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Towards New Teaching in Mathematics

Peter Baptist

Simplify Mathematics Education

7

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Dagmar Raab
(Eds.)

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Simplify Mathematics Education

Stimulating Acts: What Do We Learn from SINUS?

"Math – Revolution. SINUS has changed mathematics teaching in German classrooms. An example from Brandenburg shows how."

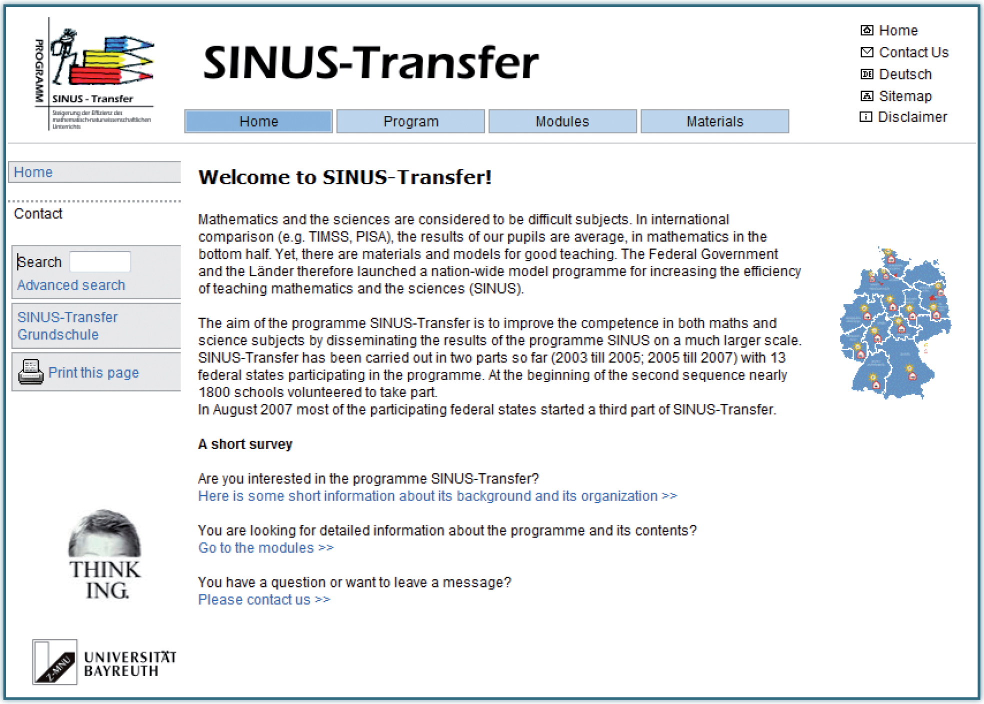
This was the headline of an article in the German weekly newspaper DIE ZEIT. Very enthusiastically a journalist reported on math lessons he had monitored. The school in the city of Brandenburg was one of more than 1800 schools that had participated in a very successful project with the bulky title Increasing Efficiency in Mathematics and Science Education. The reasons for once choosing the abbreviation SINUS for the project title are a mystery. Maybe because we find the five letters of the word SINUS in the complete title and it is short enough to remember.

Basic Information on SINUS and SINUS-Transfer

Some basic information on the project in short:

- As a consequence of the bad results of the TIMS-Study SINUS started with 180 secondary schools in 1998.
- The Institute for the Pedagogy of Natural Sciences (IPN) at the University of Kiel was responsible for science education and coordinationof SINUS.
- The Centre of Mathematics and Science Education (Z-MNU) at the University of Bayreuth was responsible for the improvement of teaching and learning of mathematics in SINUS.
- In 2003 SINUS-Transfer was born. The aim was to disseminate the successful project SINUS. We expanded to 1800 schools.
- We have organized local networks with up to ten schools. One of these schools has a "pilot-function". There has also been cooperation between school-networks in different German federal states.
- Eleven so-called modules set priorities where to start with the improvement of teaching and learning. Examples are:
  - development of problem based culture
  - learning from mistakes
  - securing basic knowledge
  - cumulative learning
  - students' cooperation
  - autonomous learning
  - assessment of competence gain.

A central server informs on the SINUS-project (www.sinus-transfer.eu). It contains materials for teachers (e.g. databases for problems and dynamic worksheets, in-service training). Information and teaching materials are available in German and English.



SINUS-Server: www.sinus-transfer.eu

What's Special about Mathematics Teaching in SINUS Schools?

First we have to realize, there is not a single way of successful mathematics teaching. In SINUS we have said goodbye to the teacher-centered dominant lecturing style in classroom. If learning is to be successful, the students must be able to tie their own think-nets, they must get the chance to go individual ways in their learning process. We should allow students to use their own informal problem-solving strategies, at least initially, and then we have to guide their mathematical thinking towards more effective strategies and advanced understandings.
There are certain basic guiding principles that are characteristic for our teaching at SINUS schools. Here are some examples:

- Less focus on passing factual knowledge to students, more focus on independent problem solving.
- Less focus on mere computing and manipulating formulas, more focus on understanding.
- Not only focus on acquiring particular math skills and results, but also focus on the necessary learning processes and strategies.

The implementation of these guiding principles leads to an experimental access to mathematics. If there is actually a formula or a rule, then it is at the end of the learning process, not at the beginning.

Don't Preach Facts, Stimulate Acts

Where do we start when we want to change teaching and learning mathematics? What is in the beginning?
As an educated person in Germany you will perhaps think of Johann Wolfgang Goethe's (1749–1832) famous drama Faust when you hear the keyword "beginning". In his study Faust speculates whether the word or the deed was in the beginning.

'Tis written: "In the beginning was the Word!"
Here now I'm balked! Who'll put me in accord?
It is impossible, the Word so high to prize,
I must translate it otherwise
If I am rightly by the Spirit taught.
'Tis written: In the beginning was the Thought!
Consider well that line, the first you see,
That your pen may not write too hastily!
Is it then Thought that works, creative, hour by hour?
Thus should it stand: In the beginning was the Power!
Yet even while I write this word, I falter,
For something warns me, this too I shall alter.
The Spirit's helping me! I see now what I need
And write assured: In the beginning was the Deed!

It is a hard inner struggle until he finally decides that in the beginning there was the deed. We immediately realize: Dr. Faust was definitely no mathematics teacher, because a mathematics teacher would never spend time on these doubts and speculations. For him or her it is absolutely clear:

In the beginning there was the problem.

Problems characterize mathematics lessons, problems run through our text books. Pages full of isolated problems like the following are familiar to all of us:

Exercises

Find the value of each expression. (8–1)
1. $\sqrt{16}$ **2.** $\sqrt{0.16}$
3. $\sqrt{81}$ **4.** $\sqrt{121}$
5. $\sqrt{15^2}$ **6.** 9^2
7. $\sqrt{16}+\sqrt{49}$ **8.** 10 squared
9. 6 squared **10.** Square of 6
11. Square of 7 **12.** Square root of 64

Find the indicated information.
13. Name five numbers that are perfect squares. (8–1)

Use the table of squares and square roots on page 475 to find the value of each expression. (8–2)
14. $\sqrt{38}$ **15.** $\sqrt{73}$
16. $\sqrt{54}$ **17.** $\sqrt{324}$
18. $\sqrt{39.69}$ **19.** $\sqrt{4356}$
20. 34^2 **21.** 7.7^2
22. Square of 34 **23.** 34 squared
24. Square root of 34
25. Square root of 69

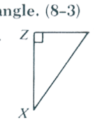
Use the table of squares and square roots on page 475 to complete each statement. (8–2)
26. $\sqrt{175}=\sqrt{25}\cdot\sqrt{7}=5\sqrt{7}\approx$ 5(?) ≈ ? .
27. $\sqrt{117}=\sqrt{9}\cdot\sqrt{13}=3\sqrt{13}\approx$ 3(?) ≈ ?

