L^q-solutions to the Cosserat spectrum in bounded and exterior domains

by

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November 2005

Contents

Abstract

In the present paper we consider for $\lambda \in \mathbb{R}$ the existence of non-trivial solutions of the following problems:

1. **classical Cosserat spectrum**

$$\underline{u} \in C^2(G)^n \cap C^0(\overline{G})^n$$
$$\Delta \underline{u} = \lambda \nabla \operatorname{div} \underline{u}$$
$$\underline{u}\Big|_{\partial G} = 0$$

2. **weak Cosserat spectrum**

$$\underline{u} \in \widehat{H}^{1,q}_{\bullet}(G)^n$$
$$\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G \;=\; \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \quad \forall\, \underline{\phi} \in \widehat{H}^{1,q'}_{\bullet}(G)^n$$

where $G \subset \mathbb{R}^n$ is a bounded or an exterior domain and $\widehat{H}^{1,q}_{\bullet}(G)$ (cf. Definition 2.1) is the suitable space for weak solutions.

This problem was investigated firstly by Eugene and Francois Cosserat. It is a special case of the Lame equation and describes the displacement of a homogeneous isotropic linear static elastic body without exterior forces.

In this paper we characterize the weak Cosserat spectrum for bounded or exterior domains $G \subset \mathbb{R}^n$ $(n \geq 2)$ and $1 < q < \infty$ (for the definition of $\widehat{H}^{2,q}_{\bullet}(G)$ cf. Definition 3.1)

Theorem 14.1. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be a bounded or an exterior domain with $\partial G \in C^{k+2}$.

1. The set

$$W := \Big\{ \lambda \in \mathbb{R} \;:\; \text{there is } 0 \neq \underline{u} \in \widehat{H}^{1,q}_{\bullet}(G)^n, \text{ such that for all } \underline{\phi} \in \widehat{H}^{1,q'}_{\bullet}(G)^n \text{ holds } \left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G \;=\; \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \Big\}$$

is finite or countably infinite.

2. For $\lambda \in \mathbb{R} \backslash \{1, 2\}$ the space

$$V_\lambda := \left\{ \underline{u} \in \widehat{H}^{1,q}_{\bullet}(G)^n \;:\; \left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G \;=\; \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \;\forall\, \underline{\phi} \in \widehat{H}^{1,q'}_{\bullet}(G)^n \right\}$$

is finite dimensional.

3. For every sequence $(\lambda_m) \subset W$ with $\lambda_m \neq \lambda_l$ for $m \neq l$ holds

$$\lambda_m \to 2 \qquad (m \to \infty)$$

4. $$\{\nabla s \;:\; s \in \widehat{H}^{2,q}_{\bullet}(G)\} \;\subset\; V_1$$

Therefore $\lambda = 1$ is an eigenvalue of infinite multiplicity and $\lambda = 2$ is an accumulation point of eigenvalues of finite multiplicity.
In this generality the result is new.

E. and F. Cosserat [Co1-Co9] studied the classical Cosserat spectrum for certain types of domains like a ball, a spherical shell or an ellipsoid. In chapter 16 we use their approach for explicit solutions.
General results are due to Mikhlin [Mi, 1973], who investigated the Cosserat spectrum for $n = 3$ and $q = 2$, and Kozhevnikov [Ko2, 1993], who treated bounded domains in the case $n = 3$ and $q = 2$. Kozhevnikov's proof is based on the theory of pseudodifferential operators.
Faierman, Fries, Mennicken and Möller [FFMM, 2000] gave a direct proof for bounded domains, $n \geq 2$ and $q = 2$.
Michel Crouzeix gave 1997 a simple proof for bounded domains, in case $n = 2, 3$ and $q = 2$.

In this paper we use the idea of Crouzeix to proof the results for bounded and exterior domains, $n \geq 2$ and $1 < q < \infty$.

The following regularity theorem is new:

Theorem 15.5. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^{k+3}$. Assume that $\underline{u} \in \widehat{H}^{1,q}_{\bullet}(G)$, $\lambda \in \mathbb{R} \backslash \{1, 2\}$ and

$$\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G \ = \ \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}^{1,q'}_{\bullet}(G)^n$$

Then

1. $\underline{u} \in \widehat{H}^{1,\tilde{q}}_{\bullet}(G)^n$ and $\nabla \underline{u} \in H^{k,\tilde{q}}(G)^{n^2}$ for all $1 < \tilde{q} < \infty$,
2. $\underline{u} \in \overline{C}^k(G)$,
3. $\Delta \underline{u} \ = \ \lambda \nabla \operatorname{div} \underline{u}$

It is amazing, that the eigenspaces of eigenvalues $\lambda \notin \{1, 2\}$ don't depend on q! Further we get important results for the classical Cosserat spectrum: $\lambda = 2$ is an accumulation point of eigenvalues, too. $\lambda = 1$ is also a classical eigenvalue, because for $s \in C_0^\infty(G)$ with $\underline{u} := \nabla s$ holds $\Delta \underline{u} = \nabla \operatorname{div} \underline{u}$.

Now we like to describe, how we proved these results.

Starting point was the paper [Si] of Christian G. Simader. He proved, that in the case of the upper half space $H = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ there exists exactly two eigenvalues, namely $\lambda = 1$ and $\lambda = 2$. Simader used the paper [MueR] and the decomposition (cf. Theorem 4.2)

$$L^q(H) = A^q(H) \oplus B^q(H)$$

He was able to solve explicitly in $\widehat{H}^{1,q}_{\bullet}(H)^n$ the equation

$$\left\langle \nabla \underline{T}_q(p), \nabla \underline{\phi} \right\rangle_H \ = \ \left\langle p, \operatorname{div} \underline{\phi} \right\rangle_H \quad \forall \ \underline{\phi} \in \widehat{H}^{1,q'}_{\bullet}(H)^n$$

for a given $p \in L^q(H)$, and proved by direct calculations, that for $p_0 \in A^q(H)$ holds

$$\operatorname{div} \underline{T}_q(p_0) = p_0$$

and for $p_h \in B^q(H)$

$$\operatorname{div} \underline{T}_q(p_h) = \frac{1}{2}\, p_h$$

We tried to carry over these results to the slightly perturbated half space $H_w = \{(x', x_n) \in \mathbb{R}^n : x_n > w(x')\}$ (for $w \in C_0^2(\mathbb{R}^{n-1})$) and to domains with compact boundary. Therefore we considered for suitable $\mu > 0$ and

$$\rho_\mu \in C_0^\infty(\mathbb{R}), \quad 0 \le \rho_\mu \le 1, \quad \rho_\mu(t) = \begin{cases} 0 & \text{, if } t \ge 4\mu \\ 1 & \text{, if } t \le 2\mu \end{cases}$$

the isomorphism

$$f \,:\, H_w \to H, \qquad f(x) = \big(x', x_n - w(x')\rho_\mu(x_n)\big)$$

and the Piola transform (cf. [Cia, p.37PP])

$$P \,:\, \widehat{H}_\bullet^{1,q}(H_w)^n \to \widehat{H}_\bullet^{1,q}(H)^n, \qquad (P\underline{v})(y) = \big[\det f'(f^{-1}(y))\big]^{-1} f'(f^{-1}(y))\, \underline{v}(f^{-1}(y))$$

and converted the inner products

$$\big\langle \nabla P^{-1}\underline{v}, \nabla P^{-1}\underline{\phi} \big\rangle_{H_w} \quad \text{respectively} \quad \big\langle \operatorname{div}(P^{-1}\underline{v}), \operatorname{div}(P^{-1}\underline{\phi}) \big\rangle_{H_w}$$

by means of the transformation rule in inner products

$$\big\langle \nabla \underline{v}, \nabla \underline{\phi} \big\rangle_H + B_1(\underline{v}, \underline{\phi}) \quad \text{respectively} \quad \big\langle \operatorname{div} \underline{v}, \operatorname{div} \underline{\phi} \big\rangle_H + B_2(\underline{v}, \underline{\phi})$$

By the properties of Piola's transform B_2 defines a compact operator, if $\operatorname{div} \underline{v} \in B^q(H)$, but we are not able to prove that for B_1 too.

As another approach we searched for a relationship of Green's function $\mathcal{G}$ of the Laplace operator (cf. Definition 17.4) to the reproducing kernel $\mathcal{R}$ in $B^q(G)$ (cf. Definition 18.2), because with

Definition 11.1. Let $n \ge 2$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^1$.

1. Let $\underline{T}_q : L^q(G) \to \widehat{H}_\bullet^{1,q}(G)^n$ be defined by (cf. Theorem 2.9)

$$\big\langle \nabla \underline{T}_q(p), \nabla \underline{\phi} \big\rangle_G \;=\; \big\langle p, \operatorname{div} \underline{\phi} \big\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}_\bullet^{1,q'}(G)^n$$

2. Let $Z_q : L^q(G) \to L^q(G), \qquad Z_q(p) := \operatorname{div}(\underline{T}_q p)$

holds

Theorem 11.3. Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^1$. Assume $\lambda \in \mathbb{R}$. Then there is $\underline{u} \in \widehat{H}^{1,q}_\bullet(G)^n$ with

$$\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G = \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}^{1,q'}_\bullet(G)^n$$

if and only if there is $p \in L^q(G)$ with

$$\lambda\, Z_q(p) = p$$

In this case one can choose $p = \operatorname{div} \underline{u}$.

If $G \subset \mathbb{R}^n$ is a bounded domain, for $p \in L^q(G)$ formally holds with $\underline{u} := \underline{T}_q(p)$

$$\underline{u}(x) = \int_G \mathcal{G}(x,y)\,(-\Delta \underline{u})(y)\,dy = \int_G \mathcal{G}(x,y)\,(-\nabla p)(y)\,dy$$

$$\begin{aligned}
Z_q(p)(x) &= \operatorname{div} \underline{u}(x) \;=\; -\sum_{i=1}^n \int_G (\partial_{x_i}\mathcal{G})(x,y)\,(\partial_{y_i} p)(y)\,dy \\
&= -\sum_{i=1}^n \int_{\partial G} \underbrace{(\partial_{x_i}\mathcal{G})(x,y)}_{=0}\, p(y)\, N_i(y)\, d\omega_y + \int_G p(y) \sum_{i=1}^n \partial_{y_i}\partial_{x_i}\mathcal{G}(x,y)\,dy \\
&= \int_G p(y) \sum_{i=1}^n \partial_{y_i}\partial_{x_i}\mathcal{G}(x,y)\,dy
\end{aligned}$$

If therefore

$$\sum_{i=1}^n \partial_{y_i}\partial_{x_i}\mathcal{G}(x,y) - \frac{1}{2}\mathcal{R}(x,y)$$

was a compact operator, the assertion about the Cosserat spectrum would follow by the spectral theorem for compact Hermitian operators. We couldn't prove that directly. There are results about the relationship of Green's function of the bilaplace operator Δ^2 to the reproducing kernel in $B^2(G)$ (see [ELPP, Theorem 4.3, p.113]), but we couldn't find the relationship above.
After solving the Cosserat spectrum in another way we can prove the relationship indirectly:

Theorem 19.1 Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k > 1 + \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{2+k}$. Let

$$\mathcal{G}(x,y) = S(x-y) + h(x,y)$$

be Green's function of the Laplace operator in G and let $\mathcal{R}$ be the reproducing kernel in $B^q(G)$. Then

$$Z_q(p)(x) \;=\; p(x) + \sum_{i=1}^{n} \int_G p(y)\, \partial_{y_i}\partial_{x_i} h(x,y)\, dy \qquad \text{a.e. for } p \in B^q(G)$$

Therefore

$$\sum_{i=1}^{n} \partial_{y_i}\partial_{x_i} h(x,y) \;+\; \frac{1}{2}\,\mathcal{R}(x,y)$$

is a compact operator.

For the unit sphere B_1 (chapter 20) and for the half space one can prove this result directly. In this cases reproducing kernel and Green's function are known explictly. It is an interesting question, whether it is possible to prove this directly in general, too.

Finally we found the paper [Cr] of Michel Crouzeix. His sketch of a proof for *bounded* domains and $q = 2$ is very short. He proved, that for $p \in B^2(G)$ holds[1]

$$\|Z_2(p) - \frac{1}{2}\,p\|_{1,2;G} \;\leq\; C\,\|p\|_{2;G} \qquad (*)$$

1. It suffices to prove $(*)$ for $p \in H^{k,2}(G) \cap B^2(G)$, because $H^{k,2}(G) \cap B^2(G)$ is dense in $B^2(G)$ with respect to $\|\cdot\|_{2;G}$.
2. Obviously
$$\|Z_2(p) - \frac{1}{2}\,p\|_{2;G} \;\leq\; C\,\|p\|_{2;G}$$
3. Choose a sufficiently smooth $\zeta : \mathbb{R}^n \to \mathbb{R}$ with
$$\zeta\Big|_{\partial G} = 0, \qquad \nabla\zeta\Big|_{\partial G} = N \text{ (outer unitary normal)}$$

[1] Crouzeix considered only $p \in L^2(G)$ with $\int_G p\,dx = 0$. That means no loss of generality, because for arbitrary $p \in L^2(G)$ for $\underline{\phi} \in H_0^{1,2}(G)$ holds:

$$\langle p - \int_G p\,dx, \operatorname{div}\underline{\phi}\rangle_G = \Big\langle p, \operatorname{div}\underline{\phi}\Big\rangle_G$$

Furthermore only for $p \in L^q(G)$ with $\int_G p\,dx = 0$ there is a constant $C > 0$, which doesn't depend on p, such that

$$\|p\|_{q;G} \;\leq\; C \sup_{0 \neq \underline{\phi} \in H_0^{1,q'}(G)} \frac{\Big\langle p, \operatorname{div}\underline{\phi}\Big\rangle}{\|\nabla\underline{\phi}\|_{q'}}$$

(see [St, Satz 8.2.1, p.256]). For our purpose the restriction to L^q-functions with mean value zero is not necessary.

4. and define for $p \in H^{k,2}(G) \cap B^2(G)$ and $\underline{u} := \underline{T}_q(p) \in H_0^{1,2}(G)^n$

$$w := \underline{u}\,\nabla\zeta \,-\, \frac{1}{2}\,p\,\zeta$$

5. One can show that $w \in H_0^{1,2}(G)$ and

$$\Delta w = 2\nabla\underline{u}\cdot\nabla\nabla\zeta + \underline{u}\cdot\nabla\Delta\zeta - \frac{1}{2}p\Delta\zeta \;\in L^2(G)$$

   Therefore $w \in H^{2,2}(G)$ and

$$\|w\|_{2,2;G} \;\le\; C\,\|p\|_{2;G}$$

6. Furthermore

$$\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p) \in H_0^{1,2}(G)$$

7. Then

$$\begin{aligned}\|\nabla(\operatorname{div}\underline{u} - \frac{1}{2}\,p)\|_{2;G} \;&\le\; \|\nabla(\nabla w\nabla\zeta)\|_{2;G} + \|\nabla\big[\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\big]\|_{2;G}\\ &\le\; \tilde{C}\,\|p\|_{2;G}\end{aligned}$$

   and finally $(*)$.

8. Then for bounded domains by Rellich's imbedding theorem $Z_2 - \frac{1}{2}I$ is a compact operator, and by the spectral theorem for compact self-adjoint operators the assertion follows.

Now we describe for each part of the proof, which additional work was necessary to carry over the results to exterior domains and the case $1 < q < \infty$.

1. is proved in chapter 9. We need elliptic regularity theorems (Theorem 7.7 and 7.8) and for exterior domains the asymptotic behavior of harmonic functions (Lemma 8.9).

2. follows immediately by Definition 11.1 and Theorem 2.8.

3. For $\zeta \in C_0^k(\mathbb{R}^n)$ to hold we need in Theorem 6.1 $\partial G \in C^{k+1}$. It is possible, that it suffices to assume that $\partial G \in C^k$. But our proof is very elementary.

4. The definition of w was the ingenious idea of Crouzeix.

5. is proved in Lemma 12.2 respectively 13.2. We need $p \in C^0(\overline{G})$ and $\underline{u} \in \overline{C}^1(G)^n$. This is shown in Lemma 12.1 respectively 13.1 by means of Sobolev's imbedding theorems. Therefore we must assume $\partial G \in C^{k+2}$ with $k > \frac{n}{q}$ in Theorem 14.1.

6. In Lemma 12.3 respectively 13.3 we need again the regularity of p and $\underline{u}$ and a few theorems about differentiable functions (chapter 5).

7. Then the estimate follows by Theorem 2.8.

8. Analogously to Rellich's imbedding theorem for bounded domains the imbedding $H^{1,q}(G) \cap B^q(G)$ in $B^q(G)$ is compact in exterior domains (Theorem 10.1). The proof is based on the asymptotic behavior of harmonic functions (Theorem 8.7). For real Banach spaces the spectral theorem B.9 is applicable. Finally we derive Theorem 14.1.

The regularity of the solutions (Theorem 15.5) can be proved as follows: If $\underline{u} \in \widehat{H}_{\bullet}^{1,q}(G)^n$, $\lambda \in \mathbb{R}\backslash\{1,2\}$ and

$$\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G \;=\; \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}_{\bullet}^{1,q'}(G)^n$$

then with $p := \operatorname{div} \underline{u}$ by Theorem 11.3 and 11.4 holds

$$p \in B^q(G), \qquad \lambda\, Z_q(p) = p$$

With $\mu := \frac{\lambda}{1-\frac{\lambda}{2}} \in \mathbb{R}$ we derive by $(*)$ respectively Theorem 12.4 and 13.4

$$p \;=\; \mu \left(Z_q(p) \;-\; \frac{1}{2}\, p \right) \;\in H^{1,q}(G)$$

For $1 < q < n$ with $q^* = \frac{nq}{n-q}$ holds $p \in L^{q^*}(G)$ and then

$$p \;=\; \mu \left(Z_{q*}(p) \;-\; \frac{1}{2}\, p \right) \;\in H^{1,q^*}(G)$$

By induction we derive $p \in H^{1,s}(G)$ for a certain $n < s < \infty$, whence $p \in C^0(\overline{G})$. Because of the asymptotic behavior of B^q-functions in exterior domains (Theorem 8.12) we derive further

$$p \in H^{1,\tilde{q}}(G) \cap C^0(\overline{G}) \qquad \forall\, 1 < q < \infty$$

and

$$\nabla \underline{u} \in H^{1,\tilde{q}}(G)^{n^2} \cap \overline{C}^1(G) \qquad \forall\, 1 < q < \infty$$

For the regularity of higher derivatives we use use the density of $H^{k,q}(G) \cap B^q(G)$ in $B^q(G)$ with respect to $\|\cdot\|_{q;G}$ (Theorem 9.1 and 9.2) and the inequality (Lemma 15.3)

$$\|Z_q(\pi) - \frac{1}{2}\, \pi\|_{k,q;G} \;\leq\; C_k\, \|\pi\|_{k-1,q;G}$$

Then we can prove

$$p \;=\; \mu^k \left(Z_s \;-\; \frac{1}{2} I \right)^k \, p \;\in H^{k,s}(G)$$

for all $n < s < \infty$. Therefore Theorem 15.5 holds.

Part I: Preliminaries

1 Notations

For $x_0 \in \mathbb{R}^n$, $0 < r < R$ we denote

$$B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\} \qquad B_r := \{x \in \mathbb{R}^n : |x| < r\}$$

$$A_{r,R}(x_0) := \{x \in \mathbb{R}^n : r < |x - x_0| < R\} \qquad A_{r,R} := \{x \in \mathbb{R}^n : r < |x| < R\}$$

Further we define for an open $G \subset \mathbb{R}^n$ and $k \in \mathbb{N}$

$$\begin{aligned} \overline{C}^k(G) &:= \{f \in C^k(G) : \text{ for } \alpha \in \mathbb{N}_0^n,\ |\alpha| \le k \text{ there is } f^{(\alpha)} \in C^0(\overline{G}) \\ &\qquad \text{with } f^{(\alpha)}|_G = D^\alpha f\} \end{aligned}$$

$$C_0^k(G) := \{f \in C^k(G) : \operatorname{supp}(f) \subset G\}$$

If $1 < q < \infty$ we always use the notation

$$q' := \frac{q}{q-1}$$

For $f \in L^q(G)$ denote

$$\|f\|_{q;G} := \left(\int_G |f|^q \, dx\right)^{\frac{1}{q}}$$

Usually we don't strictly distinguish between a function and the corresponding equivalence class in $L^q(G)$. For example the notation

$$f \in L^q(G) \cap C^0(G)$$

means, that there is a (unique) continuous representative.

Let

$$H^{k,q}(G) := \{u : G \to \mathbb{R} \mid u \text{ measurable, } D^\alpha u \in L^q(G) \text{ for all } |\alpha| \le k\}$$

With

$$\|u\|_{k,q;G} := \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_{q;G}^q\right)^{\frac{1}{q}} \qquad \text{for } u \in H^{k,q}(G)$$

$H^{k,q}(G)$ is a Banach space. Let

$$H_0^{k,q}(G) := \overline{C_0^\infty(G)}^{\|\cdot\|_{k,q;G}}$$

Underlined terms always denote vectors

$$\underline{u} := (u_1, \ldots, u_n)$$

Often we use the notation

$$\underline{u} \in H^{1,q}(G) \text{ instead of } \underline{u} \in H^{1,q}(G)^n$$

$$\text{or} \quad \nabla u \in L^q(G) \text{ instead of } \nabla u \in L^q(G)^n$$

if no confusion could arise. Further we use the notations

$$
\begin{aligned}
\langle f, g\rangle_G &:= \int_G f g \, dx \\
\langle \nabla f, \nabla g\rangle_G &:= \sum_{i=1}^n \int_G (\partial_i f)\, (\partial_i g)\, dx \\
\left\langle \underline{f}, \underline{g}\right\rangle_G &:= \sum_{i=1}^n \int_G f_i \, g_i \, dx \\
\left\langle \nabla \underline{f}, \nabla \underline{g}\right\rangle_G &:= \sum_{i,j=1}^n \int_G (\partial_i f_j)\, (\partial_i g_j)\, dx \\
\left\langle \nabla^2 f, \nabla^2 g\right\rangle_G &:= \sum_{i,j=1}^n \int_G (\partial_i \partial_j f)\, (\partial_i \partial_j g)\, dx
\end{aligned}
$$

and

$$
\begin{aligned}
\|\underline{f}\|_{q;G} &:= \left(\sum_{i=1}^n \|f_i\|_{q;G}^q \right)^{\frac{1}{q}} \\
\|\nabla f\|_{q;G} &:= \left(\sum_{i=1}^n \|\partial_i f\|_{q;G}^q \right)^{\frac{1}{q}} \\
\|\nabla^2 f\|_{q;G} &:= \left(\sum_{i,j=1}^n \|\partial_i \partial_j f\|_{q;G}^q \right)^{\frac{1}{q}} \\
\|\nabla \underline{f}\|_{q;G} &:= \left(\sum_{i,j=1}^n \|\partial_j f_i\|_{q;G}^q \right)^{\frac{1}{q}}
\end{aligned}
$$

if the expressions are well defined.

The inner product in $\mathbb{R}^n$ we denote most of the time by

$$x\, y := \sum_{i=1}^n x_i \, y_i \qquad \text{if } x,\ y \in \mathbb{R}^n$$

but sometimes we write

$$\langle x, y\rangle := \sum_{i=1}^n x_i \, y_i \qquad \text{if } x,\ y \in \mathbb{R}^n$$

An **exterior domain** is a domain $G \subset \mathbb{R}^n$ with $\mathbb{R}^n \backslash G$ compact and $0 \in \mathbb{R}^n \backslash \overline{G}$.

$A \subset\subset B$ means: A, $B \subset \mathbb{R}^n$ open, $\overline{A}$ bounded and $\overline{A} \subset B$

Let

$$\|f\|_{\infty;G} := \sup_{x \in G} |f(x)|$$

and

$$S_{n-1} := \{x \in \mathbb{R}^n : |x| = 1\} = \partial B_1 \qquad \omega_n := |S_{n-1}|_{n-1}$$

If X is a real normed vector space, we denote by

$$X^* := \left\{ F^* : X \to \mathbb{R} \,\middle|\, \sup_{\|x\| \neq 0} \frac{F^*(x)}{\|x\|} < \infty \right\}$$

its dual space.

The property (GA) denotes for $G \subset \mathbb{R}^n$:

$$\text{(GA)} \qquad \text{There is an open } \emptyset \neq K \subset \mathbb{R}^n \text{ with } G = \mathbb{R}^n \backslash \overline{K}$$

2 The space $\widehat{H}_\bullet^{1,q}(G)$

Definition 2.1. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

$$\begin{aligned}\widehat{H}_\bullet^{1,q}(G) \quad := \quad & \{u : G \to \mathbb{R} \,|\, u \text{ measurable},\ u \in L^q(G \cap B_R)\ \forall\, R > 0,\\ & \nabla u \in L^q(G) \text{ and for each } \eta \in C_0^\infty(\mathbb{R}^n) \text{ holds } \eta u \in H_0^{1,q}(G)\}\end{aligned}$$

Definition 2.2. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

$$\begin{aligned}\widehat{H}_0^{1,q}(G) \quad := \quad & \{u : G \to \mathbb{R} \text{ measurable} \mid u \in L^q(G \cap B_R)\ \forall\, R > 0,\ \nabla u \in L^q(G)\\ & \text{and there exists a sequence } (u_i) \subset C_0^\infty(G) \text{ so that}\\ & \|u - u_i\|_{q,G\cap B_R} \to 0 \quad \forall\, R > 0 \quad \text{and } \|\nabla u - \nabla u_i\|_{q,G} \to 0\}\end{aligned}$$

Theorem 2.3. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

(a) $H_0^{1,q}(G) \subset \widehat{H}_0^{1,q}(G) \subset \widehat{H}_\bullet^{1,q}(G)$

(b) For $u \in \widehat{H}_\bullet^{1,q}(G)$ by $\|\nabla u\|_{q,G}$ a norm is defined on $\widehat{H}_\bullet^{1,q}(G)$.

(c) Equipped with $\|\nabla \cdot\|_{q,G}$-norm $\widehat{H}_\bullet^{1,q}(G)$ is a Banach space being reflexive for $1 < q < \infty$. If $q = 2$ then $\widehat{H}_\bullet^{1,2}(G)$ is a Hilbert space with inner product $\langle \nabla u, \nabla v\rangle$ for $u, v \in \widehat{H}_\bullet^{1,2}(G)$.

(d) $\widehat{H}_0^{1,q}(G)$ is a closed subspace of $\widehat{H}_\bullet^{1,q}(G)$ and $\widehat{H}_0^{1,q}(G) = \overline{C_0^\infty(G)}^{\|\nabla\cdot\|_{q,G}}$

Proof. see [Si/So, Theorem I.2.2, p.27] □

Theorem 2.4. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

$$\begin{aligned}\widehat{H}_\bullet^{1,q}(G) \quad = \quad & \{u : G \to \mathbb{R} \text{ measurable} \mid u \in L^q(G \cap B_R)\ \forall\, R > 0,\ \nabla u \in L^q(G)\\ & \text{and there exists a sequence } (u_i) \subset C_0^\infty(G) \text{ so that}\\ & \|u - u_i\|_{q,G\cap B_R} + \|\nabla u - \nabla u_i\|_{q,G\cap B_R} \to 0 \quad \forall\, R > 0\}\end{aligned}$$

Proof. see [Si/So, Theorem I.2.4, p.29] □

Theorem 2.5. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ be open and bounded. Then

$$\widehat{H}_\bullet^{1,q}(G) = H_0^{1,q}(G)$$

Proof. Easy consequence of Definition 2.1 □

Theorem 2.6. Let $2 \le n \le q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

$$\widehat{H}_{\bullet}^{1,q}(G) = \widehat{H}_0^{1,q}(G)$$

Proof. see [Si/So, Theorem I.2.7, p.31] □

Theorem 2.7. Let $n \ge 2$ and let $G \subset \mathbb{R}^n$ be an exterior domain. Suppose $1 \le q < n$. Choose $r > 0$ with $\mathbb{R}^n \backslash G \subset B_r$ and let

$$\varphi_r \in C^\infty(\mathbb{R}^n), \quad 0 \le \varphi_r \le 1, \quad \varphi_r(x) = \begin{cases} 0 & \text{, if } |x| \le r \\ 1 & \text{, if } |x| \ge 2r \end{cases}$$

Then

1. $\widehat{H}_0^{1,q}(G) \subset \widehat{H}_{\bullet}^{1,q}(G)$ and $\widehat{H}_0^{1,q}(G) \ne \widehat{H}_{\bullet}^{1,q}(G)$
2. $\widehat{H}_{\bullet}^{1,q}(G) = \widehat{H}_0^{1,q}(G) \oplus \{\alpha\varphi_r \ : \ \alpha \in \mathbb{R}\}$ in the sense of a direct decomposition.

Proof. see [Si/So, Theorem I.2.16, p.36] □

Theorem 2.8 (Variational inequality in $\widehat{H}_{\bullet}^{1,q}(G)$). Let $G \subset \mathbb{R}^n$ $(n \ge 2)$ be either a bounded or an exterior domain and let $\partial G \in C^1$. Let $1 < q < \infty$. Then there exists a constant $C_q = C(q, G) > 0$ so that

$$\|\nabla u\|_{q,G} \ \le \ C_q \sup_{0 \ne \phi \in \widehat{H}_{\bullet}^{1,q'}(G)} \frac{\langle \nabla u, \nabla \phi \rangle}{\|\nabla \phi\|_{q',G}} \qquad \forall\, u \in \widehat{H}_{\bullet}^{1,q}(G)$$

Proof. see [Si/So, Theorem II.1.1, p.45] □

Theorem 2.9 (Functional representation in $\widehat{H}_{\bullet}^{1,q}(G)$). Let $G \subset \mathbb{R}^n$ $(n \ge 2)$ be either a bounded or an exterior domain and let $\partial G \in C^1$. Let $1 < q < \infty$. Then for every $F^* \in \widehat{H}_{\bullet}^{1,q'}(G)^*$ there exists a unique $u \in \widehat{H}_{\bullet}^{1,q}(G)$ so that

$$F^*(\phi) = \langle \nabla u, \nabla \phi \rangle \qquad \forall\, \phi \in \widehat{H}_{\bullet}^{1,q'}(G)$$

Furthermore with C_q by Theorem 2.8 holds

$$C_q^{-1}\|\nabla u\|_{q,G} \le \sup\Big\{F^*(\phi) \ : \ \phi \in \widehat{H}_{\bullet}^{1,q'}(G) \text{ and } \|\nabla\phi\|_{q',G} \le 1\Big\} \le \|\nabla u\|_{q,G}$$

Proof. see [Si/So, Theorem II.1.2, p.45] □

3 The space $\widehat{H}^{2,q}_{\bullet}(G)$

Definition 3.1. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

$$\begin{aligned}\widehat{H}^{2,q}_{\bullet}(G) \;:=\; &\{u : G \to \mathbb{R} \text{ measurable} \mid u,\ \nabla u \in L^q(G \cap B_R)\ \forall\, R > 0, \\ &\nabla^2 u \in L^q(G) \text{ and for each } \eta \in C_0^\infty(\mathbb{R}^n) \text{ holds } \eta u \in H_0^{2,q}(G)\}\end{aligned}$$

Definition 3.2. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

$$\begin{aligned}\widehat{H}^{2,q}_{0}(G) \;:=\; &\{u : G \to \mathbb{R} \mid u,\ \nabla u \in L^q(G \cap B_R)\ \forall\, R > 0,\ \nabla^2 u \in L^q(G) \\ &\text{and there exists a sequence } (u_i) \subset C_0^\infty(G) \text{ so that} \\ &\|u - u_i\|_{q,G\cap B_R} + \|\nabla u - \nabla u_i\|_{q,G\cap B_R} \to 0 \quad \forall\, R > 0 \\ &\text{and } \|\nabla^2 u - \nabla^2 u_i\|_{q,G} \to 0\}\end{aligned}$$

Theorem 3.3. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

(a) $H_0^{2,q}(G) \subset \widehat{H}^{2,q}_0(G) \subset \widehat{H}^{2,q}_{\bullet}(G)$

(b) For $u \in \widehat{H}^{2,q}_{\bullet}(G)$ by $\|\nabla^2 u\|_{q,G}$ a norm is defined on $\widehat{H}^{2,q}_{\bullet}(G)$.

(c) Equipped with $\|\nabla^2 \cdot\|_{q,G}$-norm $\widehat{H}^{2,q}_{\bullet}(G)$ is a Banach space being reflexive for $1 < q < \infty$. If $q = 2$ then $\widehat{H}^{2,2}_{\bullet}(G)$ is a Hilbert space with inner product $\langle \nabla^2 u, \nabla^2 v\rangle$ for $u, v \in \widehat{H}^{2,2}_{\bullet}(G)$.

(d) $\widehat{H}^{2,q}_0(G)$ is a closed subspace of $\widehat{H}^{2,q}_{\bullet}(G)$ and $\widehat{H}^{2,q}_0(G) = \overline{C_0^\infty(G)}^{\|\nabla^2\cdot\|_{q,G}}$

Proof. see [MueR, Satz II.1, p.126] □

Theorem 3.4. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

$$\begin{aligned}\widehat{H}^{2,q}_{\bullet}(G) \;=\; &\{u : G \to \mathbb{R} \mid u,\ \nabla u \in L^q(G \cap B_R)\ \forall\, R > 0,\ \nabla^2 u \in L^q(G) \\ &\text{and there exists a sequence } (u_i) \subset C_0^\infty(G) \text{ so that} \\ &\|u - u_i\|_{q,G\cap B_R} + \|\nabla u - \nabla u_i\|_{q,G\cap B_R} + \|\nabla^2 u - \nabla^2 u_i\|_{q,G\cap B_R} \to 0 \\ &\text{for all } R > 0\}\end{aligned}$$

Proof. see [MueR, Lemma II.3, p.129] □

Theorem 3.5. Let $n \geq 2,\ 1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ be open and bounded. Then

$$\widehat{H}^{2,q}_{\bullet}(G) = H_0^{2,q}(G)$$

Proof. Easy consequence of Definition 3.1 □

Theorem 3.6. Let $2 \le n \le q < \infty$ and let $G \subset \mathbb{R}^n$ satisfy (GA). Then

$$\widehat{H}^{2,q}_{\bullet}(G) = \widehat{H}^{2,q}_0(G)$$

Proof. see [MueR, Satz II.3, p.133] □

Theorem 3.7. Let $n \ge 2$ and let $G \subset \mathbb{R}^n$ be an exterior domain. Suppose $1 \le q < n$. Choose $r > 0$ with $\mathbb{R}^n \backslash G \subset B_r$ and let

$$\varphi_r \in C^\infty(\mathbb{R}^n), \quad 0 \le \varphi_r \le 1, \quad \varphi_r(x) = \begin{cases} 0 & \text{, if } |x| \le r \\ 1 & \text{, if } |x| \ge 2r \end{cases}$$

Furthermore define $\psi_{ri}(x) := \varphi_r(x) x_i$ for all $i = 1, .., n$. Then

1. $\varphi_r,\ \psi_{ri} \in \widehat{H}^{2,s}_{\bullet}(G)$ for all $1 < s < \infty$,
2. $\widehat{H}^{2,q}_0(G) \subset \widehat{H}^{2,q}_{\bullet}(G)$ and $\widehat{H}^{2,q}_0(G) \ne \widehat{H}^{2,q}_{\bullet}(G)$,
3. $\widehat{H}^{2,q}_{\bullet}(G) = \widehat{H}^{2,q}_0(G) \oplus \{\alpha\varphi_r \ :\ \alpha \in \mathbb{R}\} \oplus \{\sum_{i=1}^n \beta_i \psi_{ri} \ :\ \beta_i \in \mathbb{R}\}$ for $1 < q < \frac{n}{2}$,
4. $\widehat{H}^{2,q}_{\bullet}(G) = \widehat{H}^{2,q}_0(G) \oplus \{\sum_{i=1}^n \beta_i \psi_{ri} \ :\ \beta_i \in \mathbb{R}\}$ for $\frac{n}{2} \le q < n$.

Proof. see [MueR, Lemma II.8, p.140] and [MueR, Satz II.4, p.144] □

Theorem 3.8 (Variational inequality in $\widehat{H}^{2,q}_{\bullet}(G)$). Let $G \subset \mathbb{R}^n$ $(n \ge 2)$ be either a bounded or an exterior domain and let $\partial G \in C^2$. Let $1 < q < \infty$. Then there exists a constant $C_q = C(q, n, G) > 0$ so that

$$\|\Delta u\|_{q,G} \ \le\ C_q \sup_{0 \ne \phi \in \widehat{H}^{2,q'}_{\bullet}(G)} \frac{\langle \Delta u, \Delta\phi \rangle}{\|\Delta\phi\|_{q',G}} \qquad \forall\ u \in \widehat{H}^{2,q}_{\bullet}(G)$$

Proof. see [MueR, Hauptsatz, p.191] □

Theorem 3.9 (Functional representation in $\widehat{H}^{2,q}_{\bullet}(G)$). Let $G \subset \mathbb{R}^n$ $(n \ge 2)$ be either a bounded or an exterior domain and let $\partial G \in C^2$. Let $1 < q < \infty$. Then for every $F^* \in \widehat{H}^{2,q'}_{\bullet}(G)^*$ there exists a unique $u \in \widehat{H}^{2,q}_{\bullet}(G)$ so that

$$F^*(\phi) = \langle \nabla^2 u, \nabla^2 \phi \rangle \qquad \forall \phi \in \widehat{H}^{2,q'}_{\bullet}(G)$$

Furthermore there exists a constant $D_q = D(q, G)$ with

$$D_q^{-1} \|\nabla^2 u\|_{q,G} \le \sup \Big\{ F^*(\phi) \ :\ \phi \in \widehat{H}^{2,q'}_{\bullet}(G) \text{ and } \|\nabla^2 \phi\|_{q',G} \le 1 \Big\} \le \|\nabla^2 u\|_{q,G}$$

Proof. see [MueR, Lemma III.15, p.164] and Theorem 3.8 □

Lemma 3.10. Let $G \subset \mathbb{R}^n$ ($n \geq 2$) be either a bounded or an exterior domain and let $1 < q < \infty$. Then

$$\left\langle \nabla^2 u, \nabla^2 \phi \right\rangle = \langle \Delta u, \Delta \phi \rangle \qquad \text{for all } u \in \widehat{H}^{2,q}_{\bullet}(G),\ \phi \in \widehat{H}^{2,q'}_{\bullet}(G)$$

Proof. (a) Consider φ_r, ψ_{ri} from Theorem 3.7. It holds

$$\partial_j \varphi_r(x) = \partial_k \partial_j \varphi_r(x) = 0 \qquad \text{for } |x| \leq r \text{ and for } |x| \geq 2r$$

Furthermore

$$\partial_j \psi_{ri}(x) = \partial_j \varphi_r(x) x_i + \varphi_r(x) \delta_{ij}$$
$$\partial_k \partial_j \psi_{ri}(x) = \partial_k \partial_j \varphi_r(x) x_i + \partial_j \varphi_r(x) \delta_{ik} + \partial_k \varphi_r(x) \delta_{ij}$$

Therefore

$$\partial_j \partial_k \psi_{ri},\ \partial_j \partial_k \varphi_r \in C_0^\infty(A_{\frac{r}{2},3r})$$

(b) Suppose $G \subset \mathbb{R}^n$ is an exterior domain, $\mathbb{R}^n \backslash G \subset B_{\frac{r}{2}}$. By Theorem 3.6 and 3.7 for $u \in \widehat{H}^{2,q}_{\bullet}(G)$ there are $v \in \widehat{H}^{2,q}_0(G)$ and $f \in C^\infty(\mathbb{R}^n)$ such that

$$\partial_i \partial_j f \in C_0^\infty(A_{\frac{r}{2},3r}) \subset C_0^\infty(G) \qquad \text{and} \qquad u = v + f$$

There is by Definition 3.2 a sequence $(v_k) \subset C_0^\infty(G)$ with $\|\nabla^2(v_k - v)\|_{q;G} \to 0$. Let $\phi \in \widehat{H}^{2,q'}_{\bullet}(G)$. Then

$$\begin{aligned}
\left\langle \nabla^2 u, \nabla^2 \phi \right\rangle_G &= \sum_{i,j=1}^n \int_G \partial_i \partial_j v\ \partial_i \partial_j \phi\ dx\ +\ \sum_{i,j=1}^n \int_G \underbrace{\partial_i \partial_j f}_{\in C_0^\infty(G)}\ \partial_i \partial_j \phi\ dx \\
&= \lim_{k\to\infty} \sum_{i,j=1}^n \int_G \partial_i \partial_j v_k\ \partial_i \partial_j \phi\ dx\ -\ \sum_{i,j=1}^n \int_G \partial_j (\underbrace{\partial_i \partial_i f}_{\in C_0^\infty(G)}) \partial_j \phi\ dx \\
&= \lim_{k\to\infty} \sum_{i,j=1}^n \int_G \partial_i \partial_i v_k\ \partial_j \partial_j \phi\ dx\ +\ \sum_{i,j=1}^n \int_G \partial_i \partial_i f\ \partial_j \partial_j \phi\ dx \\
&= \langle \Delta v, \Delta \phi \rangle_G + \langle \Delta f, \Delta \phi \rangle_G = \langle \Delta u, \Delta \phi \rangle_G
\end{aligned}$$

(c) Suppose $G \subset \mathbb{R}^n$ is a bounded domain. Then $\widehat{H}^{2,q}_{\bullet}(G) = H^{2,q}_0(G)$ and the assertion follows as in (b) with $f = 0$. □

Theorem 3.11 (Functional representation in $\widehat{H}^{2,q}_\bullet(G)$). Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be either a bounded or an exterior domain and let $\partial G \in C^2$. Let $1 < q < \infty$. Then for every $F^* \in \widehat{H}^{2,q'}_\bullet(G)^*$ there exists a unique $u \in \widehat{H}^{2,q}_\bullet(G)$ so that

$$F^*(\phi) = \langle \Delta u, \Delta \phi \rangle \qquad \forall \phi \in \widehat{H}^{2,q'}_\bullet(G)$$

Furthermore there exists a constant $K_q = K(n, q, G)$ with

$$K_q^{-1} \|\Delta u\|_{q,G} \leq \sup \left\{ F^*(\phi) \ : \ \phi \in \widehat{H}^{2,q'}_\bullet(G) \text{ and } \|\Delta \phi\|_{q',G} \leq 1 \right\} \leq \|\Delta u\|_{q,G}$$

Proof. Let $F^* \in \widehat{H}^{2,q'}_\bullet(G)^*$ be given. By Theorem 3.9 there exists a unique $u \in \widehat{H}^{2,q}_\bullet(G)$ and a constant $D_q = D(q, G) > 0$ with

$$F^*(\phi) = \langle \nabla^2 u, \nabla^2 \phi \rangle \qquad \forall \phi \in \widehat{H}^{2,q'}_\bullet(G)$$

$$D_q^{-1} \|\nabla^2 u\|_{q,G} \leq \sup \left\{ F^*(\phi) \ : \ \phi \in \widehat{H}^{2,q'}_\bullet(G) \text{ and } \|\nabla^2 \phi\|_{q',G} \leq 1 \right\} \leq \|\nabla^2 u\|_{q,G}$$

By Lemma 3.10 it holds

$$F^*(\phi) = \langle \Delta u, \Delta \phi \rangle \qquad \forall \phi \in \widehat{H}^{2,q'}_\bullet(G)$$

and

$$\begin{aligned}
D_q^{-1} \|\Delta u\|_{q,G} &\leq K_n \, D_q^{-1} \|\nabla^2 u\|_{q,G} \\
&\leq K_n \sup \left\{ F^*(\phi) \ : \ \phi \in \widehat{H}^{2,q'}_\bullet(G) \text{ and } \|\nabla^2 \phi\|_{q',G} \leq 1 \right\} \\
&= K_n \sup_{0 \neq \phi \in \widehat{H}^{2,q'}_\bullet(G)} \frac{F^*(\phi)}{\|\nabla^2 \phi\|_{q',G}} \\
&\leq K_n^2 \sup_{0 \neq \phi \in \widehat{H}^{2,q'}_\bullet(G)} \frac{F^*(\phi)}{\|\Delta \phi\|_{q',G}} \\
&= \sup \left\{ F^*(\phi) \ : \ \phi \in \widehat{H}^{2,q'}_\bullet(G) \text{ and } \|\Delta \phi\|_{q',G} \leq 1 \right\}
\end{aligned}$$

□

Remark By Theorem 3.9 and Lemma 3.10 we derive, that $\|\nabla^2 \cdot\|_{q;G}$ and $\|\Delta \cdot\|_{q;G}$ are equivalent norms on $\widehat{H}^{2,q}_\bullet(G)$.

