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Towards New Teaching in Mathematics

Volker Ulm

# Teaching Mathematics – Opening up Individual Paths to Learning

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3 / 2011
ISSN 2192-7596
University of Bayreuth
www.sinus-international.net

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**Issue 3**
Volker Ulm
Teaching Mathematics – Opening up Individual Paths to Learning
Bayreuth, Germany
2011

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**Publisher**
Centre for Mathematics and Science Education (Z-MNU)
Chair of Mathematics and Mathematics Education
University of Bayreuth
95440 Bayreuth
Germany

**Translator**
Reineke-Team, Heidelberg
www.reineke-heidelberg.de

**Layout**
Carsten Miller
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**www.sinus-international.net**

**ISSN 2192-7596**

SINUS international

UNIVERSITÄT BAYREUTH

SPONSORED BY THE
Federal Ministry of Education and Research

Volker Ulm

# Teaching Mathematics – Opening up Individual Paths to Learning

## Contents

# O. Introduction

*A student is mature when he has learned enough*
*to be in a position to learn for himself.*

*(W. v. Humboldt)*

## What is the purpose of this text?

This text addresses teachers of mathematics at all types of school offering secondary level education. It offers ample food for thought about how to teach mathematics. Here you will find ideas and strategies for everyday instruction, plus numerous tried and tested problems/tasks and materials that you can put to immediate use in the classroom.
Of course, you are free to consider the text all on your own and just for yourself. However, its impact will be greater if you discuss the content with fellow teachers and relate it to developments taking place at your school. Everyday classroom experience forms the basis for an exchange of ideas among colleagues. It enables them to analyze the problems involved in teaching mathematics and to design hands-on strategies for the further improvement of mathematical instruction at their respective school. The purpose of this text is to provide support and guidance in this endeavor.
Changes at a school cannot be achieved in the short term, nor can they be brought about by "lone wolves". Instead, we need cooperation over a longer period of time involving practically all members of the math teaching staff. The ideas thus established will be grounded on a broad fund of experience and correspondingly sustainable in their effect. Working with this text will enable math teachers to make progress towards classroom teaching at its best.

## Guiding strategy

This text focuses on students' attitudes to learning in mathematics classes. The approaches explained and discussed below form an integral part of the central strategy *of enabling students to find their own independent approaches to learning*. The individual chapters consider the teaching of mathematics from different perspectives.
Chapter 1 considers problems besetting conventional mathematics teaching, i.e. problems of the kind diagnosed in international assessment studies. Chapter 2 casts a glance at teaching methods that allow students latitude for individual approaches to learning. On that basis, chapter 3 shows that open-ended problems offer sustainable approaches to the effective enhancement of students' independent and cooperative learning processes in everyday math classes. Finally, chapter 4 introduces station learning and educational projects as superordinate forms of open-ended teaching.

## Advanced teacher training for the math faculty

How can this text assist you in getting things moving at your school? One thing you can do is to join up with the other members of the math faculty and organize advanced training sessions devoted to the issues discussed in the individual chapters. The text supplies you with a host of suggestions on how to set about discussing mathematics instruction and how to explore new avenues in the teaching and learning of mathematics. After such sessions, you should test worthwhile ideas in your own classes. The experience you gain can subsequently be fed into the overall school-development process.
After some time, the opportunity will present itself to organize and share with the members of the math faculty advanced training sessions devoted to the issues discussed in the individual chapters. You will also begin exchanging materials that you and your colleagues have elaborated (unless this has already been standard practice among the teaching staff).

## The SINUS project

Many of the ideas, materials, and examples presented in the following texts are based on the SINUS pilot study “Increasing the Efficiency of Mathematics and Science Instruction”. Between 1998 and 2007, almost 1,800 German schools set about developing and exploring new paths towards teaching and learning in mathematics and science classes.

# 1. Defining Where We Stand

The first chapter casts a glance at problem areas in mathematics teaching. This gives you food for thought about mathematics as a subject and ideas for discussions with your colleagues at school.

## 1.1 For starters: two problems taken from assessment studies

As a result of international assessment studies such as TIMSS, PISA, etc., schools and education in general have hit the headlines and become a subject for public and even political discussion.
These studies attempt to analyze the educational system and have shown - just like everyday classroom experience - that mathematical instruction often fails to produce the results intended or desired. Students lack a basic comprehension of mathematics. Their problem-solving skills and/or the ability to work independently are inadequate.
Let us start with two sample problems typically addressed by assessment studies of this type. They are taken from PISA 2000 for students of the age of fifteen (cf. Deutsches PISA-Konsortium 2001, OECD o.J.).

**Glassworks**

A glassworks produces 8,000 bottles a day.
2 % of the bottles are faulty. How many bottles are below standard?

- ☐ 16 bottles
- ☐ 80 bottles
- ☐ 400 bottles
- ☐ 40 bottles
- ☐ 160 bottles

**Surface area of a continent**

This is a map of the Antarctic. Use the scale of the map to estimate its surface area.

Write down your result and say how you arrived at it.

Let us be clear in our minds as to what is being tested here. The first task is a test of basic knowledge. The second one requires students to tackle a non-standard problem courageously. They do not have a ready-made formula for calculating the area of the Antarctic but are required to use their basic knowledge about surface areas and other available mathematical "tools" to find their own path to a solution.
Such tests are expressly *not* designed to check age-appropriate knowledge based on curricula, as is the case with achievement testing at school. Rather, they are based on the assumption that after years of mathematical instruction students should have acquired a certain degree of basic mathematical comprehension that they can put to active use.
This is exactly where the second task differs from the conventional teaching of mathematics. The test calls for solutions that had not been taught in exactly this way! Of course, all the relevant aspects had been dealt with in the classroom at some time. But the students had not learned to keep these fundamentals available at all times and to work on problems in unusual contexts based on mathematical comprehension and solution-oriented thinking. This is why the findings came as no surprise.

> Discuss the following statement with your colleagues: "Students have not learned to keep fundamental knowledge available at all times and to tackle non-standard problems courageously."
> What do you feel to be unsatisfactory in your lessons?

Let us have a look at conventional math instruction.

## 1.2 The skewed perception of mathematical instruction

**Dominant role of the question-and-answer approach**

Speaking generally, but none the less appositely, one can fairly say that conventional mathematical instruction takes place mainly in small stages and in the form of questions and answers. Of course, there are many positive new approaches for the enhancement of mathematical instruction. But so far they have failed to make any major effect.
Mathematical instruction normally takes place in the form of small stages in which the teacher leads the way and the students (ideally) follow step by step. In the usual question-and-answer game, demanding and complex problems are regularly whittled down to a sequence of short questions and simple answers and spoon-fed to the students in small helpings. No time is "wasted" in acquainting the students with the subject matter in hand. The material is presented in a streamlined form. Obstacles and digressions are avoided with the help of the teacher.
This type of instruction may be justified and effective in certain specific situations. But if it is the predominant teaching method, then we have a problem. What it does is to keep students on an inappropriately tight rein. All they are asked to do is to accept and reproduce what they have been taught. This restricts the students' independence and mental flexibility and impedes effective and individual development of cross-linked knowledge.
Moreover, the students gradually develop a tendency toward inertia and lethargy (because ultimately everything will be done by the teacher). When confronted with non-standard or more broadly conceived problems, they display an incapacity for independent thinking and react helplessly. A typical reaction would be: "We haven't done this yet!" or "I can't do this!"

**Segmentation**

This method of teaching on the basis of close guidance is accompanied and reinforced by a high degree of segmentation in teaching content. One topic after another, one chapter in the textbook after another are dealt with thoroughly, carefully studied, tested - and forgotten. Temporary islands of knowledge are created where students usually store the material presented to them over a maximum of two months. Only very few students will build up a sound, well-organized knowledge base over the years and develop fundamental mathematical comprehension enabling them to apply mathematical concepts to the solution of problems in varying contexts. For most of the class, mathematics will be nothing other than the automatic application of formulas they have hardly understood and the rehearsal of facts they have superficially memorized.

**Teaching tradition**

No one is to blame for this situation. It has developed in the classroom and has been supported by basic educational conditions and curricular structures. Both teachers and students have *adjusted* their working practices *to this situation*.

Current assessment studies should therefore not be taken as a starting point for criticizing teachers or students. They analyze educational systems and assess their efficiency by asking: Which sustainable mathematical skills have students been taught by individual educational systems?

**Two fundamental questions**

Today schools are frequently called upon to change. In this context, however, we have to answer two fundamental questions:

- "Where do we want to go?" and
- "How can we get there?"

There is no simple answer to the "how" question. In the following text you will find new ideas and food for thought. First of all, however, the "where to" question should also be considered.

> What are the higher goals we want to achieve with mathematical instruction? What do we want to teach our students in the long term?
> Discuss these questions with your colleagues!

We will come back to these questions at a later point (see postscript).

## 2. Teaching Methods Put to the Test

> In this chapter, we take a look at teaching methods. A "typical" classroom situation will help us locate the methodological problems posed by mathematical instruction. We discuss how learning can be organized to provide sufficient scope for individual learning processes.

The fundamental question raised in the first chapter is: How can mathematical instruction respond to the deficits listed there? How can the desired goals be achieved?

Schools and learning are complex fields, and there are no simple answers. The following appeal made by the American mathematician Paul Halmos indicates one possible direction:

> *The best way to learn is to do – to ask and to do.*
> *The best way to teach is to make students ask and do.*
> *Don't preach facts – stimulate acts.*

This brings us to the central topic of this text. What matters is independent and autonomous learning. Students must find their own independent approaches to learning.

## 2.1 Working independently

The basic consideration here is the fact that learning is a profoundly individual process, a process actively devoted to the construction of knowledge. From the outside it can be only be controlled to a limited extent. Learning processes as such take place inside individuals, and those individuals weave their own personal thinking networks. Individuals must generate knowledge and comprehension by themselves. You cannot fill brains with knowledge the same way you transfer data to a USB stick.

If we perceive learning as an activity, we must think about the conditions under which this activity and associated processes in the brain can best be performed (cf. Spitzer 2003).

A common idea of how mathematical instruction takes place is the following: The teacher plans and organizes, explains, asks questions and makes corrections, provides structure and visualization, expounds the problems, presents solutions to those problems, accepts responsibility – and is made responsible for everything.

The most firmly established teaching method is as described in 1.2, i.e. small steps consisting of questions and answers. The teacher takes the lead, the students follow (in the ideal case), teacher shows how, students repeat (in the ideal case).

Good lectures delivered by teachers and well-thought-out, strictly organized lecture teaching may of course be justified in math instruction and provide meaningful elements for inclusion in the learning process. There is no denying this! But learning in school also requires guidance from the teacher as a specialist in the relevant activity. Guidance in this context does not mean that students be regarded as passive recipients of knowledge. Instead, it means that the teacher creates a learning situation in which students deal with mathematics in an active manner, finding independent approaches to learning.

This comparison illustrates the point. No one learns to play the piano by watching a master pianist for years or by being left alone with the instrument. What a piano pupil needs is a teacher who chooses suitable piano pieces for practising, provides instruction and help, addresses inaccuracies, shows how to make progress, and imbues the learning atmosphere with his/her own personality. In such a setting, learning to play the instrument involves a high degree of independent work to be performed by the piano student.