Find the indicated information.
28. Use the fact that $41^2 = 1681$ to complete each statement. (8–2)
a. $\sqrt{1681}=$? **b.** $\sqrt{16.81}=$?
c. $4.1^2=$? **d.** $0.41^2=$?

Locate each square root between two consecutive whole numbers. (8–2)
29. $\sqrt{19}$ is between ? and ? .
30. $\sqrt{52}$ is between ? and ? .
31. $\sqrt{71}$ is between ? and ? .
32. $\sqrt{95}$ is between ? and ? .

State whether each expression is in simplest radical form. (8–2)
33. $\sqrt{15}$ **34.** $3\sqrt{2}$
35. $\sqrt{27}$ **36.** $\sqrt{49}$
37. $\sqrt{60}$ **38.** $4\sqrt{8}$

Express in simplest radical form. (8–2)
39. $\sqrt{40}$ **40.** $\sqrt{48}$
41. $\sqrt{63}$ **42.** $3\sqrt{12}$

Identify the hypotenuse and legs of each triangle. (8–3)
43.


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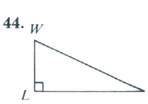
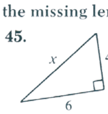


Write an equation that can be used to find the missing length in each triangle. (8–3)
45.


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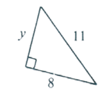


Exercises from a text book

But that is not our way of teaching in the SINUS project. We introduce mathematics in the context of carefully chosen problems. In the process of trying to solve these problems the students develop mathematics. We follow my American colleague Paul Halmos (1916–2006) who demands:

"Don't preach facts, stimulate acts."

That means: The teacher is not an entertainer, the student is not only a consumer. We do not present ready to consume mathematics. Teachers must help students to understand the concepts of mathematics, not just the mechanics of how to solve a certain problem.
"Stimulating acts" means to encourage students to develop their own informal methods for doing mathematics. We ask them

- to explore,
- to observe,
- to discover,
- to assume,
- to explain,
- to prove.

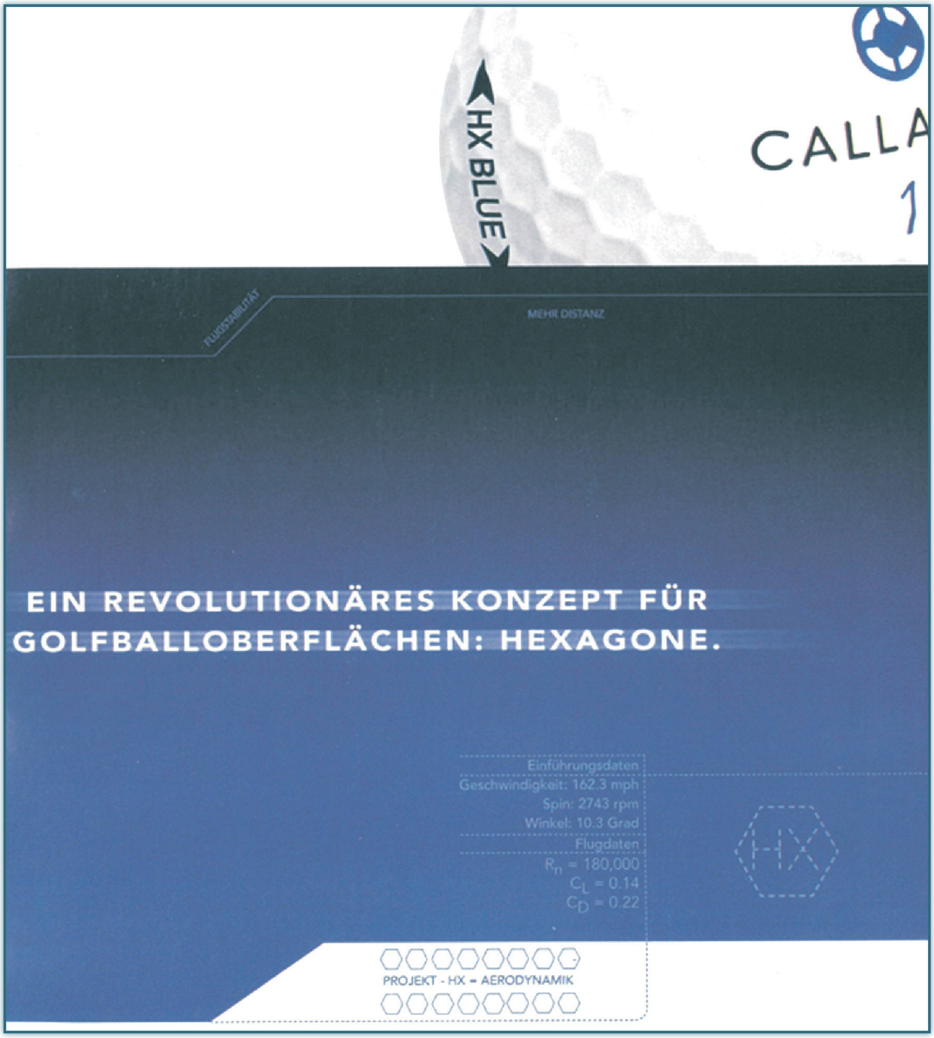
This sequence of activities exactly describes what we understand by Experimental Mathematics in SINUS. We do not start with formulas or rules, we get them at most at the end of the learning process.
Actually this method characterizes the way how to do mathematics in research. Why shouldn't we work in the same way in the classroom?

Stimulating Acts – Surface of a Golfball

Experimental mathematics is closely connected with problem-based or inquiry-based teaching and learning. We also try to combine real-life knowledge with mathematical knowledge. Thus we enable students to get a better access to the mathematical topic and to develop a deeper understanding.

But I do not want to preach facts, I want to stimulate acts. Therefore we have a look at the surface of a golfball. It is – in contrast to a table tennis ball – not smooth but fairly uneven. The dimples and their arrangement differ from brand to brand. A lot of research work is done in that field.
What is the reason for all these efforts? The players want their balls to go high, long and straight. The descend phase should be as long as possible and after landing the balls are supposed to roll many yards.
Why do I tell you all this? The reason is an advertisement in a magazine for an innovative golfball. Some years ago Callaway published an ad with the following statement:

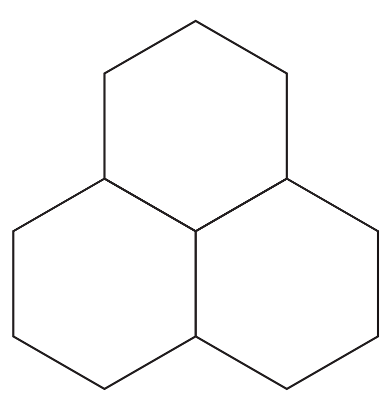
A revolutionary concept for golfball-surfaces: HEXAGONS.



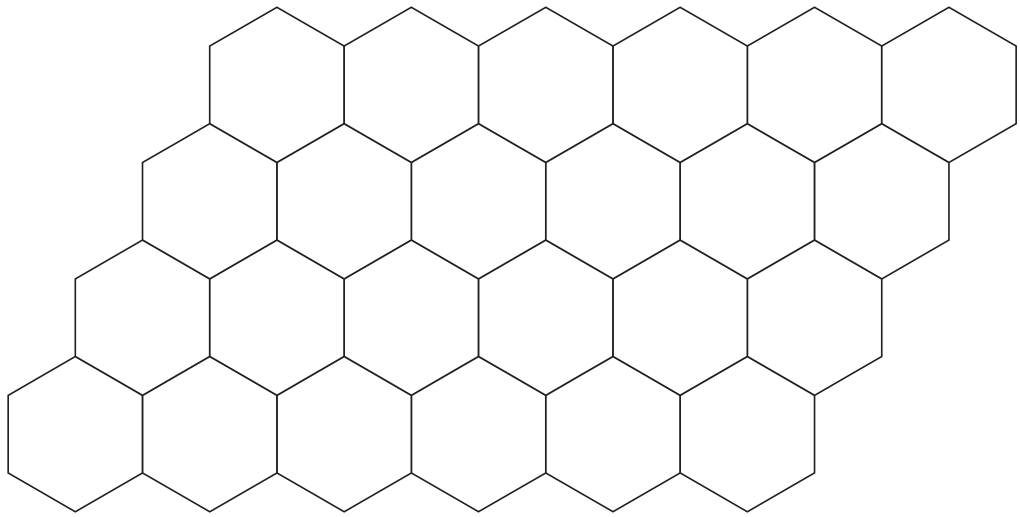
Callaway Golf advertisement

In this ad we could read: “Our goal was to develop the most progressive and most aerodynamic golfball in the world. Its patented structure of hexagons covers 100% of the surface of the ball. ...” According to the ad the surface of the golfball consists of a grid of hexagons. What does this mean?

