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Dynamic Geometry with Polygon Pantographs

Students Explore "Virtual Similarity Machines"

Dynamic geometry offers the possibility of copying the mechanical arrangement of rods of a pantograph in a virtual reality. This makes it possible for students to understand the mode of operation of this device for enlargement or reduction and, additionally, to invent and implement (in a virtual reality) their own variants of pantographs. They learn in this process how flexible or rigid polygons can be and they think intensively about the underlying principle: about the similarity of polygons.

Dynamic Triangles and Hinged Quadrangles

Three determinants are necessary to distinctively construct a triangle, and a quadrangle requires five of those. If one of these determinants is left out in each case, it will no longer be possible to distinctively construct the triangle and quadrangle, respectively. In Euclidean geometry (thus in constructions with compass and ruler), these under-determined cases are not interesting anymore in most cases. It is entirely different when dynamic geometry is practised on the computer. On leaving out a necessary condition, the students obtain a complete category of polygons, the representatives of which they vary dynamically and examine for characteristics.

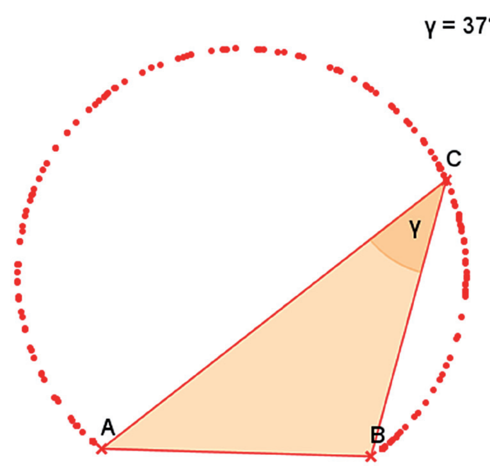


Fig. 1: circumferencial arc

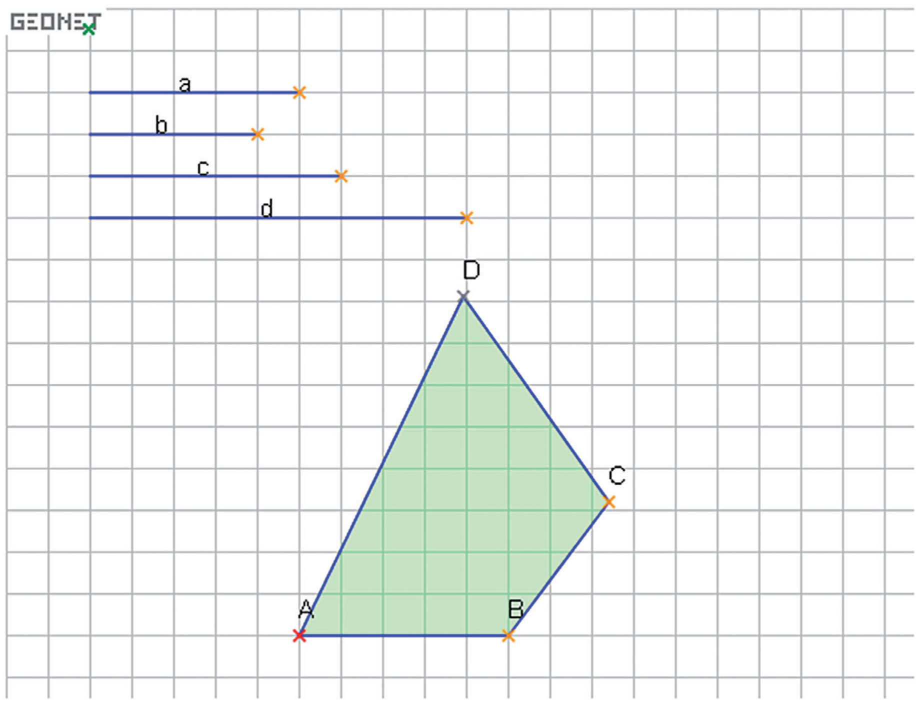
If one side (e.g. c) and the opposite angle (e.g. γ) are for example predefined in the case of a triangle construction, this will result in a category of triangles that is known from another context: c is a chord of one of the two arcs for the circumference subtending the angle γ (fig. 1). The intercept theorem is a special case (γ = 90°).

It becomes even more interesting in the case of quadrangle constructions: A quadrangle cannot be distinctively constructed on the basis of its four sides – one obtains thus the class of all quadrangles with predetermined side lengths. This category can be explored with concrete material: To that end, the students can connect four strips of cardboard or four bars from a building block system flexibly on the ends by means of hinges. Several quadrangle shapes can be formed out with such a *hinged quadrangle*.