Accordingly, there must be a shift in focus if mathematical instruction is to be successful: less guidance from the teacher, more independent, active, and constructive learning by the students. This text largely addresses the latter aspect, while of course conceding that guidance of the right kind may nevertheless be justified for a balanced teaching approach.

There are many ways of letting students find their own approaches to learning. Large-scale methods for working independently are instructional projects or station learning. These types of open education provide students with a setting for independent, self-organized, and cooperative work over lengthy periods of time. This will be dealt with in chapter 4. However, such large-scale methods are usually not the main component in mathematical instruction. Also, they assume that the students already possess self-monitoring and methodological skills, but such skills have to be acquired beforehand in a series of small stages.
In everyday math instruction, it is therefore important to create many small *islands for independent approaches to learning*. These islands give the students scope to work independently on what is traditionally done under the close guidance of the teacher.
This will be our initial focus. Here, even minor changes may prove very effective.

## 2.2 A typical teaching situation in class

Let us now leave the theoretical sphere and reconstruct an actual teaching situation in the 5th year of secondary school. We draw upon role play to make it even more trenchant.

> Re-enact the following teaching situation with your colleagues. One colleague is the teacher, the others the students. The teacher should try to convey ideas and guidance to the students in virtually the same way as described below.

The textbook contains the following standard task, which is designed to rehearse and consolidate the teaching material with reference to a workaday example:

**Paving a path**

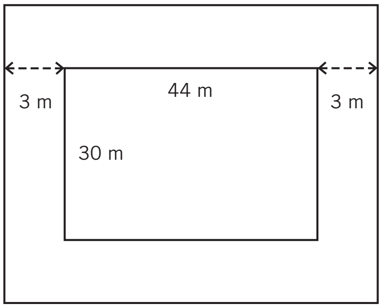
A path 3 m wide is to be placed around a rectangular lawn 44 m long and 30 m wide. What is the surface area of this path?

*Teacher:* Who wants to read out the problem?

*The problem is read aloud by a student.*

*Teacher:* Let us draw a sketch of this task.

*Teacher draws the following sketch on the blackboard:*



*Teacher:* The task is to calculate the surface area of the strips of path around the lawn in the middle. Any suggestions?

*Student:* …

*A classroom discussion develops. The first suggestion likely to lead to the solution is taken up and developed. Example:*

*Teacher:* So the path consists of four strips 3 m wide.

*Teacher highlights the strips in the sketch.*

*Teacher:* How long are the left- and the right-hand strip?

*Student:* 30 m.

*Teacher:* And what is the length of the upper and the lower strip?

*Student:* 47 m.

*Teacher:* Okay, but in the center we have 44 m plus 3 m each on the left and the right. That means that the upper strip is how long?

*Student:* 50 m.

*Teacher:* Okay, then let us write down our results.

*Teacher turns to the blackboard and writes down the following:*

| | |
|---|---|
| Upper/lower side: | 50m × 3m = |
| Left-/right-hand side: | 30m × 3m = |
| <u>Total</u> | A = |

*Teacher:* And now solve the rest of the problem on your own!

Discuss this short teaching sequence with your colleagues.

This is an alarming example because it shows that the so-called independence of students is in fact nothing but “pseudo-independence”! By drawing the sketch himself, the teacher is already anticipating many aspects of the problem. This also means that a fundamental step is withheld from the students, that of developing the mathematical model. First of all, students must understand the text, form a notion of the situation described, translate it step by step into a geometrical drawing, and find their own approaches to the solution. Absolutely fundamental activities are required right from the outset, and these must be performed by the students on their own. Otherwise they will not derive any real benefit from the problem.

From the students' perspective, many of them experience the problem as follows: Even before the text has been understood completely, a sketch has appeared on the blackboard. So they copy the sketch line by line while at the same time the question-and-answer game starts – largely without any involvement on their part. When they have finished copying the sketch, they start to write down the findings written on the blackboard by the teacher, although they may not have really understood them. Then all they have to do is the simple task of calculating 50 m × 3 m and 30 m × 3 m.

The usual procedure is to "approach" a problem by discussing not only the problem but also its solution. With a little help from the vocal "hard workers" in the class, the teacher hastily explains and structures the problem and indicates possible approaches to its solution. This procedure takes no account of the fact that students need time and silence to tackle the problem on their own. Many students are cheated out of the manifold approaches to learning that a problem can offer. Teachers should therefore restrain themselves from time to time. Too much input from the teacher means too little independence for the students.

The following is a second variant of the teaching situation. Here, importance is attached to independent and self-organized work.

*Teacher:* Who would like to read out the problem?

*A student reads the problem aloud.*

*Teacher:* You've got 10 minutes to solve this problem. If you get stuck, you can ask your neighbour for help.

*During the first three minutes, the teacher sits at his desk. Then he walks through the classroom, addresses individual students and provides support if needed.*
*After roughly 10 minutes (depends on the situation in the classroom):*

*Teacher:* Okay, the 10 minutes are up. Who would like to present his or her ideas on the blackboard?

*Four or five different ideas are presented, discussed, and compared.*
*Depending on the course of discussion, it may be necessary for the teacher to make sure that the solution is presented completely and clearly on the blackboard. If you think that the students have developed a broad comprehension of the problem, you can do without such a presentation.*

As a matter of fact, such a presentation of results cannot involve all the students to the same extent. However, if teaching phases like these take place in the classroom on a regular basis, "justice" can be brought about over time.
Decisive is the approach taken to a problem from the textbook. The second variant offers students a far greater chance of exploring the problem on their own and finding independent approaches to learning, including a degree of meandering or "straying", which in fact can be useful.

Comparing the first teaching situation with the second should deliver enough material for discussion. Discuss your ideas and thoughts with your colleagues.

**Time as an objection**

One possible objection to the second of these teaching styles might be that it is too time-consuming, i.e. that in this way one would never manage "to get through" the material specified in the curriculum. However, experience gained from the German SINUS-project for innovations in the classroom has shown that in fact the opposite is the case (see also 2.3 and 2.4).

Discuss the following questions:

- What makes better sense: to "work on" three problems during a lesson using a strongly teacher-focused approach, or to have two problems explored by the students as independently as possible in the same time?
- How much knowledge from practice phases is still present after two months, after one year, and after leaving school?

## 2.3 Japanese mathematics classes: basic approaches

Let us take a brief look at Japan, whose students did so well in TIMSS and PISA. (The following is not meant to suggest that we should choose Japan as a role model or should imitate Japanese lessons. In this regard the cultural differences are much too large. However, this should not prevent us from keeping our eyes open for inspirations for *our own* teaching.) A technical and didactic analysis of the TIMSS video study revealed the following approach to be a basic pattern for Japanese mathematics classes. While it is certainly not characteristic of all classes, it is typical of very many.

1. Pose a problem and ascertain that the problem presented has been understood completely.
2. Have students work independently, either individually or in small groups.
3. Collect the different solutions and discuss them.

Of course, this is a simplified description of the methodological design, but it does highlight the crucial features.

This design corresponds to the second teaching variant described in the previous section. The essential difference between the first and second variant already becomes apparent in the progress from item 1 to item 2. Once the problem has been understood, students begin to work on their own. They find their own independent approaches to learning. There is no discussion of possible approaches to a solution, nor does the teacher provide any guidance. Typical stimuli given in traditional classes are absent (e.g. "Let us draw a sketch…"; "Let us consider what this is about…"; "How could we go about solving the problem…"; "Who has an idea"…; "Suppose we assume….“). The students work on the problem seriously without outside assistance. A process like this supports autonomy on the part of the students. In most cases it will automatically generate different approaches to a solution (cf. Baptist 1998).

This way of teaching contrasts sharply with an approach dominated by suggestive questions and answers. In such a narrowly conceived question-and-answer game, most students are frequently confined to "studying" a problem by inserting single words or piecemeal ideas into a pre-determined train of thought. Or they are no more than passive observers.

Can this Japanese approach to mathematics classes provide inspiration for your teaching?

Here is an additional example taking us in a different direction.

Work on the following problem with your colleagues!

This chair was included in the *Guinness Book of Records* as the world's largest chair.

1. How tall would a giant have to be to use this chair?
2. Discuss your ideas with your neighbour.
3. Together with your neighbour, tell your fellow-students about your ideas and the results you've come up with.

Foto: XXXLutz St. Florian, © XXXLutz

When presented with this problem in a session with colleagues, you will presumably go through the following stages in your solution-finding process:

- First, you study the problem for yourself and figure out your own ideas on how to arrive at a solution.
- Next, you exchange ideas with your neighbour, compare your results, and discuss how the two of you went about solving the problem.
- Finally, you compare approaches and results with all members of your faculty.

In this way, you can personally experience a very natural process of learning and problem-solving.

Let us explore this structure at a more general level.

## 2.4 I–you–we: a principle for learning and working in mathematics classes

The problem about the giant chair referred to above illustrates the I-you-we triad. Peter Gallin and Urs Ruf from Switzerland introduced this concept. In the last few years, the two authors have published a number of books about how to teach German and mathematics. One example is *"Dialogisches Lernen in Sprache und Mathematik"*, volumes 1 and 2 (cf. Literature). Their "I–you–we" concept shows how learning and working in school can be organized and structured in a way that triggers effective and sustainable individual learning processes.

| |
|---|
| **I: Working individually**<br>Every student sets out on her/his own to explore an issue or a problem. He/she relates the task at hand to him-/herself, to his/her existing knowledge and follows his/her own course toward a solution. |
| **YOU: Learning with a peer**<br>Every student communicates with a peer, explains his/her ideas, and reconstructs the ideas of his partner. This makes for a deeper appreciation of the issue in hand. The peers continue to cooperate in working out a solution. |
| **WE: Communicating within the class team**<br>The working groups present the results they have obtained to the entire class, who then discuss them. Everyone's contribution is considered, and the final outcome is the product of genuinely concerted effort. |

**The "I" phase: working on one's own**

The path that leads to comprehension is the path we choose ourselves. First, students need to comprehend the task at hand. They must find their own orientation and develop a feeling for the challenge posed by the task. The next thing for students to do is to find out how the issue relates to their pre-existing knowledge, develop strategies and ideas for a solution, and finally implement them.

Analysis of such orientation and working processes has indicated that they are profoundly individual. In his reflection processes every student follows his own routine (existing knowledge, thinking patterns, problem-solving strategies etc.) at his own speed. So it obviously makes better sense and produces better results if every student goes through this individual working phase on their own.

**The "YOU" phase: learning with a peer**

Here, the focus is on cooperation and communication with a neighbour, the "YOU". (In bigger classes, working groups of three or four students are also conceivable.) The students are asked to present their ideas and results in an understandable way and, vice-versa, to consider the ideas of their peer(s).

Such an exchange encourages subject-related learning in several ways. Active communication results in profounder appreciation of the subject matter. In addition, the peer can help rectify

faulty comprehension or assist in activating basic knowledge, developing further ideas, and coping with problems as they arise.
Cooperation of this kind also enhances social skills. It induces students to listen to each other, collaborate, give mutual support, discuss things, deal with diverging views, and make compromises.