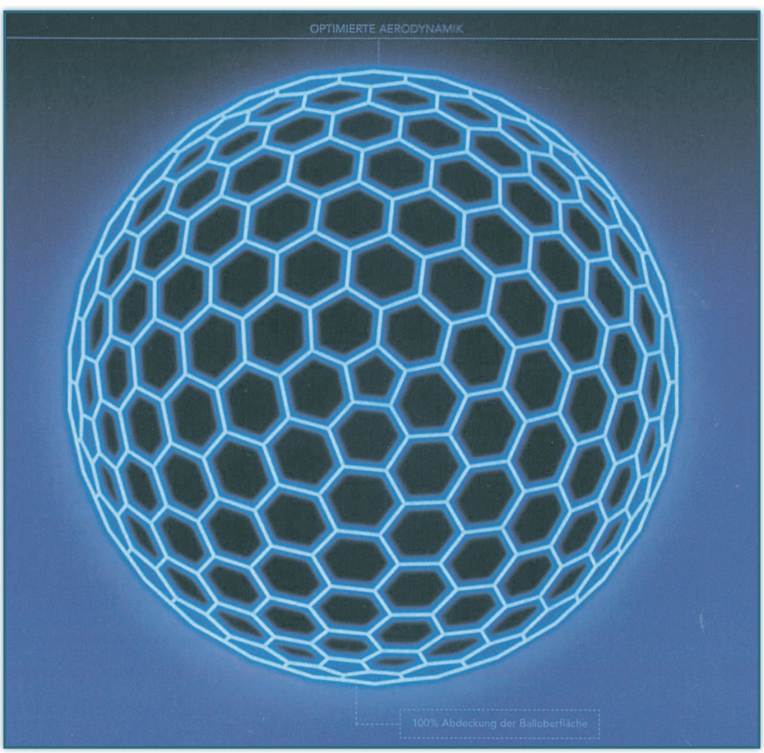
First, by hexagon Callaway understands a regular hexagon. Starting from a vertex we add further congruent regular hexagons. It turns out that on each vertex three hexagons exactly fit together.



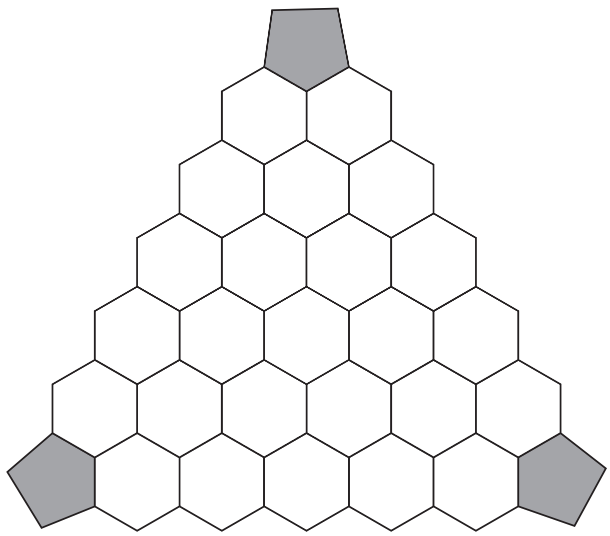
By proceeding in all directions we get a complete covering of the plane with no overlappings, a so-called tessellation.



But now we have a serious problem. How can we arch such a flat grid? How can we get the spherical surface of a golfball? Is it a kind of magic by the Callaway people? A look at the second part of the ad reveals the secret.



Callaway Golf advertisement

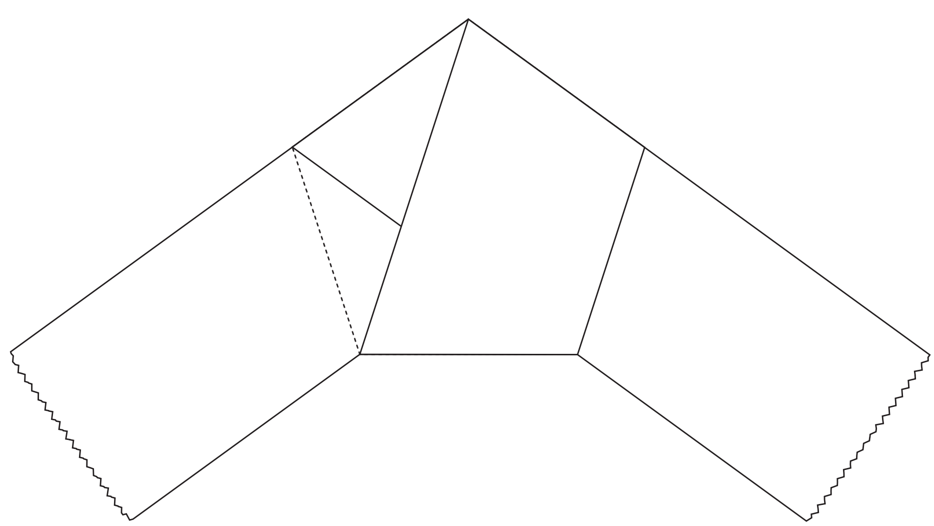


We see the grid of the golfball. Now let's have a closer look. Besides the hexagons we recognize some regular pentagons. The text of the ad doesn't mention these special 5-gons. We have a really interesting pattern. The surface of the golfball is covered by spherical triangles. In the vertices of the triangles we see pentagons, each side is formed by five regular hexagons.

