Nodal surfaces in $\mathbb{P}^3$ and coding theory

Sascha Kurz
Dept. of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany
sascha.kurz@uni-bayreuth.de

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Abstract

To each nodal hypersurface one can associate a binary linear code. Here we show that the binary linear code associated to sextics in $\mathbb{P}^3$ with the maximum number of 65 nodes, as e.g. the Barth sextic, is unique. We also state possible candidates for codes that might be associated with a hypothetical septic attaining the currently best known upper bound for the maximum number of nodes.

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1 Introduction

An irreducible hypersurface S of degree s in a complex projective space $\mathbb{P}^n := \mathbb{P}^n(\mathbb{C})$ is the zero set of an irreducible homogeneous polynomial $f(x_0, \ldots, x_n)$. A singularity is a point on the hypersurface where all partial derivatives vanish. A generic hypersurface is smooth i.e. it does not contain any singularities, so that it is natural to ask which combinations of singularities can occur on a hypersurface of given degree s in $\mathbb{P}^n$. For $s = 1$ there cannot be any singularity and for $s = 2$ there can be at most one isolated singularity which has to be an ordinary double point, or node for brevity, i.e. a singularity where the Hessian matrix is invertible. The possible combinations of singularities of cubics in $\mathbb{P}^3$ have been classified by Schläfli in 1863 [Sch63] and the classification of all quartic surfaces in $\mathbb{P}^3$ with respect to their occurring singularities has been completed in 1997 [Yan97]. In $\mathbb{P}^2$ the maximum number of isolated singularities of a plane curve of degree s is given by $\binom{s}{2}$ and attained by s lines in general position. In higher dimensions no such result is known yet. As the classification problem seems to be quite complex for larger parameters it makes sense to add further restrictions. A hypersurface is called nodal if all of its singularities are nodes. So, let $\mu(s)$ denote the maximum number of nodes of a nodal surface in $\mathbb{P}^3$ that can be described by a polynomial of degree s. Clearly we have $\mu(1) = 0$ and $\mu(2) = 1$. The values $\mu(3) = 4$ and $\mu(4) = 16$ are attained by the Cayley cubic [Cay69] and a Kummer surface [Kum64], respectively. In 1979 Beauville applied coding theory to determine $\mu(5) = 31$ [Bea79] and thereby improve a general upper bound of Basset [Bas06]. To this end let $\pi \colon \tilde{S} \to S$ be a minimal resolution of singularities. A set N of nodes on S is called even if there exists a divisor Q on $\tilde{S}$ such that $2Q \sim \pi^{-1}(N)$. The even sets of nodes on S comprise the codewords of a binary linear code C, which we call the code associated to S. In [JR97] Jaffe and Ruberman showed $\mu(6) < 66$ by excluding the existence of certain binary linear codes, so that $\mu(6) = 65$ due to the existence of the Barth sextic

[Bar96]. As mentioned by Jaffe and Ruberman it would be very interesting to exactly describe the associated code for the known examples of nodal surfaces with many nodes. For quintic surfaces this has been accomplished by Beauville [Bea79] and for generalized Kummer surfaces by Catanese [Cat96]. While an explicit equation of the Barth sextic was known its associated code was first determined by showing the uniqueness of the possible codes of sextics with 65 nodes [Kur20], cf. [CCF$^+$22] for a more direct derivation. Due to the importance of the problem, since currently none of the two mentioned preprints is published, and in order to popularize the problem we would like to give a streamlined proof of the uniqueness of the code associated to a nodal sextic with 65 nodes in this paper. Additionally we give some information on the next open case of degree $d = 7$.

The remaining part of this paper is structured as follows. In Section 1 we introduce the necessary preliminaries from algebraic geometry and coding theory. The uniqueness of the code associated to a sextic in $\mathbb{P}^3$ with the maximum number of nodes is then concluded in Section 3. We close with a conclusion and a few remarks on further open problems, including the maximum number of nodes of septics in $\mathbb{P}^3$, in Section 4.