Dynamic geometry software also offers the possibility of constructing, varying such hinged quadrangles and exploring its mobility. Furthermore, the geometry on the screen also has the advantage that the students can simply vary the lengths of the four sides and can thus explore different types of hinged quadrangles. This is realised in the dynamic worksheet of figure 2 (from Baptist 2004). Choose for example two side lengths equal in size; you will obtain a deltoid or parallelogram. The hinged quadrangle has already been constructed in the shown teaching materials. The students can thus immediately study the respective category of quadrangles without any prior special technical knowledge. If they are already familiar with the use of a DGS, then they could of course receive the request of constructing a hinged quadrangle made out of four segment lengths – by the way, this is quite a challenging task.

Experiments with Hinged Quadrangles

A quadrangle ABCD is made out of the abovementioned side lengths a, b, c and d.

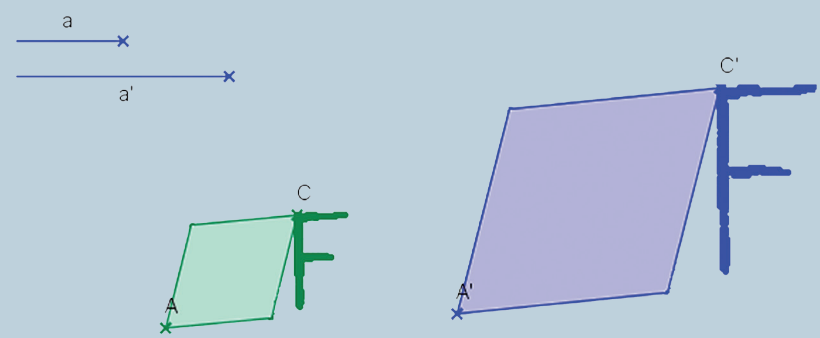


Drag on B, C and A. Write down what changes for the quadrangle and what stays the same. Why do they call it a hinged quadrangle?
Position opposite sides equally long and drag again on B and C. Write down any assumptions in your study journal.

From Similar Rhombi to Pantographs

Rhombi are the result of selecting all four sides of a hinged quadrangle to be of the same length. Such rhombi with hinges can be used for the purpose of producing enlargements or reductions. Students can explore this subject matter by means of the dynamic constructions and the accompanying work assignments described in the following sections. The GEONExT files are available for downloading at the URL specified at the end.

Box 1: Similar Rhombi – Uncoupled



- Explore the construction and describe your observations in your book.
- Draw a figure with point C. What observations do you make?
- What is the significance of the ratio of the lengths a and a'?
- Try to justify your observations.

In the framework of the dynamic worksheet, both rhombi in box 1 are constructed such that they are always similar to one another. If point C of the first rhombus is moved, then point C' of the second rhombus will move accordingly. If a figure is drawn from C (e.g. the letter F), then C' will show that the figure is larger or smaller in size. Students can recognise best when points C and C' are in the trace mode.

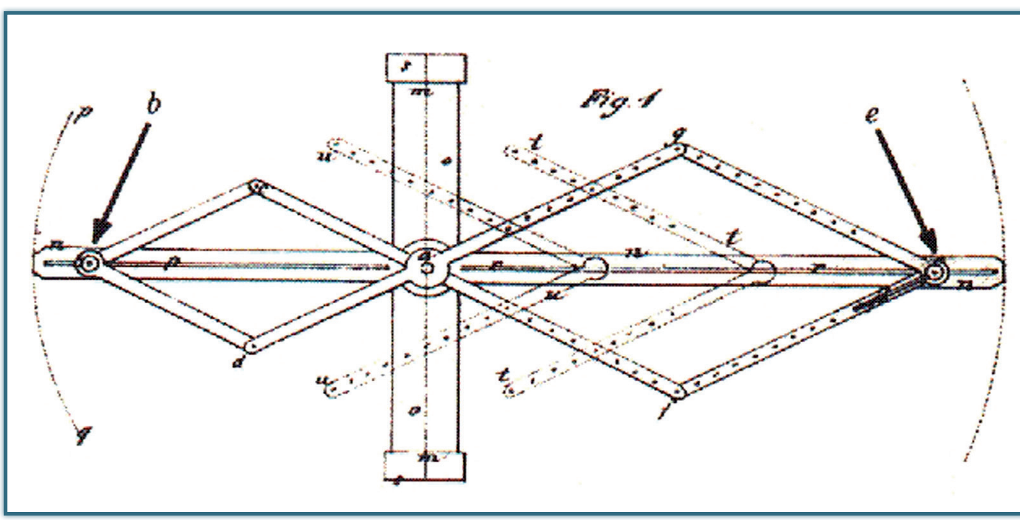


Fig. 3: Pantograph (Image from: learning mathematics, see bibliography)