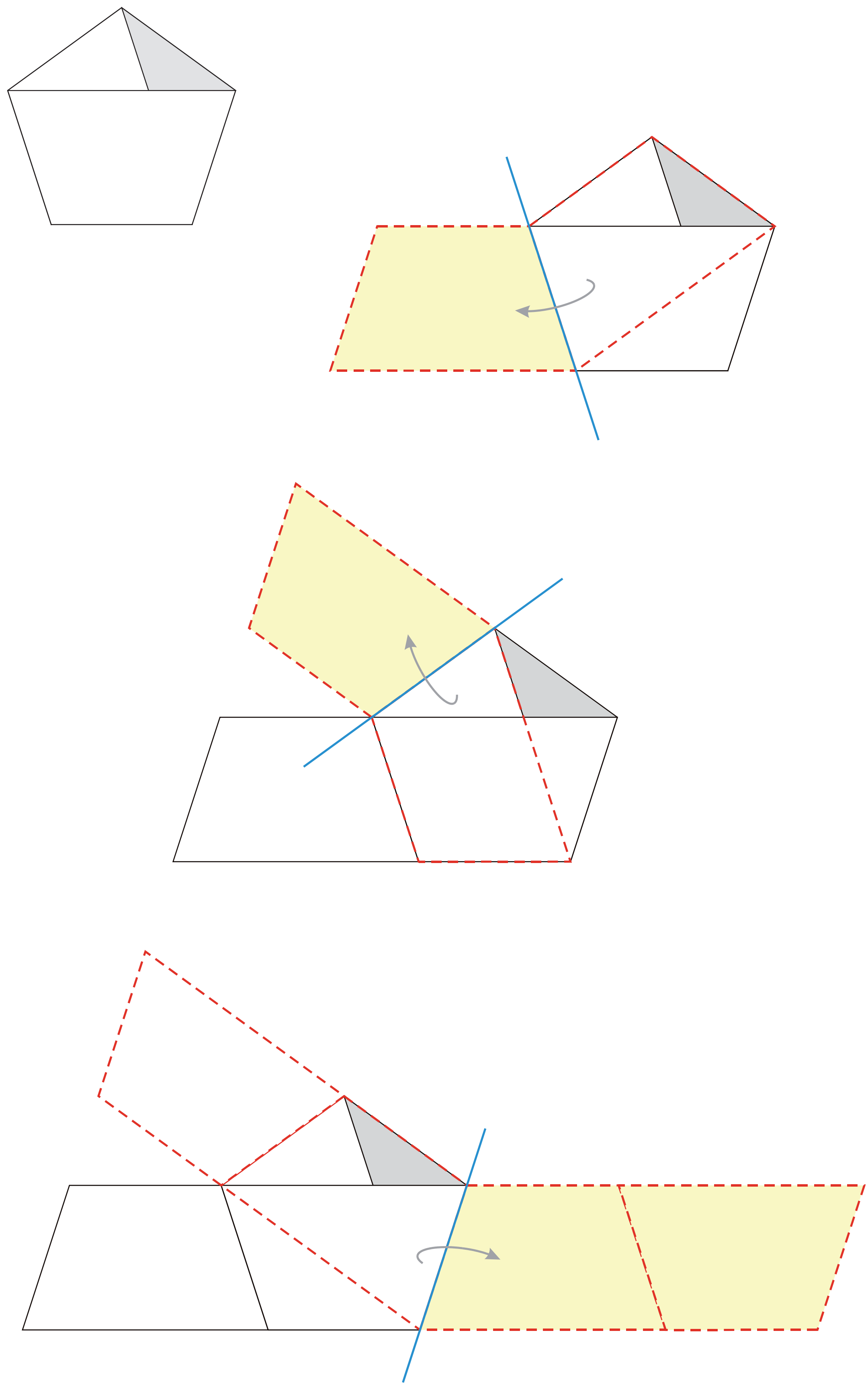
Now we ask: Why is this golfball ad so attractive for mathematics teaching? Well, we have a situation that sounds interesting and can be understood very easily. The students are challenged to develop their own ideas. Furthermore the problem directly leads to standard topics as

- constructing regular polygons,
- tessellation of the plane with regular polygons,
- Platonic and Archimedian solids.

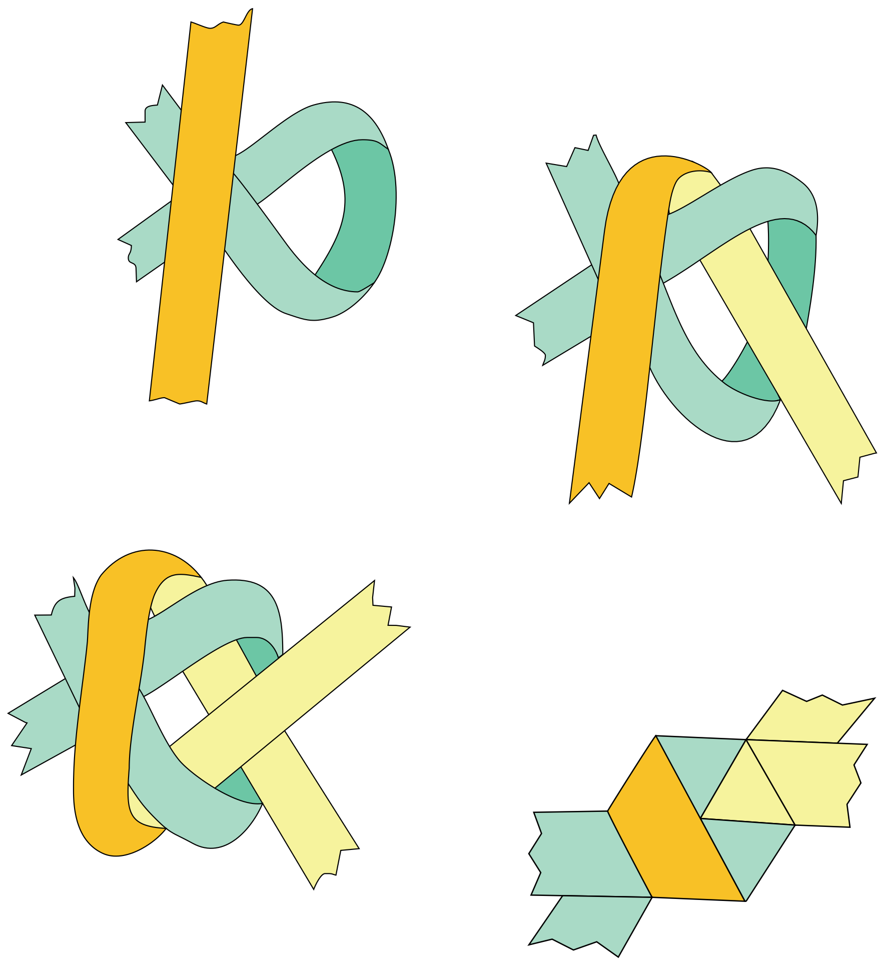
We do not study ruler and compass constructions of regular polygons, we only work with one or two strips of paper. Our golfball surface shows regular pentagons and hexagons. I am sure, you all know our first construction. We tie an ordinary knot with a strip of paper. Then we carefully tighten it as we press it flat. And a regular pentagon appears.



Now we have to find out the reason for that. And this is an interesting geometrical problem.

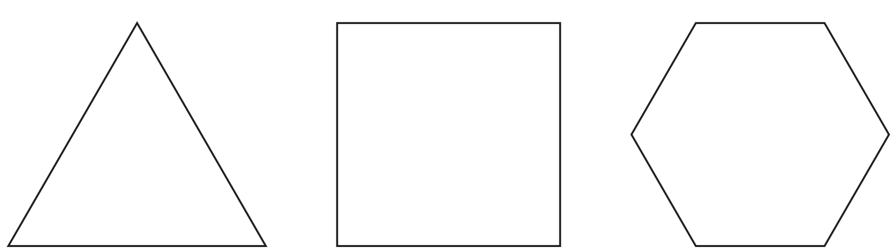


For a regular hexagon we need two strips of paper. Here we have the folding instructions:

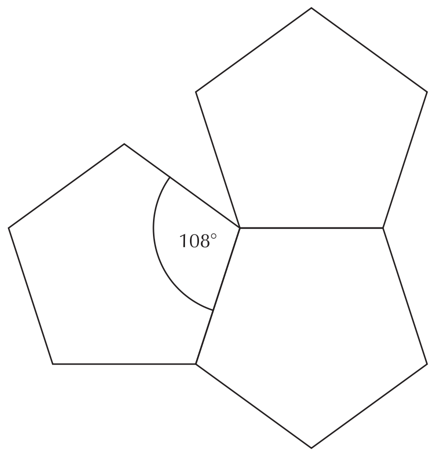


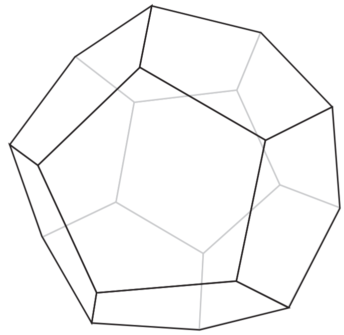
Regular Tessellations

Now we ask: Which regular tessellations of the plane are possible? Stimulating acts means making experiments either with concrete materials or intellectually in the mind. We have to consider what inner angles of regular polygons are possible. For a non-overlapping covering they must divide 360°. The astounding result: There is an infinite number of regular polygons, but there are only three regular tessellations. Johannes Kepler (1571–1630) was the first to recognize that only equilateral triangles, squares and regular hexagons have this property.