4 The spaces $A^q(G)$ and $B^q(G)$

Definition 4.1. Let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^2$. Let $1 < q < \infty$. Then

$$\begin{aligned} A^q(G) &:= \{\Delta u \,:\, u \in \widehat{H}^{2,q}_{\bullet}(G)\} \\ B^q(G) &:= \{h \in L^q(G) \,:\, \langle h, \Delta\phi\rangle_G = 0 \quad \forall\, \phi \in \widehat{H}^{2,q'}_{\bullet}(G)\} \end{aligned}$$

Theorem 4.2. Let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^2$. Let $1 < q < \infty$. Then

$$L^q(G) = A^q(G) \oplus B^q(G)$$

in the sense of a direct decomposition

Proof. see [MueR, Satz IV.2.1, p.201] □

Remark By Weyl's Lemma for $h \in B^q(G)$ holds (for a representative) $\Delta h = 0$. For bounded domains even holds

$$B^q(G) = \{h \in L^q(G) : \Delta h = 0\}$$

For exterior domains on the other hand there are harmonic L^q-functions, which are **not** in $B^q(G)$ (see [MueR] or Lemma 4.4).

Lemma 4.3. Let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^2$. Let $1 < q < \infty$ and $h \in B^q(G) \cap H^{1,q}(G)$. Then

$$\langle \nabla h, \nabla\phi\rangle_G = 0$$

for each $\phi \in \widehat{H}^{1,q'}_{\bullet}(G)$

Proof. (a) For $\phi \in \widehat{H}^{1,q'}_0(G)$ there is a sequence $(\phi_k) \subset C_0^\infty(G)$ such that

$$\|\nabla\phi_k - \nabla\phi\|_{q';G} \to 0$$

Then

$$\langle \nabla h, \nabla\phi\rangle_G = \lim_{k\to\infty} \langle \nabla h, \nabla\phi_k\rangle_G = -\lim_{k\to\infty} \langle h, \Delta\phi_k\rangle_G = 0$$

(b) For $\phi = \varphi_r$ by Theorem 2.7 holds $\nabla\varphi_r \in C_0^\infty(G)^n$ and $\varphi_r \in \widehat{H}^{2,q'}_{\bullet}(G)$. Then

$$\langle \nabla h, \nabla\varphi_r\rangle_G = -\langle h, \Delta\varphi_r\rangle_G = 0$$

(c) By Theorem 2.5 - 2.7 the assertion follows. □

Lemma 4.4. Let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^2$. Let φ_r, ψ_{ri} as in Theorem 3.7. Let

$$S(z) := \begin{cases} \frac{1}{(n-2)\omega_n} |z|^{2-n} & , \quad z \neq 0,\ n \geq 3 \\ -\frac{1}{2\pi} \ln |z| & , \quad z \neq 0,\ n = 2 \\ 0 & , \quad z = 0,\ n \geq 2 \end{cases}$$

Then for all $i, j = 1, \ldots, n$ holds

$$\langle \partial_j S, \Delta \psi_{ri} \rangle_G \ = \ -\delta_{ij}$$

Proof. For $z \neq 0$ holds

$$\partial_j S(z) \ = \ -\frac{1}{\omega_n} \frac{z_j}{|z|^n}$$

Then

$$\begin{aligned} \langle \partial_j S, \Delta \psi_{ri} \rangle_G \quad &= \quad \langle \partial_j S, \Delta (x_i \varphi_r) \rangle_G \\ &= \quad -\big\langle S, \Delta \big[\delta_{ij} \varphi_r + x_i (\partial_j \varphi_r) \big] \big\rangle_G \\ &\underset{\partial_m \varphi_r \in C_0^\infty(G)}{=\!=} \quad \delta_{ij} \langle \nabla S, \nabla \varphi_r \rangle_G - \langle \Delta S, x_i (\partial_j \varphi_r) \rangle_G \\ &\underset{\Delta S = 0}{=\!=} \quad \delta_{ij} \langle \nabla S, \nabla \varphi_r \rangle_{G \cap B_{2r}} \\ &= \quad -\delta_{ij} \langle \Delta S, \varphi_r \rangle_{G \cap B_{2r}} \\ &\qquad + \delta_{ij} \int_{\partial B_{2r}} \sum_{l=1}^n (\partial_l S)(z) \underbrace{\varphi_r(z)}_{=1} \frac{z_l}{|z|} \, d\omega_z \\ &= \quad -\delta_{ij} \frac{1}{\omega_n} \int_{\partial B_{2r}} \sum_{l=1}^n \frac{z_l}{|z|^n} \frac{z_l}{|z|} \, d\omega_z \ = \ -\delta_{ij} \end{aligned}$$

□

5 Theorems about differentiable functions

Lemma 5.1. Let $n \geq 2,\ R > 0,\ h > 0$ and

$$\begin{aligned}
Z^+_{R,h} &:= \{x = (x', x_n) \in \mathbb{R}^n \ : \ |x'| < R,\ 0 < x_n < h\} \\
Z^-_{R,h} &:= \{x = (x', x_n) \in \mathbb{R}^n \ : \ |x'| < R,\ -h < x_n < 0\} \\
Z_{R,h} &:= \{x = (x', x_n) \in \mathbb{R}^n \ : \ |x'| < R,\ |x_n| < h\} \\
E_R &:= \{x = (x', x_n) \in \mathbb{R}^n \ : \ |x'| < R,\ x_n = 0\}
\end{aligned}$$

Suppose

$$f \in C^0\left(\overline{Z^+_{R,h}}\right) \cap \overline{C}^1\left(Z^+_{R,h}\right)$$

Let

$$F(x) := \begin{cases} f(x', x_n) & , \quad \text{if } 0 \leq x_n < h \\ -3f(x', -x_n) + 4f(x', -\frac{x_n}{2}) & , \quad \text{if } -h < x_n < 0 \end{cases}$$

Then

$$F \in C^1(Z_{R,h}), \quad F|_{Z^+_{R,h}} = f$$

Proof. (a) There are $f_i \in C^0\left(\overline{Z^+_{R,h}}\right)$, $i = 1 \ldots n$ with $f_i|_{Z^+_{R,h}} = \partial_i f$

(b) For $i = 1 \ldots n-1$ and $|x'| < R$ let $(h_k) \subset \mathbb{R}$ with $0 < |h_k| < R - |x'|$, $h_k \to 0$. Let $0 < x_n < h$. Then $(x' + h_k e_i, x_n) \in Z^+_{R,h}$ and

$$f(x' + h_k e_i, x_n) - f(x', x_n) = \int_0^{h_k} (\partial_i f)(x' + te_i, x_n)\, dt$$

For $x_n \to 0$ we get

$$\begin{aligned}
f(x' + h_k e_i, 0) - f(x', 0) &= \int_0^{h_k} f_i(x' + te_i, 0)\, dt \\
&\underset{\text{mean value theorem}}{=} f_i(x' + \zeta_k e_i, 0) h_k
\end{aligned}$$

with ζ_k between 0 and h_k (this implies $|\zeta_k| \to 0$)
Therefore

$$\lim_{h \to 0} \frac{f(x' + he_i, 0) - f(x', 0)}{h} = f_i(x', 0)$$

(c) For $i = n$ and $|x'| < R,\ 0 < x_n < \frac{h}{2}$ let $(h_k) \subset \mathbb{R}$ with $0 < h_k < \frac{h}{2}$, $h_k \to 0$. Then

$$f(x', x_n + h_k) - f(x', x_n) = \int_0^{h_k} (\partial_n f)(x', x_n + t)\, dt$$

For $x_n \to 0$ we get

$$f(x', h_k) - f(x', 0) = \int_0^{h_k} f_n(x', t)\, dt = f_n(x', \zeta_k) h_k$$

with $0 < \zeta_k < h_k$. Therefore

$$\lim_{\substack{h\to 0\\ h>0}} \frac{f(x',h)-f(x',0)}{h} = f_n(x',0)$$

(d) Obviously

$$F|_{Z^+_{R,h}} \in C^1(Z^+_{R,h}) \qquad F|_{Z^-_{R,h}} \in C^1(Z^-_{R,h})$$

(e) Let $i = 1 \dots n-1$ and $|x'| < R$. Then

$$\lim_{h\to 0} \frac{F(x'+he_i,0)-F(x',0)}{h} = \lim_{h\to 0} \frac{f(x'+he_i,0)-f(x',0)}{h} \underset{(b)}{=} f_i(x',0)$$

(f) Let $|x'| < R$. Then

$$\lim_{\substack{h\to 0\\ h>0}} \frac{F(x',h)-F(x',0)}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{f(x',h)-f(x',0)}{h} \underset{(c)}{=} f_n(x',0)$$

Furthermore

$$\begin{aligned}
\lim_{\substack{h\to 0\\ h<0}} \frac{F(x',h)-F(x',0)}{h} &= \lim_{\substack{h\to 0\\ h<0}} \frac{-3f(x',-h)+4f(x',-\frac{h}{2})-f(x',0)}{h} \\
&= \lim_{\substack{h\to 0\\ h<0}} \frac{-3f(x',-h)+3f(x',0)}{h} + \frac{4f(x',-\frac{h}{2})-4f(x',0)}{h} \\
&= \lim_{\substack{h'\to 0\\ h'>0}} \frac{3f(x',h')-3f(x',0)}{h'} - \frac{2f(x',h')-2f(x',0)}{h'} \\
&\underset{(c)}{=} f_n(x',0)
\end{aligned}$$

(g) So for $i = 1, \dots, n-1$ holds

$$\partial_i F(x) = \begin{cases} f_i(x',x_n) & , \ x_n \geq 0 \\ -3f_i(x',x_n) + 4f_i(x',-\frac{x_n}{2}) & , \ x_n < 0 \end{cases}$$

and

$$\partial_n F(x) = \begin{cases} f_n(x',x_n) & , \ x_n \geq 0 \\ 3f_n(x',x_n) - 2f_n(x',-\frac{x_n}{2}) & , \ x_n < 0 \end{cases}$$

That is

$$\nabla F \in C^0\left(Z_{R,h}\right)^n$$

□

Theorem 5.2. Let $G \subset \mathbb{R}^n$ be open and bounded with $\partial G \in C^1$.
Then $f \in \overline{C}^1(G)$ if and only if there exists $\tilde{f} \in C^1_0(\mathbb{R}^n)$ such that $\tilde{f}|_G = f$

Proof. Let $f \in \overline{C}^1(G)$ (the inversion is trivial). We denote again by $f \in C^0(\overline{G})$ the continuation of f. There are $f_i \in C^0(\overline{G})$ with $f_i|_G = \partial_i f$

(a) For $x_0 \in \partial G$ there exists an open $V_{x_0} \subset \mathbb{R}^n$ with $x_0 \in V_{x_0}$. Further there is $R > 0$ and a diffeomorphism $\phi_{x_0} : V_{x_0} \to Z_{R,R}$ such that holds: $\phi_{x_0}(V_{x_0} \cap \partial G) = E_R$ and $\phi_{x_0}(V_{x_0} \cap G) = Z^+_{R,R}$.
Let $\widetilde{V}_{x_0} := \phi_{x_0}^{-1}(Z_{\frac{R}{2},\frac{R}{4}})$

(b) With $\widetilde{V}_x$ by (a) $(x \in \partial G)$ holds:

$$\partial G \subset \bigcup_{x \in \partial G} \widetilde{V}_x$$

Because of the compactness of ∂G there are $x_1, \ldots, x_N \in \partial G$ such that with $\widetilde{V}_i := \widetilde{V}_{x_i}$ holds

$$\partial G \subset \bigcup_{i=1}^N \widetilde{V}_i$$

Let $\tilde{V}_0 := G$. Then

$$\overline{G} \subset \bigcup_{i=0}^N \widetilde{V}_i$$

Choose $\varphi_i \in C_0^\infty(\widetilde{V}_i)$ for $i = 0, 1, \ldots, N$ such that

$$\sum_{i=0}^N \varphi_i(x) = 1 \qquad \text{for all } x \in \overline{G}$$

Define

$$g_i := \varphi_i f$$

Then

$$g_0 \in C_0^1(\mathbb{R}^n)$$

and for $i, j = 1, \ldots, N$ holds

$$\partial_j g_i \;=\; (\partial_j \varphi_i) f + \varphi_i(\partial_j f) \;=\; g_j^{(i)}|_G$$

with $g_j^{(i)} := (\partial_j \varphi_i) f + \varphi_i f_j \;\in C^0(\overline{G})$

Define for $i = 1, \ldots, N$

$$h_i(x) \;:=\; g_i\left(\phi_i^{-1}(x)\right) \qquad \text{for all } x \in Z^+_{R_i,R_i} \cup E_{R_i} = \phi_i(V_i \cap \overline{G})$$

Then

$$h_i \in C^0\left(\overline{Z^+_{\frac{R_i}{2},\frac{R_i}{4}}}\right)$$

and

$$\partial_j h_i\Big|_{Z^+_{\frac{R_i}{2},\frac{R_i}{4}}} = \sum_{k=1}^n \left[g_k^{(i)}\left(\phi_i^{-1}\right)\right]\left[\partial_j \left(\phi_i^{-1}\right)_k\right]\Big|_{Z^+_{\frac{R_i}{2},\frac{R_i}{4}}}$$

By Lemma 5.1 there is a

$$\tilde{h}_i \in C^1(Z_{\frac{R_i}{2},\frac{R_i}{4}}) \qquad \text{such that} \qquad \tilde{h}_i\Big|_{Z^+_{\frac{R_i}{2},\frac{R_i}{4}}} = h_i$$

By the Definition of the continuation in Lemma 5.1 we see

$$\tilde{h}_i \in C^1_0(Z_{R_i,R_i})$$

Let

$$\tilde{f}_i(y) := \tilde{h}_i\left(\phi_i(y)\right) \qquad \text{for all } y \in V_i = \phi_i^{-1}(Z_{R_i,R_i})$$

Then

$$\tilde{f}_i \in C^1_0(V_i) \qquad \text{and} \qquad \tilde{f}_i\Big|_{G\cap V_i} = g_i$$

At least

$$\tilde{f} := g_0 + \sum_{i=1}^{N} \tilde{f}_i \ \in C^1_0(\mathbb{R}^n), \qquad \tilde{f}\Big|_G = f$$

□

Lemma 5.3. Let $f \in C^1(\mathbb{R})$ with $f(0) = 0$ and $|f'(t)| \le L$ for all $t \in \mathbb{R}$. Further let $G \subset \mathbb{R}^n$ be open and $1 < q < \infty$. Then for $u \in H^{1,q}(G)$ holds

$$f(u) \in H^{1,q}(G) \qquad \nabla f(u) = f'(u)\nabla u$$

Proof. see [SiDGL, Satz 6.14] □

Lemma 5.4. Let $G \subset \mathbb{R}^n$ be open and bounded. Let $u \in H^{1,q}(G)$ (strictly: let u be a representative) and

$$Z(u) \ := \ \{x \in G \ : \ u(x) = 0\}$$

Then

$$\partial_i u(x) = 0 \qquad \text{for almost all } x \in Z(u) \quad \text{and for all } i = 1, \dots, n$$

Proof. see [SiDGL, Satz 6.15] □

Theorem 5.5. Let $1 \le q < \infty$ and let $G \subset \mathbb{R}^n$ be open and bounded. Suppose $u \in C^0(\overline{G}) \cap H^{1,q}(G)$ and $u\big|_{\partial G} = 0$. Then

$$u \in H^{1,q}_0(G)$$

Proof. (a) Choose

$$\varphi \in C^\infty(\mathbb{R}), \quad 0 \le \varphi \le 1, \quad \varphi(t) = \varphi(-t), \quad \varphi(t) = \begin{cases} 0 & ,\text{if } |t| \le 1 \\ 1 & ,\text{if } |t| \ge 2 \end{cases}$$

For $k \in \mathbb{N}$ let

$$f_k(t) := \int_0^t \varphi(ks)\, ds$$

Then

$$f_k(t) = 0 \qquad \forall\, |t| \le \frac{1}{k}$$

and

$$|t - f_k(t)| \le \int_0^{|t|} (1 - \varphi(ks))\, ds \le \min\left(|t|, \frac{2}{k}\right)$$

Furthermore

$$f_k'(t) = \varphi(kt) \underset{(k\to\infty)}{\longrightarrow} \begin{cases} 0 & ,\text{if } t = 0 \\ 1 & ,\text{if } t \ne 0 \end{cases}$$

(b) Let

$$u_k(x) := f_k(u(x))$$

By Lemma 5.3 holds

$$u_k \in H^{1,q}(G)$$

Because $u\big|_{\partial G} = 0$, $u \in C^0(\overline{G})$ and ∂G compact, there are $G_k \subset\subset G$ with $|u(x)| \le \frac{1}{k}$ for all $x \in G \backslash G_k$
Therefore

$$u_k(x) = 0 \qquad \forall\, x \in G \backslash G_k$$

and

$$u_k \in H_0^{1,q}(G)$$

(c) Now

$$|u_k(x) - u(x)| \;=\; |f_k(u(x)) - u(x)| \underset{\text{(a)}}{\le} \min\left(|u(x)|, \frac{2}{k}\right) \to 0$$

for all $x \in G$ and therefore by the dominated convergence theorem

$$\|u_k - u\|_{q;G} \to 0 \qquad (k \to \infty)$$

(d) By Lemma 5.4 for almost every $x \in Z(u)$ holds

$$\partial_i u(x) - \partial_i u_k(x) = \partial_i u(x)\Big[1 - \varphi(k\, u(x))\Big] = 0$$

For $x \in G \backslash Z(u)$ we get by (a):

$$\partial_i u(x) - \partial_i u_k(x) \to 0$$

Again by the dominated convergence theorem

$$\begin{aligned}\int_G |\partial_i u - \partial_i u_k|^q \, dx &= \int_G \Big|\partial_i u(x)\Big[1 - \varphi(k\,u(x))\Big]\Big|^q \, dx \\ &= \int_{G\setminus Z(u)} \Big|\partial_i u(x)\Big[1 - \varphi(k\,u(x))\Big]\Big|^q \, dx \to 0\end{aligned}$$

(e) By the closedness of $H_0^{1,q}(G)$ in $H^{1,q}(G)$ it follows

$$u \in H_0^{1,q}(G)$$

□

Theorem 5.6. Let $G \subset \mathbb{R}^n$ be open with $\partial G \in C^1$. Suppose $u \in C^1(\mathbb{R}^n)$ and $u\big|_{\partial G} = 0$.
Then there is $\lambda : \partial G \to \mathbb{R}$ with

$$(\nabla u)(x) = \lambda(x) N(x) \qquad \text{for all } x \in \partial G$$

where $N(x)$ is the outer normal in $x \in \partial G$

Proof. Let $x \in \partial G$. Then there are open $U \subset \mathbb{R}^n$ ($x \in U$) and $O \subset \mathbb{R}^{n-1}$, and there is a one to one map $\phi : O \to \partial G \cap U$ with $\phi \in C^1(O;\mathbb{R}^n)$ and $\phi(s) = x$ with a suitable $s \in O$. That is

$$u(\phi(t)) = 0 \qquad \forall\, t \in O$$

Differentiating this equation we derive by the chain rule

$$0 = \left\langle (\nabla u)(\phi(t)), \frac{\partial \phi}{\partial t_i}(t) \right\rangle \qquad \forall\, t \in O$$

Especially for $t = s$

$$0 = \left\langle (\nabla u)(x), \frac{\partial \phi}{\partial t_i}(s) \right\rangle$$

It is known that for the tangent space holds

$$T_x(\partial G) = \operatorname{span}\left(\frac{\partial \phi}{\partial t_1}(s), \ldots, \frac{\partial \phi}{\partial t_{n-1}}(s)\right)$$

and

$$N(x) \perp T_x(\partial G)$$

Therefore there is a $\lambda(x) \in \mathbb{R}$ such that

$$(\nabla u)(x) = \lambda(x) N(x)$$

□

Lemma 5.7. Let $1 < q < \infty$ and let $H := \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ be the upper half space. Suppose $u \in H_0^{1,q}(H) \cap C^0(\overline{H})$. Then

$$u\Big|_{\partial H} = 0$$

Proof. Let $u_k \in C_0^\infty(H)$ with $\|u-u_k\|_{1,q;H} \to 0$. Then

$$u_k(x',x_n) = u_k(x',x_n) - \underbrace{u_k(x',0)}_{=0} = \int_0^{x_n} (\partial_n u_k)(x',t)dt$$

As a convergent sequence (u_k) is bounded

$$\|u_k\|_{1,q;G} \le C \qquad \forall\, k \in \mathbb{N}$$

For $x_n \le a$ we get by Hölder's inequality

$$\begin{aligned}
|u_k(x',x_n)| &\le \left[\int_0^a |(\partial_n u_k)(x',t)|^q dt\right]^{\frac{1}{q}} a^{\frac{1}{q'}} \\
\int_{|x'|\le r} |u_k(x',x_n)|^q dx' &\le \int_{|x'|\le r}\int_0^a |(\partial_n u_k)(x',t)|^q dt\, dx'\; a^{\frac{q}{q'}} \\
&\le \int_{\mathbb{R}^{n-1}}\int_0^\infty |(\partial_n u_k)(x',t)|^q dt\, dx'\; a^{\frac{q}{q'}} \\
&\le C^q\, a^{q-1} \\
\int_0^a \int_{|x'|\le r} |u_k(x',x_n)|^q dx' dx_n &\le C^q\, a^q \qquad \forall\, r>0
\end{aligned}$$

Therefore

$$\begin{aligned}
\left[\int_0^a\!\!\int_{|x'|\le r} |u(x',x_n)|^q dx' dx_n\right]^{\frac{1}{q}} &\le \left[\int_0^a\!\!\int_{|x'|\le r} |u_k(x',x_n)|^q dx' dx_n\right]^{\frac{1}{q}} + \\
&\quad + \left[\int_0^a\!\!\int_{|x'|\le r} |(u-u_k)(x',x_n)|^q dx' dx_n\right]^{\frac{1}{q}} \\
&\le Ca + \|u_k - u\|_{q;H}
\end{aligned}$$

For $(k\to\infty)$ we derive

$$\left[\int_0^a \int_{|x'|\le r} |u(x',x_n)|^q dx' dx_n\right]^{\frac{1}{q}} \le Ca \qquad \forall\, a>0 \quad \forall\, r>0$$

Define for $r>0$ and $0 \le x_n \le \infty$

$$f_r(x_n) := \int_{|x'|\le r} |u(x',x_n)|^q dx'$$

Then f_r is continuous in $[0,\infty[$ for $r>0$ and

$$\int_0^a f_r(x_n)\, dx_n \le C^q a^q \qquad \forall\, a>0$$

Therefore

$$\begin{aligned}\int_{|x'|\le r} |u(x',0)|^q\,dx' &= f_r(0) = \frac{1}{a}\int_0^a f_r(0)\,dx_n \\ &= \frac{1}{a}\int_0^a (f_r(0)-f_r(x_n))\,dx_n + \frac{1}{a}\int_0^a f_r(x_n)\,dx_n \\ &\le \max_{0\le x_n\le a} |f_r(0)-f_r(x_n)| + C^q a^{q-1}\end{aligned}$$

For $a \to 0$ we get

$$\int_{|x'|\le r} |u(x',0)|^q dx' \le 0 \qquad \forall\, r > 0$$

Because u is continuous by a standard argument we finally derive

$$u(x',0) = 0 \qquad \forall\, x' \in \mathbb{R}^{n-1}$$

□

Theorem 5.8. Let $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be open with $\partial G \in C^1$. Suppose $f \in \overline{C}^1(G) \cap H_0^{1,q}(G)$. Then

$$f\Big|_{\partial G} = 0$$

Proof. Let $x_0 \in \partial G$ be arbitrary but fixed. There is an open $x_0 \in V \subset \mathbb{R}^n$ and a C^1-diffeomorphism $\phi : V \to Z_{R,R}$, $(R > 0)$ with $\phi(V \cap \partial G) = E_R$, $\phi(V \cap G) = Z^+_{R,R}$ (for the definition of $Z_{R,R}$, E_R, $Z^+_{R,R}$ see Lemma 5.1). Without loss of generality we can assume $\phi \in \overline{C}^1(V;\mathbb{R}^n)$ and $\phi^{-1} \in \overline{C}^1(Z_{R,R};\mathbb{R}^n)$.

Choose $\varphi \in C_0^\infty(V)$ such that $\varphi(x_0) = 1$. By Lemma A.13 we get

$$g := \varphi f \in H_0^{1,q}(G \cap V)$$

and obviously by Theorem 5.2

$$g \in C_0^1(V)$$

By Lemma A.14 holds

$$h := g \circ \phi^{-1} \in H_0^{1,q}(Z^+_{R,R}) \cap C_0^1(Z_{R,R})$$

Define

$$\tilde{h}(x) := \begin{cases} h(x) & , \quad x \in Z_{R,R} \\ 0 & , \quad x \in \mathbb{R}^n \backslash Z_{R,R} \end{cases}$$

Then

$$\tilde{h} \in H_0^{1,q}(H) \cap C^0(\overline{H})$$

By Lemma 5.7 follows

$$\tilde{h}\Big|_{\partial H} = 0$$

Therefore

$$0 = \tilde{h}(\phi(x_0)) = h(\phi(x_0)) = g(x_0) = \varphi(x_0) f(x_0) = f(x_0)$$

□

6 Existence of $\quad \zeta \in C_0^k(\mathbb{R}^n), \quad \zeta\Big|_{\partial G} = 0, \quad \nabla\zeta\Big|_{\partial G} = N$

Theorem 6.1. Let $n \geq 2$, $k \in \mathbb{N}$ and let $G \subset \mathbb{R}^n$ either be a bounded or an exterior domain with $\partial G \in C^{k+1}$. Then there is

$$\zeta \in C_0^k(\mathbb{R}^n), \quad \zeta\Big|_{\partial G} = 0, \quad \nabla\zeta\Big|_{\partial G} = N$$

where N is the unitary outer normal of G.

Proof. (a) Denote $M := \partial G$. For $x_0 \in M$ there is an open $x_0 \in \widetilde{V} \subset \mathbb{R}^n$ and $\tilde{h} : \widetilde{V} \to \mathbb{R}$, $\tilde{h} \in C^{k+1}(\widetilde{V})$ such that $(\nabla\tilde{h})(x_0) \neq 0$ and

$$M \cap \widetilde{V} = \{x \in \widetilde{V} : \tilde{h}(x) = 0\}$$

Like in Theorem 5.6 one can prove

$$(\nabla\tilde{h})(x) = \pm|(\nabla\tilde{h})(x)|N(x) \qquad \forall\, x \in M \cap \widetilde{V}$$

Because $(\nabla\tilde{h})(x_0) \neq 0$ there is an open $x_0 \in V \subset \widetilde{V}$ such that

$$(\nabla\tilde{h})(x) \neq 0 \qquad \forall\, x \in V$$

$\nabla\tilde{h}$ and $|\nabla\tilde{h}|\, N$ both are continuous in $V \cap M$ So we can assume without loss of generality

$$(\nabla\tilde{h})(x) = +|(\nabla\tilde{h})(x)|N(x) \qquad \forall\, x \in M \cap V$$

Define

$$h := \frac{\tilde{h}}{|\nabla\tilde{h}|} \in C^k(V)$$

Then

$$h(x) = 0 \qquad \forall\, x \in M \cap V$$

and for $x \in M \cap V$

$$(\nabla h)(x) = \underbrace{\tilde{h}(x)}_{=0}\, \nabla\left(\frac{1}{|\nabla\tilde{h}(x)|}\right) + \underbrace{\frac{\nabla\tilde{h}}{|\nabla\tilde{h}|}}_{=N(x)} = N(x)$$

(b) Because M is compact and by (a), there are open $V_1, \ldots, V_m \subset \mathbb{R}^n$ and $h_i \in C^k(V_i)$ such that

$$(\nabla h_i)(x) \neq 0 \qquad \forall\, x \in V_i \quad \forall i = 1, \ldots, m$$

and

$$M \cap V_i = \{x \in V_i : h_i(x) = 0\}$$

Furthermore

$$M \subset \bigcup_{i=1}^m V_i$$

and

$$\nabla h_i(x) = N(x) \qquad \forall\, x \in M \cap V_i$$

Choose $\phi_k \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \leq \phi_k \leq 1, \quad \mathrm{supp}(\phi_k) \subset V_k, \quad \sum_{k=1}^{m} \phi_k(x) = 1 \quad \forall\, x \in M$$

Define

$$f_j := \phi_j h_j \ \in C_0^k(V_j) \subset C_0^k(\mathbb{R}^n)$$

and

$$f := \sum_{j=1}^{m} f_j \ \in C_0^k(\mathbb{R}^n)$$

For $x \in M$ holds

$$f(x) = \sum_{\substack{j=1 \\ x \in V_j}}^{m} \phi_j(x)\, \underbrace{h_j(x)}_{=0} = 0$$

and

$$\begin{aligned} (\nabla f)(x) &= \sum_{\{j : x \in V_j\}} \nabla \left[\phi_j(x) h_j(x)\right] = \sum_{\{j : x \in V_j\}} \Big[(\nabla \phi_j)(x)\, \underbrace{h_j(x)}_{=0} + \phi_j(x)\, \underbrace{(\nabla h_j)(x)}_{=N(x)}\Big] \\ &= \sum_{\{j : x \in V_j\}} \phi_j(x) N(x) = \sum_{j=1}^{m} \phi_j(x) N(x) = N(x) \end{aligned}$$

□

7 Regularity theorems

Theorem 7.1. Let $k \in \mathbb{Z},\ k \geq 0$ and let $G \subset \mathbb{R}^n$ $(n \geq 2)$ satisfy (GA) and $\partial G \in C^{2+k}$. Let $1 < q < \infty$ and let $p \in \widehat{H}^{1,q}_{\bullet}(G)$. Let $x_0 \in \partial G$ and $R_0 > 0$ and assume that there is $f \in H^{k,q}(G \cap B_{R_0}(x_0))$ so that

$$\langle \nabla p, \nabla \phi \rangle = \langle f, \phi \rangle \qquad \text{for all } \phi \in \widehat{H}^{1,q'}_{\bullet}(G \cap B_{R_0}(x_0))$$

Then there exists $0 < R_k < R_0$ so that $p\big|_{G \cap B_{R_k}(x_0)} \in H^{2+k,q}(G \cap B_{R_k}(x_0))$ and with a constant $C_k = C(k, R_k, R_1, G, q) > 0$

$$\|p\|_{2+k,q;G\cap B_{R_k}(x_0)} \ \leq\ C_k \left(\|f\|_{k,q;G\cap B_{R_0}(x_0)} + \|p\|_{1,q;G\cap B_{R_0}(x_0)} \right)$$

Proof. see [Si/So, Theorem II.8.8, p.91] □

Theorem 7.2. Let $1 < q < \infty$, $x_0 \in \mathbb{R}^n$, $R > 0$ and $p \in H^{1,q}(B_R(x_0))$. Let $k \in \mathbb{Z},\ k \geq 0$ and let $f \in H^{k,q}(B_R(x_0))$. Further assume

$$\langle \nabla p, \nabla \phi \rangle = \langle f, \phi \rangle \qquad \text{for all } \phi \in C_0^\infty(B_R(x_0))$$

Then for $0 < R_1 < R$ holds $p\big|_{B_{R_1}(x_0)} \in H^{2+k,q}(B_{R_1}(x_0))$ and there is a constant $C_k = C(k, R, R_1, q) > 0$ so that

$$\|p\|_{2+k,q;B_{R_1}(x_0)} \ \leq\ C_k \left(\|f\|_{k,q;B_R(x_0)} + \|p\|_{1,q;B_R(x_0)} \right)$$

Proof. see [Si/So, Theorem II.8.9, p.91] □

Theorem 7.3. Let $R > 0$, $k \in \mathbb{Z},\ k \geq 0$ and $1 < q < \infty$. Let $p \in L^q(B_s \backslash B_R)$ for all $s > R$ and $\nabla p \in L^q(\mathbb{R}^n \backslash B_R)$. Further assume that there is $f \in H^{k,q}(\mathbb{R}^n \backslash B_R)$ so that

$$\langle \nabla p, \nabla \phi \rangle = \langle f, \phi \rangle \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n \backslash B_R)$$

Then there exists $R < R_k < \infty$ so that $\partial_j \partial_i p \in H^{k,q}(\mathbb{R}^n \backslash B_{R_k})$ for $i, j = 1, \ldots, n$ and with $C_k = C(n, k, R, R_k, q) > 0$ holds

$$\sum_{i,j=1}^{n} \|\partial_i \partial_j p\|_{k,q;\mathbb{R}^n \backslash B_{R_k}} \ \leq\ C_k \left(\|f\|_{k,q;\mathbb{R}^n \backslash B_R} + \|p\|_{1,q;B_{2R_k} \backslash B_R} \right)$$

Proof. see [Si/So, Corollary II.8.12, p.94] □

Theorem 7.4. Let $k \in \mathbb{N}_0$, $k \geq 0$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{k+2}$. Suppose $1 < q < \infty$. Assume $u \in H_0^{1,q}(G)$ and that there is $f \in H^{k,q}(G)$ such that

$$\langle \nabla u, \nabla \phi \rangle_G = \langle f, \phi \rangle_G \qquad \forall\, \phi \in C_0^\infty(G)$$

Then $u \in H^{2+k,q}(G)$ and there exists a constant $C_k = C(G, q, k, n) > 0$ such that

$$\|u\|_{2+k,q;G} \;\leq\; C_k \; (\|f\|_{k,q;G} + \|u\|_{1,q;G})$$

Proof. (a) Let $x \in \partial G$. Then

$$\langle \nabla u, \nabla \phi \rangle_G = \langle f, \phi \rangle_G \qquad \forall\, \phi \in C_0^\infty(G \cap B_1(x)) \subset C_0^\infty(G)$$

Because $\widehat{H}_\bullet^{1,q'}(G \cap B_1(x)) = H_0^{1,q'}(G \cap B_1(x)) = \overline{C_0^\infty(G \cap B_1(x))}^{\|\cdot\|_{1,q';G\cap B_1(x)}}$

$$\langle \nabla u, \nabla \phi \rangle_G = \langle f, \phi \rangle_G \qquad \forall\, \phi \in \widehat{H}_\bullet^{1,q'}(G \cap B_1(x))$$

and therefore by Theorem 7.1 there is $0 < R_x < 1$ and a constant $C_1 = C_1(k, q, G, x)$ such that

$$u\Big|_{G\cap B_{R_x}(x)} \in H^{2+k,q}(G \cap B_{R_x}(x))$$

and

$$\begin{aligned}\|u\|_{2+k,q;G\cap B_{R_x}(x)} &\leq C_1 \left(\|f\|_{k,q;G\cap B_1(x)} + \|u\|_{1,q;G\cap B_1(x)}\right)\\ &\leq C_1 \; (\|f\|_{k,q;G} + \|u\|_{1,q;G})\end{aligned}$$

(b) Let $x \in G$. Then there is $R_x > 0$ such that $B_{2R_x}(x) \subset G$. Obviously

$$u\Big|_{B_{2R_x}(x)} \in H^{1,q}(B_{2R_x}(x))$$

and

$$\langle \nabla u, \nabla \phi \rangle_G = \langle f, \phi \rangle_G \qquad \forall\, \phi \in C_0^\infty(B_{2R_x}(x))$$

By Theorem 7.2 we derive

$$u\Big|_{B_{R_x}(x)} \in H^{2+k,q}(B_{R_x}(x))$$

and there is a constant $C_2 = C_2(k, q, x, G) > 0$ such that

$$\|u\|_{2+k,q;B_{R_x}(x)} \;\leq\; C_2 \; (\|f\|_{k,q;G} + \|u\|_{1,q;G})$$

(c) Because $\overline{G}$ is compact we find $M, N \in \mathbb{N}$ and open balls $V_i \subset \mathbb{R}^n$ as in (a) and $U_j \subset G$ as in (b) such that

$$\partial G \subset \bigcup_{i=1}^{N} V_i \qquad \overline{G} \subset \bigcup_{i=1}^{N} V_i \;\cup\; \bigcup_{j=1}^{M} U_j$$

Choose $\varphi_j, \in C_0^\infty(U_j)$, $\phi_i \in C_0^\infty(V_i)$ with $0 \le \varphi_j, \phi_i \le 1$ and

$$\sum_{j=1}^{M} \varphi_j(x) + \sum_{i=1}^{N} \phi_i(x) = 1 \qquad \forall\, x \in \overline{G}$$

By (a) and (b) holds

$$u\Big|_{U_j} \in H^{2+k,q}(U_j), \quad \|u\|_{2+k,q;U_j} \le C_{2,j}\ (\|f\|_{k,q;G} + \|u\|_{1,q;G})$$

and

$$u\Big|_{G\cap V_i} \in H^{2+k,q}(G\cap V_i), \quad \|u\|_{2+k,q;G\cap V_i} \le C_{1,i}\ (\|f\|_{k,q;G} + \|u\|_{1,q;G})$$

(d) Let now $z \in C_0^\infty(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le 2+k$. Then

$$\int_G u\,(D^\alpha z)\ dx = \int_G u\ D^\alpha \left(\sum_{j=1}^{M} \varphi_j z + \sum_{i=1}^{N} \phi_i z \right)\ dx$$

$$= \sum_{j=1}^{M} \int_{U_j} u\ D^\alpha\ \underbrace{(\varphi_j z)}_{\in C_0^\infty(U_j)}\ dx + \sum_{i=1}^{N} \int_{G\cap V_i} u\ D^\alpha\ \underbrace{(\phi_i z)}_{\in C_0^\infty(G\cap V_i)}\ dx$$

$$= (-1)^{|\alpha|} \sum_{j=1}^{M} \int_{U_j} D^\alpha \left(u\Big|_{U_j} \right)\ \varphi_j z\, dx + (-1)^{|\alpha|} \sum_{i=1}^{N} \int_{G\cap V_i} D^\alpha \left(u\Big|_{G\cap V_i} \right)\ \phi_i z\, dx$$

Define

$$g_j(x) := \begin{cases} D^\alpha \left(u\Big|_{U_j} \right) & , \quad x \in U_j \\ 0 & \quad\text{else} \end{cases}$$

and

$$h_i(x) := \begin{cases} D^\alpha \left(u\Big|_{G\cap V_i} \right) & , \quad x \in G\cap V_i \\ 0 & \quad\text{else} \end{cases}$$

Then

$$g_j,\ h_i \in L^q(G)$$

and with

$$h := \sum_{i=1}^{N} \phi_i h_i + \sum_{j=1}^{M} \varphi_j g_j\ \in L^q(G)$$

holds

$$\int_G u\,(D^\alpha z)\ dx = (-1)^{|\alpha|} \int_G hz\, dx$$

Therefore

$$D^\alpha u = h \in L^q(G) \qquad \forall\ |\alpha| \le 2+k$$

that is

$$u \in H^{2+k,q}(G)$$

and by (c)

$$\|u\|_{2+k,q;G} \;\le\; \left(\sum_{j=1}^{M} C_{2,j}^{q} + \sum_{i=1}^{N} C_{1,i}^{q}\right)^{\frac{1}{q}} \left(\|f\|_{k,q;G} + \|u\|_{1,q;G}\right)$$

□

Theorem 7.5. Let $k \in \mathbb{N}_0$, $k \ge 0$ and let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^{k+2}$ and $\mathbb{R}^n \backslash G \subset B_{R_0}$. Suppose $1 < q < \infty$. Assume $u \in \widehat{H}_{\bullet}^{1,q}(G)$ and that there is $f \in H^{k,q}(G)$ such that

$$\langle \nabla u, \nabla \phi \rangle_G = \langle f, \phi \rangle_G \qquad \forall\, \phi \in C_0^\infty(G)$$

Then $\nabla u \in H^{1+k,q}(G)$ and there exists a constant $C_k = C(G, q, k, n) > 0$ and $R_k > R_0$ such that

$$\|\nabla u\|_{1+k,q;G} \;\le\; C_k \left(\|f\|_{k,q;G} + \|u\|_{q;G\cap B_{R_k}} + \|\nabla u\|_{q;G}\right)$$

Proof. (a) As in Theorem 7.4 one can show

$$\nabla u\Big|_{G\cap B_R} \in H^{1+k,q}(G \cap B_R) \qquad \forall\, R > R_0$$

and

$$\|\nabla u\|_{1+k,q;G\cap B_R} \;\le\; C_1 \left(\|f\|_{k,q;G\cap B_{2R}} + \|u\|_{1,q;G\cap B_{2R}}\right)$$

