigma_2 * [a, b, c] := [a, bcb^{-1}, b].$$

This action commutes with the diagonal action of an automorphism $\alpha \in \mathrm{Aut}(G)$ given by

$$\alpha * [a, b, c] := [\alpha(a), \alpha(b), \alpha(c)].$$

Thus, we obtain a well-defined action of $\mathrm{Aut}(G) \times \mathcal{B}_3$ on $\mathcal{S}(G)$ by

$$(\alpha, \delta) * S := \alpha * (\delta * S).$$

Proposition 3.7. *[BCG05, Proposition 2.3] Let G be a finite group and $S, S' \in \mathcal{S}(G)$ be two generating triples of G. Then, the following are equivalent:*

1. *There is a twisted covering isomorphism between $f_S \colon C_S \to \mathbb{P}^1$ and $f_{S'} \colon C_{S'} \to \mathbb{P}^1$.*
2. *The generating triples S and S' are in the same $\mathrm{Aut}(G) \times \mathcal{B}_3$ orbit.*

Remark 3.8. Proposition 3.7 tells us that the collection of triangle G-covers modulo twisted covering isomorphisms is in bijection with the quotient

$$\mathcal{T}(G) := \mathcal{S}(G)/(\mathrm{Aut}(G) \times \mathcal{B}_3).$$

For this reason, several authors put effort into the development of an efficient algorithm to compute these quotients, see [CGP23] and [Pau23]. In [CGP23] a database is set up, which contains a representative for any orbit of hyperbolic generating triples for a fixed genus $g \leq 64$ and group order $d \leq 2000$.

Proposition 3.9. *Let X and X' be Beauville n-folds given by the unmixed Beauville structures $[S_1, \ldots, S_n]$ and $[S'_1, \ldots, S'_n]$ for G, where $S_i \in S(G/K_i)$ and $S'_i \in S(G/K'_i)$. Then X and X' are biholomorphic if and only if there exist an automorphism $\alpha \in \mathrm{Aut}(G)$, braids $\delta_1, \ldots, \delta_n \in \mathcal{B}_3$ and a permutation $\tau \in \mathfrak{S}_n$, such that*

$$K'_i = \alpha(K_{\tau(i)}) \qquad \textit{and} \qquad S'_i = (\overline{\alpha}, \delta_i) * S_{\tau(i)}, \qquad \textit{where} \qquad \overline{\alpha}\colon G/K_{\tau(i)} \to G/K'_i.$$

Proof. By uniqueness of the minimal realization (Theorem 2.8), every biholomorphism $f\colon X \to X'$ between the quotients lifts to a biholomorphism

$$\hat{f}\colon C_{S_1} \times \ldots \times C_{S_n} \to C_{S'_1} \times \ldots \times C_{S'_n}.$$

As explained in Proposition 2.4, this map must be of the form

$$\hat{f}(z_1, \ldots, z_n) = (u_1(z_{\tau(1)}), \ldots, u_n(z_{\tau(n)})),$$

for some permutation $\tau \in \mathfrak{S}_n$. Such a map descends to the quotient level if and only if there exists an automorphism $\alpha \in \mathrm{Aut}(G)$ such that

$$\psi'_i(\alpha(g)) \circ u_i = u_i \circ \psi_{\tau(i)}(g) \qquad \text{for all} \qquad g \in G. \tag{3.1}$$

Now the action $\psi_{\tau(i)}\colon G \to \mathrm{Aut}(C_{S_{\tau_i}})$ factors through the faithful action

$$\overline{\psi}_{\tau(i)}\colon G/K_{\tau(i)} \to \mathrm{Aut}(C_{S_{\tau_i}})$$

and similarly ψ'_i descends to $\overline{\psi'}_i\colon G/K'_i \to \mathrm{Aut}(C_{S'_i})$. Thus Equation (3.1) is equivalent to

$$\overline{\psi'_i}(\overline{\alpha}(\overline{g})) \circ u_i = u_i \circ \overline{\psi}_{\tau(i)}(\overline{g}), \qquad \text{where} \qquad \overline{g} \in G/K_{\tau(i)}.$$

Here, $\overline{\alpha}\colon G/K_{\tau(i)} \to G/K'_i$ denotes the induced isomorphism. In other words, we have twisted covering isomorphisms

$$\begin{array}{ccc} C_{S_{\tau_i}} & \xrightarrow{u_i} & C_{S'_i} \\ \downarrow & & \downarrow \\ \dfrac{C_{S_{\tau_i}}}{G/K_{\tau(i)}} & \xrightarrow[v_i]{} & \dfrac{C_{S'_i}}{G/K'_i} \end{array}$$

The claim follows now from Proposition 3.7, that we use to translate into the language of spherical generating triples. The converse direction is clear. □

Remark 3.10. The group $\mathrm{Aut}(G) \times (\mathcal{B}_3 \wr \mathfrak{S}_n)$ acts on $\mathcal{UB}_n(G)$ by

$$\big(\alpha, \delta_1, \ldots, \delta_n, \tau\big) * (S_1, \ldots, S_n) := \big((\overline{\alpha}, \delta_1) * S_{\tau^{-1}(1)}, \ldots, (\overline{\alpha}, \delta_n) * S_{\tau^{-1}(n)}\big),$$

where $\overline{\alpha}\colon G/K_i \to G/\alpha(K_i)$ are the induced isomorphisms. According to Proposition 3.9, the orbits of this action are in one-to-one correspondence with the biholomorphism classes of unmixed Beauville n-folds with group G.

To avoid a complicated notation, we will from now on denote $\overline{\alpha}$ by α.