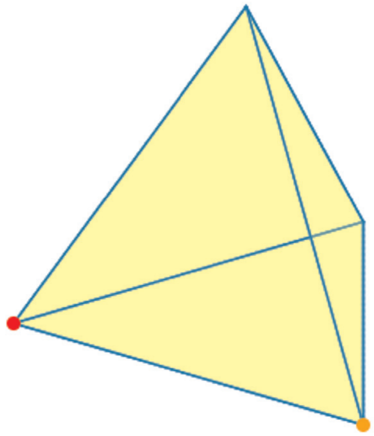
A regular tessellation with pentagons is not possible, because the interior angle measures 108° and that is not a divider of 360°.

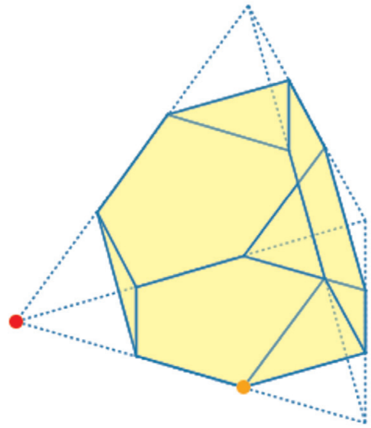




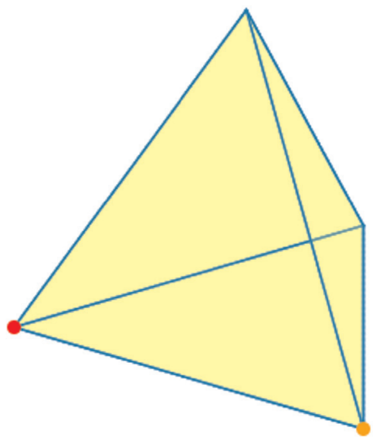
But we can form a three-dimensional vertex with three regular pentagons. Adding further regular pentagons (find out how many?) we get a special regular solid, a dodecahedron.

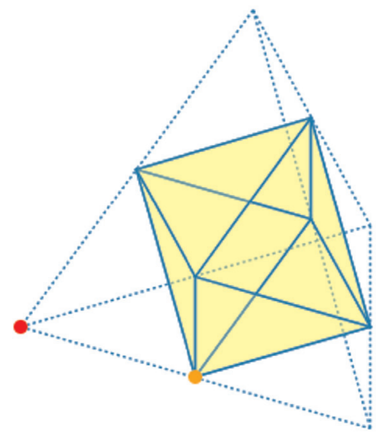
It belongs to a famous family, the Platonic solids. Definitely the best known member of this family is the cube. Now we consider three equilateral triangles that form a three-dimensional vertex. Adding a forth triangle we get a tetrahedron, another Platonic solid.
What happens if we cut off the vertices? Let me remind you of Paul Halmos: Don't preach facts, stimulate acts. We use the power of dynamic geometry, e. g. GEONExT (free download under www.geonext.de). If we cut one third of each edge, the faces of our new solid are regular triangles and hexagons, we get an Archimedian solid.





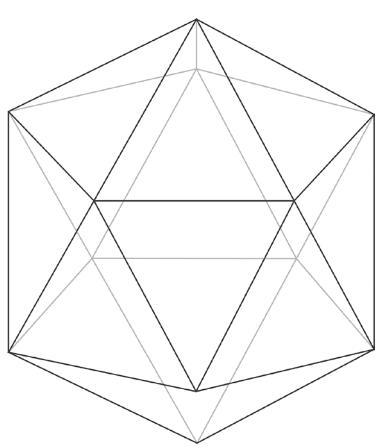
If we bisect the edges we get the octahedron, also a Platonic solid.

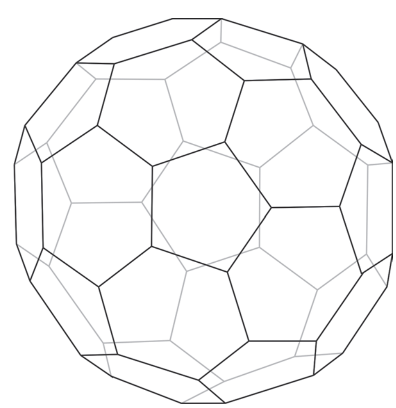




When using these GEONExT constructions one doesn't have to explain the advantage of a dynamic construction, because you really see the advantage. I cannot imagine a better visualization.

Another astounding fact: In the plane there is an infinite number of regular polygons. Going one dimension higher we only have five regular solids. Up to now the icosahedron with its 12 vertices and 30 edges and 20 faces is missing in our list. If we cut the vertices while trisecting the edges we get an Archimedian solid with 60 vertices and with regular pentagons and regular hexagons as faces. But this time it is not a Callaway golfball, we get a soccer ball.





Metric Standard Paper Size and Irrational Numbers

We skip further inverstigations on Platonic and Archimedian solids, leave geometry and jump into numbers. We want to introduce irrational numbers. Again we combine real-life knowledge with mathematical knowledge. Again we stimulate acts.
ISO A is a well known metric standard paper size. It is defined by the following two properties:

- By halving the sheet we get from ISO A n to ISO A n+1.
- All sizes are similar to each other. That means: The ratio of the longer side to the shorter side is always the same.

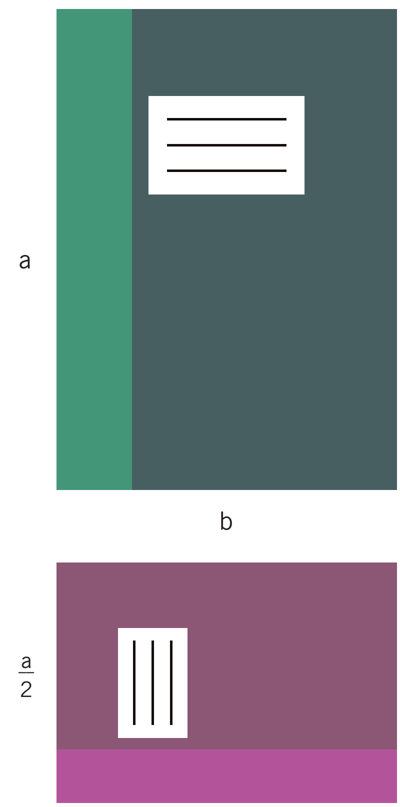
With the help of these properties we are able to compute the exact ratio of the lengths.

$$\frac{a}{b} = \frac{b}{\frac{a}{2}}$$

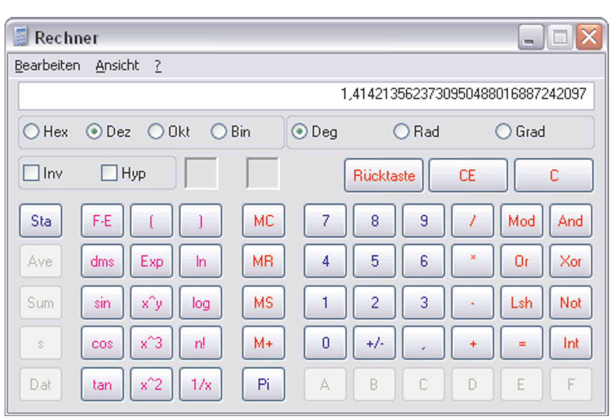
$$a^2 = 2b^2$$

$$\frac{a^2}{b^2} = 2$$

$$\frac{a}{b} = \sqrt{2}$$



At first we are not familiar with the symbol $\sqrt{2}$. What kind of number is it? Many people think that all numbers are fractions. They are in good company, because the ancient Pythagoreens believed that, too. They said: “Everything is number!” For them numbers were natural numbers and ratios of natural numbers. Nowadays we have pocket calculators. The input $\sqrt{2}$ shows:



We ask: Does this decimal fraction end? Or is it periodic? If yes, we would have a fraction, a rational number. But perhaps this decimal fraction never ends and is not periodic. Who knows?
Too bad, the pocket calculator is no help for finding out the nature of $\sqrt{2}$. Therefore we join the majority and assume that $\sqrt{2}$ is a fraction. But we do not know this fraction. Therefore we choose a general expression:

$\sqrt{2} = \frac{a}{b}$, where a and b are natural numbers having no common factor other than 1.

