Additive codes attaining the Griesmer bound

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December 19, 2024

Abstract

Additive codes may have better parameters than linear codes. However, still very few cases are known and the explicit construction of such codes is a challenging problem. Here we show that a Griesmer type bound for the length of additive codes can always be attained with equality if the minimum distance is sufficiently large. This solves the problem for the optimal parameters of additive codes when the minimum distance is large and yields many infinite series of additive codes that outperform linear codes.

Keywords: additive codes · linear codes · Griesmer bound · Galois geometry

I am grateful to Simeon Ball for feedback on earlier drafts and generously sharing his insights on additive codes. Further thanks go to Ferdinand Ihringer.

1 Introduction

For a finite set $\mathcal{A}$, called *alphabet*, a code C of *length* n and *minimum distance* d is a subset of $\mathcal{A}^n$ such that any two elements, called *codewords*, differ in at least d positions. Given parameters n, d, and the alphabet size $\#\mathcal{A}$, the aim is to maximize the code size $\#C$. For some prime power q consider the finite field $\mathbb{F}_q$ as alphabet. An $[n,k,d]_q$ code C is a k-dimensional subspace of the vector space $\mathbb{F}_q^n$ with minimum distance d. The number of codewords is given by $\#C = q^k$. We also say that C is *linear* over $\mathbb{F}_q$, since C is linearly closed, i.e., for every $u, v \in C$ and every $\alpha, \beta \in \mathbb{F}_q$ we have $\alpha u + \beta v \in C$. The parameters of an $[n,k,d]_q$ code C are related by the so-called *Griesmer bound* [50, 107]

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil =: g_q(k,d). \tag{1}$$

Interestingly enough, this bound can always be attained with equality if the minimum distance d is sufficiently large and a nice geometric construction was given by Solomon and Stiffler [107].

If a code C is additively closed for alphabet $\mathcal{A} = \mathbb{F}_q$, i.e., $u + v \in C$ for all $u, v \in C$, we say that C is *additive* over $\mathbb{F}_q$. There indeed exist parameters n, d, q such that each linear code C_1 with length n, minimum distance d, and alphabet $\mathbb{F}_q$ satisfies $\#C_1 < \#C_2$ for a suitable additive code C_2 with length n, minimum distance d, and alphabet $\mathbb{F}_q$. In this situation some authors say that additive codes outperform linear codes. Typically $k := \log_q \#C_2$ is fractional, so that no $[n,k,d]_q$ code can exist. Indeed, very few cases where additive codes outperform linear codes and k is integral are known. In [37] an additive code with length $n = 21$, minimum distance $d = 18$, alphabet $\mathcal{A} = \mathbb{F}_9$, and size 9^3 is given. However, the largest linear code with length $n = 21$, minimum distance $d = 18$, and alphabet $\mathcal{A} = \mathbb{F}_9$ has size 9^2. More precisely, the largest n such that an $[n,3,n-3]_9$ code exists is $n = 17$ [8]. So, no $[21,3,18]_9$ code exists. Instead of maximizing the size of the code we can also minimize its length while fixing the other parameters. In [51] several additive codes over $\mathbb{F}_4$ that outperform the best known linear codes were constructed using cyclic codes. E.g. for length $n = 63$ and minimum distance $d = 45$ size 4^5 can be achieved while the existence of an $[63,5,45]_4$ code is unknown. Recently, in [85] four infinite series and five sporadic examples of additive codes, with size 4^4 and alphabet $\mathbb{F}_4$, that outperform linear codes with the same length and minimum distance were constructed.

There is some renewed interest in additive codes due to applications in the construction of quantum codes, see e.g. [38, 39, 49, 89].

The aim of this paper is to show that a Griesmer type bound for additive codes, see [7, Theorem 12] or Lemma 14 for details, can always be attained with equality if the minimum distance d is sufficiently large. This explains the mentioned four infinite series and gives many more such examples. The underlying construction generalizes the Solomon–Stiffler construction and will be formulated in geometric terms. For relatively small minimum distances the problem of the determination of the optimal parameters of additive codes is widely open and a challenging research direction, as it is for linear codes. Here we restrict our considerations to linear and additive codes over finite fields. However, similar questions also arise for codes over chain rings, see e.g. [71].

For other constructions attaining the Griesmer bound for linear codes we refer e.g. to [9, 36, 60, 61, 68]. For constructions of additive codes with good parameters we refer e.g. to [2, 51, 52, 93, 99]. Many constructions are based on cyclic codes or generalizations thereof, see e.g. [53, 101]. For bounds and constructions of codes over general alphabets we refer e.g. to [23, 24]. Additive codes have e.g. applications in quantum information [31, 72, 104, 108], computer memory systems [34, 35], deep space communication [55], storage systems [18, 19], secret sharing [74], and distance-regular graphs [102].

The remaining part of this paper is structured as follows. In Section 2 we introduce the necessary preliminaries. In Section 3 we generalize the Solomon–Stiffler construction for linear codes to additive codes. Our main result is Theorem 4 stating that a Griesmer upper bound for additive codes can always be attained with equality if the minimum distance d is sufficiently large. We list some parameterized series of improvements for additive codes over linear codes in Table 2. More extensive data is moved to appendices B and C. Relations between our problem and linear equation systems over $\mathbb{Z}$ are outlined in Section 4. Results on optimal additive codes for small parameters are summarized in Section 5. Our generalization of the Solomon–Stiffler construction involves a rather strong technical assumption which is relaxed in Appendix A. A relation of some bounds for additive codes to divisible codes is briefly outlined in Appendix D. Additive two-weight codes are considered in Appendix E. Examples of additive codes that have been found by computer searches are stated explicitly in Section F.

2 Preliminaries

In this section we collect the necessary preliminaries. I.e., we introduce the coding theoretic notation in Subsection 2.1 and the geometric notation in Subsection 2.2. Basic constructions and bounds are summarized in Subsection 2.3. None of this is essentially new. However, since different notions have been used in the literature and in order to keep the paper self-contained, we provide short proofs or explanations. In Subsection 2.4 we formalize the idea of asymptotic results which hold when the minimum distance is sufficiently large.

2.1 Coding theoretic notation

Let $\mathbb{F}_q$ denote the finite field with q elements, where $q = p^l$ is a prime power. We call the prime p the characteristic of $\mathbb{F}_q$. An additive code C of length n over the alphabet $\mathcal{A} = \mathbb{F}_{q'}$ is a subset of $\mathbb{F}_{q'}^n$ such that $u+v \in C$ for all $u,v \in C$. It turns out that each code C that is additive over $\mathbb{F}_{q'}$ is linear over some subfield $\mathbb{F}_q \le \mathbb{F}_{q'}$, i.e., $\alpha u + \beta v \in C$ for all $u,v \in C$ and all $\alpha,\beta \in \mathbb{F}_q$ [6, 7]. So, we use the notation $[n,r/h,d]_q^h$ for an additive code C that is linear over $\mathbb{F}_q$ and has length n, minimum distance d, alphabet $\mathcal{A} = \mathbb{F}_{q^h}$, and size q^r, where $r \in \mathbb{N}$. We also call $k = r/h \in \mathbb{Q}$ the *dimension* of C, so that $\#C = \#\mathcal{A}^k$ and an $[n,k,d]_q^1$ additive code is an $[n,k,d]_q$ linear code. Note that k can be fractional.

An $[n,k,d]_q$ linear code C can be defined as the rowspace of a $k \times n$ matrix with entries in $\mathbb{F}_q$, called a *generator matrix* for C. Similarly, an $[n,r/h,d]_q^h$ additive code C can be defined as the $\mathbb{F}_q$-space spanned by the rows of an $r \times n$ matrix G with entries in $\mathbb{F}_{q^h}$, again called a *generator matrix* for C. Let $\mathcal{B}$ be a basis for $\mathbb{F}_{q^h}$ over $\mathbb{F}_q$ and write out the elements of G over the basis $\mathcal{B}$ to obtain an $r \times nh$ matrix $\widetilde{G}$ with entries from $\mathbb{F}_q$. Here we assume that the columns of $\widetilde{G}$ are grouped together into n groups of h columns and say that $\widetilde{G}$ is a *subfield generator matrix* for C.

Example 1. *Write $\mathbb{F}_4 \simeq \mathbb{F}_2[\omega]/(\omega^2+\omega+1)$ and consider the linear code C with generator matrix*

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \end{pmatrix}.$$

It can easily be checked that C is a $[5,2,4]_4$ code. If we interprete C as an $[5,4/2,4]_4^2$ additive code a generator matrix is e.g. given by

$$G = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & \omega & \omega & \omega & \omega \\ 1 & 0 & 1 & \omega & \omega^2 \\ \omega & 0 & \omega & \omega^2 & 1 \end{pmatrix}.$$

Here we have

$$\widetilde{G} = \begin{pmatrix} 00 & 10 & 10 & 10 & 10 \\ 00 & 01 & 01 & 01 & 01 \\ 10 & 00 & 10 & 01 & 11 \\ 01 & 00 & 01 & 11 & 10 \end{pmatrix}$$

choosing the basis $\mathcal{B} = (1,\omega)$ and using $\omega^2 = 1+\omega$.

Given a subfield generator matrix $\widetilde{G}$ for an additive code C, the column space of each block of h columns defines an $\mathbb{F}_q$-subspace of dimension at most h. By $\mathcal{X}_G(C)$ we define the multiset of the n subspaces spanned by the n blocks of h columns of $\widetilde{G}$ in this way. We say that C is *faithful* if all elements of $\mathcal{X}_G(C)$ have dimension h and *unfaithful* otherwise. This is indeed a property of the code C and does not depend on the choice of a generator matrix G or the choice of a basis $\mathcal{B}$, see e.g. [7], so that we also write $\mathcal{X}(C)$. We remark that a linear code C, i.e. an additive code with $h=1$ is unfaithful iff an arbitrary generator matrix for C contains a column consisting entirely of zeroes. An unfaithful additive code for $h=2$ is given in Example 3.

Given a linear code C the *weight* of a codeword $c \in C$ is the number of nonzero entries in c, i.e. there is exactly one codeword of weight zero and besides that the minimum possible weight equals the minimum distance of the code.[1] We call C Δ-*divisible* if the weight of each codeword is divisible by $\Delta \in \mathbb{N}$. The *maximum weight* of C is the maximum of the weights of the codewords of C. We say that C is a t-*weight* code if only t different nonzero weights occur. one-weight codes are repetitions of simplex codes, i.e. exactly those codes that attain the upper bound from Lemma 15, see Theorem 2. A linear code C is *projective* if the minimum distance of its dual code is at least 3, i.e., if each pair of columns of a generator matrix of C is linearly independent. In geometric terms the latter property means that each point in the corresponding multiset of points has multiplicity at most one, see the subsequent subsection. Projective two-weight codes have received at lot of attention, see e.g. [30, 32].

Lemma 1. *([40, Corollary 2]) Let $0 < w_1 < w_2$ be the two nonzero weights of a projective linear code over $\mathbb{F}_q$. Then, there exist positive integers u, t such that $w_1 = up^t$ and $w_2 = (u+1)p^t$, where p is the characteristic of $\mathbb{F}_q$.*

A similar statement also holds for nonprojective two-weight codes, see [113, Theorem 3] and [84]. General two-weight codes were e.g. studied in [28]. If C is an $[n,k]_q$ code where all nonzero weights are contained in $\{w_1,\dots,w_t\}$, then we also speak of an $[n,k,\{w_1,\dots,w_t\}]_q$ code. There is a vast literature on linear codes with few weights, but rather little seems to be known for additive codes with few weights, see e.g. [93]. We remark that a faithful projective $h-(n,r,s,\mu)_q$ system with type $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$, see Definition 1 and Definition 11, where t of the ε_i are nonzero, corresponds to an additive $(t+1)$-weight code. In Section 3 we give constructions for such codes. A few more observations on the special case of additive two-weight codes are given in Section E in the appendix.

2.2 Geometric notation

The set of all subspaces of $\mathbb{F}_q^r$, ordered by the incidence relation $\subseteq$, is called $(r-1)$-*dimensional projective geometry over* $\mathbb{F}_q$ and denoted by $\operatorname{PG}(r-1,q)$. Employing this algebraic notion of dimension instead of the geometric one, we will use the term i-space to denote an i-dimensional vector subspace of $\mathbb{F}_q^r$. To highlight the important geometric interpretation of subspaces we will call 1-, 2−, and $(r-1)$-spaces points, lines, and hyperplanes, respectively. For two subspaces S and S' we write $S \le S'$ if S is contained in S'. Moreover, we say that S and S' are *incident* iff $S \le S'$ or $S \ge S'$. Let $[i]_q := \frac{q^i-1}{q-1}$ denote the number of points of an arbitrary i-space in $\operatorname{PG}(r-1,q)$ where $r \ge i$. By convention we set $[0]_q := 0$. We have the following obvious but useful observations.

[1]The same is true for additive codes, see [6, Lemma 3].

Lemma 2. *For $b \geq a \geq 1$ we have*

$$[a]_q[b-1]_q - [a-1]_q[b]_q = q^{a-1} \cdot [b-a]_q, \tag{2}$$

using the convention $[0]_q = 0$, and

$$\gcd\Big([a]_q, [b]_q\Big) = \big[\gcd(a,b)\big]_q. \tag{3}$$

A *premultiset of points* $\mathcal{M}$ of $\mathrm{PG}(r-1,q)$ is a mapping from the set of points of $\mathrm{PG}(r-1,q)$ to $\mathbb{Z}$. The value $\mathcal{M}(P)$ is called the *multiplicity* of $\mathcal{M}$. We extend this notation additively to each subspace S via $\mathcal{M}(S) = \sum_{P \leq S} \mathcal{M}(P)$. So, the cardinality of $\mathcal{M}$ equals $\mathcal{M}(V)$, where V denotes the ambient space. For a hyperplane H the number of points of $\mathcal{M}$ in H is given by $\mathcal{M}(H)$. If $\mathcal{M}(P) \in \mathbb{N}$ for each point P, we say that $\mathcal{M}$ is a *multiset of points*. For each subspace S we denote its *characteristic function* by χ_S, i.e. $\chi_S(P) = 1$ if $P \leq S$ and $\chi_S(P) = 0$ otherwise.

Multisets of points can be generalized as follows, cf. [7, Definition 4].

Definition 1. *A projective $h-(n,r,s)_q$ system is a multiset $\mathcal{S}$ of n subspaces of $\mathrm{PG}(r-1,q)$ of dimension at most h such that each hyperplane contains at most s elements of $\mathcal{S}$ and some hyperplane contains exactly s elements of $\mathcal{S}$. We say that $\mathcal{S}$ is faithful if all elements have dimension h. A projective $h-(n,r,s)_q$ system $\mathcal{S}$ is a projective $h-(n,r,s,\mu)_q$ system if each point is contained in at most μ elements from $\mathcal{S}$ and there is some point that is contained in exactly μ elements from $\mathcal{S}$.*

Note that the elements of $\mathcal{S}$ span the entire ambient space $\mathrm{PG}(r-1,q)$ iff $s < n$. It is well known that a projective $1-(n,r,s)_q$ system $\mathcal{S}$ with $s < n$ is in one-to-one correspondence to a linear $[n,r,n-s]_q$ code C which has full length iff $\mathcal{S}$ is faithful, see e.g. [109, §1.1.2] or [44].[2] Each hyperplane H that contains i elements of $\mathcal{S}$ corresponds to $q-1$ codewords of weight $n-i$. In general, projective $h-(n,r,s)_q$ systems (with $s < n$) are in one-to-one correspondence to additive codes:

Theorem 1. *[7, Theorem 5] If C is an additive $[n,r/h,d]_q^h$ code with generator matrix G, then $\mathcal{X}_G(C)$ is a projective $h-(n,r,n-d)_q$ system $\mathcal{S}$, and conversely, each projective $h-(n,r,s)_q$ system $\mathcal{S}$ defines an additive $[n,r/h,n-s]_q^h$ code C.*

As mentioned before, C is faithful iff $\mathcal{S}$ is faithful. Whenever the specific choice of a generator matrix G or a subfield generator matrix $\widetilde{G}$ is irrelevant, we write $\mathcal{S} = \mathcal{X}(C)$ or $C = \mathcal{X}^{-1}(\mathcal{S})$. We also use the notion $C = \mathcal{X}^{-1}(\mathcal{M})$ for a multiset of points $\mathcal{M}$ interpreting $\mathcal{M}$ as a faithful projective $1-(n,r,s)_q$ system. In the case where $n = s > 0$ let R be the r' space spanned by the points of positive multiplicity in $\mathcal{M}$ and consider the multiset of points $\mathcal{M}'$ in $\mathrm{PG}(r'-1,q)$ as the restriction

[2] Every $\mathbb{F}_q$-linear code is also equivalent to a multi-twisted code [90]. There is also a relation to subsets of vector spaces with pairwise different linear combinations, see [94, Theorem 3.1] for details and [94, Definition 3.2] for the concept of an S_h-linear set.

of $\mathcal{M}$ to K instead. We say that a (pre-)multiset of points $\mathcal{M}$ in $\mathrm{PG}(r-1,q)$ is Δ-*divisible* for some positive integer Δ iff

$$\#\mathcal{M} \equiv \mathcal{M}(H) \pmod{\Delta} \qquad (4)$$

for every hyperplane H in $\mathrm{PG}(r-1,q)$ if $r \ge 2$ and for $r = 1$ iff $\#\mathcal{M} \equiv 0 \pmod{\Delta}$. Note that $\mathcal{M}$ is Δ-divisible iff the linear code $\mathcal{X}^{-1}(\mathcal{M})$ is Δ-divisible. We collect a few basic properties in the following lemma and refer e.g. to the surveys [81, 112] for more details.

Lemma 3. *Let $\mathcal{M}, \mathcal{M}'$ be two Δ-divisible (pre-)multisets of points in $\mathrm{PG}(r-1,q)$ for $r \ge 2$. Then, $\mathcal{M}+\mathcal{M}'$, $\mathcal{M}-\mathcal{M}'$ are also Δ-divisible and $\lambda \cdot \mathcal{M}$ is $\Delta \cdot \lambda$-divisible for every positive integer λ. Moreover, the characteristic function χ_S of every i-space S is q^{i-1}-divisible.*

Note that each projective $h-(n,r,s)_q$ system $\mathcal{S}$ can be modified to a faithful projective $h-(n,r,\le s)_q$ system $\mathcal{S}'$ by replacing each element $S \in \mathcal{S}$ with dimension less than h by an arbitrary h-space containing S if $r \ge h$. We remark that it is also possible to obtain a faithful projective $h-(n,r,s)_q$ system $\mathcal{S}'$ if we choose the replacing h-spaces carefully. On the other side, the knowledge that a projective system is unfaithful sometimes allows to deduce tighter bounds on its parameters, see e.g. the proof of Lemma 37.

To each faithful projective $h-(n,r,s)_q$ system we can also associate a multiset of points and a linear code over $\mathbb{F}_q$ with certain properties.

Definition 2. *For a faithful projective $h-(n,r,s,\mu)_q$ system $\mathcal{S}$ let $\mathcal{P}(\mathcal{S})$ denote the multiset of points that we obtain by replacing each element of $\mathcal{S}$ by its contained $[h]_q$ points.*

Lemma 4. *Let $\mathcal{S}$ be a faithful projective $h-(n,r,s,\mu)_q$ system. Then $\mathcal{P}(\mathcal{S})$ is a faithful projective $1-(n',r,s',\mu)_q$ system, where $n' = n[h]_q$ and $s' = n \cdot [h-1]_q + s \cdot q^{h-1}$. Moreover, $C := \mathcal{X}^{-1}(\mathcal{P}(\mathcal{S}))$ is a faithful q^{h-1}-divisible linear $[n',r,d']_q$ code C with maximum weight at most $n \cdot q^{h-1}$, where $d' = q^{h-1} \cdot (n-s)$.*

Proof. As an abbreviation we set $\mathcal{S}' := \mathcal{P}(\mathcal{S})$. By construction we have $n' = \#\mathcal{S}' = \#\mathcal{S} \cdot [h]_q = n[h]_q$. If a hyperplane H contains $0 \le i \le s$ elements from $\mathcal{S}$, then H contains $i \cdot [h]_q + (n-i) \cdot [h-1]_q = n \cdot [h-1]_q + i \cdot q^{h-1}$ elements from $\mathcal{S}'$, so that $s' = n \cdot [h-1]_q + s \cdot q^{h-1}$. Moreover, each hyperplane H with $\mathcal{S}'(H) = n \cdot [h-1]_q + i \cdot q^{h-1}$ corresponds to $q-1$ codewords of weight $q^{h-1} \cdot (n-i)$, so that C is q^{h-1}-divisible, has minimum weight d', and a maximum weight of at most $n \cdot q^{h-1}$. ∎

Example 2. *Consider the $[5,4/2,4]_4^2$ additive code given by the generator matrix G from Example 1 and the corresponding projective $2-(5,4,1)_2$ system $\mathcal{S}$. The linear $[15,4,8]_2$ code C constructed from $\mathcal{S}$ in Lemma 4 has e.g.*

$$\begin{pmatrix} 000 & 101 & 101 & 101 & 101 \\ 000 & 011 & 011 & 011 & 011 \\ 101 & 000 & 101 & 011 & 110 \\ 011 & 000 & 011 & 110 & 101 \end{pmatrix}$$

as a generator matrix, where we group the columns into blocks of size $[h]_q = 3$ corresponding to the points of the five lines. Here the maximum weight equals the minimum distance, i.e. every hyperplane contains exactly one element from $\mathcal{S}$.

Sometimes the stated restrictions on the weights of the linear code C in Lemma 4 can be used to prove tailored upper bounds like e.g. $n_3(8,3;3) \le 20$, see Lemma 37.

For $h > 1$ it is an interesting question whether for a given multiset of points $\mathcal{M}$ in $\mathrm{PG}(r-1,q)$ a faithful projective $h-(n,r,s,\mu)_q$ system $\mathcal{S}$ with $\mathcal{P}(\mathcal{S}) = \mathcal{M}$ exists, see e.g. [75] for some special cases. In general, this is a very hard problem and we can only give an "asymptotic" answer, see Subsection 2.4 for definitions, Theorem 3 for an explicitly parameterized solution in a special case, and Lemma 20 for a characterization based on the solvability of linear equation systems over $\mathbb{Z}$. However, in the case of existence the parameters of $\mathcal{S}$ can be computed from $\mathcal{M}$.

Definition 3. *Let $\mathcal{M}$ be a multiset of points in* $\mathrm{PG}(r-1,q)$*. We say that $\mathcal{M}$ is h-*partitionable *if there exists a faithful projective $h-(n,r,s,\mu)_q$ system $\mathcal{S}$ with $\mathcal{P}(\mathcal{S}) = \mathcal{M}$, i.e. $\mathcal{M} = \sum_{S\in\mathcal{S}} \chi_S$. We also say that $\mathcal{S}$ has* type $\mathcal{M}$*.*

Lemma 5. *Let $\mathcal{M}$ be a multiset of points in* $\mathrm{PG}(r-1,q)$ *and $\mathcal{S}$ be a faithful projective $h-(n,r,s,\mu)_q$ system with $\mathcal{P}(\mathcal{S}) = \mathcal{M}$, then we have*

$$n = \#\mathcal{M}/[h]_q, \tag{5}$$

$$s = \frac{\left(\max_{H:\dim(H)=r-1} \mathcal{M}(H)\right) \cdot [h]_q - \#\mathcal{M} \cdot [h-1]_q}{q^{h-1} \cdot [h]_q}, \tag{6}$$

$$d := n - s = \frac{\#\mathcal{M} - \max_{H:\dim(H)=r-1} \mathcal{M}(H)}{q^{h-1}}, \tag{7}$$

$$\mu = \max_{P:\dim(P)=1} \mathcal{M}(P), \tag{8}$$

and

$$\#\mathcal{M} \equiv \mathcal{M}(H) \pmod{q^{h-1}} \tag{9}$$

for each hyperplane H in $\mathrm{PG}(r-1,q)$*.*

Proof. Consider $\mathcal{M}$ as a faithful projective $1-(n',r,s',\mu')_q$ system with $s' = \max_{H:\dim(H)=r-1} \mathcal{M}(H)$, $\mu' = \max_{P:\dim(P)=1} \mathcal{M}(P)$, $n' = \#\mathcal{M}$, and set $d' := n' - s'$. Then, the stated formulas for the parameters n, s, d, μ as well as condition (9) can be easily verified using Lemma 4. ∎

The geometric lattice $\mathrm{PG}(r-1,q)$ admits duality, i.e., for an i-space S we denote the orthogonal subspace with respect to some fixed nondegenerate bilinear form by $S^\perp$. The $(r-i)$-space is also called dual of S and the dual $\mathcal{S}^\perp$ of a projective $h-(n,r,s,\mu)_q$ system $\mathcal{S}$ is obtained from $\mathcal{S}$ by replacing each element $S \in \mathcal{S}$ by its dual $S^\perp$. Directly from the definition we conclude:

Lemma 6. *The dual $\mathcal{S}^\perp$ of a faithful projective $h-(n,r,s,\mu)_q$ system $\mathcal{S}$ is a faithful projective $(r-h)-(n,r,\mu,s)_q$ system.*

Proof. The dual of an h-space in $\mathrm{PG}(r-1,q)$ is an $(r-h)$-space. Let P be an arbitrary point and H be an arbitrary hyperplane, so that $P^\perp$ is a hyperplane and $H^\perp$ is a point. Since at most μ elements of $\mathcal{S}$ contain P, at most μ elements of $\mathcal{S}^\perp$ are contained in $P^\perp$. Moreover, equality occurs for some point. Similarly, at most s elements of $\mathcal{S}$ are contained in H, so that at most s elements of $\mathcal{S}^\perp$ contain $H^\perp$. Again, equality occurs for some hyperplane. ∎

Definition 4. *By $n_q(r,h;s)$ we denote the maximum number n such that a projective $h-(n,r,s)_q$ system exists.*

Lemma 7. *Let $\mathcal{S}_1$ be a projective $h-(n_1,r,s_1,\mu_1)_q$ system and $\mathcal{S}_2$ be a projective $h-(n_2,r,s_2,\mu_2)_q$ system. Then there exists a projective $h-(n_1+n_2,r,\le s_1+s_2,\le \mu_1+\mu_2)_q$ system $\mathcal{S}$. If $\mathcal{S}_1,\mathcal{S}_2$ are faithful with types $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, then $\mathcal{S}$ is faithful with type $\mathcal{M}_1+\mathcal{M}_2$. Moreover, if $r\ge h$, then there exists a faithful $h-(1,r,1,1)_q$ system $\mathcal{S}'$.*

Proof. $\mathcal{S}$ can be constructed as the multiset union of $\mathcal{S}_1$ and $\mathcal{S}_2$. If $r\ge h$ then $\mathcal{S}'$ can be chosen as an arbitrary single h-space in $\mathrm{PG}(r-1,q)$. ∎

Corollary 1. *We have $n_q(r,h;s_1+s_2)\ge n_q(r,h;s_1)+n_q(r,h;s_2)$. If $r\ge h$, then we have $n_q(r,h;s+1)\ge n_q(r,h;s)+1$.*

If $r\le h$ we can consider an arbitrary number of copies of the unique r-space in $\mathrm{PG}(r-1,q)$, none lying in a hyperplane, so that $n_q(r,h;s)=\infty$ by definition and we will always assume $r>h$ in the following.

Lemma 8. *Let $\mathcal{S}$ be a (faithful) projective $h-(n,r,s,\mu)_{q^l}$ system. Then there exists a (faithful) projective $hl-(n,rl,s,\mu)_q$ system $\mathcal{S}'$.*

Proof. The vector space $\mathbb{F}_q^l$ is isomorphic to $\mathbb{F}_{q^l}$ when viewed as a vector space over $\mathbb{F}_q$. Under this isomorphism, we get a map Ψ from the i-spaces in $\mathrm{PG}\big(r-1,q^l\big)$ to the (li)-spaces in $\mathrm{PG}(rl-1,q)$. Applying Ψ to the elements of $\mathcal{S}$ for $i=h$ gives $\mathcal{S}'$. ∎

Corollary 2. *We have $n_q(lr,lh;s)\ge n_{q^l}(r,h;s)$ for all positive integers l.*

The map Ψ is called the field reduction map in [88] and indeed very widely used in Galois geometry. A very prominent example is a *Desarguesian spread* of l-spaces of $\mathrm{PG}(lr-1,q)$ as the image of the points of $\mathrm{PG}\big(r-1,q^l\big)$ under Ψ. Example 1 is obtained in this way and in Example 2 we can see the partition of the $[4]_2=15$ points of $\mathrm{PG}(3,2)$ into $[2]_4=5$ lines. By a projection argument we can also obtain constructions from field reduction when the dimension is not divisible by l.

Lemma 9. *We have $n_q(r-1,h,s) \ge n_q(r,h,s)$ for $r \ge 2$.*

Proof. Let $\mathcal{S}$ be a projective $h-(n,r,s)_q$ system and P an arbitrary point in $\mathrm{PG}(r-1,q)$. Projection through a point P yields $\mathrm{PG}(r-1,q)/(P) \cong \mathrm{PG}(r-2,q)$ and $\mathcal{S}$ is mapped to a projective $h-(n,r-1,s)_q$ system $\mathcal{S}'$. ■

We remark that the elements $S \in \mathcal{S}$ that contain P are mapped to $(\dim(S)-1)$-spaces S' so that $\mathcal{S}'$ may not be faithful even if $\mathcal{S}$ is.

Example 3. *Consider the faithful projective $2-(5,4,1)_2$ system $\mathcal{S}$ that corresponds to the $[5,4/2,4]_4^2$ additive code from Example 1. If we project $\mathcal{S}$ through an arbitrary point P of $\mathrm{PG}(3,2)$, we obtain an unfaithful projective $2-(5,3,1)_2$ system $\mathcal{S}'$ consisting of four lines and one point in $\mathrm{PG}(2,2)$.*

Definition 5.

$$\overline{n}_q(r,h;s) := n_{q^h}(\lceil r/h \rceil, 1; s) \tag{10}$$

In words, $\overline{n}_q(r,h;s)$ is the size of the largest projective $h-(n,r,s)_q$ system that we can obtain starting from a linear code over $\mathbb{F}_q$ via Theorem 1 and Lemma 8 by an iterative application of Lemma 9. Whenever $\overline{n}_q(r,h;s) < n_q(r,h;s)$ we say that additive codes outperform linear codes for the corresponding parameters, which is especially interesting if r/h is integral.

2.3 Basic constructions and bounds

We start to prepare a few basic constructions of projective $h-(n,r,s,\mu)_q$ systems.

Definition 6. *A* vector space partition *of $\mathrm{PG}(r-1,q)$ is a multiset $\mathcal{V}$ of subspaces with dimension at most $(r-1)$ such that every point of $\mathrm{PG}(r-1,q)$ is contained in exactly one element of $\mathcal{V}$. We say that $\mathcal{V}$ has type $1^{t_1}2^{t_2}\dots(r-1)^{t_{r-1}}$ if exactly t_i elements of $\mathcal{V}$ have dimension i for all $1 \le i \le r-1$.*

Example 4. *The set $\mathcal{S}$ of all points of $\mathrm{PG}\left(r-1,q^h\right)$ is a projective $1-\left([r]_{q^h}, r, [r-1]_{q^h}, 1\right)_{q^h}$ system. Applying Lemma 8 to $\mathcal{S}$ gives a vector space partition of $\mathrm{PG}(rh-1,q)$ of type h^{t_h} with $t_h = [r]_{q^h} = [rh]_q/[h]_q$. Here one also speaks of a h-*spread *of $\mathrm{PG}(rh-1,q)$, cf. Example 2.*

A set of matrices $M \subseteq \mathbb{F}_q^{m\times n}$ with $\mathrm{rk}(A-B) \ge \delta$ for all $A, B \in M$ with $A \ne B$ is called a *rank metric code* with minimum rank distance δ. A Singleton-type upper bound gives $\#M \le q^{\max\{m,n\}\cdot(\min\{m,n\}-\delta+1)}$. Rank metric codes attaining this bound are called *MRD* codes. They exist for all parameters with $\delta \le \min\{m,n\}$, even if one additionally requires that M is linearly closed, see e.g. [100] for a survey.

Lemma 10. *For $r > 2h$ there exists a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1,q)$ of type $h^{t_h}(r-h)^1$ where $t_h = q^{r-h}$.*

Proof. Let $M \subseteq \mathbb{F}_q^{h\times(r-h)}$ be an MRD code with minimum rank distance h and cardinality q^{r-h}. Prepending a $h \times h$ unit matrix to the elements of M gives generator matrices of h-spaces in $\mathrm{PG}(r-1, q)$ that are pairwise disjoint and disjoint to an $(r-h)$-space S. ∎

Of course, this lifting type construction is well known. Another construction, based on field reduction, is e.g. given in [10, Theorem 4.2]. For $r = 2h$ there exists a vector space partition of type h^{t_h}, i.e. a spread of h-spaces.

Lemma 11. *For $r > a > h$ with $r \equiv a \pmod h$ there exists a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1, q)$ of type $h^{t_h}(a)^1$ where $t_h = q^a \cdot \frac{q^{r-a}-1}{q^h-1}$.*

Proof. We prove by induction over r. Let $\mathcal{V}$ be the vector space partition obtained from Lemma 10 and let $S \in \mathcal{V}$ be the unique $(r-h)$-dimensional element. If $a = r-h$, which is indeed the case for all $r < 3h$, then $\mathcal{V}$ is the desired vector space partition. Otherwise we identify S with $\mathrm{PG}(r-h-1, q)$ and replace S by a vector space partition of $\mathrm{PG}(r-h-1, q)$ type $h^{t'_h}a^1$, which exists by induction. ∎

Lemma 12. *For $r > a > h$ with $r \equiv a \pmod h$ let $\mathcal{S}$ be the set of h-dimensional elements of a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1, q)$ of type $h^{t_h}a^1$ and A be the unique a-dimensional element in $\mathcal{V}$. Then, $\mathcal{S}$ is a faithful projective $h - (t_h, r, s, 1)_q$ system where $t_h = q^a \cdot \frac{q^{r-a}-1}{q^h-1}$ and $s = q^{a-h} \cdot \frac{q^{r-a}-1}{q^h-1}$. Moreover, each hyperplane that contains A contains $s - q^{a-h}$ elements from $\mathcal{S}$.*

Proof. Let H be an arbitrary hyperplane of $\mathrm{PG}(r-1, q)$. Note that every i-space intersects H in either $[i]_q$ or $[i-1]_q$ points and that the elements of $\mathcal{S}$ partition the points outside of A. Counting points yields that H contains

$$\frac{[r-1]_q - [a-1]_q - t_h \cdot [h-1]_q}{q^{h-1}} = q^{a-h} \cdot \frac{q^{r-a}-1}{q^h-1} = s$$

elements from $\mathcal{S}$ if $A \not\leq H$ and

$$\frac{[r-1]_q - [a]_q - t_h \cdot [h-1]_q}{q^{h-1}} = \frac{[r-1]_q - q^{a-1} - [a-1]_q - t_h \cdot [h-1]_q}{q^{h-1}} = s - q^{a-h}$$

elements from $\mathcal{S}$ if $A \leq H$. ∎

Lemma 13. *(Cf. [51, Lemma 6]) Let $\mathcal{S}_1$ be a faithful projective $h-(n_1, r, s_1)_q$ system, A be an a-space such that each hyperplane that contains A contains at most s_0 elements from $\mathcal{S}$, and $\mathcal{S}_2$ be a faithful projective $h-(n_2, a, s_2)_q$ system. Then, we have $n_q(r, h; \max\{s_1 + s_2, s_0 + n_2\}) \geq n_1 + n_2$.*

Proof. We identify A with $\mathrm{PG}(a-1, q)$ and insert $\mathcal{S}_2$ into the subspace A of $\mathcal{S}_1$. ∎

In analogy with linear codes, the authors of [51, 52] speak of the "additive construction X". We give an application in Lemma 26.

Next we discuss a few basic upper bounds for $n_q(r,h;s)$.

Lemma 14. *[7, Theorem 12] Let $r > h$ and $\mathcal{S}$ be a projective $h-(n,r,s)_q$ system. Then, we have*

$$g_q\Big(r, q^{h-1}\cdot(n-s)\Big) \le [h]_q \cdot n \le n_q\Big(r,1;n\cdot[h-1]_q + s\cdot q^{h-1}\Big). \tag{11}$$

Proof. W.l.o.g. we assume that $\mathcal{S}$ is faithful and that we have $n > s$. Let $n' = [h]_q \cdot n$, $s' = n\cdot[h-1]_q + s\cdot q^{h-1}$, and $d' = q^{h-1}\cdot(n-s)$. By C we denote the $[n', r, d']_q$ code and by $\mathcal{S}'$ the projective $1-(n', r, s')_q$ system constructed in the proof of Lemma 4. The parameters of $\mathcal{S}'$ give the right hand side of Inequality (11) and applying the Griesmer bound from Inequality (1) to C yields the left hand side of Inequality (11). ■

Indeed, Lemma 14 is well-known for $q = h = 2$, see e.g. [16, Lemma 1] or [51, Lemma 3].

Definition 7. *The* coding upper bound *for $n_q(r,h;s)$ is the largest integer n such that $[h]_q \cdot n \le n_q(r,1;n\cdot[h-1]_q + s\cdot q^{h-1})$. The* Griesmer upper bound *for $n_q(r,h;s)$ is the largest integer n such that $g_q\Big(r, q^{h-1}\cdot(n-s)\Big) \le [h]_q \cdot n$.*

s	Griesmer upper bound	coding upper bound	$n_2(8,2;s)$
3	9	7	5
4	12		10
5	17		17
6	22	18	18
7	25	23	23
8	30	28	28
9	33		33
10	38	36	35–36
11	43	40	40
12	44		44
13	49		49
14	54		54
15	59	57	55–57
24	94		92–94
27	107		106–107
28	110		108–110
29	115		113–115

Table 1: The Griesmer and coding upper bound for $n_2(8,2;s)$.

Example 5. *The Griesmer upper bound for $n_2(8,2;8)$ is 30 and the coding upper bound is 28. I.e., the Griesmer bound implies that no $[93,8,46]_2$ code exists but cannot rule out the existence of a $[90,8,44]_2$ code, so that $n_2(8,2;8) \le 30$ is the sharpest upper bound we can deduce from the Griesmer bound (for linear codes). However, since the existence of a $[84,8,40]_2$ code and the nonexistence of a $[87,8,42]_2$ code is known, we obtain $n_2(8,2;8) \le 28$. In Table 1 we list the Griesmer and the coding upper bound for $n_2(8,2;s)$ for $s \le 15$ and all cases were either the coding upper bound is strictly less than the Griesmer upper bound or the value of $n_2(8,2;s)$ is unknown, see [85] and [21] for $s \in \{3,4\}$. We do not display the coding upper bound when it coincides with the Griesmer upper bound.*

For the current knowledge on $n_q(r,1;s)$ or corresponding bounds for linear codes we refer to `www.codetables.de`, `http://mars39.lomo.jp/opu/griesmer.htm`, and `http://web.mat.upc.edu/simeon.michael.ball/codebounds.html`.

Lemma 15. *We have*

$$n_q(r,h;s) \le \frac{[r]_q \cdot s}{[r-h]_q}, \tag{12}$$

where the right hand side is an integer iff s is divisible by $[r-h]_q/[\gcd(r,h)]_q$. In the case of equality each point is contained in exactly $\mu = \frac{[h]_q \cdot s}{[r-h]_q}$ elements.

Proof. Let $\mathcal{S}$ be a faithful projective $h-(n,r,s)_q$ system with $n = n_q(r,h;s)$. Since each element $S \in \mathcal{S}$ is contained in $[r-h]_q$ hyperplanes and there are $[r]_q$ hyperplanes in total, we conclude $n \le \frac{[r]_q \cdot s}{[r-h]_q}$. From Equation (3) we conclude $\gcd([r]_q,[r-h]_q) = [\gcd(r,h)]_q$, so that the right hand side of Inequality (12) is an integer iff s is divisible by $[r-h]_q/[\gcd(r,h)]_q$. If $n = \frac{[r]_q \cdot s}{[r-h]_q}$, then each hyperplane contains exactly s elements from $\mathcal{S}$, so that each point is contained in the same number μ of elements by duality. Thus, we have $\mu = n \cdot [h]_q/[r]_q = \frac{[h]_q \cdot s}{[r-h]_q}$. ∎

We remark that the Griesmer upper bound is always as least as good as this explicit upper bound and both coincide iff the right hand side of Inequality (12) is an integer. Of course this bound is known, see e.g. [7, Proof of Theorem 9]. If we know that a given projective system is unfaithful, then we can typically improve the stated upper bound a bit, see e.g. the proof of Lemma 37.

By projection through a subspace K instead of a point P we can improve Lemma 9.

Lemma 16. *(Cf. [13, Proposition 1]) Let $\mathcal{S}$ be a projective $h-(n,r,s)_q$ system and K be an r'-space in $\mathrm{PG}(r-1,q)$ containing s' elements from $\mathcal{S}$. Then, projection through K yields a projective $h-(n-s',r-r',s-s')_q$ system $\mathcal{S}'$.*

Via this projection we essentially just look at all hyperplanes that contain K and can obtain general upper bounds by choosing a suitable subspace K.

Lemma 17. $n_q(r,h;s) \le t + n_q(r-ht,h;s-t)$ *for all* $t \le r/h - 1$.

Proof. Consider a projective $h-(n,r,s)_q$ system with $n = n_q(r,h;s)$ and let K be a subspace spanned by t elements of $\mathcal{S}$. Applying Lemma 16 to $\mathcal{S}$ and K gives a projective $h-(n-s',r-r',s-s')_q$ system $\mathcal{S}'$ with $s' \geq t$ and $r' \leq ht$, so that

$$n_q(r,h;s) \leq s' + n_q(r-r',h;s-s') \overset{\text{Lemma 9}}{\leq} s' + n_q(r-ht,h;s-s') \overset{\text{Corollary 1}}{\leq} t + n_q(r-ht,h;s-t).$$

■

Remark 1. *In [7, Theorem 9] Lemma 17 is written down as an explicit upper bound for $n_q(r,h;s)$, applying Lemma 15 to $\mathcal{S}'$, together with an analysis of the optimal choice of t. The bound from [7, Theorem 9] is quite effective and it is shown that it surpasses the Griesmer upper bound in many cases, see [7, Theorem 13]. As an example, we mention that $n_3(7,2;3) \leq 15$ can be concluded from $n_3(3,2;1) = 13$ via Lemma 17 or directly via [7, Theorem 9] while the Griesmer upper bound is $n_3(7,2;3) \leq 18$.*[3]

We can easily characterize asymptotically optimal projective $h-(n,r,s)_q$ systems, cf. [17].[4] I.e., if the right hand side of Inequality (12) is an integer, then the upper bound from Lemma 15 can indeed be attained.

Theorem 2. *(Cf. [64, page 83], [45, Corollary 8], or [79, Lemma 2]) For each pair of integers $r \geq h \geq 1$ a faithful projective $h-(n,r,s,\mu)_q$ system with $n = \frac{[r]_q}{[\gcd(r,h)]_q}$, $s = \frac{[r-h]_q}{[\gcd(r,h)]_q}$, and $\mu = \frac{[h]_q}{[\gcd(r,h)]_q}$ exists.*

Proof. Due to Lemma 8 we can assume $\gcd(r,h) = 1$. We prove by double induction on h and r. The set of points of $\mathrm{PG}(r-1,q)$ gives an example for $h = 1$ and arbitrary r, so that we assume $h \geq 2$ in the following. If $r = h$ or $r = 2h$, then we have $h = 1$. If $h < r < 2h$, then a projective $(r-h)-(n,r,\mu,s)_q$ system exists by induction and we can apply Lemma 6. If $r > 2h$, then we apply Lemma 11 to construct a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1,q)$ of type $h^{t_r}(r-h)^1$ and denote the special $(r-h)$-space by A. To construct the desired example we take a μ-fold copy of $\mathcal{V}$ and replace A by a projective $h-(n',r-h,s',\mu')_q$ system with $n' = [r-h]_q$, $s' = [r-2h]_q$, and $\mu' = [h]_q$. ■

Corollary 3. *Let $\mathcal{S}$ be a faithful projective $h-(n,r,s,\mu)_q$ system. Then there exists a faithful projective*

$$h-\left(n+t\cdot\frac{[r]_q}{[\gcd(r,h)]_q}, r, s+t\cdot\frac{[r-h]_q}{[\gcd(r,h)]_q}, \mu+t\cdot\frac{[h]_q}{[\gcd(r,h)]_q}\right)_q$$

system $\mathcal{S}_t$ for each $t \in \mathbb{N}$.

The parameters of the sequence $\mathcal{S}_t$ tend towards the upper bound from Lemma 15. In the subsequent section we will show that the (typically) stronger Griesmer upper bound can always be attained if s is sufficiently large.

[3] The Griesmer bound implies that the maximum minimum distance of an $[72,7]_3$-code is at most 45, which is still the best known upper bound and there exists a $[72,7,43]_3$-code.

[4] For the special case $q = 2$, $h = 2$, and arbitrary r of Theorem 2 we also refer to [52, Lemma 3 & Remark 1].