2 Preliminaries

An $[n,k]_2$ code C, or binary linear code with length n and dimension k, is a k-dimensional subspace of $\mathbb{F}_2^n$. The elements of C are called codewords. The dual code $C^\perp$ of C is the subspace that is perpendicular to C, i.e. $C^\perp$ is an $[n, n-k]_2$ code. The weight $\operatorname{wt}(c)$ of a codeword $c \in C$ is the number of non-zero entries in c and the minimum weight d of C is the minimum weight over of the non-zero codewords in C, so that we also speak of an $[n,k,d]_2$ code. If the minimum weight $d^\perp$ of the dual code $C^\perp$ is at least 3 we say that C is projective. If the non-zero weights of codewords in C are contained in $\{w_1, \dots, w_l\}$, then we say that C is a Δ-divisible $[n, k, \{w_1, \dots, w_l\}]_2$ code if w_i is divisible by Δ for all $1 \le i \le l$. The polynomial $W_C(x) := \sum_{c \in C} x^{\operatorname{wt}(c)} y^{n-\operatorname{wt}(c)}$ is called the weight enumerator of C. The MacWilliams identity determines the weigh enumerator of the dual code via $W_{C^\perp}(x,y) = \frac{1}{2^k} \cdot W_C(x+y, x-y)$ [Mac62, Mac63]. Each $k \times n$ matrix over $\mathbb{F}_2$ whose row span equals C is called a generator matrix of C. If C contains a codeword with a non-zero entry in position i for all $1 \le i \le n$, which is equivalent to $d^\perp \ge 2$, then C has full length and n is called the effective length of C. In order to ease the notation we only consider $[n,k]_2$ codes of full length. For a given codeword c of an $[n,k]_2$ code the support $\operatorname{supp}(c)$ is given by $\{1 \le i \le n \,:\, c_i \neq 0\}$, i.e., the cardinality of its support equals its weight. With this the residual code $\operatorname{Res}(C; c)$ of an $[n,k]_2$ code C with respect to a non-zero codeword c is the restriction of the codewords to $\{1, \dots, n\} \setminus \operatorname{supp}(c)$, which has effective length $n - \operatorname{wt}(c)$.

The code associated to the Cayley cubic is an $[4, 1, \{4\}]_2$ code with weight enumerator $W_C(x,y) = x^0y^4 + x^4y^0$ and the code associated with a Kummer surface is an $[16, 5, \{8, 16\}]_2$ code with weight enumerator $W_C(x,y) = x^0y^{16} + 30x^8y^8 + x^{16}y^0$. Geometrically the latter code corresponds to the 16 points of an affine solid and a generator matrix is given by

$$\begin{pmatrix} 0000000011111111 \\ 0000111100001111 \\ 0011001100110011 \\ 0101010101010101 \\ 1111111111111111 \end{pmatrix}.$$

The code associated to a quintic with 31 nodes is given by a $[31, 5, \{16\}]_2$ code with weight enumerator $W_C(x,y) = x^0y^{31} + 31x^{16}y^{15}$. Geometrically such a code corresponds to the 31 points of the projective geometry PG$(4,2)$ and a generator matrix is given by

$$\begin{pmatrix} 0000000111111110000000011111111 \\ 0001111000011110000111100001111 \\ 0110011001100110011001100110011 \\ 1010101010101010101010101010101 \\ 0000000000000001111111111111111 \end{pmatrix}.$$

The following general properties of the associated $[n,k]_2$ code C of a nodal surface with degree s and m nodes are known. For the length we have $n \le m$ and for the dimension a general argument of Beauville [Bea79] gives $k \ge m - \lceil s^3/2 \rceil + 2s^2 - 3s + 1$, see [JR97, Proposition 4.3]. If d is odd, then C is 4-divisible and 8-divisible otherwise, see [Cat81, Proposition 2.11]. The minimum distance d satisfies $d \ge 2\lceil s(s-2)/2 \rceil$, see [End98, Theorem 1.10]. So, for sextics with 65 nodes the associated code is a 8-divisible $[n,k,24]_2$ code with $n \le 65$ and $k \ge 12$.

3 Uniqueness of the code associated to a sextic with 65 nodes

First we want to conclude some further properties of 8-divisible $[n,12,24]_2$ codes (of full length) by purely coding theoretic arguments. First we observe that for a q^r-divisible linear code over $\mathbb{F}_q$ with $r \ge 1$ each residual code is q^{r-1}-divisible, see e.g. [HKK18, Lemma 7]. Note that we have $C \subseteq C^\perp$ for each 4-divisible $[n,k]_2$ code, so that $k \le n/2$. Setting $W_C(x,y) = \sum_i a_i x^i y^{n-i}$ and $W_{C^\perp}(x,y) = \sum_i a_i^* x^i y^{n-i}$ the equations for the coefficients y^0, y^1, y^2, and y^3 in the MacWilliams identity can be rewritten to

$$\sum_{i>0} a_i = 2^k - 1, \qquad (1)$$

$$\sum_{i\ge 0} i a_i = 2^{k-1} n, \qquad (2)$$

$$\sum_{i\ge 0} i^2 a_i = 2^{k-1}(a_2^* + n(n+1)/2), \qquad (3)$$

$$\sum_{i\ge 0} i^3 a_i = 2^{k-2}(3(a_2^* n - a_3^*) + n^2(n+3)/2). \qquad (4)$$

We also speak of the first four MacWilliams identities. In this special form, those equations are also known as the first four (Pless) power moments [Ple63].