**The "WE" phase: communicating within the class team**
Two things happen in the "we" phase. The working groups present their thought processes and results to the entire class and develop a joint solution with the guidance of the teacher acting as a knowledgeable expert. The solution brings together the outcomes produced by the students, possibly expands on them, and situates them in the subject-related context.
The students practise talking about mathematics, presenting results, and speaking in front of an audience, i.e. their classmates. Many students are ill at ease when asked to do the latter, preferring to hold back and stay silent. To alleviate such anxieties it is, of course, necessary to foster a community spirit and a feeling of mutual trust in the class. Another necessary prerequisite is for the teacher to regularly create situations in which students can experience success in public speaking in class.
Of course, no one can assume that the student presentations will always be perfect in terms of content, presentation skills, and understandability. However, this is not a deficiency. We can learn from mistakes. This sounds like a truism, but the operative assumption is that mistakes are allowed and have their place in the classroom, In this way, students need not fear bad marks or ridicule and laughter from their classmates.
If these matters are discussed in a considerate way, for example by asking the regular, standardized question: "What was good, what could have been better?", any criticism voiced will not be hurtful but will help improve not only mathematical comprehension but also the presentation skills of all students involved.
Once students have explored, intensely and on their own, a novel course of going about solving problems or marshalling their thoughts, we can finalize the "I–you–we" triad by summarizing or expanding the suggestions made by the students and presenting a joint result achieved by concerted effort. The students will then be "mature" enough to appreciate the assurance of their results with reference to mathematical conventions, the broader subject-related framework, and considerations associated with the curriculum.

**So what does the teacher do?**
Classes structured in this way mean that the teacher needs to assume a different, but not necessarily an easier role. During the "I" and "you" stages, the teacher establishes the modus operandi and mostly acts as a consultant approachable for individual students needing his assistance and helping them to help themselves.
During the "we" phase, the teacher's initial role is that of a moderator organizing the students' presentation of results and the ensuing discussions. Here it is important for *students to communicate with each other* and that discussions do not focus on the teacher.
Ultimately, the teacher's guidance will enable the students to combine and/or expand their contributions to produce an overall result that is mathematically sound. Central ideas and findings are placed on a broader basis and remain available to fall back on in future.

**Summary**

Of course, this "I–you–we" strategy is only one method among many. It is appropriate whenever student skills need to be established or reinforced, both in training phases and when new subject matter is involved. However, it would be exaggerated to design all work in class along these lines. So what is so special and desirable about this idea?
Students develop their *own individual* "reflection routine" before a "sample" solution is discussed with the entire class. They integrate the new material into the knowledge they already have before the teacher starts structuring and explaining. This differs essentially from the closely guided, question-and-answer type of instruction. In this sense, the "I-you-we" principle is a method that empowers students to explore *their own* approaches to learning.

**"I–you–we" and the acquisition of new knowledge**

The last two examples ("pave path", "largest chair") served to apply and to deepen the understanding of subject matter that was already familiar. The following example shows that the "I–you–we" concept may also be appropriate for gaining access to problems of a kind hitherto unknown.
Here students are introduced to trapezoids and develop methods of their own for figuring out their area. The formula for finding the area of a trapezoid is relatively complex compared with other formulae for area-finding taught in school geometry. Hence it is doubly important for students to achieve a basic understanding of the principles involved in determining the area of a trapezoid. Otherwise they will be superficially – and often wrongly – applying a formula they have hardly understood.

**Specific Quadrilaterals**

1. In your notebook, draw sketches of as many different quadrilaterals as you can think up in which two of the four sides are parallel.
   Invent a name for such quadrilaterals.
   Calculate the area of the quadrilaterals you have sketched.
   Think up the easiest general method for determining the area of such quadrilaterals. Write down your train of thought in your notebook.
2. Explain your considerations to your neighbour. Discuss with him/her the results you have arrived at and integrate your results into a joint solution.
3. In the classroom, present your thoughts and results to your classmates. Integrate the presentations made by other groups into your own work.

After so much that is new, it makes good sense to pause for a phase of reflection.

> Compare the Swiss principle of "I–you–we" with the basic design of Japanese math lessons as described in 2.3. Where do you see commonalities and differences?
>
> Test the "I–you–we" principle with the students in your own classes!

## 3. Open-Ended Approaches: Working Independently in Class

So far, we have focused on questions of methodology in teaching mathematics classes. Let us now turn to actual concrete examples of tasks/problems. When we ask students to think and work independently, their behaviour is, for the most part, determined by the problems posed, i.e. in the form of homework or classroom exercises. Accordingly, there is a great deal of "mileage" to be got out of alternative ways of presenting tasks or problems.
Our concern here is to make problems approachable and in our everyday teaching to find easy ways of inducing students to take "small steps" toward working independently and/or cooperating with others.

The "biggest chair in the world" problem casts light on another fundamental aspect. Students may feel it to be unsatisfactory that a problem does not have one single "correct" solution.
Tasks specifically promoting independent approaches to learning are open-ended. This gives students a degree of independence or autonomy in working on them. Open-ended tasks do not have one single correct solution. They outline a situation that we can discuss in mathematical terms. They invite students to engage with the mathematical side of it, and they offer several paths for achieving different solutions of equal quality.
Let us look at some examples to get a feeling for what open-endedness means. This will also give you the opportunity to identify strategies for generating open-ended tasks to be used in everyday mathematics teaching.

### 3.1 Asking questions

One way to generate open-ended learning situations is to have students ask questions about situations describable in mathematical terms. Pictures can be a good starting point. They offer students attractive access to a specific subject. Here is one example:

**Soccer balls**

1. Starting from this illustration, ask as many mathematical questions as you can think of.
2. Discuss your ideas and results with your neighbour!
3. Together with your neighbour, discuss the most interesting results with the rest of the class.

Foto: Franz Beckenbauer, Oliver Bierhoff, © Deutsche Postbank AG

Some questions that spring to mind are:
- How big is a soccer ball?
- How big is a football field?
- How many balls fit into this football field?

Of course, students are not expected to be able to say with certainty how big a soccer ball or a football field are. However, mathematics lessons do (and should) confront students with general ideas about sizes and dimensions. That means that they should be able to estimate the diameter of a football to be around 25 cms. and the size of a football field to be approximately 100 m. by 50 m. On the basis of such assumptions it is no longer difficult to estimate the number of balls in the field. This provides students with a degree of success in applying mathematics to the situation shown in the picture and gives them the courage and confidence to tackle more complex questions such as:
- What is the total value of all the soccer balls?
- How long would a person need to pump them all up?
- How long would a person need to place the balls on the field?
- How many trucks would be needed to take all the balls away?
- What arrangement would be best for getting as many soccer balls on the field as possible?
- …

Answering questions like these involve substantial mathematical thinking. For this task again, the suitable method to pursue is the "I–you–we" principle described in section 2.4. For the "you" phase (working in small groups), it is up to the teacher to decide whether to go for same-level (homogeneous) or different-level (heterogeneous) achievement groups.
But it doesn't have to be pictures. Ordinary, everyday selections of data (train schedules, restaurant menus, statistics, etc.) can be just as useful in posing and studying mathematical issues. The following example supplies students with the prices of a ski-lift as a subject for consideration and at the same time as a starting point for mathematical work. The idea underlying this problem was developed and tested at a Baden-Württemberg (Germany) secondary school in the framework of the SINUS-project (cf. LEU Stuttgart 2001, p. 40f).

The Miller family wants to go skiing during the February recess.
They collect information on prices.

**Downhill skiing Winter 2002/2003**

| **Lift tickets** | **Adults** | **Children** | |
|---|---|---|---|
| 5-day ticket | 105 € | 75 € | |
| 3-day ticket | 72 € | 51 € | |
| 1-day ticket | 27 € | 19 € | |
| Afternoon ticket | 16 € | 11 € | (from 12.30 p.m.) |

**Family package** (Lift pass valid for all family members)

| | |
|---|---|
| 5 days | 333 € |
| 3 days | 222 € |

Consider which questions to ask and then answer them.

The task is not closely defined. It merely provides the "bones" of a situation. The students must properly understand the situation before they can devise questions and do any arithmetic. As posed, the problem makes it difficult for students to embark on more or less arbitrary calculation routines, cross-linking the numbers in the figure in a haphazard manner.

Which questions could be discussed on the basis of this situation?

The following problems are conceivable:

- How much do you save if you buy a 3-day ticket as opposed to three single-day tickets?
- Why is a 3-day ticket cheaper than three single-day tickets?
- Is the family package good value for money?
- What is the minimum number of members required for the family package to pay off?
- What would the lift ticket cost for the respective student's own family?
- Would it pay to go skiing only in the afternoons?
- What are the best tickets to buy if the family plans to stay for 6 days?
- What is the best buy for a stay of 7, 8, 9, ... days?

Traditionally, the teaching of mathematics involves very detailed questions prodding the student into give the answers the teacher wants to hear. In such instances, the activities undertaken by the students are often restricted to seeking and applying those problem-solving procedures that can be deduced from the current instruction sequence.
However, we also should get our students used to more open-ended tasks or problems. After all, the problems they will have to solve later in life will rarely come with detailed instructions on how to solve them.
In structuring classes, teachers could either go by the "I-you-we" principle or design their classroom lessons along Japanese lines. Such approaches also offer many opportunities for "internal differentiation" in a class (grouping students into fast- or slow-track performers). Weaker students will ask easier questions, more advanced students will tackle more complex problems.
Data material usually offers an excellent basis for mathematical considerations, posing questions, and initiating discussions. Here is a task to present to more advanced secondary education students.

| Year | Population in bn. |
|---|---|
| 1900 | 1.65 |
| 1910 | 1.75 |
| 1920 | 1.86 |
| 1930 | 2.07 |
| 1940 | 2.30 |
| 1950 | 2.52 |
| 1960 | 3.02 |
| 1970 | 3.70 |
| 1980 | 4.44 |
| 1990 | 5.27 |
| 2000 | 6.06 |

**World population development**

The United Nations homepage contains all kinds of information about the development of the world population (www.un.org/popin). Some of those data are shown in this table.

1. Think up interesting questions to ask about this topic and try to answer them.
2. Use the web to gather information about world population development. Assemble your data in a clearly arranged form and present them to your classmates.

What considerations might be prompted by these data?

You can also use a text with implicit mathematical content to serve as a starter, as in the next example (cf. Herget. Scholz 1998. p. 113). The wording is taken from an achievement test presented to 6th graders. As long as open-ended tasks constitute an integral part of mathematical instruction, their inclusion in achievement tests is fully justified.

> The following article is taken from the German daily *Braunschweiger Zeitung*.
>
> **1,575 steps climbed in eleven minutes**
>
> NEW YORK. American mountain climber Al Waquie is the winner of this year's Empire State Building run-up.
> The 32-year-old took 11 minutes and 29 seconds to climb 1,575 steps and 86 flights of stairs.
>
> Think of a mathematical question and answer it.