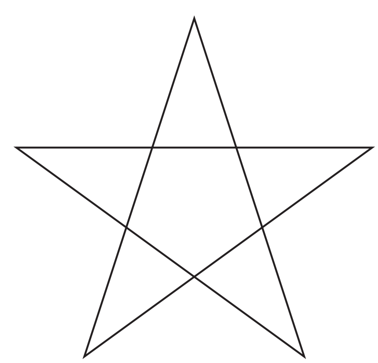
$$2 = \frac{a^2}{b^2} \Leftrightarrow 2b^2 = a^2 \Rightarrow \begin{array}{l} a^2 \text{ even,} \\ a \text{ even,} \\ a = 2n. \end{array}$$

$$2b^2 = 4n^2 \Leftrightarrow b^2 = 2n^2 \Rightarrow \begin{array}{l} b^2 \text{ even,} \\ b \text{ even,} \\ b = 2m. \end{array}$$

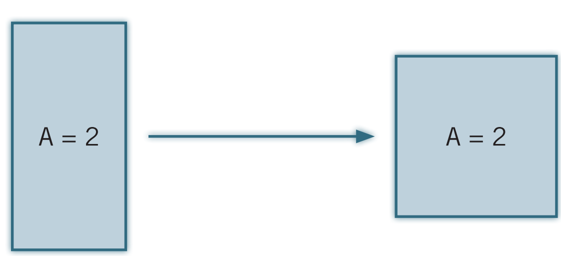
The implications here finally lead to a contradiction. Therefore $\sqrt{2}$ cannot be a rational number.

Please note: In terms of calculation, the mathematical content of this proof is completely elementary. It is the train of thoughts that is spectacular. Therefore students should get to know this proof and should try to understand its content and its beauty.

Further Activities



- Historical excursion: How, where and when did people meet irrational numbers for the first time? We have to go back to the Pythagoreens, especially to Hippasos of Metapont (450 BC) and we have to consider the intersection of the diagonals in a pentagram.
- How do we get the single digits of the decimal expansion of $\sqrt{2}$?
  A very clear geometrical method for this is the so-called HERON-algorithm. We remain in the ancient world, but about five hundred years after the Pythagoreens. The idea of the algorithm is simple. A rectangle with area 2 is transferred into a square with the same area.

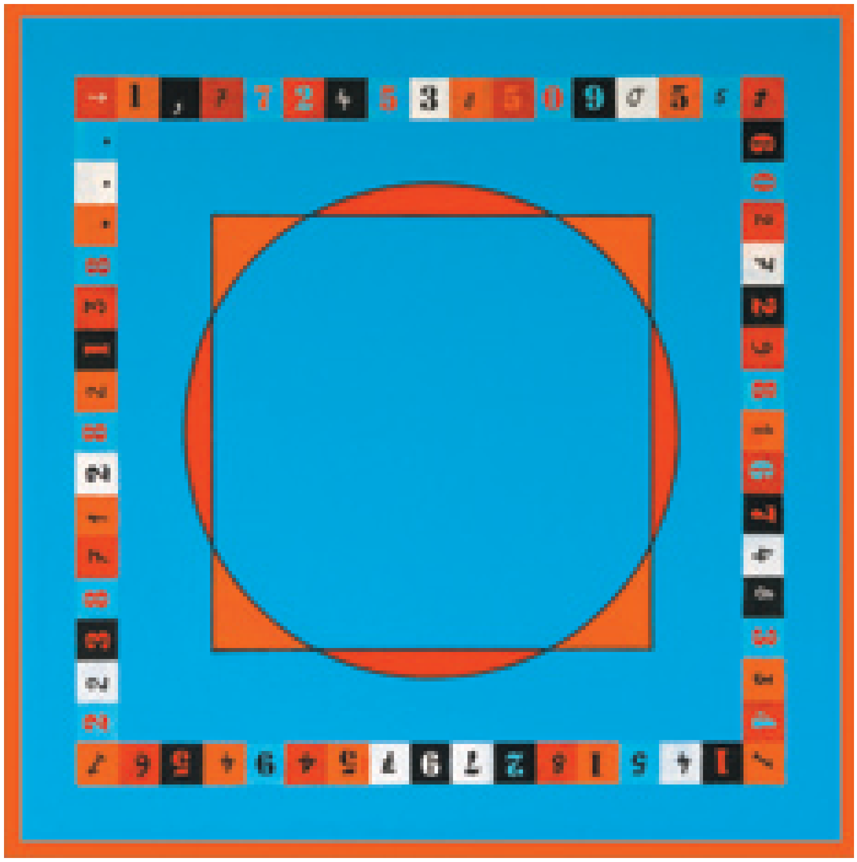


How does it work? The length of one side of the new rectangle is the meanvalue of the two sides of the old one. We choose the other sidelength so that the area of the rectangle remains 2. By repeating this process several times the rectangle step by step becomes a square. We have the quadrature of a rectangle.
More famous than the quadrature of the rectangle is the proverbial quadrature of the circle. Even the ancient Greeks tried to solve this problem. Here we meet another irrational number and that is the number π.

This painting of the Swiss artist Eugen Jost shows the first 285 decimal digits of π. Today we know more than 200 billion digits and still we are looking for a pattern in this decimal expansion. A verse from the Old Testament (Book of Kings) forms the frame of the π-painting. This verse tells us that the biblical value of π equals 3.

> *"And he made a molten sea of ten cubits from brim to brim, round in compass and five cubits the height thereof; and a line of thirty cubits did compass it round about."*

A circle of radius 1 has the area π. Squaring this circle means to construct a square with the same area by means of ruler and compass. The sidelength of this square must be $\sqrt{\pi}$. Here you see this classic problem in the interpretation of Eugen Jost. This time the frame is formed by the decimal digits of $\sqrt{\pi}$.



Experimental Mathematics Meets Experimental Art

In my opinion these paintings simultaneously show the beauty of art and the beauty of mathematics. The artist Eugen Jost likes to experiment with signs and numbers and patterns. Creative mathematicians work in exactly the same way. In this connection one likes to quote G.H. Hardy (1877 – 1947):

> *"A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. ... The mathematician's patterns, like the painter's or poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. ..."*

There is a corresponding painting by Eugen Jost with the title *Hardy's Taxi*.

We can discover a lot of interesting numbers and sequences like the Fibonacci sequence, triangle-, quadratic- and perfect numbers. This painting contains stimulus and a lot of ideas for mathematical excursions.

Of course we also have to mention the taxi driver: Godfrey Harold Hardy, a brilliant mathematician and an eccentric. He was a good-looking man, but he hated to see his face. Therefore he had no mirrors in his apartment. His first action in a hotel room: He covered all mirrors with towels.

An interesting story is the mathematical collaboration between Hardy and the Indian genius Srinivasa Ramanujan (1887 – 1920). For further details I recommend the fantastic book *The Man Who Knew Infinity* by Robert Kanigel. There you find the following anecdote. Ramanujan was ill a lot, and therefore often in hospital. Hardy visited him and because he had no other topic of conversation he told him that his taxi to the hospital had the number 1729, a rather boring number in his opinion. "No, no Hardy", said Ramanujan, "1729 is very interesting, it is the smallest number that can be written as sum of two cubes in two different ways." Ramanujan had no proof, but the right inspiration. Daily he produced new theorems that turned out to be true.