Lemma 1. *Let C be a binary 8-divisible linear code with minimum distance $d \ge 24$, dimension $k = 12$ and effective length $n \le 65$, then $a_{40} \ge 1$ and $n \ge 63$.*

Proof. Solving the first four MacWilliams identities for a_{24}, a_{32}, a_{40}, and a_{48} gives

$$a_{40} = \frac{205}{2}n^2 - 6808n - \frac{1}{2}n^3 + (208-3n)a_2^* + 3a_3^* + 6a_{56} + 20a_{64} + 147420$$

and

$$a_{40} + a_{48} = 71n^2 - \frac{14504}{3}n - \frac{1}{3}n^3 + (144-2n)a_2^* + 2a_3^* + 2a_{56} + 10a_{64} + 106470.$$

Since $a_2^*, a_3^*, a_{56}, a_{64} \ge 0$, $208 - 3n \ge 0$, $144 - 2n \ge 0$ we have

$$a_{40} \ge \frac{205}{2}n^2 - 6808n - \frac{1}{2}n^3 + 147420$$

and

$$a_{40} + a_{48} \geq 71n^2 - \frac{14504}{3}n - \frac{1}{3}n^3 + 106470.$$

For $54 \leq n \leq 60$ we have $a_{40} + a_{48} < 0$, which is impossible. If either $n \leq 53$ or $61 \leq n \leq 65$, then $a_{40} \geq 1$. Thus, $a_{40} \geq 1$. Consider the residual code C' of a codeword of weight 40. C' has dimension 11 and is 4-divisible, so that its length is at least 23, see e.g. [Gab96, Section VIII]. $\square$

We remark that there even exists an 8-divisible $[64, 13, 24]_2$ code C with generator matrix

$$\begin{pmatrix}
1111111111111111111111111111111111111111111111111111111111111111\\
0000000000000000000000000000000011111111111111111111111111111111\\
0000000000000000111111111111111100000000000000001111111111111111\\
0000000011111111000000001111111100000000111111110000000011111111\\
0000111100001111000011110000111100001111000011110000111100001111\\
0011001100110011001100110011001100110011001100110011001100110011\\
0101010101010101010101010101010101010101010101010101010101010101\\
0000000000001111001100110011110000111100001100111111000011111111\\
0001001000011101000100100001110101001000010001110100100001000111\\
0000011001100000001110101010001101011100110001011001111111111001\\
0000000001101001000000001001011001011010110011000101101000110011\\
0001001000011101011110110111010000101110110111101011100001001000\\
0000001101010110011001011100111100000011101010011001101011001111
\end{pmatrix}.$$

With these properties the code is unique, as shown by exhaustive enumeration in [Kur20]. Its automorphism group has order 23224320 and its weight enumerator is given by $W_C(x, y) = x^0y^{64} + 1008x^{24}y^{40} + 6174x^{32}y^{32} + 1008x^{40}y^{24} + x^{64}y^0$. The code was obtained in [DG75] and has the following nice description, see [JR97]: It is a subcode of the second order Reed-Muller code $R(2, 6)$ containing the first order Reed-Muller code $R(1, 6)$ as a subcode. The cosets of $R(1, 6)$ in it correspond to the symplectic forms B_a in $\mathbb{F}_{64}$, given by $B_a(x, y) = \operatorname{tr}((ax^4 + a^{16}x^{16})y)$. So, subcodes of this code give examples of 8-divisible $[n, 12, 24]_2$ codes for $n = 63$ and $n = 64$. We remark that an $[59, 12, 24]_2$ code exists while the existence of a $[58, 12, 24]_2$ code is currently unknown. So the 8-divisibility of the code is a severe restriction.