It is interesting that teaching mathematics in grade school usually involves getting students to think up their own questions and answers with reference to given situations. This is reflected in textbooks displaying pictures or data material and asking students to consider pertinent questions. Either that or there are word problems with no questions at all. To illustrate this, we present two examples taken from textbooks for 2nd and 4th graders respectively.


(taken from: Kolbinger. K.-H. u.a.: Nussknacker. Unser Rechenbuch für Klasse 2. Ausgabe Bayern. Klett Verlag. Stuttgart 2001. p. 87) – (Our mathematics textbook for the 2nd grade, Bavarian edition)

Consider questions to ask and answer them:

- Nicola wants to purchase the following items: a hiking tent (85 €), an airbed (15.90 €), and a sleeping bag (79 €). She draws the amount required from her savings account, which previously showed a balance of 484 €.
- The German radio station SWR charges a minimum of 4.79 € per second for commercials. The "Alles klar" cleansing agent commercial takes 9 seconds. The bookings are 5 times each day, Mondays through Fridays, 7 times for Saturdays, and none on Sundays.

(cf. Keller. K.-H.. Pfaff. P.: Das Mathebuch 4. Mildenberger Verlag. Offenburg 2002. p. 57 and p. 59)

Here, no explicit questions are asked. Students are challenged to get to the bottom of the situation first, to explore the mathematical content and to ask themselves what would be of interest in the given situation. All this precedes arithmetic proper. Students develop such skills as early as grade school. They should not be allowed to wither away in secondary education.

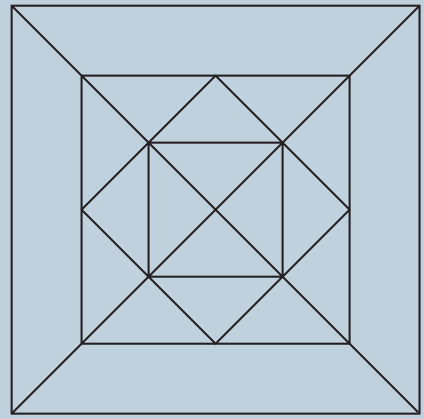
> I've heard colleagues voice the following objection: "It doesn't matter **who** poses the questions, I myself or the students. What really matters is that the students do their arithmetic."
> What do you think about this statement?

## 3.2 Exploring objects in mathematical terms

Open-ended tasks are any tasks where students are asked to explore objects and to discover and investigate their mathematical properties.
The following examples can be used for teaching subject matter at different grade levels: symmetry, triangles, quadrilaterals, circumference, area content, percentage calculation, central dilatation, ... (cf. Baptist 2000 and Wurz 1998).

**Balcony railing**

Here we see the pattern of a balcony railing.



1. Explore this figure and discover as many mathematical properties as you can.
2. Pairing up with your neighbour, discuss the results you've come up with.
3. Together with your neighbour, discuss the most intriguing results with the rest of the class.

Geometrical figures are ubiquitous in math textbooks. Mostly, the figures shown are accompanied by questions couched in very concrete terms. The teacher can easily use them to create open-ended teaching situations without too much effort by initially ignoring the textbook questions. Instead, the teacher asks students to investigate the geometrical figures as such, to discover and explore their mathematical potential.

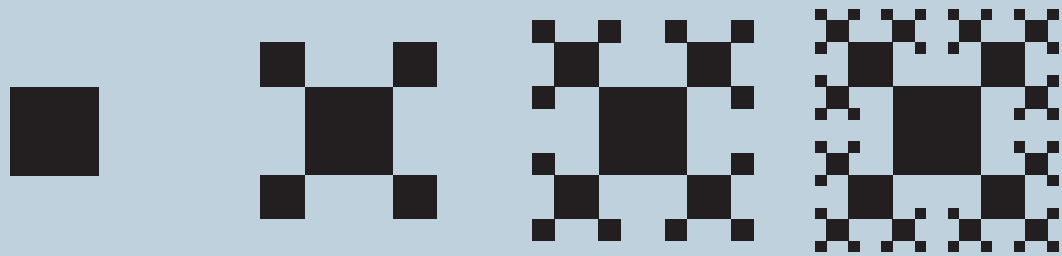
The example points in various directions.

- How many triangles can you find? How many quadrilaterals?
- What ratios do you find between triangles and quadrilaterals? What similarities are there? What percentage of the total area is taken up by the inner square?
- How long are all the rods in the pattern together?
- What is the weight of the railing element if its original size is 1 $m^2$? Could you lift it on your own?
- ...

The best thing to do would be to spend some time on this task and to explore it at several different levels. In the long term, "in-depth engagement" will be much more effective than just ticking off isolated problems.
The following geometrical patterns are easily grasped in relation to the educational objective reflected. But they also offer potential for in-depth exploration and discovery.

**Patterns resulting from squares**

Squares are used to generate a sequence of patterns. We start from a big square and add smaller squares at the four corners. Step by step, the side lengths of the squares diminish by a constant factor.



Pairing up with your neighbour, explore this sequence of patterns in as many ways as you can think of. Present to the class what the two of you have come up with.

A brief glance suffices to indicate that these patterns are an invitation to embark on mathematical considerations, pose questions, explore, try out, and discover things, in short to delve into mathematics. Depending on their imagination, interest and talent, students can explore multiple aspects:

- number of squares
- area content of patterns
- circumference of figures
- pattern overlaps or contacts
- symmetrical properties
- number of corner points
- smallest square including the $n^{th}$ pattern
- smallest square including all patterns
- behavior of the sequence of area content
- behavior of the sequence of circumferences
- computer program to generate such patterns

A lot of variations again give rise to a host of interesting questions. In place of squares, students might consider triangles, pentagons, or other polygons and assemble them in different ways. Students could expand the principle of how such patterns are combined to form three-dimensional objects by designing analogous objects made from cubes, tetrahedrons, etc., and exploring their volumes and surfaces, etc.

As they progress with their consideration of these patterns, students gain an intuitive idea of an "inferior limit" at the end of a converging sequence. But what really is the limiting point of such a sequence of geometrical figures? In what space do you really encounter a process with limiting values? Such questions result in substantial in-depth exploration for particularly enthusiastic and talented students.

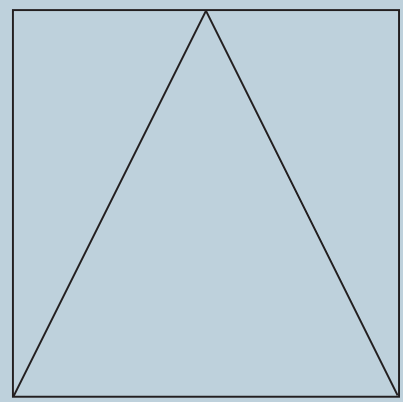
(The *Hausdorff metric* measures how far two subsets of a metric space are from each other. It turns the set of non-empty compact subsets of a metric space into a metric space in its own right. In this space, the present sequence of figures is a Cauchy sequence and subsequently

convergent. The limit value represents the end of the ascending union of all figures.)
At first sight, the following example is another rather common-or-garden geometrical figure that can serve as a starter for mathematical exploration and discovery. This idea was developed in the context of the SINUS program implemented at the Jacob-Grimm-Schule, Rotenburg, Germany.

**Triangle inscribed in a square**

Here you see a square with a triangle “fitted” into it.

1. Make at least five mathematical statements pertaining to the figure (e.g. area content, angles, ...).
2. Drag the upper corner of the triangle so as to obtain an equilateral triangle. What is the height of this new triangle? What percentage of the square is taken up by that triangle?
3. How can you drag the upper corner of the triangle so as to obtain a triangle taking up a quarter of the area of the square?



One example for use in advanced secondary education function analysis shows that we need not necessarily adhere to the rigid pattern of how to discuss curves. There is a way to open up such tasks.

**Explore functions**

Explore the family of functions

$$f_a(x) = \sqrt{x\,(a-x)}\ ,\ \ x \in \mathbb{D}_{max}$$

Given the family of functions with parameter $a \in \mathbb{R}^+$,
discuss as many properties of this family of functions as you can think of.

At first sight, this sentence leaves open what is to be done or if there is anything to be done at all. But taking a closer look at the problem, we find that the functions represent a family of semicircles. This can be discovered, described and demonstrated.
A classical approach would probably take the following course: “Discuss the maximum domain of definition, monotony, calculate the first and second derivatives, extremes, flex points, and sketch the graph for $a = 2$.”

Of course, such detailed instructions may be necessary depending on the teaching situation involved. However, it would be a shame if discussing functions always had to follow such closely described paths, if students were not regularly granted the freedom to explore their own learning paths.
A comparison with other school subjects shows that elsewhere open-ended tasks are used regularly. When students in a current affairs class are asked to discuss the emergence of the European Union, this involves describing a rather complex context, at least to some extent.
This also applies to the mathematical task we are looking at. Students are required to explore and fathom a mathematical object and to present the results achieved in a structured way. If a student determines the domain of definition and calculates derivatives but fails to recognize that the graphs are semicircles, then he/she will have missed a crucial factor.

The last of our tasks exploring the mathematical content of a situation links geometry with analysis. Here, elementary geometric considerations about the surface and the volume of cones lead to some attractive and rich functions for exploration with the help of analysis tools.

**Circular cone**

A sector of a circle is used to form a cone. Discuss the dependence of the dimensions of the cone (e.g. height, surface, volume) on the dimensions of the sector.

Students can consolidate and deepen their geometric insights by considering the coherence between the mathematical magnitudes for the circle sector (radius, central angle, arc length) and the magnitudes for the cone (generating line, altitude, base circle radius, base circle circumference) and establishing elementary connections (e. g. for the cone volume or the cone lateral area dependent on the sector dimensions), exploring them with the methods provided by analysis. For more advanced students, the topic also offers profounder challenges. The extremum problem (for which central angle of the sector – at a constant radius – does the respective pyramid volume achieve its maximum?) will prompt them to obtain a function taking the following form: $f(x) = x^2 \sqrt{1 - x^2}$ . An extended discussion of this function requires no more than the standard teaching of analysis in upper secondary education. However, a lot of mathematical understanding is required to do the actual calculations. Mirroring the graphs at the axes will give an algebraic curve described by the following equation: $y^2 = x^4 (1 - x^2)$.

Come up with further examples inducing students to explore and discover mathematical correlations.

## 3.3 Discussion

In Section 1.2 you were asked to consider the question of what our long-term aims are in teaching mathematics. You may have discovered that it is desirable to present mathematical correlations in verbal form or to evaluate the facts of a matter in terms of the mathematical factors operative in them. Such competencies are part and parcel of mathematical literacy (cf. Postscript). One especially good way of promoting them is by presenting students with open-ended tasks. The following example (after Herget, Scholz 1998, p. 32) was part of an achievement test carried out at the Ehrenbürg grammar school (*Gymnasium*), Forchheim.

> This brief report was found in the *Norderneyer Badezeitung*:
>
> > Some years ago, every tenth driver exceeded the speed limit at some time or other. Nowadays it is only every fifth. But even five percent of drivers are too many. So speed checks are still with us, and drivers exceeding the speed limit will have to pay up.
>
> Discuss the mathematical implications of this piece of news.

