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Towards New Teaching in Mathematics

Peter Baptist & Carsten Miller

Three – Four – Five – Many

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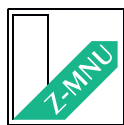
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Three – Four – Five – Many

Three vertices

The triangle is one of the simplest figures among the geometrical shapes, but nevertheless it plays a dominant role in school geometry. August Leopold Crelle (1780–1855), the founder of the distinguished *Journal for Pure and Applied Mathematics*, names one possible reason for this:

> *It is indeed remarkable that a simple figure like the triangle is so inexhaustible in its properties.*

Genuine problems can hardly be tackled in any other field of mathematics with so few resources (e.g. congruence, similarity, perimeter angle theorem). Examples can be found in the problems of the *Bundeswettbewerb Mathematik (National German Competition for Mathematics)* and the *Internationale Mathematikolympiade (International Mathematical Olympiad)*. Such problems can be used as starting points for further questions and occasionally lead to fundamental studies.

We would like to designate the 'act of being surprised' as the beginning of any science. Triangle geometry often gives opportunity for that, e.g. the points of concurrency of transversals. We have learnt from Hans Freudenthal (1905–1990):

> *The most decisive childhood experience of any mathematician who wrote autobiographies was the proposition of the perpendicular bisectors in a triangle that meet at a common point. It is indeed a wonderful proposition. Students find it easy when the problem is formulated in a less symmetrical manner.*

The same was reported by Martin Wagenschein (1896–1988) about Albert Einstein (1879–1955) who, as a child, felt amazement about the common point of concurrency of the three internal angle bisectors of a triangle.

Today, students are able to discover the different transversals and their properties with appropriate dynamic mathematics software. These visualisation tools were not available in the days of Freudenthal and Einstein. The following worksheet can be used in classroom for introduction or for recapitulation, too:
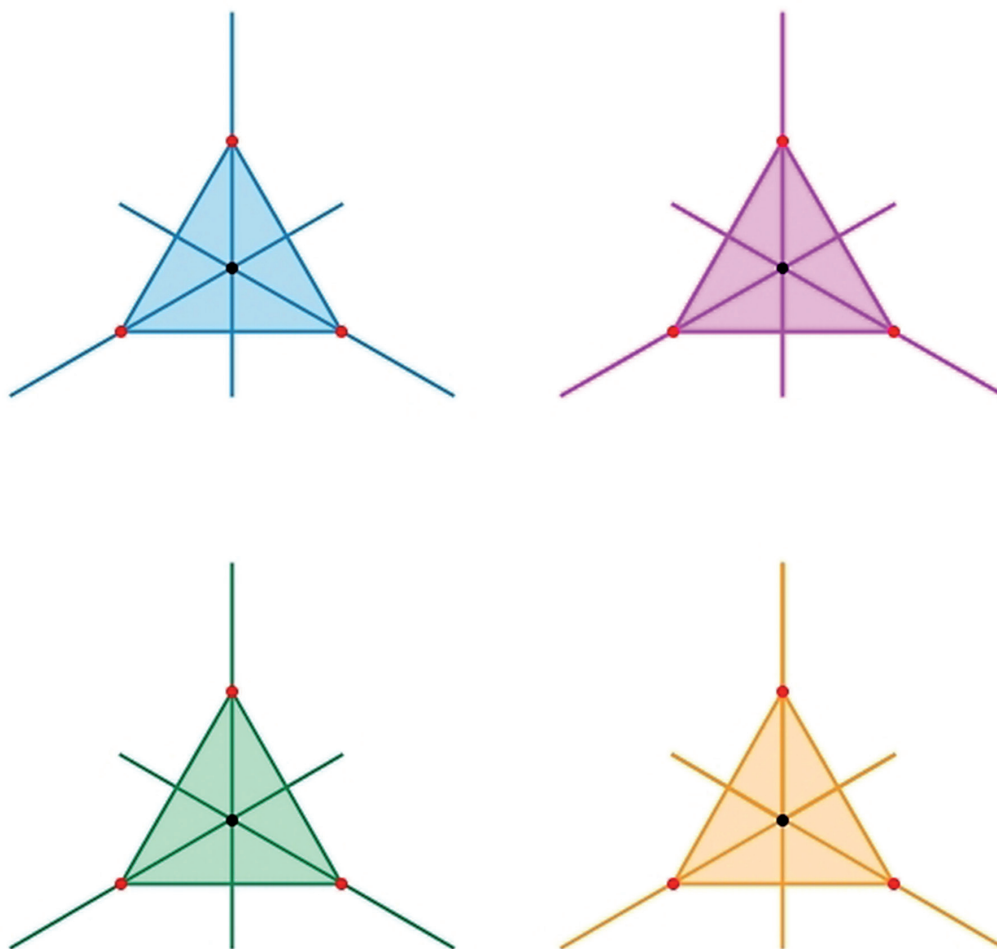
Fig. 1: Triangle transversals

Equal but yet not equal

▶ Change the shape of each triangle and try to find out the meaning of the lines in the triangles.
▶ Write down the results in your study journal (including sketches).

