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Towards New Teaching in Mathematics

Volker Ulm

Squares – Simple and Comprehensive

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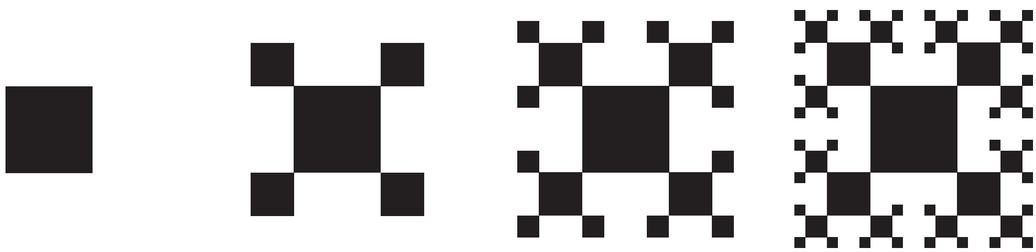
Squares – Simple and Comprehensive

Continue Patterns

1.Patterns consisting of squares

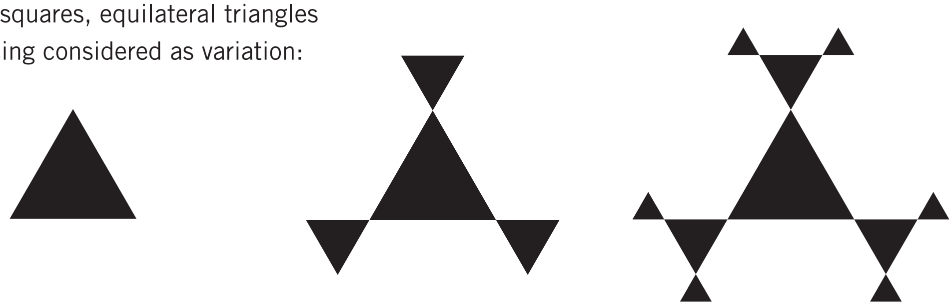
A sequence of patterns is created with squares: Small squares are in each case attached to free vertices. The side lengths of the squares are made smaller step by step with constant factor k.



a. Consider a variety of mathematical questions on this sequence of patterns and write these down.
b. Exchange your ideas with your neighbour.
c. Discuss your questions with your neighbour.
d. Present your ideas and results jointly in class.

2.Patterns consisting of triangles

Instead of squares, equilateral triangles are now being considered as variation:



a. Consider a variety of mathematical questions on this sequence of patterns as in assignment 1 and write these down.
b. Discuss your ideas with your neighbour.
c. Discuss your problems with your neighbour.
d. Present your ideas and results jointly in class.

3.Additional variations

Vary your considerations further and deal with these jointly with your neighbour.

Geometric Patterns as a Playground for Mathematical Research and Discovery

A pattern consisting of squares, which appears unimpressive at first sight, harbours extreme mathematical depth. It offers students easy approach, facilitates asking questions and virtually provokes mathematical research and discovery at widely varying levels.

A Learning Environment with Squares

What can be discovered with squares in the later years of secondary school? Aren't squares sufficiently known from a geometrical point of view and haven't these been "dealt with" algebraically with the development of formulas for the area and circumference? Not at all! The master copy "Continue patterns" is an example for how unexpected new depths can develop from something unimpressive. It dates from an idea from the textbook "mathbu.ch 9+" (Affolter, W. u.a. 2004) for 9th grade. Based on a sequence of geometrical patterns, which can be easily comprehended from an optical point of view, an abundance of problem bundles emerges abruptly due to the strategies of "Ask Questions" and "Vary the Situation" which leads to research, trials and discovery, discussion, argumentation and presentation, in short: the study of mathematics. The aspiration level extends thereby beyond all years of entire secondary school and to college mathematics. One should only get into mathematics!

At first, didactical and methodical suggestions as well as practical experiences are presented to design a teaching unit with this learning environment and afterwards the mathematical content of the subject matter is analysed.

Asking Questions

At first, the students should occupy themselves independently with the patterns on the worksheet and thus gain understanding of the subject matter. Even such approach must be developed in such a way that preferably *all* learners get an introduction, so that they already perceive feelings of success in the initial phase and do not immediately feel despondent and give up at the first mathematical hurdles.