Getting students to “evaluate” or “discuss” things is frequently neglected when it comes to the subject of mathematics in general or the design of achievement tests in that subject. All too often, problems are presented in such a way that students just have to think a little and then do their arithmetic. Solutions take the form of calculations and perhaps some answers in the form of sentences. Unfortunately, it is very rare for students to be asked to use words to express more complex thoughts.

If we want to convey more to our students than mere technical skills, i.e. if we want to provide for their mathematical education, then we must regularly draw upon open-ended approaches that invite students to evaluate, debate, and voice arguments.

Of course, it takes some time for students to develop such mathematical literacy. But discovering a real mistake in an actual newspaper item is likely to make them more talkative.

The following four problems show how a simple question like: “What do you think about it?” can induce students to do their arithmetic and discuss the subject matter. The second task was devised at the Söhre school, Lohfelden, in Hesse, Germany, the third was contributed by the Ebingen Gymnasium, and the fourth example was taken from the journal “mathematik lehren”, “Die etwas andere Aufgabe” (1999), No. 97, p. 66. The latter was also used in the framework of the SINUS-project for Hesse, Germany.

> **Ski jumping**
>
> A sports commentator reporting on a ski-jumping contest:
>
> > … At the start, the ski-jumping hill has an incline of 100%. For ski-jumpers this is almost equivalent to free fall.
>
> What do you think about this comment?

**Rolling wheels**

Svenja has overslept. She jumps on her bicycle and pedals as fast as she can. While pedaling, she thinks: If my wheels were twice as big, I could make it to school in half the time.
Is she right? What do you think?

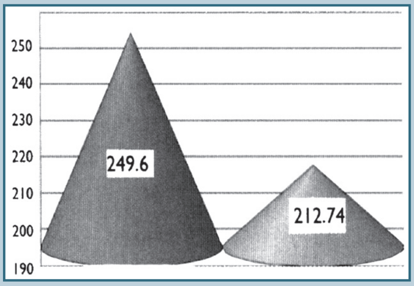
**Buns**

Jessica goes to the baker's and buys four whole-wheat buns. The saleswoman asks for 1.69 euros. Jessica pauses for a moment and then says: "I think you've made a mistake."
What do you think?

**$CO_2$ emission**

This diagram was published in the "Tag und Nacht" customer magazine of the Wetzlar utilities. Its purpose is to motivate customers to switch from fuel oil to natural gas.

1. What do you think about it? What is the percentage difference between the natural gas cone and the fuel cone?
2. Compare this with the figures shown in the graph.
3. Try to find an appropriate presentation of the figures in the form of cones (or other geometric forms).




Reduction of $CO_2$ emissions in tons per year after switching from oil to gas

Devise tasks of your own in which the simple question "What do you think?" will invite students to think mathematically and judge situations in mathematical terms.

Keep your eyes open for everyday material (newspaper items, advertisements, ...) that can serve as a starting point for student responses based on the mathematics of a situation. Exchange such material with your colleagues (the exchange needn't end there!)

## 3.4 Estimation

**Starting with an image**

In comparison with texts, images very often convey a message more graphically. They can be an inspiration to explore many different ways of thinking and working mathematically.
Here is one example (after Herget et al. 2001, p. 32):

**Hot-air balloon**

How much air will it take to fill this hot-air balloon?

Foto: Kropsoq, commons.wikimedia.org

There are obvious similarities with the PISA test question “surface area of a continent” (section 1.1) and “biggest chair” (section 2.3). Based on the (approximate) size of the people in the basket, one can go on to estimate the dimensions of the balloon. Next comes the form of the balloon. There are several ways of arriving at an approximate result. Even students in the sixth or seventh grade can solve this problem. They do their arithmetic (“length by width by height”). At a more sophisticated level, the balloon could be considered as a hemisphere with a cone attached. But that will not improve the result.
Only at first sight is it the final result that counts. Much more important are the multi-layered processes leading to that result: model-building processes. Students need to analyze the situation, simplify it, and translate it into the language of mathematics. They must study the mathematical problem they have discovered with the tools available to them, and then they must translate the result into the given situation. This gives them experience of some very fundamental calculation techniques.
In particular, students will gain the courage and self-confidence to go about tackling a problem on their own and explore their own individual paths in mathematics.

**Information left out**

When it comes to the subject of mathematics, tasks are generally formulated in such a way that they contain precisely the information needed to solve the problem. The same is true in the natural sciences. More information on top of that will tend to unsettle the students. And if there is information missing, students will tend to classify the task as unsolvable.
Of course, these are absolutely artificial situations that we only find in school. Realistic problems are rarely defined in such comprehensive detail.

Here is an example of a conventional text problem.

The total area of a parking lot is 5,000 $m^2$. Every parking space is 3 m wide and 5 m long. 40 % of the area is required for access routes.
How many cars can park in the lot?

This problem leads to the one single solution that the teacher has at the back of his mind. Here, it is particularly important to deduce the right combination of the figures given. One strategy to create open-ended approaches from such textbook problems is to omit the figures. The resulting problem was made part of an achievement test for the 6th grade.

> A parking lot is about the same size as a football field. Approximately how many cars can it accommodate?
> Explain your thinking!

With a little effort we have produced an open-ended problem that leaves a lot of latitude for mathematical work. This task is certainly no easier that the preceding one, and it requires a broad range of competencies. It requires students to find correlations between their everyday knowledge and mathematics. They need an adequate perception of dimensions. They must estimate dimensions, make assumptions, and come to decisions. They must do their arithmetic and finally explain how they arrived at their answer.

Let us take a look at a second example taken from a 6th grade-textbook.

> How many $m^2$ of cloth will Gisela need for a rectangular table 0.82 m wide and 1.13 m long, if the tablecloth is to have an overhang of 15 cm on every side?

Let us consider this question and its wording for a moment. It's the kind of question every math teacher is familiar with.

- Must the teacher stipulate that the unit of measurement is to be $m^2$? Is it meaningful at all?
- Doesn't stipulating the dimensions of the table make the problem unnecessarily boring? Don't the students have enough tables in their immediate environment?
- The students know that a tablecloth forms an overhang at the sides. Can't you leave it to them to estimate the size of the overhang? What size of overhang will look nicest?
- What about the keywords used: "rectangular", "wide", and "long". Isn't this the same as saying that what we really want you to do is calculate the size of a rectangle and all the rest of the verbiage is just hot air?

Let us again leave out all numerical values. This will give us an open-ended and lively approach with the added benefit of a much richer approach to the situation.

> How much cloth would you need to cover a table in your school with a tablecloth? Illustrate your ideas with sketches in your exercise book.

Perhaps one of the students will voice the justified objection that a result expressed in $m^2$ does not make much sense because the cloth has to be cut from a roll of fabric 2 m wide.

A discussion will ensue about how to cut the requisite fabric from a roll, how much waste there will be, etc. Open-ended approaches encourage students to dwell on a given theme and to consider it from many different angles.

The next task does not contain any numerical values either. It invites students to investigate everyday objects in mathematical terms.

> Just by looking at a CD you can recognize which parts have been written on. (Writing takes place from the inside outwards.) When is a CD half full?

(This problem was developed as part of the SINUS project at the Obersberg comprehensive school (*Gesamtschule*), Bad Hersfeld, Germany.)

Take a textbook you normally use in your lessons and scan it for tasks you can open up like this by leaving out bits of information.

**Fermi questions**

Enrico Fermi (1901–1954), an American Nobel Prize laureate for physics, was noted for confronting his students with very special questions motivating them to tackle unfamiliar problems that at first sight appeared to be unsolvable.

Here is one example that was presented to various classes. It spawned a whole lot of estimation and calculation options and resulted in many surprises.

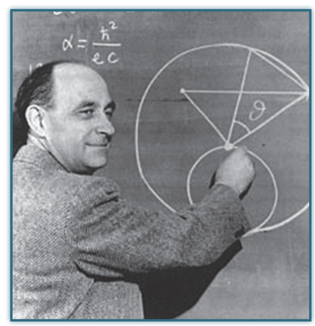


Foto: commons.wikimedia.org

> Hair grows very slowly. During the course of today's math class every single hair on your head will grow a little bit.
> Imagine all these little bits being placed next to each other. What is the total length of the hair growing from your head during this class?

The situation is easy to grasp. It is closely related to the students' lives. Yet, they have not the slightest idea what the answer might be. In addition, the task gives them the feeling of not having sufficient information to find any kind of solution at all.
But all it requires is the courage to tackle it, to activate general everyday knowledge, and to battle one's way from one sub-question to the next.

- How often do I have my hair cut?
- What length of hair is cut off by the barber?
- What is the length of hair growth per hour?
- How many hairs grow on 1 $cm^2$ of scalp?
- How much scalp is covered by hair?
- How many hairs do I have on my head?

It's easy to imagine how much fun approaching mathematics in this way can be both for teachers and students. And all the while students practice estimating, calculating time-spans, lengths, and areas, plus multiplication and division of decimals.

Here are some additional suggestions for Fermi problems.

- How many dentists are there in Europe?
- What is the total number of grades given at your school every year?
- How much of your life do you spend in front of the TV?
- How much of your life do you spend in the bathroom?
- How much of your life do you spend asleep?
- How many kilometers have you covered in your life so far?
- What is the total volume of air you have inhaled in your life?
- How many heartbeats have there been in your life so far?
- What is the total annual consumption of drinking water at your home?
- How often does the letter "e" appear in a big book?
- Can the total number of this school's students fit into your classroom?
- How many balloons will fit into your classroom?
- What is the total length of the toothpaste squeezed from the tube?
- Car tires display a certain amount of wear over time. On average, how many atoms are left on the road during the rotation of one wheel?
- How many gummi bears fit into a bus?

Here, it is equally important for the students to use creativity and imagination and to do hard mathematical work.

Think up more Fermi questions and work on them with your students.

## 3.5 Inventing problems

Here is a segment taken from a textbook for $8^{th}$-grade students.