(b) By Theorem 7.3 there is $R_k > R_0$ and $C_2 > 0$ with

$$\nabla u\Big|_{\mathbb{R}^n\backslash B_{R_k}} \in H^{1+k,q}(\mathbb{R}^n \backslash B_{R_k})$$

and

$$\|\nabla u\|_{1+k,q;\mathbb{R}^n\backslash B_{R_k}} \;\le\; C_2 \left(\|f\|_{k,q;\mathbb{R}^n\backslash B_{R_0}} + \|u\|_{1,q;B_{2R_k}\backslash B_{R_0}}\right)$$

(c) Finally by (a),(b) and Lemma A.5 the assertion follows. □

Theorem 7.6. Let $k \in \mathbb{N}_0$, $k \ge 0$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^{2+k}$. Let $u \in \widehat{H}_{\bullet}^{1,q}(G)$ and $\Delta u \in H^{k,q}(G)$. Then

$$\nabla u \in H^{k+1,q}(G)$$

and there is a constant $C_k = C(k, G, q, n) > 0$ and a $R_k > 0$ such that

$$\|\nabla u\|_{1+k,q;G} \;\le\; C_k \left(\|\Delta u\|_{k,q;G} + \|u\|_{q;G\cap B_{R_k}} + \|\nabla u\|_{q;G}\right)$$

Proof. For $\phi \in C_0^\infty(G)$ holds

$$\langle \nabla u, \nabla \phi \rangle_G = -\langle \Delta u, \phi \rangle_G$$

and the assertion follows by Theorem 7.4 and Theorem 7.5. □

Theorem 7.7. Let $k \in \mathbb{N}_0$, $k \geq 0$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{4+k}$. Suppose $1 < q < \infty$. Assume $u \in H_0^{2,q}(G)$ and that there is $f \in H^{k,q}(G)$ such that

$$\langle \Delta u, \Delta \phi \rangle_G = \langle f, \phi \rangle_G \qquad \text{for all } \ \phi \in C_0^\infty(G)$$

Then

$$u \in H^{4+k,q}(G)$$

Proof. see [SiLec, Theorem 9.12, p.157] with $m = 2$. □

Theorem 7.8. Let $k \in \mathbb{N}_0$, $k \geq 0$ and let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^{4+k}$. Suppose $1 < q < \infty$. Assume $u \in \widehat{H}_\bullet^{2,q}(G)$ and that there is $f \in H^{k,q}(G)$ such that

$$\langle \Delta u, \Delta \phi \rangle_G = \langle f, \phi \rangle_G \qquad \forall \ \phi \in C_0^\infty(G)$$

Then for every $r > 0$ holds

$$u\Big|_{G \cap B_r} \in H^{4+k,q}(G \cap B_r)$$

Proof. The Proof is exactly the same as in [SiLec, Theorem 9.12 respectively 9.11, p.156]. □

Theorem 7.9. Let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^1$. Let $1 < s < \infty$ and assume that $p \in \widehat{H}_\bullet^{1,s}(G) = H_0^{1,s}(G)$. Let $1 < q < \infty$ and assume that

$$S_q(p) := \sup_{0 \neq \phi \in C_0^\infty(G)} \frac{\langle \nabla p, \nabla \phi \rangle}{\|\nabla \phi\|_{q';G}} < \infty$$

Then $p \in H_0^{1,q}(G)$ and with $C_q > 0$ by Theorem 2.8 holds

$$\|\nabla p\|_{q;G} \ \leq \ C_q \, S_q(p)$$

Proof. see [Si/So, Theorem II.5.1, p.66] □

Theorem 7.10. Let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^1$. Let $1 < s < \infty$ and suppose $p \in \widehat{H}_{\bullet}^{1,s}(G)$. Let $1 < q < \infty$ and assume that

$$S_q(p) := \sup_{0 \neq \phi \in C_r^\infty(G)} \frac{\langle \nabla p, \nabla \phi \rangle}{\|\nabla \phi\|_{q';G}} < \infty$$

(where $C_r^\infty(G) := \{\phi_0 + c\varphi_r \,:\, \phi_0 \in C_0^\infty(G),\, c \in \mathbb{R}\}$ with φ_r as in Theorem 2.7)

Then $p \in \widehat{H}_{\bullet}^{1,q}(G)$ and with $C_q > 0$ by Theorem 2.8 holds

$$\|\nabla p\|_{q;G} \;\leq\; C_q \, S_q(p)$$

Proof. see [Si/So, Theorem II.5.3, p.67] □

8 The asymptotic behavior of harmonic functions in exterior domains

Lemma 8.1. Let $U \subset \mathbb{R}^n$ be open. Assume that $\rho > 0$, $x \in \mathbb{R}^n$ such that $\overline{B_\rho(x)} \subset U$. Let $p : U \to \mathbb{R}$ be a harmonic function. Then

$$|(\partial_j p)(x)| \;\le\; \frac{n(n+1)}{\omega_n} \, \frac{1}{\rho^{n+1}} \int_{B_\rho} |p(x+y)| \, dy$$

***Proof*.** With $\Delta p = 0$ holds $\Delta \partial_j p = 0$ too. Let $0 < \varepsilon < \rho$. Because of the mean value property we derive

$$\begin{aligned}
(\partial_j p)(x) &= \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} (\partial_j p)(x+y) \, dy \overset{\text{Gauß}}{=} \frac{1}{|B_\varepsilon|} \int_{\partial B_\varepsilon} p(x+y) \, \frac{y_j}{|y|} \, d\omega_y \\
&= \frac{n}{\varepsilon^n \omega_n} \varepsilon^{n-1} \int_{S_{n-1}} p(x+\varepsilon\xi) \, \xi_j \, d\omega_\xi \;=\; \frac{n}{\varepsilon \, \omega_n} \int_{S_{n-1}} p(x+\varepsilon\xi) \, \xi_j \, d\omega_\xi
\end{aligned}$$

Therefore

$$\varepsilon \frac{\omega_n}{n} (\partial_j p)(x) = \int_{S_{n-1}} p(x+\varepsilon\xi) \, \xi_j \, d\omega_\xi$$

and

$$\int_0^\rho \varepsilon^n \, \frac{\omega_n}{n} (\partial_j p)(x) \, d\varepsilon = \int_0^\rho \varepsilon^{n-1} \int_{S_{n-1}} p(x+\varepsilon\xi) \, \xi_j \, d\omega_\xi \, d\varepsilon$$

Next

$$\frac{1}{n+1} \, \rho^{n+1} \, \frac{\omega_n}{n} (\partial_j p)(x) \;=\; \int_{B_\rho} p(x+y) \frac{y_j}{|y|} \, dy$$

and finally

$$|(\partial_j p)(x)| \;\le\; \frac{n(n+1)}{\omega_n} \, \frac{1}{\rho^{n+1}} \int_{B_\rho} |p(x+y)| \, dy$$

□

Lemma 8.2. Let $G \subset \mathbb{R}^n$ be open and let $p \in L^q(G)$ $(1 \le q < \infty)$ be harmonic in G. Assume that $G_1 \subset G$ such that $d := \operatorname{dist}(G_1, \partial G) > 0$. Then

$$\|\partial_i p\|_{q;G_1} \;\le\; \frac{n+1}{R} \, \|p\|_{q;G} \qquad \text{for all } 0 < R < \frac{d}{2}$$

***Proof*.** By Lemma 8.1 we have for $x \in G_1$ and $0 < R < \frac{d}{2}$

$$\begin{aligned}
|(\partial_j p)(x)| &\le \frac{n+1}{R\,|B_R|} \int_{B_R} |p(x+y)| \, dy \\
&\le \frac{n+1}{R} \, |B_R|^{-1} \left(\int_{B_R} |p(x+y)|^q \, dy \right)^{\frac{1}{q}} |B_R|^{\frac{q-1}{q}} \\
&= \frac{n+1}{R} \, |B_R|^{-\frac{1}{q}} \left(\int_{B_R} |p(x+y)|^q \, dy \right)^{\frac{1}{q}}
\end{aligned}$$

This implies

$$\begin{aligned}
\left(\int_{G_1} |(\partial_j p)(x)|^q \, dx\right)^{\frac{1}{q}} &\leq \frac{n+1}{R} |B_R|^{-\frac{1}{q}} \left(\int_{G_1} \int_{B_R} |p(x+y)|^q \, dy \, dx\right)^{\frac{1}{q}} \\
&= \frac{n+1}{R} |B_R|^{-\frac{1}{q}} \left(\int_{B_R} \int_{G_1} |p(x+y)|^q \, dx \, dy\right)^{\frac{1}{q}} \\
&\leq \frac{n+1}{R} |B_R|^{-\frac{1}{q}} \left(\int_{B_R} \int_{G} |p(z)|^q \, dz \, dy\right)^{\frac{1}{q}} \\
&= \frac{n+1}{R} \|p\|_{q;G}
\end{aligned}$$

□

Lemma 8.3. Let $U \subset \mathbb{R}^n$ be open. Assume that $\rho > 0$, $x \in \mathbb{R}^n$ such that $\overline{B_\rho(x)} \subset U$. Let $p : U \to \mathbb{R}$ be a harmonic function. Then for all $i, j, k = 1, \ldots, n$ holds

$$|(\partial_i \partial_j p)(x)| \leq \frac{n(n+1)^2}{\omega_n} \frac{2^{n+2}}{\rho^{n+2}} \int_{B_\rho} |p(x+y)| \, dy$$

and

$$|(\partial_i \partial_j \partial_k p)(x)| \leq \frac{n(n+1)^3}{\omega_n} 2^n \frac{3^{n+3}}{\rho^{n+3}} \int_{B_\rho} |p(x+y)| \, dy$$

Proof. (a) By Lemma 8.1 we derive

$$|(\partial_i \partial_j p)(x)| \leq \frac{n(n+1)}{\omega_n} \frac{2^{n+1}}{\rho^{n+1}} \int_{B_{\frac{\rho}{2}}(x)} |(\partial_j p)(y)| \, dy$$

and for $y \in B_{\frac{\rho}{2}}(x)$

$$|(\partial_j p)(y)| \leq \frac{n(n+1)}{\omega_n} \frac{2^{n+1}}{\rho^{n+1}} \int_{B_{\frac{\rho}{2}}(y)} |p(z)| \, dz$$

Then

$$\begin{aligned}
|(\partial_i \partial_j p)(x)| &\leq \frac{n^2(n+1)^2}{{\omega_n}^2} \frac{2^{2(n+1)}}{\rho^{2(n+1)}} \int_{B_{\frac{\rho}{2}}(x)} \int_{B_{\frac{\rho}{2}}(y)} |p(z)| \, dz \, dy \\
&\leq \frac{n^2(n+1)^2}{{\omega_n}^2} \frac{2^{2(n+1)}}{\rho^{2(n+1)}} \int_{B_{\frac{\rho}{2}}(x)} \int_{B_\rho(x)} |p(z)| \, dz \, dy \\
&= \frac{n^2(n+1)^2}{{\omega_n}^2} \frac{2^{2(n+1)}}{\rho^{2(n+1)}} \left|B_{\frac{\rho}{2}}(x)\right| \int_{B_\rho(x)} |p(z)| \, dz \\
&= \frac{n(n+1)^2}{\omega_n} \frac{2^{n+2}}{\rho^{n+2}} \int_{B_\rho(x)} |p(z)| \, dz
\end{aligned}$$

(b) By Lemma 8.1 and (a) we get

$$|(\partial_i\partial_j\partial_k p)(x)| \;\leq\; \frac{n(n+1)^2}{\omega_n}\,\frac{3^{n+2}}{\rho^{n+2}}\int_{B_{\frac{2\rho}{3}}(x)} |(\partial_k p)(y)|\,dy$$

and for $y \in B_{\frac{2\rho}{3}}(x)$

$$|(\partial_k p)(y)| \;\leq\; \frac{n(n+1)}{\omega_n}\,\frac{3^{n+1}}{\rho^{n+1}}\int_{B_{\frac{\rho}{3}}(y)} |p(z)|\,dz$$

Therefore

$$\begin{aligned}
|(\partial_i\partial_j\partial_k p)(x)| \;&\leq\; \frac{n^2(n+1)^3}{{\omega_n}^2}\,\frac{3^{2n+3}}{\rho^{2n+3}}\int_{B_{\frac{2\rho}{3}}(x)}\int_{B_{\frac{\rho}{3}}(y)} |p(z)|\,dz\,dy \\
&\leq\; \frac{n^2(n+1)^3}{{\omega_n}^2}\,\frac{3^{2n+3}}{\rho^{2n+3}}\int_{B_{\frac{2\rho}{3}}(x)}\int_{B_{\rho}(x)} |p(z)|\,dz\,dy \\
&=\; \frac{n^2(n+1)^3}{{\omega_n}^2}\,\frac{3^{2n+3}}{\rho^{2n+3}}\,\frac{\omega_n}{n}\left(\frac{2\rho}{3}\right)^n\int_{B_{\rho}(x)} |p(z)|\,dz \\
&=\; \frac{n(n+1)^3}{\omega_n}\,2^n\,\frac{3^{n+3}}{\rho^{n+3}}\int_{B_{\rho}(x)} |p(z)|\,dz
\end{aligned}$$

□

Lemma 8.4. Let $n \geq 2$, $R > 0$, $1 \leq q < \infty$ and let $u \in L^q(\mathbb{R}^n \backslash B_R)$ be harmonic in $\mathbb{R}^n \backslash B_R$. Let

$$v(x) := |x|^{2-n}\,u\left(\frac{x}{|x|^2}\right) \qquad \text{for all } x \in B_{\frac{1}{R}} \backslash \{0\}$$

be the Kelvin transform. Then v ist harmonic in $B_{\frac{1}{R}} \backslash \{0\}$ and

$$\int_{B_{\frac{1}{R}}} |v(y)|^q\,|y|^{q(n-2)-2n}\,dy \;=\; \int_{\mathbb{R}^n \backslash B_R} |u(y)|^q\,dy$$

Proof. (a) For $\Delta v = 0$ see e.g. [ABR, p.62].

(b) Let $r > R$. Then by Lemma A.6 holds

$$\begin{aligned}
\int_{A_{R,r}} |u(x)|^q\,dx \;&\overset{x=\frac{y}{|y|^2}}{=\!=}\; \int_{A_{\frac{1}{r},\frac{1}{R}}} \left|u\left(\frac{y}{|y|^2}\right)\right|^q \frac{1}{|y|^{2n}}\,dy \\
&=\; \int_{A_{\frac{1}{r},\frac{1}{R}}} \left|u\left(\frac{y}{|y|^2}\right)|y|^{2-n}\,|y|^{n-2}\right|^q \frac{1}{|y|^{2n}}\,dy \\
&=\; \int_{A_{\frac{1}{r},\frac{1}{R}}} |v(y)|^q\,|y|^{q(n-2)-2n}\,dy
\end{aligned}$$

For $r \to \infty$ the assertion follows. $\square$

Lemma 8.5. Let $n \geq 2$, $R > 0$, $1 \leq q < \infty$ and let $u \in L^q(\mathbb{R}^n \backslash B_R)$ be harmonic in $\mathbb{R}^n \backslash B_R$. Let

$$v(x) := |x|^{2-n}\, u\left(\frac{x}{|x|^2}\right) \qquad \text{for all } x \in B_{\frac{1}{R}} \backslash \{0\}$$

be the Kelvin transform. Then there is a constant $C(n,q) > 0$ such that for all $0 < r < \frac{1}{R}$

$$\begin{aligned} \int_{B_r} |v(y)|\, dy &\leq C(n,q)\, r^{\frac{n}{q}+2} \left(\int_{B_r} |v(y)|^q\, |y|^{q(n-2)-2n}\, dy\right)^{\frac{1}{q}} \\ &= C(n,q)\, r^{\frac{n}{q}+2} \left(\int_{\mathbb{R}^n \backslash B_{\frac{1}{r}}} |u(x)|^q\, dx\right)^{\frac{1}{q}} \end{aligned}$$

Proof. (a) Assume $1 < q < \infty$. Then

$$\begin{aligned} \int_{B_r} |v(y)|\, dy &= \int_{B_r} |v(y)|\, |y|^{\frac{q(n-2)-2n}{q}}\, |y|^{\frac{2n-q(n-2)}{q}}\, dy \\ &\leq \left(\int_{B_r} |v(y)|^q\, |y|^{q(n-2)-2n}\, dy\right)^{\frac{1}{q}} \left(\int_{B_r} |y|^{\frac{2n-q(n-2)}{q-1}}\, dy\right)^{\frac{q-1}{q}} \\ &\overset{8.4}{=} \left(\int_{\mathbb{R}^n \backslash B_{\frac{1}{r}}} |u(x)|^q\, dx\right)^{\frac{1}{q}} C(n,q) \left(r^{n+\frac{2n-q(n-2)}{q-1}}\right)^{\frac{q-1}{q}} \\ &= \left(\int_{\mathbb{R}^n \backslash B_{\frac{1}{r}}} |u(x)|^q\, dx\right)^{\frac{1}{q}} C(n,q)\, r^{\frac{n(q-1)+2n-q(n-2)}{q}} \\ &= \left(\int_{\mathbb{R}^n \backslash B_{\frac{1}{r}}} |u(x)|^q\, dx\right)^{\frac{1}{q}} C(n,q)\, r^{\frac{n+2q}{q}} \end{aligned}$$

(b) Assume $q = 1$. Then

$$\begin{aligned} \int_{B_r} |v(y)|\, dy &= \int_{B_r} |v(y)|\, |y|^{(n-2)-2n}\, |y|^{2+n}\, dy \\ &\leq r^{2+n} \int_{B_r} |v(y)|\, |y|^{(n-2)-2n}\, dy \\ &\overset{8.4}{=} r^{2+n} \int_{\mathbb{R}^n \backslash B_{\frac{1}{r}}} |u(x)|\, dx \end{aligned}$$

$\square$

Lemma 8.6. Let $n \geq 2$, $R > 0$, $1 \leq q < \infty$ and let $u \in L^q(\mathbb{R}^n \backslash B_R)$ be harmonic in $\mathbb{R}^n \backslash B_R$. Let

$$v(x) := |x|^{2-n}\, u\left(\frac{x}{|x|^2}\right) \qquad \text{for all } x \in B_{\frac{1}{R}} \backslash \{0\}$$

be the Kelvin transform. Then there is a $\tilde{v} \in C^\infty(B_{\frac{1}{R}})$ such that

$$\Delta\tilde{v} = 0 \text{ in } B_{\frac{1}{R}} \quad \text{and} \quad \tilde{v}|_{B_{\frac{1}{R}} \backslash \{0\}} = v$$

Proof. (a) Let

$$\rho \in C^\infty(\mathbb{R}), \quad 0 \leq \rho \leq 1, \quad \rho(t) = \begin{cases} 0 & ,\text{if } |t| \leq 1 \\ 1 & ,\text{if } |t| \geq 2 \end{cases}$$

Let

$$\rho_k(x) := \rho\left(k|x|\right) \qquad \forall\, k \in \mathbb{N}$$

Then

$$\rho_k \in C^\infty(\mathbb{R}^n), \quad 0 \leq \rho_k \leq 1, \quad \rho_k(x) = \begin{cases} 0 & ,\text{if } |x| \leq \frac{1}{k} \\ 1 & ,\text{if } |x| \geq \frac{2}{k} \end{cases}$$

and there is a constant $C > 0$ such that

$$\|\nabla\rho_k\|_\infty \leq k\,C \quad \|\partial_i\partial_j\rho_k\|_\infty \leq k^2 C$$

(b) Let $\phi \in C_0^\infty(B_{\frac{1}{R}})$. Then $\rho_k\phi \in C_0^\infty(B_{\frac{1}{R}} \backslash \{0\})$ and

$$0 = \langle v, \Delta(\rho_k\phi)\rangle_{B_{\frac{1}{R}}} = \langle v, \phi\Delta\rho_k\rangle_{B_{\frac{1}{R}}} + 2\langle v, \nabla\rho_k\nabla\phi\rangle_{B_{\frac{1}{R}}} + \langle v, \rho_k\Delta\phi\rangle_{B_{\frac{1}{R}}}$$

We estimate

$$\begin{aligned} \left|\langle v, \phi\Delta\rho_k\rangle_{B_{\frac{1}{R}}}\right| &\leq \|\phi\|_\infty C k^2 \int_{B_{\frac{2}{k}}} |v(x)|\,dx \\ &\underset{\text{Lemma 8.5}}{\leq} \|\phi\|_\infty C\; k^2\, C(n,q)\left(\frac{2}{k}\right)^{\frac{n}{q}+2} \|u\|_{q;\mathbb{R}^n\backslash B_R} \\ &= K(n,q,\phi,u)\left(\frac{1}{k}\right)^{\frac{n}{q}} \to 0 \quad (k \to \infty) \end{aligned}$$

Similarly we derive

$$\langle v, \nabla\rho_k\nabla\phi\rangle_{B_{\frac{1}{R}}} \to 0 \quad (k \to \infty)$$

(c) Obviously

$$\rho_k v\Delta\phi \to v\Delta\phi \qquad \text{a.e. in } B_{\frac{1}{R}}$$

and

$$|\rho_k v\Delta\phi| \;\leq\; \|\Delta\phi\|_\infty|v| \underbrace{\in L^1(B_{\frac{1}{R}})}_{\text{by Lemma 8.5}}$$

By the dominated convergence theorem we get

$$\langle v, \rho_k \Delta\phi \rangle_{B_{\frac{1}{R}}} \;\to\; \langle v, \Delta\phi \rangle_{B_{\frac{1}{R}}} \quad (k \to \infty)$$

By (b) we derive for $(k \to \infty)$

$$\langle v, \Delta\phi \rangle_{B_{\frac{1}{R}}} \;=\; 0$$

and finally by Weyl's Lemma the assertion. □

Theorem 8.7. Let $n \geq 2$, $R > 0$, $1 \leq q < \infty$. Then there is a constant $C = C(n,q,R) > 0$ such that for each $x \in \mathbb{R}^n \backslash B_{2R}$ and for every harmonic $u \in L^q(\mathbb{R}^n \backslash B_R)$ holds

$$|u(x)| \;\leq\; \left\{ \begin{array}{ll} C\,\|u\|_{1;\mathbb{R}^n\backslash B_R}\,|x|^{-3} & \text{,if } q = 1 \\ C\,\|u\|_{q;\mathbb{R}^n\backslash B_R}\,|x|^{-2} & \text{,if } 1 < q \leq 2 \\ C\,\|u\|_{q;\mathbb{R}^n\backslash B_R}\,|x|^{-1} & \text{,if } 2 < q < \infty \end{array} \right\} \quad \text{if } n = 2$$

$$|u(x)| \;\leq\; \left\{ \begin{array}{ll} C\,\|u\|_{1;\mathbb{R}^n\backslash B_R}\,|x|^{-n-1} & \text{,if } q = 1 \\ C\,\|u\|_{q;\mathbb{R}^n\backslash B_R}\,|x|^{-n} & \text{,if } 1 < q \leq \frac{n}{n-1} \\ C\,\|u\|_{q;\mathbb{R}^n\backslash B_R}\,|x|^{1-n} & \text{,if } \frac{n}{n-1} < q \leq \frac{n}{n-2} \\ C\,\|u\|_{q;\mathbb{R}^n\backslash B_R}\,|x|^{2-n} & \text{,if } \frac{n}{n-2} < q < \infty \end{array} \right\} \quad \text{if } n \geq 3$$

Proof. (a) Let

$$v(x) := |x|^{2-n}\, u\left(\frac{x}{|x|^2}\right) \qquad \text{for all } x \in B_{\frac{1}{R}} \backslash \{0\}$$

be the Kelvin transform. By Lemma 8.6 there is a harmonic continuation of v in $B_{\frac{1}{R}}$, which we again denote by v. Because of the mean value property and Lemma 8.5 holds for $0 < r < \frac{1}{R}$

$$\begin{aligned} |v(0)| \;&\leq\; \frac{1}{|B_r|} \int_{B_r} |v(y)|\, dy \\ &\leq\; \frac{1}{|B_r|}\, C(n,q)\, r^{\frac{n}{q}+2} \left(\int_{\mathbb{R}^n \backslash B_{\frac{1}{r}}} |u(x)|^q\, dx \right)^{\frac{1}{q}} \\ &=\; \frac{n}{\omega_n}\, r^{-n}\, C(n,q)\, r^{\frac{n}{q}+2} \underbrace{\left(\int_{\mathbb{R}^n \backslash B_{\frac{1}{r}}} |u(x)|^q\, dx \right)^{\frac{1}{q}}}_{\to 0\ (r \to 0)} \end{aligned}$$

which tends to 0 for $(r \to 0)$, if

$$\frac{n}{q} + 2 - n \;\geq\; 0 \qquad \text{that is} \qquad n \geq q(n-2)$$

So we derive

$$v(0) = 0 \qquad \text{if } n = 2$$

$$v(0) = 0 \qquad \text{if } n \geq 3 \text{ and } q \leq \frac{n}{n-2}$$

(b) Similarly by Lemma 8.1 and Lemma 8.5 we get

$$\begin{aligned} |(\partial_j v)(0)| &\leq \frac{n(n+1)}{\omega_n} \frac{1}{r^{n+1}} \int_{B_r} |v(y)|\, dy \\ &\leq \frac{n(n+1)}{\omega_n} \frac{1}{r^{n+1}} C(n,q)\, r^{\frac{n}{q}+2} \underbrace{\left(\int_{\mathbb{R}^n \setminus B_{\frac{1}{r}}} |u(x)|^q\, dx \right)^{\frac{1}{q}}}_{\to 0\ (r \to 0)} \end{aligned}$$

which tends to 0 for $(r \to 0)$, if

$$\frac{n}{q} + 2 - (n+1) \;\geq\; 0 \qquad \text{that is} \qquad n \geq q(n-1)$$

So we derive

$$(\partial_j v)(0) = 0 \qquad \text{if } n \geq 2 \text{ and } q \leq \frac{n}{n-1}$$

(c) Similarly by Lemma 8.3 and Lemma 8.5 we get

$$\begin{aligned} |(\partial_i \partial_j v)(0)| &\leq \frac{n(n+1)^2}{\omega_n} \frac{1}{r^{n+2}} \int_{B_{2r}} |v(y)|\, dy \\ &\leq \frac{n(n+1)^2}{\omega_n} \frac{1}{r^{n+2}} C(n,q)\, r^{\frac{n}{q}+2} \underbrace{\left(\int_{\mathbb{R}^n \setminus B_{\frac{1}{2r}}} |u(x)|^q\, dx \right)^{\frac{1}{q}}}_{\to 0\ (r \to 0)} \end{aligned}$$

which tends to 0 for $(r \to 0)$, if

$$\frac{n}{q} + 2 - (n+2) \;\geq\; 0 \qquad \text{that is} \qquad 1 \geq q$$

Therefore

$$(\partial_i \partial_j v)(0) = 0 \qquad \text{if } n \geq 2 \text{ and } q = 1$$

(d) For $x \in B_{\frac{1}{R}}$ there exists by Taylor's theorem a_x, b_x, $c_x \in B_{\frac{1}{R}}$ such that

$$\begin{aligned} v(x) &= v(0) + \langle \nabla v(a_x), x \rangle \\ v(x) &= v(0) + \langle \nabla v(0), x \rangle + \frac{1}{2} \langle (\mathrm{Hess} v)(b_x)\, x, x \rangle \\ v(x) &= \sum_{|\alpha| \le 2} \frac{(D^\alpha v)(0)}{\alpha!}\, x^\alpha + \sum_{|\alpha|=3} \frac{(D^\alpha v)(c_x)}{\alpha!}\, x^\alpha \end{aligned}$$

(e) Let $n \ge 3$ and $\frac{n}{n-2} < q < \infty$. Let $x \in B_{\frac{1}{2R}}$. Then by the mean value property and Lemma 8.5

$$\begin{aligned} |v(x)| &\le \frac{1}{\left| B_{\frac{1}{2R}} \right|} \int_{B_{\frac{1}{2R}}(x)} |v(y)|\, dy \\ &\le K_1(n, R) \int_{B_{\frac{1}{R}}} |v(y)|\, dy \\ &\le K_2(n, R, q)\, \|u\|_{q; \mathbb{R}^n \setminus B_R} \end{aligned}$$

For $y \in \mathbb{R}^n \setminus B_{2R}$ therefore holds

$$\begin{aligned} |u(y)| &= |y|^{2-n}\, v\left(\frac{y}{|y|^2} \right) \\ &\le |y|^{2-n}\, K_2(n, R, q)\, \|u\|_{q; \mathbb{R}^n \setminus B_R} \end{aligned}$$

(f) Let $n \ge 3$ and $\frac{n}{n-1} < q \le \frac{n}{n-2}$ or let $n = 2$ and $2 < q < \infty$. Then by (a) $v(0) = 0$ and for $x \in B_{\frac{1}{2R}}$ holds by Lemma 8.1 and Lemma 8.5

$$\begin{aligned} |(\partial_j v)(x)| &\le \frac{n(n+1)}{\omega_n} (2R)^{n+1} \int_{B_{\frac{1}{2R}}(x)} |v(y)|\, dy \\ &\le K_3(n, R) \int_{B_{\frac{1}{R}}} |v(y)|\, dy \\ &\le K_4(n, R, q)\, \|u\|_{q; \mathbb{R}^n \setminus B_R} \end{aligned}$$

and therefore by (d)

$$\begin{aligned} |v(x)| &= |\underbrace{v(0)}_{=0} + \langle \nabla v(a_x), x \rangle| \\ &\le |\nabla v(a_x)|\, |x| \;\le\; K_5(n, R, q)\, \|u\|_{q; \mathbb{R}^n \setminus B_R}\, |x| \end{aligned}$$

For $y \in \mathbb{R}^n \setminus B_{2R}$ therefore holds

$$\begin{aligned} |u(y)| &= |y|^{2-n}\, v\left(\frac{y}{|y|^2} \right) \\ &\le |y|^{1-n}\, K_4(n, R, q)\, \|u\|_{q; \mathbb{R}^n \setminus B_R} \end{aligned}$$

(g) Let $n \geq 2$ and $1 < q \leq \frac{n}{n-1}$. Then by (a) and (b) $v(0) = 0$, $\nabla v(0) = 0$ and for $x \in B_{\frac{1}{2R}}$ holds by Lemma 8.3 and Lemma 8.5

$$\begin{aligned}
|(\partial_i \partial_j v)(x)| &\leq \frac{n(n+1)^2}{\omega_n}(4R)^{n+2} \int_{B_{\frac{1}{2R}}(x)} |v(y)|\, dy \\
&\leq K_6(n,R) \int_{B_{\frac{1}{R}}} |v(y)|\, dy \\
&\leq K_7(n,R,q)\, \|u\|_{q;\mathbb{R}^n \setminus B_R}
\end{aligned}$$

and therefore by (d)

$$\begin{aligned}
|v(x)| &= |\underbrace{v(0)}_{=0} + \underbrace{\langle \nabla v(0), x\rangle}_{=0} + \frac{1}{2}\langle (\mathrm{Hess} v)(b_x)\, x, x\rangle| \\
&\leq \frac{1}{2}\,|x|^2 \left(\sum_{i,j=1}^{n} |(\partial_i \partial_j v)(b_x)|^2 \right)^{\frac{1}{2}} \leq K_8(n,R,q)\, \|u\|_{q;\mathbb{R}^n \setminus B_R}\, |x|^2
\end{aligned}$$

For $y \in \mathbb{R}^n \setminus B_{2R}$ therefore holds

$$\begin{aligned}
|u(y)| &= |y|^{2-n}\, v\left(\frac{y}{|y|^2}\right) \\
&\leq |y|^{-n}\, K_8(n,R,q)\, \|u\|_{q;\mathbb{R}^n \setminus B_R}
\end{aligned}$$

(h) Let $n \geq 2$ and $q = 1$. Then by (a), (b) and (c) $v(0) = 0$, $\partial_i v(0) = 0$ and $\partial_i \partial_j v(0) = 0$, and for $x \in B_{\frac{1}{2R}}$ holds by Lemma 8.3 and Lemma 8.5

$$\begin{aligned}
|(\partial_i \partial_j \partial_k v)(x)| &\leq \frac{n(n+1)^3}{\omega_n}\, 2^n 3^{n+3}\, (2R)^{n+3} \int_{B_{\frac{1}{2R}}(x)} |v(y)|\, dy \\
&\leq K_9(n,R) \int_{B_{\frac{1}{R}}} |v(y)|\, dy \\
&\leq K_{10}(n,R)\, \|u\|_{1;\mathbb{R}^n \setminus B_R}
\end{aligned}$$

and therefore by (d) analogously to (f) and (g)

$$|v(x)| \leq K_{11}(n,R)\, \|u\|_{1;\mathbb{R}^n \setminus B_R}\, |x|^3$$

For $y \in \mathbb{R}^n \setminus B_{2R}$ therefore holds

$$\begin{aligned}
|u(y)| &= |y|^{2-n}\, v\left(\frac{y}{|y|^2}\right) \\
&\leq |y|^{-n-1}\, K_{11}(n,R)\, \|u\|_{1;\mathbb{R}^n \setminus B_R}
\end{aligned}$$

□

Theorem 8.8. Let $n \ge 2$, $R > 0$, $1 \le q < \infty$. Then there is a constant $C = C(n,q,R) > 0$ and a function $f_q \in L^q(\mathbb{R}^n \backslash B_{2R})$ such that for each $x \in \mathbb{R}^n \backslash B_{2R}$ and for every harmonic $u \in L^q(\mathbb{R}^n \backslash B_R)$ holds

$$|u(x)| \;\le\; C \, \|u\|_{q;\mathbb{R}^n \backslash B_R} \, f_q(x)$$

Proof. For $k \in \mathbb{Z}$ holds

$$\begin{aligned} |\cdot|^{k-n} \in L^q(\mathbb{R}^n \backslash B_{2R}) \;\; &\Leftrightarrow \;\; \int_{2R}^{\infty} r^{n-1+q(k-n)}\, dr \;<\; \infty \\ &\Leftrightarrow \;\; n-1+q(k-n) < -1 \\ &\Leftrightarrow \;\; n < q(n-k) \end{aligned}$$

Therefore

$$\begin{aligned} n = 2: \quad & |\cdot|^{-3} \in L^q(\mathbb{R}^n \backslash B_{2R}) && \Leftrightarrow \quad q > \tfrac{2}{3} \\ n = 2: \quad & |\cdot|^{-2} \in L^q(\mathbb{R}^n \backslash B_{2R}) && \Leftrightarrow \quad q > 1 \\ n = 2: \quad & |\cdot|^{-1} \in L^q(\mathbb{R}^n \backslash B_{2R}) && \Leftrightarrow \quad q > 2 \\ n \ge 3: \quad & |\cdot|^{-n-1} \in L^q(\mathbb{R}^n \backslash B_{2R}) && \Leftrightarrow \quad q > \tfrac{n}{n+1} \\ n \ge 3: \quad & |\cdot|^{-n} \in L^q(\mathbb{R}^n \backslash B_{2R}) && \Leftrightarrow \quad q > 1 \\ n \ge 3: \quad & |\cdot|^{1-n} \in L^q(\mathbb{R}^n \backslash B_{2R}) && \Leftrightarrow \quad q > \tfrac{n}{n-1} \\ n \ge 3: \quad & |\cdot|^{2-n} \in L^q(\mathbb{R}^n \backslash B_{2R}) && \Leftrightarrow \quad q > \tfrac{n}{n-2} \end{aligned}$$

and the assertion follows by Theorem 8.7 □

Lemma 8.9. Let $G \subset \mathbb{R}^n$ be open and let $p \in L^q(G)$ $(1 \le q < \infty)$ be harmonic in G. Assume that $G_1 \subset G$ such that $d := \operatorname{dist}(G_1, \partial G) > 0$. Then

$$p\Big|_{G_1} \in H^{\infty,q}(G_1)$$

Proof. (a) It is well known that $p \in C^\infty(G)$. We prove by induction

$$p\Big|_{G'} \in H^{k,q}(G')$$

for every $k \in \mathbb{N}$ and every $G' \subset G$ with $\operatorname{dist}(G', \partial G) > 0$.

(b) $\underline{\underline{k=1:}}$ By Lemma 8.2 we derive for $G' \subset G$ with $\mathrm{dist}(G', \partial G) > 0$

$$\|\partial_i p\|_{q;G'} \;\leq\; \frac{4(n+1)}{\mathrm{dist}(G', \partial G)} \, \|p\|_{q;G} < \infty$$

(c) $\underline{\underline{k \to k+1:}}$ For $k \in \mathbb{N}$ may hold

$$D^\alpha p \in L^q(G')$$

for every $|\alpha| = k$ and every $G' \subset G$ with $\mathrm{dist}(G', \partial G) > 0$.

Let $G_0 \subset G$ with $\mathrm{dist}(G_0, \partial G) > 0$. Choose

$$G' := \{x \in G : \mathrm{dist}(x, \partial G) > \frac{1}{2}\mathrm{dist}(G_0, \partial G)\}$$

Then $G_0 \subset G' \subset G$, $\mathrm{dist}(G', \partial G) > 0$ and $\mathrm{dist}(G_0, \partial G') > 0$.

Because $\Delta D^\alpha p = 0$ in G' and by assumption $D^\alpha p \in L^q(G')$ we derive by Lemma 8.2

$$\|\partial_i D^\alpha p\|_{q;G_0} \;\leq\; \frac{4(n+1)}{\mathrm{dist}(G_0, \partial G')} \, \|D^\alpha p\|_{q;G'} < \infty$$

□

Lemma 8.10. Let $n \geq 2$, $C > 0$, $R > 1$ and let p be harmonic in $\mathbb{R}^n \backslash B_{\frac{R}{2}}$.

1. If $n \geq 3$ and
$$|p(x)| \;\leq\; C\,|x|^{2-n}$$
for all $|x| \geq R$, then there is a constant $\tilde{C} > 0$ such that
$$|\partial_i p(x)| \;\leq\; \tilde{C}\,|x|^{1-n}$$
for all $|x| \geq 2R$ and all $i = 1, \ldots, n$.

2. If $n \geq 2$ and
$$|p(x)| \;\leq\; C\,|x|^{1-n}$$
for all $|x| \geq R$, then there is a constant $\tilde{C} > 0$ such that
$$|\partial_i p(x)| \;\leq\; \tilde{C}\,|x|^{-n}$$
for all $|x| \geq 2R$ and all $i = 1, \ldots, n$.

3. If $n \geq 2$ and
$$|p(x)| \;\leq\; C\,|x|^{-n}$$
for all $|x| \geq R$, then there is a constant $\tilde{C} > 0$ such that
$$|\partial_i p(x)| \;\leq\; \tilde{C}\,|x|^{-n-1}$$
for all $|x| \geq 2R$ and all $i = 1, \ldots, n$.

Proof. (a) Let $n \geq 3$ and

$$|p(x)| \leq C\,|x|^{2-n}$$

for all $|x| \geq R$. Let

$$s := \frac{n}{n-2}$$

There is $s < q < \infty$ with

$$n \geq (n-3)q$$

and

$$p \in L^q(\mathbb{R}^n \backslash B_{\frac{3R}{4}})$$

Then

$$p \in L^s(A_{r,4r})$$

for all $r > R$. By Lemma 8.2 we derive

$$\begin{aligned}
\|\partial_i p\|_{s;A_{2r,3r}} \quad &\leq \quad \frac{4(n+1)}{r}\,\|p\|_{s;A_{r,4r}} \;=\; \frac{4(n+1)}{r}\left[\int_{A_{r,4r}} |p|^s\,dx\right]^{\frac{1}{s}} \\
&\underset{\text{with } \frac{q}{s}>1}{\overset{\text{Hölder}}{\leq}} \quad \frac{4(n+1)}{r}\left[\left(\int_{A_{r,4r}} |p|^q\,dx\right)^{\frac{s}{q}} |A_{r,4r}|^{\frac{q-s}{q}}\right]^{\frac{1}{s}} \\
&= \quad \frac{C_1(n)}{r}\,r^{\frac{n(q-s)}{qs}}\,\|p\|_{q;A_{r,4r}}
\end{aligned}$$

Because

$$\begin{aligned}
\frac{n(q-s)}{sq} \leq 1 \iff n(q-s) \leq sq \iff nq - ns \leq sq \iff nq \leq s(q+n) \\
\iff \frac{nq}{n+q} \leq \frac{n}{n-2} \iff q(n-2) \leq n+q \iff q(n-3) \leq n
\end{aligned}$$

for $r > R > 1$ holds

$$\|\partial_i p\|_{s;A_{2r,3r}} \leq C_1(n)\,\|p\|_{q;A_{r,4r}}$$

and

$$\begin{aligned}
\|p\|^q_{q;A_{r,4r}} \quad &\leq \quad C^q \int_{A_{r,4r}} |x|^{q(2-n)}\,dx \;=\; C^q\,\omega_n \int_r^{4r} t^{n-1+q(2-n)}\,dt \\
&= \quad \frac{C^q \omega_n}{n+q(2-n)}\left[(4r)^{n+q(2-n)} - r^{n+q(2-n)}\right] \;=\; C_2(n,q)\,r^{n+q(2-n)}
\end{aligned}$$

Therefore

$$\|\partial_i p\|^s_{s;A_{2r,3r}} \leq C_3(n,q)\,r^{\frac{s(n+q(2-n))}{q}}$$

and

$$\begin{aligned}
\|\partial_i p\|^s_{s;\mathbb{R}^n\setminus B_{2R}} &= \sum_{m=1}^{\infty} \|\partial_i p\|^s_{s;A_{\left(\frac{3}{2}\right)^{m-1}2R,\left(\frac{3}{2}\right)^{m}2R}} = \sum_{m=1}^{\infty} \|\partial_i p\|^s_{s;A_{2\left(\frac{3}{2}\right)^{m-1}R,3\left(\frac{3}{2}\right)^{m-1}R}} \\
&\le \sum_{m=1}^{\infty} C_3(n,q)\left(\left(\frac{3}{2}\right)^{m-1} R\right)^{\frac{s(n+q(2-n))}{q}} \\
&= \sum_{m=0}^{\infty} C_4(n,q,R)\Bigg[\underbrace{\left(\frac{2}{3}\right)^{\frac{s}{q}\overbrace{(q(n-2)-n)}^{>0}}}_{<1}\Bigg]^m \;<\; \infty
\end{aligned}$$

So we get

$$\partial_i p \in L^{\frac{n}{n-2}}(\mathbb{R}^n\setminus B_{2R})$$

and $\partial_i p$ is harmonic in $\mathbb{R}^n\setminus B_{\frac{R}{2}}$. Therefore

$$\partial_i p \in L^{\frac{n}{n-2}}(\mathbb{R}^n\setminus B_{R})$$

and by Theorem 8.7 there is a constant $C_5 = C_5(n,q,R,p) > 0$ such that

$$|(\partial_i p)(x)| \;\le\; C_5\,|x|^{1-n} \qquad \text{for all } |x| \ge 2R$$

(b) Let $n \ge 2$ and

$$|p(x)| \;\le\; C\,|x|^{1-n}$$

for all $|x| \ge R$. Let

$$s := \frac{n}{n-1}$$

There is $s < q < \infty$ with

$$n \ge (n-2)q$$

and

$$p \in L^q(\mathbb{R}^n\setminus B_{\frac{3R}{4}})$$

Similarly to (a) follows

$$|(\partial_i p)(x)| \;\le\; \tilde{C}\,|x|^{-n} \qquad \text{for all } |x| \ge 2R$$

(c) Let $n \ge 2$ and

$$|p(x)| \;\le\; C\,|x|^{-n}$$

for all $|x| \ge R$. Let

$$s := 1$$

There is $1 = s < q < \infty$ with

$$n \ge (n-1)q$$

and

$$p \in L^q(\mathbb{R}^n\setminus B_{\frac{3R}{4}})$$

Similarly to (a) follows

$$|(\partial_i p)(x)| \;\le\; \tilde{C}\,|x|^{-n-1} \qquad \text{for all } |x| \ge 2R$$

□

Lemma 8.11. Let $n \geq 2$, $G \subset \mathbb{R}^n$ be an exterior domain, and let u be harmonic in G. Let $r > 0$, φ_r and ψ_{ri} as in Theorem 3.7.