The dynamic worksheet shows four look-alike equilateral triangles with transversals. The title already leads one to assume that the equality only exists at first sight. When changing the triangles by dragging the vertices, differences are revealed right away. Subsequently, it is detailed work. The students have to find out by way of experiments which transversals are shown in each case. They describe their method and justify their findings in their study journals.

Four vertices

Our dynamic excursion to the standard topic of "triangle transversals" suggests to consider certain transversals in connection with quadrangles. We send the students on an expedition, they have to work independently. Now the figures have one additional vertex; instead of considering equilateral triangles, we start with squares. The initial situation certainly shows to be entirely different. Equality of the figures is out of question in this case.
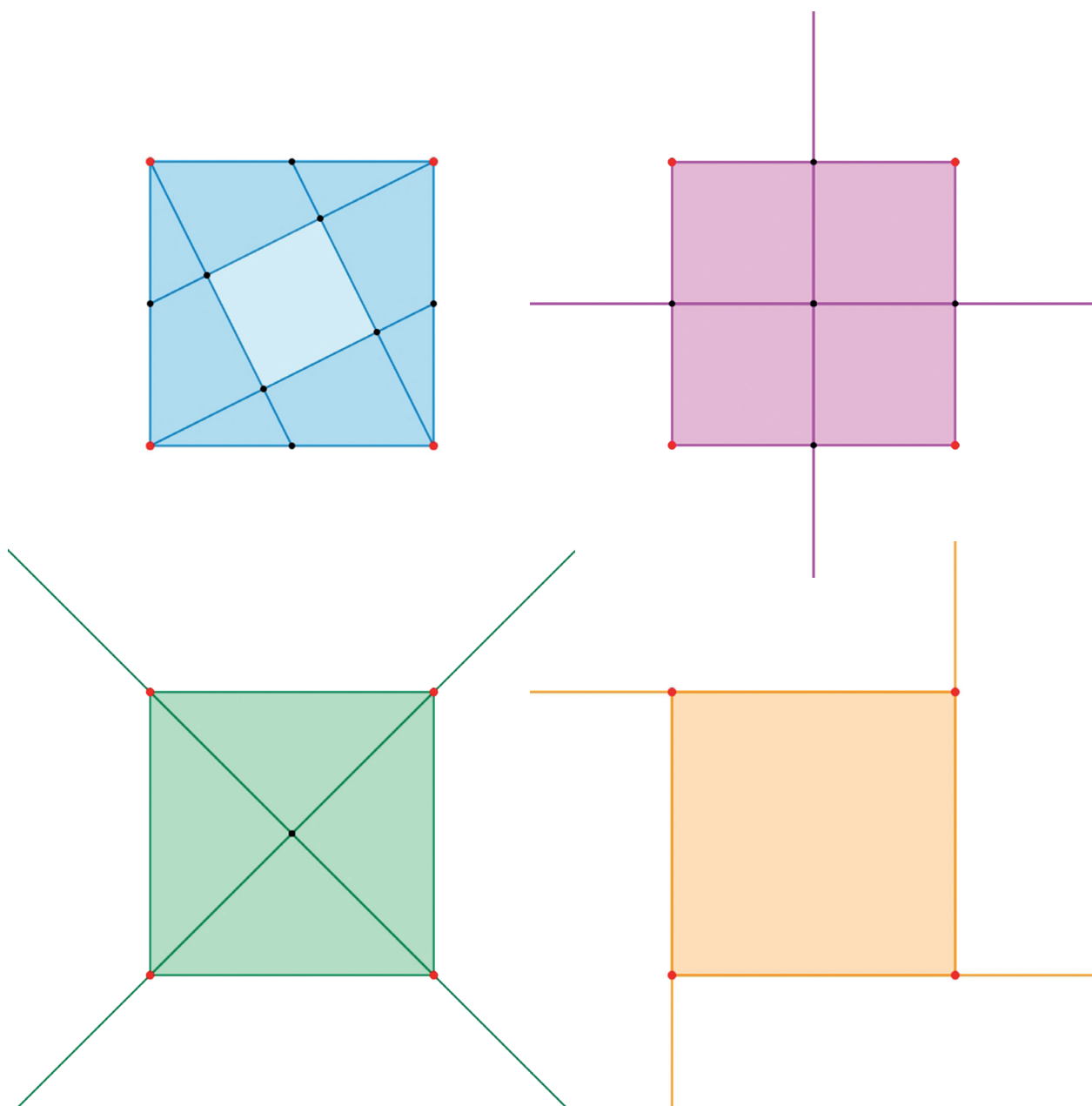