4. The orbits of $\mathcal{UB}_n(G)$ modulo $\mathrm{Aut}(G) \times (B_3 \wr \mathfrak{S}_n)$

In this section, we will provide a detailed explanation on how to count the number of elements in the quotient

$$Beau_n(G) := \frac{\mathcal{UB}_n(G)}{\mathrm{Aut}(G) \times (\mathcal{B}_3 \wr \mathfrak{S}_n)} \quad \text{by utilizing} \quad \mathcal{T}(G/K_i) = \frac{\mathcal{S}(G/K_i)}{\mathrm{Aut}(G/K_i) \times \mathcal{B}_3}.$$

As discussed in Remark 3.8, the latter may be determined by using the database from [CGP23]. We break down the problem into several steps. First we exploit the natural action of $\mathrm{Aut}(G)$ on the set of potential kernels

$$\{(K_1, \ldots, K_n) \mid K_1 \cap \ldots \cap \widehat{K_i} \cap \ldots \cap K_n = \{1_G\}, \quad K_i \trianglelefteq G\},$$

which is defined by

$$\alpha * (K_1, \ldots, K_n) := (\alpha(K_1), \ldots, \alpha(K_n)) \quad \text{for} \quad \alpha \in \mathrm{Aut}(G).$$

For each orbit, let $\mathcal{K} = (K_1, \ldots, K_n)$ be a representative. We use the permutations in $\mathfrak{S}_n$ to put equal kernels side by side. This allows us to assume and write $\mathcal{K} = (K_1, \ldots, K_n) = (N_1^{n_1}, \ldots, N_k^{n_k})$ with pairwise distinct N_i. Let $\mathrm{Aut}_{\mathcal{K}}(G)$ be the stabilizer of $\mathcal{K}$, then we can form the quotients

$$Beau_n(G, \mathcal{K}) := \frac{\prod_{i=1}^k \mathcal{S}(G/N_i)^{n_i}}{\mathrm{Aut}_{\mathcal{K}}(G) \times \prod_{i=1}^k (\mathcal{B}_3 \wr \mathfrak{S}_{n_i})} \tag{4.1}$$

Running over all $\mathcal{K}$, we obtain the elements of $Beau_n(G)$, by selecting the classes in $Beau_n(G,\mathcal{K})$ such that the condition on the intersection of the stabilizer sets from Remark 3.3 (3) holds. Thus it suffices to achieve a description of $Beau_n(G,\mathcal{K})$ in terms of $\mathcal{T}(G/N_i)$.

Remark 4.1.

1. Since any automorphism in $\mathrm{Aut}_{\mathcal{K}}(G)$ induces an automorphism in $\mathrm{Aut}\,(G/N_i)$, there is a natural surjective map

$$\eta\colon Beau_n(G,\mathcal{K}) \to \frac{\prod_{i=1}^k \mathcal{T}(G/N_i)^{n_i}}{\prod_{i=1}^k \mathfrak{S}_{n_i}}$$

   Following [Fal24], we solve the problem of counting (4.1) by counting the elements in the fibers of η.
2. Dropping the action of the symmetric group, we obtain another surjection

$$\pi\colon \frac{\prod_{i=1}^n S(G/K_i)}{\mathrm{Aut}_{\mathcal{K}}(G)\times \mathcal{B}_3^n} \to \prod_{i=1}^n \mathcal{T}(G/K_i).$$

3. For a point $x = \big([S_1],\dots,[S_n]\big) \in \prod_{i=1}^n \mathcal{T}(G/K_i)$ the fibers $\eta^{-1}\big([x]\big)$ and $\pi^{-1}(x)$ are related modulo the action of the stabilizer of x. More precisely

$$\mathrm{Stab}(x) \le \prod_{i=1}^k \mathfrak{S}_{n_i} \qquad \text{acts on} \qquad \pi^{-1}(x)$$

   and we obtain a bijection $\eta^{-1}\big([x]\big) \simeq \pi^{-1}(x)/\,\mathrm{Stab}(x)$.

Thus we can break down the enumeration problem of the fibers of η in two steps:

1. Describe the fibers $\pi^{-1}(x)$.
2. Count the orbits of the $\mathrm{Stab}(x)$-action on $\pi^{-1}(x)$.

We start with the first step. By construction of π, the fiber is given by

$$\pi^{-1}(x) = \{[\alpha_1 * S_1,\dots,\alpha_n * S_n] \mid \alpha_i \in \mathrm{Aut}(G/K_i)\}.$$

The problem is that different tuples of automorphisms

$$(\alpha_1,\dots,\alpha_n),\ (\beta_1,\dots,\beta_n) \in \prod_{i=1}^n \mathrm{Aut}(G/K_i)$$

may lead to same point in the fiber $\pi^{-1}(x)$. To deal with this ambiguity, we use the concept of automorphisms of *braid type*, introduced in [Fal24]:

Definition 4.2. Let S be a generating triple of the finite group G. Then the group of automorphisms of *braid type* on S is defined as

$$\mathcal{B}\,\mathrm{Aut}(G,S) := \{\varphi \in \mathrm{Aut}(G)\colon \exists\, \sigma \in \mathcal{B}_3 \text{ such that } \varphi * S = \sigma * S\}.$$

Remark 4.3. Since the action of an automorphism of G commutes with the action of a braid, it follows that $\mathcal{B}\,\mathrm{Aut}(G,S)$ is a subgroup of $\mathrm{Aut}(G)$.