A very simple method of enabling students introduction at different levels is to prompt them to ask their own questions on situations with a mathematical content. Particularly when the objects dealt with, such as the present patterns consisting of squares, are easily accessible, those not performing as well can also develop ideas successfully. This phase of the collection process does not yet involve calculations or justifications. On the one hand, the objective is for students to find individual approach to the learning environment at their respective level and for them to develop initial understanding for the subject matter. On the other hand, entire set of questions generally result from these activities. These questions could be the motivation and basis for further mathematical research and discovery.

The learning environment “Continue patterns” of the master copy was already worked on with several classes of 10^{th} and 11^{th} grade. The students were thereby required to use their notebook in the sense of a study journal and preferably to write down all their ideas and thoughts, approaches and meanders in their notebook. The analysis of these notes (partially) enabled a reconstruction of the thoughts and work of the students. The questions formulated on the patterns of squares were decidedly highly varied. The sequence of the following enumeration reflects the frequency of said aspects:

- Number of squares,
- Area of the patterns,
- smallest square which includes the n^{th} pattern,
- Overlappings and tangencies, respectively, in the pattern,
- Circumferences of the figures,
- Behaviour of the sequence of areas,
- smallest square which includes all patterns,
- Behaviour of the sequence of circumferences,
- Symmetry characteristics,
- Shifts so that simple figures with equal area can emerge,
- Number of vertices,
- Computer program to generate the pattern.

Questions from a study journal

Patterns made from squares

1. *Describe how the figures are created.*
2. *How large is the surrounding square?*
3. *How do the squares proliferate?*
4. *How is k selected?*
5. *Circumference*
6. *Area*
7. *Do the squares have mutual contact at any time?*
8. *Draw the 4^{th} figure.*
9. *The smallest square of the 5^{th} figure must be the size of $\frac{1}{4}$ cell. How big does the 1^{st} square have to be. $k = \frac{1}{2}$*

Tools

When dealing with the patterns of squares, the students continue to encounter sequences of real numbers and their limits (e.g. number of squares, areas, circumferences, ...). Depending on the grade and the mathematical prior knowledge of the student, these sequences can be explored at varying levels and with different tools.

Electronic calculator

An inductive approach almost comes to mind in many questions on the patterns. The electronic calculators allow the students to calculate the first members of the number sequence directly and thus gain insight to the underlying structures.

Spreadsheet calculation

The calculation with a larger number of sequence members with an electronic calculator is exhausting. A spreadsheet calculation simplifies this. Even with these tools, the students can explore the convergence of the sequence impressively and numerically. Nevertheless, it is far from even with the request of appropriate software. In order for the students to be able to actually use the spreadsheet calculation in a productive manner, they must have intensively dealt with the geometrical patterns beforehand. They must for instance have recognised the recursive structures of the sequence in order to convert these recursions in the user interface in cell references.

Internet, literature

Currently, sequences and series have largely disappeared from the mathematics curricula as independent field of study. These are, however, linked to other contents (e.g. growth processes, approximations, real numbers, convergence of functions,...). Based on such implicit anchoring of the sequence concept in mathematical studies, the students are not unfamiliar with sequences in concrete situations of their work, even when they didn't become explicitly acquainted with definitions and theorems. The learning environment of the master copy repeatedly leads to geometric sequences and series. The strong performers should at least receive a tip from the teacher to study this field that is made available relatively easily through Internet research (e.g. Wikipedia) or based on reference work (e.g. collection of formulas, mathematical thesaurus). Ultimately, understanding of the summation formula

$$1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

and the limiting process $n \to \infty$ suffices for an analytical exploration of the patterns of squares. The justification of this equation through multiplication with the denominator and subsequent simplification is easy.

In order to gain a feeling for the mathematical content of the learning environment and the above discussed questions, we will look into a few aspects in what follows in more detail. The presentation thereby does not reflect the expected student responses in detail, but instead should in fact give the teacher useful background to classify and promote the creative work of the students.

No Overlapping

An issue, which is frequently discussed by students, on the sequence of patterns of squares was the fact whether individual "branches" would touch at any time or squares would overlap one another. As for the images on the worksheet, it could be assumed that there is no overlapping just yet. In contrast, as for factor k = 1, the third figure already shows signs of overlapping. It is assumed that the question of overlapping depends on factor k.