Lemma 2. *Let C be an 8-divisible $[65, 12, 24]_2$ code. Then C does not contain a codeword of weight 64 or 56.*

Proof. The residual code of a codeword of weight 64 would be a 4-divisible code of length 1, which obviously cannot exist. Similarly, the residual code of a codeword of weight 56 would be a 4-divisible code of length 9. However, no such code exists [KK20]. $\square$

We remark that it is also possible to exclude the existence of a codeword of weight 48 in an 8-divisible $[65, 12, 24]_2$ code C by theoretical arguments. For a codeword c of weight 48 the residual code $\operatorname{Res}(C; c)$ is a $[17, k, \{4, 8, 12\}]_2$ code, so that $k \leq 8$ [Gab96, Section VIII]. The restriction of the codewords of C to the coordinates in the support of c gives a $[48, k', \{24, 48\}]_2$ code C'. Codes whose only occurring non-zero weights are the minimum distance d and $2d$ are completely characterized in [JT18]. In our situation C' is a uniquely defined $[48, k', \{24, 48\}]_2$ code with $k' \in \{4, 5\}$, so that $k \in \{7, 8\}$. Up to isomorphism there are only five $[17, 7, \{4, 8, 12\}]_2$ codes and a unique $[17, 8, \{4, 8, 12\}]_2$ code [DFG+11, Mil]. For a computer-free classification for the possibilities for $\operatorname{Res}(C; c)$ one may consider the decomposition of $\operatorname{Res}(C; c)$ into subcodes spanned by codewords of weight 4, which is completely characterized in a more general setting [KK23]. Having the explicit possible choices for C' and $\operatorname{Res}(C; c)$ at hand one can easily exclude the existence of C with a codeword c of weight 48. However, since those arguments are quite lengthy, due to many case differentiations, and such a result can be easily obtained by exhaustive computer enumeration within seconds we do not go into details.

Lemma 3. *Let C be an 8-divisible $[n, 12, 24]_2$ code with $n \leq 65$. Then the non-zero weights of C are contained in $\{24, 32, 40, 64\}$.*

Proof. From Lemma 1 we know that $n \geq 63$ and that C contains a codeword c of weight 40. The residual code then is a 4-divisible $[n-40, 11]_2$ code. Using the software package `LinCode` [BBK21] we have exhaustively enumerated all possibilities for $\operatorname{Res}(C; c)$ up to isomorphism. There are exactly 11 $[23, 11]_2$, 83 $[24, 11]_2$, and 215 $[25, 11]_2$ 4-divisible codes cf. [DFG+11, Mil]. Starting from these residual codes we have exhaustively enumerated all possibilities for C using `LinCode`. Up to isomorphism there are unique $[n, 12, 24]_2$ codes for $n \in \{63, 65\}$ and eight $[64, 12, 24]_2$ codes (assuming 8-divisibility) and in none of these cases codewords of weight 48 or 56 occur. (Generator matrices for these ten cases are given in [Kur20].) □

We remark that the initial classification of the 8-divisible $[\leq 65, 12, 24]_2$ codes in [Kur20] took almost 1000 single-core CPU hours on a computing cluster. There are e.g. 978528 8-divisible $[63, 10, 24]_2$ and 704571 8-divisible $[64, 11, 24]_2$ codes. Constructing the 8-divisible $[\leq 65, 12, 24]_2$ codes via the residual codes of a codeword of weight 40 allows to keep the numbers of intermediate codes in the enumeration process much smaller. The determination problem of possible codes associated with a sextic with 65 nodes was indeed the initial motivation to develop a new enumeration algorithm for linear codes. Over the time there were algorithmic improvements, see [Kur20, Kur24] for details, so that the computation underlying Lemma 3 can now be performed in less than two hours on a single core.

In [JR97] it was shown that the code associated to a sextic with 65 nodes cannot contain a codeword with weight 48 or 64. This boils down the number of possible codes to three:

Lemma 4. *Let C be an $[n, 12, \{24, 32, 40, 48, 56\}]_2$ code with $n \leq 65$. Then C is isomorphic to one of the three cases:*

(1) $[63, 12, 24]_2$

$$\begin{pmatrix}
001100001110000000111110100011111110010010100100001100000000000 \\
101001111111100000001101110101001101000110110000000010000000000 \\
000100111011100011110111001001000001100001101101000001000000000 \\
010001111111111001100110000011001000100011010000001000100000000 \\
110001110000001011100111101101110010001100001010000000010000000 \\
000000011000110111110001101010111001000101111000000000001000000 \\
010011110001111101010000110110001110111011111111110000000100000 \\
001000110111110110000111111000000110011000011110000000000010000 \\
000111110001100011000000000111111100001100000111100000000001000 \\
000000001111100000011111111100000011111100000001100000000000100 \\
000000000000011111111111111100000000000001111111110000000000010 \\
000000000000000000000000000011111111111111111111111000000000001
\end{pmatrix}$$