1. For all $R > 2r$ and all $k = 1, \ldots, n$ holds

$$\int_{G \cap B_R} u\, \Delta\varphi_r \, dx \;=\; -\sum_{i=1}^{n} \int_{\partial B_R} (\partial_i u)(x)\, \frac{x_i}{|x|}\, d\omega_x$$

and

$$\int_{G \cap B_R} u\, \Delta\psi_{rk} \, dx \;=\; \int_{\partial B_R} u(x)\, \frac{x_k}{|x|}\, d\omega_x \;-\; \sum_{i=1}^{n} \int_{\partial B_R} (\partial_i u)(x)\, \frac{x_i x_k}{|x|}\, d\omega_x$$

2. If there are $R > r$, $C > 0$ with

$$|u(x)| \;\leq\; C\, |x|^{1-n}$$

for all $|x| \geq R$, then

$$\int_{G} u\, \Delta\varphi_r \, dx \;=\; 0$$

3. If there are $R > r$, $C > 0$ with

$$|u(x)| \;\leq\; C\, |x|^{-n}$$

for all $|x| \geq R$, then for $k = 1, \ldots, n$ holds

$$\int_{G} u\, \Delta\psi_{rk} \, dx \;=\; 0$$

Proof. (a) Let $R > 2r$. Then by Gauß' Theorem

$$\begin{aligned}
\int_{G \cap B_R} u\, \Delta\varphi_r \, dx &= \int_{G \cap B_R} (\nabla u)\, (\nabla\varphi_r) \, dx \\
&= \int_{G \cap B_R} \underbrace{(\Delta u)}_{=0}\, \varphi_r \, dx \;-\; \sum_{i=1}^{n} \int_{\partial B_R} (\partial_i u)(x)\, \underbrace{\varphi_r(x)}_{=1}\, \frac{x_i}{|x|}\, d\omega_x \\
&= -\sum_{i=1}^{n} \int_{\partial B_R} (\partial_i u)(x)\, \frac{x_i}{|x|}\, d\omega_x
\end{aligned}$$

(b) Let $R > 2r$ and $k = 1, \ldots, n$. Again by Gauß' Theorem we derive

$$\int_{G \cap B_R} u\Delta\psi_{rk} \, dx \;=\; -\int_{G \cap B_R} \sum_{i=1}^{n} (\partial_i u)(x) \partial_i [x_k \varphi_r(x)] \, dx + \int_{\partial B_R} u(x) \sum_{i=1}^{n} \partial_i [x_k \varphi_r(x)] \frac{x_i}{|x|}\, d\omega_x$$

$$= \int\limits_{G\cap B_R} \underbrace{(\Delta u)(x)}_{=0}\, x_k\varphi_r(x)\, dx \;-\; \int\limits_{\partial B_R} \sum_{i=1}^{n} (\partial_i u)(x)\, x_k\, \underbrace{\varphi_r(x)}_{=1}\, \frac{x_i}{|x|}\, d\omega_x +$$

$$+ \int\limits_{\partial B_R} u(x) \sum_{i=1}^{n} \delta_{ik}\, \underbrace{\varphi_r(x)}_{=1}\, \frac{x_i}{|x|}\, d\omega_x + \int\limits_{\partial B_R} u(x) \sum_{i=1}^{n} x_k\, \underbrace{(\partial_i\varphi_r)(x)}_{=0}\, \frac{x_i}{|x|}\, d\omega_x =$$

$$= -\sum_{i=1}^{n} \int\limits_{\partial B_R} (\partial_i u)(x)\, \frac{x_i x_k}{|x|}\, d\omega_x + \int\limits_{\partial B_R} u(x)\, \frac{x_k}{|x|}\, d\omega_x$$

(c) Assume that for all $|x| \geq R$ holds

$$|u(x)| \;\leq\; C\,|x|^{1-n}$$

By Lemma 8.10 therefore

$$|(\partial_i u)(x)| \;\leq\; \tilde{C}\,|x|^{-n}$$

for all $|x| \geq 2R$. This implies

$$\left|\int\limits_G u\,\Delta\varphi_r\, dx\right| \;=\; \left|\int\limits_{G\cap B_{2R}} u\,\Delta\varphi_r\, dx\right| \overset{(a)}{=} \left|\sum_{i=1}^{n} \int\limits_{\partial B_{2R}} (\partial_i u)(x)\, \frac{x_i}{|x|}\, d\omega_x\right|$$

$$\leq\; \sum_{i=1}^{n} (2R)^{n-1}\,\tilde{C}\,(2R)^{-n}\omega_n$$

For $R \to \infty$ then

$$\int\limits_G u\,\Delta\varphi_r\, dx \;=\; 0$$

(d) Assume that for all $|x| \geq R$ holds

$$|u(x)| \;\leq\; C\,|x|^{-n}$$

By Lemma 8.10 therefore

$$|(\partial_i u)(x)| \;\leq\; \tilde{C}\,|x|^{-n-1}$$

for all $|x| \geq 2R$. This implies

$$\left|\int\limits_G u\,\Delta\psi_{rk}\, dx\right| \;=\; \left|\int\limits_{G\cap B_{2R}} u\,\Delta\psi_{rk}\, dx\right|$$

$$\overset{(b)}{=} \left|-\sum_{i=1}^{n} \int\limits_{\partial B_R} (\partial_i u)(x)\, \frac{x_i x_k}{|x|}\, d\omega_x + \int\limits_{\partial B_R} u(x)\, \frac{x_k}{|x|}\, d\omega_x\right|$$

$$\leq\; (2R)^{n-1}\,C\,(2R)^{-n}\omega_n + \sum_{i=1}^{n} (2R)^{n-1}\,\tilde{C}\,(2R)^{-n-1}R\,\omega_n$$

For $R \to \infty$ therefore

$$\int_G u\,\Delta\psi_{rk}\,dx \;=\; 0$$

$\square$

Theorem 8.12. Let $n \geq 2$, $1 < q < \infty$, $G \subset \mathbb{R}^n$ be an exterior domain, and let $u \in L^q(G)$ be harmonic in G. Then

$$u \in B^q(G)$$

if and only if there are $C > 0$, $R > 1$ with

$$|u(x)| \;\leq\; C\,|x|^{-n}$$

for all $|x| \geq R$.

Proof. (a) Assume that for all $|x| \geq R$ holds

$$|u(x)| \;\leq\; C\,|x|^{-n}$$

By Lemma 8.11 then for $k = 1, \ldots, n$ holds

$$\int_G u\,\Delta\psi_{rk}\,dx \;=\; \int_G u\,\Delta\varphi_r\,dx \;=\; 0$$

Further for $\phi \in C_0^\infty(G)$

$$0 \;=\; \int_G (\Delta u)\,\phi\,dx \;=\; \int_G u\,\Delta\phi\,dx$$

Therefore also for $\phi \in \widehat{H}_0^{2,q'}(G)$

$$0 \;=\; \int_G u\,\Delta\phi\,dx$$

By Theorem 2.6 and 2.7 then

$$0 \;=\; \int_G u\,\Delta\phi\,dx \qquad \forall\,\phi \in \widehat{H}_\bullet^{2,q'}(G)$$

and therefore

$$u \in B^q(G)$$

(b) Assume that

$$u \in B^q(G)$$

Let $R > 0$ with $\mathbb{R}^n \backslash G \subset B_{\frac{R}{2}}$. Let

$$v(x) := |x|^{2-n}\,u\left(\frac{x}{|x|^2}\right) \qquad \text{for each } x \in B_{\frac{1}{R}} \backslash \{0\}$$

be the Kelvin transform. By Lemma 8.6 there is a harmonic continuation of v in 0. This continuation we again denote by v. By Taylor's formula for every $x \in \overline{B_{\frac{1}{2R}}}$ there is $a_x \in \overline{B_{\frac{1}{2R}}}$ such that

$$v(x) \;=\; v(0) + \sum_{i=1}^{n} (\partial_i v)(a_x) x_i$$

Define

$$w(x) \;:=\; v(x) - v(0) \;=\; \sum_{i=1}^{n} (\partial_i v)(a_x) x_i$$

Because $\partial_i v$ is bounded in $\overline{B_{\frac{1}{2R}}}$ there is a $M > 0$ with

$$|w(x)| \;\leq\; M\,|x| \qquad \forall\,|x| \leq \frac{1}{2R}$$

Let

$$(Kw)(x) := |x|^{2-n}\, w\left(\frac{x}{|x|^2}\right) \qquad \text{for each } x \in \mathbb{R}^n \backslash B_{2R}$$

be the Kelvin transform of w. Then for $|x| \geq 2R$ holds

$$|(Kw)(x)| \;\leq\; |x|^{2-n}\, M\, \frac{1}{|x|} \;=\; M\,|x|^{1-n}$$

Further by [ABR, S.62]

$$\Delta K w = 0$$

So we derive by Lemma 8.11

$$\int_G Kw\, \Delta\varphi_r\, dx \;=\; 0$$

But also holds

$$(Kw)(x) \;=\; u(x) - |x|^{2-n}\, v(0)$$

and therefore

$$\begin{aligned}
0 \;&=\; \int_G Kw\,\Delta\varphi_r\,dx \;=\; \underbrace{\int_G u\,\Delta\varphi_r\,dx}_{=0} - v(0)\int_G |x|^{2-n}\,(\Delta\varphi_r)(x)\,dx \\
&=\; v(0) \int_{G\cap B_{2r}} \sum_{i=1}^{n} \left(\partial_{x_i}|x|^{2-n}\right)\,(\partial_i\varphi_r)(x)\,dx \\
&=\; -v(0) \int_{G\cap B_{2r}} \varphi_r(x)\,\underbrace{\Delta|x|^{2-n}}_{=0}\,dx + v(0)\int_{\partial B_{2r}} \sum_{i=1}^{n} \left(\partial_{x_i}|x|^{2-n}\right)\,\underbrace{\varphi_r(x)}_{=1}\,\frac{x_i}{|x|}\,d\omega_x \\
&=\; v(0) \int_{\partial B_{2r}} \sum_{i=1}^{n} (2-n)\,x_i\,|x|^{-n}\,\frac{x_i}{|x|}\,d\omega_x \\
&=\; v(0)\,(2-n)\,(2r)^{n-1}(2r)^{1-n}\omega_n \;=\; v(0)\,(2-n)\,\omega_n
\end{aligned}$$

For $n \geq 3$ follows

$$v(0) = 0$$

For $n = 2$ by part (a) of the proof of Theorem 8.7 holds

$$v(0) = 0$$

By Taylor's formula for every $x \in \overline{B_{\frac{1}{2R}}}$ there is $b_x \in \overline{B_{\frac{1}{2R}}}$ such that

$$v(x) \;=\; \underbrace{v(0)}_{=0} + \sum_{i=1}^{n} (\partial_i v)(0) x_i + \frac{1}{2} \langle (\text{Hess} v)(b_x)\, x, x \rangle$$

Define

$$h(x) \;:=\; v(x) - \sum_{i=1}^{n} (\partial_i v)(0) x_i \;=\; \frac{1}{2} \langle (\text{Hess} v)(b_x)\, x, x \rangle$$

Because $\partial_i \partial_j v$ is bounded in $\overline{B_{\frac{1}{2R}}}$ there is $C > 0$ with

$$|h(x)| \;\leq\; C\,|x|^2 \qquad \forall\, |x| \leq \frac{1}{2R}$$

Let

$$(Kh)(x) := |x|^{2-n}\, h\left(\frac{x}{|x|^2}\right) \qquad \text{for each } x \in \mathbb{R}^n \backslash B_{2R}$$

be the Kelvin transform of h. Then for $|x| \geq 2R$ holds

$$|(Kh)(x)| \;\leq\; |x|^{2-n}\, C\, \frac{1}{|x|^2} \;=\; C\,|x|^{-n}$$

Further by [ABR, S.62]

$$\Delta Kh = 0$$

So we derive by Lemma 8.11

$$\int\limits_G Kh\, \Delta\psi_{rk}\, dx \;=\; 0$$

But also holds

$$(Kh)(x) \;=\; u(x) - |x|^{2-n} \sum_{i=1}^{n} (\partial_i v)(0) \frac{x_i}{|x|^2} \;=\; u(x) - \sum_{i=1}^{n} (\partial_i v)(0) \frac{x_i}{|x|^n}$$

and therefore for all $k = 1, \ldots, n$

$$\begin{aligned} 0 \;&=\; \int\limits_G Kh\, \Delta\psi_{rk}\, dx \;=\; \underbrace{\int\limits_G u\, \Delta\psi_{rk}\, dx}_{=0} - \sum_{i=1}^{n} (\partial_i v)(0) \int\limits_G \frac{x_i}{|x|^n}\, (\Delta\psi_{rk})(x)\, dx \\ &\overset{4.4}{=}\; -\omega_n \sum_{i=1}^{n} (\partial_i v)(0)\, \delta_{ik} \;=\; -\omega_n\, (\partial_k v)(0) \end{aligned}$$

Then

$$\nabla v(0) \;=\; 0$$

That is

$$u(x) \;=\; (Kh)(x) \qquad \forall\, |x| \geq 2R$$

and for $|x| \geq 2R$ holds

$$|u(x)| \;=\; |(Kh)(x)| \;\leq\; C\, |x|^{-n}$$

$\square$

9 The density of $H^{k,q}(G) \cap B^q(G)$ in $B^q(G)$

Theorem 9.1. Let $n \geq 2$, $k \in \mathbb{N}$, $k \geq 2$, $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{2+k}$. Suppose $p \in B^q(G)$.
Then there is a sequence $(h_m) \subset H^{k,q}(G) \cap B^q(G)$ such that

$$\|h_m - p\|_{q;G} \to 0 \quad (m \to \infty)$$

Proof. (a) Consider Friedrichs' mollifier p_ε for $\varepsilon > 0$. Because G is bounded, we have

$$p_\varepsilon \in C_0^\infty(\mathbb{R}^n) \subset H^{\infty,q}(G) \qquad \forall\, \varepsilon > 0$$

By Theorem 3.11 there is $s^{(\varepsilon)} \in \widehat{H}^{2,q}_\bullet(G)$ such that

$$\left\langle \Delta s^{(\varepsilon)}, \Delta\phi \right\rangle_G = \langle p_\varepsilon, \Delta\phi\rangle_G \qquad \forall\, \phi \in \widehat{H}^{2,q'}_\bullet(G)$$

Because $\partial G \in C^{2+k}$, $k \geq 2$, $p_\varepsilon \in H^{\infty,q}(G)$ and

$$\left\langle \Delta s^{(\varepsilon)}, \Delta\phi \right\rangle_G = \langle \Delta p_\varepsilon, \phi\rangle_G \qquad \forall\, \phi \in C_0^\infty(G)$$

we derive by Theorem 7.7

$$\Delta s^{(\varepsilon)} \in H^{k,q}(G)$$

(b) For $\phi \in \widehat{H}^{2,q'}_\bullet(G)$ we also get

$$\begin{aligned} \left\langle \Delta s^{(\varepsilon)}, \Delta\phi \right\rangle_G &= \langle p_\varepsilon, \Delta\phi\rangle_G - \underbrace{\langle p, \Delta\phi\rangle_G}_{=0} \\ &= \langle p_\varepsilon - p, \Delta\phi\rangle_G \end{aligned}$$

and therefore by Theorem 3.8

$$\begin{aligned} \|\Delta s^{(\varepsilon)}\|_{q;G} &\leq C_q \sup_{0 \neq \phi \in \widehat{H}^{2,q'}_\bullet(G)} \frac{\left\langle \Delta s^{(\varepsilon)}, \Delta\phi \right\rangle}{\|\Delta\phi\|_{q'}} \\ &\leq C_q \|p_\varepsilon - p\|_{q;G} \;\to 0 \quad (\varepsilon \to 0) \end{aligned}$$

(c) Let

$$h^{(\varepsilon)} := p_\varepsilon - \Delta s^{(\varepsilon)}$$

Then by (a)

$$h^{(\varepsilon)} \in H^{k,q}(G) \cap B^q(G)$$

and by (b)

$$\begin{aligned} \|h^{(\varepsilon)} - p\|_{q;G} &\leq \|p_\varepsilon - p\|_{q;G} + \|\Delta s^{(\varepsilon)}\|_{q;G} \\ &\leq (1 + C_q)\, \|p_\varepsilon - p\|_{q;G} \;\to 0 \quad (\varepsilon \to 0) \end{aligned}$$

□

Theorem 9.2. Let $n \geq 2$, $k \in \mathbb{N}$, $k \geq 2$, $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^{2+k}$. Suppose $p \in B^q(G)$.
Then there is a sequence $(h_m) \subset H^{k,q}(G) \cap B^q(G)$ such that

$$\|h_m - p\|_{q;G} \to 0 \quad (m \to \infty)$$

Proof. (a) Consider Friedrichs' mollifier p_ε for $\varepsilon > 0$. Let $\mathbb{R}^n \backslash G \subset B_{\frac{R}{2}}$ for $R > 0$. Then

$$\operatorname{dist}(\mathbb{R}^n \backslash B_R, \partial G) > \operatorname{dist}(\mathbb{R}^n \backslash B_R, \partial B_{\frac{R}{2}}) = \frac{R}{2}$$

Therefore by Lemma A.12 for each $x \in \mathbb{R}^n \backslash B_R$ and for every $0 < \varepsilon < \frac{R}{2}$ holds

$$p_\varepsilon(x) = p(x)$$

and $p_\varepsilon \in L^q(\mathbb{R}^n \backslash B_R)$ is harmonic in $\mathbb{R}^n \backslash B_R$. By Lemma 8.9 we derive

$$p_\varepsilon \Big|_{\mathbb{R}^n \backslash B_{2R}} \in H^{\infty,q}(\mathbb{R}^n \backslash B_{2R})$$

Because $p_\varepsilon \in C^\infty(\mathbb{R}^n)$ obviously for every $r > 0$ holds

$$p_\varepsilon \Big|_{B_r} \in H^{\infty,q}(B_r)$$

Finally for every $0 < \varepsilon < \frac{R}{2}$ holds

$$p_\varepsilon \in H^{\infty,q}(\mathbb{R}^n)$$

(b) By Theorem 3.11 there is $s^{(\varepsilon)} \in \widehat{H}^{2,q}_\bullet(G)$ such that

$$\left\langle \Delta s^{(\varepsilon)}, \Delta\phi \right\rangle_G = \langle p_\varepsilon, \Delta\phi \rangle_G \qquad \forall \phi \in \widehat{H}^{2,q'}_\bullet(G)$$

Because $\partial G \in C^{2+k}$, $k \geq 2$, $p_\varepsilon \in H^{\infty,q}(G)$ and

$$\left\langle \Delta s^{(\varepsilon)}, \Delta\phi \right\rangle_G = \langle \Delta p_\varepsilon, \phi \rangle_G \qquad \forall \phi \in C_0^\infty(G)$$

we derive by Theorem 7.8 for each $r > 0$ and for every $0 < \varepsilon < \frac{R}{2}$

$$\Delta s^{(\varepsilon)} \Big|_{G \cap B_r} \in H^{k,q}(G \cap B_r)$$

Let $\phi \in C_0^\infty(\mathbb{R}^n \backslash B_R) \subset \widehat{H}^{2,q'}_\bullet(G)$. Then for every $0 < \varepsilon < \frac{R}{2}$ holds by (a)

$$\left\langle \Delta s^{(\varepsilon)}, \Delta\phi \right\rangle_G = \langle p_\varepsilon, \Delta\phi \rangle_G = 0$$

Therefore by Weyl's Lemma $\Delta s^{(\varepsilon)} \in L^q(G)$ is harmonic in $\mathbb{R}^n \backslash B_R$. Again by Lemma 8.9 we derive

$$\Delta s^{(\varepsilon)} \Big|_{\mathbb{R}^n \backslash B_{2R}} \in H^{\infty,q}(\mathbb{R}^n \backslash B_{2R})$$

By Lemma A.5 we get for every $0 < \varepsilon < \frac{R}{2}$

$$\Delta s^{(\varepsilon)} \in H^{k,q}(G)$$

(c) For $\phi \in \widehat{H}^{2,q'}_{\bullet}(G)$ we also get

$$\begin{aligned} \left\langle \Delta s^{(\varepsilon)}, \Delta\phi \right\rangle_G &= \langle p_\varepsilon, \Delta\phi \rangle_G - \underbrace{\langle p, \Delta\phi \rangle_G}_{=0} \\ &= \langle p_\varepsilon - p, \Delta\phi \rangle_G \end{aligned}$$

and therefore by Theorem 3.8

$$\begin{aligned} \|\Delta s^{(\varepsilon)}\|_{q;G} &\leq C_q \sup_{0 \neq \phi \in \widehat{H}^{2,q'}_{\bullet}(G)} \frac{\left\langle \Delta s^{(\varepsilon)}, \Delta\phi \right\rangle}{\|\Delta\phi\|_{q'}} \\ &\leq C_q \|p_\varepsilon - p\|_{q;G} \ \to 0 \quad (\varepsilon \to 0) \end{aligned}$$

(d) Let

$$h^{(\varepsilon)} := p_\varepsilon - \Delta s^{(\varepsilon)}$$

Then by (a) and (b)

$$h^{(\varepsilon)} \in H^{k,q}(G) \cap B^q(G) \qquad \forall\, 0 < \varepsilon < \frac{R}{2}$$

and by (c)

$$\begin{aligned} \|h^{(\varepsilon)} - p\|_{q;G} &\leq \|p_\varepsilon - p\|_{q;G} + \|\Delta s^{(\varepsilon)}\|_{q;G} \\ &\leq (1 + C_q)\, \|p_\varepsilon - p\|_{q;G} \ \to 0 \quad (\varepsilon \to 0) \end{aligned}$$

□

10 The compactness of the imbedding $H^{1,q}(G) \cap B^q(G) \subset B^q(G)$ in exterior domains

Theorem 10.1 Let $n \geq 2$, $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^1$.
Then the imbedding

$$H^{1,q}(G) \cap B^q(G) \to B^q(G)$$

is compact.

Proof. (a) $H^{1,q}(G)$ is a reflexive space. Every closed subspace of a reflexive space is reflexive (see e.g. [Alt, Satz 6.8, p.216]). Therefore $H^{1,q}(G) \cap B^q(G)$ is reflexive. By the Hahn-Banach theorem one can easily prove

$$\Big(H^{1,q}(G) \cap B^q(G)\Big)^* \;=\; H^{1,q}(G)^*\Big|_{H^{1,q}(G) \cap B^q(G)}$$

So by Lemma B.5 it suffices to show: For a sequence $(h_k) \subset H^{1,q}(G) \cap B^q(G)$ with $F^*(h_k) \to 0$ for all $F^* \in H^{1,q}(G)^*$ holds $\|h_k\|_{q;G} \to 0$.

(b) Let $R > 0$ such that $\mathbb{R}^n \backslash G \subset B_{\frac{R}{2}}$. Then

$$G_r := G \cap B_r$$

for $r > R$ is bounded and

$$\partial G_r \in C^1$$

Let $F^* \in H^{1,q}(G_r)^*$ be given. Define

$$\tilde{F}^*(h) := F^*\left(h\big|_{G_r}\right) \qquad \forall\, h \in H^{1,q}(G)$$

Then obviously $\tilde{F}^* \in H^{1,q}(G)^*$ and therefore

$$\tilde{F}^*(h_k) \to 0$$

that is

$$F^*(h_k\big|_{G_r}) \to 0$$

Because the imbedding $H^{1,q}(G_r) \to L^q(G_r)$ is compact (see e.g. [Alt, Satz 8.8, p.314]) we derive

$$\|h_k\|_{q;G\cap B_r} \to 0 \qquad (k \to \infty) \qquad \forall\, r > R$$

(c) Let $x \in \mathbb{R}^n \backslash B_{2R}$. Define

$$F^*(h) := h(x) \qquad \forall\, h \in H^{1,q}(G) \cap B^q(G)$$

Then by the mean value property

$$\begin{aligned}|F^*(h)| &= |h(x)| \le \frac{1}{|B_R|}\int_{B_R} |h(x+y)|\,dy \\ &\le |B_R|^{-1}\,\|h\|_{q;G}\,|B_R|^{\frac{q-1}{q}} \\ &\le |B_R|^{-\frac{1}{q}}\,\|h\|_{1,q;G}\end{aligned}$$

Therefore

$$F^* \in H^{1,q}(G)^*$$

that is

$$h_k(x) = F^*(h_k) \to 0 \qquad \forall\, x \in \mathbb{R}^n\backslash B_{2R}$$

(d) By Theorem 8.8 there is an $f_q \in L^q(\mathbb{R}^n\backslash B_{2R})$ such that for each $x \in \mathbb{R}^n\backslash B_{2R}$ holds

$$|h_k(x)| \le C\,\|h_k\|_{q;\mathbb{R}^n\backslash B_R}\, f_q(x)$$

By assumption there is an $C' > 0$ with

$$\|h_k\|_{q;G} \le C' \qquad \forall\, k \in \mathbb{N}$$

Therefore by (c) and the dominated convergence theorem follows

$$\|h_k\|_{q;\mathbb{R}^n\backslash B_{2R}} \to 0 \qquad (k \to \infty)$$

(e) By (b) and (d) we finally get

$$\|h_k\|_{q;G} \to 0 \qquad (k \to \infty)$$

□

Part II: The Cosserat spectrum

11 Definition of the operator Z_q and its fundamental properties

Definition 11.1. Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^1$.

1. Let $\underline{T}_q : L^q(G) \to \widehat{H}^{1,q}_{\bullet}(G)^n$ be defined by (cf. Theorem 2.9)
$$\big\langle \nabla \underline{T}_q(p), \nabla \underline{\phi} \big\rangle_G \;=\; \big\langle p, \operatorname{div} \underline{\phi} \big\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}^{1,q'}_{\bullet}(G)^n$$

2. Let $Z_q : L^q(G) \to L^q(G), \qquad Z_q(p) := \operatorname{div}(\underline{T}_q p)$

Theorem 11.2. Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^1$. Then

1. $Z_q\Big|_{A^q(G)} : A^q(G) \to A^q(G), \qquad Z_q(p_0) = p_0 \quad \text{for each } p_0 \in A^q(G)$

2. $Z_q\Big|_{B^q(G)} : B^q(G) \to B^q(G)$

Proof. (a) Let $p_0 \in A^q(G)$. Then $p_0 = \Delta s$ with $s \in \widehat{H}^{2,q}_{\bullet}(G)$. Like in the proof of Lemma 3.10 there are $v \in \widehat{H}^{2,q}_0(G)$ and $f \in C^\infty(\mathbb{R}^n)$ such that

$$\partial_i \partial_j f \in C_0^\infty(G) \qquad \text{and} \qquad s = v + f$$

There is by Definition 3.2 a sequence $(v_k) \subset C_0^\infty(G)$ with $\|\nabla^2(v_k - v)\|_{q;G} \to 0$. Let $\underline{\phi} \in \widehat{H}^{1,q'}_{\bullet}(G)^n$. Then

$$\begin{aligned}
\big\langle \nabla\nabla s, \nabla \underline{\phi} \big\rangle_G &= \sum_{i,j=1}^n \int_G \partial_i\partial_j v \; \partial_i \phi_j \; dx \;+\; \sum_{i,j=1}^n \int_G \underbrace{\partial_i \partial_j f}_{\in C_0^\infty(G)} \; \partial_i \phi_j \; dx \\
&= \lim_{k\to\infty} \sum_{i,j=1}^n \int_G \partial_i\partial_j v_k \; \partial_i \phi_j \; dx \;-\; \sum_{i,j=1}^n \int_G (\partial_j \underbrace{\partial_i \partial_i f}_{\in C_0^\infty(G)}) \phi_j \; dx \\
&= \lim_{k\to\infty} \sum_{i,j=1}^n \int_G \partial_i\partial_i v_k \; \partial_j \phi_j \; dx \;+\; \sum_{i,j=1}^n \int_G \partial_i \partial_i f \; \partial_j \phi_j \; dx \\
&= \big\langle \Delta v, \operatorname{div} \underline{\phi} \big\rangle_G + \big\langle \Delta f, \operatorname{div} \underline{\phi} \big\rangle_G \;=\; \big\langle p_0, \operatorname{div} \underline{\phi} \big\rangle_G
\end{aligned}$$

Therefore
$$\nabla s = \underline{T}_q(p_0)$$
and
$$Z_q(p_0) = \operatorname{div}(\underline{T}_q p_0) = \Delta s = p_0$$

(b) Let $p_h \in B^q(G)$. Let $\varphi \in \widehat{H}_\bullet^{2,q'}(G)$. Then $\nabla\varphi \in \widehat{H}_\bullet^{1,q'}(G)^n$. Like in (a) there are $v \in \widehat{H}_0^{2,q'}(G)$ and $f \in C^\infty(\mathbb{R}^n)$ such that

$$\partial_i\partial_j f \in C_0^\infty(G) \qquad \text{and} \qquad \varphi = v + f$$

There is by Definition 3.2 a sequence $(v_k) \subset C_0^\infty(G)$ with $\|\nabla^2(v_k - v)\|_{q;G} \to 0$.Then

$$\begin{aligned}
0 = \langle p_h, \Delta\varphi\rangle_G &= \langle p_h, \operatorname{div}\nabla\varphi\rangle_G = \left\langle \nabla\underline{T}_q(p_h), \nabla\nabla\varphi\right\rangle_G \\
&= \lim_{k\to\infty} \left\langle \nabla\underline{T}_q(p_h), \nabla\nabla v_k\right\rangle_G + \langle\nabla\underline{T}_q(p_h), \underbrace{\nabla\nabla f}_{\in C_0^\infty(G)}\rangle_G \\
&= -\lim_{k\to\infty} \left\langle \underline{T}_q(p_h), \nabla\Delta v_k\right\rangle_G - \langle\underline{T}_q(p_h), \nabla \underbrace{\Delta f}_{\in C_0^\infty(G)}\rangle_G \\
&= \left\langle \operatorname{div}(\underline{T}_q(p_h)), \Delta\varphi\right\rangle_G
\end{aligned}$$

Therefore

$$Z_q(p_h) = \operatorname{div}(\underline{T}_q(p_h)) \in B^q(G)$$

□

Theorem 11.3. Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^1$. Assume $\lambda \in \mathbb{R}$. Then there is $\underline{u} \in \widehat{H}_\bullet^{1,q}(G)^n$ with

$$\left\langle\nabla\underline{u}, \nabla\underline{\phi}\right\rangle_G = \lambda\left\langle\operatorname{div}\underline{u}, \operatorname{div}\underline{\phi}\right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}_\bullet^{1,q'}(G)^n$$

if and only if there is $p \in L^q(G)$ with

$$\lambda\, Z_q(p) = p$$

In this case one can choose $p = \operatorname{div}\underline{u}$.

Proof. (a) Assume that there is $\underline{u} \in \widehat{H}_\bullet^{1,q}(G)^n$ with

$$\left\langle\nabla\underline{u}, \nabla\underline{\phi}\right\rangle_G = \lambda\left\langle\operatorname{div}\underline{u}, \operatorname{div}\underline{\phi}\right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}_\bullet^{1,q'}(G)^n$$

Let

$$p := \operatorname{div}\underline{u}$$

Then

$$\lambda\,\underline{T}_q(p) = \underline{u}$$

and

$$\lambda\, Z_q(p) = \operatorname{div}\underline{u} = p$$

(b) Assume that there is $p \in L^q(G)$ with

$$\lambda\, Z_q(p) = p$$

Let

$$\underline{u} := \underline{T}_q(p)$$

Then

$$\begin{aligned} \left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G &= \left\langle p, \operatorname{div} \underline{\phi} \right\rangle_G = \lambda \left\langle Z_q(p), \operatorname{div} \underline{\phi} \right\rangle_G \\ &= \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}^{1,q'}_{\bullet}(G)^n \end{aligned}$$

□

Theorem 11.4. Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^1$. Assume that for $\lambda \in \mathbb{R}$ and $p \in L^q(G)$ holds

$$Z_q(p) = \lambda\, p$$

Then $\lambda = 1$ or $p \in B^q(G)$

Proof. Decompose by Theorem 4.2

$$p = p_0 + p_h, \qquad p_0 \in A^q(G), \quad p_h \in B^q(G)$$

Then by Theorem 11.2

$$\lambda\, p_0 + \lambda\, p_h \;=\; Z_q(p_0) + Z_q(p_h) \;=\; p_0 + Z_q(p_h)$$

Therefore

$$\underbrace{(\lambda - 1)p_0}_{\in A^q(G)} + \underbrace{\lambda\, p_h - Z_q(p_h)}_{\in B^q(G)} = 0$$

Because the decomposition in Theorem 4.2 is direct we get

$$(\lambda - 1)p_0 = 0$$

that is $\lambda = 1$ or $p_0 = 0$ □

12 The eigenvalues in $B^q(G)$ for exterior domains

This chapter base on the idea of Michel Crouzeix [Cr], that is mentioned in the introduction.

Lemma 12.1. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$ and let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^{k+2}$. Suppose $p \in B^q(G) \cap H^{k,q}(G)$ and $\underline{u} := \underline{T}_q(p)$. Then

1. $\nabla \underline{u} \in H^{k,q}(G)^{n^2}$ and $\|\nabla \underline{u}\|_{k,q;G} \leq C(G,k,q,n)\,\|p\|_{k,q;G}$
2. $\Delta \underline{u} = \nabla p$
3. $\underline{u} \in \overline{C}^1(G)^n$ (if $k > \frac{n}{q}$)
4. $p \in C^0(\overline{G})$ (if $k > \frac{n}{q}$)

Proof. (a) For $\phi \in C_0^\infty(G)^n$ we have

$$\langle \nabla \underline{u}, \nabla \underline{\phi} \rangle_G \;=\; \langle p, \operatorname{div} \underline{\phi} \rangle_G \;=\; -\langle \nabla p, \underline{\phi} \rangle_G$$

and for $\varphi \in C_0^\infty(G)$ and $i = 1, \ldots, n$

$$\langle \nabla u_i, \nabla \varphi \rangle_G \;=\; -\langle \partial_i p, \varphi \rangle_G$$

Because $\partial G \in C^{k+2}$ by Theorem 7.5, Lemma A.15 and Theorem 2.8 we derive

$$\nabla \underline{u} \in H^{k,q}(G)^{n^2}$$

and

$$\begin{aligned}\|\nabla \underline{u}\|_{k,q;G} \;&\leq\; C_1(G,k,q,n)\left[\|\nabla p\|_{k-1,q;G} + \|\underline{u}\|_{q;G\cap B_{R_k}} + \|\nabla \underline{u}\|_{q;G}\right] \\ &\leq\; C_2(G,k,q,n)\,\|p\|_{k,q;G}\end{aligned}$$

(b) For $\phi \in C_0^\infty(G)^n$ holds

$$\langle \Delta \underline{u}, \underline{\phi} \rangle_G \;=\; -\langle \nabla \underline{u}, \nabla \underline{\phi} \rangle_G \;=\; \langle \nabla p, \underline{\phi} \rangle_G$$

Therefore

$$\Delta \underline{u} = \nabla p$$

(c) Suppose $k > \frac{n}{q}$. Let $R > 0$ large enough such that $\partial(G \cap B_R) \in C^{k+2}$. Then

$$\underline{u} \in H^{k+1,q}(G \cap B_r)^n \qquad \forall\, r > R$$

and therefore by Sobolev's imbedding theorem (see e.g. [Alt, Satz 8.13, p.319])

$$\underline{u} \in \overline{C}^1(G \cap B_r)^n \qquad \forall\, r > R$$

Finally also holds

$$\underline{u} \in \overline{C}^1(G)^n$$

Similarly one can prove

$$p \in C^0(\overline{G})$$

□

Lemma 12.2. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^{k+2} \cap C^4$. Suppose

$$p \in B^q(G) \cap H^{k,q}(G) \qquad \underline{u} := \underline{T}_q(p)$$

Further let

$$\zeta \in C_0^3(\mathbb{R}^n), \quad \zeta\Big|_{\partial G} = 0, \quad \nabla\zeta\Big|_{\partial G} = N$$

Define

$$w := \underline{u}\,\nabla\zeta - \frac{1}{2}\,p\,\zeta$$

Then

$$w \in H_0^{1,q}(G) \cap H^{2,q}(G) \cap C_0^0(\mathbb{R}^n)$$

and there is a constant $C = C(n,q,G,\zeta) > 0$ such that

$$\|w\|_{2,q;G} \;\leq\; C\,\|p\|_{q;G}$$

Proof. (a) Let $R > 0$ such that $\mathrm{supp}(\zeta) \subset B_R$ and $\mathbb{R}^n \backslash G \subset B_R$. Choose $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta(x) = 1$ for $x \in \mathrm{supp}(\zeta)$. By the definition of $\widehat{H}_\bullet^{1,q}(G)$ holds

$$\eta\underline{u} \in H_0^{1,q}(G)^n$$

By Lemma A.3 follows

$$\underline{u}\,\nabla\zeta \;=\; (\eta\underline{u})\,\nabla\zeta \in H_0^{1,q}(G)$$

Because (by Theorem 5.5)

$$\zeta \in H_0^{1,q}(G)$$

and

$$\|\zeta\|_\infty + \|\nabla\zeta\|_\infty < \infty$$

we derive by Lemma A.4

$$p\zeta \in H_0^{1,q}(G)$$

and by Lemma 12.1 and Theorem 5.2

$$w \in H_0^{1,q}(G) \cap C_0^0(\mathbb{R}^n)$$

(b) We have

$$\partial_i w \;=\; \sum_{j=1}^n \underbrace{\partial_i u_j}_{H^{1,q}}\underbrace{\partial_j\zeta}_{H^{1,q}} + \sum_{j=1}^n \underbrace{u_j}_{H^{1,q}(G\cap B_R)}\underbrace{\partial_i\partial_j\zeta}_{C_0^1(\mathbb{R}^n)} - \frac{1}{2}\underbrace{\partial_i p}_{C^\infty(G)}\underbrace{\zeta}_{C^\infty} - \frac{1}{2}\underbrace{p}_{H^{1,q}}\underbrace{\partial_i\zeta}_{H^{1,q}}$$

Therefore by Lemma A.1 and by Lemma 12.1

$$\begin{aligned}\Delta w \;&=\; \underbrace{\Delta\underline{u}}_{=\nabla p}\cdot\nabla\zeta + 2\nabla\underline{u}\cdot\nabla\nabla\zeta + \underline{u}\cdot\nabla\Delta\zeta - \frac{1}{2}\zeta\underbrace{\Delta p}_{=0} - \nabla p\nabla\zeta - \frac{1}{2}p\Delta\zeta \\ &=\; 2\nabla\underline{u}\cdot\nabla\nabla\zeta + \underline{u}\cdot\nabla\Delta\zeta - \frac{1}{2}p\Delta\zeta\end{aligned}$$

By Lemma A.15 and Theorem 2.9 we derive

$$\begin{aligned}\|\Delta w\|_{q;G} &\leq C_1(\zeta)\|\nabla \underline{u}\|_{q;G} + C_2(\zeta)\|\underline{u}\|_{q;G\cap B_R} + C_3(\zeta)\|p\|_{q;G} \\ &\leq C_4(\zeta, n, q, G, R)\|p\|_{q;G}\end{aligned}$$

(c) Again by Lemma A.15 and Theorem 2.9 holds

$$\begin{aligned}\|w\|_{q;G} &\leq C_5(\zeta)\|\underline{u}\|_{q;G\cap B_R} + C_6(\zeta)\|p\|_{q;G} \\ &\leq C_7(\zeta, q, G, R)\|p\|_{q;G}\end{aligned}$$

(d) We have $w \in H_0^{1,q}(G)$ and $w(x) = 0$ for $|x| > R$. Therefore

$$w \in H_0^{1,q}(G \cap B_{2R})$$

and by Theorem 2.8 and Poincare's Lemma

$$\begin{aligned}\|\nabla w\|_{q;G} &= \|\nabla w\|_{q;G\cap B_{2R}} \leq C_q \sup_{0\neq\phi\in C_0^\infty(G\cap B_{2R})} \frac{\langle \nabla w, \nabla \phi\rangle}{\|\nabla\phi\|_{q'}} \\ &= C_q \sup_{0\neq\phi\in C_0^\infty(G\cap B_{2R})} \frac{\langle \Delta w, \phi\rangle}{\|\nabla\phi\|_{q'}} \\ &\leq C_8(q, G, R)\,\|\Delta w\|_{q;G\cap B_{2R}} = C_8(q, G, R)\,\|\Delta w\|_{q;G}\end{aligned}$$