The vertices C and A‘ of both rhombi were put on top of one another in the construction in box 2. For this reason, students can explore the functional principle of a mechanical arrangement of rods to enlarge or reduce drawings of a pantograph (fig. 3). Such “enlargement and reduction machines” are still used today in practice: as drafting tools or in engraving machines (fig. 4).

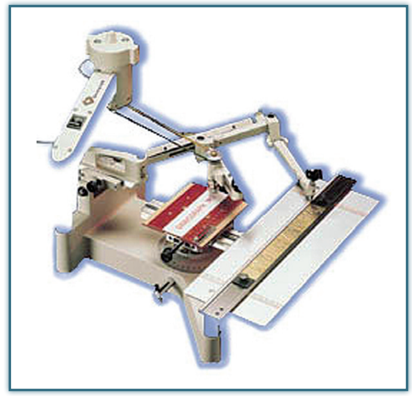
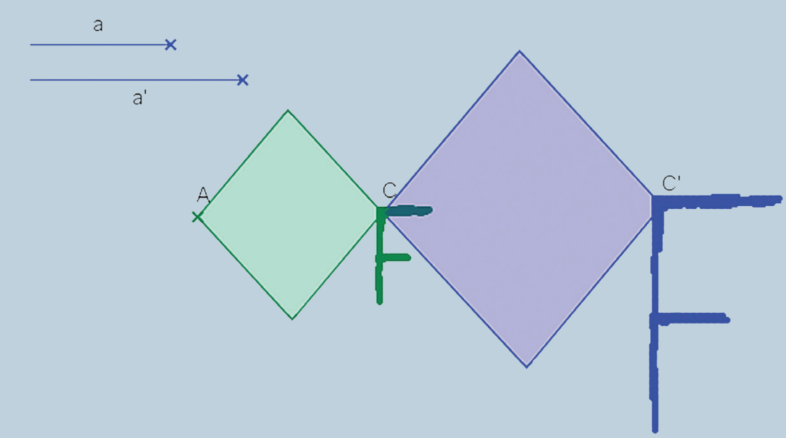


Fig. 4: Pantograph (Image from www.gravograph.com/Deutsch/Graviermaschinen/IM3.php)

Box 2: Similar Rhombi – Coupled



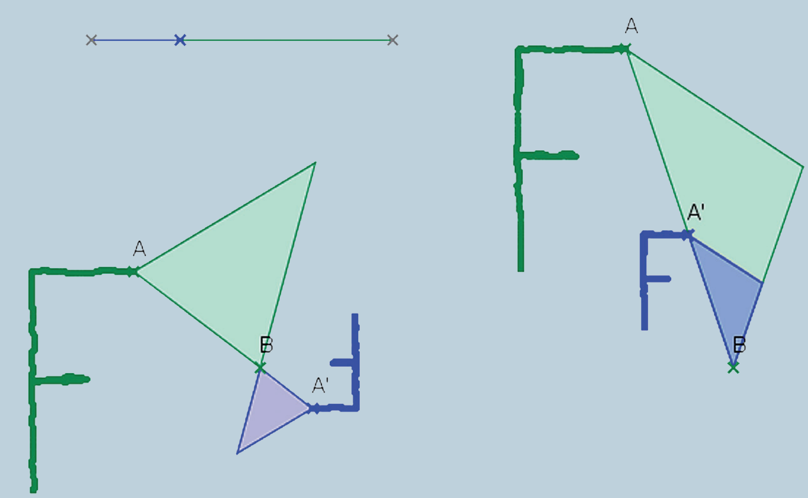
- Explore the construction and describe your observations in your book. In which way is this construction equal to the previous one and what are the differences?
- Draw a figure both with point A and point C. What observations do you make?
- What is the significance of the ratio of the lengths a and a’?
- Try to justify your observations.
- Build a mechanical arrangement of rods that possesses exactly the properties of the construction.

Triangle Pantographs

Rhombi are used in actual pantographs to easily realise the mechanical arrangement of rods. But is it actually absolutely necessary to use rhombi? Doesn't it also work with other similar polygons?