Proposition 4.4. *Two tuples of automorphisms*

$$(\alpha_1,\dots,\alpha_n),\ (\beta_1,\dots,\beta_n) \in \prod_{i=1}^{n} \mathrm{Aut}(G/K_i)$$

yield the same point in the fiber $\pi^{-1}(x)$ *if and only if there exists* $\alpha \in \mathrm{Aut}_{\mathcal{K}}(G)$ *and* $\gamma_i \in \mathcal{B}\,\mathrm{Aut}(G/K_i,S_i)$ *such that*

$$\beta_i = \alpha \circ \alpha_i \circ \gamma_i \qquad \textit{for all} \qquad i=1,\dots,n.$$

Proof. Assume that

$$[\alpha_1 * S_1,\dots,\alpha_n * S_n] = [\beta_1 * S_1,\dots,\beta_n * S_n] \in \pi^{-1}(x).$$

Then there exists an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{K}}(G)$ and braids $\delta_i \in \mathcal{B}_3$ such that

$$\beta_i * S_i = (\alpha\circ\alpha_i) * (\delta_i * S_i), \qquad \text{so that} \qquad (\alpha_i^{-1}\circ\alpha^{-1}\circ\beta_i) * S_i = \delta_i * S_i.$$

This shows that $\gamma_i := \alpha_i^{-1}\circ\alpha^{-1}\circ\beta_i$ belongs to $\mathcal{B}\,\mathrm{Aut}(G/K_i,S_i)$. Assume conversely that $\alpha \in \mathrm{Aut}_{\mathcal{K}}(G)$ and $\gamma_i \in \mathcal{B}\,\mathrm{Aut}(G/K_i,S_i)$ be such that $\beta_i = \alpha\circ\alpha_i\circ\gamma_i$. Let $\delta_i \in \mathcal{B}_3$ be a braid fulfilling $\gamma_i * S_i = \delta_i * S_i$, then we have

$$\beta_i * S_i = (\alpha\circ\alpha_i) * (\delta_i * S_i).$$

This shows that the corresponding points in the fiber agree. □

Remark 4.5. We have obtained a generalization of [Fal24, Thm. 2.18 and Cor. 2.20] in higher dimension $n > 2$. More precisely, the following hold:

1. The group

$$\mathrm{Aut}_{\mathcal{K}}(G) \times \prod_{i=1}^{n} \mathcal{B}\,\mathrm{Aut}(G/K_i,S_i) \qquad \text{acts on} \qquad \prod_{i=1}^{n} \mathrm{Aut}(G/K_i)$$

   via the rule

$$(\alpha, \gamma_1, \ldots, \gamma_n) * (\alpha_1, \ldots, \alpha_n) := (\alpha \circ \alpha_1 \circ \gamma_1^{-1}, \ldots, \alpha \circ \alpha_n \circ \gamma_n^{-1})$$

By Proposition 4.4, the quotient, which is denoted by $Q\left(\prod_{i=1}^n \mathrm{Aut}(G/K_i)\right)_{S_1,\ldots,S_n}$, is in bijection with $\pi^{-1}(x)$ via

$$\psi\colon Q\left(\prod_{i=1}^n \mathrm{Aut}(G/K_i)\right)_{S_1,\ldots,S_n} \to \pi^{-1}(x), \quad [(\alpha_1, \ldots, \alpha_n)] \mapsto [\alpha_1 \cdot S_1, \ldots, \alpha_n \cdot S_n]$$

By definition the bijection ψ depends on the choices of representatives S_i of the classes $[S_i]$.

2. To achieve a description of the fibers of η, we need to understand the induced action of $\mathrm{Stab}(x)$ on the quotient defined in (1). Let

$$x = (x_1, \ldots, x_k) \in \prod_{i=1}^k \mathcal{T}(G/N_i)^{n_i},$$

then up to exchanging the order of the factors within the product $T(G/N_i)^{n_i}$, we may assume that

$$x_i = \left([S_{i,1}]^{m_{i,1}}, \ldots, [S_{i,l_i}]^{m_{i,l_i}}\right) \in \mathcal{T}(G/N_i)^{n_i}$$

with pairwise distinct classes

$$[S_{i,1}], \ldots, [S_{i,l_i}] \qquad \text{and} \qquad m_{i,1} + \ldots + m_{i,l_i} = n_i.$$

In this notation, the stabilizer of x is given by

$$\mathrm{Stab}(x) = \prod_{i=1}^k \left(\prod_{j=1}^{l_i} \mathfrak{S}_{m_{i,j}} \right) < \prod_{i=1}^k \mathfrak{S}_{n_i}.$$

Clearly, the natural action of $\tau \in \mathrm{Stab}(x)$ on $\pi^{-1}(x)$ induces an action on the quotient

$$Q\left(\prod_{i=1}^k \mathrm{Aut}(G/N_i)^{n_i}\right)_{S_{1,1}^{m_{1,1}},\ldots,S_{1,l_1}^{m_{1,l_1}},\ldots,S_{k,1}^{m_{k,1}},\ldots,S_{k,l_k}^{m_{k,l_k}}}$$

which is compatible with ψ and defined by the following rule:

$$\tau * [(\alpha_1, \ldots, \alpha_n)] := [(\alpha_{\tau^{-1}(1)}, \ldots, \alpha_{\tau^{-1}(n)})].$$

This action is of course only well defined if we choose the same representative for all of the identical classes that occur in x.

Remark 4.6. Following the approach presented in this section, we wrote a MAGMA script that takes as input a group G, a sequence of kernels $K = (K_1, \ldots, K_n)$ of G and a sequence of classes

$$x = ([S_1], \ldots, [S_n]) \in \prod_{i=1}^{n} \mathcal{T}(G/K_i).$$

It returns the fiber $\eta^{-1}([x])$, namely all unmixed Beauville n-folds with group G and kernels K defined by $[S_1], \ldots, [S_n]$.

5. The Beauville dimension of a group

In this section we discuss the Beauville dimension of a finite group G, that has been introduced by Carta and Fairbairn in [CF22]. In their article Beauville manifolds of unmixed type and their groups are studied from a combinatorial and group theoretic perspective under the assumption that the G-action is absolutely faithful. They define the Beauville dimension of a finite group G as the minimal length of a sequence $[S_1, \ldots, S_n]$ of generating triples for G, such that

$$\Sigma_{S_1} \cap \ldots \cap \Sigma_{S_n} = \{1_G\}.$$

If no such sequence exists then the Beauville dimension of G is set to be 1, see [CF22, Definition 1.3]. According to their definition, the groups of Beauville dimension two are precisely the classical Beauville surface groups. Using computers they determine all finite groups G of order less than or equal to 1023 with Beauville dimension 2, 3 and 4.