(e) By Theorem 7.6 we finally derive $w \in H^{2,q}(G)$ and

$$\begin{aligned}\|w\|_{2,q;G} &\leq C_9(G, q, n)\ (\|\Delta w\|_{q;G} + \|w\|_{1,q;G}) \\ &\underset{(b),(c),(d)}{\leq} C_{10}(G, q, n, \zeta, R)\,\|p\|_{q;G}\end{aligned}$$

□

Lemma 12.3. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^{k+2}$. Suppose

$$p \in B^q(G) \cap H^{k,q}(G)$$

Then

$$Z_q(p) - \frac{1}{2}\,p \in B^q(G) \cap H^{k,q}(G)$$

and there is a constant $C = C(n, q, G) > 0$ such that

$$\|Z_q(p) - \frac{1}{2}\,p\|_{1,q;G} \leq C\,\|p\|_{q;G}$$

Proof. (a) With $\underline{u} := \underline{T}_q(p)$ holds $Z_q(p) - \frac{1}{2}\,p = \operatorname{div}\underline{u} - \frac{1}{2}\,p$. By Lemma 12.1 and Theorem 11.2 we get

$$Z_q(p) - \frac{1}{2}\,p \in B^q(G) \cap H^{k,q}(G)$$

and by Theorem 2.9 holds

$$\|Z_q(p) - \frac{1}{2}\,p\|_{q;G} \;\le\; C_1(n,q)\,\|p\|_{q;G}$$

(b) Let now by Theorem 6.1

$$\zeta \in C_0^3(\mathbb{R}^n), \quad \zeta\Big|_{\partial G} = 0, \quad \nabla\zeta\Big|_{\partial G} = N$$

and define

$$w :=\; \underline{u}\,\nabla\zeta \;-\; \frac{1}{2}\,p\,\zeta$$

Then by Lemma 12.2

$$w \in H_0^{1,q}(G) \cap H^{2,q}(G) \cap C_0^0(\mathbb{R}^n)$$

By Lemma A.1 therefore

$$\nabla w \nabla\zeta \in H^{1,q}(G)$$

and

$$\partial_i w \;=\; \sum_{j=1}^n (\partial_i u_j)(\partial_j\zeta) + \sum_{j=1}^n u_j(\partial_i\partial_j\zeta) - \frac{1}{2}(\partial_i p)\zeta - \frac{1}{2}p(\partial_i\zeta)$$

$$\begin{aligned}
\nabla w \nabla\zeta \;\;&=\;\; \sum_{i,j=1}^n (\partial_i u_j)(\partial_j\zeta)(\partial_i\zeta) + \sum_{i,j=1}^n u_j(\partial_i\partial_j\zeta)(\partial_i\zeta)\\
&\quad - \sum_{i=1}^n \frac{1}{2}(\partial_i p)\,\zeta\,(\partial_i\zeta) - \sum_{i=1}^n \frac{1}{2}p(\partial_i\zeta)(\partial_i\zeta)
\end{aligned}$$

Therefore

$$\begin{aligned}
\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p) \;\;&=\;\; \left[\sum_{i,j=1}^n (\partial_i u_j)(\partial_j\zeta)(\partial_i\zeta) - \operatorname{div}\underline{u}\right]\\
&\quad + \sum_{i,j=1}^n u_j(\partial_i\partial_j\zeta)(\partial_i\zeta)\\
&\quad - \sum_{i=1}^n \frac{1}{2}(\partial_i p)\,\zeta\,(\partial_i\zeta) + \frac{1}{2}p\left[1 - \sum_{i=1}^n (\partial_i\zeta)(\partial_i\zeta)\right]\\
&=:\;\; f_1 + f_2 + f_3 + f_4
\end{aligned}$$

(c) Because by Lemma 12.1 $\underline{u} \in \widehat{H}_{\bullet}^{1,q}(G)^n \cap \overline{C}^1(G)^n$ we derive by Theorem 5.8

$$\underline{u}\Big|_{\partial G} = 0$$

and by Theorem 5.6 for $x \in \partial G$

$$(\nabla u_i)(x) = \lambda_i(x) N(x)$$

For $x \in \partial G$ therefore holds

$$\begin{aligned}
\sum_{i,j=1}^{n} (\partial_i u_j)(x)(\partial_j \zeta)(x)(\partial_i \zeta)(x) &= \sum_{i,j=1}^{n} \lambda_j(x) N_i(x) N_j(x) N_i(x) \\
&= |N(x)|^2 \sum_{j=1}^{n} \lambda_j(x) N_j(x) \;=\; \sum_{j=1}^{n} (\partial_j u_j)(x) \\
&= \operatorname{div} \underline{u}(x)
\end{aligned}$$

Therefore

$$f_1 \in C^0(\mathbb{R}^n) \cap H^{1,q}(G), \qquad f_1\Big|_{\partial G} = 0$$

For $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\eta) \subset B_R$, $\mathbb{R}^n \backslash G \subset B_R$ holds $\eta f_1 \in H^{1,q}(G \cap B_R) \cap C^0(\overline{G \cap B_R})$ and

$$\eta f_1\Big|_{\partial G \cup \partial B_R} = 0$$

Therefore by Theorem 5.5

$$\eta f_1 \in H_0^{1,q}(G \cap B_R) \subset H_0^{1,q}(G)$$

and

$$f_1 \in \widehat{H}_\bullet^{1,q}(G)$$

(d) We have $u_j \in \widehat{H}_\bullet^{1,q}(G)$ and $(\partial_i \partial_j \zeta)(\partial_i \zeta) \in C_0^1(\mathbb{R}^n)$. Let $\operatorname{supp}(\zeta) \subset B_R$, $\mathbb{R}^n \backslash G \subset B_R$. Choose $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta(x) = 1$ for $x \in B_R$. Then $\eta u_j \in H_0^{1,q}(G)$ and by Lemma A.3

$$f_2 = \sum_{i,j=1}^{n} u_j (\partial_i \partial_j \zeta)(\partial_i \zeta) = \sum_{i,j=1}^{n} \eta u_j (\partial_i \partial_j \zeta)(\partial_i \zeta) \in H_0^{1,q}(G)$$

(e) By Theorem 5.5 holds $\zeta \in H_0^{1,q}(G)$ and $\partial_i p \in H^{1,q}(G)$, $\partial_i \zeta \in H^{1,q}(G)$. By Lemma A.4 we derive

$$f_3 \in H_0^{1,q}(G)$$

(f) It holds

$$1 - \sum_{i=1}^{n} (\partial_i \zeta)(\partial_i \zeta)\Big|_{\partial G} = 1 - |N|^2\Big|_{\partial G} = 0$$

Therefore

$$f_4 \in H^{1,q}(G) \cap C^0(\overline{G}), \qquad f_4\Big|_{\partial G} = 0$$

and by Theorem 5.5

$$f_4 \in \widehat{H}_\bullet^{1,q}(G)$$

(g) By (c)-(f) holds

$$\nabla w \nabla \zeta - (\operatorname{div} \underline{u} - \frac{1}{2}\, p) \in \widehat{H}^{1,q}_{\bullet}(G)$$

and therefore by Theorem 2.8

$$\|\nabla \Big[\nabla w \nabla \zeta - (\operatorname{div} \underline{u} - \frac{1}{2}\, p) \Big] \|_{q;G} \ \leq$$

$$\leq\ C_q \sup_{0 \neq \phi \in \widehat{H}^{1,q'}_{\bullet}(G)} \frac{\Big\langle \nabla \Big[\nabla w \nabla \zeta - (\operatorname{div} \underline{u} - \frac{1}{2}\, p) \Big], \nabla \phi \Big\rangle}{\|\nabla \phi\|_{q'}} \ =$$

$$\underset{\text{Lemma 4.3}}{=} C_q \sup_{0 \neq \phi \in \widehat{H}^{1,q'}_{\bullet}(G)} \frac{\Big\langle \nabla \Big[\nabla w \nabla \zeta \Big], \nabla \phi \Big\rangle}{\|\nabla \phi\|_{q'}} \ \leq\ C_q \|\nabla \Big[\nabla w \nabla \zeta \Big] \|_{q;G}$$

Finally by Lemma 12.2

$$\begin{aligned} \|\nabla(\operatorname{div} \underline{u} - \frac{1}{2}\, p)\|_{q;G} \ &\leq\ \|\nabla \Big[\nabla w \nabla \zeta - (\operatorname{div} \underline{u} - \frac{1}{2}\, p) \Big] \|_{q;G} + \|\nabla \Big[\nabla w \nabla \zeta \Big] \|_{q;G} \\ &\leq\ (1 + C_q)\, \|\nabla \Big[\nabla w \nabla \zeta \Big] \|_{q;G} \ \leq\ C_2(q, \zeta)\, \|w\|_{2,q;G} \\ &\leq\ C_3(n, q, G, \zeta) \|p\|_{q;G} \end{aligned}$$

(h) By (a) and (g) the assertion follows. □

Theorem 12.4. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^{k+2}$. Suppose

$$p \in B^q(G)$$

Then

$$Z_q(p) - \frac{1}{2}\, p \in B^q(G) \cap H^{1,q}(G)$$

and there is a constant $C = C(n, q, G) > 0$ such that

$$\|Z_q(p) - \frac{1}{2}\, p\|_{1,q;G} \ \leq\ C\, \|p\|_{q;G}$$

Proof. (a) By Theorem 9.2 there is a sequence $(p_m) \subset B^q(G) \cap H^{k,q}(G)$ such that

$$\|p_m - p\|_{q;G} \to 0$$

By Lemma 12.3 holds

$$\|\big(Z_q(p_m) - \frac{1}{2}\, p_m\big) - \big(Z_q(p_{m'}) - \frac{1}{2}\, p_{m'}\big)\|_{1,q;G} \ \leq\ C\, \|p_{m'} - p_m\|_{q;G} \ \to 0$$

By the completeness of $H^{1,q}(G)$ there exists a $u \in H^{1,q}(G)$ with

$$\|(Z_q(p_m) - \frac{1}{2}\,p_m) - u\|_{1,q;G} \;\to 0$$

Because

$$\|Z_q(p_m) - \frac{1}{2}\,p_m\|_{1,q;G} \;\leq\; C\,\|p_m\|_{q;G}$$

we get for $(m \to \infty)$

$$\|u\|_{1,q;G} \;\leq\; C\,\|p\|_{q;G}$$

(b) By Theorem 11.2 we derive

$$Z_q(p) - \frac{1}{2}\,p \in B^q(G)$$

and by Theorem 2.9 and the definition of Z_q there exists $C_1(n,q) > 0$ such that

$$\|Z_q(\pi) - \frac{1}{2}\,\pi\|_{q;G} \;\leq\; C_1(n,q)\,\|\pi\|_{q;G} \qquad \forall\,\pi \in B^q(G)$$

Therefore

$$\|\big(Z_q(p_m) - \frac{1}{2}\,p_m\big) - \big(Z_q(p) - \frac{1}{2}\,p\big)\|_{q;G} \;\leq\; C_1\,\|p - p_m\|_{q;G} \;\to 0$$

We get

$$u = Z_q(p) - \frac{1}{2}\,p$$

and the assertion follows by (a) $\square$

Theorem 12.5. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be an exterior domain with $\partial G \in C^{k+2}$. Then

$$Z_q \;-\; \frac{1}{2}\,I \;:\; B^q(G) \to B^q(G)$$

is a compact operator.

Proof. By Theorem 12.4 for all $p \in B^q(G)$ holds

$$\|Z_q(p) - \frac{1}{2}\,p\|_{1,q;G} \;\leq\; C\,\|p\|_{q;G}$$

Let $(p_m) \subset B^q(G)$ with $\|p_m\|_{q;G} \leq \tilde{C} < \infty$ for each $m \in \mathbb{N}$. Then

$$\|Z_q(p_m) - \frac{1}{2}\,p_m\|_{1,q;G} \;\leq\; C\,\tilde{C} < \infty$$

Because the imbedding $B^q(G) \cap H^{1,q}(G) \to B^q(G)$ is compact (Theorem 10.1) there is a subsequence $(p_{m_l}) \subset (p_m)$ such that $(Z_q(p_{m_l}) - \frac{1}{2}\,p_{m_l})$ is Cauchy in $B^q(G)$. $\square$

13 The eigenvalues in $B^q(G)$ for bounded domains

This chapter base on the idea of Michel Crouzeix [Cr], that is mentioned in the introduction.

Lemma 13.1. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{k+2}$. Suppose $p \in B^q(G) \cap H^{k,q}(G)$ and $\underline{u} := \underline{T}_q(p)$. Then

1. $\nabla \underline{u} \in H^{k,q}(G)^{n^2}$ and $\|\nabla \underline{u}\|_{k,q;G} \leq C(G,k,q,n)\, \|p\|_{k,q;G}$
2. $\Delta \underline{u} = \nabla p$
3. $\underline{u} \in \overline{C}^1(G)^n$ (if $k > \frac{n}{q}$)
4. $p \in C^0(\overline{G})$ (if $k > \frac{n}{q}$)

Proof. (a) For $\phi \in C_0^\infty(G)^n$ we have

$$\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G \;=\; \left\langle p, \operatorname{div} \underline{\phi} \right\rangle_G \;=\; -\left\langle \nabla p, \underline{\phi} \right\rangle_G$$

and for $\varphi \in C_0^\infty(G)$ and $i = 1, \ldots, n$

$$\langle \nabla u_i, \nabla \varphi \rangle_G \;=\; -\langle \partial_i p, \varphi \rangle_G$$

Because $\partial G \in C^{k+2}$ by Theorem 7.4, Lemma A.15 and Theorem 2.8 we derive

$$\nabla \underline{u} \in H^{k,q}(G)^{n^2}$$

and

$$\begin{aligned} \|\nabla \underline{u}\|_{k,q;G} &\leq C_1(G,k,q,n)\; [\|\nabla p\|_{k-1,q;G} + \|\underline{u}\|_{q;G} + \|\nabla \underline{u}\|_{q;G}] \\ &\leq C_2(G,k,q,n)\, \|p\|_{k,q;G} \end{aligned}$$

(b) For $\phi \in C_0^\infty(G)^n$ holds

$$\left\langle \Delta \underline{u}, \underline{\phi} \right\rangle_G \;=\; -\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G \;=\; \left\langle \nabla p, \underline{\phi} \right\rangle_G$$

Therefore

$$\Delta \underline{u} = \nabla p$$

(c) Suppose $k > \frac{n}{q}$. By (a) we get

$$\underline{u} \in H^{k+1,q}(G)^n$$

and therefore by Sobolev's imbedding theorem (see e.g. [Alt, Satz 8.13, p.319])

$$\underline{u} \in \overline{C}^1(G)^n$$

Similarly one can prove

$$p \in C^0(\overline{G})$$

□

Lemma 13.2. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{k+2} \cap C^4$. Suppose

$$p \in B^q(G) \cap H^{k,q}(G) \qquad \underline{u} := \underline{T}_q(p)$$

Further let

$$\zeta \in C_0^3(\mathbb{R}^n), \quad \zeta\Big|_{\partial G} = 0, \quad \nabla\zeta\Big|_{\partial G} = N$$

Define

$$w := \underline{u}\,\nabla\zeta - \frac{1}{2}\,p\,\zeta$$

Then

$$w \in H_0^{1,q}(G) \cap H^{2,q}(G) \cap C_0^0(\mathbb{R}^n)$$

and there is a constant $C = C(n, q, G, \zeta) > 0$ such that

$$\|w\|_{2,q;G} \;\leq\; C\,\|p\|_{q;G}$$

Proof. (a) By Lemma A.3 follows

$$\underline{u}\,\nabla\zeta \in H_0^{1,q}(G)$$

Because (by Theorem 5.5)

$$\zeta \in H_0^{1,q}(G)$$

we derive by Lemma A.4

$$p\zeta \in H_0^{1,q}(G)$$

and altogether

$$w \in H_0^{1,q}(G) \cap C_0^0(\mathbb{R}^n)$$

(b) We have

$$\partial_i w \;=\; \sum_{j=1}^{n} \underbrace{\partial_i u_j}_{H^{1,q}} \underbrace{\partial_j \zeta}_{H^{1,q}} + \sum_{j=1}^{n} \underbrace{u_j}_{H^{1,q}} \underbrace{\partial_i \partial_j \zeta}_{H^{1,q}} - \frac{1}{2} \underbrace{\partial_i p}_{C^\infty(G)} \underbrace{\zeta}_{C^\infty} - \frac{1}{2} \underbrace{p}_{H^{1,q}} \underbrace{\partial_i \zeta}_{H^{1,q}}$$

Therefore by Lemma A.1 and by Lemma 13.1

$$\begin{aligned} \Delta w \;&=\; \underbrace{\Delta \underline{u}}_{=\nabla p} \cdot \nabla\zeta + 2\nabla\underline{u} \cdot \nabla\nabla\zeta + \underline{u} \cdot \nabla\Delta\zeta - \frac{1}{2}\zeta \underbrace{\Delta p}_{=0} - \nabla p \nabla\zeta - \frac{1}{2} p \Delta\zeta \\ &=\; 2\nabla\underline{u} \cdot \nabla\nabla\zeta + \underline{u} \cdot \nabla\Delta\zeta - \frac{1}{2} p \Delta\zeta \end{aligned}$$

By Poincare's Lemma and Theorem 2.9 we derive

$$\begin{aligned} \|\Delta w\|_{q;G} \;&\leq\; C_1(\zeta)\|\nabla\underline{u}\|_{q;G} + C_2(\zeta)\|\underline{u}\|_{q;G} + C_3(\zeta)\|p\|_{q;G} \\ &\leq\; C_4(\zeta, n, q, G)\|p\|_{q;G} \end{aligned}$$

(c) Again by Poincare's Lemma and Theorem 2.9 holds

$$
\begin{aligned}
\|w\|_{q;G} &\leq C_5(\zeta)\|\underline{u}\|_{q;G} + C_6(\zeta)\|p\|_{q;G} \\
&\leq C_7(\zeta, q, G)\|p\|_{q;G}
\end{aligned}
$$

(d) We have $w \in H_0^{1,q}(G)$ and therefore by Theorem 2.8 and Poincare's Lemma

$$
\begin{aligned}
\|\nabla w\|_{q;G} &\leq C_q \sup_{0 \neq \phi \in C_0^\infty(G)} \frac{\langle \nabla w, \nabla \phi \rangle}{\|\nabla \phi\|_{q'}} \\
&= C_q \sup_{0 \neq \phi \in C_0^\infty(G)} \frac{\langle \Delta w, \phi \rangle}{\|\nabla \phi\|_{q'}} \\
&\leq C_8(q, G)\, \|\Delta w\|_{q;G}
\end{aligned}
$$

(e) By Theorem 7.6 we finally derive $w \in H^{2,q}(G)$ and

$$
\begin{aligned}
\|w\|_{2,q;G} &\leq C_9(G, q, n)\, \left(\|\Delta w\|_{q;G} + \|w\|_{1,q;G}\right) \\
&\underset{(b),(c),(d)}{\leq} C_{10}(G, q, n, \zeta)\, \|p\|_{q;G}
\end{aligned}
$$

□

Lemma 13.3. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{k+2}$. Suppose

$$p \in B^q(G) \cap H^{k,q}(G)$$

Then

$$Z_q(p) - \frac{1}{2}\, p \in B^q(G) \cap H^{k,q}(G)$$

and there is a constant $C = C(n, q, G) > 0$ such that

$$\|Z_q(p) - \frac{1}{2}\, p\|_{1,q;G} \leq C\, \|p\|_{q;G}$$

Proof. (a) With $\underline{u} := \underline{T}_q(p)$ holds $Z_q(p) - \frac{1}{2}\, p = \operatorname{div} \underline{u} - \frac{1}{2}\, p$. By Lemma 13.1 and Theorem 11.2 we get

$$Z_q(p) - \frac{1}{2}\, p \in B^q(G) \cap H^{k,q}(G)$$

and by Theorem 2.9 holds

$$\|Z_q(p) - \frac{1}{2}\, p\|_{q;G} \leq C_1(n, q)\, \|p\|_{q;G}$$

(b) Let now by Theorem 6.1

$$\zeta \in C_0^3(\mathbb{R}^n), \quad \zeta\Big|_{\partial G} = 0, \quad \nabla\zeta\Big|_{\partial G} = N$$

and define

$$w := \underline{u}\,\nabla\zeta \, - \, \frac{1}{2}\,p\,\zeta$$

Then by Lemma 13.2

$$w \in H_0^{1,q}(G) \cap H^{2,q}(G) \cap C_0^0(\mathbb{R}^n)$$

By Lemma A.1 therefore

$$\nabla w \nabla\zeta \in H^{1,q}(G)$$

and

$$\partial_i w \;=\; \sum_{j=1}^n (\partial_i u_j)(\partial_j\zeta) + \sum_{j=1}^n u_j(\partial_i\partial_j\zeta) - \frac{1}{2}(\partial_i p)\zeta - \frac{1}{2}p(\partial_i\zeta)$$

$$\begin{aligned}
\nabla w \nabla\zeta \;&=\; \sum_{i,j=1}^n (\partial_i u_j)(\partial_j\zeta)(\partial_i\zeta) + \sum_{i,j=1}^n u_j(\partial_i\partial_j\zeta)(\partial_i\zeta) \\
&\quad - \sum_{i=1}^n \frac{1}{2}(\partial_i p)\,\zeta\,(\partial_i\zeta) - \sum_{i=1}^n \frac{1}{2}p(\partial_i\zeta)(\partial_i\zeta)
\end{aligned}$$

Therefore

$$\begin{aligned}
\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p) \;&=\; \left[\sum_{i,j=1}^n (\partial_i u_j)(\partial_j\zeta)(\partial_i\zeta) - \operatorname{div}\underline{u}\right] \\
&\quad + \sum_{i,j=1}^n u_j(\partial_i\partial_j\zeta)(\partial_i\zeta) \\
&\quad - \sum_{i=1}^n \frac{1}{2}(\partial_i p)\,\zeta\,(\partial_i\zeta) + \frac{1}{2}p\left[1 - \sum_{i=1}^n (\partial_i\zeta)(\partial_i\zeta)\right] \\
&=: \; f_1 + f_2 + f_3 + f_4
\end{aligned}$$

(c) Because by Lemma 13.1 $\underline{u} \in H_0^{1,q}(G)^n \cap \overline{C}^1(G)^n$ we derive by Theorem 5.8

$$\underline{u}\Big|_{\partial G} = 0$$

and by Theorem 5.6 for $x \in \partial G$

$$(\nabla u_i)(x) = \lambda_i(x)N(x)$$

For $x \in \partial G$ therefore holds

$$\begin{aligned}
\sum_{i,j=1}^n (\partial_i u_j)(x)\,(\partial_j\zeta)(x)\,(\partial_i\zeta)(x) \;&=\; \sum_{i,j=1}^n \lambda_j(x)N_i(x)N_j(x)N_i(x) \\
&=\; |N(x)|^2 \sum_{j=1}^n \lambda_j(x)N_j(x) \;=\; \sum_{j=1}^n (\partial_j u_j)(x) \\
&=\; \operatorname{div}\underline{u}(x)
\end{aligned}$$

Therefore by Theorem 5.2

$$f_1 \in C^0(\mathbb{R}^n) \cap H^{1,q}(G), \qquad f_1\Big|_{\partial G} = 0$$

and by Theorem 5.5

$$f_1 \in H_0^{1,q}(G)$$

(d) We have $u_j \in H_0^{1,q}(G)$ and $(\partial_i\partial_j\zeta)(\partial_i\zeta) \in C_0^1(\mathbb{R}^n)$. Then by Lemma A.3

$$f_2 = \sum_{i,j=1}^{n} u_j(\partial_i\partial_j\zeta)(\partial_i\zeta) \in H_0^{1,q}(G)$$

(e) By Theorem 5.5 holds $\zeta \in H_0^{1,q}(G)$ and $\partial_i p \in H^{1,q}(G)$, $\partial_i\zeta \in H^{1,q}(G)$. By Lemma A.4 we derive

$$f_3 \in H_0^{1,q}(G)$$

(f) It holds

$$1 - \sum_{i=1}^{n}(\partial_i\zeta)(\partial_i\zeta)\Big|_{\partial G} = 1 - |N|^2\Big|_{\partial G} = 0$$

Therefore

$$f_4 \in H^{1,q}(G) \cap C^0(\overline{G}), \qquad f_4\Big|_{\partial G} = 0$$

and by Theorem 5.5

$$f_4 \in H_0^{1,q}(G)$$

(g) By (c)-(f) holds

$$\nabla w \nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p) \in H_0^{1,q}(G)$$

and therefore by Theorem 2.8

$$\|\nabla\Big[\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\Big]\|_{q;G} \;\leq$$

$$\leq\; C_q \sup_{0\neq\phi\in\widehat{H}_\bullet^{1,q'}(G)} \frac{\Big\langle \nabla\Big[\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\Big], \nabla\phi\Big\rangle}{\|\nabla\phi\|_{q'}} \;=$$

$$\underset{\text{Lemma 4.3}}{=}\; C_q \sup_{0\neq\phi\in\widehat{H}_\bullet^{1,q'}(G)} \frac{\Big\langle \nabla\Big[\nabla w\nabla\zeta\Big], \nabla\phi\Big\rangle}{\|\nabla\phi\|_{q'}} \;\leq\; C_q\|\nabla\Big[\nabla w\nabla\zeta\Big]\|_{q;G}$$

Finally by Lemma 13.2

$$\begin{aligned}
\|\nabla(\operatorname{div}\underline{u} - \frac{1}{2}\,p)\|_{q;G} \;&\leq\; \|\nabla\Big[\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\Big]\|_{q;G} + \|\nabla\Big[\nabla w\nabla\zeta\Big]\|_{q;G} \\
&\leq\; (1 + C_q)\,\|\nabla\Big[\nabla w\nabla\zeta\Big]\|_{q;G} \;\leq\; C_2(q,\zeta)\,\|w\|_{2,q;G} \\
&\leq\; C_3(n,q,G,\zeta)\|p\|_{q;G}
\end{aligned}$$

(h) By (a) and (g) the assertion follows. □

Theorem 13.4. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{k+2}$. Suppose

$$p \in B^q(G)$$

Then

$$Z_q(p) - \frac{1}{2}\, p \in B^q(G) \cap H^{1,q}(G)$$

and there is a constant $C = C(n, q, G) > 0$ such that

$$\| Z_q(p) - \frac{1}{2}\, p \|_{1,q;G} \;\leq\; C\, \| p \|_{q;G}$$

Proof. (a) By Theorem 9.1 there is a sequence $(p_m) \subset B^q(G) \cap H^{k,q}(G)$ such that

$$\| p_m - p \|_{q;G} \to 0$$

By Lemma 13.3 holds

$$\| \big(Z_q(p_m) - \frac{1}{2}\, p_m \big) - \big(Z_q(p_{m'}) - \frac{1}{2}\, p_{m'} \big) \|_{1,q;G} \;\leq\; C\, \| p_{m'} - p_m \|_{q;G} \;\to 0$$

By the completeness of $H^{1,q}(G)$ there exists a $u \in H^{1,q}(G)$ with

$$\| (Z_q(p_m) - \frac{1}{2}\, p_m) - u \|_{1,q;G} \;\to 0$$

Because

$$\| Z_q(p_m) - \frac{1}{2}\, p_m \|_{1,q;G} \;\leq\; C\, \| p_m \|_{q;G}$$

we get for $(m \to \infty)$

$$\| u \|_{1,q;G} \;\leq\; C\, \| p \|_{q;G}$$

(b) By Theorem 11.2 we derive

$$Z_q(p) - \frac{1}{2}\, p \in B^q(G)$$

and by Theorem 2.9 and the definition of Z_q

$$\| Z_q(\pi) - \frac{1}{2}\, \pi \|_{q;G} \;\leq\; C_1(n, q)\, \| \pi \|_{q;G} \qquad \forall\, \pi \in B^q(G)$$

Therefore

$$\| \big(Z_q(p_m) - \frac{1}{2}\, p_m \big) - \big(Z_q(p) - \frac{1}{2}\, p \big) \|_{q;G} \;\leq\; C_1\, \| p - p_m \|_{q;G} \;\to 0$$

We get

$$u = Z_q(p) - \frac{1}{2}\, p$$

and the assertion follows by (a) $\square$

Theorem 13.5. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{k+2}$. Then

$$Z_q - \frac{1}{2} I : B^q(G) \to B^q(G)$$

is a compact operator.

Proof. By Theorem 13.4 for all $p \in B^q(G)$ holds

$$\|Z_q(p) - \frac{1}{2}\, p\|_{1,q;G} \;\leq\; C\, \|p\|_{q;G}$$

Let $(p_m) \subset B^q(G)$ with $\|p_m\|_{q;G} \leq \tilde{C} < \infty$ for each $m \in \mathbb{N}$. Then

$$\|Z_q(p_m) - \frac{1}{2}\, p_m\|_{1,q;G} \;\leq\; C\, \tilde{C} < \infty$$

Because the imbedding $H^{1,q}(G) \to L^q(G)$ is compact (see e.g. [Alt, Satz 8.9, p.314]) there is a subsequence $(p_{m_l}) \subset (p_m)$ such that $(Z_q(p_{m_l}) - \frac{1}{2}\, p_{m_l})$ is Cauchy in $B^q(G)$.
$\square$

14 The Cosserat spectrum

Theorem 14.1. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^{k+2}$.

1. For $s \in \widehat{H}^{2,q}_\bullet(G)$ and $\underline{u}_0 := \nabla s$ holds

$$\left\langle \nabla \underline{u}_0, \nabla \underline{\phi} \right\rangle_G = \left\langle \operatorname{div} \underline{u}_0, \operatorname{div} \underline{\phi} \right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}^{1,q'}_\bullet(G)$$

2. If $\lambda \in \mathbb{R}$ and $\underline{u} \in \widehat{H}^{1,q}_\bullet(G)$ with

$$\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G = \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}^{1,q'}_\bullet(G)$$

   then $\lambda = 1$ or $\operatorname{div} \underline{u} \in B^q(G)$

3. The set

$$W := \Big\{ \lambda \in \mathbb{R} \ : \ \text{there is } 0 \neq \underline{u} \in \widehat{H}^{1,q}_\bullet(G) \text{ such that for all } \underline{\phi} \in \widehat{H}^{1,q'}_\bullet(G) \text{ holds } \left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G = \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \Big\}$$

   is finite or countably infinite.

4. For $\lambda \in \mathbb{R} \backslash \{1, 2\}$ the vector space

$$V_\lambda := \left\{ \underline{u} \in \widehat{H}^{1,q}_\bullet(G) : \left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G = \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \text{ for all } \underline{\phi} \in \widehat{H}^{1,q'}_\bullet(G) \right\}$$

   is finite-dimensional.

5. For every sequence $(\lambda_m) \subset W$ with $\lambda_m \neq \lambda_l$ for $m \neq l$ holds

$$\lambda_m \to 2 \qquad (m \to \infty)$$

Proof. (a) Let $s \in \widehat{H}^{2,q}_\bullet(G)$. By Theorem 2.5 - 2.7 we derive for $\underline{\phi} \in \widehat{H}^{1,q'}_\bullet(G)$

$$\left\langle \nabla \nabla s, \nabla \underline{\phi} \right\rangle_G = \left\langle \Delta s, \operatorname{div} \underline{\phi} \right\rangle_G = \left\langle \operatorname{div} \nabla s, \operatorname{div} \underline{\phi} \right\rangle_G$$

(b) Let $\lambda \in \mathbb{R}$ and $\underline{u} \in \widehat{H}^{1,q}_\bullet(G)$ with

$$\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G = \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}^{1,q'}_\bullet(G)$$

By Theorem 11.3 with $p := \operatorname{div} \underline{u}$ holds

$$\lambda Z_q(p) = p$$

If $\lambda = 0$, then $\underline{u} = 0$ and $\operatorname{div} \underline{u} \in B^q(G)$. If $\lambda \neq 0$, we derive by Theorem 11.4

$$\lambda = 1 \quad \text{or} \quad \operatorname{div} \underline{u} \in B^q(G)$$

(c) By Theorem 11.3 and (b) holds: $\lambda \in W \backslash \{0, 1\}$ if and only if there is $p \in B^q(G)$ with $Z_q(p) = \frac{1}{\lambda} p$

By Theorem 12.5, Theorem 13.5 and Theorem B.9 $\sigma_p^{(\mathbb{R})}(Z_q)$ is finite or countably infinite. Therefore

$$W \backslash \{0,1\} \;=\; \left\{ \mu \in \mathbb{R} \;:\; \frac{1}{\mu} \in \sigma_p^{(\mathbb{R})}(Z_q) \right\}$$

is finite or countably infinite too.

(d) Let $\lambda \in W\backslash\{0,1,2\}$. For $\underline{u} \in V_\lambda$ by Theorem 11.3 and (b) holds

$$\operatorname{div} \underline{u} \;\in\; N\left(\frac{1}{\lambda} I - Z_q\right)$$

Let $m \in \mathbb{N}$ and let $\underline{u}^{(1)}, \ldots, \underline{u}^{(m)} \in V_\lambda$ be linearly independent. Let $a_1, \ldots, a_m \in \mathbb{R}$ be given with

$$\sum_{i=1}^{m} a_i \operatorname{div} \underline{u}^{(i)} \;=\; 0$$

Then for $\underline{\phi} \in \widehat{H}_\bullet^{1,q'}(G)$

$$\sum_{i=1}^{m} a_i \left\langle \nabla \underline{u}^{(i)}, \nabla \underline{\phi} \right\rangle_G \;=\; \sum_{i=1}^{m} a_i \, \lambda \left\langle \operatorname{div} \underline{u}^{(i)}, \operatorname{div} \underline{\phi} \right\rangle_G \;=\; 0$$

By Theorem 2.8 follows

$$\sum_{i=1}^{m} a_i \, \underline{u}^{(i)} \;=\; 0$$

and by assumption

$$a_1 = \ldots = a_m = 0$$

So we derive

$$\dim V_\lambda \;\leq\; \dim N\left(\frac{1}{\lambda} I - Z_q\right)$$

But by Theorem 12.5, Theorem 13.5 and Theorem B.9 holds

$$\dim N\left(\frac{1}{\lambda} I - Z_q\right) \;<\; \infty$$

(e) Let $(\lambda_m) \subset W$ with $\lambda_m \neq \lambda_l$ for $m \neq l$. Without loss of generality we can assume $\lambda_m \notin \{0,1,2\}$. Then

$$\left(\frac{1}{\lambda_m}\right) \subset \sigma_p^{(\mathbb{R})}(Z_q) \Big\backslash \left\{\frac{1}{2}\right\}$$

is a sequence of distinct eigenvalues of Z_q. By Theorem 12.5, Theorem 13.5 and Theorem B.9 holds

$$\frac{1}{\lambda_m} \longrightarrow \frac{1}{2} \qquad (m \to \infty)$$

That is

$$\lambda_m \to 2 \qquad (m \to \infty)$$

□

15 Regularity of the solutions

Theorem 15.1. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^{k+2}$. Assume that $p \in B^q(G)$, $\lambda \in \mathbb{R} \backslash \{\frac{1}{2}\}$ such that

$$Z_q(p) \; = \; \lambda \, p$$

Then for all $1 < \tilde{q} < \infty$ holds

$$p \in H^{1,\tilde{q}}(G) \cap C^0(\overline{G})$$

and

$$Z_q(p) \; = \; Z_{\tilde{q}}(p)$$

Proof. (a) By Theorem 12.4 respectively 13.4 with $\mu := \lambda - \frac{1}{2} \neq 0$ holds

$$p \; = \; \frac{1}{\mu} \left(Z_q(p) \; - \; \frac{1}{2} \, p \right) \; \in H^{1,q}(G)$$

(b) If $q = n$, by Hölder's inequality and Theorem 8.7 follows

$$p \in H^{1,\hat{q}}(G)$$

for a $1 < \hat{q} < n$.

(c) If $1 < q < n$, with

$$q^* \; = \; \frac{nq}{n-q} \; > \; q$$

we derive by Sobolev's imbedding theorems (see e.g. [Alt, Satz 8.9, p.314])

$$p \in L^{q^*}(G \cap B_r) \qquad \forall \, r > 0$$

By Theorem 8.7 also holds

$$p \in L^{q^*}(G)$$

Therefore by Theorem 7.9 respectively 7.10

$$\underline{T}_q(p) \in \widehat{H}_{\bullet}^{1,q^*}(G)^n$$

and

$$\left\langle \nabla \underline{T}_q(p), \nabla \underline{\phi} \right\rangle_G \; = \; \left\langle p, \operatorname{div} \underline{\phi} \right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}_{\bullet}^{1,(q^*)'}(G)^n$$

Because of the uniqueness in Theorem 2.9 we derive

$$\underline{T}_q(p) \; = \; \underline{T}_{q^*}(p)$$

and therefore

$$Z_q(p) \; = \; Z_{q^*}(p)$$

Then by Theorem 12.4 respectively 13.4

$$p \; = \; \frac{1}{\mu} \left(Z_q(p) \; - \; \frac{1}{2} \, p \right) \; = \; \frac{1}{\mu} \left(Z_{q^*}(p) \; - \; \frac{1}{2} \, p \right) \; \in H^{1,q^*}(G)$$

One can easily show

$$(q^*)^* \;=\; \frac{nq}{n-2q}, \qquad\qquad q^{\overbrace{*\cdots*}^{m-\text{times}}} \;=\; \frac{nq}{n-mq}$$

By induction there is a $n < s < \infty$ with

$$p \in H^{1,s}(G)$$

(d) Assume that for a $n < s < \infty$ holds: $p \in H^{1,s}(G)$. By Sobolev's imbedding theorems (see e.g. [Alt, Satz 8.13, p.319]) holds

$$p \in C^0(\overline{G \cap B_r}) \qquad \forall\, r > 0$$

Therefore

$$p \in C^0(\overline{G})$$

By Hölder's inequality we derive

$$p \in L^{\tilde q}(G \cap B_r) \qquad \forall\, r > 0 \quad \forall\, 1 < \tilde q < \infty$$

and by Theorem 8.12

$$p \in L^{\tilde q}(G) \qquad \forall\, 1 < \tilde q < \infty$$

As in (c) one can prove

$$Z_q(p) \;=\; Z_{\tilde q}(p) \qquad \forall\, 1 < \tilde q < \infty$$

and by Theorem 12.4 respectively 13.4 follows

$$p \;=\; \frac{1}{\mu}\left(Z_q(p) \;-\; \frac{1}{2}\,p\right) \;=\; \frac{1}{\mu}\left(Z_{\tilde q}(p) \;-\; \frac{1}{2}\,p\right) \in H^{1,\tilde q}(G) \qquad \forall\, 1 < \tilde q < \infty$$

□

Lemma 15.2. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^{k+4}$. Assume that

$$p \in B^q(G) \cap H^{k,q}(G) \qquad \underline{u} := \underline{T}_q(p)$$

Further let

$$\zeta \in C_0^{k+3}(\mathbb{R}^n), \quad \zeta\Big|_{\partial G} = 0, \quad \nabla\zeta\Big|_{\partial G} = N$$

Define

$$w := \;\underline{u}\,\nabla\zeta \;-\; \frac{1}{2}\,p\,\zeta$$

Then

$$w \in H_0^{1,q}(G) \cap H^{2+k,q}(G) \cap C_0^0(\mathbb{R}^n)$$

and there is a constant $C_k = C_k(k,n,q,G,\zeta) > 0$ such that

$$\|w\|_{2+k,q;G} \;\leq\; C_k\,\|p\|_{k,q;G}$$

Proof. All assumptions of Lemma 12.2 respectively 13.2 are satisfied. Therefore we can use all equalities and inequalities which appear in these proofs. By part (b) of the proof of Lemma 12.2 respectively 13.2 holds

$$\Delta w \;=\; 2\nabla \underline{u} \cdot \nabla\nabla\zeta + \underline{u} \cdot \underbrace{\nabla\Delta\zeta}_{\in H^{k,q}(G)} - \frac{1}{2} p\Delta\zeta$$

By Lemma 12.1 respectively 13.1 und Lemma A.15 respectively Poincare's Lemma follows

$$\begin{aligned} \|\Delta w\|_{k,q;G} &\leq C_1(\zeta)\|\nabla \underline{u}\|_{k,q;G} + C_2(\zeta)\|\underline{u}\|_{k,q;G\cap B_R} + C_3(\zeta)\|p\|_{k,q;G} \\ &\leq C_4(\zeta, n, q, G, R)\|p\|_{k,q;G} \end{aligned}$$

So by Theorem 7.6

$$w \in H^{2+k,q}(G)$$

and

$$\begin{aligned} \|w\|_{2+k,q;G} &\leq C_5(G, q, n, k)\, (\|\Delta w\|_{k,q;G} + \|w\|_{1,q;G}) \\ &\leq C_6(G, q, n, \zeta, R, k)\, \|p\|_{k,q;G} \end{aligned}$$

by part (c) and (d) of the proof of Lemma 12.2 respectively 13.2. □

Lemma 15.3. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k > 1 + \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^{k+3}$. Assume that

$$p \in B^q(G) \cap H^{k,q}(G)$$

Then

$$Z_q(p) - \frac{1}{2}\, p \in B^q(G) \cap H^{k,q}(G)$$

and there is a constant $C_k = C(k, n, q, G) > 0$ such that

$$\|Z_q(p) - \frac{1}{2}\, p\|_{k,q;G} \;\leq\; C_k\, \|p\|_{k-1,q;G}$$

Proof. All assumptions of Lemma 12.3 respectively 13.3 are satisfied. Therefore we can use all equalities and inequalities which appear in these proofs. Further we can choose there because of Theorem 6.1

$$\zeta \in C_0^{k+2}(\mathbb{R}^n)$$

With the notations of the proof of Lemma 12.3 respectively 13.3 by part (g) holds

$$\nabla w \nabla\zeta - \underbrace{(\operatorname{div} \underline{u} - \frac{1}{2}\, p)}_{\in B^q(G)} \in \widehat{H}^{1,q}_{\bullet}(G)$$

Furthermore

$$\Delta\big[\nabla w \nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\big] \;=\; \Delta(\nabla w \nabla\zeta)$$

By Lemma 15.2 applied to $\tilde{k} = k-1$ and $p \in B^q(G) \cap H^{\tilde{k},q}(G)$ holds

$$\Delta(\nabla w \nabla\zeta) \in H^{k-2,q}(G)$$

and

$$\|\Delta(\nabla w \nabla\zeta)\|_{k-2,q;G} \;\le\; C_1(\zeta)\,\|w\|_{1+k,q;G} \;\le\; C_2(\zeta,n,k,q,G)\,\|p\|_{k-1,q;G}$$

By Theorem 7.6 follows

$$\|\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\|_{k,q;G} \;\le\;$$

$$\le\; C_2(\zeta,n,k,q,G)\,\Big[\|\Delta(\nabla w \nabla\zeta)\|_{k-2,q;G} + \|\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\|_{1,q;G}\Big]$$

By part (a) and (g) of the proof of Lemma 12.3 respectively 13.3 we can see

$$\|\operatorname{div}\underline{u} - \frac{1}{2}\,p\|_{q;G} \;\le\; C_4(n,q)\,\|p\|_{q;G}$$

and

$$\|\nabla\Big[\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\Big]\|_{q;G} \;\le\; C_5(n,q,G,\zeta)\,\|p\|_{q;G}$$

By Lemma 12.2 respectively 13.2 further holds

$$\|\nabla w\nabla\zeta\|_{1,q;G} \;\le\; C_6(\zeta)\,\|w\|_{2,q;G} \;\le\; C_7(\zeta,n,q,G)\,\|p\|_{q;G}$$

So we derive

$$\|\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\|_{k,q;G} \;\le\; C_8(\zeta,n,q,G,k)\,\|p\|_{k-1,q;G}$$

As one can see above, also holds

$$\|\nabla w\nabla\zeta\|_{k,q;G} \;\le\; C_9(\zeta)\,\|w\|_{k+1,q;G} \;\le\; C_{10}(\zeta,n,q,G,k)\,\|p\|_{k-1,q;G}$$

Altogether we get

$$\begin{aligned}\|\operatorname{div}\underline{u} - \frac{1}{2}\,p\|_{k,q;G} \;&\le\; \|\nabla w\nabla\zeta\|_{k,q;G} + \|\nabla w\nabla\zeta - (\operatorname{div}\underline{u} - \frac{1}{2}\,p)\|_{k,q;G}\\ &\le\; C_{11}(\zeta,n,q,G,k)\,\|p\|_{k-1,q;G}\end{aligned}$$

□

Theorem 15.4. Let $n \ge 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \ge 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^{k+3}$. Assume that $p \in B^q(G)$, $\lambda \in \mathbb{R}\backslash\{\frac{1}{2}\}$ such that

$$Z_q(p) \;=\; \lambda\,p$$

Then for all $1 < \tilde{q} < \infty$ holds

$$p \in H^{k,\tilde{q}}(G)$$

Proof. By Theorem 15.1 for all $1 < \tilde{q} < \infty$ holds

$$p \in H^{1,\tilde{q}}(G) \cap C^0(\overline{G})$$

and

$$Z_q(p) \;=\; Z_{\tilde{q}}(p)$$

Choose an arbitrary $1 < s < \infty$ with $s > n$. By Theorem 9.1 respectively 9.2 there is a sequence $(p_m) \subset H^{k,s}(G) \cap B^s(G)$ with

$$\|p_m - p\|_{s;G} \;\to\; 0 \qquad (m \to \infty)$$

By Lemma 15.3 (observe $k \geq 2 > 1 + \frac{n}{s}$) holds

$$(Z_s - \frac{1}{2}I)^l p_m \in H^{k,s}(G) \cap B^s(G) \qquad \forall\, l \in \mathbb{N}$$

and

$$\begin{aligned}
\|(Z_s - \frac{1}{2}I)^k (p_m - p_{m'})\|_{k,s;G} \;&\leq\; C_k \,\|(Z_s - \frac{1}{2}I)^{k-1} (p_m - p_{m'})\|_{k-1,s;G} \\
&\leq\; \ldots \\
&\leq\; C_k\, C_{k-1} \ldots C_2 \,\|(Z_s - \frac{1}{2}I)(p_m - p_{m'})\|_{1,s;G} \\
&\underset{13.4}{\overset{12.4}{\leq}}\; C_k\, C_{k-1} \ldots C_2\, C \,\|p_m - p_{m'}\|_{s;G} \;\to\; 0
\end{aligned}$$

Because $H^{k,s}(G)$ is complete, there exists $g \in H^{k,s}(G)$ such that

$$\|g - (Z_s - \frac{1}{2}I)^k p_m\|_{k,s;G} \;\to\; 0 \qquad (m \to \infty)$$

Because $(Z_s - \frac{1}{2}I)^k$ is a bounded operator, also holds

$$\|(Z_s - \frac{1}{2}I)^k p - (Z_s - \frac{1}{2}I)^k p_m\|_{s;G} \;\to\; 0 \qquad (m \to \infty)$$

Therefore

$$(Z_s - \frac{1}{2}I)^k p \;=\; g \in H^{k,s}(G)$$

With $\mu := \lambda - \frac{1}{2} \neq 0$ holds

$$Z_s(p) \;-\; \frac{1}{2}\,p \;=\; \mu\, p$$

so we derive

$$(Z_s - \frac{1}{2}I)^k p \;=\; \mu^k\, p$$

and also

$$p \;=\; \frac{1}{\mu^k} \left(Z_s - \frac{1}{2}I\right)^k p \,\in H^{k,s}(G)$$

for all $1 < s < \infty$ with $s > n$.