We remark that for a faithful projective $h-(n,r,s,\mu)_q$ system as in Theorem 2 each hyperplane contains exactly s and each point is contained in exactly μ elements from $\mathcal{S}$. The cases where $\mu = 1$ correspond to spreads of h-spaces of $\mathrm{PG}(r-1,q)$.

2.4 Asymptotic formulations

Definition 8. *Let $\mathcal{M}$ be a premultiset of points in $\mathrm{PG}(r-1,q)$ and V the ambient space. We say that $\sigma[r]-\mathcal{M}$ is h-*partitionable over $\mathbb{F}_q$ *if there exists a faithful projective $h-(n,r,s,\mu)_q$ system $\mathcal{S}$ with $\mathcal{P}(\mathcal{S}) = \sigma\cdot\chi_V - \mathcal{M}$, i.e. $\sigma\cdot\chi_V - \mathcal{M} = \sum_{S\in\mathcal{S}}\chi_S$. We also say that $\mathcal{S}$ has* type $\sigma[r]-\mathcal{M}$.

Lemma 18. *Let $\mathcal{M}$ be a premultiset of points $\mathcal{M}$ in $\mathrm{PG}(r-1,q)$. If $\mathcal{S}$ is a faithful projective $h-(n,r,s,\mu)_q$ system with type $\sigma[r]-\mathcal{M}$, then we have*

$$n = \left(\sigma[r]_q - \#\mathcal{M}\right)/[h]_q, \tag{13}$$

$$s = \left(\sigma[r-h]_q - \frac{[h-1]_q\cdot\#\mathcal{M} - [h]_q\cdot\min_{H:\dim(H)=r-1}\mathcal{M}(H)}{q^{h-1}}\right)/[h]_q, \tag{14}$$

$$d := n-s = \sigma q^{r-h} - \frac{\#\mathcal{M} - \min_{H:\dim(H)=r-1}\mathcal{M}(H)}{q^{h-1}}, \tag{15}$$

$$\mu = \sigma - \min_{P:\dim(P)=1}\mathcal{M}(P), \tag{16}$$

and

$$\#\mathcal{M} \equiv \mathcal{M}(H) \tag{17}$$

for each hyperplane. Moreover, we have $\#\mathcal{M}\equiv 0 \pmod{[\gcd(r,h)]_q}$ and $\sigma \equiv \frac{\#\mathcal{M}}{[\gcd(r,h)]_q} \pmod{\frac{[h]_q}{[\gcd(r,h)]_q}}$.

Proof. Using $\#\chi_V \equiv \chi_V(H)$ for each hyperplane H, the stated equations follow from Lemma 5. Since $n\in\mathbb{N}$, Equation (3) gives $\#\mathcal{M}\equiv 0 \pmod{[\gcd(r,h)]_q}$ and $\sigma \equiv \frac{\#\mathcal{M}}{[\gcd(r,h)]_q} \pmod{\frac{[h]_q}{[\gcd(r,h)]_q}}$. ■

Lemma 19. *Let $\mathcal{M}$ be a premultiset of points in $\mathrm{PG}(r-1,q)$. If $x[r]-\mathcal{M}$ is h-partitionable over $\mathbb{F}_q$ for $x\in\{\sigma,\sigma'\}$ then*

$$\left(\sigma + t\cdot\frac{[h]_q}{[\gcd(r,h)]_q}\right)\cdot[r] - \mathcal{M}$$

is h-partitionable over $\mathbb{F}_q$ for all $t\ge 0$ and we have $\sigma\equiv\sigma' \pmod{\frac{[h]_q}{[\gcd(r,h)]_q}}$.

Proof. Due to Theorem 2 we have that $\frac{[h]_q}{[\gcd(r,h)]_q}\cdot[r]$ is h-partitionable over $\mathbb{F}_q$, so that we can apply Lemma 7. The second statement is obviously true if $\sigma=\sigma'$, so that we assume $\sigma'>\sigma$ w.l.o.g. Let $\mathcal{S}$ be a faithful projective $h-(n,r,s)_q$ system with type $\sigma[r]-\mathcal{M}$ and $\mathcal{S}'$ be a faithful projective $h-(n',r,s')_q$ system with type $\sigma'[r]-\mathcal{M}$. Then, we have

$$\mathbb{Z}\ni n'-n = (\sigma'-\sigma)\cdot\frac{[r]_q}{[h]_q},$$

which implies $\sigma \equiv \sigma' \pmod{\frac{[h]_q}{[\gcd(r,h)]_q}}$ by Equation (3). ■

For those situations where we are not interested in the smallest possible value σ such that $\sigma[r] - \mathcal{M}$ is h-partitionable over $\mathbb{F}_q$ we introduce the following notion:

Definition 9. *Let $\mathcal{M}$ be a premultiset of points in* $\mathrm{PG}(r-1,q)$*. We say that $\star[r] - \mathcal{M}$ is h-partitionable over $\mathbb{F}_q$ if there exists an integer σ such that $\sigma[r] - \mathcal{M}$ is h-partitionable over $\mathbb{F}_q$.*

By using linear algebra we can decide whether $\star[r] - \mathcal{M}$ is h-partitionable over $\mathbb{F}_q$ for some premultiset of points $\mathcal{M}$ in $\mathrm{PG}(r-1,q)$. To this end we utilize the incidence matrix between points and h-spaces in $\mathrm{PG}(r-1,q)$ and the set of solutions of a linear equation system over $\mathbb{Z}$.

Definition 10. *For each $1 \le h \le r$ let $\left[{r \atop h}\right]_q$ denote the number of h-spaces in* $\mathrm{PG}(r-1,q)$*. The* incidence vector *of a faithful projective $h-(n,r,s,\mu)_q$ system $\mathcal{S}$ is a column vector $x \in \mathbb{N}^{\left[{r \atop h}\right]_q}$ whose entries $x_K \in \mathbb{N}$ equal the number of occurrences of h-spaces K in $\mathcal{S}$. Similarly, the* incidence vector *of a premultiset $\mathcal{M}$ in* $\mathrm{PG}(r-1,q)$ *is a column vector $x \in \mathbb{Z}^{[r]_q}$ whose entries $x_P \in \mathbb{Z}$ equal the point multiplicity $\mathcal{M}(P)$ for every point P in* $\mathrm{PG}(r-1,q)$*. We say that an a-space A and a b-space B are* incident *if either $A \le B$ or $A \ge B$. Let $A^{a,b;r,q} \in \{0,1\}^{\left[{r \atop a}\right]_q \times \left[{r \atop b}\right]_q}$ be the* incidence matrix *between the $\left[{r \atop a}\right]_q$ a-spaces A and the $\left[{r \atop b}\right]_q$ b-spaces B in* $\mathrm{PG}(r-1,q)$*, i.e., the entries of $A^{a,b;r,q}$ are given by $A^{a,b;r,q}_{A,B} = 1$ if A is incident with B and $A^{a,b;r,q}_{A,B} = 0$ otherwise.*

The notion $\left[{r \atop h}\right]_q$ extends $[r]_q$, counting the number of points or hyperplanes in $\mathrm{PG}(r-1,q)$, since $[r]_q = \left[{r \atop 1}\right]_q = \left[{r \atop r-1}\right]_q$. When working with those incidence matrices and incidence vectors for $\mathrm{PG}(r-1,q)$ we will always assume a fixed, but arbitrary, enumeration of the h-spaces for each $1 \le h \le r$.

Lemma 20. *Let $\mathcal{M}$ be a premultiset of points in* $\mathrm{PG}(r-1,q)$ *and z its incidence vector. Let v be the incidence vector of χ_V for the ambient space V. Then, $\star[r] - \mathcal{M}$ is h-partitionable over $\mathbb{F}_q$ iff there exists a unique integer $0 \le \sigma < \frac{[h]_q}{[\gcd(r,h)]_q}$ with $\sigma \equiv \frac{\#\mathcal{M}}{[\gcd(r,h)]_q} \pmod{\frac{[h]_q}{[\gcd(r,h)]_q}}$ and $A^{1,h;r,q} \cdot x = \sigma v - z$ admits a solution $x \in \mathbb{Z}^{\left[{r \atop h}\right]_q}$.*

Proof. If $\mathcal{S}$ is a faithful projective $h-(n,r,s,\mu)_q$ system with type $\sigma'[r] - \mathcal{M}$, then let $x' \in \mathbb{N}^{\left[{r \atop h}\right]_q}$ denote the incidence vector of $\mathcal{S}$. Due to Theorem 2 there exists a faithful projective $h-(n',r,s',\mu')_q$ system $\mathcal{S}'$ with type $\frac{[h]_q}{[\gcd(r,h)]_q} \cdot [r]$ and $v' \in \mathbb{N}^{\left[{r \atop h}\right]_q}$ its incidence vector. Now let y be the unique integer such that $0 \le \sigma < \frac{[h]_q}{[\gcd(r,h)]_q}$ for $\sigma := \sigma' - y \cdot \frac{[h]_q}{[\gcd(r,h)]_q}$, which also implies uniqueness for σ. Lemma 18 gives $\#\mathcal{M} \equiv 0 \pmod{[\gcd(r,h)]_q}$ and $\sigma' \equiv \frac{\#\mathcal{M}}{[\gcd(r,h)]_q} \pmod{\frac{[h]_q}{[\gcd(r,h)]_q}}$, so that also $\sigma \equiv \frac{\#\mathcal{M}}{[\gcd(r,h)]_q} \pmod{\frac{[h]_q}{[\gcd(r,h)]_q}}$. Since $A^{1,h;r,q} \cdot x' = \sigma' v - z$ we have $A^{1,h;r,q} \cdot (x' - y \cdot v') = \sigma v - z$, so that we set $x := x' - y \cdot v' \in \mathbb{Z}^{\left[{r \atop h}\right]_q}$.

For the other direction we assume that $x \in \mathbb{Z}^{\left[{r \atop h}\right]_q}$ is a solution of $A^{1,h;r,q} \cdot x = \sigma v - z$ with the stated constraints for σ. If $x \in \mathbb{N}^{\left[{r \atop h}\right]_q}$, then the corresponding faithful projective system has type $\sigma[r] - \mathcal{M}$ (noting $\sigma \in \mathbb{N}$). Otherwise we let $y < 0$ be the minimum entry of x. Let $\mathcal{S}''$ be the faithful projective $h - (n'', r, s'', \mu'')_q$ system with type $\sigma''[r]$ that consists of all h-spaces in $\operatorname{PG}(r-1, q)$ and $a \in \mathbb{N}^{\left[{r \atop h}\right]_q}$ its incidence vector. With this we set $x'' := x - y \cdot a \in \mathbb{N}^{\left[{r \atop h}\right]_q}$ and consider the corresponding faithful projective system with type $(\sigma - y \cdot \sigma'') \cdot [r] - \mathcal{M}$. ∎

Admittedly, Lemma 20 is not a very strong characterization result and looks rather technical. However, using the *Smith normal form* we can characterize solvability of linear equation systems over $\mathbb{Z}$. In order the keep the paper self-contained we give a brief exposition in Section 4. In Section 3 we consider a special class of premultisets of points $\mathcal{M}$ that are parameterized by parameters $\varepsilon_1, \ldots, \varepsilon_{r-1}$, which are connected to the Griesmer bound and the Solomon–Stiffler construction. Theorem 3 gives an explicit characterization of all parameters $\varepsilon_1, \ldots, \varepsilon_{r-1}$ such that $\star[r] - \mathcal{M}$ is h-partitionable.

3 A generalization of the Solomon–Stiffler construction

In [107] Solomon and Stiffler constructed $[n, k, d]_q$ codes with $n = g_q(k, d)$ for all parameters with sufficiently large minimum distance d. Here we want to show the generalization that the Griesmer upper bound for $n_q(r, h; s)$ can always be attained if s is sufficiently large.[5]

Using a specific parameterization of the minimum distance d the Griesmer bound in Inequality (1) can be written more explicitly:

Lemma 21. *Let k and d be positive integers. Write d as*

$$d = \sigma q^{k-1} - \sum_{i=1}^{k-1} \varepsilon_i q^{i-1}, \tag{18}$$

where $\sigma \in \mathbb{N}_0$ and the $0 \le \varepsilon_i < q$ are integers for all $1 \le i \le k-1$. Then, Inequality (1) is satisfied with equality iff

$$n = \sigma[k]_q - \sum_{i=1}^{k-1} \varepsilon_i [i]_q, \tag{19}$$

which is equivalent to

$$n - d = \sigma[k-1]_q - \sum_{i=1}^{k-1} \varepsilon_i [i-1]_q. \tag{20}$$

Remark 2. *Given k and d Equation (18) always determines σ and the ε_i uniquely. This is different for Equation (20) given k and $n - d = s$. Here it may happen that no solution with $0 \le \varepsilon_i \le q-1$ exists. By relaxing to $0 \le \varepsilon_i \le q$ we can ensure existence and uniqueness is enforced by additionally requiring*

[5] This property is the reason why we speak of *the* Griesmer upper bound in Definition 7 when referring to Lemma 14 which is just *an* application of the Griesmer bound.

$\varepsilon_j = 0$ *for all* $j < i$ *where* $\varepsilon_i = q$ *for some* i. *The same is true for Equation (19) given* k *and* n. *For more details we refer to [47, Chapter 2] which also gives pointers to Hamada's work on minihypers.*

Lemma 22. *Let* $S_1, \ldots, S_l$ *be a collection of subspaces of* $\mathrm{PG}(r-1, q)$ *such that exactly* ε_i *subspaces have dimension* i *for* $1 \le i \le r-1$ *and* V *be the* r*-dimensional ambient space. If*

$$\mathcal{M} = \sigma \cdot \chi_V - \sum_{i=1}^{l} \chi_{S_i} \tag{21}$$

is a multiset of points, i.e., if we have $\mathcal{M}(P) \in \mathbb{N}$ *for all points* P, *then* $\mathcal{M}$ *corresponds to a projective* $1-(n, r, \le s, \le \sigma)_q$ *system with*

$$n = \sigma \cdot [r]_q - \sum_{j=1}^{r-1} \varepsilon_j \cdot [j]_q \quad \textit{and} \quad s = \sigma \cdot [r-1]_q - \sum_{j=1}^{r-1} \varepsilon_j \cdot [j-1]_q. \tag{22}$$

Proof. Since $\#\chi_V = [r]_q$ and $\#\chi_{S_i} = [j]_q$ if S_i has dimension j, we have $n = \#\mathcal{M} = \sigma \cdot [r]_q - \sum_{j=1}^{r-1} \varepsilon_j \cdot [j]_q$. For each hyperplane H we have $\chi_V(H) = [r-1]_q$ and $\chi_{S_i}(H) \in \left\{[j-1]_q, [j]_q\right\}$ if S_i has dimension j, so that $\mathcal{M}(H) \le \sigma \cdot [r-1]_q - \sum_{j=1}^{r-1} \varepsilon_j \cdot [j-1]_q$. ■

Given parameters k, d, and q, the Solomon-Stiffler construction consists of choosing $\varepsilon_1, \ldots, \varepsilon_{k-1}$, and σ as in Equation (18). If σ is sufficiently large then a list of subspaces $S_1, \ldots, S_l$ (going in line with $\varepsilon_1, \ldots, \varepsilon_{k-1}$) can be chosen such that $\mathcal{M}$ as in Equation (21), where $r = k$, is a multiset of points. With this, the linear code C corresponding to $\mathcal{M}$ is an $[n, k, d]_q$ code with $n = g_q(k, d)$.

For a construction of a general faithful projective $h-(n, r, s)_q$ system $\mathcal{S}$ with $n > s$ we aim to reverse Lemma 4 where we have associated a linear code C to $\mathcal{S}$. To this end we consider a q^{h-1}-divisible linear $[n', r, d']_q$ code C with maximum weight at most $n \cdot q^{h-1}$, where $n' = [h]_q \cdot n$ and $d' = q^{h-1} \cdot (n-s)$. If we can partition the multiset of points $\mathcal{M} = \mathcal{X}(C)$ associated with C into the multiset union of h-spaces, we obtain a faithful projective $h-(n, r, s')_q$ system, where we hopefully have $s' = s$ (or $s' \le s$). There are a few technical obstacles to overcome. The existence of a suitable partition of a given multiset $\mathcal{M}$ of points into h-spaces is formalized in Definition 11. Here $\mathcal{M}$ is described by parameters σ and $\varepsilon_1, \ldots, \varepsilon_{r-1}$. In Lemma 28 we show how to compute the parameters of a possible faithful projective $h-(n, r, s, \mu)_q$ system from this data and deduce necessary conditions for the existence of a partition for the parameters $\varepsilon_1, \ldots, \varepsilon_{r-1}$. An additional condition for σ is concluded in Lemma 30. In Theorem 3 we show that these conditions are also sufficient.

Definition 11. *Let* $\sigma \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_{r-1} \in \mathbb{Z}$, *and let* V *denote the* r*-dimensional ambient space* $\mathrm{PG}(r-1, q)$. *We say that a faithful projective* $h-(n, r, s)_q$ *system* $\mathcal{S}$ *has* type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ *if there exist* *subspaces* $S_1 \le \cdots \le S_{r-1}$ *with* $\dim(S_i) = i$ *and*

$$\sum_{S \in \mathcal{S}} \chi_S = \sigma \cdot \chi_V - \sum_{i=1}^{r-1} \varepsilon_i \cdot \chi_{S_i}. \tag{23}$$

We say that $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ *is* h-partitionable over $\mathbb{F}_q$ *if a faithful projective* $h-(n,r,s)_q$ *system with type* $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ *exists for suitable parameters* n *and* s.

Note that all chains $S_1 \le \cdots \le S_{r-1}$ are isomorphic, so that the notion of being h-partitionable does not depend on the choice of the subspaces $S_1, \dots, S_{r-1}$. However, the chosen restriction usually causes rather large values of σ and does not cover the full generality of the Solomon-Stiffler construction. On the other hand, a more general definition causes several technical complications as we will briefly outline in Appendix A.

Next we give a few constructions.

Lemma 23. *For* $r > a \ge h$ *with* $r \equiv a \pmod h$ *and* $\sigma \in \mathbb{N}_{\ge 1}$ *we have that* $\sigma[r] - \sigma[a]$ *is h-partitionable over* $\mathbb{F}_q$.

Proof. If $a > h$, then Lemma 12 yields the existence of a faithful projective $h-(n,r,s)_q$ system $\mathcal{S}$ with type $[r]-[a]$ and we can use σ copies thereof. For $a = h$ we replace $\mathcal{S}$ by a spread of h-spaces of $\mathrm{PG}(r-1,q)$ where we remove an arbitrary element. ■

Lemma 24. *For* $1 \le j \le h$ *and* $r \ge 2h+1-j$

$$[j]_q \cdot [r] - 1 \cdot [r-h-1+j] - \left([j]_q - 1\right) \cdot [r-h-1] \tag{24}$$

is h-partitionable over $\mathbb{F}_q$.

Proof. Let A be an $(r-h-1)$-space and $B \ge A$ be an $(r-h-1+j)$-dimensional subspace of $\mathrm{PG}(r-1,q)$. By $K_1, \dots, K_l$ we denote the $l := [j]_q$ $(r-h)$-spaces with $A \le K_i \le B$. For $1 \le i \le l$ let $\mathcal{V}_i$ be a vector space partition of $\mathrm{PG}(r-1,q)$ of type $h^{t_h}(r-h)^1$ where the special $(r-h)$-space coincides with K_i, see Lemma 10. The desired faithful projective $h-(n,r,s)_q$ system is then given by the union of the t_h h-dimensional non-special subspaces of the l vector space partitions $\mathcal{V}_i$. ■

For $h = 2$ the construction of Lemma 24 is stated as construction $\mathcal{L}^\star$ in [85], see also [16, Lemma 3].

Lemma 25. *If* $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ *and* $\sigma'[r] - \sum_{i=1}^{r-1} \varepsilon'_i[i]$ *are h-partitionable over* $\mathbb{F}_q$, *then* $(\sigma+\sigma') \cdot [r] - \sum_{i=1}^{r-1} \left(\varepsilon_i + \varepsilon'_i\right) \cdot [i]$ *is h-partitionable over* $\mathbb{F}_q$.

Proof. Fix some subspaces $S_1 \le \cdots \le S_{r-1}$ as in Definition 11. Let $\mathcal{S}$ be a faithful projective $h-(n,r,s,\mu)_q$ system with type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ and $\mathcal{S}'$ be a faithful projective $h-(n',r,s',\mu')_q$

system with type $\sigma'[r]-\sum_{i=1}^{r-1}\varepsilon'_i[i]$, then the multiset union of the elements of $\mathcal{S}$ and $\mathcal{S}'$ is a faithful projective $h-(n+n',r,s+s',\mu+\mu')_q$ system with type $(\sigma+\sigma')\cdot[r]-\sum_{i=1}^{r-1}\left(\varepsilon_i+\varepsilon'_i\right)\cdot[i]$. ∎

Lemma 26. *For $r>h\ge 2$ with $r\equiv 1 \pmod h$ we have that $[h-1]_q\cdot[r]+q^{h-1}\cdot[1]$, $\left([h]_q-1\right)\cdot[r]-q\cdot[h-1]$, and $\left([h]_q-1\right)\cdot[r]+[h]-q\cdot[h-1]$ are h-partitionable over $\mathbb{F}_q$.*

Proof. Let us first consider the case $r=h+1$. Let $\mathcal{S}$ be the faithful projective $h-(n,r,s,\mu)_q$ system that consists of all h-spaces that contain point S_1 (as in Definition 11), so that $n=[h]_q$. Then S_1 is contained in $[h]_q$ elements and every other point is contained in $[h-1]_q$ elements, i.e. $\mathcal{S}$ has type $[h-1]_q\cdot[r]+q^{h-1}\cdot[1]$. Let $\mathcal{S}'$ be the faithful projective $h-(n',r,s',\mu')_q$ system that consists of all h-spaces that do not contain S_{h-1} (as in Definition 11), so that $\mathcal{S}'$ has type $\left([h]_q-1\right)\cdot[r]-q\cdot[h-1]$. Adding S_h (as in Definition 11) to $\mathcal{S}'$ gives type $\left([h]_q-1\right)\cdot[r]+[h]-q\cdot[h-1]$.

If $r>h+1$, then Lemma 23 shows that $\sigma[r]-\sigma[h+1]$ is h-partitionable for each $\sigma\in\mathbb{N}$, so that the statement follows from Lemma 25. ∎

We remark that the first construction of Lemma 26 is also described in e.g. [17, Theorem 4] and [52, Lemma 10] for $q=h=2$. Geometrically it is the smallest covering of the set of points of $\mathrm{PG}(r-1,q)$ by lines.

From Lemma 8, based on field reduction, we conclude:

Lemma 27. *If $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_{q^l}$, so is $\sigma[rl]+\sum_{i=1}^{r-1}\varepsilon_i[il]$.*

If $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_q$, then we can compute the parameters of a corresponding faithful projective $h-(n,r,s,\mu)_q$ system $\mathcal{S}$ as well as of its dual $\mathcal{S}^\perp$ from this data. Moreover, we obtain some necessary conditions on the parameters $\varepsilon_1,\dots,\varepsilon_{r-1}$. Later on we will see in Theorem 3 that they are also sufficient for the existence of $\mathcal{S}$ for a sufficiently large σ satisfying an additional modulo constraint, see Lemma 30. So, Lemma 5 specializes to:

Lemma 28. *If $\mathcal{S}$ is a faithful projective $h-(n,r,s,\mu)_q$ system with type $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$, then we have*

$$n=\left(\sigma[r]_q-\sum_{i=1}^{r-1}\varepsilon_i[i]_q\right)/[h]_q, \tag{25}$$

$$s=\max_{1\le j\le r}\left(s_1-\sum_{i=1}^{j-1}\varepsilon_i q^{i-h}\right), \tag{26}$$

where

$$s_1=\left(\sigma[r-h]_q-\sum_{i=h}^{r-1}\varepsilon_i[i-h]_q+\sum_{i=1}^{h-1}\varepsilon_i q^{i-h}[h-i]_q\right)/[h]_q, \tag{27}$$

and

$$\mu=\max_{1\le j\le r}\left(\sigma-\sum_{i=1}^{j-1}\varepsilon_i\right). \tag{28}$$

Moreover, ε_i is divisible by q^{h-i} for all $1 \le i \le h-1$ and

$$\sum_{i=1}^{r-1} \varepsilon_i [i]_q \equiv 0 \pmod{[\gcd(r,h)]_q}. \tag{29}$$

The dual $\mathcal{S}^\perp$ of $\mathcal{S}$ is a faithful projective $h'-(n,r,\mu,s)_q$ system with type $\sigma'[r] - \sum_{i=1}^{r-1} \varepsilon'_i[i]$, where $h' = r-h$, $\sigma' = s_1$, and $\varepsilon'_i = \varepsilon_{r-i} \cdot q^{h'-i}$ for all $1 \le i \le r-1$.

Proof. Let $\mathcal{M}$ be the multiset of points covered by the elements of $\mathcal{S}$ and $S_1 \le \cdots \le S_{r-1}$ be subspaces as in Definition 11. Since $\mathcal{M}$ has cardinality

$$\sigma[r]_q - \sum_{i=1}^{r-1} \varepsilon_i [i]_q$$

and one h-space contains $[h]_q$ points, we conclude Equation (25).

For an arbitrary point P let $1 \le j \le r$ denote the minimal integer such that $P \not\le S_j$, where we set $j = r$ if $P \le S_{r-1}$. With this we have

$$\mathcal{M}(P) = \sigma - \sum_{i=1}^{j-1} \varepsilon_i, \tag{30}$$

which implies Equation (28).

For an arbitrary hyperplane H let $1 \le j \le r$ denote the minimal integer such that $S_j \not\le H$, where we set $j = r$ if $H = S_{r-1}$. Counting points gives

$$\mathcal{M}(H) = \sigma[r-1]_q - \sum_{i=1}^{j-1} \varepsilon_i[i]_q - \sum_{i=j}^{r-1} \varepsilon_i[i-1]_q = \sigma[r-1]_q - \sum_{i=1}^{r-1} \varepsilon_i[i-1]_q - \sum_{i=1}^{j-1} \varepsilon_i q^{i-1}.$$

The number s_j of elements of $\mathcal{S}$ contained in H is given by $\left(\mathcal{M}(H) - n \cdot [h-1]_q\right)/q^{h-1}$, so that

$$\begin{aligned}
s_j &= \left(\sigma[r-1]_q - \sum_{i=1}^{r-1} \varepsilon_i[i-1]_q - \sum_{i=1}^{j-1} \varepsilon_i q^{i-1} - \left(\sigma[r]_q - \sum_{i=1}^{r-1} \varepsilon_i[i]_q\right) \cdot \frac{[h-1]_q}{[h]_q}\right)/q^{h-1} \\
&= \left(\sigma \cdot \left([h]_q[r-1]_q - [h-1]_q[r]_q\right) - \sum_{i=1}^{r-1} \varepsilon_i \cdot \left([h]_q[i-1]_q - [h-1]_q[i]_q\right)\right) / \left(q^{h-1} \cdot [h]_q\right) \\
&\quad - \sum_{i=1}^{j-1} \varepsilon_i q^{i-h} \\
&\overset{(2)}{=} \left(\sigma[r-h]_q - \sum_{i=h}^{r-1} \varepsilon_i[i-h]_q + \sum_{i=1}^{h-1} \varepsilon_i q^{i-h}[h-i]_q\right)/[h]_q - \sum_{i=1}^{j-1} \varepsilon_i q^{i-h}.
\end{aligned}$$

This verifies Equation (27) and yields

$$s_j = s_1 - \sum_{i=1}^{j-1} \varepsilon_i q^{i-h} \tag{31}$$

for $2 \le j \le r$, which implies Equation (26). Since $s_j \in \mathbb{N}$ for $1 \le j \le r$, we can recursively conclude that ε_i is divisible by q^{h-i} for $i = 1, \ldots, h-1$. By Equation (3) we have $\gcd\big([r]_q, [h]_q\big) = [\gcd(r,h)]_q$, so that Equation (25) implies Equation (29).

By Lemma 6 $\mathcal{S}^\perp$ is a faithful projective $h' - (n, r, s, \mu)_q$ system for $h' := r - h$. If x elements of $\mathcal{S}$ are contained in a hyperplane H, then x elements of $\mathcal{S}^\perp$ contain the point $H^\perp$. Note that $(S_{r-1})^\perp \le \cdots \le (S_1)^\perp$ are the subspaces as in Definition 11 for $\mathcal{S}^\perp$ and that we have $\dim\big((S_i)^\perp\big) = r-i$ for all $1 \le i \le r-1$. For every hyperplane $H \not\ge S_1$ we have $H^\perp \not\le (S_1)^\perp$, so that $\sigma' = s_1$. For every integer $1 \le i \le r-1$ we have

$$\varepsilon'_{r-i} = s_i - s_{i+1} = \varepsilon_i q^{i-h},$$

which is equivalent to $\varepsilon'_i = \varepsilon_{r-i} q^{h'-i}$. ■

Corollary 4. *If all ε_i are nonnegative, then $s = s_1$ and $\mu = \sigma$ (using the notation from Lemma 28).*

Corollary 5. *If $\mathcal{S}_t$ is a faithful projective $h - (n_t, r, s_r)_q$ system with type $\left(\sigma + t \cdot \frac{[h]_q}{[\gcd(r,h)]_q}\right) \cdot [r] - \sum_{i=1}^{r-1} \varepsilon_i [i]$, where $\varepsilon_1 = \cdots = \varepsilon_{h-1} = 0$ and $\varepsilon_h, \ldots, \varepsilon_{r-1} \in \mathbb{N}$, then we have*

$$n_t = t \cdot \frac{[r]_q}{[\gcd(r,h)]_q} + \left(\sigma [r]_q - \sum_{i=h}^{r-1} \varepsilon_i [i]_q\right) / [h]_q, \tag{32}$$

$$s_t = t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q} + \left(\sigma [r-h]_q - \sum_{i=h}^{r-1} \varepsilon_i [i-h]_q\right) / [h]_q, \tag{33}$$

and

$$n_t - s_t = t \cdot \frac{[h]_q}{[\gcd(r,h)]_q} \cdot q^{r-h} + \sigma \cdot q^{r-h} - \sum_{i=h}^{r-1} \varepsilon_i \cdot q^{i-h}. \tag{34}$$

Next we show that the conditions for $\varepsilon_1, \ldots, \varepsilon_{r-1}$ imply the corresponding conditions for $\varepsilon'_1, \ldots, \varepsilon'_{r-1}$ when defined formally, i.e. without assuming the existence of a faithful projective system or its dual.

Lemma 29. *Let q be a prime power, $r > h \ge 1$, $\varepsilon_1, \ldots, \varepsilon_{r-1} \in \mathbb{Z}$ such that q^{h-i} divides ε_i for all $1 \le i \le h-1$ and*

$$\sum_{i=1}^{r-1} \varepsilon_i [i]_q \equiv 0 \pmod{[\gcd(r,h)]_q}.$$

Setting $h' := r - h$ and $\varepsilon'_i := \varepsilon_{r-i} \cdot q^{h'-i}$ for $1 \le i \le r-1$, we have $\varepsilon'_1, \ldots, \varepsilon'_{r-1} \in \mathbb{Z}$, q^{h-i} divides ε'_i for all

$1 \le i \le h-1$, *and*

$$\sum_{i=1}^{r-1} \varepsilon'_i [i]_q \equiv 0 \pmod{[\gcd(r,h')]_q}. \tag{35}$$

Proof. First we observe that $\varepsilon'_i = \varepsilon_{r-i} \cdot q^{h'-i}$ is an integer and divisible by $q^{h'-i}$ for all $1 \le i \le h'$. For all $h'+1 \le i \le r-1$ we have $\varepsilon'_i = \varepsilon_{r-i} \cdot q^{h'-i} \in \mathbb{Z}$ since $1 \le r-i \le r-h'-1 = h-1$ and ε_{r-i} is divisible by $q^{h-r+i} = q^{i-h'}$.

In order to verify Equation (35) we observe $g := \gcd(r,h) = \gcd(r,h')$, so that it suffices to show $\varepsilon'_i \cdot [i]_q + \varepsilon_{r-i} \cdot [r-i]_q \equiv 0 \pmod{[g]_q}$ for all $1 \le i \le r-1$.

For $h' \ge i$ we have

$$\varepsilon'_i \cdot [i]_q + \varepsilon_{r-i} \cdot [r-i]_q = \varepsilon_{r-i} \cdot \left(q^{h'-i}[i]_q + [r-i]_q\right) \equiv \varepsilon_{r-i} \cdot [h']_q \equiv 0 \pmod{[g]_q}.$$

For $i > h'$ we have

$$\varepsilon'_i \cdot [i]_q + \varepsilon_{r-i} \cdot [r-i]_q = q^{h'-i}\varepsilon_{r-i} \cdot \left([i]_q + [r-i]_q \cdot q^{i-h'}\right) \equiv q^{h'-i}\varepsilon_{r-i} \cdot [h]_q \equiv 0 \pmod{[g]_q}.$$

■

Lemma 19 specializes to:

Lemma 30. *If* $x[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ *is h-partitionable over* $\mathbb{F}_q$ *for* $x \in \{\sigma, \sigma'\}$ *then*

$$\left(\sigma + t \cdot \frac{[h]_q}{[\gcd(r,h)]_q}\right) \cdot [r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$$

is h-partitionable over $\mathbb{F}_q$ *for all* $t \ge 0$ *and we have* $\sigma \equiv \sigma' \pmod{\frac{[h]_q}{[\gcd(r,h)]_q}}$.

In other words, if $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_q$, then $\sigma'[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_q$ for all $\sigma' \ge \sigma$ if the latter yields an integer in Equation (25). For those situations where we are not interested in the smallest possible value σ such that $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_q$ for given parameters $\varepsilon_1, \ldots, \varepsilon_{r-1}$ we introduce the following notion:

Definition 12. *We say that* $\star[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ *is h-partitionable over* $\mathbb{F}_q$ *if there exists an integer* σ *such that* $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ *is h-partitionable over* $\mathbb{F}_q$.

Lemma 31. *If* $\star[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ *is h-partitionable over* $\mathbb{F}_q$, *so is* $\star[r] + \sum_{i=1}^{r-1} \varepsilon_i[i]$.

Proof. Fix some subspaces $S_1 \le \cdots \le S_{r-1}$ as in Definition 11. Consider a faithful projective $h-(n,r,s)_q$ system $\mathcal{S}$ with type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ for a suitable σ. Let $\mathcal{S}'$ be the faithful projective $h-(n',r,s')_q$ system with type $\sigma'[r]$ that consists of all h-spaces of $\mathrm{PG}(r-1,q)$ and let μ' be the maximal number of occurrences of an element in $\mathcal{S}$. Then, μ' copies of $\mathcal{S}'$ give the desired partition after removing the elements of $\mathcal{S}$ (with their respective multiplicity). ■

It is an interesting problem to determine for which parameters $\varepsilon_1,\dots,\varepsilon_{r-1}$ we have that $\star[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_q$. Clearly we need $r\ge h$ and $\sum_{i=1}^{r-1}|\varepsilon_i|=0$ if $r=h$. Additionally we have the packing condition (29) and that ε_i has to be divisible by q^{h-i} for all $1\le i\le r-1$, see Lemma 28. These conditions are indeed sufficient.

Lemma 32.

(i) Let $\star[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ be h-partitionable over $\mathbb{F}_q$, $h\le a<j$ with $a\equiv j\pmod h$, $\varepsilon'_i=\varepsilon_i$ for all $1\le i\le r-1$ with $i\notin\{a,j\}$, $\varepsilon'_j=\varepsilon_j+1$, and $\varepsilon'_a=\varepsilon_a-1$. Then, $\star[r]-\sum_{i=1}^{r-1}\varepsilon'_i[i]$ is h-partitionable over $\mathbb{F}_q$.

(ii) If $\star[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_q$, then $\star[r+th]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_q$ for all integers $t\ge 1$.

Proof. Let $\mathcal{S}$ be a faithful projective $h-(n,r,s)_q$ system with type $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ for a suitably large σ. Fix some subspaces $S_1\le\dots\le S_{r-1}$ as in Definition 11. By $\mathcal{S}'$ we denote the faithful projective $h-(n',r,s')_q$ system with type $\sigma'[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ that arises as the multiset union of the elements of $\mathcal{S}$ and the set of all h-spaces in $\mathrm{PG}(r-1,q)$.

By Lemma 23 $[j]-[a]$ is h-partitionable over $\mathbb{F}_q$, so that we identify S_j with $\mathrm{PG}(j-1,q)$ to construct a faithful projective $h-(n'',r,s'')_q$ system $\mathcal{S}''$ with type $0[r]+[j]-[a]$. Since $\mathcal{S}'$ contains every h-space in $\mathrm{PG}(r-1,q)$ at least once and $\mathcal{S}''$ contains every h-space in $\mathrm{PG}(r-1,q)$ at most once, removing the elements in $\mathcal{S}''$ from the elements of $\mathcal{S}'$ gives a faithful projective system with type $\sigma'[r]-\sum_{i=1}^{r-1}\varepsilon'_i[i]$, which shows (i).

Lemma 23 yields the existence of a faithful projective $h-(n''',r,s''')_q$ system $\mathcal{S}'''$ with type $\sigma[r+th]-\sigma[r]$. Fix some subspaces $S'_1\le\dots\le S'_{r+th-1}$ as in Definition 11. Embedding $\mathcal{S}$ in S'_r and taking the multiset union with the elements of $\mathcal{S}'''$ gives a faithful projective system with type $\sigma[r+th]-\sum_{i=1}^{r-1}\varepsilon_i[i]$, which shows (ii). ∎

Theorem 3. *Let q be a prime power, $r>h\ge 1$, $g:=\gcd(r,h)$, and $\varepsilon_1,\dots,\varepsilon_{r-1}\in\mathbb{Z}$ such that q^{h-i} divides ε_i for all $1\le i<h$ and*

$$\sum_{i=1}^{r-1}\varepsilon_i\cdot[i]_q\equiv 0\pmod{[g]_q}. \tag{36}$$

Then $\star[r]-\sum\limits_{i=1}^{r-1}\varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_q$.

Proof. We prove by induction over h. The statement is obvious for $h=1$, so that we assume $h\ge 2$ in the following.

Due to Theorem 2 $\star[r]+[h]_q\cdot[i]$ is h-partitionable over $\mathbb{F}_q$ for each $h+1\le i\le r-1$, so that also $\star[r]-[h]_q\cdot[i]$ is h-partitionable over $\mathbb{F}_q$ for each $h+1\le i\le r-1$ by Lemma 31. Using

Lemma 25 we can assume $\varepsilon_i \geq 0$ for all $h+1 \leq i \leq r-1$ since these operations do not violate Equation (36).

By iteratively applying Lemma 32.(i) with $a = j - h$ we can assume $\varepsilon_j = 0$ for all $2h \leq j \leq r-1$ because these operations do not violate Equation (36) since $[j]_q \equiv [a]_q \mod [h]_q$. Due to Lemma 32.(ii) we can additionally assume $r < 3h$ noting that we still have $\gcd(r, h) = g$.

If $2h \leq r < 3h$, then by using Lemma 25 with Lemma 24 we can assume $\varepsilon_i = 0$ for all $r - h \leq i \leq r-1$, since these operations do not violate Equation (36). Due to Lemma 32.(ii) we can additionally assume $r \leq 2h$ noting that we still have $\gcd(r, h) = g$.

For $h < r < 2h$ we have $1 \leq r - h < h$ and we let $\mathcal{S}$ denote the desired faithful projective $(n, r, s)_q$ system with type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$. Using Lemma 29 and Lemma 28 we can deduce the existence of $\mathcal{S}^\perp$ for a sufficiently large value of σ from the induction hypothesis.

It remains to consider the case $r = 2h$ where $g = h$. Since $\gcd(r-1, h) = 1$ we can conclude the existence of a faithful projective $h - (n', r-1, s')_q$ system $\mathcal{S}'$ with type $t[r-1] - \sum_{i=1}^{r-2} \varepsilon_i[i]$, for some suitably large integer $t \geq -\varepsilon_{r-1}$, from the previous part of the proof. By applying Lemma 28 to $\mathcal{S}'$ we conclude

$$\sum_{i=1}^{r-2} \varepsilon_i[i]_q \equiv t[r-1]_q \pmod{[h]_q} \tag{37}$$

from Equation (25). From Equation (36) we conclude

$$\sum_{i=1}^{r-2} \varepsilon_i[i]_q \equiv -\varepsilon_{r-1}[r-1]_q \pmod{[h]_q}, \tag{38}$$

so that

$$(t + \varepsilon_{r-1}) \cdot [r-1]_q \equiv 0 \pmod{[h]_q}.$$

Since $\gcd(r-1, h) = 1$ Equation (3) implies $t + \varepsilon_{r-1} \equiv 0 \pmod{[h]_q}$, so that Theorem 2 yields that $(t + \varepsilon_{r-1}) \cdot [r-1]$ as well as $0 \cdot [r] + (t + \varepsilon_{r-1}) \cdot [r-1]$ are h-partitionable over $\mathbb{F}_q$. From Lemma 31 we conclude the existence of a faithful projective $h - (n'', r, s'')_q$ system $\mathcal{S}''$ with type $\sigma''[r] - (t + \varepsilon_{r-1}) \cdot [r-1]$. Fix some subspaces $S_1 \leq \cdots \leq S_{r-1}$ as in Definition 11 for $\mathrm{PG}(r-1, q)$. Embedding $\mathcal{S}'$ in S_{r-1}, such that $S_1 \leq \cdots \leq S_{r-2}$ are the subspaces as in Definition 11, and taking the multiset union with the elements from $\mathcal{S}''$ gives a faithful projective system with type $\sigma''[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$. ■

Note that the stated conditions only ensure that $t[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is h-partitionable if t is sufficiently large. Relatively small values of t can sometimes ruled out by some kind of "dimension arguments", see the example in Remark 13.

Example 6. *We want to explicitly show that* $\star[8] - [7] - [6] - [5] - [4] - [3]$ *is* 2*-partitionable over* $\mathbb{F}_2$ *going along the lines of the proof of Theorem 3. In our situation we start with* $r = 8$, $h = 2$ *and* *already have $\varepsilon_i \ge 0$ for all $1 \le i \le r-1$, so that we start by reducing to the situation where $\varepsilon_i = 0$ for all $2h \le i \le r-1$, i.e., we need to show that $\star[8] - 3[3] - 2[2]$ is 2-partitionable over $\mathbb{F}_2$. By reducing r by multiples of h it remains to show that $\star[4] - 3[3] - 2[2]$ is 2-partitionable over $\mathbb{F}_2$. From Lemma 24 we know that $[4] - [2]$ and $3[4] - [3] - 2[1]$ are 2-partitionable over $\mathbb{F}_2$, so that it remains to show that $\star[4] + 6[1]$ is 2-partitionable over $\mathbb{F}_2$. Here we do not reduce r by h since we need $r > h$. Indeed, $\star[2] + 6[1]$ is not 2-partitionable over $\mathbb{F}_2$. So, we are in the situation $r = 2h$ and first try to determine a t such that $t[3] + 6[1]$ is 2-partitionable over $\mathbb{F}_2$, where we have $h < r < 2h$. Via duality we need to show that $\star[3] + 3[2]$ is 1-partitionable over $\mathbb{F}_2$. Clearly, we can construct a faithful projective $1-(9,3,9,3)_2$ system $\mathcal{S}$ with type $0[3] + 3[2]$. By Lemma 28 $\mathcal{S}^\perp$ is a faithful projective $2-(9,3,3,9)_2$ system with type $3[3] + 6[1]$, i.e. we have $t = 3$. More directly, $\mathcal{S}^\perp$ arises by taking each line in $\mathrm{PG}(2,2)$ that contains point S_1 (as in Definition 11) three times. From Theorem 2 we know that $0[4] + 3[3]$ is 2-partitionable over $\mathbb{F}_2$. Since $\mathrm{PG}(3,2)$ contains 35 lines we conclude that $105[4] - 3[3]$ is 2-partitionable over $\mathbb{F}_2$, so that also $105[4] + 6[1]$ is 2-partitionable over $\mathbb{F}_2$. Since $[4] - [2]$ and $3[4] - [3] - 2[1]$ are 2-partitionable over $\mathbb{F}_2$ we conclude that also $116[4] - 3[3] - 2[2]$ is 2-partitionable over $\mathbb{F}_2$. From Lemma 32 we get that also $116[8] - 3[3] - 2[2]$ and $116[8] - [7] - [6] - [5] - [4] - [3]$ are 2-partitionable over $\mathbb{F}_2$. Lemma 30 yields that $\sigma[8] - [7] - [6] - [5] - [4] - [3]$ is 2-partitionable over $\mathbb{F}_2$ for all $\sigma \ge 116$ with $\sigma \equiv 2 \pmod 3$.*

Remark 3. *When we apply the constructive proof of Theorem 3 to compute σ such that $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i$ is h-partitionable over $\mathbb{F}_q$, the obtained value will usually be rather huge, so that we refrain from stating explicit upper bounds.*

For the case of Example 6 we remark that Lemma 24 also shows that $3[8] - [7] - 2[5]$ is 2-partitionable over $\mathbb{F}_2$, so that Lemma 32 implies that $3[8] - [7] - [5] - [3]$ is 2-partitionable over $\mathbb{F}_2$. Since $[8] - [6]$ and $[8] - [4]$ are 2-partitionable over $\mathbb{F}_2$ by Lemma 23, Lemma 25 implies that $5[8] - [7] - [6] - [5] - [4] - [3]$ is 2-partitionable over $\mathbb{F}_2$.