Remark 5.1. In their definition Carta and Fairbairn do not assume that the generating triples are hyperbolic i.e. they also allow elliptic curves and even projective lines in the product. Hence the Beauville manifolds corresponding to the groups in their tables with Beauville dimension 3 and 4 might not be of general type. In these cases the rigidity of the corresponding quotient manifolds does not follow from the condition that the curves are triangle curves, because there are further rigidity conditions involved, see [BGK25, Corollary 1.4]. Since they claim the rigidity in [CF22, Definition 4] but do not verify these extra conditions, we decided to investigate their groups in detail using the computer algebra system MAGMA:

1. None of the 27 groups of Beauville dimension 3 admit a rigid action on an elliptic curve or on $\mathbb{P}^1$, i.e. all of them correspond to a Beauville threefold of general type.
2. There are 153 groups of Beauville dimension 4. None of them admit a rigid action on $\mathbb{P}^1$ and only 43 of them a rigid action on an elliptic curve. The elliptic curve is always the Fermat cubic curve and the branching signature is always equal to $[3, 3, 3]$. None of these 43 groups correspond to a Beauville 4-fold of general type, or with Kodaira dimension 1 or 2.

(a) Two out of the 43 groups, namely $\mathbb{Z}_3^2$ and He(3), yield Beauville 4-folds of Kodaira dimension 0. They are quotients of a product $E_1 \times E_2 \times E_3 \times E_4$ of four Fermat elliptic curves $E_i = E$. In this case the extra rigidity condition

$$\left(H^1(\omega_{E_i}^{\otimes 2}) \otimes H^0(\omega_{E_j})\right)^G = 0 \qquad \text{for all} \qquad i \neq j$$

can be fulfilled for both groups $G = \mathbb{Z}_3^2$ and $G = \mathrm{He}(3)$ by suitable choices of generating triples. According to [BG21, Theorem 1.7] there is a unique Beauville 4-fold for each of these groups. The two 4-folds are topologically distinct.

(b) The remaining 41 groups yield Beauville 4-folds

$$X = (E \times C_1 \times C_2 \times C_3)/G$$

of Kodaira dimension 3, i.e. the curves C_i have genus at least two and E is the Fermat cubic curve as explained above. The extra rigidity condition to be checked is

$$\left(H^1(\omega_E^{\otimes 2}) \otimes H^0(\omega_{C_i})\right)^G = 0, \qquad \text{for all} \qquad 1 \leq i \leq 3.$$

By [BGK25, Corollary 3.10] this condition holds true since $E \to E/G \simeq \mathbb{P}^1$ is branched with signature $[3, 3, 3]$. Below is a table with a structural description of these groups as subgroups of $\mathrm{Aut}(E)$, i.e. as semidirect products of the form $A \rtimes \mathbb{Z}_3$, where A is an abelian group of translations and $\mathbb{Z}_3$ acts as a group of rotations. The column Id contains the MAGMA identifier: $\langle a, b\rangle$ denotes the b-th group of order a in the MAGMA *Database of Small Groups*.