Semi-rhombi are isosceles triangles. They suffice for creation of "triangle pantographs" with dynamic geometry. Based on the constructions and accompanying work assignments mapped in box 3, the students can explore this functional principle and at the same time improve their understanding of dilation centres and the intercept theorem. When the triangles are on the same side of the centre of dilation, it is possible to enlarge and reduce without the image being turned 180°.

Box 3: Triangle Pantographs



- Explore the construction and describe your observations in your book.
- How does the principle work in case of enlargement and reduction, respectively?
- Where does the fundamental difference between both pantographs come from?
- Can you build a mechanical arrangement of rods which possesses exactly the properties of the construction?

Polygon Pantographs

The pantograph principle applies wherever similar figures show up. A particularly nice example appears in the regular pentagon. If all diagonals are drawn, one will obtain another regular pentagon on the inside. The contribution of Baptist and Miller also refers to this pentagram figure in this book. Hereinafter is the presentation of a collection of dynamic worksheets in parts with which students explore similarities in pentagon and hexagon figures and which can be used for the purpose of enlargement and reduction, respectively. The materials are available for downloading at the URL specified at the end of this article.

The first two (not shown here) dynamic worksheets are used to study the pentagram figure. The students have to determine any incident angle, segment ratios and thus also segment factors. A vertex of the outer pentagon is freely movable in the third dynamic worksheet (fig. 5). As it is almost the case for a kaleidoscope, five figures reduced in size are shown on the inside, which each appear to be rotated by an angle of 72° against one another. Consequently, the students also experience the X5 rotational symmetry in the regular pentagon in addition to the pantograph characteristic.

...almost like a kaleidoscope

The points O, P, Q, M, N generate a figure like a kaleidoscope when B is moved.
Create a clean model and then contemplate which symmetry the figure marked in grey possesses.

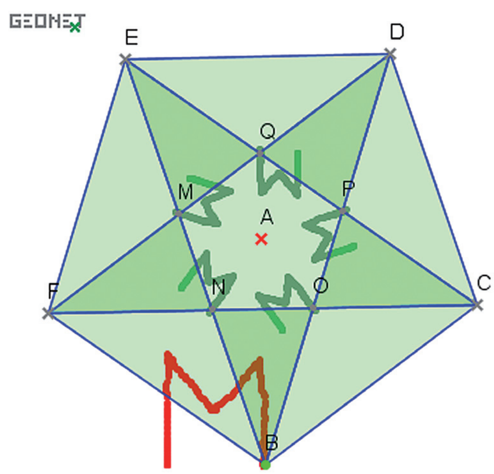


Fig. 5: Experiments with the pentagon pantograph

The following dynamic worksheet varies the geometrical situation (fig. 6). The students have to study a hexagon pantograph. In order to determine the rotation angle, they can insert artificial lines and discuss about angles in equilateral triangles. In order to calculate the reduction factor, they can for instance put the auxiliary straight line on the side of a hexagon and run along that with the freely movable vertex A. The internal points will then move along the inner sides of the hexagon. Finally, all that is left to calculate is the ratio of inner to outer side of the hexagon. The reduction factor $\frac{1}{\sqrt{3}}$ is obtained with Pythagoras theorem or with trigonometry.

Variation: Experiment with the Hexagon

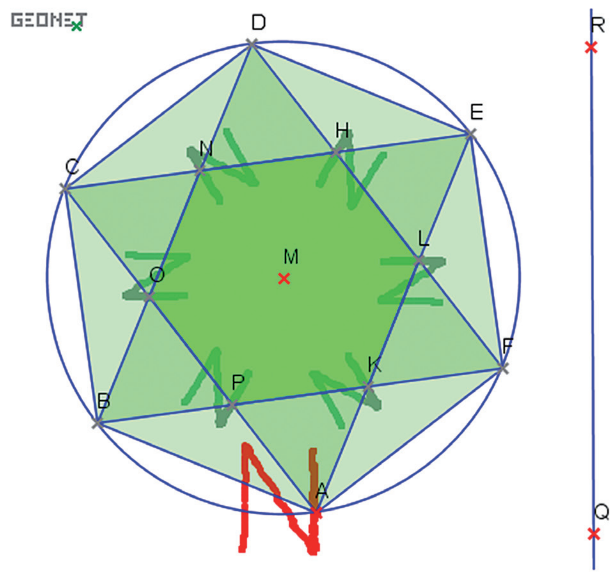


Fig. 6: Experiments with the hexagon pantograph

Observe the behaviour of the internal points while the red point is moving.
Use point A to draw the letter F and examine which rotation angles occur compared to the original figure.
Calculate the reduction factor for the grey image figure in comparison with the red original figure.