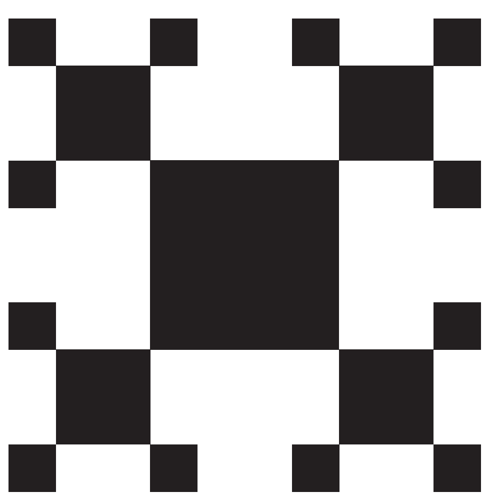
I hope my examples have made clear one of the key characteristics of mathematics: We will never come to an end. One solved problem is the starting point of new problems. Mathematics is a never ending story. Our examples show: Mathematics is much more than mere computing, mathematics is a part of our culture. Therefore historical aspects must be integrated into our teaching. Abe Shenitzer from Toronto underlines this aspect when saying:

> *“One can invent mathematics without knowing much of its history. One can use mathematics without knowing much, if any, of its history. But one cannot have a mature appreciation of mathematics without a substantial knowledge of its history.”*

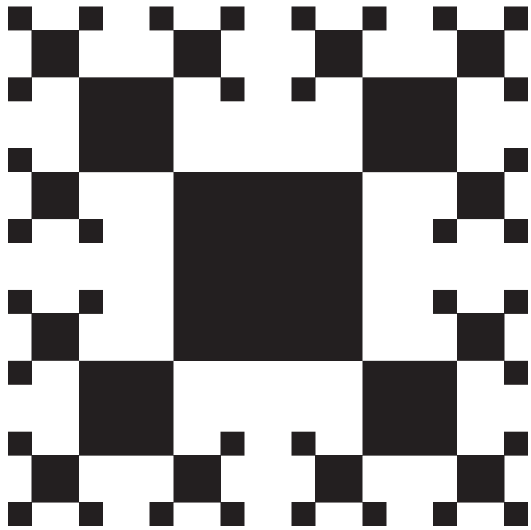
Experimental Mathematics – Fractals of Regular Polygons

A regular polygon is a convex polygon that is both equilateral and equiangular. We begin with the best known regular polygon, the square. At the four vertices of a square we attach squares whose side lengths are reduced by a certain factor f ($f<1$) in comparison to the first square.

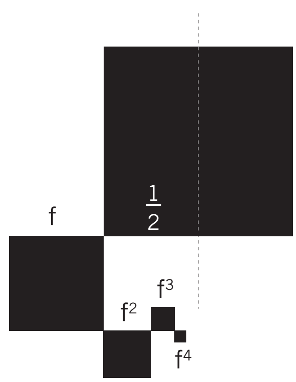
At the free vertices of the attached squares we repeat this procedure, i.e. this time we draw squares with side length f^2.



Step by step we get more and more detailed branches.



If we choose the reduction factor f comparatively large, the branches will overlap after some iteration steps. If we choose the factor f comparatively small, there will be gaps between the branches. It is an interesting problem to find out just that factor for which the separate branches will touch in the limiting case.



For symmetry reasons we only need to consider one of the four possible touching points. The side length of the first square be 1. Then the following squares have side lengths f, f^2, f^3, … etc. The demand that both sequences meet at the midpoint means: The sequence must have side length $\frac{1}{2}$ in the limiting case.
That is

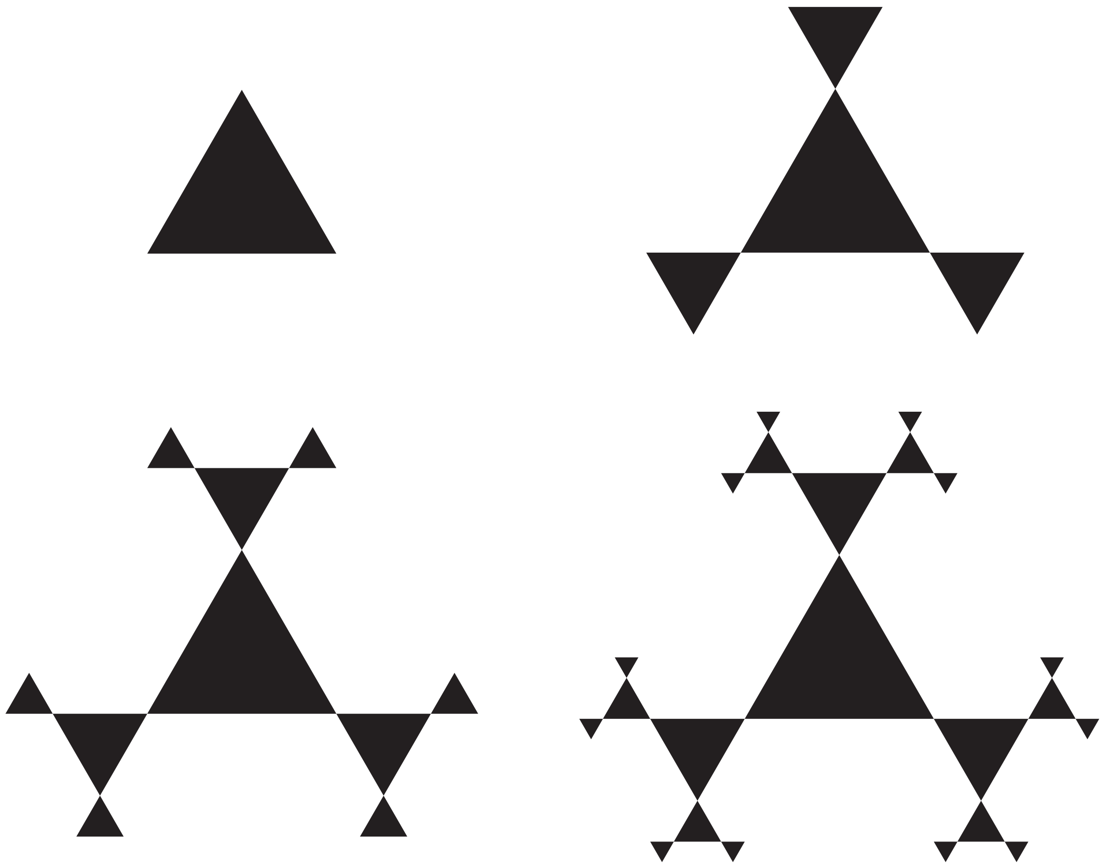
$$\frac{1}{2} = f^2 + f^3 + f^4 + \ldots = f^2\,(1 + f + f^2 + \ldots).$$

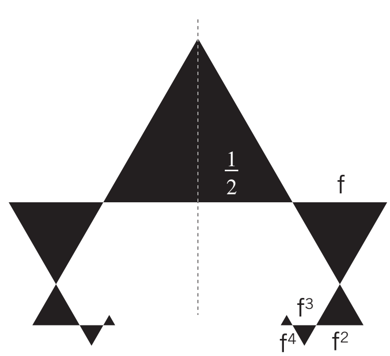
With the sum of the geometric series we get

$$1 = \frac{2\,f^2}{1 - f}\,, \quad \text{and finally} \quad 2\,f^2 + f - 1 = 0.$$

This quadratic equation has one positive solution, namely $f = \frac{1}{2}$.
Let’s summarize: If we start with a square of side length 1 and choose the reduction factor $\frac{1}{2}$, then the separate branches of the fractal touch in the limiting case.