$W_C(x, y) = x^0 y^{63} + 630x^{24}y^{39} + 3087x^{32}y^{31} + 378x^{40}y^{23}$

$\# \operatorname{Aut}(C) = 362880$

(2) $[64, 12, 24]_2$

$$\begin{pmatrix}
0000110001101110000100100100000110110000110110111110100000000000 \\
1011111000010011001000000011010000111101001110111000010000000000 \\
1010110001001010110010000000111111100000001100101011001000000000 \\
1111100000000110000000101001111101000011011011101000000100000000 \\
0111000000000101011011000110011000000110111100110011000010000000 \\
0000000100001001111110011010101001101010101010101010000001000000 \\
0101011111010000001000111100001100010010000001011000000000100000 \\
0011010011001000001111111001111101111101001101110000000000010000 \\
0000101111000110000000000001111110000110011110000011000000001000 \\
0000011111000000111111111111000011111100000001111110000000000100 \\
0000000000111111111111111111000000000011111111111110000000000010 \\
0000000000000000000000000000011111111111111111111111000000000001
\end{pmatrix}$$

$W_C(x, y) = x^0 y^{64} + 502x^{24}y^{40} + 3087c^{32}y^{32} + 506x^{40}y^{24}$

$\# \operatorname{Aut}(C) = 5760$

(3) $[65, 12, 24]_2$

$$\begin{pmatrix}
10000100000000011011001000110100111101010001011110010100000000000 \\
10100100011000000100100011011101111111001000001100011010000000000 \\
01000010011100011000000010011001100000011111011110001001000000000 \\
11110100000111011010000000111010000010111000000100001000100000000 \\
01101011000001100011010001000110010100011110000010111000010000000 \\
00101001110111101011000000100000001101110010010001000000001000000 \\
00011000111111110000011111000001000100010101010011000000000100000 \\
00000111001011100111110001010000000111100110000000110000000010000 \\
00011111000111100000000011111011111110000111001111110000000001000 \\
00000000111111110000000000001000111000000000111111110000000000100 \\
00000000000000011111111111111100000001111111111111110000000000010 \\
00000000000000000000000000000000111111111111111111111110000000001
\end{pmatrix}$$

$W_C(x,y) = x^0y^{65} + 390x^{24}y^{41} + 3055x^{32}y^{33} + 650x^{40}y^{25}$
$\#\,\mathrm{Aut}(C) = 15600$

All three codes are projective.

Theorem 5. *If S is a sextic with 65 nodes in $\mathbb{P}^3$ then its associated code is given by Lemma 4.(3).*

Proof. Let C be the code associated to a sextic in $\mathbb{P}^3$. In [End98] the existence of a code $C' \supsetneq C$ with $\dim(C') = \dim(C) + 1$ such that the codewords in $C' \backslash C$ have weights contained in $\{16, 28, 32, 36, \dots\}$ was shown. First we use `LinCode` to check that none of the cases of Lemma 4 can be extended to a $[\le 66, 13, \{24, 32, 40, 48, 56\}]_2$ code, so that there are just three choices for C. Then we used `LinCode` to extend C to C'. □

We remark that the code C' in the proof of Theorem 5 is unique. A generator matrix its given by

$$\begin{pmatrix}
100000000011000100000110100000001000110001111111001111110011110010 10\\
\end{pmatrix}$$

The automorphism group of C' has order 15600 and the weight enumerator of $C' \backslash C$ is given by $26x^{16}y^{50} + 650x^{28}y^{38} + 1690x^{32}y^{34} + 1300x^{36}y^{30} + 300x^{40}y^{26} + 130x^{44}y^{22}$. For a nice description of the associated code C of a sextic with 65 nodes and its automorphism group we refer to [CCF+22, Section 4]. Since the residual codes of codewords of weight 40 played an important role in the determination of C we mention that they all are unique with weight enumerator $W_{\mathrm{Res}(C;c)}(x,y) = x^0y^{25} + 3x^4y^{21} + 258x^8y^{17} + 1278x^{12}y^{13} + 493x^{16}y^9 + 15x^{20}y^5$ and an automorphism group of order 4608. A generator matrix is given by

$$\begin{pmatrix}
0000100100100100000000000\\
0100011010100111000000000\\
1010010000101110100000000\\
1100100010101010010000000\\
1111101111001010001000000\\
0101111000010100000100000\\
0011101100010010000010000\\
0011100001101010000001000\\
0000100000000110000000100\\
0000011100011110000000010\\
0000000011111110000000001
\end{pmatrix}.$$

While Theorem 5 shows the uniqueness of the code associated to a sextic with the maximum number of nodes, the sextic itself is far from being unique. In [Pet98, Theorem 5.5.9] a 3-parameter family of sextics with 65 nodes, including the Barth sextic, was obtained; cf. [CCF+22, Theorem 196].