The last dynamic worksheet (not shown here) of the learning environment is created in a very open manner and appeals especially to those performing better. It might be useful to apply a time buffer during the course of the studies in order to keep pace with the working speed of the students. They can try to construct pantographs by themselves in an empty GEONExT plotting area. The equilateral triangle or square pantograph leads itself to that for instance. The students encounter concrete questions when designing such constructions. These questions include:

- How can reduction pantographs be created just like enlargement pantographs?
- How movable do the polygons have to be?
- Are there any pantographs that are constructed with irregular, movable polygons?

Shouldn't the students also be able to produce functional pantographs after trying for a longer period of time. As a teacher, the following helpful tips can be given:

- Construct two similar polygons in which one moves along as a function of the other.
- Use two equilateral triangles that are linked up at a vertex (fig. 7).
- Construct a square and another square across its diagonals (fig. 8).

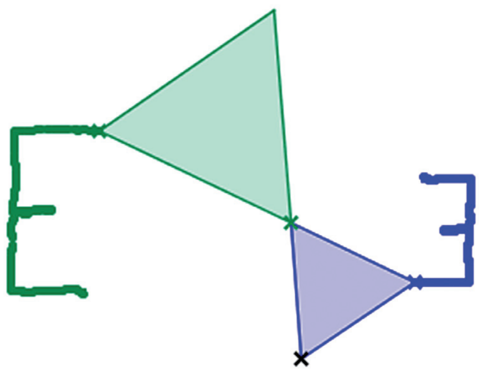


Fig. 7: Equilateral triangles

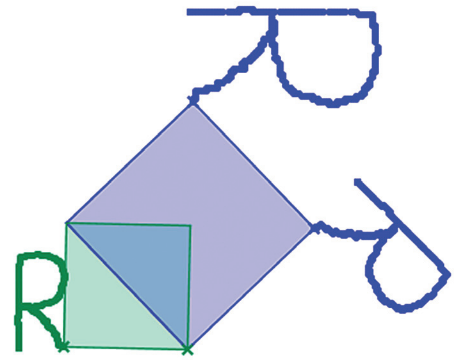


Fig. 8: Square across diagonals

The Rubber Band Pantograph

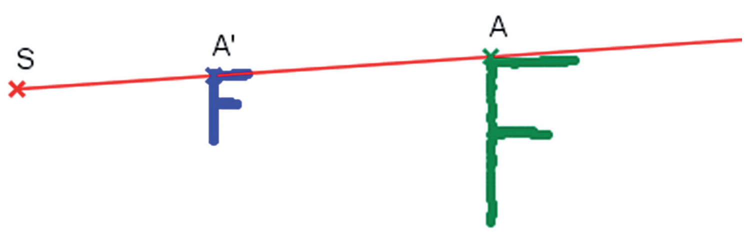


Fig. 9: Slider on a segment

Finally, the following question arises during the concrete design of (virtual) pantographs: Are similar figures even necessary or can a similar enlargement or reduction simply be realised on a segment? Such a "segment pantograph" offers many dynamic geometry programs (e.g. GEONExT) automatically based on its programming method: It suffices to place a slider A' on a half line SA (fig. 9). The software must now decide *how* point A' should move as a function of A. In principle, the programs decide on a dilation centre resulting in: If a figure is drawn with A, slider A' creates an enlarged or reduced copy.

For this purpose, there is a nice and real experiment which students can easily conduct themselves: You take a rubber band and make a knot in it somewhere (e.g. in the middle). One end is held. A piece of chalk is held on the other end and runs across the table in such a way that the knot separates a given figure. A figure emerges that is enlarged accordingly (fig. 10).

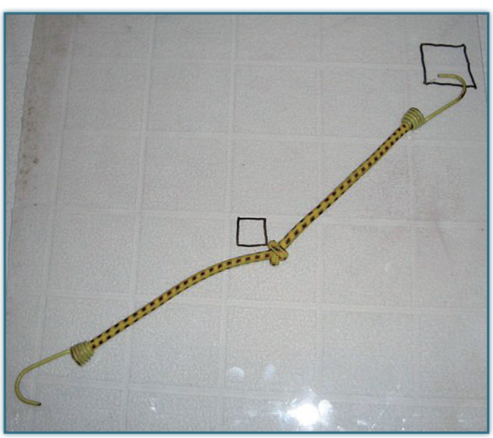


Fig. 10: A rubber band pantograph

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- GEONExT-Files: www.sinus-transfer.eu

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