By Hölder's inequality holds

$$p \in H^{k,\tilde{q}}(G \cap B_r) \qquad \forall\, r > 0 \quad \forall\, 1 < \tilde{q} < \infty$$

We already know that

$$p \in H^{1,\tilde{q}}(G) \qquad \forall\, 1 < \tilde{q} < \infty$$

By Lemma 8.10 and Theorem 8.7 finally follows

$$p \in H^{k,\tilde{q}}(G) \qquad \forall\, 1 < \tilde{q} < \infty$$

□

Theorem 15.5. Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k \geq 2$, $k > \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^{k+3}$. Assume that $\underline{u} \in \widehat{H}_{\bullet}^{1,q}(G)$, $\lambda \in \mathbb{R} \backslash \{1, 2\}$ and

$$\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G \;=\; \lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \qquad \text{for all } \underline{\phi} \in \widehat{H}_{\bullet}^{1,q'}(G)^n$$

Then

1. $\underline{u} \in \widehat{H}_{\bullet}^{1,\tilde{q}}(G)^n$ and $\nabla \underline{u} \in H^{k,\tilde{q}}(G)^{n^2}$ for all $1 < \tilde{q} < \infty$,
2. $\underline{u} \in \overline{C}^k(G)$,
3. $\Delta \underline{u} \;=\; \lambda \nabla \operatorname{div} \underline{u}$

Proof. By Theorem 11.3 and Theorem 11.4 with $p := \operatorname{div} \underline{u}$ holds

$$p \in B^q(G)$$

and

$$\lambda Z_q(p) \;=\; p$$

If $\lambda = 0$ holds $\underline{u} = 0$. If $\lambda \neq 0$ we derive by Theorem 15.1 and 15.4 for all $1 < \tilde{q} < \infty$

$$p \in H^{k,\tilde{q}}(G), \qquad Z_q(p) \;=\; Z_{\tilde{q}}(p)$$

By Lemma 12.1 respectively 13.1 then

$$\nabla \underline{u} \in H^{k,\tilde{q}}(G)^{n^2} \qquad \forall\, 1 < \tilde{q} < \infty$$

By Theorem 7.9 respectively 7.10 also holds

$$\underline{u} \in \widehat{H}_{\bullet}^{1,\tilde{q}}(G)^n \qquad \forall\, 1 < \tilde{q} < \infty$$

By Sobolev's imbedding theorems (see e.g. [Alt, Satz 8.13, p.319]) follows

$$\underline{u} \in \overline{C}^k(G)$$

Because of $k \geq 1$ for $\underline{\phi} \in C_0^\infty(G)^n$ holds

$$\left\langle \Delta \underline{u}, \underline{\phi} \right\rangle_G \;=\; -\left\langle \nabla \underline{u}, \nabla \underline{\phi} \right\rangle_G \;=\; -\lambda \left\langle \operatorname{div} \underline{u}, \operatorname{div} \underline{\phi} \right\rangle_G \;=\; \lambda \left\langle \nabla \operatorname{div} \underline{u}, \underline{\phi} \right\rangle_G$$

Therefore

$$\Delta \underline{u} \;=\; \lambda \nabla \operatorname{div} \underline{u}$$

□

16 Explicit solutions for B_1 and $\mathbb{R}^n \backslash \overline{B_1}$

Lemma 16.1. Let $n \geq 2$, $k \in \mathbb{Z}$, $k \neq -\frac{n}{2}$ and let $\Omega = B_1$ or $\Omega = \mathbb{R}^n \backslash \overline{B_1}$. Assume that $f \in C^0(\overline{\Omega})$ is harmonic in Ω and that for all $\lambda > 0$, $x \in \Omega$ with $\lambda x \in \Omega$ holds

$$f(\lambda x) = \lambda^k f(x)$$

Define

$$v(x) := \frac{1}{2n + 4k} \left(|x|^2 - 1 \right) f(x)$$

Then

1. $v \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$,
2. $\Delta v(x) = f(x)$ for all $x \in \Omega$,
3. $v\Big|_{\partial\Omega} = 0$.

Proof. (a) Obviously

$$v \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$$

and

$$v\Big|_{\partial\Omega} = 0$$

(b) So we derive for $x \in \Omega$ by Euler's identity

$$\begin{aligned}
\Delta v(x) &= \frac{1}{2n + 4k} \Big[f(x) \, \Delta \left(|x|^2 - 1 \right) + 2 \, (\nabla f)(x) \, \nabla \left(|x|^2 - 1 \right) \\
&\qquad + \left(|x|^2 - 1 \right) \underbrace{\Delta f(x)}_{=0} \Big] \\
&= \frac{1}{2n + 4k} \Big[f(x) \, 2n + 2 \sum_{i=1}^{n} 2 \, (\partial_i f)(x) \, x_i \Big] \\
&= \frac{1}{2n + 4k} \Big[2n \, f(x) + 4k \, f(x) \Big] = f(x)
\end{aligned}$$

□

Theorem 16.2. Let $n \geq 2$, $k \in \mathbb{Z}$, $k \neq -\frac{n}{2} + 1$ and let $\Omega = B_1$ or $\Omega = \mathbb{R}^n \backslash \overline{B_1}$. Assume that $p \in \overline{C}^1(\Omega)$ is harmonic in Ω and that for all $\lambda > 0$, $x \in \Omega$ with $\lambda x \in \Omega$ holds

$$p(\lambda x) \;=\; \lambda^k \, p(x)$$

Define

$$\underline{u}(x) \;:=\; \frac{1}{2n + 4(k-1)} \, \big(|x|^2 - 1\big) \; (\nabla p)(x)$$

Then

1. $\underline{u} \in C^0(\overline{\Omega})^n \cap C^\infty(\Omega)^n$,
2. $\Delta \underline{u} \;=\; \nabla p$,
3. $\underline{u}\Big|_{\partial\Omega} \;=\; 0$,
4. $\operatorname{div} \underline{u} = \begin{cases} \frac{1}{2}\, p & , \quad n = 2 \\[2ex] \frac{1}{2 + \frac{n-2}{k}}\, p & , \quad n \geq 3 \end{cases}$

Proof. (a) For $\lambda > 0$, $x \in \Omega$ with $\lambda x \in \Omega$ holds

$$p(\lambda x) \;=\; \lambda^k \, p(x)$$

Differentiating by x_i leads to

$$(\partial_i p)(\lambda x)\, \lambda \;=\; \lambda^k \, (\partial_i p)(x)$$

that is

$$(\partial_i p)(\lambda x) \;=\; \lambda^{k-1} \, (\partial_i p)(x)$$

$\partial_i p \in C^0(\overline{\Omega})$ is harmonic in Ω. Therefore by Lemma 16.1

$$\underline{u} \in C^0(\overline{\Omega})^n \cap C^\infty(\Omega)^n$$

and

$$\Delta \underline{u} \;=\; \nabla p, \qquad \underline{u}\Big|_{\partial\Omega} \;=\; 0$$

(b) Further for $x \in \Omega$

$$\begin{aligned} \operatorname{div} \underline{u}(x) \;&=\; \frac{1}{2n + 4(k-1)} \Big[\sum_{i=1}^n 2\, (\partial_i p)(x)\, x_i \;+\; \big(|x|^2 - 1\big) \underbrace{\sum_{i=1}^n (\partial_i \partial_i p)(x)}_{=0} \Big] \\ &=\; \frac{1}{n + 2(k-1)} \underbrace{\sum_{i=1}^n (\partial_i p)(x)\, x_i}_{=kp(x)} \;=\; \frac{k}{n + 2(k-1)}\, p(x) \end{aligned}$$

□

Remark 16.3. (a) In the case of the unit ball $B_1 \subset \mathbb{R}^n$ we consider harmonic homogeneous polynomials of degree $k \geq 1$. We denote a sequence of such polynomials by (p_k) (for example one can choose $p_k(x) = x_1^k$). Theorem 16.2 is applicable by p_k and we define

$$\underline{u}_k(x) \;:=\; \frac{1}{2n+4(k-1)} \,\big(|x|^2 - 1\big)\, (\nabla p_k)(x)$$

Then by Theorem 16.2

$$\underline{u}_k \in C^0(\overline{B_1})^n \cap C^\infty(B_1)^n, \qquad \Delta \underline{u}_k \;=\; \nabla p_k \quad \text{in } B_1$$

and

$$\underline{u}_k\Big|_{\partial B_1} \;=\; 0, \qquad \operatorname{div} \underline{u}_k = \begin{cases} \frac{1}{2}\, p_k & , \quad n = 2 \\ \frac{1}{2+\frac{n-2}{k}}\, p_k & , \quad n \geq 3 \end{cases} \qquad \text{in } B_1$$

Therefore for $k \in \mathbb{N}$ holds

$$\Delta \underline{u}_k \;=\; 2\, \nabla \operatorname{div} \underline{u}_k \qquad \text{if } n = 2$$

and

$$\Delta \underline{u}_k \;=\; \left(2 + \frac{n-2}{k}\right) \nabla \operatorname{div} \underline{u}_k \qquad \text{if } n \geq 3$$

If $n \geq 3$ therefore there is a sequence $(\underline{u}_k)$ of classical eigenfunctions for eigenvalues, which tends to 2.

(b) Because obviously

$$\underline{u}_k \in C^\infty(\mathbb{R}^n), \qquad \underline{u}_k\Big|_{\partial B_1} \;=\; 0$$

by Theorem 5.5 we derive for each $1 < q < \infty$

$$\underline{u}_k \in H_0^{1,q}(B_1)^n \;=\; \widehat{H}_\bullet^{1,q}(B_1)^n$$

and for $\underline{\phi} \in C_0^\infty(B_1)$

$$\begin{aligned} \big\langle \nabla \underline{u}_k, \nabla \underline{\phi} \big\rangle_{B_1} \;&=\; -\big\langle \Delta \underline{u}_k, \underline{\phi} \big\rangle_{B_1} \;=\; -\left(2 + \frac{n-2}{k}\right) \big\langle \nabla \operatorname{div} \underline{u}_k, \underline{\phi} \big\rangle_{B_1} \\ &=\; \left(2 + \frac{n-2}{k}\right) \big\langle \operatorname{div} \underline{u}_k, \operatorname{div} \underline{\phi} \big\rangle_{B_1} \end{aligned}$$

Because

$$H_0^{1,q'}(B_1)^n \;=\; \widehat{H}_\bullet^{1,q'}(B_1)^n$$

also holds

$$\big\langle \nabla \underline{u}_k, \nabla \underline{\phi} \big\rangle_{B_1} \;=\; \left(2 + \frac{n-2}{k}\right) \big\langle \operatorname{div} \underline{u}_k, \operatorname{div} \underline{\phi} \big\rangle_{B_1} \qquad \forall\, \underline{\phi} \in \widehat{H}_\bullet^{1,q'}(B_1)^n$$

and $(\underline{u}_k)$ is a sequence of weak eigenfunctions for the same eigenvalues.

Remark 16.4. (a) In the case of the exterior of the unit ball $\mathbb{R}^n\backslash\overline{B_1}$ we consider the fundamental solution

$$S(z) := \begin{cases} \frac{1}{(n-2)\omega_n}\,|z|^{2-n} & , \quad z \neq 0,\ n \geq 3 \\ -\frac{1}{2\pi}\,\ln|z| & , \quad z \neq 0,\ n = 2 \\ 0 & , \quad z = 0,\ n \geq 2 \end{cases}$$

and its derivatives. If $n \geq 3$ we define for $\alpha \in \mathbb{N}_0^n$

$$\underline{u}_\alpha(x) \ := \ \frac{1}{2n + 4(2 - n - |\alpha| - 1)}\,\big(|x|^2 - 1\big)\ (\nabla D^\alpha S)(x)$$

If $n = 2$, we consider only $|\alpha| \geq 1$. By Theorem 16.2 for the considered α holds

$$\underline{u}_\alpha \in C^0(\mathbb{R}^n\backslash B_1)^n \cap C^\infty(\mathbb{R}^n\backslash\overline{B_1})^n, \qquad \underline{u}_\alpha\Big|_{\partial B_1} \ = \ 0$$

and

$$\Delta\underline{u}_\alpha \ = \ \left(2 + \frac{n-2}{2 - n - |\alpha|}\right)\nabla\mathrm{div}\,\underline{u}_\alpha \qquad \text{in } \mathbb{R}^n\backslash\overline{B_1}$$

(b) As one can easily check, for the considered α holds

$$\underline{u}_\alpha \in L^q(B_r\backslash\overline{B_1})^n \qquad \forall\, 1 < q < \infty \quad \forall\, r > 0$$

and

$$\nabla\underline{u}_\alpha \in L^q(\mathbb{R}^n\backslash\overline{B_1})^{n^2} \quad \text{for} \begin{cases} 1 < q < \infty & , \quad \text{if } |\alpha| \geq 2,\ n \geq 2 \\ \frac{n}{n-1} < q < \infty & , \quad \text{if } |\alpha| = 1,\ n \geq 2 \\ \frac{n}{n-2} < q < \infty & , \quad \text{if } \alpha = 0,\ n \geq 3 \end{cases}$$

For $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(\eta) \subset B_R$ $(R > 1)$ for the above q holds

$$\eta\underline{u}_\alpha \in H^{1,q}(B_R\backslash\overline{B_1})^n \ \cap\ C^0(\overline{B_R}\backslash B_1)^n$$

and

$$\eta\underline{u}_\alpha\Big|_{\partial B_R \cup \partial B_1} \ = \ 0$$

Therefore by Theorem 5.5

$$\eta\underline{u}_\alpha \in H_0^{1,q}(B_R\backslash\overline{B_1})^n$$

and also

$$\eta\underline{u}_\alpha \in H_0^{1,q}(\mathbb{R}^n\backslash\overline{B_1})^n$$

That is

$$\underline{u}_\alpha \in \widehat{H}_\bullet^{1,q}(\mathbb{R}^n\backslash\overline{B_1})^n \quad \text{for} \begin{cases} 1 < q < \infty & , \quad \text{if } |\alpha| \geq 2,\ n \geq 2 \\ \frac{n}{n-1} < q < \infty & , \quad \text{if } |\alpha| = 1,\ n \geq 2 \\ \frac{n}{n-2} < q < \infty & , \quad \text{if } \alpha = 0,\ n \geq 3 \end{cases}$$

(c) If $\alpha \in \mathbb{N}_0^n$, $|\alpha| \geq 2$, for $\underline{\phi} \in C_0^\infty(\mathbb{R}^n \backslash \overline{B_1})$ holds

$$\begin{aligned}\left\langle \nabla \underline{u}_\alpha, \nabla \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}} &= -\left\langle \Delta \underline{u}_\alpha, \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}} \\ &= -\left(2 + \frac{n-2}{2-n-|\alpha|}\right) \left\langle \nabla \mathrm{div}\, \underline{u}_\alpha, \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}} \\ &= \left(2 + \frac{n-2}{2-n-|\alpha|}\right) \left\langle \mathrm{div}\, \underline{u}_\alpha, \mathrm{div}\, \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}}\end{aligned}$$

Therefore we also derive

$$\left\langle \nabla \underline{u}_\alpha, \nabla \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}} = \left(2 + \frac{n-2}{2-n-|\alpha|}\right) \left\langle \mathrm{div}\, \underline{u}_\alpha, \mathrm{div}\, \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}}$$

for all $\underline{\phi} \in \widehat{H}_0^{1,q'}(\mathbb{R}^n \backslash \overline{B_1})^n$.

(d) Let φ_r as in Theorem 2.7 (that is $r > 1$). Then for $|\alpha| \geq 2$ and $i = 1, \dots, n$

$$\langle \nabla u_{\alpha i}, \nabla \varphi_r \rangle_{\mathbb{R}^n \backslash \overline{B_1}} = \lim_{\rho \to \infty} \langle \nabla u_{\alpha i}, \nabla \varphi_r \rangle_{B_\rho \backslash \overline{B_1}} =$$

$$= \lim_{\rho \to \infty} \left[-\langle \Delta u_{\alpha i}, \varphi_r \rangle_{B_\rho \backslash \overline{B_1}} + \int\limits_{\partial B_\rho} \sum_{j=1}^n (\partial_j u_{\alpha i})(z) \underbrace{\varphi_r(z)}_{=1\,(\rho > 2r)} \frac{z_j}{|z|}\, d\omega_z \right]$$

$$\overset{(a)}{=} \lim_{\rho \to \infty} \left[-\left(2 + \frac{n-2}{2-n-|\alpha|}\right) \langle \partial_i \mathrm{div}\, u_\alpha, \varphi_r \rangle_{B_\rho \backslash \overline{B_1}} + \int\limits_{\partial B_\rho} \sum_{j=1}^n (\partial_j u_{\alpha i})(z) \frac{z_j}{|z|}\, d\omega_z \right]$$

$$= \lim_{\rho \to \infty} \left[\left(2 + \frac{n-2}{2-n-|\alpha|}\right) \langle \mathrm{div}\, u_\alpha, \partial_i \varphi_r \rangle_{B_\rho \backslash \overline{B_1}} - \right.$$

$$\left. - \left(2 + \frac{n-2}{2-n-|\alpha|}\right) \int\limits_{\partial B_\rho} \mathrm{div}\, u_\alpha \frac{z_i}{|z|}\, d\omega_z + \int\limits_{\partial B_\rho} \sum_{j=1}^n (\partial_j u_{\alpha i})(z) \frac{z_j}{|z|}\, d\omega_z \right]$$

Because for large $|z|$ and $l, m = 1, \dots, n$ holds

$$|(\partial_l u_{\alpha m})(z)| \leq C(n,\alpha)\, |z|^2\, |z|^{2-n-2-|\alpha|} = C(n,\alpha)\, |z|^{2-n-|\alpha|}$$

we derive

$$\lim_{\rho \to \infty} \int\limits_{\partial B_\rho} |(\partial_l u_{\alpha m})(z)|\, d\omega_z \leq \tilde{C}(n,\alpha) \lim_{\rho \to \infty} \rho^{n-1+2-n-|\alpha|} = 0 \quad (\text{for } |\alpha| \geq 2)$$

Therefore

$$\langle \nabla u_{\alpha i}, \nabla \varphi_r \rangle_{\mathbb{R}^n \backslash \overline{B_1}} = \left(2 + \frac{n-2}{2-n-|\alpha|}\right) \langle \mathrm{div}\, u_\alpha, \partial_i \varphi_r \rangle_{\mathbb{R}^n \backslash \overline{B_1}}$$

(e) By Theorem 2.5 - 2.7 we derive by (c) and (d) for $|\alpha| \geq 2$

$$\left\langle \nabla \underline{u}_\alpha, \nabla \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}} = \left(2 + \frac{n-2}{2-n-|\alpha|}\right) \left\langle \mathrm{div}\, \underline{u}_\alpha, \mathrm{div}\, \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}}$$

for all $\underline{\phi} \in \widehat{H}_{\bullet}^{1,q'}(\mathbb{R}^n \backslash \overline{B_1})^n$.

Therefore $\{\underline{u}_\alpha : |\alpha| \geq 2\}$ is a countable set of weak eigenfunctions for eigenvalues, which have only 2 as aa accumulation point. Because of

$$2 + \frac{n-2}{2-n-|\alpha|} \neq 1 \qquad \forall\, |\alpha| \geq 1$$

and by Theorem 11.3 and 11.4 follows

$$\operatorname{div} \underline{u}_\alpha \in B^q(\mathbb{R}^n \backslash \overline{B_1}) \qquad \forall\, |\alpha| \geq 2 \quad \forall\, 1 < q < \infty$$

and therefore

$$D^\alpha S \in B^q(\mathbb{R}^n \backslash \overline{B_1}) \qquad \forall\, |\alpha| \geq 2 \quad \forall\, 1 < q < \infty$$

(f) Let $n \geq 2$ and $\frac{n}{n-1} < q < \infty$. Then by Lemma 4.4 for $j = 1, \ldots, n$ holds

$$\partial_j S \notin B^q(\mathbb{R}^n \backslash \overline{B_1}), \qquad \partial_j S \in L^q(\mathbb{R}^n \backslash \overline{B_1})$$

For $|\alpha| = 1$ therefore holds

$$\operatorname{div} \underline{u}_\alpha \in L^q(\mathbb{R}^n \backslash \overline{B_1}) \Big\backslash B^q(\mathbb{R}^n \backslash \overline{B_1})$$

By (b) we have

$$\underline{u}_\alpha \in \widehat{H}_{\bullet}^{1,q}(\mathbb{R}^n \backslash \overline{B_1})^n$$

Assume for contradiction that for a $\lambda \in \mathbb{R}$ and for all $\underline{\phi} \in \widehat{H}_{\bullet}^{1,q'}(\mathbb{R}^n \backslash \overline{B_1})^n$ holds

$$\left\langle \nabla \underline{u}_\alpha, \nabla \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}} \;=\; \lambda \left\langle \operatorname{div} \underline{u}_\alpha, \operatorname{div} \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}}$$

Then for $\underline{\phi} \in C_0^\infty(\mathbb{R}^n \backslash \overline{B_1})^n$

$$\left\langle \Delta \underline{u}_\alpha, \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}} \;=\; \lambda \left\langle \nabla \operatorname{div} \underline{u}_\alpha, \underline{\phi} \right\rangle_{\mathbb{R}^n \backslash \overline{B_1}}$$

and therefore

$$\Delta \underline{u}_\alpha \;=\; \lambda \nabla \operatorname{div} \underline{u}_\alpha$$

By (a) we derive

$$2 + \frac{n-2}{2-n-|\alpha|} \;=\; \lambda$$

But by Theorem 11.3 and 11.4 holds

$$\lambda \;=\; 1$$

That is

$$\frac{n-2}{2-n-|\alpha|} \;=\; 1$$

and so $|\alpha| = 0$, a contradiction!

So $\underline{u}_\alpha$ is for $|\alpha| = 1$ a classical solution of the eigenvalue problem, but it is not a weak solution.

Part III: Green's function and reproducing kernels

17 Existence of Green's function

Lemma 17.1. Let $n \geq 2$ and let

$$S(z) := \begin{cases} \frac{1}{(n-2)\omega_n} |z|^{2-n} & , \; z \neq 0, \; n \geq 3 \\ -\frac{1}{2\pi} \ln |z| & , \; z \neq 0, \; n = 2 \\ 0 & , \; z = 0, \; n \geq 2 \end{cases}$$

be the **fundamental solution** of the Laplace operator. Then for each $R > 0$ and for all $f \in C^0(\overline{B_R})$ holds

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} |S(z)\, f(z)| \, d\omega_z &= 0 \\ \lim_{\varepsilon \to 0} \sum_{i=1}^n \int_{\partial B_\varepsilon} (\partial_i S)(z)\, f(z)\, \frac{z_i}{|z|} \, d\omega_z &= -f(0) \\ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{B_\varepsilon} |S(z)| \, dz &= 0 \\ \lim_{\varepsilon \to 0} \int_{B_\varepsilon} |(\partial_j S)(z)| \, dz &= 0 \end{aligned}$$

Proof. (a) There is $M < \infty$ such that $|f(z)| \leq M$ for all $z \in \overline{B_R}$. Then

$$\int_{\partial B_\varepsilon} |S(z) f(z)| \, d\omega_z \;\leq\; \left\{ \begin{array}{ll} M\, \varepsilon^{n-1} \frac{1}{n-2} \varepsilon^{2-n} & , \quad n \geq 3 \\ M\, \varepsilon \Big| \ln |\varepsilon| \Big| & , \quad n = 2 \end{array} \right\} \to 0 \quad (\varepsilon \to 0)$$

(b) For every $n \geq 2$ holds

$$(\partial_i S)(z) = -\frac{1}{\omega_n} \frac{z_i}{|z|^n}$$

Therefore

$$\begin{aligned} \sum_{i=1}^n \int_{\partial B_\varepsilon} (\partial_i S)(z)\, f(z)\, \frac{z_i}{|z|} \, d\omega_z &= -\varepsilon^{n-1} \frac{1}{\omega_n} \int_{S_{n-1}} \varepsilon^{1-n}\, f(\varepsilon\xi) \, d\omega_\xi \\ &= -\frac{1}{\omega_n} \int_{S_{n-1}} f(\varepsilon\xi) \, d\omega_\xi \\ &\to -f(0) \quad (\varepsilon \to 0) \end{aligned}$$

(c) Let $n \geq 3$. Then

$$\frac{1}{\varepsilon} \int_{B_\varepsilon} |S(z)| \, dz \;=\; \frac{1}{(n-2)\varepsilon} \int_0^\varepsilon r^{n-1+2-n} \, dr \;=\; \frac{1}{(n-2)2}\, \varepsilon \;\to 0 \quad (\varepsilon \to 0)$$

(d) Let $n = 2$. Then for $0 < \varepsilon < 1$

$$\begin{aligned}\frac{1}{\varepsilon}\int_{B_\varepsilon}|S(z)|\,dz &= -\frac{1}{\varepsilon}\int_0^\varepsilon r\,\ln r\,dr \;=\; -\frac{1}{\varepsilon}\left[\frac{1}{2}r^2\,\ln r\Big|_0^\varepsilon - \int_0^\varepsilon \frac{1}{2}r^2\,\frac{1}{r}\,dr\right]\\ &= -\frac{1}{\varepsilon}\left[\frac{1}{2}\varepsilon^2\,\ln\varepsilon - \frac{1}{4}\varepsilon^2\right] \;=\; -\frac{1}{2}\varepsilon\,\ln\varepsilon + \frac{1}{4}\varepsilon \;\to 0 \quad (\varepsilon\to 0)\end{aligned}$$

(e) For $n \geq 2$ holds

$$\int_{B_\varepsilon}|(\partial_j S)(z)|\,dz \;\leq\; \int_0^\varepsilon r^{n-1+1-n}\,dr \;=\; \varepsilon \;\to 0 \quad (\varepsilon\to 0)$$

□

Theorem 17.2. Let $n \geq 2$ and let S be the fundamental solution of the Laplace operator. Then for every $z \neq 0$ holds

$$\Delta S(z) = 0$$

and for all $u \in C_0^2(\mathbb{R}^n)$ and for each $x \in \mathbb{R}^n$ holds

$$u(x) \;=\; \int_{\mathbb{R}^n} S(x-y)\,[-\Delta u(y)]\,dy$$

Proof. see [SiDGL, Satz 3.1] □

Theorem 17.3. Let $n \geq 2$ and let S be the fundamental solution of the Laplace operator. Let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^1$. Assume that there is $h : G \times \overline{G} \to \mathbb{R}$ such that for all $x, y \in G$ holds

$$h(x,\cdot) \in \overline{C}^1(G) \cap C^2(G), \qquad \Delta_y h(x,y) = 0$$

Define

$$\phi(x,y) := S(x-y) + h(x,y)$$

Suppose $u \in \overline{C}^1(G) \cap C^2(G)$ and $f := -\Delta u \in C^0(\overline{G})$. Then for each $x \in G$ holds

$$u(x) \;=\; \int_{\partial G}\left[\phi(x,y)\,\frac{\partial u(y)}{\partial N} - u(y)\,\frac{\partial \phi(x,y)}{\partial N_y}\right]\,d\omega_y + \int_G \phi(x,y)\,f(y)\,dy$$

(where $\frac{\partial}{\partial N}$ denotes differentiation along the outward normal to ∂G)

Proof. see [SiDGL, Satz 3.3] □

Theorem and Definition 17.4. Let $n \geq 2$ and let S be the fundamental solution of the Laplace operator. Let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^3$. Then there is a unique $h : G \times \overline{G} \to \mathbb{R}$ such that for all $x, y \in G$ holds

$$h(x,\cdot) \in \overline{C}^2(G), \qquad \Delta_y h(x,y) = 0$$

and for $x \in G$, $y \in \partial G$ holds

$$h(x,y) = -S(x-y)$$

We define by

$$\mathcal{G}(x,y) \;:=\; S(x-y) + h(x,y)$$

the **Green's function** of the Laplace operator in G.

Proof. (a) Choose $\rho \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \rho \leq 1$ and

$$\rho(t) = \begin{cases} 0 & , \quad |t| \leq 1 \\ 1 & , \quad |t| \geq 2 \end{cases}$$

Define for $r > 0$

$$S_r(z) := \rho\left(\frac{|z|}{r}\right) S(z)$$

Then $S_r \in C^\infty(\mathbb{R}^n)$ and therefore

$$S_r \in H^{\infty,2n}(G)$$

For $x \in G$ choose $0 < r_x \leq \frac{1}{4}\operatorname{dist}(x, G)$. By Theorem 2.9 there is $f_{r_x}(x,\cdot) \in H_0^{1,2n}(G)$ such that

$$\langle \nabla_y f_{r_x}(x,\cdot), \nabla\varphi \rangle_G \;=\; \langle \nabla_y S_{r_x}(x-\cdot), \nabla\varphi \rangle_G \qquad \forall\, \varphi \in H_0^{1,(2n)'}(G)$$

For $\varphi \in C_0^\infty(G)$ holds

$$\langle \nabla_y f_{r_x}(x,\cdot), \nabla\varphi \rangle_G \;=\; -\langle \Delta_y S_{r_x}(x-\cdot), \varphi \rangle_G$$

Therefore by Theorem 7.4 because of $\partial G \in C^3$ holds

$$f_{r_x}(x,\cdot) \in H_0^{1,2n}(G) \cap H^{3,2n}(G)$$

By $3 - \frac{n}{2n} = 2 + \frac{1}{2}$ and by Sobolev's imbedding theorem (see e.g. [Alt, Satz 8.13, p.319]) holds

$$f_{r_x}(x,\cdot) \in \overline{C}^2(G) \qquad \forall\, x \in G$$

Further by Weyl's Lemma

$$\Delta_y \Big[f_{r_x}(x,y) - S_{r_x}(x-y) \Big] \;=\; 0 \qquad \forall\, x, y \in G$$

By Theorem 5.8 also holds

$$f_{r_x}(x,y) = 0 \qquad \forall\, x \in G \quad \forall\, y \in \partial G$$

Define

$$h(x,y) \;:=\; f_{r_x}(x,y) - S_{r_x}(x-y)$$

Then all assertions are fulfilled.

(b) Let $h^{(1)}$ and $h^{(2)}$ be two functions with the above properties. For a fixed $x_0 \in G$ holds

$$h^{(1)}(x_0,\cdot)\Big|_{\partial G} = h^{(2)}(x_0,\cdot)\Big|_{\partial G}$$

and

$$\Delta_y \Big[h^{(1)}(x_0,y) - h^{(2)}(x_0,y) \Big] = 0 \qquad \forall\, y \in G$$

Therefore by the maximum principle

$$h^{(1)} = h^{(2)}$$

□

Lemma 17.5. Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^3$. Then for Green's function holds

$$\mathcal{G}(x,y) \;=\; \mathcal{G}(y,x) \qquad \text{for all } x,y \in G$$

Proof. (by [He, p. 238])

Let $x_1, x_2 \in G$, $x_1 \neq x_2$. Let $\varepsilon > 0$ with $B_\varepsilon(x_1) \subset G$, $B_\varepsilon(x_2) \subset G$ and

$$B_\varepsilon(x_1) \cap B_\varepsilon(x_2) = \emptyset$$

Then by Green's identity

$$\begin{aligned} 0 \;&=\; \int\limits_{G \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))} \big[\mathcal{G}(x_1,y)\, \underbrace{\Delta_y \mathcal{G}(x_2,y)}_{=0} - \mathcal{G}(x_2,y)\, \underbrace{\Delta_y \mathcal{G}(x_1,y)}_{=0}\big]\, dy \\ &=\; \int\limits_{\partial G} \big[\underbrace{\mathcal{G}(x_1,y)}_{=0}\, \frac{\partial}{\partial N_y}\mathcal{G}(x_2,y) \;-\; \underbrace{\mathcal{G}(x_2,y)}_{=0}\, \frac{\partial}{\partial N_y}\mathcal{G}(x_1,y)\big]\, d\omega_y \\ &\quad - \int\limits_{\partial B_\varepsilon(x_1)} \sum_{i=1}^{n} \big[\mathcal{G}(x_1,y)\, \partial_{y_i}\mathcal{G}(x_2,y)\, \frac{y_i}{|y|} \;-\; \mathcal{G}(x_2,y)\, \partial_{y_i}\mathcal{G}(x_1,y)\, \frac{y_i}{|y|}\big]\, d\omega_y \\ &\quad - \int\limits_{\partial B_\varepsilon(x_2)} \sum_{i=1}^{n} \big[\mathcal{G}(x_1,y)\, \partial_{y_i}\mathcal{G}(x_2,y)\, \frac{y_i}{|y|} \;-\; \mathcal{G}(x_2,y)\, \partial_{y_i}\mathcal{G}(x_1,y)\, \frac{y_i}{|y|}\big]\, d\omega_y \end{aligned}$$

For $\varepsilon \to 0$ we derive by Lemma 17.1

$$0 = -\mathcal{G}(x_2,x_1) + \mathcal{G}(x_1,x_2)$$

□

Lemma 17.6. Let $n \geq 2$, $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^3$. Then for Green's function $\mathcal{G}(x,y) = S(x-y) + h(x,y)$ and for all $i,j = 1,\dots,n$ holds

1. $\partial_{x_i} h \in L^q(G \times G)$
2. $\partial_{x_j}\partial_{x_i} h \in L^q(G \times G)$
3. $\partial_{y_j}\partial_{x_i} h \in L^q(G' \times G)$ for all $G' \subset\subset G$
4. $\partial_{x_i} h \in C^0(G \times \overline{G})$

Proof. (a) Let $x \in G$ and $r_x := \frac{1}{4}\operatorname{dist}(x,G)$. Then by the proof of Theorem 17.4

$$\langle \nabla_y f_{r_x}(x,\cdot), \nabla\varphi \rangle_G \;=\; \langle \nabla_y S_{r_x}(x-\cdot), \nabla\varphi\rangle_G \qquad \forall\, \varphi \in C_0^\infty(G)$$

By Theorem 7.9 follows

$$\|\nabla_y f_{r_x}(x,\cdot)\|_{q;G} \;\leq\; C_q\, \|\nabla_y S_{r_x}(x-\cdot)\|_{q;G}$$

Because $\big[(x,y) \mapsto \nabla_y S_{r_x}(x-y)\big] \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$ and G is bounded, we derive

$$\begin{aligned}
\int\limits_{G\times G} |\partial_{y_i} h(x,y)|^q \; dx\, dy \;&\leq\; \int\limits_{G\times G} \Big[|\partial_{y_i} f_{r_x}(x,y)|^q + |\partial_{y_i} S_{r_x}(x-y)|^q \Big]\, dx\, dy \\
&\leq\; (C_q^q + 1) \int\limits_{G\times G} \underbrace{|\partial_{y_i} S_{r_x}(x-y)|^q}_{\leq M}\, dx\, dy \;<\; \infty
\end{aligned}$$

Therefore by Lemma 17.5

$$\begin{aligned}
\int\limits_{G\times G} |\partial_{x_i} h(x,y)|^q \; dx\, dy \;&=\; \int\limits_{G\times G} \big|\partial_{x_i}\big[h(y,x)\big]\big|^q \; dx\, dy \\
&=\; \int\limits_{G\times G} |(\partial_{y_i} h)(y,x)|^q \; dx\, dy \\
&\underset{\text{Fubini}}{=}\; \int\limits_{G\times G} |\partial_{y_i} h(x,y)|^q \; dx\, dy \;<\; \infty
\end{aligned}$$

(b) Like in (a) with $r_x := \frac{1}{4}\operatorname{dist}(x,G)$ holds

$$\langle \nabla_y f_{r_x}(x,\cdot), \nabla\varphi \rangle_G \;=\; \langle \nabla_y S_{r_x}(x-\cdot), \nabla\varphi\rangle_G \qquad \forall\, \varphi \in C_0^\infty(G)$$

Because $f_{r_x}(x,\cdot) \in \overline{C}^2(G)$ therefore follows

$$\langle \Delta_y f_{r_x}(x,\cdot), \varphi \rangle_G \;=\; \langle \Delta_y S_{r_x}(x-\cdot), \varphi\rangle_G \qquad \forall\, \varphi \in C_0^\infty(G)$$

Because $C_0^\infty(G)$ is dense in $L^{q'}(G)$, we derive

$$\langle \Delta_y f_{r_x}(x,\cdot), \varphi \rangle_G \;=\; \langle \Delta_y S_{r_x}(x-\cdot), \varphi\rangle_G \qquad \forall\, \varphi \in L^{q'}(G)$$

Further because of the $L^q - L^{q'}$-duality, we get

$$\|\Delta_y f_{r_x}(x,\cdot)\|_{q;G} \;=\; \sup_{\varphi\in L^{q'}(G)} \frac{\langle \Delta_y f_{r_x}(x,\cdot),\varphi\rangle_G}{\|\varphi\|_{q'}} \;\le\; \|\Delta_y S_{r_x}(x-\cdot)\|_{q;G} \quad \forall\, x\in G$$

Then

$$\int\limits_{G\times G} |\Delta_y f_{r_x}(x,y)|^q \; dx\,dy \;\;\le\;\; \int\limits_{G\times G} |\Delta_y S_{r_x}(x-y)|^q \; dx\,dy \;<\; \infty$$

By Theorem 7.6 holds

$$\begin{aligned} \|f_{r_x}(x,\cdot)\|_{2,q;G} \quad &\le \quad C(G,q,n)\Big[\|\Delta_y f_{r_x}(x,\cdot)\|_{q;G} + \|f_{r_x}(x,\cdot)\|_{1,q;G}\Big] \\ &\underset{\text{Poincare}}{\le} \quad \tilde{C}(G,q,n)\Big[\|\Delta_y f_{r_x}(x,\cdot)\|_{q;G} + \|\nabla_y f_{r_x}(x,\cdot)\|_{q;G}\Big] \end{aligned}$$

So by (a)

$$\int\limits_{G\times G} \big|\partial_{y_j}\partial_{y_i} f_{r_x}(x,y)\big|^q \; dx\,dy \;<\; \infty$$

and also

$$\int\limits_{G\times G} \big|\partial_{y_j}\partial_{y_i} h(x,y)\big|^q \; dx\,dy \;<\; \infty$$

Therefore

$$\begin{aligned} \int\limits_{G\times G} \big|\partial_{x_j}\partial_{x_i} h(x,y)\big|^q \; dx\,dy \quad &= \quad \int\limits_{G\times G} \big|\partial_{x_j}\partial_{x_i}\big[h(y,x)\big]\big|^q \; dx\,dy \\ &= \quad \int\limits_{G\times G} \big|\partial_{x_j}\big[(\partial_{y_i} h)(y,x)\big]\big|^q \; dx\,dy \\ &= \quad \int\limits_{G\times G} \big|(\partial_{y_j}\partial_{y_i} h)(y,x)\big|^q \; dx\,dy \\ &\underset{\text{Fubini}}{=} \quad \int\limits_{G\times G} \big|(\partial_{y_j}\partial_{y_i} h)(x,y)\big|^q \; dx\,dy \;<\; \infty \end{aligned}$$

(c) Let $x_0\in G$, $\delta>0$ and $B_{16\delta}(x_0)\subset G$. As one can see by the uniqueness of h in the proof of Theorem 17.4

$$h(x,y) \;=\; f_{r_x}(x,y) - S_{r_x}(x-y) \qquad \forall\, 0<r_x\le \frac{1}{4}\,\mathrm{dist}(x,G)$$

For $x\in B_{4\delta}(x_0)$ therefore holds

$$h(x,y) \;=\; f_\delta(x,y) - S_\delta(x-y)$$

Let $x\in B_{2\delta}(x_0)$. Then for every $\varphi\in C_0^\infty(G)$ and every $0<|h|<\delta$ holds

$$\big\langle \nabla_y\big[f_\delta(x+he_i,\cdot) - f_\delta(x,\cdot)\big],\nabla\varphi\big\rangle_G \;=\; \big\langle \nabla_y\big[S_\delta(x+he_i-\cdot) - S_\delta(x-\cdot)\big],\nabla\varphi\big\rangle_G$$

By Theorem 7.9 follows

$$\|\nabla_y\big[f_\delta(x+he_i,\cdot)-f_\delta(x,\cdot)\big]\|_{q;G} \;\le\; C_q\,\|\nabla_y\big[S_\delta(x+he_i-\cdot)-S_\delta(x-\cdot)\big]\|_{q;G}$$

Because $S_\delta \in C^\infty(\mathbb{R}^n)$ and G is bounded, we derive by the mean value formula

$$\int_{B_{2\delta}(x_0)} \frac{1}{|h|^q}\,\|\nabla_y\big[S_\delta(x+he_i-\cdot)-S_\delta(x-\cdot)\big]\|_{q;G}^q\,dx \;\le\; C_1(\delta,x_0,G)<\infty$$

for each $0<|h|<\delta$. Therefore

$$\int\limits_{B_{2\delta}(x_0)\times G} \left|\frac{\partial_{y_j}f_\delta(x+he_i,y)-\partial_{y_j}f_\delta(x,y)}{h}\right|^q dx\,dy \;<\; C_2(\delta,x_0,G)<\infty$$

for each $0<|h|<\delta$.