Now we start with an equilateral triangle of side length 1 and construct the corresponding fractal. The triangles have side lengths 1, f (< 1), f^2, f^3, …





For the touching condition of the two sequences of triangles we get the expression

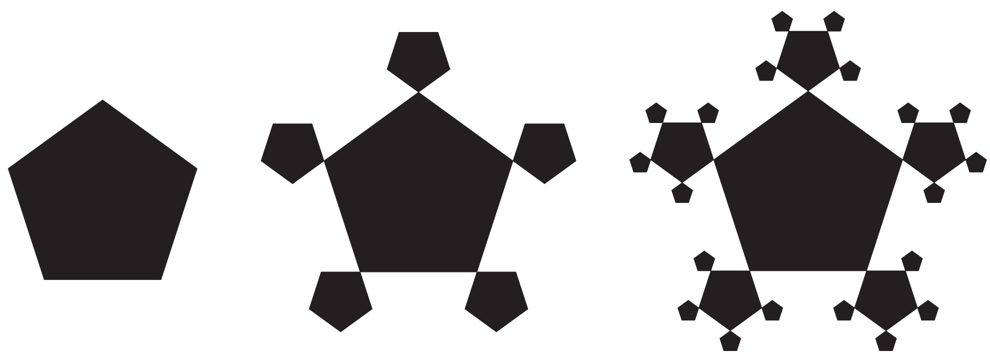
$$\frac{1}{2} + \frac{f}{2} = \frac{f^2}{2} + f^3 + f^4 + f^5 + \dots ,$$

and

$$1 + f = f^2 + 2\, f^3\, (1 + f + f^2 + \dots).$$

This equation has one positive solution $f = \frac{1}{\tau}$, where τ is the golden section, i. e. $\tau = \frac{1}{2}(1 + \sqrt{2})$. Exactly for this value for f the two sequences of triangles will touch in the limiting case.

In the next step we can consider regular pentagons, hexagons etc. We can investigate for what polygons the reduction factor f is a rational number and for what polygons it depends on the golden section.



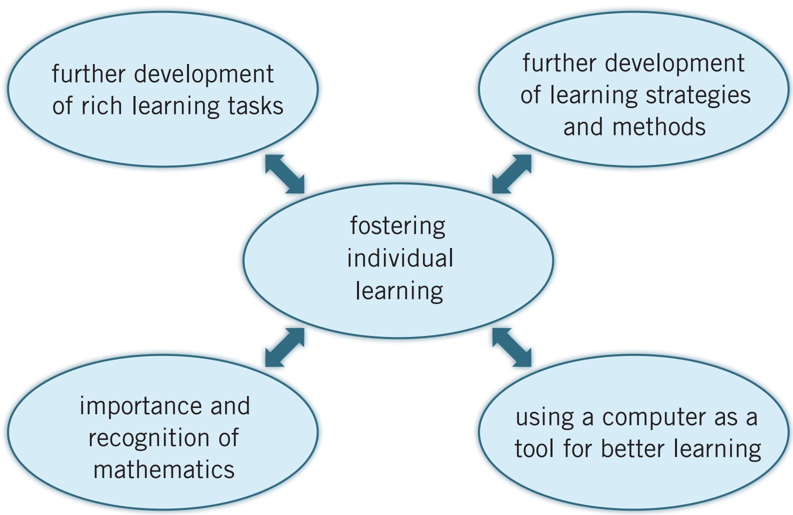
At the moment it is more important how we tackled the problem. Teaching takes place under a problem-oriented aspect. Again that means: Don’t preach facts, stimulate acts. We want to find out for what reduction factor f the branches of the polygon fractals will touch in the limiting case.
For this investigation students have to make sketches, they have to find algebraic equations for a geometrical problem. Nearly automatically, students learn to handle series (especially geometric series, conditions for convergence of series) and solve equations (quadratic and cubic equations). We do not start with formulas or theoretical considerations, we get them at most at the end of the learning process. That is what we understand by experimental mathematics.

SINUS Revisited

Back to the SINUS-project. Another article in the weekly newspaper DIE ZEIT describes very precisely:

> *"Realistic problems instead of mechanical computing, individual and independent learning instead of stubborn formula training. These are the characteristics of math teaching in SINUS. ..."*

Our main focus has been to foster individual and independent learning. To support this aim we have started further development of a problem culture and of learning strategies and methods. We have encouraged teachers to use the computer as a tool for better learning and we have stressed the importance of mathematics for our daily life and for our future.



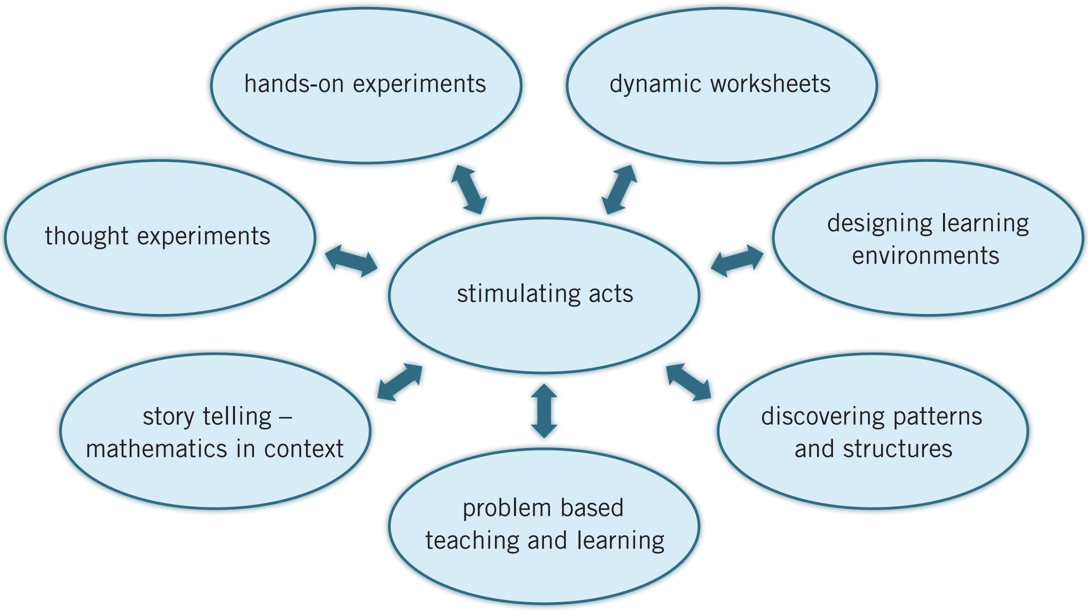
Further details can be found on our website www.sinus-transfer.eu. There also is a fantastic handbook *Teaching with Rich Learning Tasks* by Gary Flewelling. His and our teaching and learning philosophy complement each other perfectly. Strictly speaking our goal of mathematics education for the 21st century was already written down in the 19th century by Wilhelm von Humboldt (1767–1835):

> *"The student is mature, if he or she has learnt so much that he or she is able to learn independently."*

Even about 200 years old, this message is still very topical. Big ideas never loose their relevance.

Why Has Our SINUS-Project Been so Successful?

Many of our suggested ideas and methods of teaching and learning are not new, but before SINUS they were not implemented in our classrooms very often. What is the reason for the "wind of change"? Why have we finally succeeded in persuading teachers?
Changes in teaching and learning only make sense when they are accepted by the teachers and if they are built into the classroom routine. Therefore we have not preached facts to the teachers, we have stimulated acts.



Together with teachers we developed new materials and learning environments; and we discussed the new methods with them. Additionally we gave information on actual research of teaching and learning and also mathematical advice, if necessary.
I am convinced the basis of our success has been the strong integration of teachers. They have been an important part of our network but we have respected their independence simultaneously.

> Don't preach facts, stimulate acts – the same philosophy has worked for students and for teachers.

SINUS international

Towards New Teaching in Mathematics

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