Fig. 2: Quadrangle transversals

Only the perpendicular bisectors and the internal angle bisectors appear to intersect in a point. While the configuration with the altitudes is uninspiring, the dissection of the square by the median lines stimulates interesting considerations, for instance geometrical shapes and the area of certain sections.
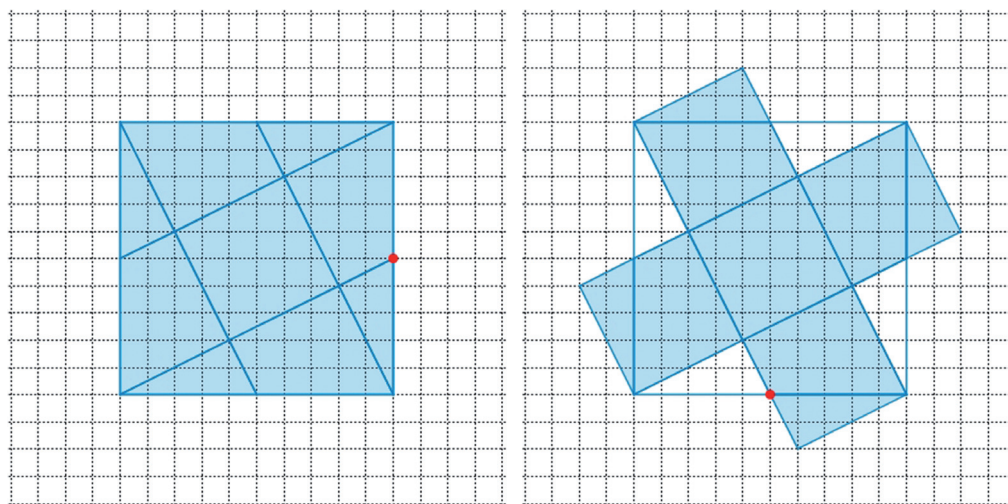

Fig. 3: Median lines in a square

Now we start with our investigations. What happens if the initial quadrilaterals are no longer squares? In order to be able to answer this question, we consider dynamic worksheets where the configuration with the squares and the corresponding transversals can be changed.

We demonstrate the method by using the example with the median lines. At first, the students are asked to make discoveries. They generate the following quadrangles by dragging on the vertices of the square: Rectangle, rhombus, parallelogram, isosceles trapezium, arbitrary quadrangle.
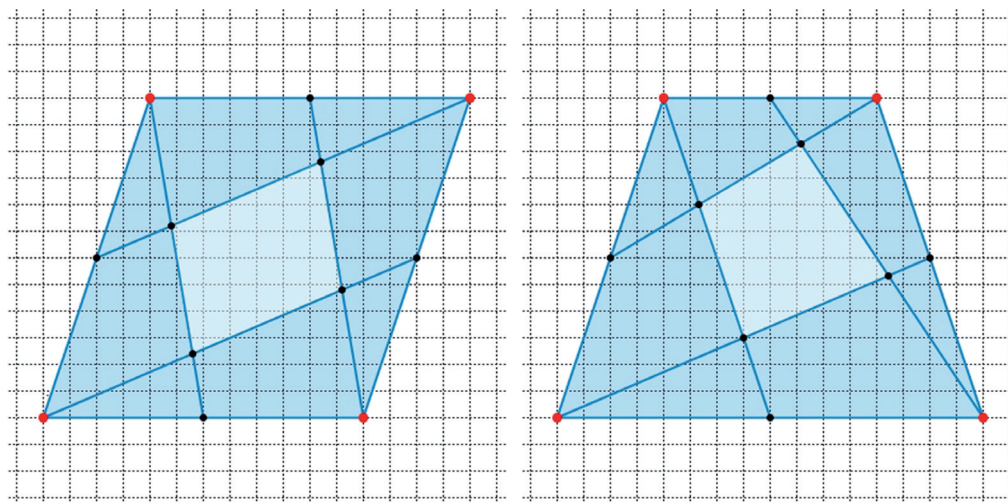

Fig. 4: Median lines in quadrangles

The underlying lattice structure makes the task easier. Subsequently, assumptions about the shape of the figure generated by the transversals must be written down in a study journal (together with significant sketches).