Because of the weak compactness of L^q there is $g_{ij}\in L^q(B_{2\delta}(x_0)\times G)$ and a sequence $(h_k)\subset\mathbb{R}$ with $0<|h_k|<\delta$ and $h_k\to 0$ $(k\to\infty)$ such that

$$\int\limits_{B_{2\delta}(x_0)\times G} \frac{\partial_{y_j}f_\delta(x+h_ke_i,y)-\partial_{y_j}f_\delta(x,y)}{h_k}\,\phi(x,y)\,dx\,dy \;\to$$

$$\to \int\limits_{B_{2\delta}(x_0)\times G} g_{ij}(x,y)\,\phi(x,y)\,dx\,dy \qquad (k\to\infty)$$

for all $\phi\in C_0^\infty(B_{2\delta}(x_0)\times G)$.

For $\phi\in C_0^\infty(B_\delta(x_0)\times G)$ holds

$$\int\limits_{B_{2\delta}(x_0)\times G} \frac{\partial_{y_j}f_\delta(x+h_ke_i,y)-\partial_{y_j}f_\delta(x,y)}{h_k}\,\phi(x,y)\,dx\,dy \;=$$

$$= \int\limits_{B_{2\delta}(x_0)\times G} \partial_{y_j}f_\delta(x,y)\,\left[\frac{\phi(x-h_ke_i,y)-\phi(x,y)}{h_k}\right]dx\,dy \;\to$$

$$\underset{\text{by (a)}}{\longrightarrow} -\int\limits_{B_{2\delta}(x_0)\times G} \partial_{y_j}f_\delta(x,y)\,\partial_{x_i}\phi(x,y)\,dx\,dy \qquad (k\to\infty)$$

Therefore

$$\int\limits_{B_\delta(x_0)\times G} g_{ij}(x,y)\,\phi(x,y)\,dx\,dy \;=\; -\int\limits_{B_\delta(x_0)\times G} \partial_{y_j}f_\delta(x,y)\,\partial_{x_i}\phi(x,y)\,dx\,dy$$

$$= \int\limits_{B_\delta(x_0)\times G} f_\delta(x,y)\,\partial_{y_j}\partial_{x_i}\phi(x,y)\,dx\,dy$$

for all $\phi\in C_0^\infty(B_\delta(x_0)\times G)$. We get

$$\partial_{y_j}\partial_{x_i}f_\delta = g_{ij}\in L^q(B_\delta(x_0)\times G)$$

and

$$\partial_{y_j}\partial_{x_i}h \in L^q(B_\delta(x_0) \times G)$$

(d) For $G' \subset\subset G$ there are $N \in \mathbb{N}$, $\delta_i > 0$ and $x_i \in G$ such that $B_{16\delta_i}(x_i) \subset G$ and

$$G' \subset \bigcup_{i=1}^{N} B_{\delta_i}(x_i)$$

respectively

$$G' \times G \subset \left(\bigcup_{i=1}^{N} B_{\delta_i}(x_i) \right) \times G$$

Then by (c) and a partition of unity $\partial_{y_j}\partial_{x_i}h \in L^q(G' \times G)$ follows.

(e) By (a), (b) and (d) holds

$$\partial_{x_i}h \in H^{1,q}(G' \times G) \qquad \forall\, G' \subset\subset G \quad \forall\, 1 < q < \infty$$

By Sobolev's imbedding theorems (see e.g. [Alt, Satz 8.13, p.319]) follows

$$\partial_{x_i}h \in C^0(G \times \overline{G})$$

□

18 Existence of reproducing kernels in $B^q(G)$

Theorem 18.1. Let $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be either a bounded domain or an exterior domain. Assume that $F^* \in B^{q'}(G)^*$. Then there is a unique $h \in B^q(G)$ such that

$$F^*(\pi) = \langle h, \pi \rangle_G \qquad \text{for all } \pi \in B^{q'}(G)$$

and with the constant C_q by Theorem 3.8 holds

$$(1 + C_q)^{-1} \|h\|_{q;G} \;\leq \sup_{0 \neq \pi \in B^{q'}(G)} \frac{F^*(\pi)}{\|\pi\|_{q'}} \;\leq\; \|h\|_{q;G}$$

Proof. By the Hahn-Banach-Theorem there is $\tilde{F}^* \in L^{q'}(G)^*$ such that

$$\tilde{F}^*\Big|_{B^{q'}(G)} = F^* \qquad \|\tilde{F}^*\| = \|F^*\|$$

There exists $f \in L^q(G)$ such that

$$\tilde{F}^*(g) = \langle f, g \rangle_G \qquad \forall\, g \in L^{q'}(G)$$

and

$$\|f\|_{q;G} = \|\tilde{F}^*\| = \|F^*\|$$

By Theorem 4.2 holds

$$f = \underbrace{\Delta s}_{\in A^q(G)} + \underbrace{h}_{\in B^q(G)}$$

and for $\pi \in B^{q'}(G)$ holds

$$F^*(\pi) \;=\; \tilde{F}^*(\pi) \;=\; \langle f, \pi \rangle \;=\; \underbrace{\langle \Delta s, \pi \rangle}_{=0\,(4.1)} + \langle h, \pi \rangle \;=\; \langle h, \pi \rangle$$

Further by Theorem 3.8

$$\|\Delta s\|_{q;G} \;\leq\; C_q \sup_{0 \neq \phi \in \widehat{H}^{2,q'}_{\bullet}(G)} \frac{\langle \Delta s, \Delta\phi \rangle}{\|\Delta\phi\|_{q'}} \;=\; C_q \sup_{0 \neq \phi \in \widehat{H}^{2,q'}_{\bullet}(G)} \frac{\langle f, \Delta\phi \rangle}{\|\Delta\phi\|_{q'}} \;\leq\; C_q \|f\|_{q;G}$$

and therefore

$$\|h\|_{q;G} \;\leq\; (1 + C_q) \|f\|_{q;G} \;=\; (1 + C_q) \|F^*\|$$

and

$$\|F^*\| \;\leq\; \|h\|_{q;G}$$

by Hölder's inequality.

(b) Let $h^{(1)}$, $h^{(2)} \in B^q(G)$ with

$$\left\langle h^{(1)}, \pi \right\rangle_G = \left\langle h^{(2)}, \pi \right\rangle_G \qquad \text{for all } \pi \in B^{q'}(G)$$

Then by Theorem 4.2

$$\left\langle h^{(1)}, g \right\rangle_G = \left\langle h^{(2)}, g \right\rangle_G \qquad \text{for all } g \in L^{q'}(G)$$

and therefore

$$h^{(1)} = h^{(2)}$$

almost everywhere. $\square$

Theorem and Definition 18.2. Let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain.

1. Let $1 < q < \infty$. Then for each $x \in G$ there is a unique

$$\mathcal{R}_q(x, \cdot) \in B^{q'}(G)$$

such that

$$p(x) \; = \; \int_G \mathcal{R}_q(x, y)\, p(y)\, dy \qquad \text{a.e. for all } p \in B^q(G)$$

The function

$$\mathcal{R}_q : G \times G \to \mathbb{R}$$

is called **reproducing kernel** of $B^q(G)$ for each $1 < q < \infty$.

2. Let $G \subset \mathbb{R}^n$ ne a bounded domain. For $1 < q, s < \infty$ and for every x, $y \in G$ holds

$$\mathcal{R}_q(x, y) \; = \; \mathcal{R}_s(x, y), \qquad \mathcal{R}_q(x, y) \; = \; \mathcal{R}_q(y, x)$$

Therefore in the case of a bounded domain we can use the notation

$$\mathcal{R} \; := \; \mathcal{R}_q$$

Proof. (a) For $p \in B^q(G)$ by Weyl's Lemma there is a unique $\tilde{p} \in C^\infty(G)$ with $\tilde{p} = p$ almost everywhere and $\Delta\tilde{p} = 0$. We identify this $\tilde{p}$ with the equivalence class $p \in B^q(G)$. In this sense the notation $p \in B^q(G) \cap C^\infty(G)$ is meaningful and unique.

Let $x \in G$. For $p \in B^q(G)$ $(\cap C^\infty(G))$ and $r < \operatorname{dist}(x, \partial G)$ by the mean value property holds

$$\begin{aligned} |p(x)| \;&\leq\; \frac{1}{|B_r|} \int\limits_{B_r(x)} |p(y)|\, dy \;\leq\; |B_r|^{-1}\, |B_r|^{\frac{q-1}{q}} \left(\int\limits_{B_r(x)} |p(y)|^q\, dy \right)^{\frac{1}{q}} \\ &\leq\; \left(\frac{\omega_n r^n}{n} \right)^{-\frac{1}{q}} \|p\|_{q;G} \end{aligned}$$

For $r \to \operatorname{dist}(x, \partial G)$ we derive

$$|p(x)| \;\leq\; \left(\frac{\omega_n}{n} \right)^{-\frac{1}{q}} \Big[\operatorname{dist}(x, \partial G) \Big]^{-\frac{n}{q}} \|p\|_{q;G}$$

Therefore for every fixed $x \in G$ the map

$$\Big[p \mapsto p(x) \Big] \; \in B^q(G)^*$$

By Theorem 18.1 there is a unique $\mathcal{R}_q(x,\cdot) \in B^{q'}(G)$ with

$$p(x) \;=\; \int_G \mathcal{R}_q(x,y)\,p(y)\,dy$$

almost everywhere for every $p \in B^q(G)$ $(\cap C^\infty(G))$

(b) For $x,\, y \in G$ holds

$$\mathcal{R}_q(x,y) \;=\; \int_G \mathcal{R}_{q'}(y,z)\,\mathcal{R}_q(x,z)\,dz \;=\; \int_G \mathcal{R}_q(x,z)\,\mathcal{R}_{q'}(y,z)\,dz \;=\; \mathcal{R}_{q'}(y,x)$$

Especially for $q = 2$

$$\mathcal{R}_2(x,y) \;=\; \mathcal{R}_2(y,x)$$

(c) Let $G \subset \mathbb{R}^n$ be a bounded domain. Then by Weyl's Lemma and Theorem 2.5 holds

$$B^q(G) \;=\; \{h \in L^q(G) \,:\, \Delta h = 0\}$$

If $2 < t < \infty$, we derive $t' < 2 = 2'$. Then by Hölder's inequality

$$\mathcal{R}_2(x,\cdot) \in B^{t'}(G)$$

and for $p \in B^t(G) \subset B^2(G)$ almost everywhere holds

$$p(x) \;=\; \int_G \mathcal{R}_2(x,y)\,p(y)\,dy$$

Because of the uniqueness in (a) follows

$$\mathcal{R}_2(x,y) \;=\; \mathcal{R}_t(x,y) \qquad \forall\, 2 \le t < \infty$$

If $1 < t < 2$, we have $2 < t' < \infty$ and therefore by (b)

$$\mathcal{R}_t(x,y) \;=\; \mathcal{R}_{t'}(y,x) \;=\; \mathcal{R}_2(y,x) \;=\; \mathcal{R}_2(x,y)$$

□

19 Relationship of Green's function to reproducing kernels

Theorem 19.1 Let $n \geq 2$, $1 < q < \infty$, $k \in \mathbb{N}$, $k > 1 + \frac{n}{q}$ and let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^{2+k}$. Let

$$\mathcal{G}(x,y) = S(x-y) + h(x,y)$$

be Green's function of the Laplace operator in G and let $\mathcal{R}$ be the reproducing kernel in $B^q(G)$. Then

$$Z_q(p)(x) \;=\; p(x) \;+\; \sum_{i=1}^{n} \int_G p(y)\, \partial_{y_i} \partial_{x_i} h(x,y)\, dy \qquad \text{a.e. for } p \in B^q(G)$$

Therefore

$$\sum_{i=1}^{n} \partial_{y_i} \partial_{x_i} h(x,y) \;+\; \frac{1}{2}\, \mathcal{R}(x,y)$$

is a compact operator.

Proof. (a) At the beginning assume that $p \in H^{k,q}(G) \cap B^q(G)$. By Lemma 13.1 holds

$$\underline{u} := \underline{T}_q(p) \in \overline{C}^1(G)^n, \qquad \nabla \underline{u} \in H^{k,q}(G)^{n^2}, \qquad \Delta \underline{u} = \nabla p$$

By Sobolev's imbedding theorem (see e.g. [Alt, Satz 8.13, p.319]) also holds

$$p \in \overline{C}^1(G), \qquad \underline{u} \in \overline{C}^2(G)^n$$

By Theorem 5.8 holds

$$\underline{u}\Big|_{\partial G} = 0$$

and therefore by Theorem 17.3

$$u_i(x) \;=\; \int_G \mathcal{G}(x,y)\, (-\Delta u_i)(y)\, dy \;=\; \int_G \Big[S(x-y) + h(x,y) \Big]\, (-\partial_i p)(y)\, dy$$

For $r > 0$ define S_r as in the proof of Theorem 17.4. Let

$$u_i^{(r)}(x) \;:=\; \int_G \Big[S_r(x-y) + h(x,y) \Big]\, (-\partial_i p)(y)\, dy$$

Then by Lemma 17.6 $u_i^{(r)} \in C^1(G)$ and

$$\partial_j u_i^{(r)}(x) \;=\; \int_G \Big[\partial_{x_j} S_r(x-y) + \partial_{x_j} h(x,y) \Big]\, (-\partial_i p)(y)\, dy$$

Further holds

$$\begin{aligned} |u_i^{(r)}(x) - u(x)| \;&\leq\; \int_G \left[1 - \rho\left(\frac{|x-y|}{r} \right) \right]\, |S(x-y)|\, |(\partial_i p)(y)|\, dy \\ &\leq\; \int_{B_{2r}(x)} |S(x-y)|\, \underbrace{|(\partial_i p)(y)|}_{\leq M}\, dy \;\leq\; M \int_{B_{2r}} |S(z)|\, dz \end{aligned}$$

By Lemma 17.1 $(u_i^{(r)})$ is uniformly convergent in G to u (for $r \to 0$). Furthermore

$$\left|\partial_j u_i^{(r)}(x) \;-\; \int_G \Big[(\partial_j S)(x-y) + \partial_{x_j} h(x,y)\Big]\,(-\partial_i p)(y)\,dy\right| \;=$$

$$= \;\left|\int_G (\partial_j S_r)(x-y)\,(\partial_i p)(y)\,dy \;-\; \int_G (\partial_j S)(x-y)\,(\partial_i p)(y)\,dy\right| \;\le$$

$$\le\; \int_{B_{2r}(x)} \frac{1}{r}\,\underbrace{\left|\rho'\left(\frac{|x-y|}{r}\right)\right|}_{\le C}\,|S(x-y)|\;\underbrace{|(\partial_i p)(y)|}_{\le M}\,dy \;+$$

$$+\;\int_{B_{2r}(x)} \underbrace{\left[1-\rho\left(\frac{|x-y|}{r}\right)\right]}_{\le 1}\,|(\partial_j S)(x-y)|\;\underbrace{|(\partial_i p)(y)|}_{\le M}\,dy \;\le$$

$$\le\; K\,\left[\frac{1}{r}\int_{B_{2r}} |S(z)|\,dz \;+\; \int_{B_{2r}} |(\partial_j S)(z)|\,dz\right]$$

Therefore by Lemma 17.1 $(\partial_j u_i^{(r)})$ is uniformly Cauchy in G (for $r \to 0$). By Lemma A.16 exists

$$\operatorname{div}\underline{u}(x) \;=\; \sum_{i=1}^n \int_G \Big[\partial_{x_i} S(x-y) + \partial_{x_i} h(x,y)\Big]\,(-\partial_i p)(y)\,dy$$

By Lemma 17.6 we can apply Gauß' Theorem and derive

$$\begin{aligned}
\operatorname{div}\underline{u}(x) \;&=\; \lim_{\varepsilon\to 0}\sum_{i=1}^n \int_{G\setminus B_\varepsilon(x)} \Big[\partial_{x_i} S(x-y) + \partial_{x_i} h(x,y)\Big]\,(-\partial_i p)(y)\,dy \\
&=\; \lim_{\varepsilon\to 0}\int_{G\setminus B_\varepsilon(x)} \Big[\underbrace{\sum_{i=1}^n \partial_{y_i}\partial_{x_i} S(x-y)}_{=-\Delta S(x-y)=0} + \sum_{i=1}^n \partial_{y_i}\partial_{x_i} h(x,y)\Big]\,p(y)\,dy \\
&\quad -\sum_{i=1}^n \int_{\partial G} \Big[\partial_{x_i} S(x-y) + \partial_{x_i} h(x,y)\Big]\,p(y)\,N_i(y)\,d\omega_y \\
&\quad +\lim_{\varepsilon\to 0}\sum_{i=1}^n \int_{\partial B_\varepsilon(x)} \Big[\partial_{x_i} S(x-y) + \partial_{x_i} h(x,y)\Big]\,p(y)\,\frac{y_i-x_i}{|y-x|}\,d\omega_y
\end{aligned}$$

By the definition of Green's functions holds

$$S(x-y) + h(x,y) = 0 \qquad \forall\, y \in \partial G \quad \forall\, x \in G$$

and therefore also

$$\partial_{x_i}\big[S(x-y) + h(x,y)\big] = 0 \qquad \forall\, y \in \partial G \quad \forall\, x \in G$$

Because $p \in C^0(G)$ and $\partial_{x_i} h \in C^0(G \times \overline{G})$ we get

$$\lim_{\varepsilon\to 0}\sum_{i=1}^n \int_{\partial B_\varepsilon(x)} \partial_{x_i} h(x,y)\,p(y)\,\frac{y_i-x_i}{|y-x|}\,d\omega_y \;=\; 0$$

Furthermore by Lemma 17.1 holds

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{n}\int_{\partial B_\varepsilon(x)}\partial_{x_i}S(x-y)\,p(y)\,\frac{y_i-x_i}{|y-x|}\,d\omega_y \;=$$

$$=\;\lim_{\varepsilon\to 0}\sum_{i=1}^{n}\int_{\partial B_\varepsilon} -(\partial_i S)(z)\,p(x-z)\,\frac{z_i}{|z|}\,d\omega_z \;=\; p(x)$$

Altogether we derive

$$Z_q(p)(x) \;=\; \operatorname{div}\underline{u}(x) \;=\; p(x)+\int\limits_G\sum_{i=1}^{n}\partial_{y_i}\partial_{x_i}h(x,y)\,p(y)\,dy$$

for every $x\in G$ and every $p\in H^{k,q}(G)\cap B^q(G)$.

(b) Let now $p\in B^q(G)$ be arbitrary. Then we identify p and the unique harmonic representative $\tilde p\in C^\infty(G)$ with $\tilde p = p$ almost everywhere in G. Define

$$F(p)(x) \;:=\; \int\limits_G\sum_{i=1}^{n}\partial_{y_i}\partial_{x_i}h(x,y)\,p(y)\,dy \qquad \forall\, x\in G$$

By Theorem 9.1 there is a sequence $(p_m)\subset H^{k,q}(G)\cap B^q(G)$ such that

$$\|p_m-p\|_{q;G}\to 0 \qquad (m\to\infty)$$

By (a) holds

$$F(p_m) \;=\; Z_q(p_m)-p_m \qquad \forall\, m\in\mathbb{N}$$

Because Z_q is a bounded operator, we derive $F(p_m)\in L^q(G)$ and

$$\|F(p_m)-F(p_{m'})\|_{q;G} \;\le\; \|Z_q(p_m)-Z_q(p_{m'})\|_{q;G}+\|p_m-p_{m'}\|_{q;G} \;\to 0$$

So there is $g\in L^q(G)$ with

$$\|F(p_m)-g\|_{q;G} \;\to 0$$

and

$$g \;=\; Z_q(p)-p$$

By the Riesz-Fischer Theorem there is subsequence (which we again denote by (p_m)) such that $F(p_m)\to g$ pointwise almost everywhere in G. Further

$$\begin{aligned}|F(p_m)(x)-F(p)(x)| \;&\le\; \int\limits_G\sum_{i=1}^{n}|\partial_{y_i}\partial_{x_i}h(x,y)|\,|p_m(y)-p(y)|\,dy\\ &\le\; \underbrace{\left(\int\limits_G\sum_{i=1}^{n}|\partial_{y_i}\partial_{x_i}h(x,y)|^{q'}\,dy\right)^{\frac{1}{q'}}}_{<\infty \text{ (Lemma 17.6)}}\|p_m-p\|_{q;G} \;\to 0\end{aligned}$$

Therefore $F(p) = g \in L^q(G)$ and $F(p) = Z_q(p) - p$, that is

$$Z_q(p)(x) \;=\; p(x) + \int\limits_G \sum_{i=1}^{n} \partial_{y_i}\partial_{x_i} h(x,y)\, p(y)\, dy$$

for almost every $x \in G$ and every $p \in B^q(G)$.

(c) By (b) for $p \in B^q(G)$ and almost every $x \in G$ holds

$$\int\limits_G \sum_{i=1}^{n} \partial_{y_i}\partial_{x_i} h(x,y)\, p(y)\, dy \;+\; \frac{1}{2}\int\limits_G \mathcal{R}(x,y)\, p(y)\, dy \;=\; Z_q(p)(x) \;-\; \frac{1}{2}\, p(x)$$

and because by Theorem 13.5 $Z_q - \frac{1}{2}I$ is a compact operator, in this sense

$$\sum_{i=1}^{n} \partial_{y_i}\partial_{x_i} h(x,y) \;+\; \frac{1}{2}\,\mathcal{R}(x,y)$$

is a compact operator. $\square$

20 Explicit calculation for B_1

Theorem 20.1. Let $n \geq 2$. Let S the fundamental solution of the Laplace operator. Then

$$\mathcal{G}_{B_1} = S(x-y) + h_{B_1}(x,y)$$

with

$$h_{B_1}(x,y) := \begin{cases} \frac{1}{(2-n)\,\omega_n} \Big[1 - 2\langle x,y\rangle + |x|^2|y|^2\Big]^{\frac{2-n}{2}} & , \text{ for } n \geq 3 \\ \frac{1}{2\pi} \ln \sqrt{1 - 2\langle x,y\rangle + |x|^2|y|^2} & , \text{ for } n = 2 \end{cases}$$

is Green's function of the Laplace operator for the unit ball B_1.

Proof. Obviously by Definition 17.4. □

Theorem 20.2. Let $n \geq 2$. Then

$$\mathcal{R}_{B_1}(x,y) = \frac{(n-4)\,|x|^4\,|y|^4 + \big(8\langle x,y\rangle - 2n - 4\big)\,|x|^2\,|y|^2 + n}{\omega_n\,\big(1 - 2\langle x,y\rangle + |x|^2|y|^2\big)^{1+\frac{n}{2}}}$$

is the reproducing kernel of $B^q(G)$ for $1 < q < \infty$.

Proof. see [ABR, Theorem 8.13, p.157] and Theorem 18.2. □

Remark 20.3. (a) We now consider explicitly the operator

$$\sum_{i=1}^{n} \partial_{y_i}\partial_{x_i} h_{B_1}(x,y) + \frac{1}{2}\,\mathcal{R}_{B_1}(x,y)$$

from Theorem 19.1. For $n \geq 2$ holds

$$\begin{aligned} \partial_{y_i} h_{B_1}(x,y) &= \frac{1}{2\omega_n} \Big[1 - 2\langle x,y\rangle + |x|^2|y|^2\Big]^{-\frac{n}{2}} \big(-2x_i + |x|^2\,2y_i\big) \\ &= \frac{1}{\omega_n} \Big[1 - 2\langle x,y\rangle + |x|^2|y|^2\Big]^{-\frac{n}{2}} \big(-x_i + |x|^2\,y_i\big) \end{aligned}$$

and further

$$\begin{aligned} \omega_n\,\partial_{x_i}\partial_{y_i} h_{B_1}(x,y) &= -\frac{n}{2}\Big[\dots\Big]^{-\frac{n}{2}-1} \big(-x_i + |x|^2\,y_i\big)\big(-2y_i + 2x_i\,|y|^2\big) + \\ &\quad + \Big[\dots\Big]^{-\frac{n}{2}} \big(-1 + 2x_i\,y_i\big) \end{aligned}$$

Therefore

$$\begin{aligned} &\omega_n \Big[1 - 2\langle x,y\rangle + |x|^2|y|^2\Big]^{\frac{n}{2}+1} \partial_{x_i}\partial_{y_i} h_{B_1}(x,y) = \\ &= \big(-x_i + |x|^2 y_i\big)\big(ny_i - nx_i|y|^2\big) + \big(-1 + 2x_iy_i\big)\big(1 - 2\langle x,y\rangle + |x|^2|y|^2\big) = \\ &= -n\,x_i\,y_i + n\,|x|^2\,y_i^2 + n\,x_i^2\,|y|^2 - n\,|x|^2\,|y|^2\,x_i\,y_i - 1 + 2x_i\,y_i + \\ &\quad + 2\langle x,y\rangle - 4x_i\,y_i\,\langle x,y\rangle - |x|^2\,|y|^2 + 2|x|^2\,|y|^2\,x_i\,y_i \end{aligned}$$

and

$$\omega_n \Big[1 - 2\langle x,y\rangle + |x|^2|y|^2\Big]^{\frac{n}{2}+1} \sum_{i=1}^{n} \partial_{x_i}\partial_{y_i} h_{B_1}(x,y) \;=$$

$$= \; -n\,\langle x,y\rangle \;+\; n\,|x|^2\,|y|^2 \;+\; n\,|x|^2\,|y|^2 \;-\; n\,|x|^2\,|y|^2\,\langle x,y\rangle \;-\; n \;+\; 2\langle x,y\rangle \,+$$
$$+\, 2n\,\langle x,y\rangle \;-\; 4\langle x,y\rangle^2 \;-\; n\,|x|^2\,|y|^2 \;+\; 2|x|^2\,|y|^2\,\langle x,y\rangle \;=$$

$$= \; (n+2)\,\langle x,y\rangle \;+\; n\,|x|^2\,|y|^2 \;+\; (2-n)\,|x|^2\,|y|^2\,\langle x,y\rangle \;-\; n \;-\; 4\langle x,y\rangle^2$$

Then

$$\omega_n \Big[1 - 2\langle x,y\rangle + |x|^2|y|^2\Big]^{\frac{n}{2}+1} \Big\{ \sum_{i=1}^{n} \partial_{x_i}\partial_{y_i} h_{B_1}(x,y) \;+\; \tfrac{1}{2}\,\mathcal{R}_{B_1}(x,y) \Big\} \;=$$

$$= \; (n+2)\,\langle x,y\rangle \;+\; n\,|x|^2\,|y|^2 \;+\; (2-n)\,|x|^2\,|y|^2\,\langle x,y\rangle \;-\; n \;-\; 4\langle x,y\rangle^2 \,+$$
$$+ (\tfrac{n}{2}-2)\,|x|^4\,|y|^4 \;+\; 4\langle x,y\rangle\,|x|^2\,|y|^2 \;-\; (n+2)\,|x|^2\,|y|^2 \;+\; \tfrac{n}{2} \;=$$

$$= \; (n+2)\langle x,y\rangle - \tfrac{n}{2} - 2|x|^2|y|^2 + (6-n)|x|^2|y|^2\langle x,y\rangle - 4\langle x,y\rangle^2 + (\tfrac{n}{2}-2)|x|^4|y|^4$$

$$= \; \Big(1 - 2\langle x,y\rangle + |x|^2|y|^2\Big)\Big(-\tfrac{n}{2} + 2\langle x,y\rangle + (\tfrac{n}{2}-2)|x|^2|y|^2\Big)$$

Altogether holds

$$\sum_{i=1}^{n} \partial_{x_i}\partial_{y_i} h_{B_1}(x,y) \;+\; \frac{1}{2}\,\mathcal{R}_{B_1}(x,y) \;=\; \frac{-\frac{n}{2} + 2\langle x,y\rangle + (\frac{n}{2}-2)|x|^2|y|^2}{\omega_n\Big(1 - 2\langle x,y\rangle + |x|^2|y|^2\Big)^{\frac{n}{2}}}$$

(b) Next we prove that the integral kernel

$$K_n(x,y) \;=\; \frac{-\frac{n}{2} + 2\langle x,y\rangle + (\frac{n}{2}-2)|x|^2|y|^2}{\omega_n\Big(1 - 2\langle x,y\rangle + |x|^2|y|^2\Big)^{\frac{n}{2}}}$$

is compact in $L^q(B_1)$. For this aim we define

$$f(x,y) \;:=\; 1 - 2\langle x,y\rangle + |x|^2|y|^2$$

and

$$g_n(x,y) \;:=\; -\frac{n}{2} + 2\langle x,y\rangle + \Big(\frac{n}{2}-2\Big)\,|x|^2|y|^2$$

(c) For $n = 2$ holds

$$g_2(x,y) \;:=\; -1 + 2\langle x,y\rangle - |x|^2|y|^2 \;=\; -f(x,y)$$

So

$$K_2(x,y) \;=\; -\frac{1}{2\pi}$$

is as a Hilbert-Schmidt kernel compact.

(d) For $x, y \in \overline{B_1}$ holds

$$g_3(x,y) = -\frac{3}{2} + 2\langle x,y\rangle - \frac{1}{2}|x|^2|y|^2 = -\frac{1}{2} + \underbrace{\langle x,y\rangle}_{\leq |x||y|} - \frac{1}{2}|x|^2|y|^2 + \langle x,y\rangle - 1$$

$$\leq -\frac{1}{2}\underbrace{(|x|\,|y|-1)^2}_{\geq 0} + \underbrace{\langle x,y\rangle - 1}_{\leq 0} \leq 0$$

and for $n \geq 4$

$$g_n(x,y) \leq -\frac{n}{2} + 2|x|\,|y| + \underbrace{\left(\frac{n}{2}-2\right)}_{\geq 0}|x|^2|y|^2 \leq -\frac{n}{2} + 2 + \left(\frac{n}{2}-2\right) = 0$$

Further holds

$$f(x,y) \geq 1 - 2|x|\,|y| + |x|^2|y|^2 = (1-|x|\,|y|)^2 \geq 0$$

Therefore for $n \geq 3$ and $x, y \in \overline{B_1}$

$$\begin{aligned}
|K_n(x,y)| &= \frac{\frac{n}{2} - 2\langle x,y\rangle + (2-\frac{n}{2})|x|^2|y|^2}{\omega_n\left(1 - 2\langle x,y\rangle + |x|^2|y|^2\right)^{\frac{n}{2}}} \\
&= \frac{1 - 2\langle x,y\rangle + |x|^2|y|^2 + \overbrace{|x|^2|y|^2 - 1}^{\leq 0} + \frac{n}{2}\left(1 - |x|^2|y|^2\right)}{\omega_n\left(1 - 2\langle x,y\rangle + |x|^2|y|^2\right)^{\frac{n}{2}}} \\
&\leq \frac{1 - 2\langle x,y\rangle + |x|^2|y|^2 + \frac{n}{2}\left(1 - |x|\,|y|\right)\overbrace{(1 + |x|\,|y|)}^{\leq 2}}{\omega_n\left(1 - 2\langle x,y\rangle + |x|^2|y|^2\right)^{\frac{n}{2}}} \\
&\leq \frac{1 - 2\langle x,y\rangle + |x|^2|y|^2 + n\left(1 - 2\langle x,y\rangle + |x|^2|y|^2\right)^{\frac{1}{2}}}{\omega_n\left(1 - 2\langle x,y\rangle + |x|^2|y|^2\right)^{\frac{n}{2}}} \\
&= \frac{1}{\omega_n\left(1 - 2\langle x,y\rangle + |x|^2|y|^2\right)^{\frac{n}{2}-1}} + \frac{n}{\omega_n\left(1 - 2\langle x,y\rangle + |x|^2|y|^2\right)^{\frac{n-1}{2}}}
\end{aligned}$$

Because of

$$\begin{aligned}
|x-y|^2 &= |x|^2 + |y|^2 - 2\langle x,y\rangle \\
&= 1 - 2\langle x,y\rangle + |x|^2|y|^2 + |x|^2 - 1 + |y|^2 - |x|^2|y|^2 \\
&= 1 - 2\langle x,y\rangle + |x|^2|y|^2 + |x|^2 - 1 + \underbrace{|y|^2}_{\leq 1}(1 - |x|^2) \\
&\leq 1 - 2\langle x,y\rangle + |x|^2|y|^2
\end{aligned}$$

holds

$$\begin{aligned}|K_n(x,y)| &\leq \frac{1}{\omega_n\,|x-y|^{n-2}} + \frac{n}{\omega_n\,|x-y|^{n-1}} \\ &= \frac{|x-y|+n}{\omega_n\,|x-y|^{n-1}} \;\leq\; \frac{2+n}{\omega_n\,|x-y|^{n-1}}\end{aligned}$$

and K_n is as a Schur kernel compact.

Appendix

A Proofs

Lemma A.1. Let $G \subset \mathbb{R}^n$ be open and let $1 < q < \infty$. Suppose $f, g \in H^{1,q}(G)$ and $\|f\|_{\infty;G} + \|\nabla f\|_{\infty;G} < \infty$. Then

$$f \cdot g \in H^{1,q}(G) \quad \text{and} \quad \partial_i(f \cdot g) = (\partial_i f)g + f(\partial_i g)$$

Proof. We have $f \cdot g \in L^q(G)$ because f is bounded.

Because $H = W$ [Me/Se] there is a sequence $(g_k) \subset C^\infty(G) \cap H^{1,q}(G)$ with

$$\|g - g_k\|_{1,q;G} \to 0$$

Let $\phi \in C_0^\infty(G)$. Then

$$\begin{aligned}
\int_G f\, g\, \partial_i \phi \, dx \quad &\underset{\text{f bounded}}{=} \quad \lim_{k\to\infty} \int_G f\, g_k\, \partial_i \phi \, dx \\
&= \quad \lim_{k\to\infty} \int_G f\, \partial_i \underbrace{(g_k \phi)}_{\in C_0^\infty(G)} \, dx - \int_G f\, (\partial_i g_k)\, \phi \, dx \\
&= \quad \lim_{k\to\infty} - \int_G (\partial_i f)\, g_k\, \phi \, dx - \int_G f\, (\partial_i g_k)\, \phi \, dx \\
&= \quad - \int_G [(\partial_i f)\, g + f\, (\partial_i g)]\, \phi \, dx
\end{aligned}$$

Therefore

$$\partial_i(f \cdot g) = (\partial_i f)g + f(\partial_i g) \in L^q(G)$$

□

Lemma A.2. Let $G \subset \mathbb{R}^n$ be open and let $1 < q < \infty$. Suppose $f \in H^{1,q}(G)$, $g \in C_0^\infty(G)$. Then

$$f \cdot g \in H_0^{1,q}(G) \quad \text{and} \quad \partial_i(f \cdot g) = (\partial_i f)g + f(\partial_i g)$$

Proof. By Lemma A.1 we have $f \cdot g \in H^{1,q}(G)$ and $\partial_i(f \cdot g) = (\partial_i f)g + f(\partial_i g)$

Because $H = W$ [Me/Se] there is a sequence $(f_k) \subset C^\infty(G) \cap H^{1,q}(G)$ with

$$\|f - f_k\|_{1,q;G} \to 0$$

Then

$$\|f_k g - fg\|_{q;G} \quad \le \quad \|g\|_{\infty;G} \|f_k - f\|_{q;G} \to 0$$

$$\begin{aligned}
\|\nabla(f_k g - fg)\|_{q;G} \quad &\le \quad \|(\nabla f_k - \nabla f)g\|_{q;G} + \|(f_k - f)\nabla g\|_{q;G} \\
&\le \quad \|g\|_{\infty;G} \|(\nabla f_k - \nabla f)\|_{q;G} + \|\nabla g\|_{\infty;G} \|(f_k - f)\|_{q;G}
\end{aligned}$$

which tends to 0 for $(k \to \infty)$. By $f_k g \in C_0^\infty(G) \subset H_0^{1,q}(G)$ and the closedness of $H_0^{1,q}(G)$ in $H^{1,q}(G)$ the assertion of the Lemma follows. □

Lemma A.3. Let $G \subset \mathbb{R}^n$ be open and let $1 < q < \infty$. Suppose $f \in H^{1,q}(G)$, $g \in H_0^{1,q}(G)$ and $\|f\|_{\infty;G} + \|\nabla f\|_{\infty;G} < \infty$. Then

$$f \cdot g \in H_0^{1,q}(G) \quad \text{and} \quad \partial_i(f \cdot g) = (\partial_i f)g + f(\partial_i g)$$

Proof. There is a sequence $(g_k) \subset C_0^\infty(G)$ with

$$\|g - g_k\|_{1,q;G} \to 0$$

Then

$$\|g_k f - fg\|_{q;G} \leq \|f\|_{\infty;G}\|g_k - g\|_{q;G} \to 0$$

$$\begin{aligned} \|\nabla(g_k f - fg)\|_{q;G} &\leq \|(\nabla g_k - \nabla g)f\|_{q;G} + \|(g_k - g)\nabla f\|_{q;G} \\ &\leq \|f\|_{\infty;G}\|(\nabla g_k - \nabla g)\|_{q;G} + \|\nabla f\|_{\infty;G}\|(g_k - g)\|_{q;G} \end{aligned}$$

which tends to 0 for $(k \to \infty)$. By $g_k f \in H_0^{1,q}(G)$ (Lemma A.2) and the closedness of $H_0^{1,q}(G)$ in $H^{1,q}(G)$ the assertion of the Lemma follows. □

Lemma A.4. Let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with $\partial G \in C^1$. Suppose $f \in H^{1,q}(G)$, $g \in \widehat{H}_\bullet^{1,q}(G)$ and $\|g\|_{\infty;G} + \|\nabla g\|_{\infty;G} < \infty$. Then

$$f \cdot g \in \widehat{H}_\bullet^{1,q}(G) \quad \text{and} \quad \partial_i(f \cdot g) = (\partial_i f)g + f(\partial_i g)$$

Proof. (a) First let G be bounded, that is $\widehat{H}_\bullet^{1,q}(G) = H_0^{1,q}(G)$ by Lemma 2.5.