No.	G	Id	No.	G	Id
1	$(\mathbb{Z}_2 \times \mathbb{Z}_6) \rtimes \mathbb{Z}_3$	$\langle 36, 11\rangle$	22	$\mathbb{Z}_{183} \rtimes \mathbb{Z}_3$	$\langle 549, 3\rangle$
2	$\mathbb{Z}_{21} \rtimes \mathbb{Z}_3$	$\langle 63, 3\rangle$	23	$(\mathbb{Z}_3 \times \mathbb{Z}_{63}) \rtimes \mathbb{Z}_3$	$\langle 567, 13\rangle$
3	$(\mathbb{Z}_3 \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3$	$\langle 81, 9\rangle$	24	$(\mathbb{Z}_8 \times \mathbb{Z}_{24}) \rtimes \mathbb{Z}_3$	$\langle 576, 1070\rangle$
4	$\mathbb{Z}_6^2 \rtimes \mathbb{Z}_3$	$\langle 108, 22\rangle$	25	$\mathbb{Z}_{201} \rtimes \mathbb{Z}_3$	$\langle 603, 3\rangle$
5	$\mathbb{Z}_{39} \rtimes \mathbb{Z}_3$	$\langle 117, 3\rangle$	26	$\mathbb{Z}_{219} \rtimes \mathbb{Z}_3$	$\langle 657, 4\rangle$
6	$(\mathbb{Z}_4 \times \mathbb{Z}_{12}) \rtimes \mathbb{Z}_3$	$\langle 144, 68\rangle$	27	$\mathbb{Z}_{15}^2 \rtimes \mathbb{Z}_3$	$\langle 675, 12\rangle$
7	$\mathbb{Z}_{57} \rtimes \mathbb{Z}_3$	$\langle 171, 4\rangle$	28	$(\mathbb{Z}_2 \times \mathbb{Z}_{114}) \rtimes \mathbb{Z}_3$	$\langle 684, 45\rangle$
8	$(\mathbb{Z}_3 \times \mathbb{Z}_{21}) \rtimes \mathbb{Z}_3$	$\langle 189, 8\rangle$	29	$\mathbb{Z}_{237} \rtimes \mathbb{Z}_3$	$\langle 711, 3\rangle$
9	$(\mathbb{Z}_3 \times \mathbb{Z}_{15}) \rtimes \mathbb{Z}_3$	$\langle 225, 5\rangle$	30	$(\mathbb{Z}_9 \times \mathbb{Z}_{27}) \rtimes \mathbb{Z}_3$	$\langle 729, 95\rangle$
10	$\mathbb{Z}_9^2 \rtimes \mathbb{Z}_3$	$\langle 243, 26\rangle$	31	$(\mathbb{Z}_6 \times \mathbb{Z}_{42}) \rtimes \mathbb{Z}_3$	$\langle 756, 117\rangle$
11	$(\mathbb{Z}_2 \times \mathbb{Z}_{42}) \rtimes \mathbb{Z}_3$	$\langle 252, 40\rangle$	32	$\mathbb{Z}_{273} \rtimes \mathbb{Z}_3$	$\langle 819, 9\rangle$
12	$\mathbb{Z}_{93} \rtimes \mathbb{Z}_3$	$\langle 279, 3\rangle$	33	$\mathbb{Z}_{273} \rtimes \mathbb{Z}_3$	$\langle 819, 10\rangle$
13	$(\mathbb{Z}_6 \times \mathbb{Z}_{18}) \rtimes \mathbb{Z}_3$	$\langle 324, 50\rangle$	34	$(\mathbb{Z}_3 \times \mathbb{Z}_{93}) \rtimes \mathbb{Z}_3$	$\langle 837, 8\rangle$
14	$\mathbb{Z}_{111} \rtimes \mathbb{Z}_3$	$\langle 333, 4\rangle$	35	$\mathbb{Z}_{291} \rtimes \mathbb{Z}_3$	$\langle 873, 3\rangle$
15	$(\mathbb{Z}_3 \times \mathbb{Z}_{39}) \rtimes \mathbb{Z}_3$	$\langle 351, 8\rangle$	36	$(\mathbb{Z}_{10} \times \mathbb{Z}_{30}) \rtimes \mathbb{Z}_3$	$\langle 900, 141\rangle$
16	$\mathbb{Z}_{129} \rtimes \mathbb{Z}_3$	$\langle 387, 3\rangle$	37	$\mathbb{Z}_{309} \rtimes \mathbb{Z}_3$	$\langle 927, 3\rangle$
17	$\mathbb{Z}_{12}^2 \rtimes \mathbb{Z}_3$	$\langle 432, 103\rangle$	38	$\mathbb{Z}_{18}^2 \rtimes \mathbb{Z}_3$	$\langle 972, 122\rangle$
18	$\mathbb{Z}_{147} \rtimes \mathbb{Z}_3$	$\langle 441, 3\rangle$	39	$\mathbb{Z}_{327} \rtimes \mathbb{Z}_3$	$\langle 981, 4\rangle$
19	$(\mathbb{Z}_7 \times \mathbb{Z}_{21}) \rtimes \mathbb{Z}_3$	$\langle 441, 12\rangle$	40	$(\mathbb{Z}_3 \times \mathbb{Z}_{111}) \rtimes \mathbb{Z}_3$	$\langle 999, 9\rangle$
20	$(\mathbb{Z}_2 \times \mathbb{Z}_{78}) \rtimes \mathbb{Z}_3$	$\langle 468, 49\rangle$	41	$(\mathbb{Z}_4 \times \mathbb{Z}_{84}) \rtimes \mathbb{Z}_3$	$\langle 1008, 409\rangle$
21	$(\mathbb{Z}_3 \times \mathbb{Z}_{57}) \rtimes \mathbb{Z}_3$	$\langle 513, 9\rangle$			

In summary our verifications show that all of the groups in the tables of Carta and Fairbairn yield Beauville manifolds. We suggest to incorporate the additional rigidity conditions into the definition of the Beauville dimension, or alternatively, restrict to hyperbolic generating triples. Moreover, we want to point out that the uniqueness of the minimal realization, which in particular allows us to attach a unique group to a given Beauville manifold X is only established if the Kodaira dimension is maximal. In case where X has Kodaira dimension zero, we can uniquely attach the holonomy group of the underlying flat Riemannian manifold, which is a quotient of G. The two 4-folds in (a) both have holonomy $\mathbb{Z}_3^2$, cf. [BG21, Remark 5.10].

As we have seen, there are Beauville manifolds of dimension $\dim(X) \geq 3$ obtained by actions which are not absolutely faithful. Thus it makes sense to generalize the definition of the Beauville dimension by allowing non-trivial kernels. In contrast to [CF22] we decided to restrict to hyperbolic generating triples.

Definition 5.2. The *Beauville dimension* $d(G)$ of a finite group G is the minimal positive integer $n \geq 2$, such that $\mathcal{UB}_n(G) \neq \emptyset$. If no such integer exists then the Beauville dimension of G is set to be equal to 1.

It is natural to ask the question if there are further groups of Beauville dimension $d(G) \geq 3$, using our definition. For this purpose we wrote a MAGMA algorithm to check if a given finite group G admits an unmixed n-fold Beauville structure. Since the presence of non-trivial kernels increases the computational difficulty drastically, we restrict to groups of order less than or equal to 255 and $n = 3$. The algorithm also determines the Beauville dimension of the respective groups according to our definition. We find:

Proposition 5.3. *The groups G of order less or equal to 255, which admit an unmixed 3-fold Beauville structure are the following:*

No.	G	Id	$d(G)$
1	$\mathbb{Z}_5^2$	$\langle 25, 2\rangle$	2
2	$\mathbb{Z}_7^2$	$\langle 49, 2\rangle$	2
3	$\mathfrak{S}_5$	$\langle 120, 34\rangle$	2
4	$\mathbb{Z}_{11}^2$	$\langle 121, 2\rangle$	2
5	$\mathrm{He}(5)$	$\langle 125, 3\rangle$	2
6	$\mathbb{Z}_5^3$	$\langle 125, 5\rangle$	3
7	$\mathbb{Z}_2^4.Q_8$	$\langle 128, 36\rangle$	2
8	$\mathrm{PSL}(2,7)$	$\langle 168, 42\rangle$	2
9	$\mathbb{Z}_{13}^2$	$\langle 169, 2\rangle$	2
10	$\mathrm{SL}(2,5) \rtimes \mathbb{Z}_2$	$\langle 240, 90\rangle$	2
11	$\mathcal{A}_5 \rtimes \mathbb{Z}_4$	$\langle 240, 91\rangle$	2
12	$\mathbb{Z}_2 \times \mathfrak{S}_5$	$\langle 240, 189\rangle$	2
13	$(\mathbb{Z}_3 \times \mathrm{He}(3)) \rtimes \mathbb{Z}_3$	$\langle 243, 3\rangle$	2
14	$(\mathbb{Z}_3 \times M_{27}) \rtimes \mathbb{Z}_3$	$\langle 243, 4\rangle$	3
15	$\mathbb{Z}_3^3 \rtimes \mathbb{Z}_9$	$\langle 243, 13\rangle$	3

Here $\mathrm{He}(p)$ *is the Heisenberg groups of order* p^3 *and* M_{27} *is the unique non-abelian group of order* 27 *which has an element of order* 9.