The students experiment on the computer screen, they observe the variing configurations, they discover properties and relationships, they write down assumptions, they discuss their results among one another. They do not look only for answers and proofs, they learn to pose questions and to formulate conjectures. The objective is that the students also think of the "why". That means they try to prove their conjectures or they debunk meanders by counter examples. Expedient use of dynamic worksheets leads automatically to the application of standards of education.

Five vertices

The Pythagoreans chose the five-pointed star, the pentagram, for their identification. These followers of Pythagoras created a type of sect which lived according to very strict rules. Mathematics was in the centre of their doctrine because in their opinion God had aligned the cosmos according to numbers. More to the point, this means: "Everything is number".
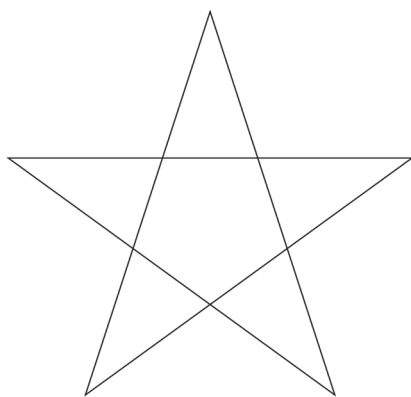

Fig. 5: Pentagram

Numbers were natural numbers and ratios of natural numbers for the Pythagoreans. Ancient mathematics was violently shaken up when around 450 B.C. one of the Pythagoreans, namely Hippasus of Metapontum, discovered that there are line segments, which do not have a common measure, they are incommensurable. It is for this reason that the foundation of the Pythagorean doctrine was questioned. It is particularly interesting that Hippasus came to his findings while regarding the pentagram, the internal "society badge". This discovery had far-reaching consequences for the development of Greek mathematics. People lost their interest in numbers and henceforth increased their study of geometry.

Even nowadays the pentagram may activate the curiosity of students. We connect the apexes of the pentagram and we get a regular pentagon with the accompanying diagonals. This famous figure is eminently suitable to go on a (guided) excursion.

A famous figure

- ▶ Which geometric shapes can be discovered in this figure?
- ▶ Which of the triangles are congruent?
- ▶ Determine all angles in the figure.
- ▶ How is a regular pentagon created?
- ▶ Investigate properties of the diagonal configuration.
- ▶ Which symmetries does the figure contain?
- ▶ How do the symmetry properties change by colouring certain sections of the figure?
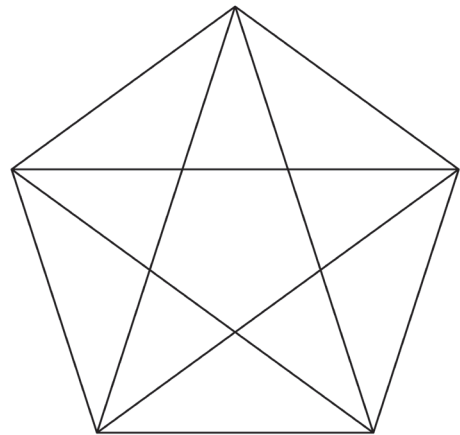
When this figure is at hand as a dynamic configuration, we will be able to travel through two thousand years in just a few seconds. By dragging the marked point with the mouse, we start with the logo of the Pythagoreans and arrive at the logo of the car manufacturer Chrysler.
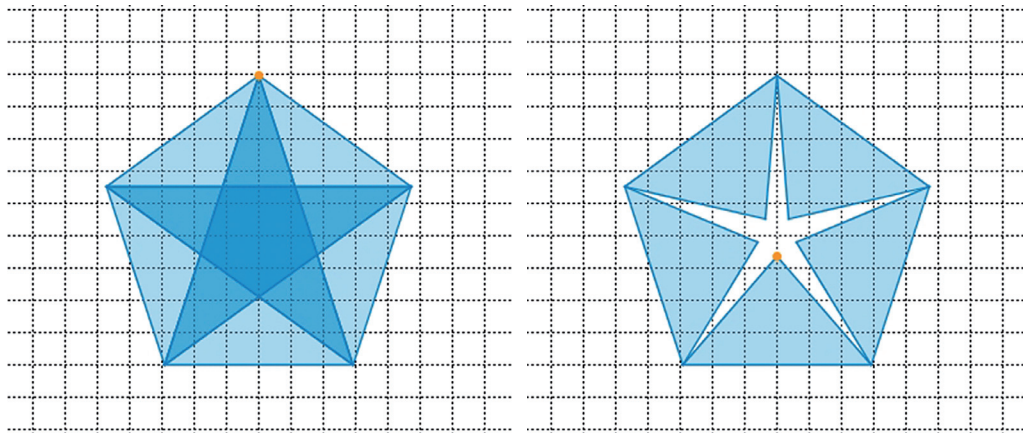
Fig. 6: Time travel