By Lemma A.1 we get $f \cdot g \in H^{1,q}(G)$ and $\partial_i(f \cdot g) = (\partial_i f)g + f(\partial_i g)$

Let an arbitrary $1 < s < \infty$ be given. Then

$$g \in L^s(G), \quad \nabla g \in L^s(G)$$

because $|G| < \infty$ and $\|g\|_{\infty;G} + \|\nabla g\|_{\infty;G} < \infty$. Therefore

$$\sup_{0 \neq \phi \in C_0^\infty(G)} \frac{\langle \nabla g, \nabla \phi \rangle}{\|\nabla \phi\|_{s';G}} \underset{\text{Hölder}}{\leq} \|\nabla g\|_{s;G} < \infty$$

By Theorem 7.9 we derive

$$g \in H_0^{1,s}(G) \qquad \forall\, 1 < s < \infty$$

(b) Now let $1 < s < q$ be fixed. Then $f \in H^{1,s}(G)$ by Hölder's inequality because G is bounded. Let $\lambda := \frac{q}{s} > 1$. Because

$$g \in H_0^{1,\frac{s\lambda}{\lambda-1}}(G)$$

there is a sequence $(g_k) \subset C_0^\infty(G)$ with

$$\|g_k - g\|_{1,\frac{s\lambda}{\lambda-1};G} \to 0$$

Because of Lemma A.2 holds

$$fg_k \in H_0^{1,s}(G)$$

and

$$\begin{aligned}
\|fg_k - fg\|_{s;G}^s \quad &= \quad \int_G |f|^s \, |g_k - g|^s \, dx \\
&\underset{\text{Hölder with } \lambda}{\leq} \quad \underbrace{\left(\int_G |f|^{s\lambda} \, dx \right)^{\frac{1}{\lambda}}}_{<\infty} \underbrace{\left(\int_G |g_k - g|^{\frac{s\lambda}{\lambda-1}} \, dx \right)^{\frac{\lambda-1}{\lambda}}}_{\to 0} \;\to\; 0
\end{aligned}$$

Further

$$\|\nabla(fg_k - fg)\|_{s;G} \;\leq\; \|(\nabla f)(g_k - g)\|_{s;G} + \|f(\nabla g_k - \nabla g)\|_{s;G}$$

and

$$\|(\nabla f)(g_k - g)\|_{s;G}^s \;\leq\; \underbrace{\left(\int_G |\nabla f|^{s\lambda} \, dx \right)^{\frac{1}{\lambda}}}_{<\infty} \underbrace{\left(\int_G |g_k - g|^{\frac{s\lambda}{\lambda-1}} \, dx \right)^{\frac{\lambda-1}{\lambda}}}_{\to 0} \;\to\; 0$$

as above. Similarly follows

$$\|f(\nabla g_k - \nabla g)\|_{s;G} \to 0$$

Altogether we derive

$$\|fg_k - fg\|_{1,s;G} \to 0$$

Therefore

$$fg \in H_0^{1,s}(G)$$

and

$$\sup_{0 \neq \phi \in C_0^\infty(G)} \frac{\langle \nabla(fg), \nabla\phi \rangle}{\|\nabla\phi\|_{q';G}} \;\leq\; \|\nabla(fg)\|_{q;G} \;<\; \infty$$

By Theorem 7.9 finally we get

$$fg \in H_0^{1,q}(G)$$

(c) Let now G be an exterior domain. Then

$$fg \in L^q(G)$$

because $f \in L^q(G)$ and g is bounded. Because $H = W$ [Me/Se] there is a sequence $(f_k) \subset C^\infty(G) \cap H^{1,q}(G)$ with

$$\|f - f_k\|_{1,q;G} \to 0$$

Let $\phi \in C_0^\infty(G)$. Then

$$\begin{aligned}
\int_G f\, g\, \partial_i \phi\, dx \quad &\underset{\text{g bounded}}{=} \quad \lim_{k\to\infty} \int_G g\, f_k\, \partial_i \phi\, dx \\
&= \lim_{k\to\infty} \int_G g\, \partial_i \underbrace{(f_k \phi)}_{\in C_0^\infty(G)}\, dx - \int_G g\, (\partial_i f_k)\, \phi\, dx \\
&= \lim_{k\to\infty} -\int_G (\partial_i g)\, f_k\, \phi\, dx - \int_G g\, (\partial_i f_k)\, \phi\, dx \\
&= -\int_G [(\partial_i f)\, g + f\, (\partial_i g)]\, \phi\, dx
\end{aligned}$$

Therefore

$$\partial_i (f \cdot g) = \underbrace{(\partial_i f)}_{\in L^q} g + \underbrace{f}_{\in L^q} (\partial_i g) \in L^q(G)$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$. Then by definition of $\widehat{H}_\bullet^{1,q}(G)$:

$$\eta g \in H_0^{1,q}(G)$$

For $K := \text{supp}(\eta)$ there is a $R > 0$ with $K \subset B_R$ and $\mathbb{R}^n \backslash G \subset B_R$. Define $U := G \cap B_R$. Then U is a bounded domain with $\partial U \in C^1$. We have

$$\eta g \Big|_U \in H_0^{1,q}(U), \qquad f\Big|_U \in H^{1,q}(U)$$

By (b) we get

$$(\eta g) f|_U \in H_0^{1,q}(U)$$

and

$$\eta(gf) \in H_0^{1,q}(G)$$

By definition of $\widehat{H}_\bullet^{1,q}(G)$ we finally get

$$fg \in \widehat{H}_\bullet^{1,q}(G)$$

□

Lemma A.5. Let $G \subset \mathbb{R}^n$ be an exterior domain. Let $u \in L^q(G \cap B_R)$ for all $R > 0$. Assume that there exist $f_i^{(R)} = \partial_i \left(u\big|_{G \cap B_R} \right) \in L^q(G \cap B_R)$ and $g_i = \partial_i \left(u\big|_{\mathbb{R}^n \backslash B_{R_1}} \right) \in L^q(\mathbb{R}^n \backslash B_{R_1})$ for $i = 1, \ldots, n$ (with $\mathbb{R}^n \backslash B_{R_1} \subset G$). Then

$$\nabla u \in L^q(G)$$

Proof. For $R' > R$ holds

$$f_i^{(R')} = f_i^{(R)} \text{ a.e. in } G \cap B_R$$

So we are able to define

$$f_i(x) := f_i^{(R)}(x) \text{ a.e. for } x \in G \cap B_R$$

Then

$$f_i \in L^q(G \cap B_R) \qquad \forall\, R > 0$$

Let $\phi \in C_0^\infty(\mathbb{R}^n \backslash B_{R_1})$ with $\mathrm{supp}(\phi) \subset G \cap B_R$ Then

$$\begin{aligned} \langle g_i, \phi \rangle_{\mathbb{R}^n \backslash B_{R_1}} &= -\langle u, \partial_i \phi \rangle_{\mathbb{R}^n \backslash B_{R_1}} = -\langle u, \partial_i \phi \rangle_{G \cap B_R} \\ &= -\langle f_i, \phi \rangle_{G \cap B_R} = \langle f_i, \phi \rangle_{\mathbb{R}^n \backslash B_{R_1}} \end{aligned}$$

Therefore

$$f_i = g_i \quad \text{a.e. in } L^q(\mathbb{R}^n \backslash B_{R_1})$$

and

$$\partial_i u = f_i \in L^q(G)$$

□

Lemma A.6. For $n \geq 2$ let $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$, $f(x) := \frac{x}{|x|^2}$. Then $f^{-1} = f$ and

$$\det f'(x) = -\frac{1}{|x|^{2n}}$$

Proof. due to C.G.Simader

Obviously $f^2 = \mathrm{id}$ and

$$\partial_j f_i(x) = \partial_j \left(\frac{x_i}{|x|^2} \right) = \frac{|x|^2 \delta_{ij} - 2x_i x_j}{|x|^4}$$

So

$$\det f'(x) = \frac{(-2)^n}{|x|^{4n}} \begin{vmatrix} x_1^2 - \frac{|x|^2}{2} & x_1 x_2 & \cdots & x_1 x_n \\ x_1 x_2 & x_2^2 - \frac{|x|^2}{2} & & \\ \vdots & & \ddots & \\ x_1 x_n & \cdots & & x_n^2 - \frac{|x|^2}{2} \end{vmatrix}$$

We now prove that

$$\begin{vmatrix} x_1^2 - \frac{|x|^2}{2} & x_1 x_2 & \cdots & x_1 x_n \\ x_1 x_2 & x_2^2 - \frac{|x|^2}{2} & & \\ \vdots & & \ddots & \\ x_1 x_n & \cdots & & x_n^2 - \frac{|x|^2}{2} \end{vmatrix} = -\frac{|x|^{2n}}{(-2)^n}$$

$\underline{\underline{n = 2:}}$ The case $n = 2$ can be derived by simple calculation

$\underline{\underline{n \to n+1:}}$ Let the assumption be true for a $n \in \mathbb{N}$, $n \geq 2$

For $x_1 = 0$ we have $|x| = |(x_2, \dots, x_{n+1})|$ and by assumption

$$\begin{vmatrix} x_1^2 - \frac{|x|^2}{2} & x_1x_2 & \cdots & x_1x_n \\ x_1x_2 & x_2^2 - \frac{|x|^2}{2} & & \\ \vdots & & \ddots & \\ x_1x_n & \cdots & & x_n^2 - \frac{|x|^2}{2} \end{vmatrix} = \begin{vmatrix} -\frac{|x|^2}{2} & 0 & \cdots & 0 \\ 0 & x_2^2 - \frac{|x|^2}{2} & & \\ \vdots & & \ddots & \\ 0 & \cdots & & x_n^2 - \frac{|x|^2}{2} \end{vmatrix}$$

$$= -\frac{|x|^2}{2}\left(-\frac{|x|^{2n}}{(-2)^n}\right) = -\frac{|x|^{2(n+1)}}{(-2)^{n+1}}$$

For $x_1 \neq 0$

$$\begin{vmatrix} x_1^2 - \frac{|x|^2}{2} & x_1x_2 & \cdots & x_1x_{n+1} \\ x_1x_2 & x_2^2 - \frac{|x|^2}{2} & & \\ \vdots & & \ddots & \\ x_1x_{n+1} & \cdots & & x_{n+1}^2 - \frac{|x|^2}{2} \end{vmatrix} =$$

$$= x_1 \begin{vmatrix} x_1 - \frac{|x|^2}{2x_1} & x_1x_2 & \cdots & x_1x_{n+1} \\ x_2 & x_2^2 - \frac{|x|^2}{2} & & \\ \vdots & & \ddots & \\ x_{n+1} & x_2x_{n+1} & \cdots & x_{n+1}^2 - \frac{|x|^2}{2} \end{vmatrix}$$

by subtracting the first column multiplied by x_j of the j-th column:

$$= x_1 \begin{vmatrix} x_1 - \frac{|x|^2}{2x_1} & \frac{x_2|x|^2}{2x_1} & \frac{x_3|x|^2}{2x_1} & \cdots & \frac{x_{n+1}|x|^2}{2x_1} \\ x_2 & -\frac{|x|^2}{2} & 0 & & 0 \\ \vdots & 0 & -\frac{|x|^2}{2} & & \vdots \\ & \vdots & & \ddots & 0 \\ x_{n+1} & 0 & 0 & \cdots & -\frac{|x|^2}{2} \end{vmatrix}$$

by multiplication of x_1 to the first row:

$$= \begin{vmatrix} x_1^2 - \frac{|x|^2}{2} & \frac{x_2|x|^2}{2} & \frac{x_3|x|^2}{2} & \cdots & \frac{x_{n+1}|x|^2}{2} \\ x_2 & -\frac{|x|^2}{2} & 0 & & 0 \\ \vdots & 0 & -\frac{|x|^2}{2} & & \vdots \\ & \vdots & & \ddots & 0 \\ x_{n+1} & 0 & 0 & \cdots & -\frac{|x|^2}{2} \end{vmatrix}$$

by dividing the last columns by $\frac{|x|^2}{2}$:

$$= \frac{|x|^{2n}}{2^n} \begin{vmatrix} x_1^2 - \frac{|x|^2}{2} & x_2 & x_3 & \cdots & x_{n+1} \\ x_2 & -1 & 0 & & 0 \\ \vdots & 0 & -1 & & \vdots \\ & \vdots & & \ddots & 0 \\ x_{n+1} & 0 & 0 & \cdots & -1 \end{vmatrix}$$

by adding the j-th column multiplied by x_j to the first column for $(j = 2, \dots, n)$:

$$= \frac{|x|^{2n}}{2^n} \begin{vmatrix} x_1^2 + x_2^2 + \ldots + x_{n+1}^2 - \frac{|x|^2}{2} & x_2 & x_3 & \cdots & x_{n+1} \\ 0 & -1 & 0 & & 0 \\ \vdots & 0 & -1 & & \vdots \\ & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

$$= \frac{|x|^{2n}}{2^n} \frac{|x|^2}{2} (-1)^n = -\frac{|x|^{2(n+1)}}{(-2)^{n+1}}$$

□

Lemma A.7. Let $G \subset \mathbb{R}^n$ be an exterior domain with $\mathbb{R}^n \backslash G \subset B_{\frac{R}{2}}$. Suppose $1 < q < \infty$. Assume $u \in \widehat{H}_{\bullet}^{2,q}(G)$ and

$$\rho \in C^\infty(\mathbb{R}^n), \quad 0 \le \rho \le 1, \quad \rho(x) = \begin{cases} 0 & \text{,if } |x| \le R \\ 1 & \text{,if } |x| \ge 2R \end{cases}$$

Then

$$\rho u \in \widehat{H}_{\bullet}^{2,q}(G)$$

and

$$\partial_i(\rho u) = (\partial_i \rho) u + \rho(\partial_i u)$$

$$\partial_j \partial_i(\rho u) = (\partial_j \partial_i \rho) u + (\partial_i \rho)(\partial_j u) + (\partial_j \rho)(\partial_i u) + \rho(\partial_j \partial_i u)$$

Proof. (a)

$$\|\rho u\|_{q;G\cap B_r} \le \|u\|_{q;G\cap B_r} < \infty \qquad \forall\, r > 0$$

(b) Let $\phi \in C_0^\infty(G)$. Then

$$\begin{aligned} \int_G \rho\, u\, \partial_i \phi\, dx &= \int_G u\, \partial_i \underbrace{(\rho\phi)}_{\in C_0^\infty(G)}\, dx - \int_G u\, (\partial_i \rho)\, \phi\, dx \\ &= -\int_G (\partial_i u)\, \rho\, \phi\, dx - \int_G u\, (\partial_i \rho)\, \phi\, dx \end{aligned}$$

Therefore

$$\partial_i(\rho u) = (\partial_i \rho) u + \rho(\partial_i u) \in L^q(G \cap B_r) \qquad \forall\, r > 0$$

(c) Let $\phi \in C_0^\infty(G)$. Then

$$\begin{aligned}
\int_G \rho\, u\, (\partial_j \partial_i \phi)\, dx &\overset{\text{(b)}}{=} -\int_G [\partial_i(\rho\, u)]\, \partial_j \phi\, dx \\
&\overset{\text{(b)}}{=} -\int_G (\partial_i u)\, \rho\, (\partial_j \phi)\, dx - \int_G u\, (\partial_i \rho)\, (\partial_j \phi)\, dx \\
&= -\int_G (\partial_i u)\, \partial_j \underbrace{(\rho \phi)}_{\in C_0^\infty(G)}\, dx + \int_G (\partial_i u)\, (\partial_j \rho)\, \phi\, dx \\
&\quad - \int_G u\, \partial_j \underbrace{[(\partial_i \rho) \phi]}_{\in C_0^\infty(G)}\, dx + \int_G u\, (\partial_j \partial_i \rho)\, \phi\, dx \\
&= \int_G \left[(\partial_j \partial_i \rho) u + (\partial_i \rho)(\partial_j u) + (\partial_j \rho)(\partial_i u) + \rho(\partial_j \partial_i u) \right] \phi\, dx
\end{aligned}$$

Therefore

$$\partial_j \partial_i(\rho u) = (\partial_j \partial_i \rho) u + (\partial_i \rho)(\partial_j u) + (\partial_j \rho)(\partial_i u) + \rho(\partial_j \partial_i u) \in L^q(G)$$

(d) For $\eta \in C_0^\infty(\mathbb{R}^n)$ we have $\eta\rho \in C_0^\infty(G)$ and therefore by definition of $\widehat{H}_\bullet^{2,q}(G)$

$$\eta(\rho u) = (\eta \rho) u \in H_0^{2,q}(G)$$

So altogether

$$\rho u \in \widehat{H}_\bullet^{2,q}(G)$$

□

Definition A.8 (Friedrichs' mollifier).

(i) Let $j \in C_0^\infty(\mathbb{R}^n)$, $j \geq 0$, $j(x) = 0$ for $|x| \geq 1$, $j(x) = j(-x)$ and $\int\limits_{\mathbb{R}^n} j(x)\, dx = 1$. For $\varepsilon > 0$ let $j_\varepsilon(x) := \varepsilon^{-n} j\left(\frac{x}{\varepsilon}\right)$. For a suitable $\tilde{j} \in C_0^\infty(\mathbb{R})$ we also assume $j(x) = \tilde{j}(|x|)$ for each $x \in \mathbb{R}^n$

(ii) Let $1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ be open and $f \in L^q(G)$. Let

$$g(x) := \begin{cases} f(x) & , \quad x \in G \\ 0 & , \quad \text{else} \end{cases}$$

Then we denote by

$$f_\varepsilon(x) := \int_{\mathbb{R}^n} j_\varepsilon(x - y) g(y)\, dy \qquad \text{for } x \in \mathbb{R}^n, \text{ and } \varepsilon > 0$$

Friedrichs' mollifier.

We could choose for example

$$\tilde{j}(t) := \begin{cases} c\, e^{-\frac{1}{1-t^2}} & , \quad |t| < 1 \\ 0 & , \quad |t| \geq 1 \end{cases}$$

with a suitable $c \in \mathbb{R}$

Lemma A.9. Let $1 < q < \infty$ and $f \in L^q(\mathbb{R}^n)$, $g \in L^{q'}(\mathbb{R}^n)$. Then

$$\langle f, g_\varepsilon \rangle = \langle f_\varepsilon, g \rangle \qquad \forall \varepsilon > 0$$

Proof. By Fubini's theorem

$$\langle f_\varepsilon, g \rangle = \int_{\mathbb{R}^n} g(x) \int_{\mathbb{R}^n} j_\varepsilon(x-y) f(y)\, dy\, dx = \langle f, g_\varepsilon \rangle$$

□

Lemma A.10. Let $1 \leq q < \infty$ and let $G \subset \mathbb{R}^n$ be open and $f \in L^q(G)$ with $\nabla f \in L^q(G)$. Let $G' \subset G$ such that $d = \operatorname{dist}(G', \partial G) > 0$. Let $0 < \varepsilon < d$. Then

$$(\partial_i f)_\varepsilon(x) = \Big[\partial_i(f_\varepsilon)\Big](x) \qquad \forall x \in G'$$

Proof. Let

$$g(x) := \begin{cases} f(x) & , \quad x \in G \\ 0 & , \quad \text{else} \end{cases}$$

Then

$$\begin{aligned} \Big[\partial_i(f_\varepsilon)\Big](x) &= \int_{B_\varepsilon(x)} \partial_{x_i} j_\varepsilon(x-y) g(y) dy \\ &= \int_{B_\varepsilon(x)} \partial_{x_i} j_\varepsilon(x-y) f(y) dy \\ &= -\int_{B_\varepsilon(x)} \partial_{y_i} j_\varepsilon(x-y) f(y) dy \\ &= \int_{B_\varepsilon(x)} j_\varepsilon(x-y) \partial_{y_i} f(y) dy \\ &= (\partial_i f)_\varepsilon(x) \end{aligned}$$

□

Lemma A.11. Let $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be open. Suppose $f \in L^q(G)$. Then

1. $\|f_\varepsilon\|_{q;\mathbb{R}^n} \leq \|f\|_{q;G} \qquad \forall\, \varepsilon > 0$
2. $\|f_\varepsilon - f\|_{q;G} \to 0 \quad (\varepsilon \to 0)$

Proof. see [SiDGL, Satz 2.5] □

Lemma A.12. Let $G \subset \mathbb{R}^n$ be open. Assume that $u : G \to \mathbb{R}$ is a harmonic function. Let $G_1 \subset G$ with $d := \operatorname{dist}(G_1, \partial G) > 0$. Then

$$u_\varepsilon(x) = u(x)$$

for each $x \in G_1$ and for all $0 < \varepsilon < d$

Proof.

$$\begin{aligned}
u_\varepsilon(x) &= \int_{B_\varepsilon(x)} j_\varepsilon(x-z)u(z)\,dz \;=\; \int_{B_\varepsilon(x)} j_\varepsilon(z-x)u(z)\,dz \\
&\underset{z=x+y}{=} \int_{B_\varepsilon(0)} j_\varepsilon(y)u(x+y)\,dy \\
&= \int_0^\varepsilon r^{n-1}\,\varepsilon^{-n}\,\tilde{j}\left(\frac{r}{\varepsilon}\right) \int_{S_{n-1}} u(x+r\xi)\,d\omega_\xi\,dr \\
&\underset{\text{mvp}}{=} \int_0^\varepsilon r^{n-1}\,\varepsilon^{-n}\,\tilde{j}\left(\frac{r}{\varepsilon}\right)\,dr\,\omega_n\,u(x) \\
&= u(x)\int_0^\varepsilon r^{n-1}\int_{S_{n-1}} j_\varepsilon(r\xi)\,d\omega_\xi\,dr \\
&= u(x)\int_{B_\varepsilon(x)} j_\varepsilon(z)\,dz \;=\; u(x)
\end{aligned}$$

□

Lemma A.13. Let $1 < q < \infty$ and let $G, V \subset \mathbb{R}^n$ be open. Suppose $f \in H_0^{1,q}(G)$ and $\varphi \in C_0^\infty(V)$. Then

$$\varphi\cdot f \in H_0^{1,q}(G \cap V)$$

Proof. By Lemma A.1 we get

$$f\cdot\varphi \in H^{1,q}(G) \quad \text{and} \quad \partial_i(f\cdot\varphi) = (\partial_i f)\varphi + f(\partial_i\varphi)$$

Let $f_k \in C_0^\infty(G)$ with

$$\|f_k - f\|_{1,q;G} \to 0 \quad (k \to \infty)$$

Define

$$g_k := \varphi f_k \in C_0^\infty(G \cap V)$$

Then

$$\|f_k\varphi - f\varphi\|_{q;G\cap V} \;\leq\; \|\varphi\|_\infty \|f_k - f\|_{q;G} \to 0$$

$$\begin{aligned}
\|\nabla(f_k\varphi - f\varphi)\|_{q;G\cap V} &\leq \|(\nabla f_k - \nabla f)\varphi\|_{q;G\cap V} + \|(f_k - f)\nabla\varphi\|_{q;G\cap V} \\
&\leq \|\varphi\|_\infty \|(\nabla f_k - \nabla f)\|_{q;G} + \|\nabla\varphi\|_\infty \|(f_k - f)\|_{q;G}
\end{aligned}$$

which tends to 0 for $(k \to \infty)$. □

Lemma A.14. Let $1 < q < \infty$ and let $G, G' \subset \mathbb{R}^n$ be open and bounded. Assume that $\phi : G \to G'$ is a C^1-diffeomorphism with $\phi \in \overline{C}^1(G;\mathbb{R}^n)$ and $\phi^{-1} \in \overline{C}^1(G';\mathbb{R}^n)$.

1. Suppose $g \in C_0^0(G)$. Then $h := g \circ \phi^{-1} \in C_0^0(G')$.
2. Suppose $g \in C^1(G) \cap H_0^{1,q}(G)$. Then $h := g \circ \phi^{-1} \in C^1(G') \cap H_0^{1,q}(G')$.

Proof. (a) Let $g \in C_0^0(G)$. Let $x \in G'$ with $h(x) \neq 0$. Then $g\left(\phi^{-1}(x)\right) \neq 0$. That is $\phi^{-1}(x) \in \operatorname{supp}(g)$. We get $x \in \phi\left(\operatorname{supp}(g)\right)$. Therefore

$$\left\{x \in G' : h(x) \neq 0\right\} \subset \phi\left(\operatorname{supp}(g)\right)$$

and

$$\operatorname{supp}(h) \subset \phi\left(\operatorname{supp}(g)\right) \subset \phi(G) = G'$$

(b) Let $g \in C^1(G) \cap H_0^{1,q}(G)$. Obviously $h \in C^1(G')$ and

$$(\partial_i h)(x) = \sum_{k=1}^{n} (\partial_k g)(\psi(x))(\partial_i \psi_k)(x)$$

where $\psi := \phi^{-1}$. Then

$$\int_{G'} |h(x)|^q \, dx \;=\; \int_G |g(y)|^q \underbrace{|\det\phi'(y)|}_{\text{bounded}} \, dy \;<\infty$$

and

$$\int_{G'} |\partial_i h(x)|^q \, dx \;=\; \int_G \Big| \sum_{k=1}^{n} (\partial_k g)(y) \underbrace{(\partial_i \psi_k)(\phi(y))}_{\text{bounded}} \Big|^q \underbrace{|\det\phi'(y)|}_{\text{bounded}} \, dy \;<\infty$$

Therefore

$$h \in H^{1,q}(G')$$

Let $g_k \in C_0^\infty(G)$ with

$$\|g_k - g\|_{1,q;G} \to 0 \quad (k \to \infty)$$

Define

$$h_k := g_k \circ \phi^{-1} \in C_0^1(G') \subset H_0^{1,q}(G')$$

We derive

$$\int_{G'} |h(x) - h_k(x)|^q \, dx \;=\; \int_G |g(y) - g_k(y)|^q \underbrace{|\det\phi'(y)|}_{\text{bounded}} \, dy \;\to 0$$

and

$$\int_{G'} |\partial_i h(x) - \partial_i h_k(x)|^q \, dx =$$

$$= \int_G \Big| \sum_{l=1}^{n} \Big[(\partial_l g)(y) - (\partial_l g_k)(y) \Big] \underbrace{(\partial_i \psi_l)(\phi(y))}_{\text{bounded}} \Big|^q \underbrace{|\det\phi'(y)|}_{\text{bounded}} \, dy \;\to 0$$

□

Lemma A.15. Let $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be an exterior domain. Then there is a constant $C = C(n,q,G) > 0$ such that for $u \in \widehat{H}^{1,q}_\bullet(G)$ and $R > 0$ holds

$$\|u\|_{q;G\cap B_R} \;\le\; C\,R^{1+\frac{n-1}{q}}\;\|\nabla u\|_{q;G\cap B_R}$$

Proof. (a) Let $\delta > 0$ with $0 \in B_\delta \subset \mathbb{R}^n \backslash \overline{G}$. Let $u \in C_0^\infty(G)$, $0 < r < R$ and $\xi \in S_{n-1}$. Then

$$u(r\xi) \;=\; u(r\xi) - u(\delta) \;=\; \int_\delta^r \;\nabla u(t\xi)\cdot\xi\,dt$$

We derive

$$|u(r\xi)|^q \;\le\; r^{q-1}\int_\delta^r \;|\nabla u(t\xi)|^q\;dt \;\le\; R^{q-1}\int_\delta^R \;|\nabla u(t\xi)|^q\;dt$$

Further holds

$$\begin{aligned}\int_{S_{n-1}} |u(r\xi)|^q\,d\omega_\xi \;&\le\; R^{q-1}\int_\delta^R \frac{t^{n-1}}{t^{n-1}}\int_{S_{n-1}} |\nabla u(t\xi)|^q\;d\omega_\xi\,dt \\ &\le\; R^{q-1}\delta^{1-n}\underbrace{\int_\delta^R \;t^{n-1}\int_{S_{n-1}} |\nabla u(t\xi)|^q\;d\omega_\xi\,dt}_{=\|\nabla u\|^q_{q;G\cap B_R}}\end{aligned}$$

Finally

$$\|u\|^q_{q;G\cap B_R} \;=\; \int_0^R \;r^{n-1}\int_{S_{n-1}} |u(r\xi)|^q\,d\omega_\xi\,dr \;\le\; \frac{1}{n}\,R^{n+q-1}\,\delta^{1-n}\,\|\nabla u\|^q_{q;G\cap B_R}$$

Therefore the assertion holds for $u \in C_0^\infty(G)$

(b) Let now $u \in \widehat{H}^{1,q}_\bullet(G)$. Choose $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta(x) = 1$ for $|x| < R$. Then $\eta u \in H_0^{1,q}(G)$ and there is a sequence $(u_k) \subset C_0^\infty(G)$ such that

$$\|\eta u - u_k\|_{1,q;G} \to 0$$

Especially

$$\|u - u_k\|_{1,q;G\cap B_R} \;=\; \|\eta u - u_k\|_{1,q;G\cap B_R} \;\le\; \|\eta u - u_k\|_{1,q;G} \to 0$$

and the assertion follows by (a). □

Lemma A.16. Let $G \subset \mathbb{R}^n$ be open. Assume that $f_k \in C^1(G)$ $(k \in \mathbb{N})$ and $f : G \to \mathbb{R}$ such that $f_k(x) \to f(x)$ for every $x \in G$ and that $(\partial_i f_k)$ is uniformly Cauchy in G for all $i = 1,\dots,n$.

Then $f \in C^1(G)$ and

$$\partial_i f = \lim_{k\to\infty} \partial_i f_k$$

Proof. Denote

$$f_i := \lim_{k\to\infty} \partial_i f_k$$

Then f_i is continuous. For $x \in G$, $\rho > 0$, $B_\rho(x) \subset G$, $h \in \mathbb{R}$ and $|h| < \rho$ holds

$$f_k(x + he_i) - f_k(x) \;=\; \int_0^h (\partial_i f_k)(x + te_i)\, dt$$

For $k \to \infty$ we derive

$$f(x + he_i) - f(x) \;=\; \int_0^h f_i(x + te_i)\, dt \;=\; h\, f_i(x + t_h e_i)$$

with t_h between 0 and h. Therefore

$$\lim_{0\neq h\to 0} \frac{f(x + he_i) - f(x)}{h} \;=\; f_i(x)$$

□

B Spectral theory of compact operators in real Banach spaces

All proofs of this section are due to [Alt].

Theorem B.1. Let X be a normed real vector space and let $Y \subset X$ be a closed subspace with $Y \neq X$. Then for every $0 < \theta < 1$ there is $x_\theta \in X$ such that

$$\|x_\theta\| = 1 \quad \text{and} \quad \theta \leq \operatorname{dist}(x_\theta, Y)$$

Proof. see [Alt, Satz 2.4, p.87] □

Lemma B.2. Let X be a normed real vector space and let $Y \subset X$ be finite-dimensional. Then Y is a closed subspace of X.

Proof. see [Alt, Lemma 2.8, p.91] □

Theorem B.3. Let X be a normed real vector space and let $\overline{B_1(0)} \subset X$ be compact. Then

$$\dim X < \infty$$

Proof. see [Alt, Satz 2.9, p.92] □

Definition and Theorem B.4. Let X and Y be real Banach spaces. Then a bounded linear operator $T : X \to Y$ is **compact**, if one of the following equivalent properties is fulfilled:

1. $\overline{T(B_1(0))}$ is compact in Y.
2. For every bounded sequence $(x_n) \subset X$ there exists a subsequence (x_{n_k}) such that $(Tx_{n_k}) \subset Y$ is convergent.

Proof. see [Alt, Definition 8.1, p.301] □

Lemma B.5. Let X and Y be real Banach spaces and let X be reflexive. Then a linear operator $T : X \to Y$ is compact, if and only if for every sequence $(x_n) \subset X$ with

$$x_n \overset{\text{weak}}{\longrightarrow} x \in X \qquad (n \to \infty)$$

holds

$$\|Tx_n - Tx\|_Y \to 0 \qquad (n \to \infty)$$

Proof. see [Alt, Lemma 8.2, p.302] □

Lemma B.6. Let X, Y and Z be real Banach spaces. Let $T_1 : X \to Y$ and $T_2 : Y \to Z$ be bounded linear operators. Assume that either T_1 or T_2 is compact. Then

$$T_2 T_1 \text{ is compact.}$$

Proof. (a) Let $(x_n) \subset X$ be a bounded sequence.

(b) Assume that T_1 is compact. Then there is a subsequence (x_{n_k}) such that $(T_1 x_{n_k}) \subset Y$ is convergent. Because T_2 is bounded, $(T_2 T_1 x_{n_k}) \subset Z$ is convergent too.

(c) Assume that T_2 is compact. Then $(T_1 x_{n_k}) \subset Y$ is a bounded sequence and therefore by Defintion B.4 there exists a subsequence $(T_1 x_{n_k})$ such that $(T_2 T_1 x_{n_k}) \subset Z$ is convergent. □

Definition B.7. Let X be a real Banach space and let $T : X \to X$ be a bounded linear operator. Then we define by

1. $N(T) := \{x \in X \,:\, Tx = 0\}$ the **nullspace** of T,
2. $R(T) := \{y \in X \,:\, \text{there is an } x \in X \text{ with } Tx = y\}$ the **range** of T,
3. $\rho^{(\mathbb{R})}(T) := \{\lambda \in \mathbb{R} \,:\, N(\lambda I - T) = \{0\} \text{ and } R(\lambda I - T) = X\}$ the **real resolvent set** of T,
4. $\sigma^{(\mathbb{R})}(T) := \mathbb{R} \backslash \rho^{(\mathbb{R})}(T)$ the **real spectrum** of T,
5. $\sigma_p^{(\mathbb{R})}(T) := \left\{\lambda \in \sigma^{(\mathbb{R})}(T) \,:\, N(\lambda I - T) \neq \{0\}\right\}$ the **set of eigenvalues** of T.

Theorem B.8. Let X be a real Banach space and let $T : X \to X$ be a compact operator. Let $A := I - T$. Then

1. $\dim N(A) < \infty$,
2. $R(A)$ is a closed subspace of X,
3. if $N(A) = \{0\}$, then $R(A) = X$.

Proof. (a) Let $x \in N(A)$ with $\|x\| < 1$. Then $\|Tx\| = \|x\| < 1$. Therefore

$$B_1(0) \cap N(A) \subset T(B_1(0))$$

and

$$\overline{B_1(0) \cap N(A)} \subset \overline{T(B_1(0))}$$

Because T is compact, $\overline{T(B_1(0))}$ is compact, and because $\overline{B_1(0) \cap N(A)}$ is closed, we derive that $\overline{B_1(0) \cap N(A)}$ is compact too. By Theorem B.3 holds

$$\dim N(A) < \infty$$

(b) Let $x \in \overline{R(A)}$. Then there are $\tilde{x}_n \in X$ with $A\tilde{x}_n \to x$. Choose $a_n \in N(A)$ such that

$$\|\tilde{x}_n - a_n\| \ \leq \ 2\,\mathrm{dist}(\tilde{x}_n, N(A))$$

Let $x_n := \tilde{x}_n - a_n$. Then $\mathrm{dist}(\tilde{x}_n, N(A)) = \mathrm{dist}(x_n, N(A))$, $A\tilde{x}_n = Ax_n$. Altogether holds

$$x_n \in X, \qquad Ax_n \to x, \qquad \|x_n\| \ \leq \ 2\,d_n \ := \ 2\,\mathrm{dist}(x_n, N(A))$$

Assume for contradiction that (d_n) is not bounded. Then there is a subsequence (d_{n_k}) with $0 < d_{n_k} \to \infty$ for $k \to \infty$. Let

$$y_k := \frac{x_{n_k}}{d_{n_k}}$$

Then

$$Ay_k = \frac{Ax_{n_k}}{d_{n_k}} \to 0$$

Because (y_k) is bounded and T is compact, there is a subsequence (y_{k_l}) with $Ty_{k_l} \to y$ for $l \to \infty$. We derive

$$y_{k_l} \ = \ Ay_{k_l} + Ty_{k_l} \ \to y$$

and by the continuity of A

$$Ay \ = \ \lim_{l \to \infty} Ay_{k_l} \ = \ 0$$

Therefore $y \in N(A)$ and

$$\|y_{k_l} - y\| \ \geq \ \mathrm{dist}(y_{k_l}, N(A)) \ = \ \mathrm{dist}\left(\frac{x_{n_{k_l}}}{d_{n_{k_l}}}, N(A)\right) \ = \ \frac{\mathrm{dist}\left(x_{n_{k_l}}, N(A)\right)}{d_{n_{k_l}}} \ = \ 1$$

which is a contradiction. So (d_n) is bounded, and therefore (x_n) is bounded too. Then there is a subsequence (x_{n_k}) such that

$$Tx_{n_k} \to z \qquad (k \to \infty)$$

Therefore

$$x \ \leftarrow \ Ax_{n_k} \ = \ A\left(Ax_{n_k} + Tx_{n_k}\right) \ \to \ A(x+z)$$

and

$$x \ = \ A(x+z) \in R(A)$$

(c) Let $N(A) = \{0\}$. Assume for contradiction that there is $x \in X \backslash R(A)$.

Assume $A^n x \in R(A^{n+1})$ for $n \geq 0$. Then there would be $y \in X$ with $A^n x = A^{n+1} y$, that is $A^n(x - Ay) = 0$. By $N(A) = \{0\}$ we would derive $x - Ay = 0$, i.e. $x \in R(A)$, which is a contradiction. Therefore

$$A^n x \in R(A^n) \backslash R(A^{n+1}) \qquad \forall\, n \geq 0$$

Further

$$A^{n+1} \ = \ (I - T)^{n+1} \ = \ I + \underbrace{\sum_{k=1}^{n+1} \binom{n+1}{k} (-T)^k}_{\text{compact by Lemma B.6}}$$

By (b) $R(A^{n+1})$ is closed. So there is $a_{n+1} \in R(A^{n+1})$ such that

$$0 \; < \; \|A^n x - a_{n+1}\| \; \leq \; 2 \operatorname{dist}\big(A^n x, R(A^{n+1})\big)$$

Consider

$$x_n \; := \; \frac{A^n x - a_{n+1}}{\|A^n x - a_{n+1}\|} \; \in R(A^n)$$

For $y \in R(A^{n+1})$ holds

$$\begin{aligned} \|x_n - y\| \; &= \; \frac{\|A^n x - (a_{n+1} + \|A^n x - a_{n+1}\| y)\|}{\|A^n x - a_{n+1}\|} \\ &\geq \; \frac{\operatorname{dist}\big(A^n x, R(A^{n+1})\big)}{\|A^n x - a_{n+1}\|} \; \geq \; \frac{1}{2} \end{aligned}$$

For $m > n$ we derive

$$\|Tx_n - Tx_m\| \; = \; \|x_n - \underbrace{(Ax_n + x_m - Ax_m)}_{\in R(A^{n+1})}\| \; \geq \; \frac{1}{2}$$

On the other hand (x_n) is bounded and T is compact, so there is a subsequence (x_{n_k}) such that

$$\|Tx_{n_k} - Tx_{n_l}\| \to 0 \qquad (k, l \to \infty)$$

which is a contradiction. □

Theorem B.9. Let X be a real Banach space and let $T : X \to X$ be a compact operator. Then

1. $\sigma^{(\mathbb{R})}(T) \backslash \{0\} \subset \sigma_p^{(\mathbb{R})}(T)$,
2. $\sigma^{(\mathbb{R})}(T)$ is finite or countably infinite,
3. $\sigma^{(\mathbb{R})}(T)$ is bounded,
4. if $\sigma^{(\mathbb{R})}(T)$ is countably infinite, for every sequence $(\lambda_k) \subset \sigma^{(\mathbb{R})}(T) \backslash \{0\}$ with $\lambda_k \neq \lambda_l$ $(k \neq l)$ holds $\lambda_k \to 0$ for $k \to \infty$,
5. $\dim N(\lambda I - T) \; < \infty$ for every $\lambda \in \mathbb{R} \backslash \{0\}$.

Proof. (a) Let $\lambda \in \mathbb{R} \backslash \{0\}$. Then by Theorem B.8 $\dim N(I - \frac{T}{\lambda}) < \infty$ and therefore

$$\dim N(\lambda I - T) \; < \infty$$

(b) Let $0 \neq \lambda \notin \sigma_p^{(\mathbb{R})}(T)$. Then $\dim N(I - \frac{T}{\lambda}) = \{0\}$ and by Theorem B.8 holds $N(I - \frac{T}{\lambda}) = X$, i.e. $\lambda \in \rho^{(\mathbb{R})}(T)$. This proves

$$\sigma^{(\mathbb{R})}(T) \backslash \{0\} \subset \sigma_p^{(\mathbb{R})}(T)$$

(c) Let $(\lambda_k) \subset \sigma^{(\mathbb{R})}(T)\backslash\{0\}$ with $\lambda_k \neq \lambda_l$ $(k \neq l)$. By (b) there are eigenvectors $e_n \in X\backslash\{0\}$ such that $Te_n = \lambda_n e_n$. Define

$$X_n := \operatorname{span}(e_1, \ldots, e_n)$$

Assume that $e_1, \ldots, e_{n-1}$ are linearly independent. Let $a_k \in \mathbb{R}$ such that

$$\sum_{k=1}^{n} a_k e_k = 0$$

If $a_n = 0$ by assumption holds $a_1 = \ldots = a_{n-1} = 0$. If $a_n \neq 0$ we derive

$$\begin{aligned} 0 &= Te_n - \lambda_n e_n = \frac{1}{a_n}(\lambda_n - T)\sum_{k=1}^{n-1} a_k e_k \\ &= \sum_{k=1}^{n-1} \frac{a_k}{a_n} \underbrace{(\lambda_n - \lambda_k)}_{\neq 0} e_k \end{aligned}$$

and therefore $a_1 = \ldots = a_{n-1} = 0$. So $a_n = 0$, which is a contradiction. By induction follows

$$\dim X_n = n \qquad \forall\, n \in \mathbb{N}$$

By Theorem B.1 and Theorem B.2 there are $x_n \in X_n$ such that

$$\|x_n\| = 1 \quad \text{and} \quad \frac{1}{2} \leq \operatorname{dist}(x_n, X_{n-1})$$

Further

$$(T - \lambda_n)x_n = (T - \lambda_n)\sum_{k=1}^{n} a_{nk} e_k = \sum_{k=1}^{n-1} a_{nk}(\lambda_k - \lambda_n)e_k \in X_{n-1} \qquad \forall\, n \in \mathbb{N}$$

and

$$Tx_n = \sum_{k=1}^{n} a_{nk}\, \lambda_k\, e_k \in X_n \qquad \forall\, n \in \mathbb{N}$$

Therefore for $m < n$ holds

$$\|T(\frac{x_n}{\lambda_n}) - T(\frac{x_m}{\lambda_m})\| = \|x_n + \underbrace{\frac{1}{\lambda_n}(Tx_n - \lambda_n x_n) - \frac{1}{\lambda_m}Tx_m}_{\in X_{n-1}}\| \geq \frac{1}{2}$$

Because T is compact, no subsequence of $(\frac{x_n}{\lambda_n})$ is bounded. Therefore

$$\frac{1}{|\lambda_n|} = \|\frac{x_n}{\lambda_n}\| \to \infty \qquad (n \to \infty)$$

and

$$\lambda_n \to 0 \qquad (n \to \infty)$$

(d) By (c) we know that $\sigma^{(\mathbb{R})}(T) \setminus [-r, r]$ is finite for every $r > 0$. Therefore $\sigma^{(\mathbb{R})}(T)$ is bounded and

$$\sigma^{(\mathbb{R})}(T)\backslash\{0\} = \bigcup_{n\in\mathbb{N}} \sigma^{(\mathbb{R})}(T) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right]$$

So $\sigma^{(\mathbb{R})}(T)$ is finite or countably infinite. $\square$

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