Definition 13. *For integers $n > s \ge 1$, $r > h \ge 1$, and a prime power q let the* surplus *be defined by*

$$\theta(n,r,s,h,q) := n \cdot [h]_q - g_q\Big(r, q^{h-1} \cdot (n-s)\Big). \tag{39}$$

So the surplus is negative iff n is larger than the Griesmer upper bound for $n_q(r,h;s)$.

Lemma 33. *Let $n > s \ge 1$, $r > h \ge 1$ be integers and q be a prime power. If $\theta(n,r,s,h,q) \ge 0$, then there exists a faithful projective $h - \left(n + t \cdot \frac{[r]_q}{[\gcd(r,h)]_q}, r, s + t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q}\right)_q$ system $\mathcal{S}_t$ for all sufficiently large t.*

Proof. Setting $d' := q^{h-1} \cdot (n-s)$ and $n' := g_q(r,d')$ we have $[h]_q > \theta(n,r,s,h,q) = n[h]_q - n' \ge 0$. Due to Lemma 21 we can choose integers $\sigma, \varepsilon_1, \ldots, \varepsilon_{r-1}$, with $\sigma \ge 0$ and $0 \le \varepsilon_i < q$ for all $1 \le i \le r-1$, such that

$$d' = \sigma q^{r-1} - \sum_{i=1}^{r-1} \varepsilon_i q^{i-1} \tag{40}$$

and

$$n' = \sigma[r]_q - \sum_{i=1}^{r-1} \varepsilon_i[i]_q. \tag{41}$$

Since d' is divisible by q^{h-1} we have $\varepsilon_i = 0$ for all $1 \le i \le h-1$. Let $\tau := \theta(n,r,s,h,q)$, $\sigma' := \sigma + \tau$, $\varepsilon'_{r-1} = \varepsilon_{r-1} + \tau q$, and $\varepsilon'_i = \varepsilon_i$ for all $1 \le i \le r-2$, so that $\varepsilon'_i \in \mathbb{N}$ for all $1 \le i \le r-1$ and $\varepsilon'_i = 0$ for all $1 \le i \le h-1$. Note that

$$d' = \sigma' q^{r-1} - \sum_{i=1}^{r-1} \varepsilon'_i q^{i-1} \tag{42}$$

and

$$n[h]_q = n' + \tau = \sigma'[r]_q - \sum_{i=1}^{r-1} \varepsilon'_i[i]_q, \tag{43}$$

so that

$$\sum_{i=1}^{r-1} \varepsilon'_i[i]_q \equiv 0 \pmod{[\gcd(r,h)]_q}. \tag{44}$$

From Theorem 3 and Lemma 30 we conclude that $\left(\sigma' + t \cdot \frac{[h]_q}{[\gcd(r,h)]_q}\right) \cdot [r] - \sum_{i=1}^{r-1} \varepsilon'_i[i]$ is h-partitionable over $\mathbb{F}_q$ for all sufficiently large t. From Corollary 5, Equation (42), and Equation (43) we compute the stated parameters of $\mathcal{S}_t$. ■

For other results and notions of asymptotically good additive codes we refer e.g. to [103].

Example 7. *Let $q = 2$, $r = 8$, $h = 2$, $n = 30$, and $s = 8$, so that $d' = q^{h-1} \cdot (n-s) = 44$. In order to compute $n' := g_2(8,44)$ we determine parameters σ, $\varepsilon_1, \dots, \varepsilon_7$, so that Equation (40) is satisfied, where $0 \le \varepsilon_1, \dots, \varepsilon_7 < 2$. Here the unique solution is given by $\sigma = 1$, $\varepsilon_7 = \varepsilon_5 = \varepsilon_3 = 1$, and $\varepsilon_i = 0$ otherwise. Via Equation (41) we can compute $n' = [8]_2 - [7]_2 - [5]_2 - [3]_2 = 90$, so that the surplus is given by $\theta(30,8,8,2,2) = n[h]_q - n' = 0$. Applying Theorem 3 shows that $\star[8] - [7] - [5] - [3]$ is 2-partitionable over $\mathbb{F}_2$. As mentioned in Remark 3 $3[8] - [7] - [5] - [3]$ is 2-partitionable over $\mathbb{F}_2$, so that $n_2(8,3;8+21t) \ge 30 + 85t$ for all $t \ge 2$.*

Remark 4. *The series of projective systems in Example 7 attains indeed the Griesmer upper bound. In general we can start with any set of parameters that do not violate the Griesmer bound in Lemma 14 and obtain a series of projective systems that tend towards the upper bound in Lemma 15.*

Remark 5. *The proof of Lemma 33 provides a more general construction. Starting from parameters $\sigma, \varepsilon_1, \dots, \varepsilon_{r-1}$ obtained from Equation (40) and Equation (41) we can modify to any nonnegative parameters $\sigma', \varepsilon'_1, \dots, \varepsilon'_{r-1}$ satisfying equations (42), (43) and the conditions from Theorem 3, where Condition (44) is automatically satisfied, see e.g. Example 8.*

Example 8. *Let $q = 2$, $r = 9$, $h = 3$, $n = 55$, and $s = 7$, so that $d' = q^{h-1} \cdot (n-s) = 192$. In order to compute $n' := g_2(9,192)$ we determine parameters σ, $\varepsilon_1, \dots, \varepsilon_8$, so that Equation (40) is* *satisfied, where* $0 \le \varepsilon_1, \ldots, \varepsilon_8 < 2$. *Here the unique solution is given by* $\sigma = 1$, $\varepsilon_7 = 1$, *and* $\varepsilon_i = 0$ *otherwise. Via Equation (41) we can compute* $n' = [9]_2 - [7]_2 = 384$, *so that the surplus is given by* $\theta(55, 9, 7, 3, 2) = n[h]_q - n' = 1$. *We have that* $\star[9] - [7]$ *is not* 3*-partitionable over* $\mathbb{F}_2$ *since Condition (29) is violated. Here we use the surplus to modify* $\star[9] - [7]$ *to* $\star[9] - 2[6]$ *(instead of* $\star[9] - 2[8]$*). Since* $2[3] - 2[2]$ *is* 1*-partitionable over* $\mathbb{F}_8$ *we have that* $2[9] - 2[6]$ *is* 3*-partitionable over* $\mathbb{F}_2$*, see Lemma 27. Thus, we have* $n_2(9, 3; 7 + 9t) \ge 55 + 73t$ *for all* $t \ge 1$.

The reason for the modification of the initial parameters $\sigma, \varepsilon_1, \ldots, \varepsilon_{r-1}$ in the proof of Lemma 33 is not only to satisfy Condition (44) but also to obtain the "right" n as shown in the following example:

Example 9. *Let* $q = 2$, $r = 7$, $h = 3$, $n = 13$, *and* $s = 2$, *so that* $d' = q^{h-1} \cdot (n - s) = 44$. *In order to compute* $n' := g_2(7, 44)$ *we determine parameters* σ, $\varepsilon_1, \ldots, \varepsilon_6$, *so that Equation (40) is satisfied, where* $0 \le \varepsilon_1, \ldots, \varepsilon_6 < 2$. *Here the unique solution is given by* $\sigma = 1$, $\varepsilon_5 = \varepsilon_3 = 1$, *and* $\varepsilon_i = 0$ *otherwise. Via Equation (41) we can compute* $n' = [7]_2 - [5]_2 - [3]_2 = 89$, *so that the surplus is given by* $\theta(13, 7, 2, 3, 2) = n[h]_q - n' = 2$. *Here it happens that* $\star[7] - [5] - [3]$ *is* 3*-partitionable over* $\mathbb{F}_2$ *nevertheless the surplus is positive. However Lemma 28 implies that whenever* $\sigma[7] - [5] - [3]$ *is* 3*-partitionable over* $\mathbb{F}_2$*, then we have* $\sigma \equiv 3 \pmod 7$ *so that we obtain faithful projective* $3 - (49 + 127t, 7, 6 + 15t)_2$ *systems for sufficiently large values of* t *instead of the desired faithful projective* $3 - (13 + 127t, 7, 2 + 15t)_2$ *systems.*

Theorem 4. *For all sufficiently large* s *we have that* $n_q(r, h; s)$ *attains the Griesmer upper bound, see Definition 7.*

Proof. Let $s_i := \frac{[r-h]_q}{[\gcd(r,h)]_q} - i$ for $0 \le i < \frac{[r-h]_q}{[\gcd(r,h)]_q}$ and n_i be the Griesmer upper bound for $n_q(r, h; s_i)$, i.e. $n_i[h]_q \ge g_k\big(r, q^{h-1} \cdot (n_i - s_i)\big)$ while $(n_i + 1) \cdot [h]_q < g_k\big(r, q^{h-1} \cdot (n_i + 1 - s_i)\big)$. Let $\sigma_i, \varepsilon_{1,i}, \ldots, \varepsilon_{r-1,i} \in \mathbb{N}$ with $\varepsilon_{j,i} < q$ for all $1 \le j \le r - 1$ be uniquely defined by

$$d_i := q^{h-1} \cdot (n_i - s_i) = \sigma_i \cdot q^{r-1} - \sum_{j=1}^{r-1} \varepsilon_{j,i} \cdot q^{j-1}, \tag{45}$$

so that

$$g_q(r, d_i) = \sigma_i \cdot [r]_q - \sum_{j=1}^{r-1} \varepsilon_{j,i} \cdot [j]_q, \tag{46}$$

using Lemma 21, and $\theta(n_i, r, s_i, h, q) \ge 0$. Similarly, let $\sigma_i', \varepsilon_{1,i}', \ldots, \varepsilon_{r-1,i}' \in \mathbb{N}$ with $\varepsilon_{j,i}' < q$ for all $1 \le j \le r - 1$ be uniquely defined by

$$d_i' := q^{h-1} + d_i = \sigma_i' \cdot q^{r-1} - \sum_{j=1}^{r-1} \varepsilon_{j,i}' \cdot q^{j-1}, \tag{47}$$

so that

$$g_q(r,d_i') = \sigma_i' \cdot [r]_q - \sum_{j=1}^{r-1} \varepsilon_{j,i}' \cdot [j]_q \tag{48}$$

and $\theta(n_i+1,r,s_i,h,q) < 0$. Now, let $s_{i,t} := s_i + t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q}$ and $n_{i,t} := n_i + t \cdot \frac{[r]_q}{[\gcd(r,h)]_q}$, so that Lemma 21 implies

$$d_{i,t} := q^{h-1} \cdot (n_{i,t} - s_{i,t}) = \left(\sigma_i + t \cdot \frac{[h]_q}{[\gcd(r,h)]_q}\right) \cdot q^{r-1} - \sum_{i=1}^{r-i} \varepsilon_i \cdot q^{i-1}, \tag{49}$$

$$g_q(r,d_{i,t}) = t \cdot \frac{[r]_q}{[\gcd(r,h)]_q} \cdot [h]_q + \sigma_i \cdot [r]_q - \sum_{j=1}^{r-1} \varepsilon_{j,i} \cdot [j]_q, \tag{50}$$

$$d_{i,t}' := q^{h-1} + d_{i,t} = \left(\sigma_i' + t \cdot \frac{[h]_q}{[\gcd(r,h)]_q}\right) \cdot q^{r-1} - \sum_{i=1}^{r-i} \varepsilon_i' \cdot q^{i-1}, \tag{51}$$

and

$$g_q\left(r,d_{i,t}'\right) = t \cdot \frac{[r]_q}{[\gcd(r,h)]_q} \cdot [h]_q + \sigma_i' \cdot [r]_q - \sum_{j=1}^{r-1} \varepsilon_{j,i}' \cdot [j]_q. \tag{52}$$

Thus we have

$$\theta(n_{i,t},r,s_{i,t},h,q) = \theta(n_i,r,s_i,h,q) \geq 0 \text{ and } \theta(n_{i,t}+1,r,s_{i,t},h,q) = \theta(n_i+1,r,s_i,h,q) < 0,$$

i.e. the Griesmer upper bound for $n_q(r,h;s_{i,t})$ is given by $n_{i,t}$ for all $t \in \mathbb{N}$. Lemma 33 yields the existence of a faithful $h-(n_{i,t},r,s_{i,t})_q$ system $\mathcal{S}_{i,t}$ for all sufficiently large t. ∎

The proof of Theorem 4 suggest the following algorithm to determine explicit formulas for $n_q(r,h;s)$ assuming that s is sufficiently large. For all $0 \leq i < \frac{[r-h]_q}{[\gcd(r,h)]_q}$ compute the Griesmer upper bound n_i for $n_q(r,h;s_i)$ where $s_i = \frac{[r-h]_q}{[\gcd(r,h)]_q} - i$. Then we have

$$n_q\left(r,h;t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q} - i\right) = t \cdot \frac{[r]_q}{[\gcd(r,h)]_q} - \left(\frac{[r]_q}{[\gcd(r,h)]_q} - n_i\right) \tag{53}$$

for all sufficiently large t. As an example we mention:

Proposition 1. *(Cf. [51, Table I],[52, Table II]) For all sufficiently large t we have*

- $n_2(7,2;31t) = 127t$;
- $n_2(7,2;31t-1) = 127t-5$;
- $n_2(7,2;31t-2) = 127t-10$;
- $n_2(7,2;31t-3) = 127t-15$;
- $n_2(7,2;31t-4) = 127t-20$;
- $n_2(7,2;31t-5) = 127t-21$;
- $n_2(7,2;31t-6) = 127t-26$;
- $n_2(7,2;31t-7) = 127t-31$;

- $n_2(7,2;31t-8) = 127t-36$;
- $n_2(7,2;31t-9) = 127t-41$;
- $n_2(7,2;31t-10) = 127t-42$;
- $n_2(7,2;31t-11) = 127t-47$;
- $n_2(7,2;31t-12) = 127t-52$;
- $n_2(7,2;31t-13) = 127t-55$;
- $n_2(7,2;31t-14) = 127t-60$;
- $n_2(7,2;31t-15) = 127t-63$;
- $n_2(7,2;31t-16) = 127t-68$;
- $n_2(7,2;31t-17) = 127t-73$;
- $n_2(7,2;31t-18) = 127t-76$;
- $n_2(7,2;31t-19) = 127t-81$;
- $n_2(7,2;31t-20) = 127t-84$;
- $n_2(7,2;31t-21) = 127t-87$;
- $n_2(7,2;31t-22) = 127t-92$;
- $n_2(7,2;31t-23) = 127t-95$;
- $n_2(7,2;31t-24) = 127t-100$;
- $n_2(7,2;31t-25) = 127t-105$;
- $n_2(7,2;31t-26) = 127t-108$;
- $n_2(7,2;31t-27) = 127t-113$;
- $n_2(7,2;31t-28) = 127t-116$;
- $n_2(7,2;31t-29) = 127t-121$;
- $n_2(7,2;31t-30) = 127t-126$.

In [52] the stated formulas of Proposition 1 were indeed shown to be true for all $t \ge 2$ and $n_2(7,2;31-i)$ was determined for all $i \in \{0,\ldots,31\}\backslash\{19,24,25\}$, referring to [83] for $i = 18$ and [51] for the previous state of the art. The final three missing cases were resolved in [85]. Having Theorem 4 at hand, the determination of $n_q(r,h;s)$ can be reduced to a finite set of cases for s for each given list of parameters q, r, and h, as the results of Griesmer [50], Solomon and Stiffler [107] imply for $h = 1$ a.k.a. linear codes. For the ease of the reader we collect such results in Section C in the appendix and mention that it is an important and difficult task to determine $n_q(r,h;s)$ for the remaining *small* values of s, even for $h = 1$. We try to summarize the current state of knowledge for small parameters, as we are aware of, in Section 5.

As observed in Remark 4 the upper bound of Lemma 15 can be reached asymptotically, i.e. we have

$$\lim_{s\to\infty} \frac{n_q(r,h;s)\cdot[r-h]_q}{s\cdot[r]_q} = 1 \tag{54}$$

for all parameters. Thus, we have

$$n_q(r,h;s) > \overline{n}_q(r,h;s) \tag{55}$$

for all sufficiently large s whenever r/h is not an integer, i.e., additive codes outperform linear codes for large enough s if r/h is fractional. For the case $r/h \in \mathbb{N}$ we provide:

q	r	h	i	$s_{i,t} := t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q} - i$	$n_q(r,h;s_{i,t})$	$n_q(r,h;s_{i,t}) - \overline{n}_q(r,h;s_{i,t})$
2	8	2	13	$21t-13$	$85t-55$	2
2	8	2	14	$21t-14$	$85t-60$	2
2	8	2	18	$21t-18$	$85t-76$	2
2	8	2	19	$21t-19$	$85t-81$	2
3	6	2	7	$10t-7$	$91t-67$	3
3	6	2	8	$10t-8$	$91t-77$	3
3	6	2	9	$10t-9$	$91t-87$	3
2	9	3	5	$9t-5$	$73t-43$	2
2	9	3	6	$9t-6$	$73t-52$	2
2	9	3	7	$9t-7$	$73t-59$	4
2	9	3	8	$9t-8$	$73t-68$	4
4	6	2	9	$17t-9$	$273t-149$	4
4	6	2	10	$17t-10$	$273t-166$	4
4	6	2	11	$17t-11$	$273t-183$	4
4	6	2	12	$17t-12$	$273t-200$	4
4	6	2	13	$17t-13$	$273t-213$	8
4	6	2	14	$17t-14$	$273t-230$	8
4	6	2	15	$17t-15$	$273t-247$	8
4	6	2	16	$17t-16$	$273t-264$	8

Table 2: Parameterized series of improvements for additive codes.

Theorem 5. *For all sufficiently large t we have the following improvements of additive codes over linear codes listed in Table 2 and the tables in Section B.*

The four series of improvements for $n_2(8,2;s)$ were already mentioned in [85]. For $(q,r,h) = (2,10,2)$ we refer to Table 7. An example for $h = 4$ is given by Table 9, which also shows that $n_q(r,h;s_{i,t}) - \overline{n}_q(r,h;s_{i,t})$ does not need to be a power of the characteristic of $\mathbb{F}_q$. An example for $q = 5$ is given by Table 8. In Theorem 9 we will determine the parameterized series of improvements $n_q(6,2;s) > \overline{n}_q(6,2;s)$ for all field sizes q. It turns out that there are $q(q-2)$ series and the improvement is divisible by q.

q	r	h	s	$n_q(r,h;s)$	$\overline{n}_q(r,h;s)$
2	8	2	9	33	31
2	8	2	10	35–36	34
2	8	2	11	40	39
2	8	2	14	54	50
2	8	2	27	106–107	103
3	6	2	3	21	17
3	6	2	8	66–68	65

Table 3: Sporadic improvements for additive codes.

In the introduction we have mentioned the improvement $n_3(6,2;3) = 21$ over $\overline{n}_3(6,2;3) = 17$ from [37]. Corollary 3 yields $n_3(6,2;10t-7) \ge 91t-70$ for all $t \ge 1$. Note that we have $\overline{n}_3(6,2;10t-7) = 91t-70$ for all $t \ge 2$. In Table 3 we collect the known sporadic improvements where $n_q(r,h;s) > \overline{n}_q(r,h;s)$ which are not part of a parametric series of improvements. So e.g. we do not mention $n_2(8,2;23) = 89 > 87 = \overline{n}_2(8,2;23)$ since Corollary 3 gives $n_2(8,2;21t-19) = 85t-81$ for $t \ge 2$, which is contained in Table 2. The data for $(q,r,h) = (2,8,2)$ descends from [85].

q	r	h	s	$n_q(r,h;s)$	$\overline{n}_q(r,h;s)$
2	10	2	18	64–68	62–66
2	22	2	26	56–90	55–74

Table 4: Temporary sporadic improvements for additive codes.

Sometimes one can construct additive codes which have better parameters than the best known linear codes. The lower bound $n_2(70,2;40) \ge 47$ and the lower bounds listed in Table 4 were obtained in [51].

4 Linear equation systems over $\mathbb{Z}$ and the Smith normal form

For each field $\mathbb{F}$ the Gauss–Jordan algorithm computes the set of all $x \in \mathbb{F}^n$ satisfying $Ax = b$ for a matrix $A \in \mathbb{F}^{m\times n}$ and a vector $b \in \mathbb{F}^m$. The Hermite normal form [62] of A is a triangular matrix and the Smith normal form (SNF) [106] for A is a diagonal matrix. Both allow to solve linear equation systems over $\mathbb{Z}$ and can be computed efficiently, see e.g. [29]. For a brief introduction into the underlying theory we refer e.g. to [97, chapters 4,5]. Known results and general techniques to (theoretically) compute SNFs for incidence matrices, e.g. in $\operatorname{PG}(r-1,q)$, are surveyed in [105]. Here we want to show the relation to our topic in an informal way by merely considering examples.

Theorem 6. *Let A be a nonzero $m \times n$ matrix over a principal ideal domain R. There exist invertible $m \times m$ and $n \times n$-matrices S, T (with entries in R) such that*

$$SAT = \begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & \alpha_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & \alpha_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{pmatrix}, \tag{56}$$

α_i divides α_{i+1} for all $1 \le i < r$, and the α_i are unique up to multiplication by a unit (in R).

The right hand side of Equation (56) is called the *Smith normal form (SNF)* of matrix A. The elements α_i are called the *elementary divisors*, *invariants*, or *invariant factors*. The matrices S and T can be algorithmically obtained by recursively applying invertible row and column operations to A till it reaches the desired diagonal form.

Example 10. *Let*

$$A = \begin{pmatrix} 1 & 1 & 5 & 7 \\ 2 & 8 & 10 & 20 \\ 3 & 3 & 45 & 51 \\ 1 & 7 & 5 & 13 \\ 2 & 2 & 40 & 44 \end{pmatrix}$$

and M be the $\mathbb{Z}$-module generated by the rows of A. The Smith normal form of A is given by

$$D := SAT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & -1 & -6 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

We have

$$M' = \left\{(z_1, z_2, z_3, z_4) \in \mathbb{Z}^4 \,:\, z_1 \equiv 0 \pmod 1, z_2 \equiv 0 \pmod 6, z_3 \equiv 0 \pmod{30}, z_4 = 0\right\}$$

for the $\mathbb{Z}$-module M' generated by the rows of D and

$$\begin{aligned} M = \Big\{ &(z_1, z_2, z_3, z_4) \in \mathbb{Z}^4 \,:\, z_1 \equiv 0 \pmod 1, -z_1 + z_2 \equiv 0 \pmod 6, \\ &-6z_1 - z_2 + z_4 \equiv 0 \pmod{30}, z_1 + z_2 + z_3 - z_4 = 0\Big\}. \end{aligned}$$

Equivalently, the linear equation system $D^\top x = b = (b_1, \dots, b_5)^\top$ has solutions over $\mathbb{Z}$ iff b_2 is divisible by 6, b_3 is divisible by 30, and we have $b_4 = b_5 = 0$. The full set of solutions is given by

$$\left\{(x_1, \dots, x_4)^\top \in \mathbb{Z}^4 \,:\, x_1 = b_1/1, x_2 = b_2/6, x_3 = b_3/30\right\}.$$

Proposition 2. *Let $A \in \mathbb{Z}^{m\times n}$ and $D = SAT$ its SNF. Then $D^\top \tilde{x} = \tilde{b}$ has a solution $\tilde{x} \in \mathbb{Z}^m$ iff $\tilde{b}_i \equiv 0 \pmod{\alpha_i}$ for all $1 \le i \le r$ and $\tilde{b}_i = 0$ for all $r < i \le n$. The set of all solutions is given by*

$$\left\{(\tilde{x}_1, \dots, \tilde{x}_m)^\top \in \mathbb{Z}^m \,:\, \tilde{x}_i = \tilde{b}_i/\alpha_i \,\forall 1 \le i \le r\right\}$$

Moreover, we have $A^\top x = b$ with $x \in \mathbb{Z}^m$ iff $D^\top \tilde{x} = \tilde{b}$ with $\tilde{x} \in \mathbb{Z}^m$, where $\tilde{b} = T^\top b$ and $\tilde{x}^\top = x^\top S^{-1}$.

Remark 6. *The case of some zero rows in the SNF of a matrix, see Example 10, is typical for our situation of the incidence matrix $A^{1,h;r,q}$, since there are more h-spaces than points in $\mathrm{PG}(r-1,q)$ when $r-1 > h > 1$. So, there exist h-spaces $S_1, \dots, S_l$ and integers $x_1, \dots, x_l$, not all equal to zero, such that $\mathcal{M} := \sum_{i=1}^{l} x_i \cdot \chi_{S_i}$ is the empty multiset of points, i.e. $\mathcal{M}(P) = 0$ for every point P. In the literature such solutions are known as (subspace)* trades*, see e.g. [78]. In $\mathrm{PG}(3,2)$ each set of pairwise disjoint lines can be completed to a line spread, so that there clearly exist two different line spreads $\mathcal{L}_1 = \{L_1, \dots, L_5\}$, $\mathcal{L}_2 = \{L_6, \dots, L_{10}\}$ with $\sum_{i=1}^{5} \chi_{L_i} = \sum_{i=6}^{10} \chi_{L_i}$.*

Example 11. *Let B be the incidence matrix between lines and points in $\mathrm{PG}(2,3)$ and T be the column transformation matrix of the Smith normal form of B, i.e.*

$$B = \begin{pmatrix} 1&0&0&1&0&0&1&0&0&1&0&0&0\\ 1&0&0&0&1&0&0&0&1&0&1&0&0\\ 1&0&0&0&0&1&0&1&0&0&0&1&0\\ 0&1&0&0&1&0&0&1&0&1&0&0&0\\ 0&1&0&0&0&1&1&0&0&0&1&0&0\\ 0&1&0&1&0&0&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1&1&0&0&0\\ 0&0&1&1&0&0&0&1&0&0&1&0&0\\ 0&0&1&0&1&0&1&0&0&0&0&1&0\\ 1&1&1&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&1&1&1&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1&0&0&0&0&0&0&0&-2&-1&-1&-2&-3\\ 0&1&0&0&0&0&0&-1&-1&-1&-2&-1&-11\\ 0&0&1&0&0&0&0&-2&0&-1&0&0&-11\\ 0&0&0&1&0&0&0&-2&0&-2&-1&0&-7\\ 0&0&0&0&1&0&0&0&-2&-2&0&-1&-3\\ 0&0&0&0&0&1&0&-1&-1&-2&-2&-2&-3\\ 0&0&0&0&0&1&1&-1&-1&-3&-3&-3&-7\\ 0&0&0&0&0&0&0&1&0&0&0&0&-3\\ 0&0&0&0&0&0&0&0&1&0&0&0&-3\\ 0&0&0&0&0&0&1&0&0&0&-1&-1&-7\\ 0&0&0&0&0&0&0&0&0&0&1&0&-3\\ 0&0&0&0&0&0&0&0&0&0&0&1&-3\\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

Here five invariant factors of B equal 3, the last equals 12, and the first seven equal 1. So, for $z = (z_1, \dots, z_{13}) \in \mathbb{Z}^{1\times[3]_3}$ we have $z \in M := \left\{x^\top B \,:\, x \in \mathbb{Z}^{[3]_3}\right\}$ iff

$$\begin{aligned} 2z_2 + z_3 + z_4 + 2z_6 + 2z_7 + z_8 &\equiv 0 \pmod{3},\\ z_1 + 2z_2 + z_5 + 2z_6 + 2z_7 + z_9 &\equiv 0 \pmod{3},\\ 2z_1 + 2z_2 + 2z_3 + z_4 + z_5 + z_6 &\equiv 0 \pmod{3},\\ 2z_1 + z_2 + 2z_4 + z_6 + 2z_{10} + z_{11} &\equiv 0 \pmod{3},\\ z_1 + 2z_2 + 2z_5 + z_6 + 2z_{10} + z_{12} &\equiv 0 \pmod{3},\\ z_2 + z_3 + 2z_4 + 2z_7 + 2z_{10} + z_{13} &\equiv 0 \pmod{3},\\ \sum_{i=1}^{13} z_i &\equiv 0 \pmod{4}, \end{aligned}$$

where we have broken up the condition of the last column of T modulo 12 into two conditions modulo 3 and modulo 4, respectively.

Example 12. *Let C be the $\mathbb{Z}_3$-code of the incidence matrix between lines and points in $\mathrm{PG}(2,3)$ B as in*

Example 11. Generator matrices of C and its dual code $C^\perp$ are e.g. given by

$$\begin{pmatrix} 1&0&0&0&0&1&0&1&0&0&0&1&0\\ 0&1&0&0&0&1&0&2&2&0&1&0&2\\ 0&0&1&0&0&1&0&0&1&0&2&2&2\\ 0&0&0&1&0&2&0&1&2&0&2&1&1\\ 0&0&0&0&1&2&0&2&1&0&1&2&0\\ 0&0&0&0&0&1&1&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1&0&0&0&0&1&0&1&0&2&2&0&2\\ 0&1&0&0&0&1&0&2&2&0&1&0&2\\ 0&0&1&0&0&1&0&0&1&0&2&2&2\\ 0&0&0&1&0&2&0&1&2&2&1&0&0\\ 0&0&0&0&1&2&0&2&1&0&1&2&0\\ 0&0&0&0&0&0&1&1&1&2&2&2&0 \end{pmatrix},$$

respectively. Clearly, we have $\dim(C) + \dim(C^\perp) = [3]_3 = 13$ *since C and $C^\perp$ are vector spaces. Now consider the incidence matrix*

$$B' = \begin{pmatrix} 0&1&1&0&1&1&0&1&1&0&1&1&1\\ 0&1&1&1&0&1&1&1&0&1&0&1&1\\ 0&1&1&1&1&0&1&0&1&1&1&0&1\\ 1&0&1&1&0&1&1&0&1&0&1&1&1\\ 1&0&1&1&1&0&0&1&1&1&0&1&1\\ 1&0&1&0&1&1&1&1&0&1&1&0&1\\ 1&1&0&1&1&0&1&1&0&0&1&1&1\\ 1&1&0&0&1&1&1&0&1&1&0&1&1\\ 1&1&0&1&0&1&0&1&1&1&1&0&1\\ 0&0&0&1&1&1&1&1&1&1&1&1&0\\ 1&1&1&0&0&0&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&0&0&0&1&1&1&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0 \end{pmatrix}$$

between affine planes and points in $\mathrm{PG}(2,3)$*. By C' we denote the corresponding $\mathbb{Z}_3$-code and observe that every row of B' is orthogonal to every row of B w.r.t. $\mathbb{Z}/3\mathbb{Z}$, i.e. $C' \subseteq C^\perp$. By e.g. computing the Hermite normal form of B' we can verify* $\dim(C') = 6$*, so that indeed $C' = C^\perp$. In the context of Example 11 this means that we can replace the six* $\bmod 3$*-conditions by* $B'z \equiv 0 \pmod 3$*, which corresponds to thirteen single* $\bmod 3$*-conditions. Of course we can also select six of these such that the corresponding rows of B' generate C'.*

Theorem 7. *(E.g. [33, Theorem 3.1].) The invariant factors of $A^{1,h;r,q}$ are all p-powers except the last, which is a p-power times $[h]_q$, where p is the characteristic of $\mathbb{F}_q$.*

The part $[h]_q$ of the last invariant factor has an easy explanation. Let $S_1, \dots, S_l$ be a list of h-spaces in $\mathrm{PG}(r-1,q)$, $x_1, \dots, x_l \in \mathbb{Z}$, and $\mathcal{M} = \sum_{i=1}^{l} x_i \cdot \chi_{S_i}$ be a premultiset of points. Then clearly, $\#\mathcal{M}$ has to be divisible by $[h]_q$ since each h-space consists of $[h]_q$ points. For the other invariant factors and their multiplicities we refer to [33, Theorem 3.3]. A corresponding basis over the *p-adic integers* is described in [33, Section 5] and [33, Theorem 7.2]. For a more geometric description, using (generalized) *Reed–Muller codes*, we refer to [3]. If the field size q itself is a prime, then the additional conditions on the premultiset of points $\mathcal{M}$ are equivalent to $\mathcal{M}(A) \equiv 0 \pmod{q^{h-1}}$ for all *affine subspaces* A of codimension 1, i.e., for all A that can be written as $H \backslash K$ where H is a hyperplane of $\mathrm{PG}(r-1,q)$ and $K \le H$ is a $(r-2)$-space. It can be checked easily that those conditions are equivalent to $\mathcal{M}(H) \equiv \#\mathcal{M} \pmod{q^{h-1}}$ for every hyperplane H, i.e., q^{h-1}-divisibility. If q is a proper p-power, then those conditions are only necessary and there are further conditions arising from so-called subfield subcodes (or Baer subspaces in geometrical terms), see [3, Subsection 5.8]. Except the first condition $\mathcal{M} \equiv 0 \pmod{[h]_q}$ all conditions are also satisfied for the set of all points of the ambient space $\mathcal{V} = \mathrm{PG}(r-1,q)$, so that we can also apply them to $\mathcal{M}$ directly when studying $\sigma \cdot \chi_V - \mathcal{M}$ for some $\sigma \in \mathbb{N}$.

Given these reformulations one can turn the conditions from Lemma 20 into fairly explicit ones and decide whether $\star[r] - \mathcal{M}$ is h-partitionable over $\mathbb{F}_q$ for a given premultiset of points $\mathcal{M}$ in $\operatorname{PG}(r-1,q)$. However, determining the smallest $\sigma \in \mathbb{N}$ such that $\sigma[r] - \mathcal{M}$ is h-partitionable over $\mathbb{F}_q$ is a significant and hard challenge. One example is given by partial spreads of h-spaces. To this end let $A_q(r, 2h; h)$ denote the maximum number of h-spaces in $\operatorname{PG}(r-1,q)$ such that each point is covered at most once. For a given partial spread $\mathcal{P}$ let $\mathcal{M}$ denote the set of uncovered points. In our notation $\mathcal{P}$ is a faithful projective $h - (\#\mathcal{P}, r, s, 1)_q$ system with type $1 \cdot [r] - \mathcal{M}$, where $\#\mathcal{P}$ and s can be computed from $\mathcal{M}$ via Lemma 18. In the other direction we have the conditions $\#\mathcal{M} \equiv [r]_q \pmod{[h]_q}$, $\mathcal{M}(P) \le 1$ for every point P, and $\mathcal{M}(H) \equiv \#\mathcal{M} \pmod{q^{h-1}}$ for every hyperplane H of $\operatorname{PG}(r-1,q)$. As an example we state $129 \le A_2(11, 8; 4) \le 132$ [80]. There cannot be a partial spread of 133 solids (4-spaces) in $\operatorname{PG}(10, 2)$ since there is no 8-divisible set of 52 points in $\operatorname{PG}(10, 2)$ while there are 8-divisible sets of cardinality 67, see e.g. [65, 67]. So, the open question is whether $[11] - \mathcal{M}$ can be partitioned into 132 solids for one of these choices for $\mathcal{M}$. Another example is given by $244 \le A_3(8, 6; 3) \le 248$. If $A_3(8, 6; 3) = 248$, then the set $\mathcal{M}$ of uncovered points has to be the unique 9-divisible set of points with cardinality 56 over $\mathbb{F}_3$, which is known as the *Hill cap* [63], see e.g. [65, Section 6] for more details.

Definition 14. *Let p be a prime, l be a positive integer, and $B \in R^{m\times n}$ a matrix where $R = \mathbb{Z}$ or $R = \mathbb{Z}/p^l\mathbb{Z}$. The* (linear) $\mathbb{Z}_{p^l}$-code *C of B is given by the row span of B w.r.t. $\mathbb{Z}/p^l\mathbb{Z}$. The matrix B is called a* generator matrix *of C. The* dual code *$C^\perp$ consists of all row vectors that are orthogonal to all elements in C (w.r.t. $\mathbb{Z}/p^l\mathbb{Z}$). We also call $C^\perp$ the* $\mathbb{Z}_{p^l}$-kernel *of B.*

Example 13. *Let B be the incidence matrix between planes and points in* $\operatorname{PG}(3, 2)$ *and T be the column transformation matrix of the Smith normal form of B, i.e.*

$$B = \begin{pmatrix} 1&0&1&0&1&0&1&0&1&0&1&0&1&0&0\\ 1&0&0&1&1&0&0&1&1&0&0&1&0&1&0\\ 1&0&1&0&0&1&0&1&0&1&0&1&1&0&0\\ 1&0&0&1&0&1&1&0&0&1&1&0&0&1&0\\ 0&1&0&1&0&1&0&1&1&0&1&0&1&0&0\\ 0&1&1&0&0&1&1&0&1&0&0&1&0&1&0\\ 0&1&0&1&1&0&1&0&0&1&0&1&1&0&0\\ 0&1&1&0&1&0&0&1&0&1&1&0&0&1&0\\ 1&1&0&0&1&1&0&0&1&1&0&0&0&0&1\\ 1&1&0&0&0&0&1&1&0&0&1&1&0&0&1\\ 0&0&1&1&0&0&1&1&1&1&0&0&0&0&1\\ 0&0&1&1&1&1&0&0&0&0&1&1&0&0&1\\ 1&1&1&1&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1 \end{pmatrix} \quad \textit{and} \quad T = \begin{pmatrix} 1&0&0&0&0&0&0&0&-1&0&0&-1&-3&-3&-13\\ 0&1&0&0&0&0&-1&-1&-1&-1&-1&-3&-1&-1&-6\\ 0&0&1&0&0&-1&0&-1&0&-1&-1&-3&-1&-2&-20\\ 0&0&0&1&0&-1&-1&0&0&0&-1&-1&-3&-2&-27\\ 0&0&0&1&1&-1&-1&-1&0&0&-1&-4&-5&-2&-41\\ 0&0&0&0&0&1&0&0&0&0&0&0&-1&0&-6\\ 0&0&0&0&0&0&1&0&0&0&0&0&-1&-1&-6\\ 0&0&0&0&0&0&0&1&0&0&0&0&-1&-1&-13\\ 0&0&0&0&1&0&0&0&1&0&0&-3&-4&-1&-34\\ 0&0&0&0&0&0&0&0&1&1&0&-1&-2&-1&-13\\ 0&0&0&0&0&0&0&0&0&1&1&-1&-2&-2&-13\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&-6\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&-1&-13\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&-6\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

Here six invariant factors of B equal 2*, three equal* 4*, the last equals* 28*, and the first five equal* 1*. Generator* *matrices for the $\mathbb{Z}_4$-code C of B and its dual code $C^\perp$ are e.g. given by*

$$\begin{pmatrix}
1&0&0&1&0&1&1&0&0&1&1&0&0&1&0\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1\\
0&0&1&1&0&0&1&1&0&0&1&1&1&1&0\\
0&0&0&2&0&0&0&2&0&0&0&2&0&2&0\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1\\
0&0&0&0&0&2&0&2&0&0&0&0&0&2&2\\
0&0&0&0&0&0&2&2&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&2&0&2&0&2&2\\
0&0&0&0&0&0&0&0&0&0&2&2&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&2&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\quad \textit{and} \quad
\begin{pmatrix}
1&0&0&1&0&1&1&0&1&0&0&1&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1\\
0&0&1&1&0&0&1&1&0&0&1&1&1&1&0\\
0&0&0&2&0&0&0&2&0&0&0&2&0&2&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&0&2&0&2&0&0&0&0&0&2&2\\
0&0&0&0&0&0&2&2&0&0&0&0&2&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&2&2&0&2\\
0&0&0&0&0&0&0&0&0&2&0&2&0&2&2\\
0&0&0&0&0&0&0&0&0&0&2&2&2&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

with invariant factors $[1,1,1,1,1,2,2,2,2,2,2,0,0,0,0]$ *and* $[1,1,1,1,2,2,2,2,2,2,0,0,0,0,0]$*, which we abbreviate as* $1^5 2^6 0^4$ *and* $1^4 2^6 0^5$*. The rows of the stated generator matrix for* $C^\perp$ *correspond to ten* $\bmod 4$*-conditions, where six can be rewritten to* $\bmod 2$*-conditions.*

5 Small parameters

In this section we want to collect results on $n_q(r,h;s)$ for small parameters. For results for $q = 2$ and $q = 3$ we refer to Subsection 5.1 and Subsection 5.2, respectively. Here we start with some parametric results. As mentioned before, we have $n_q(r,h;s) = \infty$ for $r \le h$, so that we assume $r \ge h+1$ in the following. We are especially interested in the integral cases where $r/h \in \mathbb{N}$. Ignoring the degenerated case $r = h$, the first interesting case $r = 2h$ can be completely solved:

Theorem 8.

$$n_q(2h,h;s) = \overline{n}_q(2h,h;s) = \frac{[2h]_q}{[h]_q} \cdot s = \left(q^h + 1\right) \cdot s \tag{57}$$

Proof. The lower bound follows from Theorem 2 and the upper bound from Lemma 15. ■

Clearly we have $n_q(3,1;1) = 1$. For odd q the upper bound $n_q(3,1;2) \le q+1$ was shown in [25] and an *oval* (conic) in $\mathrm{PG}(2,q)$ shows $n_q(3,1;2) \ge q+1$ for all field sizes q. Segre [98] has shown that all examples attaining $n_q(3,1;2) = q+1$ are equivalent for odd q, cf. [69]. For even q each faithful projective $1-(q+1,3,2)_q$ system can be extended to a faithful projective $1-(q+2,3,2)_q$ system, which is called *hyperoval*. The upper bound $n_q(3,1;2) \le q+2$ follows from the Griesmer bound, as shown below when setting $t = 1$, $i = q-1$.

Proposition 3. *We have*

$$n_q(3,1;(q+1)t-i) = [3]_q \cdot t - [2]_q \cdot i = \left(q^2+q+1\right) \cdot t - (q+1) \cdot i \tag{58}$$

for $0 \le i \le 1$, $t \ge 1$ *and for* $2 \le i \le q$, $t \ge 2$.

Proof. The statement for $n_q(3,1;(q+1)t)$, i.e. $i=0$, follows from Theorem 2. Let $n_{t,i}=[3]_q t-[2]_q i$ and $s_{t,i}=[2]_q t-[1]_q i$ for $0\le i\le q$ and $t\ge 1$. Since $d_{t,i}:=n_{t,i}-s_{t,i}=q^2t-qi$ we can apply the Solomon–Stiffler construction if we can find i lines in PG$(2,q)$ that cover each point at most twice for $2\le i\le q$ or at most once for $i=1$. By duality and using $n_q(3,1;2)\ge q+1$ this is indeed possible. For $t\ge 1$ and $1\le i\le q$ the Griesmer bound, see Lemma 21, shows that the length of n' of each $[n',3,d']_q$ code with minimum distance

$$d':=d_{t,i}+1=(n_{t,i}+1)-s_{t,i}=t\cdot q^2-(i-1)\cdot q-(q-1)\cdot 1$$

is at least

$$t\cdot[3]_q-(i-1)\cdot[2]_q-(q-1)\cdot[1]_q=[3]_q t-[2]_q i+2>n_{t,i},$$

i.e. $n_q(3,1;(q+1)t-i)\le[3]_q\cdot t-[2]_q\cdot i$ follows from the Griesmer upper bound. ■

We remark that Lemma 15 implies

$$n_q(3,1;(q+1)t-i)\le\left\lfloor\frac{[3]_q}{[3-1]_q}\cdot((q+1)t-i)\right\rfloor=[3]_q t-qi-\left\lceil\frac{i}{q+1}\right\rceil \tag{59}$$

for $0\le i\le q$ and $t\ge 1$, which is tight iff $i=0,1$. For $t=i=1$ the q^2 lines of PG$(2,q)$ that are disjoint to an arbitrary but fixed point yield a projective faithful $2-(q^2,3,1,q)_q$ system whose dual is a projective faithful $1-(q^2,3,q,1)_q$ system. The determination of $n_q(3,1;s)$ is a challenging problem for $3\le s\le q-1$ and was solved completely for $q\le 9$ only, see e.g. `http://mars39.lomo.jp/opu/griesmer.htm`. In Table 5 we summarize the known values for $q\le 9$. We remark that in all cases $1-\left(n_q(3,1;s),3,s,1\right)_q$ systems do exist, see `http://web.mat.upc.edu/simeon.michael.ball/codebounds.html`.

Lemma 34. *We have* $\overline{n}_q(6,2;(q^2+1)\,t-i)=\left(q^4+q^2+1\right)t-(q^2+1)\,i$ *for all* $0\le i\le q^2$ *and all* $t\ge 2$.