Side length 1 can be assumed for the first square without any limitations. The squares of the patterns have the side lengths 1, k, k^2 , k^3 , … As is evident from the sketch from a student (fig. 1), all patterns are free from overlapping when

$$\forall n \geq 2: \; k^2 + k^3 + k^4 + \ldots + k^n < \frac{1}{2}$$

$$\Leftrightarrow \quad k^2 + k^3 + k^4 \ldots \leq \frac{1}{2}$$

$$\Leftrightarrow \quad k^2 \times \frac{1}{1-k} \leq \frac{1}{2}$$

$$\Leftrightarrow \quad (2k-1)\,(k+1) \leq 0$$

$$\Leftrightarrow \quad k \leq \frac{1}{2}\text{, since k is positive.}$$

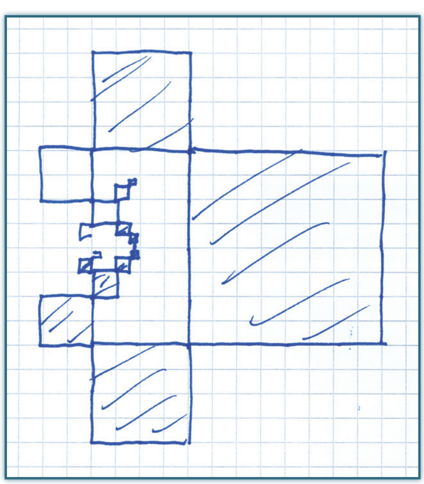


Fig. 1: Sketches from a student on overlappings

We always assume $k \in \;]\,0; \frac{1}{2}\,]$ hereinafter. Thus, it is guaranteed that the squares do not overlap in all figures.

Number of Squares

The question that is asked most frequently by the students is as follows: "How many squares does the n^{th} pattern have?" Approach is maintained by simply counting in the first figures: 1, 5, 17,… Counting directly is rather difficult for the fourth pattern. The objective is to find arithmetic rules and the first few figures are taken into account again. The first figure includes 1 square; the second one $1 + 4$; the third one $1 + 4 + 4 \times 3$; the fourth one $1 + 4 + 4 \times 3 + 4 \times 3^2$. And the underlying arithmetic pattern is already been inductively recorded. The number of new squares being added in every step triples from time to time. The n^{th} figure consists of $1 + 4 + 4 \times 3 + 4 \times 3^2 + \ldots + 4 \times 3^{n-2}$ squares. This term is simplified to

$$1 + 4 \times (1 + 3 + 3^2 + \ldots + 3^{n-2}) \;=\; 1 + 4 \times \frac{3^{n-1} - 1}{3 - 1} \;=\; 2 \times 3^{n-1} - 1$$

with the summation formula for geometric series. The number of squares grows thus exponentially.

Area

Interestingly, about twice as many students asked questions on the area of the figures compared to on the circumference. This may be due to the fact that the surface is immediately "visible" in the case of a two dimensional figure compared to the fact that higher abstraction skills are necessary to observe the circumference.

What is the area of the n^{th} pattern? Side length 1 is again assumed for the first square. The squares that are added from figure to figure have area 1, k^2, k^4, k^6, … When these are combined with the above results about the number of respective squares, one will immediately obtain $1 + 4k^2 + 4 \times 3k^4 + 4 \times 3^2 k^6 + \ldots + 4 \times 3^{n-2} k^{2n-2}$ for the area of the n^{th} pattern. This calculation hereby assumes that there is no overlapping, that is $k \leq \frac{1}{2}$.

It has already been emphasised that spreadsheet calculations can be a useful tool to examine occurring sequences by way of experiments and to gain insight to their limits. By way of example, figure 2 shows the development of the area of the patterns of squares for the case $k = \frac{1}{2}$ shown on the master copy.

The above presentation of summation of the area contains a geometric series, which is equal to $1 + 4k^2 \times (1 + 3k^2 + (3k^2)^2 + \ldots + (3k^2)^{n-2})$. The limiting process $n \to \infty$ can be completed in an easy and analytical manner. Since $(3k^2) < 1$, the sequence of areas converges towards $1 + 4k^2 \times \frac{1}{1 - 3k^2} \;=\; \frac{1 + k^2}{1 - 3k^2}$. In other words, the

Index	Area
1	1,00000
2	2,00000
3	2,75000
4	3,31250
5	3,73438
6	4,05078
7	4,28809
8	4,46606
9	4,59955
10	4,69966
...	
30	4,99905
31	4,99929
32	4,99946
33	4,99960
34	4,99970
35	4,99977
...	