1. $\frac{2x-3y}{2x-3y} - \frac{2x+3y}{2x-3y} + \frac{8x^2+18y^2}{4x^2-9y^2}$
2. $\frac{3p^2+1.5p-3}{18p^2-8} - \frac{2p+3}{15p+10} - \frac{p-1}{12p-8}$
3. $\frac{a^2+b^2}{2ab} - \frac{a}{a+b} - \frac{b}{a-b} + \frac{b^4-a^4+4a^3b}{2(a^3b-ab^3)}$
4. $\frac{x+y}{y} - \frac{x-y}{x} - \frac{4xy}{x^2+y^2}$
5. $\frac{3a-2b}{a^2-2ab+b^2} - \frac{2a+3b}{a^2-b^2}$
6. $\frac{7}{3k} - \frac{5}{k-3} + \frac{3}{k+1} + \frac{1}{k+5}$
7. $\frac{5a-6b}{4a+4b} - \frac{2a-b}{3a-3b} - \frac{a^2-37ab+28b^2}{12a^2-12b^2}$
8. $\frac{2r}{rs-s^2} - \frac{2s}{r^2-rs} + \frac{r+s}{2rs}$
9. $\frac{1}{z-1} + \frac{1}{z+1} - \frac{2}{z^2-1} - 1$
10. $\frac{1}{p^2-p} - \frac{p^2}{p+1} + \frac{1}{p} - \frac{2}{p^2-1} + p - 2$
11. $\frac{m-3}{m+4} - \frac{m^2-9m-3}{m^2+m-12} + \frac{m-5}{m-3}$
12. $\frac{a}{a-b} - \frac{b^2}{a^2+ab+b^2} - \frac{a^2b}{a^3-b^3}$

Problems of the type shown above certainly make good sense when the main concern is to establish routines in algebra classes. But it does not make much sense to invariably ask students to adhere to predefined methods or to solve given equations with the sole help of calculations they have just learned in class.
If nothing else, assessment studies such as TIMSS or PISA have shown that major emphasis on calculation methods does not occasion many long-term effects in mathematics instruction. Of course, it trains students to use methods in question, and they do indeed develop a feeling for the calculation method they are expected to use for the type of task they are faced with. But such superficial skills have no staying power, and they do not offer any help at all when students are confronted with problems they are not already familiar with.
Even in working on routine problems, teachers should attach greater significance to basic mathematical skills and the encouragement of student creativity and imagination. But how is this to be done? Again, the answer lies in the open-ended approach, e.g. by letting students vary tasks or invent their own (cf. 3.6). Here are some examples taken from the section "Calculating with Numbers and Terms: Solving Equations" (cf. LEU, Stuttgart 2000, p. 48):

| Convergent approach | Open-ended approach |
|---|---|
| Calculate $3^5, 6^3, 2^7, 12^2$ | Calculate powers you like the look of!<br>Calculate powers with a three-figure outcome. |
| Solve $12 \times 17$ | Find products with a value near 200. |
| Solve $24 \times [(9 + 8) : 2]$ | Take the numbers 24, 9, 8. Use them to calculate 5 different terms.<br><br>Using these numbers, give three terms yielding results between 0 and 10.<br><br>Find three terms yielding results between 100 and 110.<br><br>Invent calculation problems involving brackets. |

| | |
|---|---|
| Solve the equation $7x - 11 = 24$. | Devise some equations with the solution $x = 5$.<br>Devise exponential equations with the solution $x = 5$.<br>Devise quadratic equations with 1 and 5 as solutions. Describe all possible quadratic equations.<br>Devise a corresponding text problem for this equation: $7x - 11 = 24$. |
| Solve the equation system:<br>(I) $4x - y = 1$<br>(II) $x + 2y = 7$ | Devise different equation systems readily solvable by means of standard addition, identification or insertion methods (and having the solution set $\{(1;3)\}$ ).<br>(cf. Henn 1999, p. 10) |
| A rabbit eats 2 kg and 500 g of fodder in 10 days. What is its average daily consumption? | Devise a text problem containing 2 kg, 500 g and 10 d. Then solve the problem.<br>(cf. LEU Stuttgart 2001, p. 75.) |

One thing we know for sure. Neither the wordings in the right-hand or the left-hand column make real sense on their own. If the job in hand is to introduce students to a new subject area, then simple automated tasks can of course be used initially.

But it would be a shame to have training phases only consist of terms that gradually get more complicated. Experience from everyday teaching and from achievement tests shows that skills acquired in this way will remain effective for brief periods only.

Devising open-ended tasks requires teachers to see mathematics from a different viewpoint. This kind of wording looks down on the convergent approaches from an elevated level by revealing their structure and promoting creative and imaginative approaches. They induce teachers to ask themselves how the textbook author arrived at the tasks presented in the book.

**Example: Mathematical stories**

Students often regard text problems as difficult to deal with. Some youngsters even perceive them as “threatening”. Here matters can be greatly improved if the student himself invents text problems, thus enabling students to lose some of the awe they have of them. Here students must be able to follow how data can be put in a text in an understandable form. The following problems are by no means new. Problems of this kind are found in many math textbooks. What is new about them is that the following problems were devised as “math stories” by a class of 8th-grade students in the framework of the SINUS-project.

**The Backstreet Boys**

In 1998, the age of all Backstreet Boys together was 107 years. Kevin was one year older than Brian and Howie. Nick was six years and A. J. five years younger than Kevin. How old were they all?

**Forgetful Max**

Max wants to know the price of a ballpoint pen he bought together with some other things. But all he remembers is that the ballpoint pen was half as expensive as his fountain pen. And he remembers that the fountain pen cost 2 € more than the ballpoint. He also remembers that the ballpoint and the notebook had the same price tag. The notebook, the book, and the folder cost a total of 20 €. The book cost 4 € more than the notebook. The folder was priced at 4 €.

**Santa Claus**

Santa Claus is very busy during the Christmas season. This is why he has a helper. Santa Claus A is in Finland and wants to travel back to the North Pole. At 7 p.m. he sets out on his 1,150 km. journey. His travel speed is 25 km/h. Santa Claus B is at the North Pole and wants to continue going about his business in Finland. He also starts his journey at 7 p.m., travelling at a speed of 35 km/h. When will the two of them meet?

A teacher enrolled in the SINUS-project has this to say about his teaching experience with examples like these (Kassel University, Germany, 2003):

> *"Letting students invent their own problems raises the issue of how to make sure that we get the results we want. One method that has proved useful is the following: Students are asked to note the problems devised and their names on a postcard. A second card is used to note the solution (use different colors and number the cards). The teacher copies these cards (fitting 4 each of them onto a letter-size page) and binds them together to produce a small math folder. The students are free to choose which problems they want to solve and can request the solution from the author of the problem. Or they can look up the solution in an index file. Or the teacher compiles an index file from the postcards (plus a second file for the solutions) so that students can organize their work independently."*

Devise a method for getting your students to invent problems for math classes.

## 3.6 Varying tasks

Common teaching practice often takes the form of having students deal with whole "forests" of tasks found in textbooks one after the other. When one problem has been worked out, on to the next problem, and so forth! This may serve a useful purpose when the job in hand is to establish calculation routines. However, problems are bound to arise if such superficial consideration of problems starts dominating math instruction. Practical experience shows that knowledge acquired by such methods produces hardly any long-term effects, nor does it help much in transfer situations.

Just ticking off one problem after another must be replaced by intensive engagement with problem contexts. This can be initiated by asking a routine list of questions after a solution has been found (cf. Baptist 1998):

Examples are:

- What central/crucial problem did this task present?
- What strategies did we pursue?
- How can we summarize the result?
- How important is the result and what do we learn from it?
- How does this problem fit in with what we have learnt so far?
- What should we remember?
- Are there alternative paths for arriving at a solution?
- How can we extend, generalize, and vary the problem?

Let us look at the last of these questions.

One tried and tested strategy for gaining novel insights, not just in mathematics, is to proceed from things we know, vary them, and find out if anything of interest materializes in the course of such variation. In classroom mathematics, known facts or traditional textbook problems can serve as seminal material for a host of variations.

Let us look at some examples before going on to discuss ideas of implementing them in class (the first two examples are taken from Schupp, 2002).

**Initial task: set of distances**

Sketch all points at a distance of 2 cm from a line.

Ways of varying the problem

1. Sketch all points at a distance of 2 cm from a given line segment.
2. … all points at a distance of 2 cm from a given point.
3. … all points at a distance of 2 cm from a given circle.
4. … all points at a distance of 2 cm from a given square.
5. … all points at a distance of 2 cm from a given pair of segments.
6. … all points at a distance of 2 cm from two points.
7. … all points at an equal distance from a given pair of points.
8. … all points at an equal distance from a given pair of lines.
9. … all points at an equal distance of 2 cm from a given pair of line segments.
10. Identify all lines at an equal distance from two given points.
11. Identify all lines at an equal distance from three given points.
12. Identify all planes in space that are equidistant from two given points.
13. Identify all circles that are equidistant from two given points.
14. Identify all planes in space at a distance of 2 cm from a given line.
15. What is the set of all points at a distance of 2 cm from a cube?
16. …

We could continue with this list almost indefinitely. The variations suggested are intended to give you a feeling for what is meant by varying tasks. Actually, however, it is up to the student working on a task to find his own variations.

**Initial task: A calculation**

Solve $3\frac{1}{4} - 4\frac{1}{2} + 2\frac{1}{2} - 5\frac{1}{3}$

Variations

1. Does the result change when we insert brackets?
2. What is the maximum number of brackets we can insert at different places? How many different results will ensue?
3. How must the first (second, third, fourth) number change to obtain a sum of 0 (or a positive number)?
4. What will be the outcome if the fractions are neglected?
5. What will be the result if we consider the fractions only?
6. What will change if we swap two numbers?
7. How can we make the initial task more difficult (less difficult)?
8. What will change if the plus sign in the middle is replaced by a minus, multiplication or division sign?
9. Give four other numbers whose sum will produce the same value.
10. Invent a math story around this task.
11. …

This is an example showing that variations can be used to make a rather boring task interesting and revealing – with the side-effect that the routines to be practiced will be done automatically.

The following problem can be used to make students cross-link what they know about geometry.

*“A quadrilateral with four right angles is a rectangle.”*

Think of as many variations as possible, and use them to come up with new true statements.

What does varying actually mean? The stipulation is to change every important part of a given statement or a stipulated task; or to change every parameter of a mathematical problem. This requires mathematical imagination as well as firmly established mathematical literacy!
The ideas must be organized, evaluated, and explored for their feasibility. Some variations will turn out to be pointless, wrong, or too difficult. This method produces problems devised by the students themselves, problems growing out of the original task and carried on in many different directions. To solve such a bundle of problems, it may be necessary to agree on a division of labor. However, a teaching unit like this cannot be completed meaningfully without

summing up and evaluating the results achieved. This is also an occasion for demonstrating strategies that stand one in good stead in mathematical work.
Considering and engaging with a set of problems from many different angles and creating all kinds of links with prior knowledge will certainly enable students to acquire greater proficiency in mathematical thinking and coping creatively with mathematical problems than by just having them figure out isolated tasks in a sequence of short-winded and isolated problems.

Here is another example for advanced secondary students. Problems similar to 1. and 2. can be found in many analysis textbooks.

**Rectangles and cylinders plus variations**

1. $f(x) = \frac{1}{x^2 + 1}$ , $x \in \mathbb{R}$

   Discuss.

2. The area between the graph of the function f and the x-axis contains symmetrical rectangles in symmetry with the y-axis, where the corner points are plotted on the x-axis and the graph respectively.
   How does the area content of these rectangles depend on their form?
   Which rectangle has the largest area?

3. If the rectangles from 2. rotate around the y-axis, this will generate cylinders.
   What is the interdependence between the volume of these cylinders and their form?
   Which cylinder has the largest volume?

4. Vary the considerations made for 1. – 3., e.g.
   - by choosing a different function f,
   - by considering rectangles replaced by triangles,
   - by having the figures rotate around the x-axis,
   - ...