Proof. With $\tilde{q}:=q^2$ we compute $\gcd(r,h)=2$, $[\gcd(r,h)]_q=q+1$, $\frac{[r-h]_q}{[\gcd(r,h)]_q}=q^2+1=\tilde{q}+1$, and $\frac{[r]_q}{[\gcd(r,h)]_q}=q^4+q^2+1=\tilde{q}^2+\tilde{q}+1$. For the lower bounds we refer to Proposition 3. Due to Theorem 2 we can assume $1\le i\le q^2$. Let $n_{i,t}:=\left(q^4+q^2+1\right)t-(q^2+1)\,i$ and $s_{i,t}:=(q^2+1)\,t-i$, so that

$$(n_{i,t}+1)-s_{i,t}=t\cdot\tilde{q}^2-(i-1)\cdot\tilde{q}-(\tilde{q}-1)\cdot 1.$$

Since $t\cdot[3]_{\tilde{q}}-(i-1)\cdot[2]_{\tilde{q}}-(\tilde{q}-1)\cdot[1]_{\tilde{q}}=n_{i,t}+2$ Lemma 21 yields $n_{\tilde{q}}(3,1;(\tilde{q}+1)\,t-i)\le(\tilde{q}^2+\tilde{q}+1)\cdot t-(\tilde{q}+1)\,i$ for $1\le i\le\tilde{q}$. ■

Theorem 9. *Let* $0\le i\le q^2$ *and* $i=aq-b$ *where* $a,b\in\mathbb{N}$ *and* $b\le q-1$. *For each* $t\ge q^2+q$ *we have*

$$n_q(6,2;(q^2+1)t-i)=(q^4+q^2+1)t-(q^2+1)i+\max\{a-2,0\}\cdot q \tag{60}$$

q	s	$n_q(3,1;s)$	Gub	gap
2	1	1	1	
	2	4	4	
	3	7	7	
3	1	1	1	
	2	4	5	1
	3	9	9	
	4	13	13	
4	1	1	1	
	2	6	6	
	3	9	11	2
	4	16	16	
	5	21	21	
5	1	1	1	
	2	6	7	1
	3	11	13	2
	4	16	19	3
	5	25	25	
	6	31	31	
7	1	1	1	
	2	8	9	1
	3	15	17	2
	4	22	25	3
7	5	29	33	4
	6	36	41	5
	7	49	49	
	8	57	57	
8	1	1	1	
	2	10	10	
	3	15	19	4
	4	28	28	
	5	33	37	4
	6	42	46	4
	7	49	55	6
	8	64	64	
	9	73	73	
9	1	1	1	
	2	10	11	1
	3	17	21	4
	4	28	31	3
	5	37	41	4
	6	48	51	3
	7	55	61	6
	8	65	71	6
	9	81	81	
	10	91	91	

Table 5: Griesmer upper bounds and exact values for $n_q(3,1;s)$ for $1 \leq s \leq q+1$ and $q \leq 9$.

and $n_q(6,2;(q^2+1)t-i) = \overline{n}_q(6,2;(q^2+1)t-i) + \max\{\lceil i/q\rceil - 2, 0\}\cdot q$.

Proof. First we note $a = \lceil i/q \rceil$, so that the second part follows from Lemma 34 and Equation (60). Due to Theorem 2 we can assume $1 \le i \le q^2$. Let $n_{i,t} := \left(q^4+q^2+1\right)t - (q^2+1)\,i$ and $s_{i,t} := (q^2+1)\,t - i$, so that $n_{i,t} = \overline{n}_q(6,2;s_{i,t})$ for all $t \ge 2$ by Lemma 34 and

$$q\cdot\Big((n_{i,t}+1) - s_{i,t}\Big) = t\cdot q^5 - (i-1)\cdot q^3 - (q-1)\cdot q^2 - (q-1)\cdot q. \tag{61}$$

For $1 \le i \le q$ we have

$$t[6]_q - (i-1)[4] - (q-1)[3] - (q-1)[2] = t[6]_q - i[5]_q + q + 3 = n_{i,t}(q+1) + 2 > n_{i,t}(q+1),$$

so that Lemma 21 gives $n_q(6,2;s_{i,t}) \le n_{i,t}$. For $q+1 \le i \le 2q$ we have

$$q\cdot\Big((n_{i,t}+1) - s_{i,t}\Big) = t\cdot q^5 - q^4 - (i-q-1)\cdot q^3 - (q-1)\cdot q^2 - (q-1)\cdot q$$

and

$$t[6]_q - [5]_q - (i-q-1)[4] - (q-1)[3] - (q-1)[2] = t[6]_q - i[4]_q + q + 2 = n_{i,t}(q+1) + 1 > n_{i,t}(q+1),$$

so that Lemma 21 gives $n_q(6,2;s_{i,t}) \le n_{i,t}$. For $1 \le i \le 2q$ we have $n_q(6,2;s_{i,t}) \ge \overline{n}_q(6,2;s_{i,t}) = n_{i,t}$. So far we have shown $n_q(6,2;s_{i,t}) = \overline{n}_q(6,2;s_{i,t}) = n_{i,t} + \max\{a-2,0\}\cdot q$ for $a \le 2$, i.e. $0 \le i \le 2q$, and $t \ge q^2+q > 2$.

For $3 \le a \le q$, $0 \le b \le q-1$, and $i = aq - b$ we have

$$q\cdot\Big((n_{i,t} + (a-2)q) - s_{i,t}\Big) = t\cdot q^5 - (a-1)\cdot q^4 - (q-1-b)\cdot q^3 - (q+2-a)q^2$$

and

$$\begin{aligned} t\cdot[6]_q - (a-1)\cdot[5]_q - (q-1-b)\cdot[4]_q - (q+2-a)[3]_q &= t\cdot[6]_q - i[4]_q + (a-2)q(q+1) \\ &= (n_{i,t} + (a-2)q)\cdot(q+1). \end{aligned}$$

By Theorem 2 we have that $(q+1)[3]$ is 2-partitionable over $\mathbb{F}_q$. Lemma 24 shows that $[6]-[4]$ and $(q+1)[6]-[5]-q[3]$ are 2-partitionable over $\mathbb{F}_q$. Consider the corresponding constructions. Taking $(a-2)$ copies of the first, $(q-1-b)$ copies of the second, and $(a-1)$ copies of the third construction shows that $t[6]-(a-1)[5]-(q-1-b)[4]-(q+2-a)[3]$ is 2-partitionable over $\mathbb{F}_q$ for

$$t = (q+1)(a-1) + (q+1-b) = aq - b + a = i + a \le q^2 + q.$$

Since $[6]$ is 2-partitionable over $\mathbb{F}_q$ by Theorem 2 we have $n_q(6,2;s_{i,t}) \ge n_{i,t} + (a-2)q$ for all

$2q+1 \le i \le q^2$ and all $t \ge q^2+q$. For the upper bound we consider

$$q \cdot \Big((n_{i,t} + (a-2)q + 1) - s_{i,t} \Big) = t \cdot q^5 - (a-1) \cdot q^4 - (q-1-b) \cdot q^3 - (q+1-a)q^2 - (q-1)q$$

and

$$\begin{aligned} & t \cdot [6]_q - (a-1) \cdot [5]_q - (q-1-b) \cdot [4]_q - (q+1-a)[3]_q - (q-1)[2]_q \\ = \; & t \cdot [6]_q - i[4]_q + (a-2)q(q+1) + q + 2 = (n_{i,t} + (a-2)q + 1) \cdot (q+1) + 1 \\ > \; & (n_{i,t} + (a-2)q + 1) \cdot (q+1), \end{aligned}$$

so that Lemma 21 gives $n_q(6,2;s_{i,t}) \le n_{i,t} + (a-2)q$. ■

Theorem 10. *We have* $n_q(h+1,h;s) = s \cdot [h+1]_q$, $\overline{n}_q(h+1,h;s) = s \cdot \left(q^h + 1\right)$, *and* $n_q(h+1,h;s) - \overline{n}_q(h+1,h;s) = s \cdot q[h-1]_q$, *which is positive for* $h > 1$.

Proof. For $n_q(h+1,h;s)$ the upper bound follows from Lemma 15 and a construction is given by choosing each h-space s times. The value of $\overline{n}_q(h+1,h;s)$ follows from Theorem 8. ■

Proposition 4. *For even* h *we have* $n_q(h+2,h;s) = \frac{[h+2]_q}{[2]_q} \cdot s = [h/2+1]_{q^2} \cdot s$ *for all* $s \in \mathbb{N}$. *Let* $h = 2h'+1$ *where* $h' \in \mathbb{N}_{\ge 1}$. *We have*

$$n_q(h+2,h;(q+1)t-i) = [h+2]_q \cdot t - \frac{[h+2]_q + q}{[2]_q} \cdot i = [h+2]_q \cdot t - \left(q \cdot [h'+1]_{q^2} + 1\right) \cdot i \qquad (62)$$

for $0 \le i \le 1$, $t \ge 1$ *and for* $2 \le i \le q$, $t \ge 2$.

Proof. The statements for $n_q(h+2,h;s)$ with even h and for $n_q(h+2,h;(q+1)t)$ with odd h follow from Theorem 2, so that we assume $h = 2h'+1$ for a fixed $h' \in \mathbb{N}_{\ge 1}$ and $i \ge 1$ in the following. Let

$$n_{t,i} = [h+2]_q t - \frac{[h+2]_q + q}{[2]_q} \cdot i = [h+2]_q \cdot t - \left(q \cdot [h'+1]_{q^2} + 1\right) \cdot i \qquad (63)$$

and

$$s_{t,i} = (q+1)t - i \qquad (64)$$

for $0 \le i \le q$ and $t \ge 1$. Lemma 11 yields the existence of a faithful projective $2-\left(\frac{q^{h+2}-q^3}{q^2-1}, h+2, \star, 1\right)_q$ system where all elements are disjoint to a special 3-space A. Taking $s_{t,i}$ copies thereof and embedding the dual of a faithful projective $1-\left([3]_q t - (q+1)i, 3, (q+1)t - i\right)_q$ system from Proposition 3 into A gives a faithful projective $2-(n_{t,i}, h+2, \star, s_{t,i})_q$ system $\mathcal{S}_{t,i}$ for $t \ge 2$ or $t = i = 1$ since

$$\frac{q^{h+2}-q^3}{q^2-1} \cdot s_{t,i} + \left([3]_q t - (q+1)i\right) = [h+2]_q \cdot t - \frac{q^{h+2}+q^2-q-1}{q^2-1} \cdot i = [h+2]_q \cdot t + \frac{[h+2]_q + q}{[2]_q} \cdot i.$$

So, the dual $\mathcal{S}_{t,i}^{\perp}$ of $\mathcal{S}_{t,i}$ yields $n_q(h+2,h;s_{t,i}) \ge n_{t,i}$ for $0 \le i \le 1$, $t \ge 1$ and for $2 \le i \le q$, $t \ge 2$. Let

$$d_{t,i} := q^{h-1} \cdot (n_{t,i} - s_{t,i}) = q^{h-1} \cdot \left(q^2[h]_q t - q \cdot [h'+1]_{q^2} \cdot i\right) = \left([h]_q t - qi[h']_{q^2}\right) \cdot q^{h+1} - i \cdot q^h,$$

so that

$$q^{h-1} \cdot (n_{t,i} + 1 - s_{t,i}) = \left([h]_q t - qi[h']_{q^2}\right) \cdot q^{h+1} - (i-1) \cdot q^h - (q-1) \cdot q^{h-1}.$$

So, for $t \ge 1$ and $1 \le i \le q$ the Griesmer bound, see Lemma 21, shows that the length of n' of each $[n', h+2, d']_q$ code with minimum distance $d' = d_{t,i} + q^{h-1}$ is at least

$$\begin{aligned} & \left([h]_q t - qi[h']_{q^2}\right) \cdot [h+2]_q - (i-1) \cdot [h+1]_q - (q-1) \cdot [h]_q \\ = \;& [h+2]_q t \cdot [h]_q - q^3 i[h']_{q^2} \cdot [h]_q - q(q+1)i[h']_{q^2} - (iq-1)[h]_q \\ = \;& [h]_q \cdot \left([h+2]_q t - \left(q[h'+1]_{q^2} + 1\right) \cdot i\right) + (i+1)[h]_q - q(q+1)i[h']_{q^2} \\ = \;& [h]_q \cdot n_{t,i} + [h]_q - 1 > [h]_q \cdot n_{t,i}, \end{aligned}$$

i.e. the Griesmer upper bound from Definition 7 is $n_q(h+2,h;s_{t,i}) \le n_{t,i}$, see Lemma 14. ∎

We remark that Lemma 15 implies

$$n_q(h+2,h;(q+1)t-i) \le \left\lfloor \frac{[h]_q}{[2]_q} \cdot ((q+1)t-i) \right\rfloor = [h+2]_q t - q \cdot [(h+1)/2]_{q^2} \cdot i - \left\lceil \frac{i}{q+1} \right\rceil$$

for odd h, $0 \le i \le q$, and $t \ge 1$, which is tight iff $i = 0, 1$, see Proposition 3 for $h = 1$.

Proposition 5. *For odd $h \ge 3$ we have $n_q(h+2,h;1) = \frac{[h+2]_q - q^2}{[2]_q} = q^3 \cdot [(h-1)/2]_{q^2} + 1$.*

Proof. The dual of a faithful projective $h-(n,h+2,1,\mu)_q$ system is a faithful projective $2-(n,h+2,\mu,1)_q$ system, i.e. a partial line spread in $\mathrm{PG}(h+1,q)$, so that [10, Theorem 4.1, 4.2] yields the stated formula. ∎

Proposition 6. *Bounds for $n_q(5,3;s)$, where $1 \le s \le q+1$ and $q \le 9$, are summarized in Table 6.*

Proof. For $q \le s \le q+1$ we refer to Proposition 4 and for $s = 1$ Proposition 5 yields $n_q(5,3;1) = q^3 + 1$. As in the proof of Proposition 4 we can use Proposition 11 to deduce $n_q(5,3;s) \ge q^3 s + n_q(3,1;s)$. If not stated otherwise we use this lower bound with the values from Table 5. The dual of a faithful projective $3-(n,5,s,\mu)_q$ system is a faithful projective $2-(n,5,\mu,s)_q$ system $\mathcal{S}$. Due to Lemma 4 $\mathcal{P}(\mathcal{S})$ is q-divisible. By Lemma 3 $\mathcal{M} := s \cdot \chi_V - \mathcal{P}(\mathcal{S})$ is also q-divisible with cardinality $\#\mathcal{M} = s[5]_q - n[2]_q$ and maximum point multiplicity at most s, where V is the 5-dimensional ambient space. So, nonexistence results for q-divisible multisets of points in $\mathrm{PG}(4,q)$ with suitable cardinalities and maximum point multiplicities can imply upper bounds for $n_q(5,3;s)$.

As a example we will use the fact that there is no 3-divisible multiset of points in $\mathrm{PG}(4,q)$ with cardinality 6 and maximum point multiplicity at most 2, which can be verified by exhaustive

q	s	$n_q(5,3;s)$	Gub	gap
2	1	9	9	
	2	20	20	
	3	31	31	
3	1	28	28	
	2	**58**	59	1
	3	90	90	
	4	121	121	
4	1	65	65	
	2	134	134	
	3	**201**	203	2
	4	272	272	
	5	341	341	
5	1	126	126	
	2	**256**	257	1
	3	**386**	388	2
	4	**516**	519	3
	5	650	650	
	6	781	781	
7	1	344	344	
	2	**694**	695	1
	3	**1044**	1046	2
	4	**1394**	1397	3

q	s	$n_q(5,3;s)$	Gub
7	5	**1744**	1748
	6	**2094**	2099
	7	2450	2450
	8	2801	2801
8	1	513	513
	2	1034	1034
	3	1551–1555	1555
	4	2076	2076
	5	2593–2597	2597
	6	3114–3118	3118
	7	3633–3639	3639
	8	4160	4160
	9	4681	4681
9	1	730	730
	2	1468–1469	1469
	3	2204–2208	2208
	4	2944–2947	2947
	5	3682–3686	3686
	6	4422–4425	4425
	7	5158–5164	5164
	8	5897–5903	5903
	9	6642	6642
	10	7381	7381

Table 6: Griesmer upper bounds and values for $n_q(5,3;s)$ for $1 \le s \le q+1$ and $q \le 9$.

enumeration or proven theoretically, see Lemma 42. Thus, we have $n_3(5,3;2) < 59$ since $2 \cdot [5]_3 - 59 \cdot [2]_3 = 6$.

Using the software package `LinCode` [27] we have shown by exhaustive enumeration that the following multisets of points do not exist:

- There is no 3-divisible multiset of points in $\mathrm{PG}(4,3)$ with cardinality $\#\mathcal{M} = 6$ and maximum point multiplicity at most 2.
- There is no 4-divisible multiset of points in $\mathrm{PG}(4,4)$ with cardinality $\#\mathcal{M} = 13$ and maximum point multiplicity at most 3.
- There is no 5-divisible multiset of points in $\mathrm{PG}(4,5)$ with cardinality $\#\mathcal{M} \in \{20, 21, 22\}$ and maximum point multiplicity at most 4.
- There is no 7-divisible multiset of points in $\mathrm{PG}(4,7)$ with cardinality $\#\mathcal{M} \in \{35, 36, 37, 38, 42, 43, 44, 45, 4$ and maximum point multiplicity at most 6.

In Table 6 we have marked the corresponding improved upper bounds for $n_q(5,3;s)$ in bold font. For more details we refer to Section D. ∎

So far we have tried to determine parametric formulas or bounds for $n_q(r,h;s)$ for small parameters of r and h in terms of s. We may also consider the situation for fixed small values of s. Since $n_q(r,h;s) = \infty$ for $r \le h$ we assume $r \ge h+1$ in the following.

Lemma 35. *We have $n_q(r,h;s) = s$ for $hs < r$.*

Proof. Let $\mathcal{S}$ be a faithful projective $h-(n,r,s)_q$ system with $n = n_q(r,h;s)$. The span of n elements from $\mathcal{S}$ has dimension at most nh, i.e. all elements of $\mathcal{S}$ are contained in a hyperplane of $\mathrm{PG}(r-1,q)$. ∎

So, we will mostly assume $s \ge r/h$ in the following, i.e. $s = 2$ is the first interesting case.

Theorem 11. *We have $n_q(3h,h;2) = \overline{n}_q(3h,h;2)$, i.e. $n_q(3h,h;2) = q^h + 1$ if q is odd and $n_q(3h,h;2) = q^h + 2$ for even q.*

Proof. Let $\mathcal{S}$ be a faithful projective $h-(n,3h;2)_q$ system with $n = n_q(3h,h;2)$. Any two elements of $\mathcal{S}$ span a $2h$-space since otherwise we find three elements contained in a hyperplane. Denote the number of hyperplanes with i elements from $\mathcal{S}$ by a_i. Double-counting yields the equations

$$a_0 + a_1 + a_2 = [3h]_q, \tag{65}$$

$$a_1 + 2a_2 = n \cdot [2h]_q, \text{ and} \tag{66}$$

$$a_2 = \tfrac{n(n-1)}{2} \cdot [h]_q, \tag{67}$$

so that the second equation minus twice the third equation gives

$$a_1 = n[h]_q \cdot \left(q^h + 2 - n\right).$$

Since a_1 is nonnegative, we have $n_q(3h, h; 2) \le q^h + 2$. If $n = q^h + 2$, then $a_1 = 0$. However, the corresponding code would be a projective two-weight code with weight difference $2q^{h-1}$, which is not a power of the characteristic of $\mathbb{F}_q$ if q is odd – contradiction to Lemma 1 from [40]. Ovals and hyperovals in $\mathrm{PG}\left(2, q^h\right)$ give the corresponding constructions for odd q and even q, respectively. ∎

We remark that $n_3(6, 2; 2) = 10$ was shown in [7] by exhaustive enumeration.

Remark 7. *Let $\mathcal{S}$ be a faithful $h - (n, lh; l-1)_q$ system with $n = n_q(lh, h; l-1)$ for some $l \ge 2$. If there exist i elements of $\mathcal{S}$ that span a subspace S of dimension strictly less than hi for some $1 \le i \le l$, then adding any further $l - i$ elements yields the existence of a hyperplane with at least l elements, which is a contradiction. Thus, the dimension spanned by any subset of elements of $\mathcal{S}$ is congruent to 0 modulo h. In [21, Proposition 3.1] is was shown for $q = h = 2$ that this conditions ensures that $\mathcal{S}$ can be obtained from a faithful projective $1 - (n, l; l-1)_{q^h}$ system by the subfield construction. The existence of non-Desarguesian spreads of h-spaces for $q > 2$ or $h > 2$ shows that further conditions are needed in order to conclude linearity, cf. [6, Theorem 13] and [1]. For characterizations of Desarguesian spreads we refer to [95]. So, it is an interesting question, whether $n_q(lh, h; l-1) > \overline{n}_q(lh, h; l-1)$ is possible for $l > 3$.*

Lemma 36. *For each odd prime power q we have*

$$n_q(5, 2; 2) \le q^2 + q + 1. \tag{68}$$

Proof. Consider a faithful projective $2 - (n, 5, 2, \mu)_q$ system $\mathcal{S}$ and denote the number of hyperplanes that contain i elements from $\mathcal{S}$ by a_i. We will show $\mu = 1$ if $n > q^2 + 2$. The dual $\mathcal{S}^\perp$ is a faithful projective $3 - (n, 5, \mu, 2)_q$ system. Note that two planes (3-spaces) in $\mathrm{PG}(4, q)$ intersect in at least a point. If $\mathcal{S}^\perp$ contains a plane with multiplicity 2, then we have $n = 2$. If $\mathcal{S}^\perp$ contains two elements E_1, E_2 such that their intersection is a line L, then the elements in $\mathcal{S}^\perp \backslash \{E_1, E_2\}$ need to intersect E_1 outside of L, so that $n \le q^2 + 2$.

If $n > q^2 + 2$, then the elements of $\mathcal{S}$ form a partial line spread, i.e. we have $\mu = 1$. Double-counting gives

$$a_0 + a_1 + a_2 = [5]_q, \tag{69}$$
$$a_1 + 2a_2 = n[3]_q, \text{ and} \tag{70}$$
$$2a_2 = n(n-1). \tag{71}$$

If $n > [3]_q + 1$, then $a_1 < 0$, which is impossible. If $n = [3]_q + 1 = q^2 + q + 2$, then we have $a_1 = 0$, so that the corresponding code is a projective two-weight code with difference $2q$ of the occurring nonzero weights. However, the weight difference of a projective two-weight code has to be a power of the characteristic of $\mathbb{F}_q$, see Lemma 1 or [40, Corollary 2], which gives a contradiction if q is odd. ∎

Remark 8. *For $q = 2$ a vector space partition of $\mathrm{PG}(4,2)$ of type $2^6 3^1$ gives a faithful projective $2-(8,5,2)_2$ system (which is unique) and there also exists a faithful projective $2-(6,5,2)_2$ system that contains a pair of lines intersecting in a point. For $q \in \{3,5\}$ there exist a faithful projective $2-(q^2+2,5,2)_q$ system that contains a pair of lines intersecting in a point. For $q = 3$ the maximum size n of a faithful projective $2-(n,5,2)_q$ system is $12 = q^2 + q$ [7].*

A subclass of special interest are so-called *MDS codes* attaining the Singleton bound with equality, see e.g. [6, 7]. Many of these codes fall into the class of Reed–Solomon codes, but there are also other constructions see e.g. [7, Remark 27] and [70].

Proposition 7. *We have $n_q(6,2;q) \le q \cdot (q^2 - q + 1)$. If $\mathcal{S}$ is a faithful projective $2-(n,6,q,\mu)_q$ system attaining equality, then we have $\mu = 1$ and each 4-space contains either 0 or q elements from $\mathcal{S}$ and each subset of elements of $\mathcal{S}$ spans an even-dimensional subspace.*

Proof. If L_1, L_2 are two different elements in $\mathcal{S}$ that intersect in a point, then let π be the 3-space spanned by L_1, L_2 and consider the projection of $\mathcal{S}$ through π, see Lemma 9, so that Lemma 15 gives $n \le (q-2)\cdot(q^2+q+1)+2 = q\cdot(q^2-q-1)$. Thus, if $n > q\cdot(q^2-q-1)$, then we have $\mu = 1$ (noting that the case $L_1 = L_2$ leads to even stronger upper bounds for n). Assuming $\mu = 1$, let $\mathcal{S}'$ be the projection of $\mathcal{S}$ through L_1, so that $\mathcal{S}'$ is a faithful projective $2-(n-1,4;q-1,\mu')_q$ system. Lemma 15 gives $\#\mathcal{S}' = n - 1 \le (q^2+1)\cdot(q-1)$, so that $\#\mathcal{S} = n \le q\cdot(q^2-q+1)$. In the case of equality we have $\mu' = q-1$ and $\mathcal{S}'$ has type $(q-1)\cdot[4]$. Moreover, $\mathcal{S}'$ is faithful, i.e., any subset of elements of $\mathcal{S}$ spans an even-dimensional subspace. If S is an arbitrary 4-space that contains an element L from $\mathcal{S}$, then projection through L yields that the elements of $\mathcal{S}\backslash\{L\}$ cover $(q-1)\cdot(q+1)$ points from S. Since no subset of elements from $\mathcal{S}$ spans a 5-space, every element of $\mathcal{S}$ that intersects S is fully contained in S, i.e., S contains exactly q elements from $\mathcal{S}$. ∎

Remark 9. *A (multi-)set $\mathcal{S}_q$ of 2-spaces in $\mathrm{PG}(5,q)$ with cardinality $q(q^2-q+1)$ such that each 4-space contains either 0 or q elements from $\mathcal{S}$ is a special case of a so-called* perp-system*, see [37] for details. They do indeed exist for even field sizes q [37, Lemma 5.1]. The construction is based on maximal arcs in $\mathrm{PG}(2,q^2)$ – Denniston arcs to be more precise [41] – i.e. the corresponding codes are linear over $\mathbb{F}_{q^2}$. For odd q we cannot obtain such examples from maximal arcs in $\mathrm{PG}(2,q^2)$ [4], so that $\overline{n}_q(6,2;q) < q\cdot(q^2-q+1)$. For $q = 3$ an example attaining the upper bound from Proposition 7 was found by a computer search, see [37, Example 2], and for odd $q > 3$ no such example is known.*

5.1 Additive codes over the binary field

As mentioned before, we have $n_q(r,h;s) = \infty$ for $h \le r$, so that we assume $r \ge h+1$. For $n_2(3,2;s)$ we refer to Theorem 10 and for $n_2(4,2;s)$ we refer to Theorem 8.

Theorem 12. *([16]) We have*

- $n_2(5,2;7t) = 31t$ *for* $t \ge 1$*;*
- $n_2(5,2;7t-1) = 31t-5$ *for* $t \ge 1$*;*
- $n_2(5,2;7t-2) = 31t-10$ *for* $t \ge 1$*;*
- $n_2(5,2;7t-3) = 31t-15$ *for* $t \ge 1$*;*
- $n_2(5,2;7t-4) = 31t-20$ *for* $t \ge 1$*;*
- $n_2(5,2;7t-5) = 31t-23$ *for* $t \ge 1$*;*
- $n_2(5,2;7t-6) = 31t-28$ *for* $t \ge 2$ *and* $n_2(5,2;1) = 1$.

Proof. Lemma 35 gives $n_2(5,2;1) = 1$. The other upper bounds follow from the Griesmer upper bound. Due to Corollary 3 it suffices to give a construction for the first elements of the seven sequences. Theorem 2 gives $n_2(5,2;7) \ge 31$. Lemma 23 shows that $[5]-[3]$ is 2-partitionable over $\mathbb{F}_2$, so that $n_2(5,2;2) \ge 8$. From small linear codes we conclude $n_2(5,2;3) \ge \overline{n}_2(5,2;3) = 11$, $n_2(5,2;4) \ge \overline{n}_2(5,2;4) = 16$, and $n_2(5,2;5) \ge \overline{n}_2(5,2;5) = 21$. From Lemma 24 we conclude that $3[5]-[4]-2[2]$ is 2-partitionable over $\mathbb{F}_2$, so that also $3[5]-[4]$ is 2-partitionable over $\mathbb{F}_2$ and $n_2(5,2;6) \ge 26$. Corollary 1 gives $n_2(5,2;8) \ge n_2(5,2;2) + n_2(5,2;6) \ge 34$. ∎

Remark 10. *For $r > 2h$ Lemma 23 gives that $[r]-[r-h]$ is h-partitionable over $\mathbb{F}_q$, so that $n_q\left(r,h;q^{r-2h}\right) \ge q^{r-h}$. From the Griesmer upper bound we can conclude that indeed $n_q\left(r,h;q^{r-2h}\right) = q^{r-h}$. For $r > 2h$ with $r \equiv 1 \pmod h$ Lemma 26 gives that $[h-1]_q \cdot [r] + q^{h-1} \cdot [1]$ is h-partitionable over $\mathbb{F}_q$, so that $n_q\left(r,h;1+[h-1]_q \cdot \sum_{i=1}^{\lfloor (r-h)/h \rfloor} q^{r-h-ih}\right) \ge 1 + [h-1]_q \cdot \sum_{i=1}^{\lfloor r/h \rfloor} q^{r-ih}$. From the Griesmer upper bound we can conclude that this lower bound is indeed tight.*

Theorem 13. *([16, Theorem 1]) We have $n_2(6,2;s) = \overline{n}_2(6,2;s)$ for all s, i.e.*

- $n_2(6,2;5t) = 21t$ *for* $t \ge 1$*;*
- $n_2(6,2;5t-1) = 21t-5$ *for* $t \ge 1$*;*
- $n_2(6,2;5t-2) = 21t-10$ *for* $t \ge 1$ *and* $n_2(5,2;3) = 9$*;*
- $n_2(6,2;5t-3) = 21t-15$ *for* $t \ge 1$*;*
- $n_2(6,2;5t-4) = 21t-20$ *for* $t \ge 1$.

Proof. The lower bounds follow from $n_2(6,2;s) \ge \overline{n}_2(6,2;s)$ and Corollary 3. The upper bound $n_2(6,2;3) \le 9$ was shown in [21, Section 4.2]. All other upper bounds follow from the Griesmer upper bound. ∎

Theorem 14. *([85]; cf. [51, Table I],[52, Table II]) We have*

- $n_2(7,2;31t) = 127t$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-1) = 127t-5$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-2) = 127t-10$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-3) = 127t-15$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-4) = 127t-20$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-5) = 127t-21$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-6) = 127t-26$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-7) = 127t-31$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-8) = 127t-36$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-9) = 127t-41$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-10) = 127t-42$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-11) = 127t-47$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-12) = 127t-52$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-13) = 127t-55$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-14) = 127t-60$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-15) = 127t-63$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-16) = 127t-68$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-17) = 127t-73$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-18) = 127t-76$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-19) = 127t-81$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-20) = 127t-84$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-21) = 127t-87$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-22) = 127t-92$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-23) = 127t-95$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-24) = 127t-100$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-25) = 127t-105$ *for* $t \geq 1$*;*
- $n_2(7,2;31t-26) = 127t-108$ *for* $t \geq 2$ *and* $n_2(7,2;5) = 17$*;*
- $n_2(7,2;31t-27) = 127t-113$ *for* $t \geq 2$ *and* $n_2(7,2;4) = 12$*;*
- $n_2(7,2;31t-28) = 127t-116$ *for* $t \geq 2$ *and* $n_2(7,2;3) = 7$*;*
- $n_2(7,2;31t-29) = 127t-121$ *for* $t \geq 2$ *and* $n_2(7,2;2) = 2$*;*
- $n_2(7,2;31t-30) = 127t-126$ *for* $t \geq 1$*.*

Proof. Lemma 35 gives $n_2(7,2;1) = 1$ and $n_2(7,2;2) = 2$. Theorem 2 yields $n_2(7,2;31t) = 127t$ for $t \geq 1$. In [21] $n_2(7,2;3) \leq 7$ was shown. The coding upper bound implies $n_2(7,2;4) \leq 12$ and $n_2(7,2;5) \leq 17$. All other upper bounds follow from the Griesmer upper bound. Due to Corollary 3 and Corollary 1 it suffices to gives constructions for $s \in \{3,\dots,13,15,21,25,26,30\}$. Constructions for $s = 3,4$ were given in [21] and for $s = 5$ we can use $n_2(7,2;5) \geq \overline{n}_2(7,2;5) = 17$. For $s \in \{6,7,12,13\}$ examples were found using ILP searches, see [85]. For $s = 9$ an example is given by a vector space partition of PG(6,2) of type $2^{35}3^1 4^1$. For $s = 15$ an example is given in [52, Example 2]. For $s \in \{8,10,11,21,25,26,30\}$ examples can be easily constructed using the general tools provided in Section 3, see Proposition 10 for the details. ∎

Theorem 15. *([85]) For $s \geq 30$ the Griesmer upper bound for $n_2(8,2;s)$ can always be attained. For all $s \in \mathbb{N}_{>0}$ with $s \not\equiv 2,3,7,8 \pmod{21}$ and $s \notin \{9,10,11,14,15,24,27\}$ we have $n_2(8,2;s) = \overline{n}_2(8,2;s)$. More concretely:*

- $n_2(8,2;21) = 85t$ *for* $t \geq 1$;
- $n_2(8,2;21t-1) = 85t-5$ *for* $t \geq 1$;
- $n_2(8,2;21t-2) = 85t-10$ *for* $t \geq 1$;
- $n_2(8,2;21t-3) = 85t-15$ *for* $t \geq 1$;
- $n_2(8,2;21t-4) = 85t-20$ *for* $t \geq 1$;
- $n_2(8,2;21t-5) = 85t-21$ *for* $t \geq 1$;
- $n_2(8,2;21t-6) = 85t-26$ *for* $t \geq 2$ *and* $n_2(8,2;15) \in \{55,56,57\}$;
- $n_2(8,2;21t-7) = 85t-31$ *for* $t \geq 1$;
- $n_2(8,2;21t-8) = 85t-36$ *for* $t \geq 1$;
- $n_2(8,2;21t-9) = 85t-41$ *for* $t \geq 1$;
- $n_2(8,2;21t-10) = 85t-42$ *for* $t \geq 2$ *and* $n_2(8,2;11) = 40$;
- $n_2(8,2;21t-11) = 85t-47$ *for* $t \geq 2$ *and* $n_2(8,2;10) \in \{35,36\}$;
- $n_2(8,2;21t-12) = 85t-52$ *for* $t \geq 1$;
- $n_2(8,2;21t-13) = 85t-55$ *for* $t \geq 3$, $n_2(8,2;8) = 28$, *and* $n_2(8,2;29) \in \{113,114,115\}$;
- $n_2(8,2;21t-14) = 85t-60$ *for* $t \geq 3$, $n_2(8,2;7) = 23$, *and* $n_2(8,2;28) \in \{108,109,110\}$;
- $n_2(8,2;21t-15) = 85t-63$ *for* $t \geq 3$, $n_2(8,2;6) = 18$, *and* $n_2(8,2;27) \in \{106,107\}$;
- $n_2(8,2;21t-16) = 85t-68$ *for* $t \geq 2$;
- $n_2(8,2;21t-17) = 85t-73$ *for* $t \geq 2$ *and* $n_2(8,2;4) = 10$;
- $n_2(8,2;21t-18) = 85t-76$ *for* $t \geq 3$, $n_2(8,2;3) = 5$, $n_2(8,2;24) \in \{92,93,94\}$;
- $n_2(8,2;21t-19) = 85t-81$ *for* $t \geq 2$ *and* $n_2(8,2;2) = 2$;
- $n_2(8,2;21t-20) = 85t-84$ *for* $t \geq 1$.

Proof. Lemma 35 gives $n_2(8,2;1) = 1$ and $n_2(8,2;2) = 2$. Theorem 2 yields $n_2(8,2;21t) = 85$ for $t \geq 1$. In [21] $n_2(8,2;3) \leq 5$ and $n_2(8,2;4) \leq 10$ were shown. The coding upper bound implies $n_2(8,2;6) \leq 18$, $n_2(8,2;7) \leq 23$, $n_2(8,2;8) \leq 28$, $n_2(8,2;10) \leq 36$, $n_2(8,2;11) \leq 40$, and $n_2(8,2;15) \leq 57$. All other upper bounds follow from the Griesmer upper bound. The lower bound $n_2(8,2;s) \geq \overline{n}_2(8,2;s)$ matches the upper bound for all $s \in \{5,\dots,48\}\backslash\{9,10,11,14,15,23,24,27,28,29,44,45\}$. For $s \in \{9,10,11,14,23,27,45,49,50\}$ we refer to [85] for explicit examples obtained using ILP searches. For $s = 50$ also the tools from Section 3 can be used, see Proposition 11 for the details. For $s \in \{15,24,28,29\}$ the lower bound $n_2(8,2;s) \geq \overline{n}_2(8,2;s)$ still gives the best known construction. With this, all remaining constructions can be obtained using Corollary 3 and Corollary 1. ∎

A few further constructions and upper bounds for $n_2(r,2;s)$ can be found in the literature:

- $n_2(14,2;7) \leq 11$ [13];
- $n_2(9,2;5) \leq 11$ [13], [14];

- $n_2(15,2;8) \le 13$ [13];
- $n_2(10,2;6) \le 14$ [11];
- $n_2(28,2;14) \le 17$ [15];
- $n_2(35,2;18) \ge 22$ [15].

For $n_2(4,3;s)$ we refer to Theorem 10, for $n_2(5,3;s)$ we refer to Proposition 4 in combination with Table 6, and for $n_2(6,3;s)$ we refer to Theorem 8.

Theorem 16. *We have*

- $n_2(7,3;15t) = 127t$ *for* $t \ge 1$;
- $n_2(7,3;15t-1) = 127t-9$ *for* $t \ge 1$;
- $n_2(7,3;15t-2) = 127t-18$ *for* $t \ge 1$;
- $n_2(7,3;15t-3) = 127t-27$ *for* $t \ge 1$;
- $n_2(7,3;15t-4) = 127t-36$ *for* $t \ge 1$;
- $n_2(7,3;15t-5) = 127t-45$ *for* $t \ge 1$;
- $n_2(7,3;15t-6) = 127t-54$ *for* $t \ge 1$;
- $n_2(7,3;15t-7) = 127t-61$ *for* $t \ge 1$;
- $n_2(7,3;15t-8) = 127t-70$ *for* $t \ge 1$;
- $n_2(7,3;15t-9) = 127t-77$ *for* $t \ge 1$;
- $n_2(7,3;15t-10) = 127t-86$ *for* $t \ge 1$;
- $n_2(7,3;15t-11) = 127t-95$ *for* $t \ge 1$;
- $n_2(7,3;15t-12) = 127t-104$ *for* $t \ge 1$;
- $n_2(7,3;15t-13) = 127t-111$ *for* $t \ge 1$;
- $n_2(7,3;15t-14) = 127t-120$ *for* $t \ge 2$ *and* $n_2(7,3;1) = 1$.

Proof. Lemma 35 gives $n_2(7,3;1) = 1$. Theorem 2 yields $n_2(7,3;15) = 127$. All other upper bounds follow from the Griesmer upper bound. The lower bound $n_2(7,3;s) \ge \overline{n}_2(7,3;s)$ matches the upper bound for $s = 9$. For $s \in \{3,5\}$ examples have been found using ILP searches, see Section F. For $s \in \{2,6,13,14\}$ the constructions from Section 3 can be used, see Proposition 14 for the details. With this, all remaining constructions can be obtained using Corollary 3 and Corollary 1. ∎

Remark 11. *We have* $n_2(7,3;2) = 16$ *and it is not hard to show that any projective* $3-(16,7,2,\mu)_2$ *system* $\mathcal{S}$ *is indeed faithful and we have* $\mu = 1$, *i.e.* $\mathcal{S}$ *is a partial plane spread of cardinality* 16 *in* PG(6, 2). *Those have been classified in [66] and there are exactly* $37 + 3988 = 4025$ *isomorphism types. For more details on the equivalence of linear or additive codes we refer e.g. to [5].*

Lemma 37. *We have* $n_3(8,3;3) \le 20$.

Proof. Assume that $\mathcal{S}$ is a faithful projective $3-(21,8,3)_2$ system and $C := \mathcal{X}^{-1}(\mathcal{S})$ is the linear code that corresponds to the multiset of points covered by the elements of $\mathcal{S}$. By Lemma 4 C is a $[147,8,\{72,76,80,84\}]_2$ code.

If a plane S would be contained at least twice, then $\#\mathcal{S} \le 2 + [8-3]_2/[8-6]_2 \le 13$ – contradiction. If two different planes $S_1, S_2 \in \mathcal{S}$ intersect in at least a point then projection through S_1 gives a $3-(20,5,2)_2$ system $\mathcal{S}'$ by Lemma 16, where S_2 is projected to a line or a point, i.e. $\mathcal{S}'$ is unfaithful. Let $S' \in \mathcal{S}'$ be an element of dimension at most 2, so that S' is contained in at least 7 hyperplanes of $\mathrm{PG}(4,2)$. These hyperplanes contain at most one other element from $\mathcal{S}'$. Since every hyperplane contains at most two elements from $\mathcal{S}'\backslash\{S'\}$ we have

$$\#\mathcal{S}' - 1 \le \frac{7\cdot 1 + 24\cdot 2}{3} < 19,$$

which is a contradiction. Thus, no two elements of $\mathcal{S}$ intersect in a point and the linear code C is projective. Let A_i denote the number of codewords of weight i in C. From the first three MacWilliams identities we compute $A_{84} = 234 - A_{72}$, $A_{80} = 3A_{72} - 609$, and $A_{76} = 630 - 3A_{72}$.

Let $c \in C$ be a codeword of weight 84 and C' be the corresponding residual code,[6] so that C' is a projective $[63,7,\{30,32,34,36,38\}]_2$ code. Using the software package `LinCode` [27] we have verified by exhaustive enumeration that there is a unique such code and a generator matrix is given by

$$\begin{pmatrix}
111111111111111111111111000000001111000000001110000000001000000\\
111111111111111100000000111111110000111100000001110000000100000\\
000000001111111111111111111111110000000011110000001110000010000\\
000011110000111100001111000011111111111111110000000001110001000\\
011100110011000100010011000100110111011100010110110110110000100\\
101101010101011100100101001001010001000101111011011011010000010\\
000110100110101101100001011011011010001100101110100111100000001
\end{pmatrix}.$$

The corresponding weight enumerator is given by

$$1 + 76x^{30} + 31x^{32} + 20x^{38}.$$

With this we can compute the number A_i of codewords of weight i of C as $A_{72} = 183$, $A_{76} = 51 - y$, $A_{80} = 2y$, and $A_{84} = 21 - y$, where $0 \le y \le 20$. Plugging $A_{72} = 183$ into the previous equations gives $A_{84} = 51$, $A_{80} = -60$, and $A_{76} = 81$, which is a contradiction since A_{80} cannot be negative.

If C does not contain a codeword of weight 84, then we have $A_{84} = 0$, so that $A_{72} = 234$,

[6] Our coding theoretic proof of $n_3(8,3;3) \le 20$ is rather concise and we did not introduce all used concepts properly. However, all of this is quite standard when the aim is to show the nonexistence of certain linear codes see e.g. [26] or [54, Chapter 3].

$A_{80} = 93$, and $A_{76} = -72$, which is a contradiction since A_{76} cannot be negative. ■

We remark that an $[147, 8, \{72, 76, 80, 128\}]_2$ code exists and the coding upper bound gives $n_3(8, 3; 3) \le 21$.

Theorem 17. *We have*

- $n_2(8, 3; 31t) = 255t$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 1) = 255t - 9$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 2) = 255t - 18$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 3) = 255t - 27$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 4) = 255t - 36$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 5) = 255t - 43$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 6) = 255t - 52$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 7) = 255t - 59$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 8) = 255t - 68$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 9) = 255t - 75$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 10) = 255t - 84$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 11) = 255t - 91$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 12) = 255t - 100$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 13) = 255t - 109$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 14) = 255t - 118$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 15) = 255t - 127$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 16) = 255t - 134$ *for* $t \ge 2$ *and* $n_2(8, 3; 15) \in \{119, \ldots, 121\}$;
- $n_2(8, 3; 31t - 17) = 255t - 143$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 18) = 255t - 150$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 19) = 255t - 159$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 20) = 255t - 166$ *for* $t \ge 2$ *and* $n_2(8, 3; 11) \in \{87, \ldots, 89\}$;
- $n_2(8, 3; 31t - 21) = 255t - 175$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 22) = 255t - 182$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 23) = 255t - 191$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 24) = 255t - 200$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 25) = 255t - 209$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 26) = 255t - 218$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 27) = 255t - 223$ *for* $t \ge 1$;
- $n_2(8, 3; 31t - 28) = 255t - 232$ *for* $t \ge 2$ *and* $n_2(8, 3; 3) \in \{18, \ldots, 20\}$;
- $n_2(8, 3; 31t - 29) = 255t - 241$ *for* $t \ge 2$ *and* $n_2(8, 3; 2) = 10$;
- $n_2(8, 3; 31t - 30) = 255t - 250$ *for* $t \ge 2$ *and* $n_2(8, 3; 1) = 1$.

Proof. Lemma 35 gives $n_2(8,3;1) = 1$. Theorem 2 yields $n_2(8,3;31) = 255$. From Lemma 17 and $n_2(5,3;2) = 9$ we conclude $n_2(8,3;2) \leq 10$. Lemma 37 gives $n_2(8,8;3) \leq 20$. All other upper bounds follow from the Griesmer upper bound. The lower bound $n_2(8,3;s) \geq \overline{n}_2(8,3;s)$ matches the upper bound for $s \in \{2,9\}$. For $s \in \{3,5,6,7,10,19,20,22\}$ examples have been found using ILP searches, see Section F. For $s \in \{4,21,25,30,41\}$ the constructions from Section 3 can be used, see Proposition 15 for the details. With this, all remaining constructions can be obtained using Corollary 3 and Corollary 1. ∎

Remark 12. *By a heurisic search we have constructed more than eighty thousand* $[132,7,\{64,68,72,76\}]_2$ *codes. After extending six thousand of these codes we have found a unique* $[133,8,\{64,68,72,76\}]_2$ *code. The corresponding (multi-) set of points allows to choose only between* 15 *and* 17 *planes instead of* 19.