There are three groups of Beauville dimension $d(G) = 3$ in our table. The smallest of them $\mathbb{Z}_5^3$ is the only abelian and moreover the only group that does not appear in the table of Carta and Fairbairn. This means that a Beauville structure of $\mathbb{Z}_5^3$ can only exist with non-trivial kernels. This group is interesting for the following reasons:

Remark 5.4.

1. According to [BCG05] the abelian groups of Beauville dimension two are $\mathbb{Z}_n^2$, where $\gcd(n, 6) = 1$. Carta and Fairbairn extended this result to higher dimensions. Using their definition, they show that an abelian group of Beauville dimension greater than two has Beauville dimension four and is isomorphic to $\mathbb{Z}_n^2$, where $\gcd(n, 2) = 1$, see [CF22, Theorem 2.9]. In contrast the group $\mathbb{Z}_5^3$ is not 2-generated, which raises the question of a general structure theorem for abelian groups G of Beauville dimension $d(G) \geq 3$ in view of our definition.
2. Carta and Fairbairn point out that the order of the groups of Beauville dimension greater than two which are contained in their tables is always divisible by 3. They ask if this is a general fact [CF22, Problem 4.3]. Using our definition of Beauville dimension the group $\mathbb{Z}_5^3$ of Beauville dimension three gives a negative answer to this question.

Proposition 5.5. *The group* $\mathbb{Z}_n^3$ *has an unmixed* 3*-fold Beauville structure if and only if* $\gcd(n, 6) = 1$.

Proof. Assume that $\gcd(n, 6) = 1$. We choose the kernels $K_i = \langle e_i \rangle$ for $G = \mathbb{Z}_n^3 = \langle e_1, e_2, e_3 \rangle$ and the following generating triples of the quotient groups G/K_i of type $[n, n, n]$:

i	S_i	$\Sigma_{S_i} + K_i$
1	$[e_2 - e_3, e_2 + e_3, -2e_2]$	$\{(l, j, k) \in G \mid j = -k \text{ or } j = k \text{ or } j = 0\}$
2	$[e_1 + e_3, e_3 - e_1, -2e_3]$	$\{(l, j, k) \in G \mid l = k \text{ or } l = -k \text{ or } k = 0\}$
3	$[2e_1 + e_2, e_1, -3e_1 - e_2]$	$\{(l, j, k) \in G \mid l = 2j \text{ or } l = 0 \text{ or } l = 3j\}$

To verify the freeness condition, we consider an element $(l, j, k) \in \bigcap_{i=1}^3 (\Sigma_{S_i} + K_i)$ and show that it is trivial. If $j = 0$, we conclude directly from the third row that $l = 0$ and consequently from the second row $k = 0$, since $\gcd(n, 6) = 1$. On the other hand, if $j \neq 0$, the three conditions yield $0 \neq j = \pm k = \pm l = \pm m \cdot j$ with $m \in \{2, 3\}$, leading to a contradiction.

Suppose that $G = \mathbb{Z}_n^3$ has a 3-fold Beauville structure. We show that $\gcd(n, 6) = 1$.

Step 1. Reduction to the case $\mathbb{Z}_{p^k}^3$. The group G is the product of its Sylow subgroups

$$G = \bigoplus_{p \text{ prime}} G_p, \qquad \text{with} \qquad G_p = \mathbb{Z}_{p^{k_p}}^3.$$

This implies

$$G/K_i = \bigoplus_{p\ prime} G_p/(K_i \cap G_p).$$

If the group G/K_i has the generating triple $S_i := [x_i, y_i, z_i]$ (from a Beauville structure of G), then $G_p/(K_i \cap G_p)$ has the generating triple $\pi_p(S_i)$ consiting of the projections $\pi_p(x_i)$, $\pi_p(y_i)$ and $\pi_p(z_i)$. Let Σ_{S_i} and $\Sigma_{\pi_p(S_i)}$ be the corresponding stabilizer sets. We have $\Sigma_{\pi_p(S_i)} \subset \Sigma_{S_i}$ because the projections $\pi_p(x_i)$, $\pi_p(y_i)$ and $\pi_p(z_i)$ are multiples of x_i, y_i and z_i. Therefore, we have

$$\Sigma_{\pi_p(S_i)} + (K_i \cap G_p) \subset \Sigma_{S_i} + K_i.$$

Note that $K_i \cap G_p$ is a proper subgroup of G_p. Otherwise, if $K_1 \cap G_p = G_p$, then $K_2 \cap G_p = 0$ by minimality of the action and therefore

$$G_p/(K_2 \cap G_p) = G_p = \mathbb{Z}^3_{p^{k_p}}.$$

A contradiction, since the group $\mathbb{Z}^3_{p^{k_p}}$ is not 2-generated. Thus we have shown that $\mathbb{Z}^3_{p^{k_p}}$ admits a 3-fold Beauville structure for all primes p dividing n.
Step 2. Reduction to $\mathbb{Z}_p^3$. By the first step we may assume that $G = \mathbb{Z}^3_{p^k}$. Since G/K_i is 2-generated, it holds

$$G/K_i = \mathbb{Z}_{p^{a_i}} \times \mathbb{Z}_{p^{b_i}}, \quad \text{where} \quad k \geq a_i \geq b_i \geq 0.$$