**Notes**

In part 2 there is one largest rectangle, i.e. with the width of 2. However, in part 3 there is no largest cylinder! With increasing diameter the volume of the cylinder displays a strictly monotonic growth between 0 and $\pi$ . Now for the variations:
Choosing a different function can radically change the situation. The function $f(x) = \left| \frac{1}{x} \right|$ will produce rectangles of equal area content. The volume of the cylinders is directly proportional to their diameter.
You could change the exponential of x for the given function and consider $f(x) = \frac{1}{x^4 + 1}$ . The graph of the function will offer a similar picture. There is a maximum triangle and a maximum cylinder. However, the largest cylinder is by no means a result of the rotation of the cylinder. We could consider the function as a cosine function. This will take you on to trigonometric equations. Solving them will require methods of approximation.

**Potential course of classroom lessons**

H. Schupp recommends the following (ideal-type) steps in varying tasks (cf. Schupp 1999):

- assign the initial task,
- solve this task, if possible, by using several methods.

Up to here, this is a quite traditional way of teaching mathematics. However, experience shows that subsequent variations will increase after we have worked out different paths toward a solution:

- ask students to work out variations,
- refrain from comment when collecting suggestions.

In this brainstorming phase, students probe their way into the initial task. Ideas are collected on the blackboard. The teacher refrains from making any comments or remarks. Students are of course allowed to respond (as long as their remarks are not openly offensive).

- evaluate, structure, and organize the suggestions with the whole class.

Evaluating the content of the ideas is done jointly, e.g. by asking questions such as: What does not make sense? What is easy, difficult, too difficult for us? Which questions should we explore further? What do we do first, what next? Which one will we save for last? This way, a plan will evolve charting out the subsequent work to be done.

- attempt to solve selected alternatives.

In handling the variation phase, diverse grouping forms may be appropriate. A division of labor between groups will be particularly useful in coping with a broad range of tasks.

- presenting the solutions,
- more suggestions for variations, if needed,
- summarizing all efforts made (as appropriate).

Of course, a variation unit may take a completely different course. It is even advisable to plan for a different course in cases where a learning group is either supposed to master certain requisite skills (reasoning, listening to one another, concentrating for a larger part of the lesson) or is not yet capable of doing so. In these cases, it is best to use a "soft" approach: occasionally, the teacher explicitly presents an isolated variation (but not the accompanying solution) to help students acquire a feeling for the varying of tasks. Eventually this will lead to a more advanced format.
You will find a comprehensive account and discussion of how to deal with variations plus numerous examples in the German publications by Hans Schupp (cf. References). The following objections are also discussed:

**Objection: "No, it takes too long."**

Teachers sometimes fail to appreciate that variations are not simply additions to conventional teaching patterns. In fact, they are a natural part of teaching math. Variations are characterized by introductory phases and frequent repetition phases. But these always enhance the training. Strict adherence to working with formulas is replaced by a thorough perusal of problems.

**Objection: "That's nothing new! Experienced teachers have always worked with variations."**
True enough! But the teacher was always the one to include variants of an initial constellation at the lesson planning stage. The students were merely asked to solve the variants. Ultimately, the effect was that of a group of apparently linear task sequences.
It is crucial for the intended teaching format that it should be the students who compose the variations on their own. This gives them the opportunity to decide for themselves on the small area they want to explore.

1. "Add three successive natural numbers. What strikes you?"
   Solve and vary this problem. You should find at least five variants (cf. Schupp 1999).
2. Open a textbook, pick a problem, and vary it.
3. Try varying tasks with your students. The pattern described here can serve as a blueprint.

# 4. Extended Forms of Independent Work: How to Organize Learning on One's Own

This chapter focuses on learning circles and project learning. These teaching patterns enable students to work independently and to organize learning on their own. They give them the opportunity to explore their own learning paths over an extended period of time.

In the two preceding chapters the focus was on independent learning in regular, everyday teaching. By varying tasks we found easy ways of giving students latitude in finding their own paths to independent mathematical thinking and working. Mostly, the instruction segments described did not cover more than one period. In getting students accustomed to independence (and in getting teachers used to "letting go"), such "small" steps are both practical and at the same time very effective.
Let us now turn to a brief discussion of extended forms of independent work. The examples we will be concentrating on are learning circles and project learning. They are both very important for practical teaching. We should, however, also bear in mind that both subjects offer enough material to fill whole books (e.g. Frey 2002).

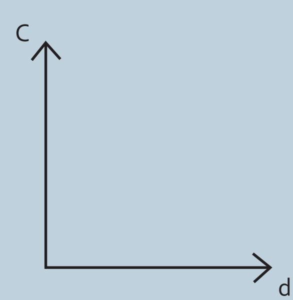
## 4.1 Learning circles and learning in stages

Learning circles offer students separate "stages". They are requested to study tasks and problems independently and on their own. Some sources use "learning circles" and "learning in stages" as synonyms, others emphasize that "stages" must always adhere to predetermined sequences.
Before we embark on a more general discussion of learning circles in the teaching of mathematics, let us look at two examples. The first set of tasks is taken from a learning circle called "measurements of a circle".

| Stage No. 2 | Circle circumference – Introduction | Difficulty: xx |
|---|---|---|

This stage features various round objects (buckets, cans, plates, saucers, CDs, records, coins) and a tape measure.

1. Measure the circumference C and the diameter d of each of the objects. Make a table and enter the values obtained. Does anything strike you?
2. Present the measurements obtained in a system of coordinates.
3. For each value-pair obtained, calculate the quotient $\frac{C}{d}$. What can you say about the correlation between circumference C and diameter d? Write your thoughts down in your exercise book.



The second example is taken from a learning circle dealing with "percentages and diagrams". Its aim is application and in-depth exploration of subject matter learnt previously.

| Stage No. 4 | Diagrams | Difficulty: xx |
|---|---|---|

1. In a survey, the student paper FUZZI found out that all students enrolled at the Friedrich-Uzzendorfer-Gymnasium had received a total of 13,495 € pocket money in October. Of this they spent 1,824 € on sweets, 2,106 € on clothing, 1,330 € on learning materials, 4,190 € on leisure activities, and 2,563 € on miscellaneous items. The remainder were savings.

2. A 250 g yoghurt cup contains 9.25 g protein, 16.5 g carbon hydrates and 7.75 g fat.
   a. Give the share of protein, carbon hydrates and fat, in percentages.
   b. Show the yoghurt ingredients in the form of a diagram.
   c. What other ingredients are present in the yoghurt? What total mass do they have?

(The tasks are taken from the SMART task database, cf. http://smart.uni-bayreuth.de)

How can we use such materials for independent learning in everyday teaching? Here are some ideas.

- **Functions in the course of classroom lessons:** Initiating learning circles is appropriate when new subject matter is to be introduced (see first example). This is particularly true when there are several approaches to a subject matter that can be reflected upon in different stages. The individual stage can offer students objects and media for "getting a grip on" new content at an enactive level.
  However, learning circles are also suitable for practice phases when students practice handling, repeating, deepening, and implementing what they learnt (see second example).

- **Presenting materials:** Students receive tasks and instructions in written form. Depending on the learning circle's character and size, it may be appropriate to hand copies of all tasks and all stages to all students, or to make only a few copies available (sheets in transparent covers, index cards, or laminated material). Opting for the latter involves two different types of organization. One is to establish fixed learning stages in the classroom with students moving from one stage to the next. The other is to display the materials at a central point, asking students to come and get them for their respective (group) working tables.

- **Checking solutions:** To enable students to check and improve their work on their own, we suggest that the solutions to the tasks should also be made available. They can be displayed for inspection on a bulletin board or on the teacher's desk. Putting them up on a board has the advantage that students can check results outside the actual math classes. Another option is to hand out problems and their solutions together (e.g. on the front and back of index cards).

When students do simple arithmetic, quick access to the solution is an advantage. This gives them immediate feedback about how successful they have been in dealing with their tasks. With more complex problems requiring more comprehensive solution strategies, a premature look at the solution may involve the risk of crucial ideas just being adopted. In addition, this would reduce the potential of a stage offering complex and difficult learning content.
At all events, a substantial part of the responsibility for student learning will be transferred from the teacher to the student.

- **Control slips:** Comprehensive learning circles should go with control slips handed to the students. They present an overview of the whole and of the individual stages. In addition, students can use them to check off the stages they have already done. The students (and the teacher) can always check the current state of achievement.

- **Internal differentiation:** Learning circles offer excellent opportunities for distinguishing between low and high achievers, slow and fast students. Students work on their own and determine their own learning speed. Obligatory stages are for all students. However, a learning circle should also include stages for advanced students. They provide an opportunity to give adequate support to faster and advanced-track students.

- **Timing:** When working independently, students are required to spend and use the time available in a meaningful way. They must check whether they are sticking to the time schedule they have worked out on their own. In traditional math classes, students are hardly ever confronted with such necessities. But such skills are enormously important when it comes to exploring and grasping larger contexts.

- **Role of the teacher:** The teacher keeps very much in the background when choosing such forms of instruction. However, he is always prepared to act as an advisor. Teachers experience these phases of instruction as quite relaxing. They are relieved of the obligation of always having to organize and “give”.
  Of course, teachers need adequate time to prepare a well-thought-out learning circle. Teachers should make copious use of the exchange of ideas with colleagues, including exchanges across schools and different types of schools.

## 4.2 Project-based learning

Enriching classroom teaching with projects is certainly the most challenging, but at the same time the most beneficial form of independent learning. It is challenging because it requires high-level skills on the part of the students, e.g. skills in applying methods, self-management, and social competence. So project-based learning should never degenerate into a teacher-centered training course where ultimately the teacher still does all the planning, structuring and organizing, prepares and procures all the materials, or even produces and presents the results.

Project-based learning is highly beneficial. It offers great latitude for letting students come up with their own initiatives, develop creativity, autonomy, and responsibility, and it offers opportunities for cooperation. This type of learning strongly promotes all these skills. On the other hand, projects also create the organizational framework in which interesting and unconventional questions can be investigated.
In the literature, various phases are suggested for organizing the course of a project. These are the ones that seem to make most sense for math projects:

1. Planning and preparing the project: Typically, a project starts with an idea, the project initiative. Ideally, it is an organic product developed in a classroom lesson and posed in the form of a question that all participants would like to explore immediately. Practically, however, it is occasionally left to the teacher to provide the initial impetus for a learning project, perhaps following up on an idea voiced by students.
   The planning phase involves collecting initial spontaneous ideas, organizing and evaluating them. Project aims evolve in this way. How to achieve these aims will be discussed in the classroom. Plans should be elaborate enough to allow for efficient work in the subsequent phases. At the same time, they should leave enough room for the students' spontaneous ideas. Some aims may be reformulated or even ditched in later phases. Others may be replaced by new aims.
   In such planning phases, it may make sense to install working groups for a division of labor in dealing with complex problems. The next thing to be established is the time frame for the project. In reality and given the external constraints that are likely to occur, this task tends to be left to the teacher.
   Generally, projects are characterized by involving students as early as possible in planning activities, defining aims, and establishing working methods. This enables them to exercise responsibility at an early stage and to decide how to proceed with a high degree of autonomy.