Remark 13. *Assume that $\mathcal{S}$ is a faithful projective $3-(n,8,3)_2$ system that matches the upper bound $n_2(8,3;11) \leq 89 =: n$. The linear code C corresponding to the multiset of points $\mathcal{M}$ covered by the elements of $\mathcal{S}$, see Lemma 4, would be a 4-divisible $[623,8,\geq 312]_2$ code. In Proposition 15 we showed that $(7t-4)\cdot[8]-[7]-[4]$ is 3-partitionable over $\mathbb{F}_2$ for all $t \geq 3$. Plugging in $t = 1$ we easily see that $3[8]-[7]-[4]$ is 1-partitionable over $\mathbb{F}_2$, which is essentially the Solomon–Stiffler construction for the code linear C. However, this code is ruled out due to the condition on the maximum distance in Lemma 4. Framed differently, if $H = S_7$ is the hyperplane that is removed (according to the notation in Definition 11), then we have $\mathcal{M}(H) \leq (3-1)\cdot[7]_2 = 254$. However, each element of $\mathcal{S}$ intersects H in at least $[2]_2 = 3$ points, which yields the contradiction $3\cdot 89 = 267 > 254$. Thus, we conclude that $3[8]-[7]-[4]$ is 3-partitionable over $\mathbb{F}_2$ nevertheless all conditions of Theorem 3 are satisfied. Such "dimension arguments" are quite common in nonexistence proofs of vector space partitions of a certain type, see e.g. [46].*

Theorem 18. *We have*

- $n_2(9,3;9t) = 73t$ *for* $t \geq 1$;
- $n_2(9,3;9t-1) = 73t-9$ *for* $t \geq 1$;
- $n_2(9,3;9t-2) = 73t-18$ *for* $t \geq 2$ *and* $n_2(9,3;7) \in \{49,\ldots,55\}$;
- $n_2(9,3;9t-3) = 73t-27$ *for* $t \geq 2$ *and* $n_2(9,3;6) \in \{42,\ldots,46\}$;
- $n_2(9,3;9t-4) = 73t-36$ *for* $t \geq 2$ *and* $n_2(9,3;5) \in \{33,\ldots,37\}$;
- $n_2(9,3;9t-5) = 73t-43$ *for* $t \geq 7$, $n_2(9,3;4) = 28$, *and* $n_2(9,3;13) \in \{101,\ldots,103\}$;
- $n_2(9,3;9t-6) = 73t-52$ *for* $t \geq 8$, $n_2(9,3;3) \in \{15,\ldots,19\}$, *and* $n_2(9,3;12) \in \{92,\ldots,94\}$;
- $n_2(9,3;9t-7) = 73t-59$ *for* $t \geq 10$, $n_2(9,3;2) = 10$, *and* $n_2(9,3;11) \in \{83,\ldots,87\}$;
- $n_2(9,3;9t-8) = 73t-68$ *for* $t \geq 11$, $n_2(9,3;1) = 1$, *and* $n_2(9,3;10) \in \{74,\ldots,78\}$.

Proof. Lemma 35 gives $n_2(9,3;1) = 1$. Theorem 2 yields $n_2(9,3;9) = 73$. We have $n_2(9,3;2) \leq n_2(8,3;1) = 10$. The coding upper bound implies $n_2(9,3;3) \leq 19$ and $n_2(9,3;4) \leq 28$. More precisely, in [26] $d \leq 33$ for each $[73,8,d]_2$ code was shown, so that $d \leq 66$ for each $[140,9,d]_2$

code and $n_2(9,3;3) < 20$. In [111] the nonexistence of a $[56,7,26]_2$ code was shown, so that no $[104,8,50]_2$ and no $[203,9,99]_2$ codes exist. With this, we conclude $n_2(9,3;4) < 29$. All other upper bounds follow from the Griesmer upper bound. For lower bounds for $n_2(9,3;9t-i)$ for $i \in \{5,6,7,8\}$ and large t we refer to Proposition 16. All other lower bounds are obtained from $n_2(9,3;s) \ge \overline{n}_2(9,3;s)$. ■

As already remarked in Table 2, we have $n_2(9,3;9t-i) > \overline{n}_2(9,3;9t-i)$ for all $i \in \{5,6,7,8\}$ and sufficiently large t. We conjecture that the lower bounds on t from Proposition 16 can be lowered substantially.

5.2 Additive codes over the ternary field

As mentioned before, we have $n_q(r,h;s) = \infty$ for $h \le r$, so that we assume $r \ge h+1$. For $n_3(3,2;s)$ we refer to Theorem 10 and for $n_3(4,2;s)$ we refer to Theorem 8.

Theorem 19. *We have*

- $n_3(5,2;13t) = 121t$ *for* $t \ge 1$*;*
- $n_3(5,2;13t-1) = 121t-10$ *for* $t \ge 1$*;*
- $n_3(5,2;13t-2) = 121t-20$ *for* $t \ge 1$*;*
- $n_3(5,2;13t-3) = 121t-30$ *for* $t \ge 1$*;*
- $n_3(5,2;13t-4) = 121t-40$ *for* $t \ge 1$*;*
- $n_3(5,2;13t-5) = 121t-50$ *for* $t \ge 1$*;*
- $n_3(5,2;13t-6) = 121t-60$ *for* $t \ge 1$*;*
- $n_3(5,2;13t-7) = 121t-67$ *for* $t \ge 1$*;*
- $n_3(5,2;13t-8) = 121t-77$ *for* $t \ge 2$ *and* $n_3(5,2;5) \in \{41,\dots,44\}$*;*
- $n_3(5,2;13t-9) = 121t-87$ *for* $t \ge 2$ *and* $n_3(5,2;4) \in \{33,34\}$*;*
- $n_3(5,2;13t-10) = 121t-94$ *for* $t \ge 1$*;*
- $n_3(5,2;13t-11) = 121t-104$ *for* $t \ge 2$ *and* $n_3(5,2;2) = 12$*;*
- $n_3(5,2;13t-12) = 121t-114$ *for* $t \ge 2$ *and* $n_3(5,2;1) = 1$*;*

Proof. Lemma 35 gives $n_3(5,2;1) = 1$ and $n_3(5,2;2) = 12$ was shown in [7] by exhaustive enumeration. The other upper bounds follow from the Griesmer upper bound. For $s \in \{9,10,20\}$ the lower bound $n_3(5,2;s) \ge \overline{n}_3(5,2;s)$ matches the upper bound. Theorem 2 gives $n_3(5,2;13) \ge 121$. Lemma 23 shows that $[5]-[3]$ is 2-partitionable over $\mathbb{F}_3$, so that $n_3(5,2;3) \ge 27$. From Lemma 24 we conclude that $4[5]-[4]-3[2]$ is 2-partitionable over $\mathbb{F}_3$, so that also $4[5]-[4]$ is 2-partitionable over $\mathbb{F}_3$ and $n_3(5,2;12) \ge 111$. Via ILP searches we found the following lower bounds: $n_3(5,2;4) \ge 33$, $n_3(5,2;5) \ge 41$, $n_3(5,2;7) \ge 61$, $n_3(5,2;8) \ge 71$, and $n_3(5,2;11) \ge 101$, see Section F. The other lower bounds follow from Corollary 3 and Corollary 1. ■

Theorem 20. *We have*

- $n_3(6,2;10t) = 91t$ *for* $t \ge 1$*;*

- $n_3(6,2;10t-1) = 91t-10$ *for* $t \geq 1$*;*
- $n_3(6,2;10t-2) = 91t-20$ *for* $t \geq 2$ *and* $n_3(6,2;8) \in \{66,\ldots,68\}$*;*
- $n_3(6,2;10t-3) = 91t-30$ *for* $t \geq 2$ *and* $n_3(6,2;7) \in \{55,\ldots,61\}$*;*
- $n_3(6,2;10t-4) = 91t-40$ *for* $t \geq 2$ *and* $n_3(6,2;6) \in \{48,49\}$*;*
- $n_3(6,2;10t-5) = 91t-50$ *for* $t \geq 2$ *and* $n_3(6,2;5) \in \{37,\ldots,41\}$*;*
- $n_3(6,2;10t-6) = 91t-60$ *for* $t \geq 2$ *and* $n_3(6,2;4) \in \{28,\ldots,31\}$*;*
- $n_3(6,2;10t-7) = 91t-67$ *for* $t \geq 8$*,* $n_3(6,2;3) = 21$*,* $n_3(6,2;13) \in \{112,113\}$*, and* $n_3(6,2;23) \in \{203,\ldots,206\}$*;*
- $n_3(6,2;10t-8) = 91t-77$ *for* $t \geq 2$ *and* $n_3(6,2;2) = 10$*;*
- $n_3(6,2;10t-9) = 91t-87$ *for* $t \geq 2$ *and* $n_3(6,2;1) = 1$*.*

Proof. Lemma 35 gives $n_3(6,3;1) = 1$ and Theorem 11 gives $n_3(6,2;2) = 10$. The coding upper bound yields $n_3(6,2;3) \leq 21$, $n_3(6,2;6) \leq 49$, $n_3(6,2;8) \leq 68$, and $n_3(6,2;13) \leq 113$. The other upper bounds follow from the Griesmer upper bound. For $s \in \{9,14,\ldots,19\}$ the lower bound $n_3(6,2;s) \geq \overline{n}_3(6,2;s)$ matches the upper bound. Theorem 2 gives $n_3(6,2;10) \geq 91$ and $n_3(6,2;3) \geq 21$ was shown in [37]. For the three series of improvements $n_3(6,2;10t-i) > \overline{n}_3(6,2;10t-i)$ for $i \in \{7,8,9\}$ we refer to Proposition 18. Via ILP searches we found the following lower bounds: $n_3(6,2;8) \geq 66$, $n_3(6,2;11) \geq 95$, and $n_3(6,2;12) \geq 105$, see Section F. The other lower bounds follow from $n_3(6,2;s) \geq \overline{n}_3(6,2;s)$, Corollary 3, and Corollary 1. ■

We remark that the function $n_3(6,1;s)$ is only partially known.

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A Generalized type of a faithful projective system

Since Definition 11 is too restricted to cover the full generality of the Solomon–Stiffler construction we provide a generalization and analyze its implications for corresponding theoretical results in this section.

Definition 15. *We say that a faithful projective* $h-(n,r,s)_q$ *system* $\mathcal{S}$ *has* generalized type $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ *if there exist subspaces* $T_1,\dots,T_l$ *with* $\#\left\{1\le j\le l \mid \dim\left(T_j\right)=i\right\}=|\varepsilon_i|$ *for* $1\le i\le r-1$ *and*

$$\sum_{S\in\mathcal{S}}\chi_S=\sigma\cdot\chi_V-\sum_{i=j}^{l}\operatorname{sgn}(\varepsilon_{\dim(T_j)})\cdot\chi_{T_i},\qquad(72)$$

where sgn *denotes the sign function. We say that* $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ *is* weakly h-partitionable over $\mathbb{F}_q$ *if a faithful projective* $h-(n,r,s)_q$ *system with generalized type* $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ *exists for suitable subspaces* $T_1,\dots,T_l$ *and parameters* n,s.

While the notion of being h-partitionable over $\mathbb{F}_q$ does not depend on the choice of the subspaces $S_1,\dots,S_{r-1}$, the notion of being weakly h-partitionable over $\mathbb{F}_q$ can depend on a careful selection of the subspaces $T_1,\dots,T_l$. Evaluating $\#\mathcal{M}$, $\min_H\mathcal{M}(H)$, $\min_P\mathcal{M}(P)$ in Lemma 18 (and an application of Equation (2)) yields:

Lemma 38. *If $\mathcal{S}$ is a faithful projective $h-(n,r,s,\mu)_q$ system with generalized type $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$, then we have*

$$n=\left(\sigma[r]_q-\sum_{i=1}^{r-1}\varepsilon_i[i]_q\right)/[h]_q \tag{73}$$

and

$$\sum_{i=1}^{r-1}\varepsilon_i[i]_q\equiv 0 \pmod{[\gcd(r,h)]_q}. \tag{74}$$

If additionally all ε_i are nonnegative, then

$$s\le\left(\sigma[r-h]_q-\sum_{i=h}^{r-1}\varepsilon_i[i-h]_q+\sum_{i=1}^{h-1}\varepsilon_i q^{i-h}[h-i]_q\right)/[h]_q \quad \text{and} \quad \mu\le\sigma. \tag{75}$$

From Lemma 8, based on field reduction, we conclude:

Lemma 39. *If $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is weakly h-partitionable over $\mathbb{F}_{q^l}$, so is $\sigma[rl]+\sum_{i=1}^{r-1}\varepsilon_i[il]$.*

Lemma 19 implies:

Corollary 6. *If $x[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is weakly h-partitionable over $\mathbb{F}_q$ for $x\in\{\sigma,\sigma'\}$ then*

$$\left(\sigma+t\cdot\frac{[h]_q}{[\gcd(r,h)]_q}\right)\cdot[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$$

is weakly h-partitionable over $\mathbb{F}_q$ for all $t\ge 0$ and we have $\sigma\equiv\sigma' \pmod{\frac{[h]_q}{[\gcd(r,h)]_q}}$.

For those situations where we are not interested in the smallest possible value σ such that $\sigma[r]-\mathcal{M}$ is weakly h-partitionable over $\mathbb{F}_q$ we specialize Definition 9:

Definition 16. *We say that $\star[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is weakly h-partitionable over $\mathbb{F}_q$ if there exists a $\sigma\in\mathbb{N}$ such that $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is weakly h-partitionable over $\mathbb{F}_q$.*

Lemma 40. *If $\star[r]-\mathcal{M}$ is h-partitionable over $\mathbb{F}_q$, so is $\star[r]+\mathcal{M}$.*

Corollary 7. *If $\star[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is weakly h-partitionable over $\mathbb{F}_q$, so is $\star[r]+\sum_{i=1}^{r-1}\varepsilon_i[i]$.*

It is an interesting, but possibly very hard, problem to determine for which parameters $\varepsilon_1,\dots,\varepsilon_{r-1}$ we have that $\star[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is weakly h-partitionable over $\mathbb{F}_q$. Clearly we need $r\ge h$ and $\sum_{i=1}^{r-1}|\varepsilon_i|=0$ if $r=h$. Additionally we have the packing condition (74), see Lemma 38. For $h=1$ this condition is trivially satisfied and we indeed have that $\star[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is weakly 1-partitionable over $\mathbb{F}_q$ for all parameters.

Lemma 41. *Let $\star[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ be weakly h-partitionable over $\mathbb{F}_q$.*

(i) If $\varepsilon_i \ge 0$ for all $1 \le i \le h-1$, then there exists a q^{h-1}-divisible multiset of points $\mathcal{M}_1$ in $\operatorname{PG}(r-1,q)$ *with cardinality* $\#\mathcal{M}_1 = \sum_{i=1}^{h-1} \varepsilon_i [i]_q$.

(ii) If $\varepsilon_i \le 0$ for all $1 \le i \le h-1$, then there exists a q^{h-1}-divisible multiset of points $\mathcal{M}_2$ in $\operatorname{PG}(r-1,q)$ *with cardinality* $\#\mathcal{M}_2 = -\sum_{i=1}^{h-1} \varepsilon_i [i]_q$.

Proof. Let $\mathcal{S}$ be a faithful projective $h-(n,r,s)_q$ system of generalized type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i [i]$ for suitable parameters and $T_1, \dots, T_l$ denote the subspaces as in Definition 11. The multiset of points covered by the elements of $\mathcal{S}$ is given by

$$\mathcal{M} = \sigma\chi_V - \sum_{i=1}^{l} \operatorname{sgn}(\varepsilon_{\dim(T_i)}) \cdot \chi_T,$$

where V denotes the r-dimensional ambient space $\operatorname{PG}(r-1,q)$, so that $\mathcal{M}$ is q^{h-1}-divisible by Lemma 3. In case (i) Lemma 3 implies that also $\mathcal{M}_1 := \sum_{1 \le i \le l \,:\, \dim(T_i) \le h-1} \chi_{T_i}$ is q^{h-1}-divisible. For case (ii) we consider the μ-complement $\mathcal{M}^{C_\mu}$ of $\mathcal{M}$, defined by $\mathcal{M}^{C_\mu}(P) = \mu - \mathcal{M}(P)$ for every point P, for a suitably large $\mu \in \mathbb{N}$. Applying Lemma 3 yields that $\mathcal{M}^{C_\mu}$ as well as $\mathcal{M}_2 := \sum_{1 \le i \le l \,:\, \dim(T_i) \le h-1} \chi_{T_i}$ is q^{h-1}-divisible. ∎

Remark 14. *The possible lengths of q^{h-1}-divisible codes over $\mathbb{F}_q$ have been completely characterized in [73, Theorem 1]. The condition in Lemma 41 is only necessary and far from being sufficient. The distribution of the ε_i with $i < h$ is not used at all. E.g. for $h = 3$ and $q = 2$ there exists a 4-divisible multiset of points $\mathcal{M}$ with cardinality 7. However, $\mathcal{M}$ has to equal the characteristic function of a 3-space, see e.g. [77, Corollary 4], so that $(\varepsilon_2, \varepsilon_1) = (2,1)$ is impossible. Partitions of Δ-divisible multisets of points into subspaces are e.g. briefly discussed in [81, Section 10.1], [82] and have e.g. applications for the so-called supertail of a vector space partition [57, 92].*

If the ε_i with $i < h$ have different signs the cardinality $\sum_{i=1}^{h-1} \varepsilon_i [i]_q$ is not sufficient to conclude claims on the nonexistence. E.g., we will see later on that $\star[r] - q \cdot [2]$ is 3-partitionable over $\mathbb{F}_q$ if $r > 3$ and $r \not\equiv 0 \pmod 3$, so that $\star[r] - q \cdot [2] + (q^2+q+1) \cdot [1]$ is weakly 3-partitionable over $\mathbb{F}_q$ by replacing one 3-space by its q^2+q+1 points, while there clearly is no q^2-divisible multiset of points of cardinality $-q \cdot [2]_q + (q^2+q+1) \cdot [1]_q = 1$.

B Parameterized series of additive codes that outperform linear codes

In this appendix we want to extend Table 2 on parameterized series of improvements $n_q(r,h;s) > \overline{n}_q(r,h;s)$ for additive codes in the integral case $r/h \in \mathbb{N}$. By Theorem 4 we just need to compare the Griesmer upper bounds for $n_q\!\left(r,h;\frac{[r-h]_q}{[h]_q} \cdot t - i\right)$ and $\overline{n}_q\!\left(r,h;\frac{[r-h]_q}{[h]_q} \cdot t - i\right) = n_{q^h}\!\left(\frac{r}{h},1;[r/h-1]_{q^h} \cdot t - i\right)$

q	r	h	i	$s_{i,t} := t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q} - i$	$n_q(r,h;s_{i,t})$	$n_q(r,h;s_{i,t}) - \overline{n}_q(r,h;s_{i,t})$
2	10	2	13	$85t-13$	$341t-55$	2
2	10	2	14	$85t-14$	$341t-60$	2
2	10	2	18	$85t-18$	$341t-76$	2
2	10	2	19	$85t-19$	$341t-81$	2
2	10	2	34	$85t-34$	$341t-140$	2
2	10	2	35	$85t-35$	$341t-145$	2
2	10	2	39	$85t-39$	$341t-161$	2
2	10	2	40	$85t-40$	$341t-166$	2
2	10	2	45	$85t-45$	$341t-183$	2
2	10	2	46	$85t-46$	$341t-188$	2
2	10	2	50	$85t-50$	$341t-204$	2
2	10	2	51	$85t-51$	$341t-209$	2
2	10	2	53	$85t-53$	$341t-215$	2
2	10	2	54	$85t-54$	$341t-220$	2
2	10	2	55	$85t-55$	$341t-223$	4
2	10	2	56	$85t-56$	$341t-228$	4
2	10	2	58	$85t-58$	$341t-236$	2
2	10	2	59	$85t-59$	$341t-241$	2
2	10	2	60	$85t-60$	$341t-244$	4
2	10	2	61	$85t-61$	$341t-249$	4
2	10	2	66	$85t-66$	$341t-268$	2
2	10	2	67	$85t-67$	$341t-273$	2
2	10	2	71	$85t-71$	$341t-289$	2
2	10	2	72	$85t-72$	$341t-294$	2
2	10	2	74	$85t-74$	$341t-300$	2
2	10	2	75	$85t-75$	$341t-305$	2
2	10	2	76	$85t-76$	$341t-308$	4
2	10	2	77	$85t-77$	$341t-313$	4
2	10	2	79	$85t-79$	$341t-321$	2
2	10	2	80	$85t-80$	$341t-326$	2
2	10	2	81	$85t-81$	$341t-329$	4
2	10	2	82	$85t-82$	$341t-334$	4

Table 7: Parameterized series of improvements for additive codes with $q = 2$, $r = 10$, and $h = 2$.

in terms of $t \in \mathbb{N}$ for all $0 \le i < \frac{[r-h]_q}{[h]_q}$. This can be easily done by a small computer program. Here we just list the cases of all such i for small parameters q, r, and h. Since $n_q(h,h;s) = s$ and $n_q(2h,h;s) = \overline{n}_q(2h,h;s)$ for all $s \in \mathbb{N}$, see Theorem 8, improvements can only occur if $r/h \ge 3$. The case $(r,h) = (6,2)$ is completely settled in Theorem 9. There are exactly $q(q-2)$ parametric improvements for each field size q. So, especially none for $q = 2$. As shown in [16], cf. Subsection 5.1, we even have $n_2(6,2;s) = \overline{n}_2(6,2;s)$. The parametric improvements for $n_q(6,2;s)$ are already stated in Table 2 when $q \in \{3,4\}$. For $q = 5$ we refer to Table 8. Since there would be at least 35 parametric improvements for $q \ge 7$ and we have an analytic solution in Theorem 9, we abstain from listing further tables for $n_q(6,2;s)$.

q	r	h	i	$s_{i,t} := t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q} - i$	$n_q(r,h;s_{i,t})$	$n_q(r,h;s_{i,t}) - \overline{n}_q(r,h;s_{i,t})$
5	6	2	11	$26t-11$	$651t-281$	5
5	6	2	12	$26t-12$	$651t-307$	5
5	6	2	13	$26t-13$	$651t-333$	5
5	6	2	14	$26t-14$	$651t-359$	5
5	6	2	15	$26t-15$	$651t-385$	5
5	6	2	16	$26t-16$	$651t-406$	10
5	6	2	17	$26t-17$	$651t-432$	10
5	6	2	18	$26t-18$	$651t-458$	10
5	6	2	19	$26t-19$	$651t-484$	10
5	6	2	20	$26t-20$	$651t-510$	10
5	6	2	21	$26t-21$	$651t-531$	15
5	6	2	22	$26t-22$	$651t-557$	15
5	6	2	23	$26t-23$	$651t-583$	15
5	6	2	24	$26t-24$	$651t-609$	15
5	6	2	25	$26t-25$	$651t-635$	15

Table 8: Parameterized series of improvements for additive codes with $q = 5$, $r = 6$, and $h = 2$.

Parametric improvements for $n_2(8,2;s)$ are contained in Table 2 and for $n_3(8,2;s)$ we refer to Table 11 and Table 12. The latter are so numerous that we do not give further tables for larger field sizes. Parametric improvements for $n_2(10,2;s)$ are listed in Table 7.

For $h = 3$ Table 2 contains the parametric improvements for $n_2(9,3;s)$ and for $n_3(9,3;s)$ we list them in Table 10. For $n_4(9,3;65t-i)$ there are parametric improvements for all $13 \le i \le 64$ and for $n_2(12,3;73t-i)$ there are parametric improvements for all $i \in \{5,\ldots,8\} \cup \{14,\ldots 17\} \cup \{23,\ldots,26\} \cup \{32,\ldots,35\} \cup \{37,\ldots,44\} \cup \{46,\ldots,53\} \cup \{55,\ldots,62\}, \cup \{64,\ldots,71\}$. The only other case being reasonably small to be fully included are the parametric improvements for $n_2(12,4;s)$, see Table 9.

The collected data suggests:

Conjecture 1. *For each prime power q and $r,h \in \mathbb{N}$ with $r \ge 3h$, $h \ge 2$, $r \equiv 0 \pmod{h}$ and $(q,r,h) \ne (2,6,2)$ there exist infinitely many $s \in \mathbb{N}$ with $n_q(r,h;s) > \overline{n}_q(r,h;s)$.*

q	r	h	i	$s_{i,t} := t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q} - i$	$n_q(r,h;s_{i,t})$	$n_q(r,h;s_{i,t}) - \overline{n}_q(r,h;s_{i,t})$
2	12	4	7	$17t-7$	$273t-117$	2
2	12	4	8	$17t-8$	$273t-134$	2
2	12	4	9	$17t-9$	$273t-147$	6
2	12	4	10	$17t-10$	$273t-164$	6
2	12	4	11	$17t-11$	$273t-179$	8
2	12	4	12	$17t-12$	$273t-196$	8
2	12	4	13	$17t-13$	$273t-213$	8
2	12	4	14	$17t-14$	$273t-230$	8
2	12	4	15	$17t-15$	$273t-245$	10
2	12	4	16	$17t-16$	$273t-262$	10

Table 9: Parameterized series of improvements for additive codes with $q = 2$, $r = 12$, and $h = 4$.

q	r	h	i	$s_{i,t} := t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q} - i$	$n_q(r,h;s_{i,t})$	$n_q(r,h;s_{i,t}) - \overline{n}_q(r,h;s_{i,t})$
3	9	3	10	$28t-10$	$757t-274$	6
3	9	3	11	$28t-11$	$757t-302$	6
3	9	3	12	$28t-12$	$757t-330$	6
3	9	3	13	$28t-13$	$757t-355$	9
3	9	3	14	$28t-14$	$757t-383$	9
3	9	3	15	$28t-15$	$757t-411$	9
3	9	3	16	$28t-16$	$757t-439$	9
3	9	3	17	$28t-17$	$757t-467$	9
3	9	3	18	$28t-18$	$757t-495$	9
3	9	3	19	$28t-19$	$757t-517$	15
3	9	3	20	$28t-20$	$757t-545$	15
3	9	3	21	$28t-21$	$757t-573$	15
3	9	3	22	$28t-22$	$757t-598$	18
3	9	3	23	$28t-23$	$757t-626$	18
3	9	3	24	$28t-24$	$757t-654$	18
3	9	3	25	$28t-25$	$757t-682$	18
3	9	3	26	$28t-26$	$757t-710$	18
3	9	3	27	$28t-27$	$757t-738$	18

Table 10: Parameterized series of improvements for additive codes with $q = 3$, $r = 9$, and $h = 3$.

q	r	h	i	$s_{i,t} := t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q} - i$	$n_q(r,h;s_{i,t})$	$n_q(r,h;s_{i,t}) - \overline{n}_q(r,h;s_{i,t})$
3	8	2	7	$91t-7$	$820t-67$	3
3	8	2	8	$91t-8$	$820t-77$	3
3	8	2	9	$91t-9$	$820t-87$	3
3	8	2	17	$91t-17$	$820t-158$	3
3	8	2	18	$91t-18$	$820t-168$	3
3	8	2	19	$91t-19$	$820t-178$	3
3	8	2	27	$91t-27$	$820t-249$	3
3	8	2	28	$91t-28$	$820t-259$	3
3	8	2	29	$91t-29$	$820t-269$	3
3	8	2	34	$91t-34$	$820t-310$	3
3	8	2	35	$91t-35$	$820t-320$	3
3	8	2	36	$91t-36$	$820t-330$	3
3	8	2	37	$91t-37$	$820t-337$	6
3	8	2	38	$91t-38$	$820t-347$	6
3	8	2	39	$91t-39$	$820t-357$	6
3	8	2	44	$91t-44$	$820t-401$	3
3	8	2	45	$91t-45$	$820t-411$	3
3	8	2	46	$91t-46$	$820t-421$	3
3	8	2	47	$91t-47$	$820t-428$	6
3	8	2	48	$91t-48$	$820t-438$	6
3	8	2	49	$91t-49$	$820t-448$	6
3	8	2	54	$91t-54$	$820t-492$	3
3	8	2	55	$91t-55$	$820t-502$	3
3	8	2	56	$91t-56$	$820t-512$	3
3	8	2	57	$91t-57$	$820t-519$	6
3	8	2	58	$91t-58$	$820t-529$	6

Table 11: Parameterized series of improvements for additive codes with $q = 3$, $r = 8$, and $h = 2$ – part 1.

Looking at the data one might also conjecture that the improvement $n_q(r,h;s) - \overline{n}_q(r,h;s)$ is always divisible by the characteristic p of $\mathbb{F}_q$ when $r/h \in \mathbb{N}$ and s is sufficiently large. It is also conspicuous that the improvements seem to come in blocks of consecutive values of s with equal improvement $n_q(r,h;s) - \overline{n}_q(r,h;s)$. So far it seems that the length of those blocks is always divisible by the field size q, but this might also be an artifact due to too few observations.

q	r	h	i	$s_{i,t} := t \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q} - i$	$n_q(r,h;s_{i,t})$	$n_q(r,h;s_{i,t}) - \overline{n}_q(r,h;s_{i,t})$
3	8	2	59	$91t - 59$	$820t - 539$	6
3	8	2	61	$91t - 61$	$820t - 553$	3
3	8	2	62	$91t - 62$	$820t - 563$	3
3	8	2	63	$91t - 63$	$820t - 573$	3
3	8	2	64	$91t - 64$	$820t - 580$	6
3	8	2	65	$91t - 65$	$820t - 590$	6
3	8	2	66	$91t - 66$	$820t - 600$	6
3	8	2	67	$91t - 67$	$820t - 607$	9
3	8	2	68	$91t - 68$	$820t - 617$	9
3	8	2	69	$91t - 69$	$820t - 627$	9
3	8	2	71	$91t - 71$	$820t - 644$	3
3	8	2	72	$91t - 72$	$820t - 654$	3
3	8	2	73	$91t - 73$	$820t - 664$	3
3	8	2	74	$91t - 74$	$820t - 671$	6
3	8	2	75	$91t - 75$	$820t - 681$	6
3	8	2	76	$91t - 76$	$820t - 691$	6
3	8	2	77	$91t - 77$	$820t - 698$	9
3	8	2	78	$91t - 78$	$820t - 708$	9
3	8	2	79	$91t - 79$	$820t - 718$	9
3	8	2	81	$91t - 81$	$820t - 735$	3
3	8	2	82	$91t - 82$	$820t - 745$	3
3	8	2	83	$91t - 83$	$820t - 755$	3
3	8	2	84	$91t - 84$	$820t - 762$	6
3	8	2	85	$91t - 85$	$820t - 772$	6
3	8	2	86	$91t - 86$	$820t - 782$	6
3	8	2	87	$91t - 87$	$820t - 789$	9
3	8	2	88	$91t - 88$	$820t - 799$	9
3	8	2	89	$91t - 89$	$820t - 809$	9

Table 12: Parameterized series of improvements for additive codes with $q = 3$, $r = 8$, and $h = 2$ – part 2.

While it is interesting to know that the Griesmer upper bound can always be reached, see Theorem 4, and that infinitely many parametric improvements $n_q\left(r,h;\frac{[r-h]_q}{[h]_q}\cdot t - i\right) > \overline{n}_q\left(r,h;\frac{[r-h]_q}{[h]_q}\cdot t - i\right)$ do indeed exist, explicit lower bounds on t would be desirable. In Section C we compute such lower bounds for t in a generic manner by using a few basic general constructions. The import and very hard problem of determining $n_q(r,h;s)$ for relatively small values of s remains widely open. We provide a few general results in Section 5 and study the cases of field sizes $q = 2$ and $q = 3$ in Subsection 5.1 and Subsection 5.2, respectively.

C Generic results

The aim of this section is to determine $n_q(r, h; s)$ for small parameters q, r, and s explicitly, assuming that s is sufficiently large. As upper bound we will always utilize the Griesmer upper bound, which can be reached due to Theorem 4. For the corresponding lower bounds we will use a few basic general constructions, see Theorem 2, Lemma 24, Lemma 26, and Lemma 23. With these we will show that $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is h-partitionable over $\mathbb{F}_q$ for suitable parameters σ and $\varepsilon_1, \ldots, \varepsilon_{r-1}$. The stated general results for $n_q\left(r, h; \frac{[r-h]_q}{[\gcd(rh,)]_q} \cdot t - i\right)$ will then follow from Corollary 3 and Corollary 1 (which we will not mention explicitly in the subsequent proofs). All those reasonings may be automated so that we speak of generic results. We will restrict ourselves to small parameters, not treating those that are covered by the general results from Section 5. Especially we restrict to field sizes $q \in \{2, 3\}$. Cf. the results for $n_q(r, h; s)$ for $q \in \{2, 3\}$ and small values of s in Subsection 5.1 and Subsection 5.2, respectively.

Proposition 8. *The Griesmer upper bound for $n_2(5, 2; s)$ is attained for all $s \ge 2$, i.e. we have*

- $n_2(5, 2; 7t) = 31t$ *for* $t \ge 1$ *via* $3t \cdot [5]$;
- $n_2(5, 2; 7t - 1) = 31t - 5$ *for* $t \ge 1$ *via* $3t \cdot [5] - [4]$;
- $n_2(5, 2; 7t - 2) = 31t - 10$ *for* $t \ge 1$ *via* $(3t - 1) \cdot [5] + [2] - 2[1]$;
- $n_2(5, 2; 7t - 3) = 31t - 15$ *for* $t \ge 1$ *via* $(3t - 1) \cdot [5] - 2[3]$;
- $n_2(5, 2; 7t - 4) = 31t - 20$ *for* $t \ge 1$ *via* $(3t - 2) \cdot [5] + 2[1]$;
- $n_2(5, 2; 7t - 5) = 31t - 23$ *for* $t \ge 1$ *via* $(3t - 2) \cdot [5] - [3]$;
- $n_2(5, 2; 7t - 6) = 31t - 28$ *for* $t \ge 2$ *via* $(3t - 2) \cdot [5] - [4] - [3]$.

Proof. By Theorem 2 we have that 3[5], [4], and [2] are 2-partitionable over $\mathbb{F}_2$. Lemma 24 shows that [5] − [3] and 3[5] − [4] − 2[2] are 2-partitionable over $\mathbb{F}_2$. By Lemma 26 we have that 2[5] + [2] − 2[1] and [5] + 2[1] are 2-partitionable over $\mathbb{F}_2$. So, also 3[5] − [4] is 2-partitionable over $\mathbb{F}_2$. ∎

Proposition 9. *The Griesmer upper bound for $n_2(6, 2; s)$ is attained for all $s \ge 8$, i.e. we have*

- $n_2(6, 2; 5t) = 21t$ *for* $t \ge 1$ *via* $t \cdot [6]$;
- $n_2(6, 2; 5t - 1) = 21t - 5$ *for* $t \ge 1$ *via* $t \cdot [6] - [4]$;
- $n_2(6, 2; 5t - 2) = 21t - 10$ *for* $t \ge 1$ *via* $t \cdot [6] - [4] + [3] + 2[1]$;
- $n_2(6, 2; 5t - 3) = 21t - 15$ *for* $t \ge 3$ *via* $t \cdot [6] - 3[4]$;
- $n_2(6, 2; 5t - 4) = 21t - 20$ *for* $t \ge 1$ *via* $(t - 1) \cdot [6] + [2]$.

Proof. By Theorem 2 we have that [6], [4], and [2] are 2-partitionable over $\mathbb{F}_2$. Lemma 24 shows that [6] − [4] is 2-partitionable over $\mathbb{F}_2$. By Lemma 26 we have that [3] + 2[1] is 2-partitionable over $\mathbb{F}_2$. ∎

Proposition 10. *The Griesmer upper bound for $n_2(7,2;s)$ is attained for all $s \geq 24$, i.e. we have*

- $n_2(7,2;31t) = 127t$ *for* $t \geq 1$ *via* $3t \cdot [7]$;
- $n_2(7,2;31t-1) = 127t-5$ *for* $t \geq 1$ *via* $3t \cdot [7] - [4]$;
- $n_2(7,2;31t-2) = 127t-10$ *for* $t \geq 1$ *via* $3t \cdot [7] - 2[4]$;
- $n_2(7,2;31t-3) = 127t-15$ *for* $t \geq 1$ *via* $3t \cdot [7] - 3[4]$;
- $n_2(7,2;31t-4) = 127t-20$ *for* $t \geq 1$ *via* $3t \cdot [7] - 2[5] + 2[1]$;
- $n_2(7,2;31t-5) = 127t-21$ *for* $t \geq 1$ *via* $3t \cdot [7] - [6]$;
- $n_2(7,2;31t-6) = 127t-26$ *for* $t \geq 1$ *via* $3t \cdot [7] - [6] - [4]$;
- $n_2(7,2;31t-7) = 127t-31$ *for* $t \geq 1$ *via* $3t \cdot [7] - [6] - 2[4]$;
- $n_2(7,2;31t-8) = 127t-36$ *for* $t \geq 2$ *via* $3t \cdot [7] - [6] - [5] - [4] + [2] - 2[1]$;
- $n_2(7,2;31t-9) = 127t-41$ *for* $t \geq 1$ *via* $(3t-1) \cdot [7] + 4[1]$;
- $n_2(7,2;31t-10) = 127t-42$ *for* $t \geq 1$ *via* $(3t-1) \cdot [7] + [2] - 2[1]$;
- $n_2(7,2;31t-11) = 127t-47$ *for* $t \geq 1$ *via* $(3t-1) \cdot [7] - 2[3]$;
- $n_2(7,2;31t-12) = 127t-52$ *for* $t \geq 1$ *via* $(3t-1) \cdot [7] - [5] + 2[1]$;
- $n_2(7,2;31t-13) = 127t-55$ *for* $t \geq 1$ *via* $(3t-1) \cdot [7] - [5] - [3]$;
- $n_2(7,2;31t-14) = 127t-60$ *for* $t \geq 2$ *via* $(3t-1) \cdot [7] - [5] - [4] - [3]$;
- $n_2(7,2;31t-15) = 127t-63$ *for* $t \geq 1$ *via* $(3t-1) \cdot [7] - 2[5]$;
- $n_2(7,2;31t-16) = 127t-68$ *for* $t \geq 1$;
- $n_2(7,2;31t-17) = 127t-73$ *for* $t \geq 2$ *via* $(3t-2) \cdot [7] - [6] - [4] - 2[3]$;
- $n_2(7,2;31t-18) = 127t-76$ *for* $t \geq 2$ *via* $(3t-1) \cdot [7] - [6] - [5] - [3]$;
- $n_2(7,2;31t-19) = 127t-81$ *for* $t \geq 2$ *via* $(3t-1) \cdot [7] - [6] - [5] - [4] - [3]$;
- $n_2(7,2;31t-20) = 127t-84$ *for* $t \geq 1$ *via* $(3t-2) \cdot [7] + 2[1]$;
- $n_2(7,2;31t-21) = 127t-87$ *for* $t \geq 1$ *via* $(3t-2) \cdot [7] - [3]$;
- $n_2(7,2;31t-22) = 127t-92$ *for* $t \geq 2$ *via* $(3t-2) \cdot [7] - [4] - [3]$;
- $n_2(7,2;31t-23) = 127t-95$ *for* $t \geq 1$ *via* $(3t-2) \cdot [7] - [5]$;
- $n_2(7,2;31t-24) = 127t-100$ *for* $t \geq 2$ *via* $(3t-2) \cdot [7] - [5] - [4]$;
- $n_2(7,2;31t-25) = 127t-105$ *for* $t \geq 2$ *via* $(3t-2) \cdot [7] - [5] - 2[4]$;
- $n_2(7,2;31t-26) = 127t-108$ *for* $t \geq 2$ *via* $(3t-2) \cdot [7] - [6] - [3]$;
- $n_2(7,2;31t-27) = 127t-113$ *for* $t \geq 2$ *via* $(3t-2) \cdot [7] - [6] - [4] - [3]$;
- $n_2(7,2;31t-28) = 127t-116$ *for* $t \geq 2$ *via* $(3t-2) \cdot [7] - [6] - [5]$;
- $n_2(7,2;31t-29) = 127t-121$ *for* $t \geq 2$ *via* $(3t-2) \cdot [7] - [6] - [5] - [4]$;
- $n_2(7,2;31t-30) = 127t-126$ *for* $t \geq 1$ *via* $3t \cdot [7] - [2]$.

Proof. By Theorem 2 we have that $3[7]$, $3[7]-[2]$, and $[4]$ are 2-partitionable over $\mathbb{F}_2$. Lemma 24 shows that $[7]-[5]$ and $3[7]-[6]-2[4]$ are 2-partitionable over $\mathbb{F}_2$. By Lemma 23 $[7]-[3]$ and $[6]-[4]$ are 2-partitionable over $\mathbb{F}_2$. By Lemma 26 we have that $2[7]+[2]-2[1]$ and $[7]+2[1]$ are 2-partitionable over $\mathbb{F}_2$. So, also $3[7]-[6]$ and $3[7]-3[4]$ are 2-partitionable over $\mathbb{F}_2$. ∎

Proposition 11. *The Griesmer upper bound for $n_2(8,2;s)$ is attained for all $s \geq 260$, i.e. we have*

- $n_2(8,2;21t) = 85t$ *for* $t \geq 1$ *via* $t \cdot [8]$;

- $n_2(8,2;21t-1) = 85t-5$ *for* $t \geq 1$ *via* $t \cdot [8] - [4]$;
- $n_2(8,2;21t-2) = 85t-10$ *for* $t \geq 2$ *via* $t \cdot [8] - 2[4]$;
- $n_2(8,2;21t-3) = 85t-15$ *for* $t \geq 3$ *via* $t \cdot [8] - 3[4]$;
- $n_2(8,2;21t-4) = 85t-20$ *for* $t \geq 1$ *via* $t \cdot [8] - [6] + [2]$;
- $n_2(8,2;21t-5) = 85t-21$ *for* $t \geq 1$ *via* $t \cdot [8] - [6]$;
- $n_2(8,2;21t-6) = 85t-26$ *for* $t \geq 2$ *via* $t \cdot [8] - [6] - [4]$;
- $n_2(8,2;21t-7) = 85t-31$ *for* $t \geq 3$ *via* $t \cdot [8] - [6] - 2[4]$;
- $n_2(8,2;21t-8) = 85t-36$ *for* $t \geq 4$ *via* $t \cdot [8] - [6] - 3[4]$;
- $n_2(8,2;21t-9) = 85t-41$ *for* $t \geq 3$ *via* $t \cdot [8] - [7] + 4[1]$;
- $n_2(8,2;21t-10) = 85t-41$ *for* $t \geq 2$ *via* $t \cdot [8] - 2[6]$;
- $n_2(8,2;21t-11) = 85t-47$ *for* $t \geq 3$ *via* $t \cdot [8] - 2[6] - [4]$;
- $n_2(8,2;21t-12) = 85t-52$ *for* $t \geq 3$ *via* $t \cdot [8] - [7] - [5] + 2[1]$;
- $n_2(8,2;21t-13) = 85t-55$ *for* $t \geq 3$ *via* $t \cdot [8] - [7] - [5] - [3]$;
- $n_2(8,2;21t-14) = 85t-60$ *for* $t \geq 4$ *via* $t \cdot [8] - [7] - [5] - [4] - [3]$;
- $n_2(8,2;21t-15) = 85t-63$ *for* $t \geq 3$ *via* $t \cdot [8] - 3[6]$;
- $n_2(8,2;21t-16) = 85t-68$ *for* $t \geq 4$ *via* $t \cdot [8] - 3[6] - [4]$;
- $n_2(8,2;21t-17) = 85t-73$ *for* $t \geq 4$ *via* $t \cdot [8] - 1[7] - [6] - [5] + 2[1]$;
- $n_2(8,2;21t-18) = 85t-76$ *for* $t \geq 4$ *via* $t \cdot [8] - [7] - [6] - [5] - [3]$;
- $n_2(8,2;21t-19) = 85t-81$ *for* $t \geq 5$ *via* $t \cdot [8] - [7] - [6] - [5] - [4] - [3]$;
- $n_2(8,2;21t-20) = 85t-84$ *for* $t \geq 1$ *via* $(t-1) \cdot [8] + [2]$.