Fig. 2: Spreadsheet calculation for area

areas of the figures remain limited. In case $k = \frac{1}{2}$, which is outlined on the worksheet, the areas converge towards value 5 which can be assumed by means of the spreadsheet calculation.

Circumference

What is the circumference of the figures? Is the sequence of circumferences limited as well? The train of thought is similar to that of areas. The squares that are added step by step have the following circumference: 4, 4k, $4k^2$, $4k^3$, … The n^{th} pattern possesses the overall circumference of $4 + 4 \times 4k + 4 \times 3 \times 4k^2 + 4 \times 3^2 \times 4k^3 + \ldots + 4 \times 3^{n-2} \times 4k^{n-1} = 4 + 16k \times (1 + 3k + (3k)^2 + (3k)^{n-2})$. The geometric series contained herein converges exactly when $3k < 1$. In the case $k < \frac{1}{3}$, the sequence of circumferences remains limited; it converges towards $4 \times \frac{1+k}{1-3k}$. If, on the other hand, the factor $k \geq \frac{1}{3}$, the sequence of circumferences grows unrestrictedly across all limits. This is particularly the case for the patterns of squares presented on the worksheet.

Smallest Squares

How big is the smallest square, which completely contains the n^{th} figure? From step to step, the figures become "wider" and "higher" by two times the side length of the new squares to be added. The smallest square, which includes the n^{th} figure, has the following side length:

$$1 + 2k + 2k^2 + \ldots + 2k^{n-1} = 2 \times \frac{1-k^n}{1-k} - 1 = \frac{1+k-2k^n}{1-k}.$$

How big is the smallest square which contains all figures? To that end, only the previous term has to be examined for convergence for $n \to \infty$. Since $k \leq \frac{1}{2} < 1$, it converges towards $\frac{1+k}{1-k}$. All figures of the sequence are positioned in a square with this (smallest possible) side length. The smallest square, which surrounds all figures, in the case $k = \frac{1}{2}$ shown on the worksheet possesses side length 3 and area 9. In comparison with that, we had already determined earlier that the sequence of areas of patterns converges towards 5.

Variations

The strategy varying mathematical situations opens up new fields for mathematical research and discovery on the basis of what is known. The subject matter of "pattern consisting of squares" can be varied in many respects. Only a few aspects of the manner of constructing this sequence need to be "tweaked":

- The squares can be dealt with differently. By way of example, the squares that are added from one sequence member to the next, could not be attached on free vertices but on free edges.
- Other polygons could be taken as a basis of the sequence of figures, e.g. equilateral triangles or other regular polygons instead of squares.

This lends itself to expansion to three-dimensional reality:

- Cubes instead of squares can be considered. Smaller cubes are attached analogically on the vertices of an initial cube. In turn, smaller cubes, etc. are placed on the free vertices of the solid objects that emerged.
- The cubes which are added from one solid object of the sequence to the next do not have to be attached on vertices. These can also be added on edges or surfaces.
- The solid objects of the sequence can also emerge from other types of solid objects, e.g. from tetrahedrons or other regular polyhedrons instead of cubes.

If in each case the number of appearing "basic building blocks" is examined and the circumferences, surfaces and areas and volumes, respectively, are determined, the uniform arithmetic basic structures are recognised and each time one will come across series of similar form.

By way of example for the varied variation possibilities, we will briefly focus our attention on the "pattern consisting of equilateral triangles". Particularly the question about no overlapping provides a surprising result.

As outlined on the worksheet, smaller equilateral triangles are added on free vertices from step to step based on an equilateral triangle. Analogously with the squares, the triangles are gradually reduced in size by factor k.