The non-zero weights occurring in a code associated to a nodal sextic in $\mathbb{P}^3$ are contained in $\{24, 32, 40, 56\}$ and all cases can indeed occur [CT07, VGZ18]. The exclusion of weight 56 in [CC97] turned out to be incorrect [CC98]. In [PT09, Lemma 2.3] it was show that a binary code with non-zero weights in $\{24, 32, 56\}$ has dimension at most 10 and in [PT09, Lemma 2.1] it was show that a binary code with non-zero weights in $\{24, 32\}$ has dimension at most 9.

Lemma 6. *Let C be a $[n, k, \{24, 32\}]_2$ code with $n \le 56$. Then, we have $k \le 8$. Moreover, if $k = 8$, then $n \in \{51, 54, 55, 56\}$. For $(n, k) = (51, 8)$ the code is projective.*

Proof. Since each $[n, 8, 24]_2$ code satisfies $n \ge 51$, see [BJV00], we have $51 \le n \le 56$. The fact that each $[n, 7, 24]_2$ code satisfies $n \ge 50$ implies that the code is projective for $(n, k) = (51, 8)$. For an $[n, 8, \{24, 32\}]_2$ code we apply the dual transform as in [BB24, Theorem 3.3] with $\alpha = \frac{1}{8}$ and $\beta = -3$. The projective dual code $D_{\alpha,\beta}(C)$ has length $128\alpha n + 255\beta$ and minimum distance $64(\alpha n + 2\beta)$ if $n < 2^k - 1$. Since no $[67, 8, 32]_2$ code and no $[83, 8, 40]_2$ code exists, see [BJV00], we have $n \notin \{52, 53\}$. Let C' be the $[56, 9, \{24, 32, 56\}]_2$ code arising from C by adding a codeword of weight 56. Then we clearly have $a_{56} = 1$ and $a_{24} = a_{32}$ for C'. Using the MacWilliams identities we compute $a_{24} = a_{32} = 255$ and $a_2^* = 10$. Let $\mathcal{P}'$ be the multiset of points of cardinality 56 in $\mathrm{PG}(8, 2)$ corresponding to C', see e.g. [DS98]. Projection of $\mathcal{P}'$ through a point of multiplicity $m \ge 1$ yields a multiset of points $\mathcal{P}$ of cardinality $56 - m$ in $\mathrm{PG}(7, 2)$ that corresponds to an $[56 - m, 8, \{24, 32\}]_2$ code, so that $m \in \{1, 2, 5\}$. Since each point of multiplicity m contributes $\binom{m}{2}$ to a_2^* the multiset $\mathcal{P}'$ consists either of one point of multiplicity 5 and 51 points of multiplicity 1 or five points of multiplicity 2 and 46 points of multiplicity 1.

Now assume that H is an $[l, 9, \{24, 32\}]_2$ code and H' the $[56, 10, \{24, 32, 56\}]_2$ code that arises from H by adding a codeword of weight 56. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be the multisets of points corresponding to H and H', respectively. For H' we compute $a_{24} = a_{32} = 511$, $a_{56} = 1$, $a_2^* = 7$, and $a_3^* = 0$. Clearly, $\mathcal{Q}'$ contains a point of multiplicity 1 and projection through this point yields a multiset of points that corresponds to a $[55, 9, \{24, 32\}]_2$ code. W.l.o.g. we assume that H was chosen as such a code, i.e., we have $l = 55$. From the classification of the possible lengths of $[\le 56, 8, \{24, 32\}]_2$ codes we conclude that the point multiplicities in $\mathcal{Q}$ are contained in $\{0, 1, 4\}$. From the MacWilliams identities we compute $a_{24} = 284$, $a_{32} = 227$, and $a_2^* = 7$ for H. Since each point of multiplicity m contributes $\binom{m}{2}$ to a_2^* this is impossible. $\square$

Up to isomorphism there are unique $[51, 8, \{24, 32\}]_2$ and $[54, 8, \{24, 32\}]_2$ codes, two $[55, 8, \{24, 32\}]_2$ codes, three two $[56, 8, \{24, 32\}]_2$ codes, and two $[56, 9, \{24, 32, 56\}]_2$ codes. Generator matrices are given by

$$\begin{pmatrix}
101100011111111111100001010111111111100111110000000 \\
010011011111111100001111010111111111100111111010000000 \\
000000100001111111111100000000111111110011001000000 \\
001111100110011001100110011001100110101010100010000 \\
00111111110000111100110011000011110001011001000010000 \\
110111100110000011101011000010111010010101100000100 \\
110101101010101010011100011100001011100110100000010 \\
000110100011001100101101111111010101010100100000001
\end{pmatrix},
\begin{pmatrix}
111111111111000000111111111111111000000000111111000000000 \\
000000000000111111111111111111111100000001000110010000000 \\
000000011111000000000000000011111111111110011111100100000 \\
000111111111111111100001111111110001111010000100010000 \\
001000111111001111001100110000110010001010111000001000 \\
00111100011100000100010111001100011001111110010000000100 \\
01111111101100011000011000110100111001001000100000000010 \\
10100111100111000000011011000100100110101110110000000001
\end{pmatrix},$$