Proof. By Theorem 2 we have that [8], [6], and [4] are 2-partitionable over $\mathbb{F}_2$. Lemma 24 shows that [8] − [6] and 3[8] − [7] − 2[5] are 2-partitionable over $\mathbb{F}_2$. By Lemma 23 [8] − [4] and [5] − [3] are 2-partitionable over $\mathbb{F}_2$. By Lemma 26 we have that [5] + 2[1] is 2-partitionable over $\mathbb{F}_2$. ∎

Proposition 12. *The Griesmer upper bound for* $n_2(9,2;s)$ *is attained for all* $s \geq 156$*, i.e. we have*

- $n_2(9,2;127t) = 511t$ *for* $t \geq 1$ *via* $3t \cdot [9]$;
- $n_2(9,2;127t-1) = 511t-5$ *for* $t \geq 1$ *via* $3t \cdot [9] - [4]$;
- $n_2(9,2;127t-2) = 511t-10$ *for* $t \geq 1$ *via* $3t \cdot [9] - 2[4]$;
- $n_2(9,2;127t-3) = 511t-15$ *for* $t \geq 1$ *via* $3t \cdot [9] - 3[4]$;
- $n_2(9,2;127t-4) = 511t-20$ *for* $t \geq 2$ *via* $3t \cdot [9]$-*4[4]*;
- $n_2(9,2;127t-5) = 511t-21$ *for* $t \geq 1$ *via* $3t \cdot [9] - [6]$;
- $n_2(9,2;127t-6) = 511t-26$ *for* $t \geq 1$ *via* $3t \cdot [9] - [6] - [4]$;
- $n_2(9,2;127t-7) = 511t-31$ *for* $t \geq 1$ *via* $3t \cdot [9] - [6] - 2[4]$;
- $n_2(9,2;127t-8) = 511t-36$ *for* $t \geq 2$ *via* $3t \cdot [9] - [6] - 3[4]$;
- $n_2(9,2;127t-9) = 511t-41$ *for* $t \geq 2$ *via* $3t \cdot [9] - [6] - 4[4]$;
- $n_2(9,2;127t-10) = 511t-42$ *for* $t \geq 1$ *via* $3t \cdot [9] - 2[6]$;
- $n_2(9,2;127t-11) = 511t-47$ *for* $t \geq 1$ *via* $3t \cdot [9] - 2[6] - [4]$;
- $n_2(9,2;127t-12) = 511t-52$ *for* $t \geq 2$ *via* $3t \cdot [9] - 2[6] - 2[4]$;
- $n_2(9,2;127t-13) = 511t-55$ *for* $t \geq 1$ *via* $3t \cdot [9] - [7] - [5] - [3]$;

- $n_2(9,2;127t-14) = 511t-60$ *for* $t \geq 2$ *via* $3t \cdot [9]-[7]-[5]-[4]-[3]$;
- $n_2(9,2;127t-15) = 511t-63$ *for* $t \geq 1$ *via* $3t \cdot [9]-[7]-2[5]$;
- $n_2(9,2;127t-16) = 511t-68$ *for* $t \geq 2$ *via* $3t \cdot [9]-[7]-[6]-2[3]$;
- $n_2(9,2;127t-17) = 511t-73$ *for* $t \geq 2$ *via* $3t \cdot [9]-3[6]-2[4]$;
- $n_2(9,2;127t-18) = 511t-76$ *for* $t \geq 2$ *via* $3t \cdot [9]-[7]-[6]-[5]-[3]$;
- $n_2(9,2;127t-19) = 511t-81$ *for* $t \geq 2$ *via* $3t \cdot [9]-[7]-[6]-[5]-[4]-[3]$;
- $n_2(9,2;127t-20) = 511t-84$ *for* $t \geq 2$ *via* $3t \cdot [9]-4[6]$;
- $n_2(9,2;127t-21) = 511t-85$ *for* $t \geq 1$ *via* $3t \cdot [9]-[8]$;
- $n_2(9,2;127t-22) = 511t-90$ *for* $t \geq 1$ *via* $3t \cdot [9]-[8]-[4]$;
- $n_2(9,2;127t-23) = 511t-95$ *for* $t \geq 1$ *via* $3t \cdot [9]-[8]-2[4]$;
- $n_2(9,2;127t-24) = 511t-100$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-3[4]$;
- $n_2(9,2;127t-25) = 511t-105$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-4[4]$;
- $n_2(9,2;127t-26) = 511t-106$ *for* $t \geq 1$ *via* $3t \cdot [9]-[8]-[6]$;
- $n_2(9,2;127t-27) = 511t-111$ *for* $t \geq 1$ *via* $3t \cdot [9]-[8]-[6]-[4]$;
- $n_2(9,2;127t-28) = 511t-116$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-[6]-2[4]$;
- $n_2(9,2;127t-29) = 511t-121$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-[6]-3[4]$;
- $n_2(9,2;127t-30) = 511t-126$ *for* $t \geq 2$ *via* $3t \cdot [9]-6[6]$;
- $n_2(9,2;127t-31) = 511t-127$ *for* $t \geq 1$ *via* $3t \cdot [9]-[8]-2[6]$;
- $n_2(9,2;127t-32) = 511t-132$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-2[6]-[4]$;
- $n_2(9,2;127t-33) = 511t-137$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-2[6]-2[4]$;
- $n_2(9,2;127t-34) = 511t-140$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-[7]-[5]-[3]$;
- $n_2(9,2;127t-35) = 511t-145$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-[7]-[5]-[4]-[3]$;
- $n_2(9,2;127t-36) = 511t-148$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-3[6]$;
- $n_2(9,2;127t-37) = 511t-153$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-3[6]-[4]$;
- $n_2(9,2;127t-38) = 511t-158$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-3[6]-2[4]$;
- $n_2(9,2;127t-39) = 511t-161$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-[7]-[6]-[5]-[3]$;
- $n_2(9,2;127t-40) = 511t-166$ *for* $t \geq 2$ *via* $3t \cdot [9]-[8]-[7]-[6]-[5]-[4]-[3]$;
- $n_2(9,2;127t-41) = 511t-169$ *for* $t \geq 1$ *via* $(3t-1) \cdot [9]-[2]+[1]$;
- $n_2(9,2;127t-42) = 511t-170$ *for* $t \geq 1$ *via* $(3t-1) \cdot [9]+[1]$;
- $n_2(9,2;127t-43) = 511t-175$ *for* $t \geq 1$ *via* $(3t-1) \cdot [9]-2[3]$;
- $n_2(9,2;127t-44) = 511t-180$ *for* $t \geq 2$ *via* $(3t-1) \cdot [9]-[4]-2[3]$;
- $n_2(9,2;127t-45) = 511t-183$ *for* $t \geq 1$ *via* $(3t-1) \cdot [9]-[5]-[3]$;
- $n_2(9,2;127t-46) = 511t-188$ *for* $t \geq 2$ *via* $(3t-1) \cdot [9]-[5]-[4]-[3]$;
- $n_2(9,2;127t-47) = 511t-191$ *for* $t \geq 1$ *via* $(3t-1) \cdot [9]-2[5]$;
- $n_2(9,2;127t-48) = 511t-196$ *for* $t \geq 2$ *via* $(3t-1) \cdot [9]-[6]-2[3]$;
- $n_2(9,2;127t-49) = 511t-201$ *for* $t \geq 2$ *via* $(3t-1) \cdot [9]-[6]-[4]-2[3]$;
- $n_2(9,2;127t-50) = 511t-204$ *for* $t \geq 2$ *via* $(3t-1) \cdot [9]-[6]-[5]-[3]$;
- $n_2(9,2;127t-51) = 511t-209$ *for* $t \geq 2$ *via* $(3t-1) \cdot [9]-[6]-[5]-[4]-[3]$;
- $n_2(9,2;127t-52) = 511t-212$ *for* $t \geq 2$ *via* $(3t-1) \cdot [9]-[6]-2[5]$;

- $n_2(9,2;127t-53) = 511t-215$ *for* $t \geq 1$ *via* $(3t-1)\cdot[9]-[7]-[3]$;
- $n_2(9,2;127t-54) = 511t-220$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[7]-[4]-[3]$;
- $n_2(9,2;127t-55) = 511t-223$ *for* $t \geq 1$ *via* $(3t-1)\cdot[9]-[7]-[5]$;
- $n_2(9,2;127t-56) = 511t-228$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[7]-[5]-[4]$;
- $n_2(9,2;127t-57) = 511t-233$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[7]-[5]-2[4]$;
- $n_2(9,2;127t-58) = 511t-236$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[7]-[6]-[3]$;
- $n_2(9,2;127t-59) = 511t-241$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[7]-[6]-[4]-[3]$;
- $n_2(9,2;127t-60) = 511t-244$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[7]-[6]-[5]$;
- $n_2(9,2;127t-61) = 511t-249$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[7]-[6]-[5]-[4]$;
- $n_2(9,2;127t-62) = 511t-254$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[7]-4[5]$;
- $n_2(9,2;127t-63) = 511t-255$ *for* $t \geq 1$ *via* $(3t-1)\cdot[9]-2[7]$;
- $n_2(9?,2;127t-64) = 511t-260$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-2[3]$;
- $n_2(9,2;127t-65) = 511t-265$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[4]-2[3]$;
- $n_2(9,2;127t-66) = 511t-268$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[5]-[3]$;
- $n_2(9,2;127t-67) = 511t-273$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[5]-[4]-[3]$;
- $n_2(9,2;127t-68) = 511t-276$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-2[5]$;
- $n_2(9,2;127t-69) = 511t-281$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[6]-2[3]$;
- $n_2(9,2;127t-70) = 511t-286$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-2[7]-3[5]$;
- $n_2(9,2;127t-71) = 511t-289$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[6]-[5]-[3]$;
- $n_2(9,2;127t-72) = 511t-294$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[6]-[5]-[4]-[3]$;
- $n_2(9,2;127t-73) = 511t-297$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[6]-2[5]$;
- $n_2(9,2;127t-74) = 511t-300$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[7]-[3]$;
- $n_2(9,2;127t-75) = 511t-305$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[7]-[4]-[3]$;
- $n_2(9,2;127t-76) = 511t-308$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[7]-[5]$;
- $n_2(9,2;127t-77) = 511t-313$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[7]-[5]-[4]$;
- $n_2(9,2;127t-78) = 511t-318$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-3[7]-2[5]$;
- $n_2(9,2;127t-79) = 511t-321$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[7]-[6]-[3]$;
- $n_2(9,2;127t-80) = 511t-326$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[7]-[6]-[4]-[3]$;
- $n_2(9,2;127t-81) = 511t-329$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[7]-[6]-[5]$;
- $n_2(9,2;127t-82) = 511t-334$ *for* $t \geq 2$ *via* $(3t-1)\cdot[9]-[8]-[7]-[6]-[5]-[4]$;
- $n_2(9,2;127t-83) = 511t-339$ *for* $t \geq 1$ *via* $(3t-2)\cdot[9]-[2]+2[1]$;
- $n_2(9,2;127t-84) = 511t-340$ *for* $t \geq 1$ *via* $(3t-2)\cdot[9]+2[1]$;
- $n_2(9,2;127t-85) = 511t-343$ *for* $t \geq 1$ *via* $(3t-2)\cdot[9]-[3]$;
- $n_2(9,2;127t-86) = 511t-348$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[4]-[3]$;
- $n_2(9,2;127t-87) = 511t-351$ *for* $t \geq 1$ *via* $(3t-2)\cdot[9]-[5]$;
- $n_2(9,2;127t-88) = 511t-356$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[5]-[4]$;
- $n_2(9,2;127t-89) = 511t-361$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[5]-2[4]$;
- $n_2(9,2;127t-90) = 511t-364$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[6]-[3]$;
- $n_2(9,2;127t-91) = 511t-369$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[6]-[4]-[3]$;

- $n_2(9,2;127t-92) = 511t-372$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[6]-[5]$;
- $n_2(9,2;127t-93) = 511t-377$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[6]-[5]-[4]$;
- $n_2(9,2;127t-94) = 511t-382$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-4[5]$;
- $n_2(9,2;127t-95) = 511t-383$ *for* $t \geq 1$ *via* $(3t-2)\cdot[9]-[7]$;
- $n_2(9,2;127t-96) = 511t-388$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[7]-[4]$;
- $n_2(9,2;127t-97) = 511t-393$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[7]-2[4]$;
- $n_2(9,2;127t-98) = 511t-398$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[7]-3[4]$;
- $n_2(9,2;127t-99) = 511t-403$ *for* $t \geq 3$ *via* $(3t-2)\cdot[9]-[7]-4[4]$;
- $n_2(9,2;127t-100) = 511t-404$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[7]-[6]$;
- $n_2(9,2;127t-101) = 511t-409$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[7]-[6]-[4]$;
- $n_2(9,2;127t-102) = 511t-414$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[7]-[6]-2[4]$;
- $n_2(9,2;127t-103) = 511t-419$ *for* $t \geq 3$ *via* $(3t-2)\cdot[9]-[7]-[6]-[5]-2[3]$;
- $n_2(9,2;127t-104) = 511t-424$ *for* $t \geq 3$ *via* $(3t-2)\cdot[9]-3[6]-[5]-2[4]$;
- $n_2(9,2;127t-105) = 511t-425$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[7]-2[6]$;
- $n_2(9,2;127t-106) = 511t-428$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[3]$;
- $n_2(9,2;127t-107) = 511t-433$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[4]-[3]$;
- $n_2(9,2;127t-108) = 511t-436$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[5]$;
- $n_2(9,2;127t-109) = 511t-441$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[5]-[4]$;
- $n_2(9,2;127t-110) = 511t-446$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-2[7]-2[5]$;
- $n_2(9,2;127t-111) = 511t-449$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[6]-[3]$;
- $n_2(9,2;127t-112) = 511t-454$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[6]-[4]-[3]$;
- $n_2(9,2;127t-113) = 511t-457$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[6]-[5]$;
- $n_2(9,2;127t-114) = 511t-462$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[6]-[5]-[4]$;
- $n_2(9,2;127t-115) = 511t-467$ *for* $t \geq 3$ *via* $(3t-2)\cdot[9]-[8]-4[5]$;
- $n_2(9,2;127t-116) = 511t-468$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[7]$;
- $n_2(9,2;127t-117) = 511t-473$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[7]-[4]$;
- $n_2(9,2;127t-118) = 511t-478$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[7]-2[4]$;
- $n_2(9,2;127t-119) = 511t-483$ *for* $t \geq 3$ *via* $(3t-2)\cdot[9]-[8]-[7]-[5]-2[3]$;
- $n_2(9,2;127t-120) = 511t-488$ *for* $t \geq 3$ *via* $(3t-2)\cdot[9]-3[7]-[5]-2[4]$;
- $n_2(9,2;127t-121) = 511t-489$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[7]-[6]$;
- $n_2(9,2;127t-122) = 511t-494$ *for* $t \geq 2$ *via* $(3t-2)\cdot[9]-[8]-[7]-[6]-[4]$;
- $n_2(9,2;127t-123) = 511t-499$ *for* $t \geq 3$ *via* $(3t-2)\cdot[9]-[8]-[7]-3[5]$;
- $n_2(9,2;127t-124) = 511t-504$ *for* $t \geq 3$ *via* $(3t-2)\cdot[9]-[8]-[7]-[6]-[5]-2[3]$;
- $n_2(9,2;127t-125) = 511t-509$ *for* $t \geq 1$ *via* $3t\cdot[9]-2[2]$;
- $n_2(9,2;127t-126) = 511t-510$ *for* $t \geq 1$ *via* $3t\cdot[9]-[2]$.

Proof. By Theorem 2 we have that $3[9]$, $[8]$, $[6]$, and $[4]$ are 2-partitionable over $\mathbb{F}_2$. Lemma 24 shows that $[9]-[7]$ and $3[9]-[8]-2[6]$ are 2-partitionable over $\mathbb{F}_2$. By Lemma 23 $[9]-[5]$, $[7]-[5]$, $[7]-[3]$, $[8]-[6]$, $[8]-[4]$, and $[6]-[4]$ are 2-partitionable over $\mathbb{F}_2$. By Lemma 26 we have that $[5]+2[1]$ and $2[5]+[1]$ are 2-partitionable over $\mathbb{F}_2$. ∎

Proposition 13. *The Griesmer upper bound for $n_2(10,2;s)$ is attained for all $s \ge 396$, i.e. we have*

- $n_2(10,2;85t) = 341t$ *for* $t \ge 1$ *via* $t \cdot [10]$*;*
- $n_2(10,2;85t-1) = 341t-5$ *for* $t \ge 1$ *via* $t \cdot [10] - [4]$*;*
- $n_2(10,2;85t-2) = 341t-10$ *for* $t \ge 2$ *via* $t \cdot [10] - 2[4]$*;*
- $n_2(10,2;85t-3) = 341t-15$ *for* $t \ge 3$ *via* $t \cdot [10] - 3[4]$*;*
- $n_2(10,2;85t-4) = 341t-20$ *for* $t \ge 4$ *via* $t \cdot [10] - 4[4]$*;*
- $n_2(10,2;85t-5) = 341t-21$ *for* $t \ge 1$ *via* $t \cdot [10] - [6]$*;*
- $n_2(10,2;85t-6) = 341t-26$ *for* $t \ge 2$ *via* $t \cdot [10] - [6] - [4]$*;*
- $n_2(10,2;85t-7) = 341t-31$ *for* $t \ge 3$ *via* $t \cdot [10] - [6] - 2[4]$*;*
- $n_2(10,2;85t-8) = 341t-36$ *for* $t \ge 4$ *via* $t \cdot [10] - [6] - 3[4]$*;*
- $n_2(10,2;85t-9) = 341t-41$ *for* $t \ge 5$ *via* $t \cdot [10] - [6] - 4[4]$*;*
- $n_2(10,2;85t-10) = 341t-42$ *for* $t \ge 2$ *via* $t \cdot [10] - 2[6]$*;*
- $n_2(10,2;85t-11) = 341t-47$ *for* $t \ge 3$ *via* $t \cdot [10] - 2[6] - [4]$*;*
- $n_2(10,2;85t-12) = 341t-52$ *for* $t \ge 4$ *via* $t \cdot [10] - 2[6] - 2[4]$*;*
- $n_2(10,2;85t-13) = 341t-55$ *for* $t \ge 3$ *via* $t \cdot [10] - [7] - [5] - [3]$*;*
- $n_2(10,2;85t-14) = 341t-60$ *for* $t \ge 4$ *via* $t \cdot [10] - [7] - [5] - [4] - [3]$*;*
- $n_2(10,2;85t-15) = 341t-63$ *for* $t \ge 3$ *via* $t \cdot [10] - [7] - 2[5]$*;*
- $n_2(10,2;85t-16) = 341t-68$ *for* $t \ge 4$ *via* $t \cdot [10] - [7] - [6] - 2[3]$*;*
- $n_2(10,2;85t-17) = 341t-73$ *for* $t \ge 5$ *via* $t \cdot [10] - [7] - 2[5] - 2[4]$*;*
- $n_2(10,2;85t-18) = 341t-76$ *for* $t \ge 4$ *via* $t \cdot [10] - [7] - [6] - [5] - [3]$*;*
- $n_2(10,2;85t-19) = 341t-81$ *for* $t \ge 5$ *via* $t \cdot [10] - [7] - [6] - [5] - [4] - [3]$*;*
- $n_2(10,2;85t-20) = 341t-84$ *for* $t \ge 4$ *via* $t \cdot [10] - [7] - [6] - 2[5]$*;*
- $n_2(10,2;85t-21) = 341t-85$ *for* $t \ge 1$ *via* $t \cdot [10] - [8]$*;*
- $n_2(10,2;85t-22) = 341t-90$ *for* $t \ge 2$ *via* $t \cdot [10] - [8] - [4]$*;*
- $n_2(10,2;85t-23) = 341t-95$ *for* $t \ge 3$ *via* $t \cdot [10] - [8] - 2[4]$*;*
- $n_2(10,2;85t-24) = 341t-100$ *for* $t \ge 4$ *via* $t \cdot [10] - [8] - 3[4]$*;*
- $n_2(10,2;85t-25) = 341t-105$ *for* $t \ge 5$ *via* $t \cdot [10] - [8] - 4[4]$*;*
- $n_2(10,2;85t-26) = 341t-106$ *for* $t \ge 2$ *via* $t \cdot [10] - [8] - [6]$*;*
- $n_2(10,2;85t-27) = 341t-111$ *for* $t \ge 3$ *via* $t \cdot [10] - [8] - [6] - [4]$*;*
- $n_2(10,2;85t-28) = 341t-116$ *for* $t \ge 4$ *via* $t \cdot [10] - [8] - [6] - 2[4]$*;*
- $n_2(10,2;85t-29) = 341t-121$ *for* $t \ge 5$ *via* $t \cdot [10] - [8] - [6] - 3[4]$*;*
- $n_2(10,2;85t-30) = 341t-126$ *for* $t \ge 6$ *via* $t \cdot [10] - [8] - [6] - [5] - [4] - 2[3]$*;*
- $n_2(10,2;85t-31) = 341t-127$ *for* $t \ge 3$ *via* $t \cdot [10] - [8] - 2[6]$*;*
- $n_2(10,2;85t-32) = 341t-132$ *for* $t \ge 4$ *via* $t \cdot [10] - [8] - 2[6] - [4]$*;*
- $n_2(10,2;85t-33) = 341t-137$ *for* $t \ge 5$ *via* $t \cdot [10] - [8] - 2[6] - 2[4]$*;*
- $n_2(10,2;85t-34) = 341t-140$ *for* $t \ge 4$ *via* $t \cdot [10] - [8] - [7] - [5] - [3]$*;*
- $n_2(10,2;85t-35) = 341t-145$ *for* $t \ge 5$ *via* $t \cdot [10] - [8] - [7] - [5] - [4] - [3]$*;*
- $n_2(10,2;85t-36) = 341t-148$ *for* $t \ge 4$ *via* $t \cdot [10] - [8] - 3[6]$*;*
- $n_2(10,2;85t-37) = 341t-153$ *for* $t \ge 5$ *via* $t \cdot [10] - [8] - 3[6] - [4]$*;*

- $n_2(10,2;85t-38) = 341t-158$ *for* $t \geq 6$ *via* $t \cdot [10]-[8]-3[6]-2[4]$;
- $n_2(10,2;85t-39) = 341t-161$ *for* $t \geq 5$ *via* $t \cdot [10]-[8]-[7]-[6]-[5]-[3]$;
- $n_2(10,2;85t-40) = 341t-166$ *for* $t \geq 6$ *via* $t \cdot [10]-[8]-[7]-[6]-[5]-[4]-[3]$;
- $n_2(10,2;85t-41) = 341t-169$ *for* $t \geq 5$ *via* $t \cdot [10]-[8]-4[6]$;
- $n_2(10,2;85t-42) = 341t-170$ *for* $t \geq 2$ *via* $t \cdot [10]-2[8]$;
- $n_2(10,2;85t-43) = 341t-175$ *for* $t \geq 3$ *via* $t \cdot [10]-2[8]-[4]$;
- $n_2(10,2;85t-44) = 341t-180$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[4]-2[3]$;
- $n_2(10,2;85t-45) = 341t-183$ *for* $t \geq 3$ *via* $t \cdot [10]-[9]-[5]-[3]$;
- $n_2(10,2;85t-46) = 341t-188$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[5]-[4]-[3]$;
- $n_2(10,2;85t-47) = 341t-191$ *for* $t \geq 3$ *via* $t \cdot [10]-2[8]-[6]$;
- $n_2(10,2;85t-48) = 341t-196$ *for* $t \geq 4$ *via* $t \cdot [10]-2[8]-[6]-[4]$;
- $n_2(10,2;85t-49) = 341t-201$ *for* $t \geq 5$ *via* $t \cdot [10]-2[8]-[6]-2[4]$;
- $n_2(10,2;85t-50) = 341t-204$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[6]-[5]-[3]$;
- $n_2(10,2;85t-51) = 341t-209$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[6]-[5]-[4]-[3]$;
- $n_2(10,2;85t-52) = 341t-212$ *for* $t \geq 4$ *via* $t \cdot [10]-2[8]-2[6]$;
- $n_2(10,2;85t-53) = 341t-215$ *for* $t \geq 3$ *via* $t \cdot [10]-[9]-[7]-[3]$;
- $n_2(10,2;85t-54) = 341t-220$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[7]-[4]-[3]$;
- $n_2(10,2;85t-55) = 341t-223$ *for* $t \geq 3$ *via* $t \cdot [10]-[9]-[7]-[5]$;
- $n_2(10,2;85t-56) = 341t-228$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[7]-[5]-[4]$;
- $n_2(10,2;85t-57) = 341t-233$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[7]-[5]-2[4]$;
- $n_2(10,2;85t-58) = 341t-236$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[7]-[6]-[3]$;
- $n_2(10,2;85t-59) = 341t-241$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[7]-[6]-[4]-[3]$;
- $n_2(10,2;85t-60) = 341t-244$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[7]-[6]-[5]$;
- $n_2(10,2;85t-61) = 341t-249$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[7]-[6]-[5]-[4]$;
- $n_2(10,2;85t-62) = 341t-254$ *for* $t \geq 6$ *via* $t \cdot [10]-2[8]-4[6]$;
- $n_2(10,2;85t-63) = 341t-255$ *for* $t \geq 3$ *via* $t \cdot [10]-3[8]$;
- $n_2(10,2;85t-64) = 341t-260$ *for* $t \geq 4$ *via* $t \cdot [10]-3[8]-[4]$;
- $n_2(10,2;85t-65) = 341t-265$ *for* $t \geq 5$ *via* $t \cdot [10]-3[8]-2[4]$;
- $n_2(10,2;85t-66) = 341t-268$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[8]-[5]-[3]$;
- $n_2(10,2;85t-67) = 341t-273$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[8]-[5]-[4]-[3]$;
- $n_2(10,2;85t-68) = 341t-276$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[8]-2[7]$;
- $n_2(10,2;85t-69) = 341t-281$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[8]-[6]-2[3]$;
- $n_2(10,2;85t-70) = 341t-286$ *for* $t \geq 6$ *via* $t \cdot [10]-[9]-[8]-[6]-[4]-2[3]$;
- $n_2(10,2;85t-71) = 341t-289$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[8]-[6]-[5]-[3]$;
- $n_2(10,2;85t-72) = 341t-294$ *for* $t \geq 6$ *via* $t \cdot [10]-[9]-[8]-[6]-[5]-[4]-[3]$;
- $n_2(10,2;85t-73) = 341t-297$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[8]-[6]-2[5]$;
- $n_2(10,2;85t-74) = 341t-300$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[8]-[7]-[3]$;
- $n_2(10,2;85t-75) = 341t-305$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[8]-[7]-[4]-[3]$;
- $n_2(10,2;85t-76) = 341t-308$ *for* $t \geq 4$ *via* $t \cdot [10]-[9]-[8]-[7]-[5]$;

- $n_2(10,2;85t-77) = 341t-313$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[8]-[7]-[5]-[4]$;
- $n_2(10,2;85t-78) = 341t-318$ *for* $t \geq 6$ *via* $t \cdot [10]-[9]-[8]-2[6]-2[5]$;
- $n_2(10,2;85t-79) = 341t-321$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[8]-[7]-[6]-[3]$;
- $n_2(10,2;85t-80) = 341t-326$ *for* $t \geq 6$ *via* $t \cdot [10]-[9]-[8]-[7]-[6]-[4]-[3]$;
- $n_2(10,2;85t-81) = 341t-329$ *for* $t \geq 5$ *via* $t \cdot [10]-[9]-[8]-[7]-[6]-[5]$;
- $n_2(10,2;85t-82) = 341t-334$ *for* $t \geq 6$ *via* $t \cdot [10]-[9]-[8]-[7]-[6]-[5]-[4]$;
- $n_2(10,2;85t-83) = 341t-339$ *for* $t \geq 1$ *via* $(t-1) \cdot [10]+2[2]$;
- $n_2(10,2;85t-84) = 341t-340$ *for* $t \geq 1$ *via* $(t-1) \cdot [10]+[2]$.

Proof. By Theorem 2 we have that [10], [8], [6], and [4] are 2-partitionable over $\mathbb{F}_2$. Lemma 24 shows that [10] − [8] and 3[10] − [9] − 2[7] are 2-partitionable over $\mathbb{F}_2$. By Lemma 23 [10] − [6], [8] − [6], [6] − [4], [9] − [7], [7] − [5], and [5] − [3] are 2-partitionable over $\mathbb{F}_2$.

By Lemma 26 we have that [5] + 2[1] is 2-partitionable over $\mathbb{F}_2$. ■

Proposition 14. *The Griesmer upper bound for* $n_2(7,3;s)$ *is attained for all* $s \geq 12$, *i.e. we have*

- $n_2(7,3;15t) = 127t$ *for* $t \geq 1$ *via* $7t \cdot [7]$;
- $n_2(7,3;15t-1) = 127t-9$ *for* $t \geq 1$ *via* $7t \cdot [7]-[6]$;
- $n_2(7,3;15t-2) = 127t-18$ *for* $t \geq 1$ *via* $(7t-1) \cdot [7]+[3]-2[2]$;
- $n_2(7,3;15t-3) = 127t-27$ *for* $t \geq 1$ *via* $(7t-1) \cdot [7]-2[5]$;
- $n_2(7,3;15t-4) = 127t-36$ *for* $t \geq 2$ *via* $(7t-1) \cdot [7]-[6]-2[5]$;
- $n_2(7,3;15t-5) = 127t-45$ *for* $t \geq 1$ *via* $(7t-2) \cdot [7]-[5]-2[4]$;
- $n_2(7,3;15t-6) = 127t-54$ *for* $t \geq 2$ *via* $(7t-1) \cdot [7]-4[5]$;
- $n_2(7,3;15t-7) = 127t-61$ *for* $t \geq 1$ *via* $(7t-3) \cdot [7]-[5]-[4]$;
- $n_2(7,3;15t-8) = 127t-70$ *for* $t \geq 2$ *via* $(7t-3) \cdot [7]-[6]-[5]-[4]$;
- $n_2(7,3;15t-9) = 127t-77$ *for* $t \geq 1$ *via* $(7t-4) \cdot [7]-[5]$;
- $n_2(7,3;15t-10) = 127t-86$ *for* $t \geq 2$ *via* $(7t-4) \cdot [7]-[6]-[5]$;
- $n_2(7,3;15t-11) = 127t-95$ *for* $t \geq 1$ *via* $(7t-5) \cdot [7]-2[4]$;
- $n_2(7,3;15t-12) = 127t-104$ *for* $t \geq 2$ *via* $(7t-5) \cdot [7]-[6]-2[4]$;
- $n_2(7,3;15t-13) = 127t-111$ *for* $t \geq 1$ *via* $(7t-6) \cdot [7]-[4]$;
- $n_2(7,3;15t-14) = 127t-120$ *for* $t \geq 2$ *via* $(7t-6) \cdot [7]-[6]-[4]$.

Proof. By Theorem 2 we have that 7[7], [6], and [3] are 3-partitionable over $\mathbb{F}_2$. Lemma 24 shows that [7] − [4], 3[7] − [5] − 2[3], and 7[7] − [6] − 6[3] are 3-partitionable over $\mathbb{F}_2$. By Lemma 26 we have that 6[7] + [3] − 2[2] is 3-partitionable over $\mathbb{F}_2$. ■

Proposition 15. *The Griesmer upper bound for* $n_2(8,3;s)$ *is attained for all* $s \geq 72$, *i.e. we have*

- $n_2(8,3;31t) = 255t$ *for* $t \geq 1$ *via* $7t \cdot [8]$;
- $n_2(8,3;31t-1) = 255t-9$ *for* $t \geq 1$ *via* $7t \cdot [8]-[6]$;
- $n_2(8,3;31t-2) = 255t-18$ *for* $t \geq 2$ *via* $7t \cdot [8]-2[6]$;
- $n_2(8,3;31t-3) = 255t-27$ *for* $t \geq 3$ *via* $7t \cdot [8]-3[6]$;
- $n_2(8,3;31t-4) = 255t-36$ *for* $t \geq 2$ *via* $7t \cdot [8]-4[6]$;

- $n_2(8,3;31t-5) = 255t-43$ *for* $t \ge 3$ *via* $(7t-1)\cdot[8]-[5]-[4]$;
- $n_2(8,3;31t-6) = 255t-52$ *for* $t \ge 1$ *via* $(7t-1)\cdot[8]-[6]-[5]-[4]$;
- $n_2(8,3;31t-7) = 255t-59$ *for* $t \ge 3$ *via* $(7t-1)\cdot[8]-[7]-[5]$;
- $n_2(8,3;31t-8) = 255t-68$ *for* $t \ge 2$ *via* $(7t-1)\cdot[8]-[7]-[6]-[5]$;
- $n_2(8,3;31t-9) = 255t-75$ *for* $t \ge 3$ *via* $(7t-2)\cdot[8]-[4]$;
- $n_2(8,3;31t-10) = 255t-84$ *for* $t \ge 1$ *via* $(7t-2)\cdot[8]-[6]-[4]$;
- $n_2(8,3;31t-11) = 255t-91$ *for* $t \ge 3$ *via* $(7t-2)\cdot[8]-[7]$;
- $n_2(8,3;31t-12) = 255t-100$ *for* $t \ge 2$ *via* $(7t-2)\cdot[8]-[7]-[6]$;
- $n_2(8,3;31t-13) = 255t-109$ *for* $t \ge 3$ *via* $(7t-2)\cdot[8]-[7]-2[6]$;
- $n_2(8,3;31t-14) = 255t-118$ *for* $t \ge 3$ *via* $(7t-3)\cdot[8]-[5]-2[4]$;
- $n_2(8,3;31t-15) = 255t-127$ *for* $t \ge 1$ *via* $(7t-3)\cdot[8]-4[5]$;
- $n_2(8,3;31t-16) = 255t-134$ *for* $t \ge 3$ *via* $(7t-3)\cdot[8]-[7]-[5]-[4]$;
- $n_2(8,3;31t-17) = 255t-143$ *for* $t \ge 2$ *via* $(7t-3)\cdot[8]-[7]-[6]-[5]-[4]$;
- $n_2(8,3;31t-18) = 255t-150$ *for* $t \ge 3$ *via* $(7t-4)\cdot[8]-2[4]$;
- $n_2(8,3;31t-19) = 255t-159$ *for* $t \ge 1$ *via* $(7t-4)\cdot[8]-[6]-2[4]$;
- $n_2(8,3;31t-20) = 255t-166$ *for* $t \ge 3$ *via* $(7t-4)\cdot[8]-[7]-[4]$;
- $n_2(8,3;31t-21) = 255t-175$ *for* $t \ge 2$ *via* $(7t-4)\cdot[8]-[7]-[6]-[4]$;
- $n_2(8,3;31t-22) = 255t-182$ *for* $t \ge 4$ *via* $(7t-4)\cdot[8]-2[7]$;
- $n_2(8,3;31t-23) = 255t-191$ *for* $t \ge 1$ *via* $(7t-5)\cdot[8]-2[5]$;
- $n_2(8,3;31t-24) = 255t-200$ *for* $t \ge 2$ *via* $(7t-5)\cdot[8]-[6]-2[5]$;
- $n_2(8,3;31t-25) = 255t-209$ *for* $t \ge 3$ *via* $(7t-5)\cdot[8]-[7]-[5]-2[4]$;
- $n_2(8,3;31t-26) = 255t-218$ *for* $t \ge 4$ *via* $(7t-5)\cdot[8]-3[6]-2[5]$;
- $n_2(8,3;31t-27) = 255t-223$ *for* $t \ge 1$ *via* $(7t-6)\cdot[8]-[5]$;
- $n_2(8,3;31t-28) = 255t-232$ *for* $t \ge 2$ *via* $(7t-6)\cdot[8]-[6]-[5]$;
- $n_2(8,3;31t-29) = 255t-241$ *for* $t \ge 3$ *via* $(7t-6)\cdot[8]-[7]-2[4]$;
- $n_2(8,3;31t-30) = 255t-250$ *for* $t \ge 4$ *via* $(7t-6)\cdot[8]-[7]-[6]-2[4]$.

Proof. By Theorem 2 we have that $7[8]$, $7[4]$, $[6]$, and $[3]$ are 3-partitionable over $\mathbb{F}_2$. Lemma 24 shows that $[8]-[5]$, $3[8]-[6]-2[4]$, and $7[8]-[7]-6[4]$ are 3-partitionable over $\mathbb{F}_2$. By Lemma 26 we have that $6[8]+[3]-2[2]$ is 3-partitionable over $\mathbb{F}_2$. Moreover, Lemma 26 implies that $[5]+2[1]$ is 2-partitionable over $\mathbb{F}_2$. With this, the corresponding dual shows that $2[5]+[4]$ is 3-partitionable over $\mathbb{F}_2$, so that also $2[8]+[4]$ is 3-partitionable over $\mathbb{F}_2$. ∎

Proposition 16. *The Griesmer upper bound for* $n_2(9,3;s)$ *is attained for all* $s \ge 663$*, i.e. we have*

- $n_2(9,3;9t) = 73t$ *for* $t \ge 1$ *via* $t\cdot[9]$;
- $n_2(9,3;9t-1) = 73t-9$ *for* $t \ge 1$ *via* $t\cdot[9]-[6]$;
- $n_2(9,3;9t-2) = 73t-18$ *for* $t \ge 2$ *via* $t\cdot[9]-2[6]$;
- $n_2(9,3;9t-3) = 73t-27$ *for* $t \ge 3$ *via* $t\cdot[9]-3[6]$;
- $n_2(9,3;9t-4) = 73t-36$ *for* $t \ge 4$ *via* $t\cdot[9]-4[6]$;
- $n_2(9,3;9t-5) = 73t-43$ *for* $t \ge 7$ *via* $t\cdot[9]-[8]-[5]-[4]$;

- $n_2(9,3;9t-6) = 73t-52$ *for* $t \geq 8$ *via* $t \cdot [9] - [8] - [6] - [5] - [4]$;
- $n_2(9,3;9t-7) = 73t-59$ *for* $t \geq 10$ *via* $t \cdot [9] - [8] - [7] - [5]$;
- $n_2(9,3;9t-8) = 73t-68$ *for* $t \geq 11$ *via* $t \cdot [9] - [8] - [7] - [6] - [5]$.

Proof. By Theorem 2 we have that [9] and 7[5] are 3-partitionable over $\mathbb{F}_2$. Lemma 24 shows that [9] − [6], 3[9] − [7] − 2[5], and 7[9] − [8] − 6[5] are 3-partitionable over $\mathbb{F}_2$. By Lemma 26 we have that 2[5] − 2[1] is 2-partitionable over $\mathbb{F}_2$. With this, the corresponding dual shows that 5[5] − [4] is 3-partitionable over $\mathbb{F}_2$. ∎

For $i \in \{5,6,7,8\}$ this gives four infinite series of improvements $n_2(9,3,9t-i) > \overline{n}_2(9,3,9t-i)$ for sufficiently large $t \in \mathbb{N}$, see Table 2.

Proposition 17. *The Griesmer upper bound for* $n_3(5,2;s)$ *is attained for all* $s \geq 22$*, i.e. we have*

- $n_3(5,2;13t) = 121t$ *for* $t \geq 1$ *via* $4t \cdot [5]$;
- $n_3(5,2;13t-1) = 121t-10$ *for* $t \geq 1$ *via* $4t \cdot [5] - [4]$;
- $n_3(5,2;13t-2) = 121t-20$ *for* $t \geq 2$ *via* $4t \cdot [5] - 2[4]$;
- $n_3(5,2;13t-3) = 121t-30$ *for* $t \geq 1$ *via* $(4t-1) \cdot [5] + [2] - 3[1]$;
- $n_3(5,2;13t-4) = 121t-40$ *for* $t \geq 1$ *via* $(4t-1) \cdot [5] - 3[3]$;
- $n_3(5,2;13t-5) = 121t-50$ *for* $t \geq 3$ *via* $(4t-1) \cdot [5] - 2[4] + [2] - 3[1]$;
- $n_3(5,2;13t-6) = 121t-60$ *for* $t \geq 3$ *via* $(4t-1) \cdot [5] - 2[4] - 3[3]$;
- $n_3(5,2;13t-7) = 121t-67$ *for* $t \geq 1$ *via* $(4t-2) \cdot [5] - 2[3]$;
- $n_3(5,2;13t-8) = 121t-77$ *for* $t \geq 2$ *via* $(4t-2) \cdot [5] - [4] - 2[3]$;
- $n_3(5,2;13t-9) = 121t-87$ *for* $t \geq 3$ *via* $(4t-2) \cdot [5] - 2[4] - 2[3]$;
- $n_3(5,2;13t-10) = 121t-94$ *for* $t \geq 1$ *via* $(4t-3) \cdot [5] - [3]$;
- $n_3(5,2;13t-11) = 121t-104$ *for* $t \geq 2$ *via* $(4t-3) \cdot [5] - [4] - [3]$;
- $n_3(5,2;13t-12) = 121t-114$ *for* $t \geq 3$ *via* $(4t-3) \cdot [5] - 2[4] - [3]$.

Proof. By Theorem 2 we have that 4[5], [4], and [2] are 2-partitionable over $\mathbb{F}_3$. Lemma 24 shows that [5] − [3] and 4[5] − [4] − 3[2] are 2-partitionable over $\mathbb{F}_3$. By Lemma 26 we have that 3[5] + [2] − 3[1] and [5] + 3[1] are 2-partitionable over $\mathbb{F}_3$. ∎

Proposition 18. *The Griesmer upper bound for* $n_3(6,2;s)$ *is attained for all* $s \geq 90$*, i.e. we have*

- $n_3(6,2;10t) = 91t$ *for* $t \geq 1$ *via* $t \cdot [6]$;
- $n_3(6,2;10t-1) = 91t-10$ *for* $t \geq 1$ *via* $t \cdot [6] - [4]$;
- $n_3(6,2;10t-2) = 91t-20$ *for* $t \geq 2$ *via* $t \cdot [6] - 2[4]$;
- $n_3(6,2;10t-3) = 91t-30$ *for* $t \geq 3$ *via* $t \cdot [6] - 3[4]$;
- $n_3(6,2;10t-4) = 91t-40$ *for* $t \geq 4$ *via* $t \cdot [6] - 4[4]$;
- $n_3(6,2;10t-5) = 91t-50$ *for* $t \geq 5$ *via* $t \cdot [6] - 5[4]$;
- $n_3(6,2;10t-6) = 91t-60$ *for* $t \geq 6$ *via* $t \cdot [6] - 6[4]$;
- $n_3(6,2;10t-7) = 91t-67$ *for* $t \geq 8$ *via* $t \cdot [6] - 2[5] - 2[3]$;
- $n_3(6,2;10t-8) = 91t-77$ *for* $t \geq 9$ *via* $t \cdot [6] - 2[5] - [4] - 2[3]$;
- $n_3(6,2;10t-9) = 91t-87$ *for* $t \geq 10$ *via* $t \cdot [6] - 2[5] - 2[4] - 2[3]$.

Proof. By Theorem 2 we have that [6], [4], and 4[3] are 2-partitionable over $\mathbb{F}_3$. Lemma 24 shows that [6] − [4] and 4[6] − [5] − 3[3] are 2-partitionable over $\mathbb{F}_3$. ∎

D q-divisible multisets of points

In Proposition 6 we have used nonexistence results for q-divisible multisets of points in PG$(4,q)$ with maximum point multiplicity at most s to conclude upper bounds for $n_q(5,3;s)$, see Table 6.

n / r	1	2	3	4	5
2	1	0	0	0	0
3	0	1	0	0	0

Table 13: Number of nonisomorphic 2-divisible spanning multisets of points in PG$(r-1,2)$ with cardinality n.

We have used the software package `LinCode` [27] to enumerate q-divisible codes over $\mathbb{F}_q$, which are in one-to-one correspondence to spanning multisets of points. The obtained enumeration results for $q \in \{2,3,4,5,7\}$ are given in tables 13-17.

n / r	1	2	3	4	5
3	1	0	0	0	0
4	0	1	0	0	0
6	1	1	0	0	0
7	0	1	1	0	0
8	0	1	1	1	0

Table 14: Number of nonisomorphic 3-divisible spanning multisets of points in PG$(r-1,3)$ with cardinality n.

The results for $q=2$ in Table 13 can be easily verified theoretically. For $q=3$ one can show that a 3-divisible multiset of points with cardinality 6 is given by $3\chi_P + 3\chi_Q$ for two points P and Q that may coincide:

Lemma 42. *There is no* 3*-divisible multiset of points in* PG$(v-1,3)$ *with maximum point multiplicity* 2 *and cardinality* 6.

Proof. Let $\mathcal{M}$ be a 3-divisible multiset of points in PG$(v-1,3)$ with cardinality $\#\mathcal{M}=6$, so that $\mathcal{M}(H) \in \{0,3,6\}$ for every hyperplane H. Due to e.g. [81, Lemma 7.7] not all points can have multiplicity at most one. Let r denote the dimension of the span of the points with positive multiplicity $\mathcal{M}(P) > 0$. Let $\mathcal{M}'$ denote the embedding in a corresponding r-space which is isomorphic to PG$(r-1,q)$, so that $\mathcal{M}'(H) \in \{0,3\}$ for every hyperplane of PG$(r-1,q)$. The following observation can be easily verified: If $r \le 2$ or we have $\mathcal{M}(P) \ge 3$ for some point then $\mathcal{M} = 6\chi_P$ or $\mathcal{M} = 3\chi_P + 3\chi_{P'}$ for some point P'. So, let P be a point with multiplicity $\mathcal{M}'(P) = 2$. Since $\mathcal{M}'(H) \le 3$ we have $\mathcal{M}'(Q) \le 1$ for every point $Q \ne P$, so that there are exactly four points of multiplicity one and $r = 3$. However, for some point Q with multiplicity $\mathcal{M}(Q) = 1$ the four lines through Q each have multiplicity 3, so that $\#\mathcal{M} = 1 + 4 \cdot 2 = 9 \neq 6$ – contradiction. ■

n / r	1	2	3	4	5
4	1	0	0	0	0
5	0	1	0	0	0
8	1	1	0	0	0
9	0	1	1	0	0
10	0	1	1	1	0
12	1	2	**2**	0	0
13	0	2	3	1	0
14	0	1	5	3	1
15	0	1	**3**	**6**	**2**

Table 15: Number of nonisomorphic 4-divisible spanning multisets of points in $\mathrm{PG}(r-1,4)$ with cardinality n.