Figure 3 provides the key to the question whether the patterns remain free from overlapping. Both outlined arms do not touch when

$$\frac{1}{2}k^2 + k^3 + k^4 \ldots \leq \frac{1}{2} + \frac{1}{2}k$$

$$\Leftrightarrow \quad k^2 \times (1 + k + k^2 + \ldots) \leq \frac{1}{2} + \frac{1}{2}k + \frac{1}{2}k^2$$

$$\Leftrightarrow \quad k^2 \times \frac{1}{1-k} \leq \frac{1}{2} + \frac{1}{2}k + \frac{1}{2}k^2$$

$$\Leftrightarrow \quad 2k^2 \leq 1 - k^3$$

$$\Leftrightarrow \quad (k^2 + k - 1)(k + 1) \leq 0 \, .$$

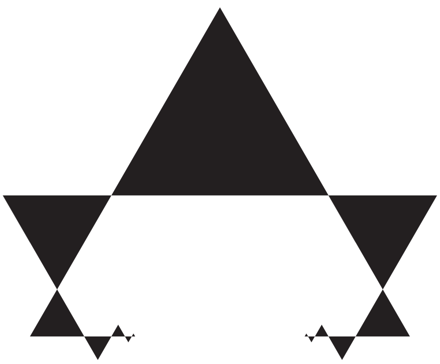


Fig. 3: Section from the pattern consisting of triangle

Since k is positive, the last inequality is equivalent to $k \leq \frac{\sqrt{5}-1}{2}$. Unexpectedly, we arrived at the limit number for no overlapping at golden ratio $\Phi = \frac{\sqrt{5}-1}{2} \approx 0{,}618$! An abundance of references to other areas of mathematics, art, architecture and nature has appeared (cf. e.g. Beutelspacher, Petri 1996).

If for instance regular pentagons instead of squares or triangles are considered as a further variation, one gets patterns which are free from overlapping for $k \leq \Phi^2$. The appropriate calculation possesses the same structure as that for squares and triangles and can be found for instance in (Kratz 1993, p. 190 et seq.).

Outlook (for Teachers)

Up to now we have considered limits of number sequences, e.g. of circumferences or areas. When the given patterns are studied to some extent, intuitive notions of the *"limit figure"* of the sequence of patterns appear. One tends to talk about the area or the circumference of the limit figure. But what can we actually understand by that? Can we sensibly speak of a *limit of a sequence of geometric figures*? That is possible as a matter of fact! The learning environment clearly transcends beyond school mathematics with this type of question. Here is a brief outlook for those who are interested:

The individual patterns of squares can be regarded as compact (that is, closed and bounded) subsets of $\mathbb{R}^2$. We designate the set of all non-empty, compact subsets of $\mathbb{R}^2$ with K. Any individual pattern of squares is an element of this set K, the sequence of patterns of squares is a sequence of elements in K.
We define a distance function, a metric, on the set K. To that end, the distance of point x from set A is defined as $d(x, A) = \min\limits_{a \in A} d(x, a)$ at first for every point $x \in \mathbb{R}^2$ and every set $A \in K$. The value $d(x, A) = \max \{\max\limits_{a \in A} d(a, B), \max\limits_{b \in B} d(b, A)\}$ is understood as the distance of two sets $A, B \in K$.
With this metric on K, the so-called Hausdorff metric, (K, d) is even a *complete metric space* (see e.g. Barnsley 1988, Edgar 1990). The usual terms such as "converging sequence", "Cauchy sequence" or "limit of a sequence" are defined in this space. The sequence of geometric figures considered in the learning environment is a Cauchy sequence in this space, which converges in each case towards a well-defined limit because of the completeness of the space.

This concept of convergence (analogous for $\mathbb{R}^3$) also offers a sustainable foundation for many geometric limit processes of standard mathematics in secondary school. Three examples:

- Inscribed or *circumscribed* polygons *approach* a circle *increasingly* more during the Archimeds method to determine the area of a circle.
- The volume formula for cylinders or cones can be attained as one *gets closer* to these objects with prisms and pyramids, respectively.
- When formulating the integral term, areas that are bordered curvilinearly *are approached more and more* by figures from rectangular strips.

In other words, sequences of geometric objects are considered to attain results concerning the geometric limit objects. In each case one has convergence in the abovementioned sense.

Summary

The learning environment “continue patterns” offers several multi-layered possibilities to study mathematics on the basis of easy to comprehend geometric basic forms. Students can experience mathematics as a field for individual and cooperative research and discovery at their respective performance level when the concept of classes is open-ended. Those performing less well may experience personal successes when they determine the number of squares in the individual patterns by way of counting and then inductively discover number sequences. At the same time, the learning environment also offers those performing well to dive deeply into mathematics based on standard content of the lessons.

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