2. Implementing phase: This phase is the core of project-based learning. It involves the implementation of initial plans. Students' knowledge, insights, and skills are increased. Therefore, certain phases of reflection should be incorporated. These enable students to reconsider what they have done so far and subsequently to vary or correct the aims set. A project with an extended time frame should include regular review points as "organizational control centers" (Frey 2002), giving students the opportunity to review the progress made in relation to the project as a whole. With their fellow students, they can exchange information about the activities they have undertaken and decide on how to proceed with the next steps.
   Once a teacher sees that a project is "under way," he should desist from getting actively involved. As an advisor he should be available only upon request. The central issue in project-based learning is enabling students to undertake learning on their own, to deal with its organization and engage in a cooperative working style. Its purpose is to get them to explore their own individual paths towards learning.

3. Presentation phase: It is in the nature of things for project-based learning to be oriented toward a product or a target. Once the implementation phase has been completed with satisfactory results, it seems highly appropriate to present the project results to

a larger group. For schools, such a publication phase can take the form of presenting the results achieved to a parallel class, to the student body as a whole, or to a larger public outside school, perhaps to the press or to local radio stations.
Organizing such a presentation is equivalent to active appreciation of the students' efforts in an appropriate setting. This enables them to experience their own work as meaningful and valuable. Using the implementation phase to float the idea of publishing the project results can be a strong motivation factor.
In addition, such presentations are an effective opportunity to make mathematics instruction more visible, to free it from the "ivory tower" of the classroom, and indicate its scope outside school. Not least, this will improve the image of mathematics and the appreciation of this subject in the class, in the school community, and among the parents and will also contribute to enhancing its image in society as a whole.

4. Evaluation phase: Evaluating school projects often takes the form of "post-mortems". Activities undertaken are reviewed, reflected upon, and evaluated. Questions to be answered include whether the targets set have been achieved. In addition, such meetings also serve to analyze the paths chosen to achieve these targets. From a distance, participants consider the effectiveness of procedures and potential for improvements. Discussing these matters with the whole class and combining this with requests for students to submit written remarks (e.g. on questionnaires) seems recommendable. Such critical reflection about the project and a review of past efforts help give "simple actions" an "educative value" (Frey 2002).

Here are some issues chosen more or less at random (as suggestions) for potential projects in mathematics instruction:

- all about circles
- pyramids
- Pythagoras
- parabolas
- surveying activities in the field
- the Earth from a mathematical viewpoint
- the surface area of the school building
- outstanding mathematicians
- mathematics and the arts
- fractals: beauty and chaos in mathematics
- What is the optimum packaging?
- How do we optimize school bus traffic?
- What entrance fees will generate the highest profit for a community outdoor swimming-pool?

My personal experience with these matters suggests that implementing an extensive project with one or two classes per year seems realistic in view of the existing framework conditions for German schools. In addition, with regard to existing potential and skills, not every class is advanced enough (yet) for meaningful project-based learning.

**Example**

Trigonometry is an obvious basis for surveying activities in the field. Here, students can experience the benefits of mathematics outside their school lives. The following article from the German daily "*Fränkischer Tag*" describes such a project conducted at the Ehrenbürg-Gymnasium, Forchheim, Germany.

### Mathematics out in the open

**Ehrenbürg-Gymnasium students conduct a surveying project**

**Forchheim.** In a three-week project, class 10a students of the Ehrenbürg-Gymnasium have been exploring the question of how to undertake surveying activities in the field.

To start with, they devised related questions, such as: What is the distance from the St. Martin Church tower to the chimney of the 4P sheet production site? What is the height of the McDonald's tower in the south of Forchheim; how far does the Walberla hill rise over Kirchehrenbach; or what is the rise of the Reifenberg chapel over the Wiesent valley?

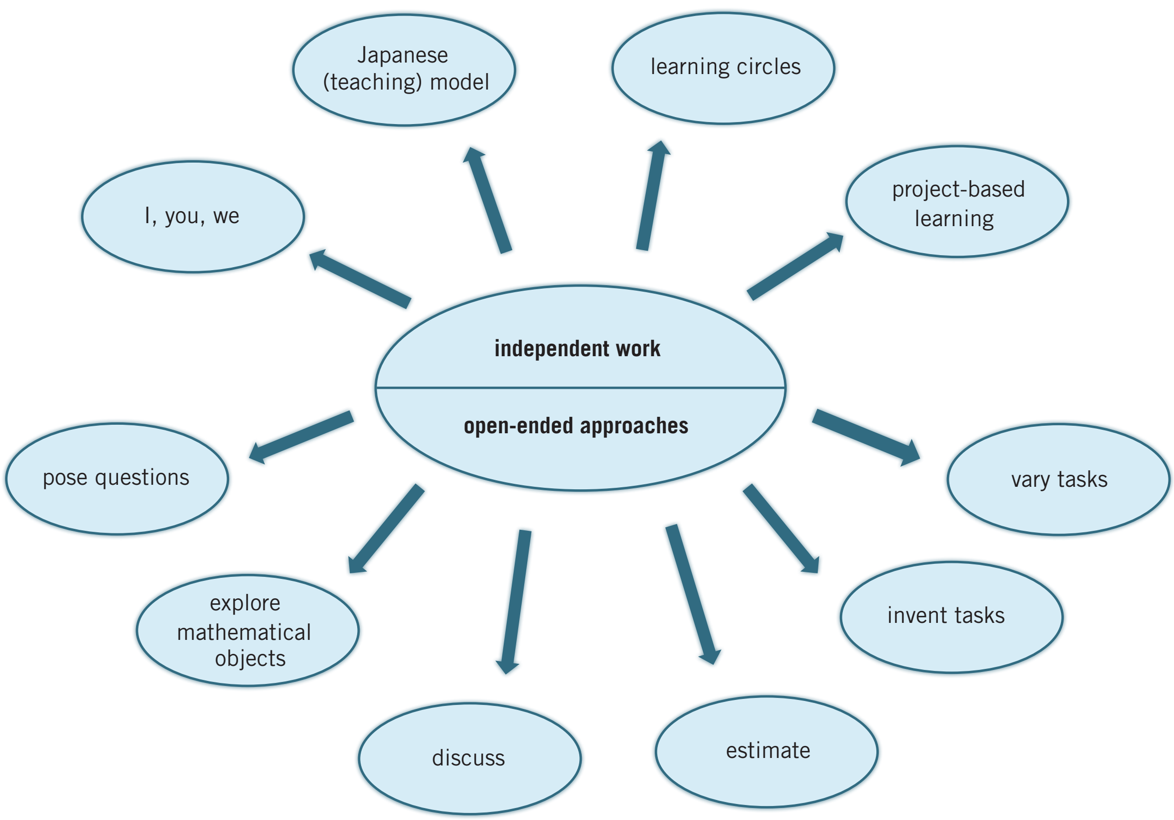
After devising the questions, hard work awaited them. The students had to devise procedures with which to determine these unknown distances by developing special skills for measuring angles. In doing this, they found that all the toil during the school year had paid off and that sine and cosine do have their extracurricular benefits.

Once the theoretical groundwork was over, mathematical skills were applied out in the field. Finally, after several excursions, the students had come up with solutions for all the surveying problems devised. To locate points at very remote distances they borrowed a special instrument, a theodolite, from the Surveying Office at Erlangen.

The project was completed with an exhibition in the school auditorium. It was a detailed documentation of how the students had organized the project and the results they obtained. The exhibition welcomes the public during regular school hours (from 10 to 12am during the summer recess until 10 August, and then from 27 August onward).

**Summary**

In the last chapters we learnt and discussed a variety of ways in which students can be given the necessary latitude for independent, responsible, and cooperative work in the course of everyday instruction. The individual elements should not be viewed as isolated from each other but as interlinked methods for offering variety in teaching mathematics and enabling students to explore their own learning paths in different situations. The diagram below summarizes essential ideas and key concepts.



## 5. Postscript: Mathematical Literacy

We have now discussed the teaching and learning of mathematics from various different angles. Let us go back to the very beginning of this paper where we posed the following question: Where do we want do go in teaching mathematics? What long-term objectives do we want to achieve with our students? For all our efforts, we should never lose sight of these overarching questions. The objectives they define can and should govern whatever we do.

In section 1.1. we looked at two tasks from PISA assessment studies. We recognized that the PISA test questions and the conventional teaching do not concur. Let us use the PISA papers to look once again at the question of where we want to go. After all, designing an assessment test must be supported by concepts related to desirable objectives in teaching mathematics.

The PISA assessment tests are based on the fundamental concept of "mathematical literacy". This involves several competencies:

- sound basic understanding and basic skills,
- flexibility in applying mathematical concepts and translating insights into mathematical content (modeling),
- evaluating matters in mathematical terms,
- mathematical communication,
- recognizing the role of mathematics in the world.

These objectives sound attractive as they stand, and we assume that you are likely to agree with the statement that our children should develop such skills at general education schools. So the question is: How can the teaching of mathematics be designed to convey to our children such a comprehensive range of mathematical literacy? The answer is simple. In the way we have designed and discussed in this paper!

- The majority of learning situations discussed in this paper require students to be flexible in applying and comprehending mathematical concepts inside and outside their school lives. The purpose of this broad range of contextual problems is to achieve variable thinking and a basic understanding of mathematics.
- General education mathematics regularly calls upon students to evaluate issues in terms of mathematics. In chapter 3 we saw that newspaper articles or diagrams are suitable for such exercises.
- Naturally, one of the broader objectives of education is to develop children's communication and team skills. How this can be done systematically in mathematical instruction was described in chapter 2, where we considered teaching methods. The "I, you, we" principle and the basic pattern of Japanese mathematics classes center around communication and cooperation with fellow students as crucial phases in the learning process. The same applies to the presentation of results and joint discussion in the classroom.
- The open-ended problems discussed in chapter 3 require students to correlate everyday knowledge with what they learnt in math. Students are asked to estimate, to conjecture, to translate real-life situations into mathematical structures (modeling), to work with real-life data, or to discuss an issue in mathematical language. The intention

is to make students comprehend that mathematics in school and outside it are not two separate and distinct worlds. In the last resort, mathematics is only one tool for exploring and comprehending the world. This insight is also supported by the project-based learning approach described in chapter 4.

The idea of mathematical literacy can also be useful in explaining to parents why their children's math problems differ from what they remember from their own school lives. Parents need to be told patiently and clearly that it is not training in, and the application of, mathematical routines alone that count. The overall objective is rather to achieve mathematical literacy for their children - as defined above. Then hardly any parent is likely to object to the changes that have taken place in the problems and requirements. Experience with the SINUS-project indicates that there has been very little discord on this point.

**Summary**
Mathematical literacy can serve as a guiding vision for the teaching of mathematics. The major issue in this essay has been to show you how you can achieve this objective by effecting systematic and continuous changes in everyday teaching situations, and to encourage you to embark on such new paths together with your math colleagues.

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- “SINUS Transfer”: http://sinus-transfer.eu

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# Towards New Teaching in Mathematics

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Federal Ministry of Education and Research

3 / 2011
ISSN 2192-7596
University of Bayreuth
www.sinus-international.net