The possible cardinalities of q^l-divisible multisets of points over $\mathbb{F}_q$, where $l \in \mathbb{N}$, have been completely characterized in [73, Theorem 1]. In our situation $l = 1$ there exists a q-divisible multiset of points with cardinality n in $\mathrm{PG}(r-1,q)$, where r is sufficiently large, iff n can be written as $n = a \cdot q + b \cdot (q+1)$ with $a, b \in \mathbb{N}$. If $n \geq q^2 - q$ this is always possible and $q^2 - q - 1$ does not admit such a representation. The union of a q-fold points and b lines gives a q-divisible multiset of points and the question arises if there are other examples. For cardinality q^2 there is e.g. the affine plane, i.e. $\chi_E - \chi_L$ for some plane E and some line $L \leq E$. In tables 13-17 we mark those entries which contain examples that do not consist of unions of q-fold points and lines in bold font and briefly discuss the obtained additional constructions in the following.

There is a unique spanning 4-divisible multiset of points in $\mathrm{PG}(2,4)$ that has cardinality 12 and maximum point multiplicity at most 3. It is a doubled oval with an automorphism group of order 138240.

There is a unique spanning 5-divisible multiset of points in $\mathrm{PG}(2,5)$ that has cardinality 23 and maximum point multiplicity at most 4. A corresponding matrix is given by

$$\begin{pmatrix} 11111111111111111110100 \\ 00000011223333333441010 \\ 01133311441112333231001 \end{pmatrix}.$$

This object is also known as a *(strong)* $(3 \mod 5)$ *arc* in $\mathrm{PG}(2,5)$ and a nice picture can be found in [96]. By [87, Theorem 10] it is in one-to-one correspondence to a $(9,1)$-*blocking set* in $\mathrm{PG}(2,5)$ with line multiplicities contained in $\{1,2,3,4\}$, i.e., (trivial) blocking sets containing a full line are excluded. It is well known that the *projective triangle* is the only possibility over $\mathbb{F}_5$, see e.g. [20, 42]. In general ($t \mod q$) arcs are the combinatorial objects that characterize when linear codes are extendable without increasing the minimum distance, which can be utilized for nonexistence results for linear codes with certain parameters, see e.g. [86].

There is a unique spanning 5-divisible multiset of points in $\mathrm{PG}(3,5)$ that has cardinality 24 and does not consist of a union of four lines. A corresponding matrix is given by

$$\begin{pmatrix} 111111111111111111101000 \\ 000000112233333334410100 \\ 011333002200033343440010 \\ 101013012401301333300001 \end{pmatrix}.$$

Although, cardinalities of q-divisible multisets of points in $\mathrm{PG}(4,q)$ whose cardinalities are multiples of $q+1$ do not play a role in the context of Proposition 6, they are interesting from another point of view. A maximal partial line spread is a set of lines which covers each point at most once and which cannot be extended by a further line without destroying this property. The set of points that are not covered by some given partial line spread $\mathcal{L}$ of $\mathrm{PG}(r-1,q)$, called holes, form a q-divisible set of points $\mathcal{M}$ in $\mathrm{PG}(r-1,q)$ with cardinality $[r]_q - \#\mathcal{L} \cdot [2]_q$. The property that $\mathcal{L}$ is maximal is equivalent to the property that $\mathcal{M}$ does not contain a full line in its support (set of points P with $\mathcal{M}(P) \ge 1$).

n / r	1	2	3	4	5
5	1	0	0	0	0
6	0	1	0	0	0
10	1	1	0	0	0
11	0	1	1	0	0
12	0	1	1	1	0
15	1	2	1	0	0
16	0	2	3	1	0
17	0	1	4	3	1
18	0	1	3	5	2
20	1	4	2	1	0
21	0	3	9	5	1
22	0	2	14	18	6
23	0	1	**109**	27	18
24	0	1	64	**35**	29

Table 16: Number of nonisomorphic 5-divisible spanning multisets of points in $\mathrm{PG}(r-1,5)$ with cardinality n.

Theorem 21. *([48, Theorem 13]) Let $\mathcal{M}$ be a q^r-divisible multiset of points with cardinality $\delta[r+1]_q$ in $\mathrm{PG}(v-1,q)$ where $v > r$. If $q > 2$ and $1 \le \delta < \varepsilon$, where $q+\varepsilon$ is the size of the smallest nontrivial* *blocking sets in* PG(2, q)*, then there exists* $(r + 1)$*-spaces* $S_1, \ldots, S_\delta$ *such that*

$$\mathcal{M} = \sum_{i=1}^{\delta} \chi_{S_i},$$

i.e., $\mathcal{M}$ *is the sum of* $(r + 1)$*-spaces.*

There are two 7-divisible spanning multisets of points in PG(2, 7) with cardinality 39 and maximum point multiplicity at most 6. Generator matrices are given by

$$\begin{pmatrix}111111111111111111111111111111111100100\\000000000111111111222333444555666611010\\000335566000023555504412255611405522001\end{pmatrix} \text{ and } \begin{pmatrix}111111111111111111111111111111111100100\\000000000111111111222333444555666611010\\001444666000344455513400022244401315001\end{pmatrix}.$$

As the previous example, it can be obtained from a (12, 1) blocking set in PG(2, 7). Again one example is given by the projective triangle and there is another (sporadic) example, see e.g. [22]. There are seventeen 7-divisible spanning multisets of points in PG(2, 7) with cardinality 47 and maximum point multiplicity at most 6. Clearly, we can start from one of the two stated examples for cardinality 39 and add the points of a line. However, there are several other examples. In [56] a maximal partial line spread in PG(3, 7) of cardinality 45 and a corresponding 7-divisible set of points of cardinality 40, that is not the union of five disjoint lines, was constructed.

The described computer enumeration results imply that each 7-divisible multiset of points $\mathcal{M}$ in PG(4, 7) with cardinality at most 38 can be written as the sum of characteristic functions of lines and seven-fold points. By restricting the maximum point multiplicity to six we can slightly extend the outlined computer enumerations of linear codes to conclude:

Proposition 19. *Let* $\mathcal{M}$ *be a* 7*-divisible multiset of points in* PG(4, 7) *with maximum point multiplicity at most* 6*. If* $\#\mathcal{M} \le 38$ *or* $\#\mathcal{M} \in \{41, \ldots, 46\}$*, then there exist lines* $L_1, \ldots, L_l$ *such that* $\mathcal{M} = \sum_{i=1}^{l} \chi_{L_i}$*.*

For small dimensions r the q-divisible multisets of points in PG($r - 1, q$) can be easily characterized. In PG(0, q) the ambient space itself is just a point so that the only q-divisible multisets of points are given by $m \cdot q \cdot \chi_P$ for any $m \in \mathbb{N}$ and the unique point P. Now let $\mathcal{M}$ be a q-divisible multiset of points in PG(1, q). If P is a point with multiplicity $\mathcal{M}(P) \ge q$, then $\mathcal{M} - q\chi_P$ is also a q-divisible multiset of points in PG(1, q), so that we can assume $\mathcal{M}(P) \le q - 1$ for all points P w.l.o.g. In PG(1, q) hyperplanes coincide with points, so that $\mathcal{M}(P) \equiv \#\mathcal{M} \pmod q$ for every point P, so that all points have the same multiplicity and $\mathcal{M}$ consists of copies of the entire ambient space, which is a line. Thus, for $r \le 2$ we have that q-divisible multisets of points in PG($r - 1, q$) are given as the union of q-fold points and lines. As already mentioned, in PG(2, q) we have the affine plane with cardinality q^2 and maximum point multiplicity one.

For a given multiset of points let p_i denote the number of points of multiplicity i and a_i denote the number of hyperplanes with multiplicity i. By double-counting we obtain:

n / r	1	2	3	4	5
7	1	0	0	0	0
8	0	1	0	0	0
14	1	1	0	0	0
15	0	1	1	0	0
16	0	1	1	1	0
21	1	2	1	0	0
22	0	2	3	1	0
23	0	1	4	3	1
24	0	1	3	5	2
28	1	5	3	1	0
29	0	3	10	5	1
30	0	2	15	19	6
31	0	1	12	29	19
32	0	1	8	43	32
35	1	7	11	4	1
36	0	6	34	32	7
37	0	3	56	124	57
38	0	2	57	329	261
39	0	1	**41**	584	973
40	0	1	18	**720**	3639

Table 17: Number of nonisomorphic 7-divisible spanning multisets of points in $\mathrm{PG}(r-1,7)$ with cardinality n.

Lemma 43. *Let $\mathcal{M}$ be a multiset of points in* $\mathrm{PG}(r-1,q)$*, where $r \ge 3$, and S an arbitrary $(r-2)$-space. For the $q+1$ hyperplanes $H_0,\dots,H_q$ that contain S we have*

$$\sum_{i=0}^{q} \mathcal{M}(H_i) = \#\mathcal{M} + q \cdot \mathcal{M}(S). \tag{76}$$

Moreover, we have

$$\sum_{i=0}^{\#\mathcal{M}} p_i = [r]_q, \qquad \sum_{i=0}^{\#\mathcal{M}} a_i = [r]_q,$$
$$\sum_{i=1}^{\#\mathcal{M}} ip_i = \#\mathcal{M} \cdot [r-1]_q, \qquad \sum_{i=1}^{\#\mathcal{M}} ia_i = \#\mathcal{M} \cdot [r-1]_q, \quad \textit{and}$$
$$\binom{\#\mathcal{M}}{2} \cdot [r-2]_q + q^{r-2} \cdot \left(\sum_{i=2}^{\#\mathcal{M}} \binom{i}{2} \cdot p_i\right) = \sum_{i=2}^{\#\mathcal{M}} \binom{i}{2} \cdot a_i.$$

Lemma 44. *Let $\mathcal{M}$ be a multiset of points in* $\mathrm{PG}(r-1,q)$*, $1 \le s \le r-2$, and S an arbitrary s-space. If each hyperplane $H \ge S$ satisfies $\mathcal{M}(H) \ge x$, then we have $\#\mathcal{M} > qx - (q-1) \cdot \mathcal{M}(S)$.*

Proof. Since each s-space is contained in $[r-s]_q$ hyperplanes and each $(s+1)$-space is contained in $[r-s-1]_q$ hyperplanes, double-counting points yields $\#\mathcal{M} \ge \mathcal{M}(S) + \frac{[r-s]_q \cdot (x-\mathcal{M}(S))}{[r-s-1]_q}$. Using $[r-s]_q/[r-s-1]_q > q$ yields the stated inequality. ∎

We can easily check that divisibility is preserved by projection through a point:

Lemma 45. *Let $\mathcal{M}$ be a (spanning) Δ-divisible multiset of points in* $\mathrm{PG}(r-1,q)$*, where $r \ge 2$. For an arbitrary point P let $\mathcal{M}'$ be the multiset of points obtained from $\mathcal{M}$ by projection through P. Then, $\mathcal{M}'$ is a (spanning) Δ-divisible multiset of points in* $\mathrm{PG}(r-2,q)$ *with cardinality $\#\mathcal{M} - \mathcal{M}(P)$.*

The multiplicities of the lines through P w.r.t $\mathcal{M}$ are in one-to-to correspondence to the point multiplicities w.r.t. $\mathcal{M}'$.

Lemma 46. *Let $\mathcal{M}$ be a multiset of points in* $\mathrm{PG}(r-1,q)$ *with cardinality $\#\mathcal{M} \le [3]_q$, where $r \ge 3$. If P is a point with positive multiplicity, then there exists an $(r-2)$-space S with $\mathcal{M}(S) = \mathcal{M}(P)$.*

Proof. For $r = 3$ the statement is trivial, so that we assume $r > 3$. Let T be a subspace with maximum possible dimension satisfying $\mathcal{M}(T) = \mathcal{M}(P)$ and $\dim(T) \le r-2$. If $\dim(T) = r-2$ we are done. Otherwise consider the $[r-\dim(T)]_q \ge [3]_q$ subspaces that have dimension $\dim(T)+1$ and contain T. Since there are at most $[3]_q$ points with positive multiplicity we conclude the existence of a subspace $T' > T$ with $\dim(T') = \dim(T+1)$ and $\mathcal{M}(T') = \mathcal{M}(T) = \mathcal{M}(P)$ – contradiction. ∎

Proposition 20. *Each 7-divisible multiset of points in* PG(4, 7) *with cardinality* 42 *has maximum point multiplicity at least* 7.

Proof. Let $\mathcal{M}$ be a spanning 7-divisible multiset of points in PG$(r-1, 7)$ with cardinality 42 and maximum point multiplicity at most 6. Due to the classification of q-divisible multisets of points in PG$(0, q)$ and PG$(1, q)$, we can assume $r \ge 3$. Since no 7-divisible multiset of points with cardinality 41 exists over $\mathbb{F}_7$, see e.g. [73, Theorem 1], we can use Lemma 45 to conclude $\mathcal{M}(P) \neq 1$ for all points P in PG$(r-1, q)$.[7] Using the notation from Lemma 43 we state $p_0 + \sum_{i=2}^{6} p_i = [r]_7$, $\sum_{i=2}^{6} ip_i = 42$, and $p_i = 0$ for $i = 1$ or $i \ge 7$. Let Q be a point with positive but minimum possible positive multiplicity, i.e. $\mathcal{M}(Q) \ge 2$. From Lemma 44 we conclude $\mathcal{M}(Q) \in \{2, 3\}$.

Let us first consider the case $r = 3$ and $\mathcal{M}(Q) = 2$. All eight lines through Q have multiplicity exactly 7, see Equation (76), and either contain a point with multiplicity 5 or a point with multiplicity 3 and another point with multiplicity 2. Thus, we have $p_3 + p_5 \ge 8$. If $p_5 \ge 8$, then we have $p_5 = 8$, $p_2 = 1$, and $p_i = 0$ otherwise. However, considering the eight lines through a point of multiplicity 5 gives a contradiction. Thus, we have $p_3 \ge 1$ and let R denote point of multiplicity 3. From the eight lines through R, seven have multiplicity 7 and one has multiplicity 14, so that $p_5 \le 1$ and $p_3 \le 4$ (using $p_1 = 0$), which contradicts $p_3 + p_5 \ge 8$.

Assume $\mathcal{M}(Q) = 2$ and let $\mathcal{M}'$ denote the projection of $\mathcal{M}$ through Q, so that $\mathcal{M}'$ is a spanning 7-divisible multiset of points in PG$(r-2, 7)$ with cardinality 40 and without a point with multiplicity 1, see Lemma 45. From our computer enumeration we know that $\mathcal{M}'$ is the sum of the characteristic functions of five lines. Since $\mathcal{M}$ does not contain a point with multiplicity 1, we have $\mathcal{M}' = 2 \cdot \chi_{L_1} + 3 \cdot \chi_{L_2}$ for two lines L_1, L_2. If $L_1 = L_2$ then we have $r = 3$, which we already have excluded before. If $|L_1 \cap L_2| = 1$, then we have $r = 4$ and $\mathcal{M}(L) \in \{2, 4, 5, 7\}$ for every line L that contains Q, so that no point has multiplicity 4 or 6. Seven of these lines have multiplicity 4, one has multiplicity 7, and seven have multiplicity 5, so that $8 \le p_2 \le 9$, $7 \le p_3 \le 8$, $0 \le p_5 \le 1$, and $p_3 + p_5 = 8$.

If P_5 is a point with multiplicity 5, P_3 a point with multiplicity 3, and L the line spanned by P_3 and P_5, then L cannot contain a point with multiplicity 2, so that $\mathcal{M}(L) \in \{8, 11, 14, 17, 20, 23, 26\}$ using Equation (76) and the 7-divisibility of $\mathcal{M}$. If $\mathcal{M}(L) = 26$, then $\mathcal{M} - 3 \cdot \chi_L$ would be a 7-divisible multiset of points in PG(3, 7) with cardinality 21 and maximum point multiplicity at most 6 – contradiction. If $\mathcal{M}(L) \in \{8, 17, 20, 23\}$, then Equation (76) cannot be satisfied. So, there exist three lines L_1', L_2', L_3' through P_5 with $\mathcal{M}(L_1') = \mathcal{M}(L_2') = 11$, $\mathcal{M}(L_3') = 14$, and all seven points of multiplicity 3 are contained on $L_1' \cup L_2' \cup L_3'$. Thus, no hyperplane $H \ge L_1'$ can satisfy $\mathcal{M}(H) = 14$, so that Equation (76) cannot be satisfied – contradiction. It remains to consider the case $p_2 = 9$, $p_3 = 8$, and $p_5 = 0$ for $|L_1 \cap L_2| = 1$. Let L' be a line spanned by two points of multiplicity 3, so that L' does not contain a point of multiplicity 2. From Equation (76) we conclude $\mathcal{M}(L') \in \{12, 18, 21\}$.

[7] Alternatively, one can consider the eight hyperplanes through an $(r-2)$-space with multiplicity 1. We remark that there is some research on sets without tangents [110] and generalizations thereof [58].

If $\mathcal{M}(L') = 21$, then L' consists of seven points of multiplicity 3. Let L'' be a line spanned by the unique point of multiplicity 3 outside of L' and a point of multiplicity 3 on L'. As before, we conclude $\mathcal{M}(L'') \in \{12, 18, 21\}$, which is impossible since there are only two points of multiplicity 3 on L''. If $\mathcal{M}(L') = 18$, then L' consists of six points of multiplicity 3. Since all eight hyperplanes through L' have multiplicity 21, we have $p_3 \ge 6 + 8 = 14$, which is a contradiction. So, all lines contain either 0, 1, or 4 points of multiplicity 3, so that $p_3 \equiv 1 \pmod 3$ – contradiction. It remains to consider the case $|L_1 \cap L_2| = 0$ where $r = 5$, $p_2 = 9$, $p_3 = 8$, and $p_5 = 0$. Moreover, each line L that contains a point of multiplicity 2 satisfies $\mathcal{M}(L) \in \{2, 4, 5\}$. For a plane π we can use Lemma 76 and the 7-divisibility of $\mathcal{M}$ to deduce $\mathcal{M}(\pi) \in \{0, 2, 3, 4, 5, 7, 10, 11, 12, 14, 18, 19, 21, 26, 28\}$. If $\mathcal{M}(\pi) = 10$, then we have $\mathcal{M}(H_i) = 14$ for $0 \le i \le 7$, so that $p_2 \ge 16$ – contradiction. If $\mathcal{M}(\pi) = 3$, then we have $\mathcal{M}(H_i) = 7$ for seven hyperplanes, so that $p_2 \ge 14$ – contradiction. If π contains a point P_3 of multiplicity 3 and $\mathcal{M}(\pi) < 19$, then there exists a line $P_3 \le L \le \pi$ with multiplicity 3. Since each plane through L has multiplicity at least 5, each hyperplane $H \ge L$ has multiplicity at least $3 + 2 \cdot 8$, i.e., multiplicity at least 21. This excludes the cases $\pi \in \{5, 7, 11\}$. Now let L directly be some line with multiplicity 3, which exists since $p_3 \ge 1$ and $p_2 + p_3 < [4]_7$. Since each plane $\pi \ge L$ has multiplicity at least 12, we have $\mathcal{M}(H) \ge 3 + 9 \cdot 8 > 35$ for each hyperplane $H \ge L$ – contradiction.

In the following we have $\mathcal{M}(Q) = 3$, so that $p_2 = 0$. If $\mathcal{M}(P) \in \{5, 6\}$ for some point P, then each hyperplane $H \ge P$ satisfies $\mathcal{M}(H) \ge 14$, so that Lemma 44 gives a contradiction. Thus, we have $p_1 = p_2 = p_5 = p_6 = 0$, $p_3 \equiv 2 \pmod 4$, and $p_4 \equiv 0 \pmod 3$. If $\mathcal{M}(P) \equiv 0 \pmod 3$, then $\frac{1}{3} \cdot \mathcal{M}$ would be a 7-divisible multiset of points in $\mathrm{PG}(r-1, 7)$ with cardinality 14 and maximum point multiplicity at most 2 – contradiction. Thus, we have $p_4 \in \{3, 6, 9\}$ and $p_3 \in \{2, 6, 10\}$. From Lemma 46 we conclude the existence of $(r-2)$-spaces S_3, S_4 with $\mathcal{M}(S_3) = 3$ and $\mathcal{M}(S_4) = 4$. Applying Equation (76) to S_3 yields that seven of the eight hyperplanes through S_3 have multiplicity 7, so that $p_4 \ge 7$. Similarly, applying Equation (76) to S_4 yields that six of the eight hyperplanes through S_4 have multiplicity 7, so that $p_3 \ge 6$ – contradiction. ∎

Corollary 8. *If $\mathcal{M}$ is a 7-divisible multiset of points in $\mathrm{PG}(4, 7)$ with cardinality 42, then there exist points $P_1, \dots, P_6$ so that $\mathcal{M} = \sum_{i=1}^{6} 7 \cdot \chi_{P_i}$.*

Proposition 21. *Each 7-divisible multiset of points in $\mathrm{PG}(4, 7)$ with cardinality 43 has maximum point multiplicity at least 7.*

Proof. Let $\mathcal{M}$ be a spanning 7-divisible multiset of points in $\mathrm{PG}(r-1, 7)$ with cardinality 43 and maximum point multiplicity at most 6. Due to the classification of q-divisible multisets of points in $\mathrm{PG}(0, q)$ and $\mathrm{PG}(1, q)$, we can assume $r \ge 3$. Since no 7-divisible multiset of points with cardinality 41 exists over $\mathbb{F}_7$, see e.g. [73, Theorem 1], we can use Lemma 45 to conclude $\mathcal{M}(P) \ne 2$ for all points P in $\mathrm{PG}(r-1, q)$.[8]

[8] Alternatively, one can consider the eight hyperplanes through an $(r-2)$-space with multiplicity 2.

Assume $\mathcal{M}(P) \neq 1$ for all points P for a moment. If P_6 is a point of multiplicity 6, then there exists a $(r-2)$-space S_6 of multiplicity 6, see Lemma 46. Since there are no points of multiplicity 1 or 2, all eight hyperplanes through S_6 have multiplicity at least 15 – contradiction. If P_3 is a point of multiplicity 3, then there exists an $(r-2)$-space S_3 of multiplicity 3, see Lemma 46. All eight hyperplanes through S_3 have multiplicity exactly 8. Since there are no points of multiplicity 1 or 2, we have $p_5 \geq 8$, so that $p_5 = 8$ and $p_3 = 1$. However, considering the eight hyperplanes through an $(r-2)$-space S_5 of multiplicity 5 yields a contradiction. Since $\#\mathcal{M}$ is not divisible by 4, there exists a point P_5 and an $(r-2)$-space S_5 of multiplicity 5. Since each point with positive multiplicity has multiplicity at least 4, all eight hyperplanes through S_5 have multiplicity at least 5 – contradiction. Thus, there exists a point P_1 of multiplicity 1. By $\mathcal{M}'$ we denote the projection of $\mathcal{M}$ through P_1. From Lemma 45 we conclude that $\mathcal{M}'$ is a spanning 7-divisible multiset of points in $\operatorname{PG}(r-2,7)$ with cardinality 42. From our previous enumeration we know that $\mathcal{M}' = \sum_{i=1}^{6} 7 \cdot \chi_{Q_i}$ for some points $Q_1, \dots, Q_6$. If S_3 is an $(r-2)$-space with multiplicity 3, then all eight hyperplanes through S_3 have multiplicity 8. If S_4 is an $(r-2)$-space with multiplicity 4, then seven hyperplanes through S_4 have multiplicity 8 and one has multiplicity 15. If S_5 is an $(r-2)$-space with multiplicity 5, then at least six hyperplanes through S_5 have multiplicity 8. Either there are two with multiplicity 15 or one with multiplicity 22. Since there is no 7-divisible multiset of points in $\operatorname{PG}(4,7)$ with cardinality 35 and maximum point multiplicity at most six, $\mathcal{M}$ cannot contain a full line in its support, i.e., every line through a point with multiplicity 1 has either multiplicity 1 or contains at least one point with multiplicity at least 3.

Assume $r = 3$ for a moment. If L is a line with $\mathcal{M}(L) > 15$, then we have $\mathcal{M}(L) = 22$ and L consists of four points of multiplicity 5 and 2 points of multiplicity 1. Considering the eight lines through a point of multiplicity 5 on L yields $p_4 = 0$, $p_5 = 4$, and $p_3 \leq 7$, so that $a_{22} = 1$. Using Lemma 43 we conclude $7a_{15} = 3p_3 - 16$, which gives a contradiction. Using $a_i = 0$ for $i > 15$, Lemma 43 gives

$$a_{15} = \frac{3p_3 + 6p_4 + 10p_5}{7} - 5.$$

Assume that L is a line consisting of three points of multiplicity 5 and five points of multiplicity 0. Considering the lines through a point P_5 with multiplicity 5 on L gives $p_4 + p_5 = 5$. The two points with multiplicity at least 4 outside of L span a line L' that meets L in a unique point, which gives a contradiction if we consider a different point of multiplicity 5 on L. Thus, no such line exists and we have $p_5 \leq 3$. Now assume that L is a line consisting of two points P_5, P_5' of multiplicity 5, one point P_4 of multiplicity 4, one point P_1 of multiplicity 1, and four points of multiplicity zero. Considering the lines through P_5 yields $p_4 + p_5 = 5$ again and the two points with multiplicity at least 4 outside of L span a line meeting L in P_5. Applying the same argument to P_5' gives a contradiction, so that $p_5 \in \{0, 1\}$. If P_5 is the unique point of multiplicity 5, then the two lines of multiplicity 15 through P_5 both contain exactly two points of multiplicity 4 and none of multiplicity 3, so that $p_4 = 4$, $p_3 \le 6$, $a_{15} = 2$, and Lemma 43 yields a contradiction. If $p_4 > 0$ and $p_5 = 0$, then the unique line of multiplicity 15 through a point of multiplicity 4 consists of three points of multiplicity 4, three points of multiplicity 1, and two points of multiplicity zero, so that $p_4 \equiv 0 \pmod 3$ and $p_3 \le 7$. With this we conclude $a_4 \le 9$ and $a_{15} \le 3$, so that Lemma 43 yields a contradiction. If $p_4 + p_5 = 0$, then we have $a_{15} = 0$ and Lemma 43 yields a contradiction again. So, we can assume $r \in \{4, 5\}$ in the remaining part.

Let P_5 be a point of multiplicity 5 and $\mathcal{M}''$ be the projection of $\mathcal{M}$ through P_5. From Lemma 45 we conclude that $\mathcal{M}''$ is 7-divisible with cardinality 38. From our previous computer enumeration we conclude the existence of three lines L', L'', L''' and two points P', P'' such that $\mathcal{M}'' = \chi_{L'} + \chi_{L''} + \chi_{L'''} + 7 \cdot \chi_{P'} + 7 \cdot \chi_{P''}$. Either we have $L' = L'' = L'''$ or there exists a point $\widetilde{P}$ with $\mathcal{M}''(\widetilde{P}) \in \{1, 2\}$. In the latter case let $L \ge P_5$ be the corresponding line with $\mathcal{M}(L) \in \{6, 7\}$, so that L contains at least one point with multiplicity 1. Since each line through a point with multiplicity 1 has a multiplicity that is congruent to 1 modulo 7, we conclude $L' = L'' = L'''$. So, for any line $L \ge P_5$ we have $\mathcal{M}(L) \in \{8, 12, 15, 19, 22\}$. Since lines through a point of multiplicity 1 have a multiplicity that is congruent to 1 modulo 7 and hyperplanes that contain a point of multiplicity 3 have multiplicity 8, we conclude $\mathcal{M}(L) \in \{8, 15, 19, 22\}$, each line $L_{19} \ge P_5$ with $\mathcal{M}(L_{19}) = 19$ consists of exactly three points of multiplicity 5, and each line $L_{22} \ge P_5$ with $\mathcal{M}(L_{22}) = 22$ consists of exactly two points of multiplicity 5. In both cases the line has to contain a point of multiplicity 4 which is contained in a hyperplane with multiplicity at least 22 – contradiction. Thus, we have $\mathcal{M}(L) \in \{8, 15\}$. However, then we conclude $P' \le L'$ and $P'' \le L'$, which contradicts $r \ge 4$, so that P_5 cannot exist and we have $p_5 = 0$.

It remains to consider the cases where $r \in \{4, 5\}$ and the maximum point multiplicity is 3 or 4. Let S be an arbitrary $(r-2)$-space. For every hyperplane $H \ge S$ we have $\mathcal{M}(H) = 8$ if H contains a point of multiplicity 3 and $\mathcal{M}(H) \in \{8, 15\}$ if H contains a point of multiplicity 4. Since every line that contains a point of multiplicity 1 has a multiplicity that is congruent to 1 modulo 7, Lemma 43 gives $\mathcal{M}(S) \in \{0, 1, 3, 4, 6, 7, 8, 12, 15, 22, 29\}$. If $\mathcal{M}(S) = 12$, then seven of the eight hyperplanes through S have multiplicity 15 and one has multiplicity 22, so that S cannot contain a point of multiplicity larger than 1. This is impossible, so that we have $\mathcal{M}(S) \ne 12$. If $\mathcal{M}(S) \in \{15, 22, 29\}$, then S cannot contain a point with multiplicity larger than 1, which is only possible if S contains a full line in its support – contradiction. If $\mathcal{M}(S) = 7$, then S consists of a point P_3 of multiplicity 3 and a point P_4 of multiplicity 4. However, not all hyperplanes through S can have multiplicity 8, so that $\mathcal{M}(S) \ne 7$. If S contains both a point of multiplicity 3 and 4, then $\mathcal{M}(S) = 8$, which is impossible. Thus, we have $p_3 \cdot p_4 = 0$. If P_4 and P_4' are two different points with multiplicity 4, then the line L spanned by P_4 and P_4' has multiplicity 8, so that $r = 4$. Let $\pi \ge L$ be a plane with multiplicity 15 and $P_1 \le \pi$ be a point with multiplicity 1. Then the two lines $\langle P_4, P_1 \rangle$ and $\left\langle P_4', P_1 \right\rangle$ have multiplicity 8 and π contains 7 points of multiplicity 1. Considering another line through two points of multiplicity 1, not being equal to P_1, in π yields a contradiction. If P_4 is the unique point with multiplicity 4, then considering the lines through a point of multiplicity 1 yields a full line in the support of $\mathcal{M}$ – contradiction. Thus, we have $p_4 = 0$ and the maximum point multiplicity of $\mathcal{M}$ equals 3. Considering the lines through a point of multiplicity 1 we conclude $p_3 \geq 6$. Let π be the subspace spanned by three points of multiplicity 3, so that $\mathcal{M}(\pi) \geq 9$ and $\dim(\pi) \leq 3$. Thus, we have $r = 4$ and π is a hyperplane, which is also impossible. ■

We remark that it is indeed possible to show that each 7-divisible multiset of points $\mathcal{M}$ in $\mathrm{PG}(r-1, q)$ with $\#\mathcal{M} \leq 38$ or $\#\mathcal{M} \in \{41, 42, 43\}$ can be written as the sum of characteristic functions of seven-fold points and lines, without using any computer enumerations. Also the full characterization for cardinalities $\#\mathcal{M} \in \{39, 40\}$ is doable by hand. The question is whether such results can be obtained in a more elegant (and general) way than indicated in the proof of Proposition 20.

E Additive two-weight codes

As mentioned at the end of Subsection 2.1, (projective) linear two-weight codes have received a lot of attention in the literature. So, here we collect a few basic observations on additive two-weight codes in the geometric framework. To this end, a faithful projective $h-(n, r, \{s_1, \ldots, s_a\}, \mu)_q$ system $\mathcal{S}$ is a faithful projective $h-(n, r, \min\{s_1, \ldots, s_a\}, \mu)_q$ system where the number of elements of $\mathcal{S}$ that are contained in a hyperplane H of $\mathrm{PG}(r-1, q)$ is contained in $\{s_1, \ldots, s_a\}$ for each hyperplane H. So, for $a = 2$, each hyperplane contains either s_1 or s_2 elements of $\mathcal{S}$. We are interested in the possible parameters of such combinatorial objects. For $r = h$ each hyperplane contains none of the elements of $\mathcal{S}$. The other cases where s_2 is not attained by at least one hyperplane, i.e. additive one-weight codes, are characterized in Theorem 2. I.e. for $r > h$ there exists a positive integer l such that $n = l \cdot \frac{[r]_q}{[\gcd(r,h)]_q}$, $s = l \cdot \frac{[r-h]_q}{[\gcd(r,h)]_q}$, and $\mu = l \cdot \frac{[h]_q}{[\gcd(r,h)]_q}$. For $r = h + 1$ we can choose $0 < l < [h+1]_q$ h-spaces with multiplicity s_1 and the other $\left([h+1]_q - l\right)$ h-spaces with multiplicity $s_2 > s_1$ for all $s_1, s_2 \in \mathbb{N}$, so that $n = s_2[h+1]_q - (s_2 - s_1)l$. In general we have:

Lemma 47. *Let $\mathcal{S}$ be a faithful projective $h-(n, r, \{s_1, \ldots, s_a\})_q$ system and $\mathcal{S}'$ be a faithful projective $h-(n', r, \{s'_1, \ldots, s'_b\})_q$ system, then $\mathcal{S} \cup \mathcal{S}'$ is a faithful projective*

$$h - \left(n + n', r, \left\{s_i + s'_j \,:\, 1 \leq i \leq a, 1 \leq j \leq b\right\}\right)_q$$

system.

So, choosing $a = 1$ and $b = 2$ e.g. gives constructions for two-weight codes using Theorem 2. Another variant is to embed an example from Theorem 2 for $\mathrm{PG}(r'-1, q)$ in $\mathrm{PG}(r-1, q)$, where $r > r'$, and choose it as $\mathcal{S}'$.

For a given faithful projective $h-(n, r, \{s_1, \dots, s_a\})_q$ system $\mathcal{S}$ we can also consider the l-fold copy of $\mathcal{S}$ for $l \in \mathbb{N}$, which is a faithful projective $h-(ln, \{ls_1, \dots, ls_a\})_q$ system. So, one might assume that $\mathcal{S}$ is a set of h-spaces and no subset of $\mathcal{S}$ partitions the points of the $\frac{[h]_q}{[\gcd(r,h)]_q}$-fold copy of the ambient space $\mathrm{PG}(r-1, q)$ as in Theorem 2.

For $r = 2h$ we can consider a partial spread $\mathcal{P}$ of h-spaces in $\mathrm{PG}(2h-1, q)$, i.e. a set of h-spaces with pairwise trivial intersection. Due to the existence of a spread of h-spaces we have constructions for all $1 \le \#\mathcal{P} \le q^h + 1$. I.e. faithful projective $h-(n, 2h, \{0, 1\})_q$ systems exist for all $1 \le n \le q^h + 1$. Using copies of a spread of h-spaces and Lemma 47 we can conclude that faithful projective $h-(n, 2h, \{s_1, s_2\})_q$ systems exist for all $s_1, s_2 \in \mathbb{N}$.

For $r = 2h+1$ and $h \ge 2$ we have that $[2h+1] - [h+1]$ and $[h-1]_q \cdot [2h+1] + q^{h-1} \cdot [1]$ are h-partitionable over $\mathbb{F}_q$, see Lemma 23 and Lemma 26, so that projective faithful $h-\left(q^{h+1}, 2h+1, \{0, q\}\right)_q$ and $h-\left([2h]_q - q^h, 2h+1, \left\{[h]_q - 1, [h]_q\right\}\right)_q$ systems do exist for all $h \ge 2$.

n	s_1	s_2	n	s_1	s_2	n	s_1	s_2	n	s_1	s_2	n	s_1	s_2
1	0	1	11	2	3	5	1	5	21	3	5	22	2	6
6	0	2	8	0	4	13	1	5	20	4	5	26	2	6
8	0	2	16	0	4	17	1	5	21	4	5	20	4	6
7	1	3	10	2	4	21	1	5	24	0	6	22	4	6
9	1	3	12	2	4	15	3	5	10	2	6	24	4	6
11	1	3	14	2	4	17	3	5	14	2	6			
10	2	3	16	2	4	19	3	5	18	2	6			

Table 18: Parameters of additive two-weight codes in $\mathrm{PG}(4, 2)$ for $s_1 < s_2 < 7$.

In Table 18 we have collected all feasible parameters of sets of n lines in $\mathrm{PG}(4, 2)$ such that each hyperplane contains either s_1 or s_2 lines, where $s_1 < s_2 < 7$. Note that Theorem 2 yields a set of 31 lines in $\mathrm{PG}(4, 2)$ such that each hyperplane contains exactly 7 lines. Since the span of two lines is at most 4-dimensional, the only possibility for $\{s_1, s_2\} = \{0, 1\}$ is a single line. The previously stated general constructions explain the cases $(n, s_1, s_2) \in \{(8, 0, 2), (11, 2, 3), (16, 0, 4), (5, 1, 5)\}$. So, there is much to explore.

Remark 15. *In [83] it was shown that no projective* $[66, 5, \{48, 56\}]_4$ *code exists. The question whether such a two-weight code exists if we only assume additivity instead of linearity over* $\mathbb{F}_4$ *remains open. A projective* $[198, 10, \{96, 112\}]_2$ *was constructed in [76].*[9] *So, can the corresponding set of* 198 *points in* $\mathrm{PG}(9, 2)$ *be partitioned into* 66 *lines such that each hyperplane contains either* 10 *or* 18 *lines?*

By double-counting we can infer some necessary existence conditions via linear equation systems. To this end let x_{s_i} denote the number of hyperplanes of $\mathrm{PG}(r-1, q)$ that contain exactly

[9] A generator matrix can be obtained from `http://www.tec.hkr.se/~chen/research/2-weight-codes/search.php`.

s_i elements of a putative faithful projective $h-(n,r,\{s_1,\ldots,s_t\})_q$ system $\mathcal{S}$. Counting the number of hyperplanes gives

$$\sum_{i=1}^{t} x_{s_i} = [r]_q. \tag{77}$$

Double-counting the incidences between elements of $\mathcal{S}$ and hyperplanes yields

$$\sum_{i=1}^{t} s_i \cdot x_{s_i} = n \cdot [r-h]_q. \tag{78}$$

In order to double-count the number of incidences between pairs of elements of $\mathcal{S}$ and hyperplanes we let y_i denote the number of pairs of elements of $\mathcal{S}$ whose span has dimension i. With this, we compute

$$\sum_{i=1}^{t} \binom{s_i}{2} \cdot x_{s_i} = \sum_{i=h}^{2h} y_i \cdot [r-i]_q, \tag{79}$$

where we additionally have

$$\sum_{i=h}^{2h} y_i = \binom{n}{2}. \tag{80}$$

Of course, all occurring variables have to be nonnegative and integral. If $\mathcal{S}$ is a set of lines, i.e. $h=2$, the linear equation system simplifies to

$$\begin{aligned} x_{s_1} + x_{s_2} &= [r]_q, \\ s_1 x_{s_1} + s_2 x_{s_2} &= n[r-2]_q, \\ \binom{s_1}{2} \cdot x_{s_1} + \binom{s_2}{2} \cdot x_{s_2} &= \binom{n}{2} \cdot [r-4]_q + q^{r-4} \cdot y_3. \end{aligned}$$

In Table 19 we collect all parameters where the equation system for x_{s_1}, s_{s_2}, and y_3 has nonnegative integer solutions but no corresponding additive two-weight code in $\mathrm{PG}(4,2)$ exists, where we restrict the parameters to $s_1 < s_2 < 7$ and sets of lines. If not stated otherwise those nonexistence results have been obtained by integer linear programming (ILP) computations directly modelling faithful projective systems (or the multiset of covered points) with given parameters.

The example attaining $n_3(6,2;3) = 21$ from [37], mentioned in the introduction and improving $\overline{n}_3(6,2;3) = 17$, is quite exceptional and corresponds to an additive two-weight code over $\mathbb{F}_9$ that is not linear. As mentioned in Remark 9 it is a special case of a so-called *perp-system*, see [37] for details. An example of a perp-system is a (multi-)set $\mathcal{S}_q$ of 2-spaces in $\mathrm{PG}(5,q)$ with cardinality $q(q^2-q+1)$ such that each 4-space contains either 0 or q elements from $\mathcal{S}$. They do indeed exist for even q or $q=3$, see Remark 9. From each perp-system we get a two-weight code [37, Theorem

n	s_1	s_2	n	s_1	s_2	n	s_1	s_2	n	s_1	s_2	n	s_1	s_2
2	0	2	12	0	4	14	2	5	18	1	6	26	5	6
4	0	2	13	1	4	17	2	5	15	3	6			
3	0	3	16	1	4	6	0	6	18	3	6			
12	0	3	5	0	5	12	0	6	18	4	6			
13	1	3	20	0	5	18	0	6	26	4	6			
4	0	4	9	1	5	13	1	6	25	5	6			

Table 19: Parameters of nonexistent additive two-weight codes in $\operatorname{PG}(4,2)$ for $s_1 < s_2 < 7$.

2.2]. For $\mathcal{S}_2$ the corresponding projective $[18,6,\{8,12\}]_2$ code is unique, see e.g. [59], and can be obtained from the unique projective $[6,3,\{4,6\}]_4$ code, i.e. a hyperoval in geometric terms. With respect to $\mathcal{S}_3$ we remark that there are at least six nonisomorphic projective $[84,6,\{54,63\}]_3$ codes, see e.g. [12, 43] for constructions. Not all of these sets of points can be partitioned into lines. The dual of $\mathcal{S}_q$ is a set of $q(q^2-q+1)$ 4-spaces that each point is contained in 0 or q of these 4-spaces. Another example would be a set $\mathcal{S}'$ of 22 6-spaces in $\operatorname{PG}(9,2)$ such that each point is contained in either 0 or 2 elements. The uncovered points correspond to a projective $[330,10,\{160,176\}]_2$ code. There exist more than 1700 nonisomorphic such two-weight codes, see e.g. [43, 76, 91]. So far it is not known whether some of these point sets can be partitioned into 4-spaces. There also exist projective $[110,5,\{80,88\}]_4$ codes, see e.g. [43], and the question arises whether some of the corresponding sets of points can be partitioned into lines.

F Computer searches

In this section we list the examples that we have found by computer searches. Faithful projective systems can be easily modeled as ILPs. To reduce the search space we prescribe subgroups of the automorphism group. Alternatively we can try to partition suitable multisets of points. Those multisets of points can again be modeled as ILPs and we may prescribe subgroups of the automorphism group. Alternatively we use the database of *best known linear codes* (BKLC) in `Magma` or enumerate suitable linear codes using `LinCode` [27]. For each case we give a list of generator matrices for the subspaces.

$n_2(7,3;3) \ge 23$: $\begin{pmatrix}0010000\\0001110\\0000001\end{pmatrix}$, $\begin{pmatrix}0110000\\0001001\\0000011\end{pmatrix}$, $\begin{pmatrix}0100000\\0000101\\0000010\end{pmatrix}$, $\begin{pmatrix}1000011\\0010010\\0001110\end{pmatrix}$, $\begin{pmatrix}1000010\\0110001\\0001001\end{pmatrix}$, $\begin{pmatrix}1000110\\0010101\\0001000\end{pmatrix}$, $\begin{pmatrix}1001001\\0111011\\0000100\end{pmatrix}$, $\begin{pmatrix}1000001\\0100011\\0000111\end{pmatrix}$, $\begin{pmatrix}1000111\\0100110\\0001100\end{pmatrix}$, $\begin{pmatrix}1000001\\0100111\\0010001\end{pmatrix}$, $\begin{pmatrix}1000011\\0101101\\0011110\end{pmatrix}$, $\begin{pmatrix}1000010\\0100010\\0011011\end{pmatrix}$, $\begin{pmatrix}1000100\\0101110\\0011100\end{pmatrix}$, $\begin{pmatrix}1001100\\0100001\\0011001\end{pmatrix}$, $\begin{pmatrix}1001000\\0100100\\0010011\end{pmatrix}$, $\begin{pmatrix}1010100\\0100001\\0001101\end{pmatrix}$, $\begin{pmatrix}1100100\\0010011\\0001011\end{pmatrix}$, $\begin{pmatrix}1010010\\0101011\\0000100\end{pmatrix}$, $\begin{pmatrix}1100111\\0010110\\0001100\end{pmatrix}$, $\begin{pmatrix}1010110\\0110101\\0001000\end{pmatrix}$, $\begin{pmatrix}1011010\\0110010\\0000110\end{pmatrix}$, $\begin{pmatrix}1000000\\0101000\\0010100\end{pmatrix}$, $\begin{pmatrix}1000000\\0101001\\0010111\end{pmatrix}$.