$$\begin{pmatrix}
1111111111111000000111111111111111000000000111101000000000 \\
00000000000001111111111111111111111000000001000110010000000 \\
000000011111100000000000000011111111111111001111000100000 \\
000111111111111111110000111111111110001111010000000010000 \\
0110011001111000011000011110000110110011001100100001000 \\
0011101010001001101011101111111100101111111111100000000100 \\
111000110111001111100011011011101011110011111100000000010 \\
1111101010010101100100011010001100001100110011000000001
\end{pmatrix},
\begin{pmatrix}
111111111111111111110000000000000000111111111111010000000 \\
111111111000000000011111111110000011111111111010100000000 \\
111111111000011111100000111111111100000011111010010000000 \\
000000000000000011100111111111001110001111111110000100000 \\
000001111001111110110000100011110011110000000011100001000 \\
1111100010100011111010110110100010001011000100000000100 \\
111110010100110100101000011110100001110000100110000000010 \\
0000001110110011010111110010010000001101011110000000001
\end{pmatrix},$$

$$\begin{pmatrix}
1111111111111111111111111100000000000000111110000010000000 \\
0000001111111111111111000000011111000000000101000000100000 \\
1111110000000011111111000000000000011111111100100000100000 \\
1111110000011111111110011111111110011110000000100000010000 \\
0011110000001100001111111100000111100110011000010000001000 \\
1100110011110000110011001100110000111111000000001000000100 \\
0001111100110011010000000000101010011110101010101110000000010 \\
1100110101111100101100010000000101011100011100010000000001
\end{pmatrix},
\begin{pmatrix}
11111111111111111111111111111111100000000000000000000010000000 \\
000000001111111111111111100000000111100001110000000001000000 \\
111111110000000011111111100000000000011111000111000000100000 \\
000011110000111100001111000000001111111100000011100010000 \\
001111110111000100010001111001011100000100100001000 \\
000100011001001100010111000001101001111110110100000000100 \\
1111001110111101011110000001111000010110111111100000010 \\
010111111110000100010111011000010010100011010100000000001
\end{pmatrix},$$

$$\begin{pmatrix}
11111111111111111111111111111100000000000000000000100000000 \\
00000000111111111111111100000001111000011100000000100000000 \\
111111110000000011111111100000000000000111100011100000100000 \\
000011110000111100001111000000001111111100000011100010000 \\
000101110011001100010001011111100010011001000111000001000 \\
111110010111000100010111101111110110011101111110010000000100 \\
011000001101010000101110101111100110100110001010000000010 \\
101010100001110101111011101111111110001101010111000000001
\end{pmatrix},
\begin{pmatrix}
11111111111111111111111111100000000000001111100000100000000 \\
00000001111111111111110000000111111000000000110100000100000000 \\
111111000000001111110000000000000011111111100100001000000 \\
11111100001111111111001111111111001111000000001000100000 \\
001111000000110000111111110000111001100110000100001000000 \\
00000101110100001100011110001110000101111110100000001000 \\
000011000110110100111000000111011111101000110110000000100 \\
110000011101000111110001100100100110001010110000000010 \\
11110111101000111100101100101111001110101111010000000001
\end{pmatrix},$$

$$\begin{pmatrix}
111111111111111111000000111100000000000111111110000000000 \\
11111111111111100000111111000000111110000011111010100000000 \\
000000000000011111111111100000000000111111111111001000000 \\
000000111111111111100011100111111110011111111100000100000 \\
0011111000111101111111101100111001111100100111100000010000 \\
01000000010111101110010001100101011110010101110000001000 \\
110000001110111101111000101011001000101100100100000000100 \\
0011111001000100001011010110010000110101111010100000000010 \\
011111101100100001010000001011101101100101010010000000001
\end{pmatrix}.$$

Corollary 7. *Let C be a $[56, k, \{24, 32, 56\}]_2$ code. Then, we have $k \le 9$.*

We remark that there exists a $[56, 10, 24]_2$ code with weight enumerator $W_C(x, y) = x^0y^{56} + 399x^{24}y^{32} + 224x^{28}y^{28} + 399x^{32}y^{24} + x^{56}y^0$.