Here $b_i = 0$ is possible, but $a_i \neq 0$ according to the remark at the end of the first step. Since G/K_i is 2-generated the kernel K_i must have an element of order p^k. We use this fact to show $a_i = k$. Assume $a_1 < k$, then $p^{k-1}G \subset K_1$. Let $g \in K_2$ be an element of order $ord(g) = p^k$, then

$$0 \neq p^{k-1}g \in (p^{k-1}G) \cap K_2 \subset K_1 \cap K_2 = 0.$$

A contradiction. Thanks to this argument we have

$$G/K_i = \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^{b_i}}, \quad \text{with} \quad k \geq b_i \geq 0.$$

This shows that

$$\frac{p^{k-1}G}{(p^{k-1}G) \cap K_i} \simeq p^{k-1} \cdot (G/K_i) \quad \simeq \quad \mathbb{Z}_p \quad \text{or} \quad \mathbb{Z}_p^2.$$

Now let $S_i = [x_i, y_i, z_i]$ be a generating triple of G/K_i, then the elements of the triple

$$[p^{k-1}x_i, p^{k-1}y_i, p^{k-1}z_i]$$

generate the group $p^{k-1} \cdot (G/K_i)$. This generating triple has at least 2 elements different from zero. It holds

$$(p^{k-1} \cdot \Sigma_{S_i}) + (K_i \cap p^{k-1}G) \subset \Sigma_{S_i} + K_i.$$

Here the set $(p^{k-1} \cdot \Sigma_{S_i})$ coincides with the stabilizer set of

$$[p^{k-1}x_i, p^{k-1}y_i, p^{k-1}z_i].$$

A priori one of the elements of this triple might be zero, but it never happens. Assume otherwise and $i = 1$, then

$$(p^{k-1} \cdot \Sigma_{S_1}) + (K_1 \cap p^{k-1}G) = p^{k-1}G \cong \mathbb{Z}_p^3.$$

We observe that the intersection of the sets $(p^{k-1} \cdot \Sigma_{S_j}) + (K_j \cap p^{k-1}G)$ for $j = 2$ and 3 must be trivial. This implies that

$$\frac{p^{k-1} \cdot G}{(K_2 \cap p^{k-1}G) + (K_3 \cap p^{k-1}G)} \cong \mathbb{Z}_p$$

is a Beauville surface group, which is a contradiction. In summary this shows that $p^{k-1} \cdot G \cong \mathbb{Z}_p^3$ admits a 3-fold Beauville structure.

Step 3. It remains to exclude the groups $G = \mathbb{Z}_p^3$ for the primes $p = 2$ and $p = 3$. This is a straightforward computation by hand or by MAGMA. □

6. Proof of the main theorems

We will now apply our implementation of the method outlined in Section 4 to prove the main theorems from the introduction.

Proof of Theorem 1.3 and 1.4. According to Proposition 5.3 we know that $\mathcal{U}B_3(G) = \emptyset$ for all groups G of order $|G| \leq 25$ except for $G = \mathbb{Z}_5^2 = \langle e_1, e_2 \rangle$. Here the types of the generating triples S_i defining X are $[5, 5, 5]$. By the Hurwitz formula, the possible genera of the curves are either 6 or 2, depending on the kernel K_i being trivial or isomorphic to $\mathbb{Z}_5$. Using Proposition 2.1 we see that the only possible values of the holomorphic Euler number are $\chi(\mathcal{O}_X) = -1$ or -5. More precisely, $\chi(\mathcal{O}_X) = -1$ occurs when two of the three kernels are trivial and $\chi(\mathcal{O}_X) = -5$ when all kernels are trivial. In the first case, we can reorder K_i in such a way that $K_1 = K_2 = \{0\}$ and only the third one is not trivial. Furthermore, since the automorphism group $\mathrm{GL}(2, 5)$ of $\mathbb{Z}_5^2$ is acting transitively on $\mathbb{Z}_5^2$, we can assume $K_3 = \langle e_2 \rangle$. There is only one $\mathrm{GL}(2, 5) \times \mathcal{B}_3$ orbit on the set of spherical system of generators of $\mathbb{Z}_5^2$ with signature $[5, 5, 5]$. It is represented by $S := [e_1, e_2, 4e_1 + 4e_2]$. Similarly for $\overline{G}_3 \cong \mathbb{Z}_5$ the set $\mathcal{T}(\overline{G}_3)$ consists of only one equivalence class represented by $S' = [e_1, e_1, 3e_1]$. Running the algorithm presented in Section 4 for $K = (\langle 0 \rangle, \langle 0 \rangle, \langle e_2 \rangle)$ and $S = [S, S, S']$ we obtain 8 biholomorphism classes of unmixed Beauville 3-folds with $\chi(\mathcal{O}_X) = -1$ represented by the Beauville structures in Table 1.

Table 1
The 8 biholomorphism classes of unmixed Beauville 3-folds with $G = \mathbb{Z}_5^2$ and $\chi(\mathcal{O}_X) = -1$ and their Hodge numbers.