The Chrysler logo offers a lot of activities (cf. also Oetterer): Describe the composition of the figure, determine the area of the star, consider different types of stars, investigate symmetries, percentage calculation.

Many vertices

The inspiration for the next worksheet originates from the Swiss designer, painter, sculptor and architect MAX BILL (1908–1994). In the years 1935–1938 he created fifteen variations on a topic about polygons (*quinze variations sur une même thème*). We choose one of these variations:
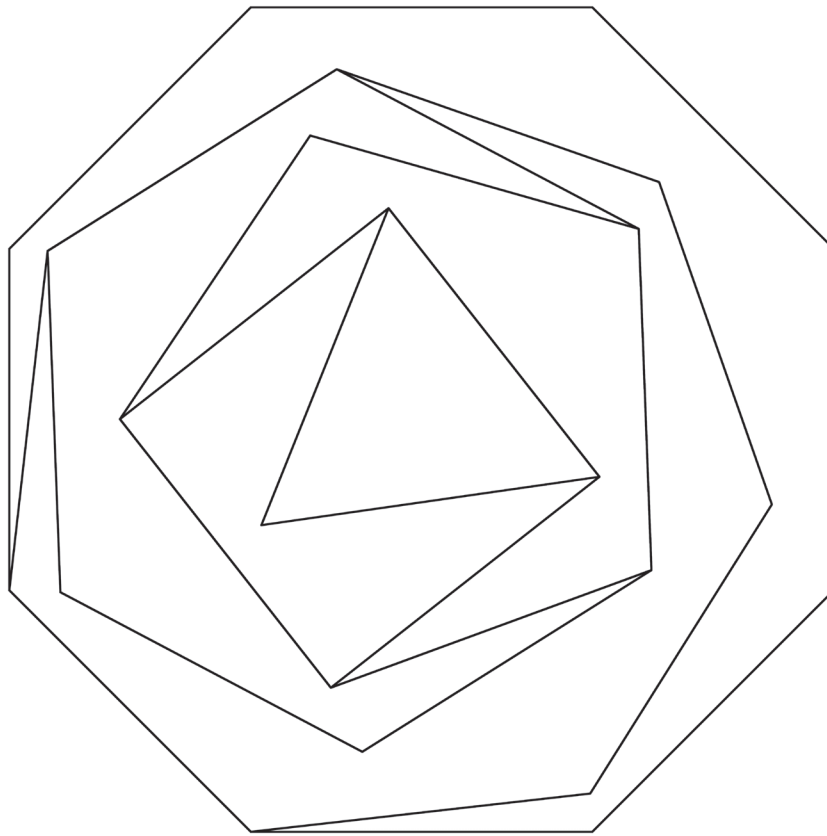

Fig. 7: Polygon spiral

The figure starts with an equilateral triangle. One of the sides becomes the side length of the surrounding square; in the same way we get the regular pentagon, hexagon, heptagon and octagon. The individual regular polygons, which emerge from one another in the shape of a spiral, have the same side length. Starting with the octagon, we recognise that the area of the larger polygon in each case is reduced by the area of the next smaller polygon.
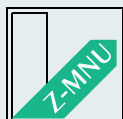
Investigation of the polygon spiral:

- ▶ Describe the figure, particularly the geometric shapes.
- ▶ Draw the figure in your study journal.
- ▶ Determine all angles in the figure.
- ▶ Determine the area of the geometric shapes.
- ▶ Formulate and solve tasks of percentage calculation that relate to the figure.

The topic of "three – four – five – many" is far from exhausted, neither from a mathematical nor from an artistic point of view. The consideration of the variations by Max Bill provides a multitude of inspirations, to the point of personal variations. Art and mathematics, a topic ad infinitum.

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