4 Conclusion and outlook

We have determined the unique code associated to a sextic in $\mathbb{P}^3$ with the maximum number of nodes in Theorem 5. For each $0 \le \mu \le 65$ there exists a sextic in $\mathbb{P}^3$ with μ nodes [Bar96, CC82]. Using similar techniques for sextics with 64 codes there currently remain just seven possible candidates for the associated code [CCF+22, Theorem 227]. However for sextics with 63 nodes such a list of candidates would be rather huge, so that there is a need for more necessary conditions on the associated codes that might by exploited using coding theory techniques.

For septics in $\mathbb{P}^3$ the currently best known bounds on the maximum number of nodes are given by $99 \le \mu(7) \le 104$ [Lab05, Lab06, Var83]. With respect to the upper bound we mention that the $[96, 10, 44]_2$ code with weight enumerator $W_C(x, y) = x^0y^{96} + 504x^{44}y^{52} + 124x^{48}y^{48} + 336x^{52}y^{44} + 56x^{60}y^{36} + 3x^{64}y^{32}$ given by the generator matrix

$$\begin{pmatrix}
100000000010000100111101100000110001001101111111000100000110101010011001110110111001111000101100\\
010000000001100010011110110000011000000110111111100100000110101010011001110110111001011100010110\\
001000000010010101110010111000111101101110100001011010001110100001111111010110101111110110100111\\
000100000000100101011100111110001111011011101000000110100011111000011111110101101011111101101 0011\\
000010000001010000110000111111011111100101100101101011101011101011001011000110000110000000101000101\\
000001000000111000011000000111110111111001011001011010111010111010110010110001100001100100010100010\\
000000100010011100100101001111011111010100001101110001110011001111111110001010111101011010011111101\\
000000010001101110010010000111101111111010000110101000111001100111111111001101011110101101001111110\\
000000001010010011110100000001100011011011111100010000011010001001100110111011100111010001011001 1\\
000000000101001001111010000000110001101101111111000100000110101010011001100110111001111100010110 01
\end{pmatrix}$$

satisfies all constraints on the weights from [End98]. In the special case where an even set of nodes is cut out by a smooth cubic surface the corresponding codeword must have weight 36, while weight 40 cannot occur [End98]. To this end, a $[94, 10, 36]_2$ code with weight enumerator $W_C(x, y) = x^0y^{94} + 120x^{36}y^{58} + 182x^{44}y^{50} + 489x^{48}y^{46} + 192x^{52}y^{42} + 14x^{56}y^{38} + 26x^{60}y^{34}$ is given by the generator matrix

$$\begin{pmatrix}
1111111111111110000000000000111110000000111111111111000000011000000000000000000000000001000000000\\
0000001111110001111110000000000000111111111111111111100011100110000000000000000000000001000000000\\
0001110001110000000000111111111111111111110000000111111000111000011000000000000000000000001000000000\\
1111111111110001111111111110000000111111000000000000011100000000001100000000000000000000000100000000\\
0001111110001110001110001110001110001110001110001111111111000000000110000000000000000000000010000000\\
0001110000000111000000011100011111111110001111110000000111111000000000000110011100000000000000001000000\\
0001110001111111111111100011111100011100000001110001110000000000000000000000111110000000000000000100000\\
0010010010010110110110010110010010110010110010010010010110111110101110101111111000000000000010000\\
1111111110000000000111111000000011100000000000111111000111000000000000000000000011100011111000000001000\\
0000110100100010011010111010100101010001010110110010110011101011010011100101101011000000000111
\end{pmatrix}.$$

So, additional restrictions on the code are needed in order to improve upon the upper bound for $\mu(7)$ using coding theoretic methods. The algorithmic tools for the exhaustive enumeration of linear codes are available. A first step might be the computation of associated codes of known hypersurfaces with many nodes.

For another type of singularities, so-called cusps, one can associate a 3-divisible code over $\mathbb{F}_3$ [BR07], see also [Lab05] for a general overview on hypersurfaces with many singularities.

On the coding theory side there are a few more sophisticated techniques that we have not applied in Section 3. If a binary linear code is 2^r-divisible then there are some restrictions on the number of codewords whose weights are divisible by 2^{r+1}, see e.g. [DGS99, Proposition 5]. The MacWilliams identities can be generalized to coordinate partitions [Sim95]. In combination with linear programming this was used quite successfully to show the non-existence of some binary linear codes or to even show uniqueness of some optimal codes, see e.g. [Jaf00]. However, applying those techniques usually comes with extensive computer calculations. So, in order to keep the paper relatively short and since all of the used computer calculations were performed in a reasonable short amount of time we have not applied those techniques to our problem.

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