	S_1	S_2	S_3	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{2,1}$
1	$[e_2, 3e_1 + 3e_2, 2e_1 + e_2]$	$[4e_1 + 3e_2, 3e_1, 3e_1 + 2e_2]$	$[3e_1, 3e_1, 4e_1]$	2	0	0	7	10
2	$[e_2, 3e_1 + 3e_2, 2e_1 + e_2]$	$[e_1 + 4e_2, 3e_1 + e_2, e_1]$	$[e_1, e_1, 3e_1]$	2	0	0	7	10
3	$[4e_1 + 3e_2, 4e_2, e_1 + 3e_2]$	$[4e_1 + e_2, 4e_1 + 4e_2, 2e_1]$	$[2e_1, 2e_1, e_1]$	3	1	0	5	9
4	$[3e_1 + e_2, 3e_1 + 4e_2, 4e_1]$	$[3e_2, 4e_1 + e_2, e_1 + e_2]$	$[4e_1, 4e_1, 2e_1]$	3	1	0	5	9
5	$[2e_1 + 2e_2, 2e_1 + 4e_2, e_1 + 4e_2]$	$[2e_2, 4e_1, e_1 + 3e_2]$	$[4e_1, 4e_1, 2e_1]$	3	1	0	5	9
6	$[e_1 + e_2, e_1 + 2e_2, 3e_1 + 2e_2]$	$[2e_2, 4e_1, e_1 + 3e_2]$	$[2e_1, 2e_1, e_1]$	3	1	0	5	9
7	$[4e_1 + 4e_2, 4e_1 + 3e_2, 2e_1 + 3e_2]$	$[e_2, 2e_1, 3e_1 + 4e_2]$	$[e_1, e_1, 3e_1]$	4	2	0	3	8
8	$[e_2, 2e_1 + 2e_2, 3e_1 + 2e_2]$	$[e_1 + 4e_2, 3e_1 + e_2, e_1]$	$[3e_1, 3e_1, 4e_1]$	4	2	0	3	8

The Hodge numbers are computed using the generating triples as explained in [FG16, Theorem 3.7]. It remains to classify the unmixed Beauville manifolds X with $\chi(\mathcal{O}_X) \in \{-5, -4, -3, -2\}$ under the assumption that the G-action is absolutely faithful. By Hurwitz's bound we have $|G| \leq 84(g_i - 1)$, where $g_i = g(C_i)$. In combination with the formula for $\chi(\mathcal{O}_X)$ from Proposition 2.1 we obtain a bound for the group order in terms of the holomorphic Euler number:

$$N := |G| \leq \lfloor 168\sqrt{-21\chi(\mathcal{O}_X)} \rfloor.$$

This already shows the finiteness of the classification, which is performed with MAGMA. To make the algorithm more efficient, we invoke some additional combinatorics, as explained in [FG16]. Since $(g_i - 1)$ is a divisor of $N \cdot \chi(\mathcal{O}_X)$ we can create for fixed value of $\chi = \chi(\mathcal{O}_X)$ a list of 4-tuples for the possibilities of the group order and the genera

$$[N, g_1, g_2, g_3].$$

For each 4-tuple we determine the possible types $T_i = [m_{i,1}, m_{i,2}, m_{i,3}]$ of the generating triples in the Beauville structures. The entries $m_{i,j} \geq 2$ are divisors of N, fulfill the Hurwitz formula (cf. Remark 3.2) and the following additional combinatorial constraints:

$$m_{i,j} | (g_{[i+1]} - 1)(g_{[i+2]} - 1) \qquad \text{and} \qquad m_{i,j} \leq 4g_i + 2,$$

see [FG16, Prop. 4.8]. This allows us to determine a list of 4-tuples

$$[N, T_1, T_2, T_3]$$

of possible group orders and types. Not all of them will occur. However, any Beauville structure $[S_1, S_2, S_3]$ attached to an unmixed Beauville threefold X with $\chi = \chi(\mathcal{O}_X)$ obtained from an absolutely faithful G-action yields a tuple

$$[|G|, T(S_1), T(S_2), T(S_3)]$$

in this list. For each $[N, T_1, T_2, T_3]$ we run through the groups G of order N and check if there is an unmixed Beauville structure $[S_1, S_2, S_3]$, with $S_i \in \mathcal{S}(G)$ and $T_i = T(S_i)$. In

Table 2
Beauville 3-folds X with $\chi(\mathcal{O}_X) \in \{-5, -4, -3, -2\}$ obtained by an absolutely faithful action.

	G	T_1	T_3	T_2	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	χ	$\mathcal{N}$
1	$\mathfrak{S}_5$	$[2,5,4]$	$[2,6,5]$	$[3,4,4]$	4	1	0	5	12	-2	1
2	$\mathrm{PSL}(2,7)$	$[2,3,7]$	$[3,3,4]$	$[7,7,7]$	6	1	0	11	24	-4	2
3	$\mathrm{PSL}(2,7)$	$[2,3,7]$	$[3,3,4]$	$[7,7,7]$	9	4	0	5	21	-4	2
4	$\mathrm{PSL}(2,7)$	$[2,3,7]$	$[4,4,4]$	$[3,3,7]$	6	1	0	7	20	-4	2
5	$\mathrm{PSL}(2,7)$	$[2,3,7]$	$[4,4,4]$	$[3,3,7]$	7	2	0	5	19	-4	2
6	$\mathfrak{S}_5$	$[2,5,4]$	$[3,4,4]$	$[3,6,6]$	8	2	0	7	24	-5	1
7	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	6	0	0	15	30	-5	2
8	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	7	1	0	13	29	-5	3
9	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	7	1	0	17	33	-5	1
10	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	8	2	0	11	28	-5	13
11	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	8	2	0	15	32	-5	3
12	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	9	3	0	9	27	-5	14
13	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	9	3	0	13	31	-5	4
14	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	10	4	0	7	26	-5	12
15	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	10	4	0	11	30	-5	8
16	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	11	5	0	5	25	-5	3
17	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	11	5	0	9	29	-5	7
18	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	12	6	0	3	24	-5	4
19	$\mathbb{Z}_5^2$	$[5,5,5]$	$[5,5,5]$	$[5,5,5]$	12	6	0	7	28	-5	3

case, we determine for each T_i a representative of each orbit of the $\mathrm{Aut}(G) \times \mathcal{B}_3$-action on the set of generating triples S_i of G with type T_i. The method from Section 4 is then used to determine the $\mathrm{Aut}(G) \times (\mathcal{B}_3 \wr \mathfrak{S}_3)$ orbits, i.e. the number $\mathcal{N}$ of biholomorphism classes. The output is summarized in Table 2. □

Data availability

No data was used for the research described in the article.

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