$n_2(7,3;5) \ge 41$: $\begin{pmatrix}0100011\\0010110\\0001111\end{pmatrix}$, $\begin{pmatrix}0100110\\0010011\\0001000\end{pmatrix}$, $\begin{pmatrix}0100011\\0011010\\0000100\end{pmatrix}$, $\begin{pmatrix}0101100\\0010101\\0000010\end{pmatrix}$, $\begin{pmatrix}0100110\\0011100\\0000001\end{pmatrix}$, $\begin{pmatrix}1000011\\0010001\\0001011\end{pmatrix}$, $\begin{pmatrix}1000100\\0010101\\0001010\end{pmatrix}$, $\begin{pmatrix}1000001\\0010100\\0001101\end{pmatrix}$, $\begin{pmatrix}1000110\\0010010\\0001001\end{pmatrix}$, $\begin{pmatrix}1000010\\0011000\\0000101\end{pmatrix}$, $\begin{pmatrix}1000101\\0100001\\0001110\end{pmatrix}$, $\begin{pmatrix}1001010\\0101000\\0000111\end{pmatrix}$, $\begin{pmatrix}1000101\\0100100\\0001100\end{pmatrix}$, $\begin{pmatrix}1001011\\0100010\\0000110\end{pmatrix}$, $\begin{pmatrix}1001001\\0100001\\0000011\end{pmatrix}$, $\begin{pmatrix}1000110\\0100101\\0011011\end{pmatrix}$, $\begin{pmatrix}1000011\\0101101\\0011010\end{pmatrix}$, $\begin{pmatrix}1001110\\0101001\\0010101\end{pmatrix}$, $\begin{pmatrix}1000111\\0101011\\0011101\end{pmatrix}$,

$$\begin{pmatrix}1001100\\0101010\\0011001\end{pmatrix}, \begin{pmatrix}1000001\\0110010\\0001011\end{pmatrix}, \begin{pmatrix}1000101\\0110001\\0001010\end{pmatrix}, \begin{pmatrix}1000010\\0110100\\0001001\end{pmatrix}, \begin{pmatrix}1001000\\0111010\\0000101\end{pmatrix}, \begin{pmatrix}1000100\\0110101\\0001101\end{pmatrix}, \begin{pmatrix}1010011\\0100101\\0001100\end{pmatrix}, \begin{pmatrix}1011000\\0101011\\0000110\end{pmatrix}, \begin{pmatrix}1010010\\0100101\\0001110\end{pmatrix}, \begin{pmatrix}1010001\\0101010\\0000111\end{pmatrix},$$

$$\begin{pmatrix}1010100\\0101001\\0000011\end{pmatrix}, \begin{pmatrix}1010011\\0110100\\0001010\end{pmatrix}, \begin{pmatrix}1011011\\0110010\\0000101\end{pmatrix}, \begin{pmatrix}1010110\\0110011\\0001011\end{pmatrix}, \begin{pmatrix}1010111\\0110001\\0001101\end{pmatrix}, \begin{pmatrix}1010101\\0110110\\0001001\end{pmatrix}, \begin{pmatrix}1100001\\0010100\\0001000\end{pmatrix}, \begin{pmatrix}1101011\\0010010\\0000100\end{pmatrix}, \begin{pmatrix}1100010\\0010111\\0001111\end{pmatrix}, \begin{pmatrix}1100100\\0011110\\0000001\end{pmatrix},$$

$$\begin{pmatrix}1101000\\0010001\\0000010\end{pmatrix}, \begin{pmatrix}1000000\\0100000\\0010000\end{pmatrix}.$$

$n_2(8,3;3) \geq 18$: $$\begin{pmatrix}01000101\\00101101\\00011111\end{pmatrix}, \begin{pmatrix}01001111\\00100001\\00011011\end{pmatrix}, \begin{pmatrix}01001010\\00100101\\00010011\end{pmatrix}, \begin{pmatrix}10011010\\00100000\\00000111\end{pmatrix}, \begin{pmatrix}10100101\\00010000\\00001110\end{pmatrix}, \begin{pmatrix}10010110\\00110000\\00001001\end{pmatrix}, \begin{pmatrix}10001110\\01010011\\00101001\end{pmatrix}, \begin{pmatrix}10001001\\01100101\\00010111\end{pmatrix},$$

$$\begin{pmatrix}10000111\\01011111\\00111110\end{pmatrix}, \begin{pmatrix}10010001\\01110100\\00000010\end{pmatrix}, \begin{pmatrix}10110010\\01101100\\00000001\end{pmatrix}, \begin{pmatrix}10010010\\01110000\\00001011\end{pmatrix}, \begin{pmatrix}10110001\\01100000\\00000110\end{pmatrix}, \begin{pmatrix}10100011\\01010000\\00001101\end{pmatrix}, \begin{pmatrix}10100001\\01011000\\00000011\end{pmatrix}, \begin{pmatrix}11000101\\00100110\\00010101\end{pmatrix}, \begin{pmatrix}11001111\\00100010\\00011101\end{pmatrix}, \begin{pmatrix}11001010\\00101010\\00010001\end{pmatrix}.$$

$n_2(8,3;5) \geq 37$: $$\begin{pmatrix}10011000\\00101011\\00000101\end{pmatrix}, \begin{pmatrix}10100010\\00010110\\00001111\end{pmatrix}, \begin{pmatrix}10010001\\00110111\\00001010\end{pmatrix}, \begin{pmatrix}10000001\\01000011\\00010001\end{pmatrix}, \begin{pmatrix}10000011\\01000010\\00110011\end{pmatrix}, \begin{pmatrix}10000100\\01001100\\00010011\end{pmatrix}, \begin{pmatrix}10001100\\01001000\\00110010\end{pmatrix}, \begin{pmatrix}10000110\\01001101\\00011100\end{pmatrix},$$

$$\begin{pmatrix}10001101\\01001011\\00111000\end{pmatrix}, \begin{pmatrix}10001010\\01000101\\00011111\end{pmatrix}, \begin{pmatrix}10000101\\01001111\\00111010\end{pmatrix}, \begin{pmatrix}10000010\\01000001\\00100010\end{pmatrix}, \begin{pmatrix}10001000\\01000100\\00100001\end{pmatrix}, \begin{pmatrix}10001011\\01000110\\00100100\end{pmatrix}, \begin{pmatrix}10001111\\01001010\\00100101\end{pmatrix}, \begin{pmatrix}10000111\\01010011\\00101111\end{pmatrix}, \begin{pmatrix}10001110\\01101000\\00011010\end{pmatrix}, \begin{pmatrix}10001001\\01010100\\00110101\end{pmatrix},$$

$$\begin{pmatrix}10010110\\01000111\\00101100\end{pmatrix}, \begin{pmatrix}10100101\\01001110\\00011000\end{pmatrix}, \begin{pmatrix}10011111\\01001001\\00110100\end{pmatrix}, \begin{pmatrix}10010011\\01100111\\00001011\end{pmatrix}, \begin{pmatrix}10110010\\01011000\\00000110\end{pmatrix}, \begin{pmatrix}10010111\\01100000\\00001110\end{pmatrix}, \begin{pmatrix}10110111\\01010000\\00001001\end{pmatrix}, \begin{pmatrix}10011100\\01100100\\00000010\end{pmatrix}, \begin{pmatrix}10111000\\01011100\\00000001\end{pmatrix}, \begin{pmatrix}10101001\\01110000\\00000111\end{pmatrix},$$

$$\begin{pmatrix}10100001\\01110100\\00001101\end{pmatrix}, \begin{pmatrix}10100100\\01111000\\00000011\end{pmatrix}, \begin{pmatrix}11000111\\00010101\\00001000\end{pmatrix}, \begin{pmatrix}11001010\\00111011\\00000100\end{pmatrix}, \begin{pmatrix}11000101\\00100110\\00001100\end{pmatrix}, \begin{pmatrix}01000000\\00101101\\00011011\end{pmatrix}, \begin{pmatrix}10000000\\00100000\\00010000\end{pmatrix}, \begin{pmatrix}10000000\\00100011\\00010010\end{pmatrix}, \begin{pmatrix}11000000\\00100111\\00011110\end{pmatrix}.$$

$n_2(8,3;6) \geq 46$: $$\begin{pmatrix}01000100\\00101101\\00010110\end{pmatrix}, \begin{pmatrix}01001100\\00100110\\00011011\end{pmatrix}, \begin{pmatrix}01001000\\00101011\\00011101\end{pmatrix}, \begin{pmatrix}10001001\\01000110\\00010101\end{pmatrix}, \begin{pmatrix}10000111\\01001101\\00111111\end{pmatrix}, \begin{pmatrix}10000111\\01010001\\00001010\end{pmatrix}, \begin{pmatrix}10001011\\01110011\\00000101\end{pmatrix}, \begin{pmatrix}10001100\\01010100\\00000011\end{pmatrix},$$

$$\begin{pmatrix}10001000\\01111100\\00000010\end{pmatrix}, \begin{pmatrix}10001110\\01001011\\00101010\end{pmatrix}, \begin{pmatrix}10010000\\01000010\\00100111\end{pmatrix}, \begin{pmatrix}10101110\\01000001\\00011110\end{pmatrix}, \begin{pmatrix}10010110\\01000001\\00101111\end{pmatrix}, \begin{pmatrix}10100111\\01000011\\00011010\end{pmatrix}, \begin{pmatrix}10011000\\01000101\\00100010\end{pmatrix}, \begin{pmatrix}10100101\\01001111\\00010001\end{pmatrix}, \begin{pmatrix}10011111\\01001011\\00100111\end{pmatrix}, \begin{pmatrix}10100100\\01000110\\00011110\end{pmatrix},$$

$$\begin{pmatrix}10011110\\01000010\\00110101\end{pmatrix}, \begin{pmatrix}10011001\\01000011\\00111001\end{pmatrix}, \begin{pmatrix}10011111\\01001010\\00110011\end{pmatrix}, \begin{pmatrix}10011100\\01001101\\00111001\end{pmatrix}, \begin{pmatrix}10010101\\01010100\\00100001\end{pmatrix}, \begin{pmatrix}10101100\\01101111\\00010011\end{pmatrix}, \begin{pmatrix}10010100\\01010110\\00100101\end{pmatrix}, \begin{pmatrix}10100011\\01100010\\00011111\end{pmatrix}, \begin{pmatrix}10010001\\01011000\\00100011\end{pmatrix}, \begin{pmatrix}10100001\\01100110\\00010010\end{pmatrix},$$

$$\begin{pmatrix}10010000\\01011100\\00100001\end{pmatrix}, \begin{pmatrix}10100011\\01101011\\00010011\end{pmatrix}, \begin{pmatrix}01010100\\00100000\\00001000\end{pmatrix}, \begin{pmatrix}10010010\\01010001\\00111010\end{pmatrix}, \begin{pmatrix}10010011\\01011101\\00110001\end{pmatrix}, \begin{pmatrix}10011000\\01011010\\00110010\end{pmatrix}, \begin{pmatrix}10000110\\01100010\\00001111\end{pmatrix}, \begin{pmatrix}10000100\\01101000\\00000001\end{pmatrix}, \begin{pmatrix}10010100\\01110101\\00001111\end{pmatrix}, \begin{pmatrix}10110110\\01100101\\00001010\end{pmatrix},$$

$$\begin{pmatrix}10101000\\01011010\\00000101\end{pmatrix}, \begin{pmatrix}01000000\\00001001\\00000111\end{pmatrix}, \begin{pmatrix}01000000\\00100000\\00010000\end{pmatrix}, \begin{pmatrix}01000000\\00101110\\00011001\end{pmatrix}, \begin{pmatrix}10000000\\00001000\\00000100\end{pmatrix}, \begin{pmatrix}10000000\\00100110\\00011101\end{pmatrix}, \begin{pmatrix}11000000\\00001011\\00000110\end{pmatrix}, \begin{pmatrix}11000000\\00101001\\00010111\end{pmatrix}.$$

$n_2(8,3;7) \geq 55$: $$\begin{pmatrix}01000110\\00101101\\00010111\end{pmatrix}, \begin{pmatrix}11000110\\00100011\\00011001\end{pmatrix}, \begin{pmatrix}01001000\\00101010\\00011011\end{pmatrix}, \begin{pmatrix}11001011\\00100100\\00010101\end{pmatrix}, \begin{pmatrix}01000010\\00110011\\00001010\end{pmatrix}, \begin{pmatrix}11001010\\00100001\\00000101\end{pmatrix}, \begin{pmatrix}01001101\\00111101\\00000010\end{pmatrix}, \begin{pmatrix}11000000\\00101000\\00000001\end{pmatrix},$$

$$\begin{pmatrix}01010111\\00100010\\00001111\end{pmatrix}, \begin{pmatrix}11000101\\00010001\\00001000\end{pmatrix}, \begin{pmatrix}01100000\\00010100\\00001111\end{pmatrix}, \begin{pmatrix}11000010\\00110000\\00001000\end{pmatrix}, \begin{pmatrix}01100101\\00010010\\00001101\end{pmatrix}, \begin{pmatrix}11000101\\00110100\\00001001\end{pmatrix}, \begin{pmatrix}10000110\\00100000\\00010110\end{pmatrix}, \begin{pmatrix}10100000\\01000011\\00010000\end{pmatrix}, \begin{pmatrix}10000011\\00101111\\00011101\end{pmatrix}, \begin{pmatrix}10101111\\01001110\\00011000\end{pmatrix},$$

$$\begin{pmatrix}10001111\\00101011\\00011100\end{pmatrix}, \begin{pmatrix}10100100\\01001000\\00011010\end{pmatrix}, \begin{pmatrix}10001000\\00111001\\00000010\end{pmatrix}, \begin{pmatrix}10101110\\01000100\\00000001\end{pmatrix}, \begin{pmatrix}10011011\\00100001\\00000111\end{pmatrix}, \begin{pmatrix}10100101\\00010011\\00001100\end{pmatrix}, \begin{pmatrix}10011010\\00110000\\00000111\end{pmatrix}, \begin{pmatrix}10110101\\01010101\\00001100\end{pmatrix}, \begin{pmatrix}10100011\\01011011\\00000110\end{pmatrix}, \begin{pmatrix}10000000\\01000111\\00010111\end{pmatrix},$$

$$\begin{pmatrix}10010000\\01000000\\00101100\end{pmatrix}, \begin{pmatrix}10000010\\01001010\\00011100\end{pmatrix}, \begin{pmatrix}10010010\\01000001\\00100101\end{pmatrix}, \begin{pmatrix}10001101\\01001110\\00010010\end{pmatrix}, \begin{pmatrix}10011111\\01001001\\00100111\end{pmatrix}, \begin{pmatrix}10000100\\01011101\\00000011\end{pmatrix}, \begin{pmatrix}10010101\\01000010\\00001110\end{pmatrix}, \begin{pmatrix}10011001\\01011001\\00000011\end{pmatrix}, \begin{pmatrix}10110101\\01100000\\00001110\end{pmatrix}, \begin{pmatrix}10001100\\01000101\\00101110\end{pmatrix},$$

$$\begin{pmatrix}10001111\\01001011\\00101001\end{pmatrix}, \begin{pmatrix}10001010\\01000011\\00111110\end{pmatrix}, \begin{pmatrix}10001001\\01001101\\00110101\end{pmatrix}, \begin{pmatrix}10000011\\01010010\\00100100\end{pmatrix}, \begin{pmatrix}10001110\\01010000\\00101110\end{pmatrix}, \begin{pmatrix}10001011\\01011000\\00100111\end{pmatrix}, \begin{pmatrix}10001110\\01010100\\00111100\end{pmatrix}, \begin{pmatrix}10010101\\01000111\\00110100\end{pmatrix}, \begin{pmatrix}10010001\\01011111\\00101100\end{pmatrix}, \begin{pmatrix}10101001\\01100111\\00010110\end{pmatrix},$$

$$\begin{pmatrix}10010100\\01011011\\00111100\end{pmatrix}, \begin{pmatrix}10000001\\01100001\\00000100\end{pmatrix}, \begin{pmatrix}10000100\\01100110\\00001011\end{pmatrix}, \begin{pmatrix}10000110\\01100011\\00011110\end{pmatrix}, \begin{pmatrix}10000101\\01101010\\00010010\end{pmatrix}, \begin{pmatrix}10010010\\01101001\\00000100\end{pmatrix}, \begin{pmatrix}10101011\\01110000\\00000110\end{pmatrix}.$$

$n_2(8,3;10) \geq 80$: $$\begin{pmatrix}00100110\\00010111\\00001010\end{pmatrix}, \begin{pmatrix}11101010\\00010011\\00000101\end{pmatrix}, \begin{pmatrix}00100001\\00011000\\00000011\end{pmatrix}, \begin{pmatrix}11100101\\00010001\\00001110\end{pmatrix}, \begin{pmatrix}01000111\\00100101\\00001011\end{pmatrix}, \begin{pmatrix}01000010\\00101001\\00000110\end{pmatrix}, \begin{pmatrix}01000011\\00100011\\00011001\end{pmatrix}, \begin{pmatrix}11001011\\00101110\\00011110\end{pmatrix},$$

$$\begin{pmatrix}01001100\\00100011\\00010000\end{pmatrix}, \begin{pmatrix}11001000\\00100110\\00011110\end{pmatrix}, \begin{pmatrix}01001111\\00100010\\00010110\end{pmatrix}, \begin{pmatrix}11001010\\00101000\\00010001\end{pmatrix}, \begin{pmatrix}01001100\\00100100\\00010100\end{pmatrix}, \begin{pmatrix}11000110\\00100110\\00010010\end{pmatrix}, \begin{pmatrix}01001001\\00100111\\00011111\end{pmatrix}, \begin{pmatrix}11001111\\00101011\\00011100\end{pmatrix}, \begin{pmatrix}01011001\\00101010\\00000101\end{pmatrix}, \begin{pmatrix}11000011\\00010101\\00001101\end{pmatrix},$$

$$\begin{pmatrix}10000100\\00101110\\00010010\end{pmatrix}, \begin{pmatrix}10100100\\01000010\\00010111\end{pmatrix}, \begin{pmatrix}10001010\\00101001\\00010001\end{pmatrix}, \begin{pmatrix}10100001\\01000101\\00011011\end{pmatrix}, \begin{pmatrix}10000010\\00110111\\00001011\end{pmatrix}, \begin{pmatrix}10100111\\01000001\\00001010\end{pmatrix}, \begin{pmatrix}10010001\\00100101\\00001111\end{pmatrix}, \begin{pmatrix}10100010\\00010101\\00001000\end{pmatrix}, \begin{pmatrix}10010101\\00110011\\00001111\end{pmatrix}, \begin{pmatrix}10110101\\01010011\\00001000\end{pmatrix},$$

$$\begin{pmatrix}10010100\\00110110\\00001001\end{pmatrix}, \begin{pmatrix}10110010\\01010001\\00001011\end{pmatrix}, \begin{pmatrix}10010001\\00111101\\00000011\end{pmatrix}, \begin{pmatrix}10110001\\01010110\\00001110\end{pmatrix}, \begin{pmatrix}10101000\\01010001\\00000100\end{pmatrix}, \begin{pmatrix}10000111\\01000011\\00001001\end{pmatrix}, \begin{pmatrix}10000100\\01000001\\00001100\end{pmatrix}, \begin{pmatrix}10000101\\01000100\\00011001\end{pmatrix}, \begin{pmatrix}10010100\\01001101\\00100010\end{pmatrix}, \begin{pmatrix}10000110\\01001000\\00011111\end{pmatrix},$$

$$\begin{pmatrix}10011111\\01000011\\00100100\end{pmatrix}, \begin{pmatrix}10001001\\01010100\\00000010\end{pmatrix}, \begin{pmatrix}10011000\\01001010\\00000001\end{pmatrix}, \begin{pmatrix}10010010\\01010100\\00001100\end{pmatrix}, \begin{pmatrix}10110001\\01100011\\00000110\end{pmatrix}, \begin{pmatrix}10010101\\01011001\\00000010\end{pmatrix}, \begin{pmatrix}10111100\\01100110\\00000001\end{pmatrix}, \begin{pmatrix}10011000\\01011010\\00000101\end{pmatrix}, \begin{pmatrix}10110100\\01100001\\00001101\end{pmatrix}, \begin{pmatrix}10000111\\01000110\\00100000\end{pmatrix},$$

$\begin{pmatrix}10000001\\01000100\\00101100\end{pmatrix}, \begin{pmatrix}10000110\\01000110\\00101101\end{pmatrix}, \begin{pmatrix}10000111\\01001000\\00101000\end{pmatrix}, \begin{pmatrix}10001101\\01001110\\00100001\end{pmatrix}, \begin{pmatrix}10000011\\01000010\\00110100\end{pmatrix}, \begin{pmatrix}10001100\\01000000\\00110010\end{pmatrix}, \begin{pmatrix}10001100\\01000101\\00110101\end{pmatrix}, \begin{pmatrix}10001000\\01000111\\00111101\end{pmatrix}, \begin{pmatrix}10001111\\01001111\\00111110\end{pmatrix}, \begin{pmatrix}10001111\\01010010\\00100001\end{pmatrix},$
$\begin{pmatrix}10001000\\01011111\\00101101\end{pmatrix}, \begin{pmatrix}10000000\\01010111\\00110110\end{pmatrix}, \begin{pmatrix}10011100\\01000000\\00111111\end{pmatrix}, \begin{pmatrix}10000000\\01010000\\00111010\end{pmatrix}, \begin{pmatrix}10010000\\01000000\\00110101\end{pmatrix}, \begin{pmatrix}10010000\\01011101\\00100000\end{pmatrix}, \begin{pmatrix}10010110\\01011110\\00100000\end{pmatrix}, \begin{pmatrix}10100000\\01101001\\00010000\end{pmatrix}, \begin{pmatrix}10100011\\01100100\\00010000\end{pmatrix}, \begin{pmatrix}10011011\\01010000\\00111011\end{pmatrix},$
$\begin{pmatrix}10011100\\01011110\\00110010\end{pmatrix}, \begin{pmatrix}10000001\\01100011\\00010011\end{pmatrix}, \begin{pmatrix}10000000\\01100100\\00010011\end{pmatrix}, \begin{pmatrix}10000100\\01101000\\00011010\end{pmatrix}, \begin{pmatrix}10001110\\01100101\\00010100\end{pmatrix}, \begin{pmatrix}10001010\\01101010\\00010111\end{pmatrix}, \begin{pmatrix}10000110\\01110101\\00001101\end{pmatrix}, \begin{pmatrix}10000010\\01111010\\00000111\end{pmatrix}, \begin{pmatrix}10010011\\01100111\\00001001\end{pmatrix}, \begin{pmatrix}10100001\\01110100\\00001010\end{pmatrix},$
$\begin{pmatrix}10101000\\01111011\\00000111\end{pmatrix}, \begin{pmatrix}10101001\\01110011\\00000100\end{pmatrix}.$

$n_2(8,3;19) \geq 155$: $\begin{pmatrix}00100100\\00010011\\00001001\end{pmatrix}, \begin{pmatrix}11100111\\00010010\\00001011\end{pmatrix}, \begin{pmatrix}00101010\\00010010\\00000111\end{pmatrix}, \begin{pmatrix}11100100\\00010101\\00001100\end{pmatrix}, \begin{pmatrix}00110100\\00001100\\00000011\end{pmatrix}, \begin{pmatrix}11100010\\00001000\\00000110\end{pmatrix}, \begin{pmatrix}01000101\\00100100\\00001110\end{pmatrix}, \begin{pmatrix}01000101\\00100110\\00001101\end{pmatrix},$
$\begin{pmatrix}01000000\\00100000\\00010000\end{pmatrix}, \begin{pmatrix}11000000\\00100000\\00010000\end{pmatrix}, \begin{pmatrix}01000110\\00100011\\00011110\end{pmatrix}, \begin{pmatrix}11001010\\00100011\\00011110\end{pmatrix}, \begin{pmatrix}01000101\\00101000\\00010001\end{pmatrix}, \begin{pmatrix}11000110\\00101101\\00010100\end{pmatrix}, \begin{pmatrix}01001001\\00100000\\00010001\end{pmatrix}, \begin{pmatrix}11000100\\00101011\\00010000\end{pmatrix}, \begin{pmatrix}01001101\\00100101\\00010001\end{pmatrix}, \begin{pmatrix}11001011\\00101001\\00011101\end{pmatrix},$
$\begin{pmatrix}01001100\\00100100\\00010010\end{pmatrix}, \begin{pmatrix}11000101\\00100110\\00010010\end{pmatrix}, \begin{pmatrix}01001001\\00101001\\00011011\end{pmatrix}, \begin{pmatrix}11001010\\00101011\\00011011\end{pmatrix}, \begin{pmatrix}01001100\\00101010\\00011100\end{pmatrix}, \begin{pmatrix}11000101\\00100110\\00010101\end{pmatrix}, \begin{pmatrix}01010011\\00100001\\00000100\end{pmatrix}, \begin{pmatrix}11000001\\00011101\\00000010\end{pmatrix}, \begin{pmatrix}01010011\\00110100\\00001011\end{pmatrix}, \begin{pmatrix}11010100\\00110110\\00001010\end{pmatrix},$
$\begin{pmatrix}01010110\\00110010\\00001111\end{pmatrix}, \begin{pmatrix}11010011\\00110010\\00001000\end{pmatrix}, \begin{pmatrix}01010111\\00110101\\00001011\end{pmatrix}, \begin{pmatrix}11010110\\00110001\\00001010\end{pmatrix}, \begin{pmatrix}01101000\\00000110\\00000001\end{pmatrix}, \begin{pmatrix}01110001\\00000100\\00000011\end{pmatrix}, \begin{pmatrix}11000001\\00001100\\00000010\end{pmatrix}, \begin{pmatrix}10000010\\00100101\\00010101\end{pmatrix}, \begin{pmatrix}10100001\\01000001\\00011101\end{pmatrix}, \begin{pmatrix}10000011\\00100111\\00011100\end{pmatrix},$
$\begin{pmatrix}10100100\\01001110\\00011100\end{pmatrix}, \begin{pmatrix}10000100\\00100001\\00011111\end{pmatrix}, \begin{pmatrix}10100101\\01000010\\00011111\end{pmatrix}, \begin{pmatrix}10000111\\00101110\\00010110\end{pmatrix}, \begin{pmatrix}10101000\\01001100\\00010111\end{pmatrix}, \begin{pmatrix}10000010\\00101111\\00011000\end{pmatrix}, \begin{pmatrix}10101101\\01000001\\00011000\end{pmatrix}, \begin{pmatrix}10001000\\00101110\\00010100\end{pmatrix}, \begin{pmatrix}10100001\\01000100\\00010111\end{pmatrix}, \begin{pmatrix}10001110\\00101100\\00010110\end{pmatrix},$
$\begin{pmatrix}10100010\\01000111\\00010110\end{pmatrix}, \begin{pmatrix}10000100\\00110111\\00001111\end{pmatrix}, \begin{pmatrix}10100110\\01000010\\00001000\end{pmatrix}, \begin{pmatrix}10001001\\01001000\\00000011\end{pmatrix}, \begin{pmatrix}10001010\\01001001\\00000101\end{pmatrix}, \begin{pmatrix}10000110\\01000111\\00010011\end{pmatrix}, \begin{pmatrix}10010001\\01000011\\00101100\end{pmatrix}, \begin{pmatrix}10000010\\01000001\\00011000\end{pmatrix}, \begin{pmatrix}10011010\\01000001\\00101111\end{pmatrix}, \begin{pmatrix}10000100\\01000010\\00011111\end{pmatrix},$
$\begin{pmatrix}10011011\\01000010\\00100001\end{pmatrix}, \begin{pmatrix}10000110\\01001111\\00010001\end{pmatrix}, \begin{pmatrix}10010100\\01000011\\00101000\end{pmatrix}, \begin{pmatrix}10000011\\01001110\\00011100\end{pmatrix}, \begin{pmatrix}10011111\\01001110\\00100111\end{pmatrix}, \begin{pmatrix}10001110\\01000111\\00010110\end{pmatrix}, \begin{pmatrix}10011000\\01000111\\00101100\end{pmatrix}, \begin{pmatrix}10001011\\01001011\\00011000\end{pmatrix}, \begin{pmatrix}10010100\\01001010\\00101010\end{pmatrix}, \begin{pmatrix}10000000\\01000000\\00100000\end{pmatrix},$
$\begin{pmatrix}10000101\\01001010\\00100010\end{pmatrix}, \begin{pmatrix}10000111\\01001000\\00100100\end{pmatrix}, \begin{pmatrix}10000111\\01001011\\00100111\end{pmatrix}, \begin{pmatrix}10001101\\01000000\\00100010\end{pmatrix}, \begin{pmatrix}10001100\\01000110\\00100011\end{pmatrix}, \begin{pmatrix}10001010\\01001111\\00100010\end{pmatrix}, \begin{pmatrix}10001101\\01001101\\00101001\end{pmatrix}, \begin{pmatrix}10000000\\01000000\\00110000\end{pmatrix}, \begin{pmatrix}10000000\\01000100\\00110001\end{pmatrix}, \begin{pmatrix}10000001\\01000100\\00111110\end{pmatrix},$
$\begin{pmatrix}10000101\\01001101\\00111101\end{pmatrix}, \begin{pmatrix}10001011\\01000100\\00110101\end{pmatrix}, \begin{pmatrix}10001100\\01000110\\00111101\end{pmatrix}, \begin{pmatrix}10001001\\01001110\\00111001\end{pmatrix}, \begin{pmatrix}10001111\\01001000\\00111001\end{pmatrix}, \begin{pmatrix}10000011\\01010010\\00100111\end{pmatrix}, \begin{pmatrix}10000001\\01011111\\00100010\end{pmatrix}, \begin{pmatrix}10000100\\01011101\\00100001\end{pmatrix}, \begin{pmatrix}10000010\\01011001\\00101111\end{pmatrix}, \begin{pmatrix}10001110\\01010100\\00100110\end{pmatrix},$
$\begin{pmatrix}10001110\\01010001\\00101100\end{pmatrix}, \begin{pmatrix}10001001\\01011111\\00101111\end{pmatrix}, \begin{pmatrix}10000110\\01010001\\00111111\end{pmatrix}, \begin{pmatrix}10011100\\01000011\\00110111\end{pmatrix}, \begin{pmatrix}10000110\\01010110\\00111111\end{pmatrix}, \begin{pmatrix}10010000\\01000011\\00111011\end{pmatrix}, \begin{pmatrix}10000001\\01011100\\00110011\end{pmatrix}, \begin{pmatrix}10011001\\01001111\\00111000\end{pmatrix}, \begin{pmatrix}10000101\\01011101\\00110110\end{pmatrix}, \begin{pmatrix}10010100\\01001101\\00111010\end{pmatrix},$
$\begin{pmatrix}10000001\\01011011\\00111100\end{pmatrix}, \begin{pmatrix}10010101\\01001111\\00111100\end{pmatrix}, \begin{pmatrix}10001010\\01010110\\00110001\end{pmatrix}, \begin{pmatrix}10010110\\01000101\\00111100\end{pmatrix}, \begin{pmatrix}10001101\\01010000\\00110110\end{pmatrix}, \begin{pmatrix}10011001\\01001001\\00110011\end{pmatrix}, \begin{pmatrix}10001011\\01011000\\00110010\end{pmatrix}, \begin{pmatrix}10011110\\01001010\\00110101\end{pmatrix}, \begin{pmatrix}10001100\\01011110\\00110101\end{pmatrix}, \begin{pmatrix}10010001\\01000110\\00111010\end{pmatrix},$
$\begin{pmatrix}10011110\\01011010\\00101110\end{pmatrix}, \begin{pmatrix}10011100\\01011010\\00100011\end{pmatrix}, \begin{pmatrix}10011110\\01010101\\00101101\end{pmatrix}, \begin{pmatrix}10011011\\01010111\\00101001\end{pmatrix}, \begin{pmatrix}10010101\\01011011\\00101000\end{pmatrix}, \begin{pmatrix}10011000\\01011010\\00101110\end{pmatrix}, \begin{pmatrix}10010000\\01010101\\00101011\end{pmatrix}, \begin{pmatrix}10010010\\01011010\\00100101\end{pmatrix}, \begin{pmatrix}10010011\\01011111\\00101000\end{pmatrix}, \begin{pmatrix}10101010\\01101101\\00011010\end{pmatrix},$
$\begin{pmatrix}10101100\\01100100\\00011101\end{pmatrix}, \begin{pmatrix}10101010\\01100110\\00010100\end{pmatrix}, \begin{pmatrix}10101001\\01100111\\00010100\end{pmatrix}, \begin{pmatrix}10100001\\01100110\\00011011\end{pmatrix}, \begin{pmatrix}10101110\\01101010\\00011001\end{pmatrix}, \begin{pmatrix}10101000\\01100011\\00011110\end{pmatrix}, \begin{pmatrix}10100011\\01100001\\00010111\end{pmatrix}, \begin{pmatrix}10100000\\01100010\\00010111\end{pmatrix}, \begin{pmatrix}10010100\\01010101\\00110011\end{pmatrix}, \begin{pmatrix}10010110\\01011111\\00110011\end{pmatrix},$
$\begin{pmatrix}10010101\\01011011\\00111100\end{pmatrix}, \begin{pmatrix}10011001\\01010111\\00110101\end{pmatrix}, \begin{pmatrix}10011101\\01010000\\00110001\end{pmatrix}, \begin{pmatrix}10011101\\01010110\\00110010\end{pmatrix}, \begin{pmatrix}10011001\\01010101\\00111001\end{pmatrix}, \begin{pmatrix}10011101\\01010100\\00111110\end{pmatrix}, \begin{pmatrix}10011101\\01011110\\00111111\end{pmatrix}, \begin{pmatrix}10000000\\01100000\\00010000\end{pmatrix}, \begin{pmatrix}10000101\\01100101\\00011010\end{pmatrix}, \begin{pmatrix}10000011\\01101101\\00010011\end{pmatrix},$
$\begin{pmatrix}10001000\\01100011\\00011001\end{pmatrix}, \begin{pmatrix}10001000\\01100010\\00011010\end{pmatrix}, \begin{pmatrix}10001100\\01100101\\00011110\end{pmatrix}, \begin{pmatrix}10001111\\01101101\\00010011\end{pmatrix}, \begin{pmatrix}10001000\\01101010\\00011011\end{pmatrix}, \begin{pmatrix}10000111\\01110110\\00001010\end{pmatrix}, \begin{pmatrix}10001001\\01110010\\00000110\end{pmatrix}, \begin{pmatrix}10010110\\01101110\\00000001\end{pmatrix}, \begin{pmatrix}10010010\\01110001\\00000101\end{pmatrix}, \begin{pmatrix}10110001\\01110011\\00001101\end{pmatrix},$
$\begin{pmatrix}10010011\\01111001\\00000101\end{pmatrix}, \begin{pmatrix}10110011\\01110101\\00001101\end{pmatrix}, \begin{pmatrix}10011010\\01111011\\00000100\end{pmatrix}, \begin{pmatrix}10110101\\01111101\\00000010\end{pmatrix}, \begin{pmatrix}10100110\\01100001\\00001001\end{pmatrix}, \begin{pmatrix}10100111\\01100011\\00001001\end{pmatrix}, \begin{pmatrix}10101100\\01100100\\00000001\end{pmatrix}.$

$n_2(8,3;20) \geq 164$: $\begin{pmatrix}01000101\\00010000\\00001010\end{pmatrix}, \begin{pmatrix}01001010\\00110000\\00000101\end{pmatrix}, \begin{pmatrix}01000101\\00100000\\00001111\end{pmatrix}, \begin{pmatrix}01000110\\00100010\\00010001\end{pmatrix}, \begin{pmatrix}01001101\\00100010\\00010001\end{pmatrix}, \begin{pmatrix}01000100\\00100110\\00011101\end{pmatrix}, \begin{pmatrix}01001100\\00100110\\00011101\end{pmatrix}, \begin{pmatrix}01000001\\00101111\\00011010\end{pmatrix},$
$\begin{pmatrix}01000011\\00101111\\00011010\end{pmatrix}, \begin{pmatrix}01000010\\00101111\\00011010\end{pmatrix}, \begin{pmatrix}01000101\\00101100\\00011000\end{pmatrix}, \begin{pmatrix}01001111\\00101100\\00011000\end{pmatrix}, \begin{pmatrix}01000100\\00101101\\00011011\end{pmatrix}, \begin{pmatrix}01001100\\00101101\\00011011\end{pmatrix}, \begin{pmatrix}01001011\\00100010\\00010001\end{pmatrix}, \begin{pmatrix}01001000\\00100110\\00011101\end{pmatrix}, \begin{pmatrix}01001000\\00101101\\00011011\end{pmatrix}, \begin{pmatrix}01001010\\00101100\\00011000\end{pmatrix},$
$\begin{pmatrix}10000111\\00100111\\00011110\end{pmatrix}, \begin{pmatrix}10001110\\00100111\\00011110\end{pmatrix}, \begin{pmatrix}10001001\\00100111\\00011110\end{pmatrix}, \begin{pmatrix}10000000\\01000000\\00011110\end{pmatrix}, \begin{pmatrix}10000000\\01000000\\00111001\end{pmatrix}, \begin{pmatrix}10000101\\01000011\\00011011\end{pmatrix}, \begin{pmatrix}10001111\\01000010\\00110110\end{pmatrix}, \begin{pmatrix}10000101\\01001101\\00010101\end{pmatrix}, \begin{pmatrix}10001111\\01001011\\00111111\end{pmatrix}, \begin{pmatrix}10000100\\01001110\\00010111\end{pmatrix},$
$\begin{pmatrix}10001100\\01001001\\00111110\end{pmatrix}, \begin{pmatrix}10001101\\01000011\\00011100\end{pmatrix}, \begin{pmatrix}10001011\\01000010\\00111000\end{pmatrix}, \begin{pmatrix}10001001\\01001110\\00010101\end{pmatrix}, \begin{pmatrix}10000111\\01001001\\00111111\end{pmatrix}, \begin{pmatrix}10000111\\01010001\\00001011\end{pmatrix}, \begin{pmatrix}10001000\\01110011\\00000110\end{pmatrix}, \begin{pmatrix}10000101\\01010010\\00001110\end{pmatrix}, \begin{pmatrix}10000110\\01110001\\00001001\end{pmatrix}, \begin{pmatrix}10000001\\01011100\\00000010\end{pmatrix},$
$\begin{pmatrix}10000010\\01111000\\00000001\end{pmatrix}, \begin{pmatrix}10010001\\01000110\\00001011\end{pmatrix}, \begin{pmatrix}10110011\\01001011\\00000110\end{pmatrix}, \begin{pmatrix}10011001\\01001000\\00000111\end{pmatrix}, \begin{pmatrix}10110111\\01000100\\00001110\end{pmatrix}, \begin{pmatrix}10010011\\01010010\\00001001\end{pmatrix}, \begin{pmatrix}10110010\\01110001\\00000111\end{pmatrix}, \begin{pmatrix}10010010\\01010100\\00001011\end{pmatrix}, \begin{pmatrix}10110001\\01111010\\00000110\end{pmatrix}, \begin{pmatrix}10000000\\01000000\\00100111\end{pmatrix},$
$\begin{pmatrix}10000110\\01000001\\00100100\end{pmatrix}, \begin{pmatrix}10001010\\01000001\\00101101\end{pmatrix}, \begin{pmatrix}10001000\\01000111\\00101001\end{pmatrix}, \begin{pmatrix}10001010\\01000110\\00101010\end{pmatrix}, \begin{pmatrix}10001110\\01000111\\00101010\end{pmatrix}, \begin{pmatrix}10000010\\01010010\\00100000\end{pmatrix}, \begin{pmatrix}10000001\\01100001\\00010000\end{pmatrix}, \begin{pmatrix}10000010\\01010110\\00100000\end{pmatrix}, \begin{pmatrix}10000001\\01101101\\00010000\end{pmatrix}, \begin{pmatrix}10000111\\01010110\\00100011\end{pmatrix},$

$\begin{pmatrix}10001110\\01101111\\00010010\end{pmatrix},\begin{pmatrix}10000101\\01010011\\00101110\end{pmatrix},\begin{pmatrix}10001111\\01101011\\00011001\end{pmatrix},\begin{pmatrix}10001000\\01010100\\00100101\end{pmatrix},\begin{pmatrix}10000100\\01100011\\00011111\end{pmatrix},\begin{pmatrix}10001000\\01010100\\00101000\end{pmatrix},\begin{pmatrix}10000100\\01101000\\00010100\end{pmatrix},\begin{pmatrix}10001011\\01011001\\00100011\end{pmatrix},\begin{pmatrix}10000110\\01100101\\00010010\end{pmatrix},\begin{pmatrix}10000011\\01010011\\00110000\end{pmatrix},$
$\begin{pmatrix}10000011\\01011011\\00110000\end{pmatrix},\begin{pmatrix}10001010\\01010110\\00110111\end{pmatrix},\begin{pmatrix}10001100\\01010010\\00111010\end{pmatrix},\begin{pmatrix}10001100\\01010100\\00111100\end{pmatrix},\begin{pmatrix}10001001\\01011010\\00110001\end{pmatrix},\begin{pmatrix}10001101\\01011111\\00110001\end{pmatrix},\begin{pmatrix}10010100\\01000111\\00100001\end{pmatrix},\begin{pmatrix}10101111\\01001110\\00010011\end{pmatrix},\begin{pmatrix}10010111\\01001001\\00100001\end{pmatrix},\begin{pmatrix}10101101\\01000111\\00010011\end{pmatrix},$
$\begin{pmatrix}10010110\\01001101\\00100010\end{pmatrix},\begin{pmatrix}10101100\\01001011\\00010001\end{pmatrix},\begin{pmatrix}10010010\\01001010\\00101100\end{pmatrix},\begin{pmatrix}10101001\\01000101\\00011000\end{pmatrix},\begin{pmatrix}10011101\\01000010\\00100110\end{pmatrix},\begin{pmatrix}10100110\\01000001\\00011101\end{pmatrix},\begin{pmatrix}10011000\\01001000\\00100011\end{pmatrix},\begin{pmatrix}10100110\\01000100\\00010010\end{pmatrix},\begin{pmatrix}10011100\\01001001\\00100101\end{pmatrix},\begin{pmatrix}10100111\\01000111\\00011111\end{pmatrix},$
$\begin{pmatrix}10011100\\01001000\\00101000\end{pmatrix},\begin{pmatrix}10101100\\01000100\\00010100\end{pmatrix},\begin{pmatrix}10010111\\01001111\\00110100\end{pmatrix},\begin{pmatrix}10011000\\01000110\\00110011\end{pmatrix},\begin{pmatrix}10011101\\01000011\\00111011\end{pmatrix},\begin{pmatrix}10011010\\01001001\\00110010\end{pmatrix},\begin{pmatrix}10011011\\01001110\\00110010\end{pmatrix},\begin{pmatrix}10011101\\01001100\\00110001\end{pmatrix},\begin{pmatrix}10011000\\01001100\\00111100\end{pmatrix},\begin{pmatrix}10011110\\01001110\\00111010\end{pmatrix},$
$\begin{pmatrix}10010110\\01010101\\00101011\end{pmatrix},\begin{pmatrix}10101011\\01101001\\00010110\end{pmatrix},\begin{pmatrix}10010010\\01011010\\00100011\end{pmatrix},\begin{pmatrix}10100011\\01100111\\00010010\end{pmatrix},\begin{pmatrix}10010011\\01011100\\00100001\end{pmatrix},\begin{pmatrix}10100001\\01101011\\00010011\end{pmatrix},\begin{pmatrix}10010100\\01011110\\00101000\end{pmatrix},\begin{pmatrix}10101000\\01101101\\00010100\end{pmatrix},\begin{pmatrix}10011011\\01010000\\00100000\end{pmatrix},\begin{pmatrix}10100110\\01100000\\00010000\end{pmatrix},$
$\begin{pmatrix}10011000\\01010001\\00101000\end{pmatrix},\begin{pmatrix}10100000\\01100111\\00010100\end{pmatrix},\begin{pmatrix}10010011\\01010110\\00110010\end{pmatrix},\begin{pmatrix}10010110\\01010111\\00111101\end{pmatrix},\begin{pmatrix}10010010\\01011110\\00110001\end{pmatrix},\begin{pmatrix}10010000\\01011110\\00111100\end{pmatrix},\begin{pmatrix}10010100\\01011011\\00111100\end{pmatrix},\begin{pmatrix}10011101\\01010000\\00110000\end{pmatrix},\begin{pmatrix}10000001\\01100100\\00000011\end{pmatrix},\begin{pmatrix}10000100\\01100010\\00001101\end{pmatrix},$
$\begin{pmatrix}10001010\\01100011\\00000111\end{pmatrix},\begin{pmatrix}10010111\\01110010\\00001100\end{pmatrix},\begin{pmatrix}10110110\\01100001\\00001000\end{pmatrix},\begin{pmatrix}10011001\\01110100\\00000011\end{pmatrix},\begin{pmatrix}10110101\\01101100\\00000010\end{pmatrix},\begin{pmatrix}10100010\\01000110\\00001101\end{pmatrix},\begin{pmatrix}10100111\\01000101\\00001001\end{pmatrix},\begin{pmatrix}10101001\\01010011\\00000100\end{pmatrix},\begin{pmatrix}10101110\\01011000\\00000001\end{pmatrix},\begin{pmatrix}10100001\\01100011\\00001110\end{pmatrix},$
$\begin{pmatrix}10100011\\01100101\\00001101\end{pmatrix},\begin{pmatrix}11010000\\00001010\\00000101\end{pmatrix},\begin{pmatrix}11110000\\00001010\\00000101\end{pmatrix},\begin{pmatrix}11010011\\00001000\\00000100\end{pmatrix},\begin{pmatrix}11110010\\00001000\\00000100\end{pmatrix},\begin{pmatrix}11000111\\00100100\\00011100\end{pmatrix},\begin{pmatrix}11001110\\00100100\\00011100\end{pmatrix},\begin{pmatrix}11000001\\00101011\\00010110\end{pmatrix},\begin{pmatrix}11000011\\00101011\\00010110\end{pmatrix},\begin{pmatrix}11000010\\00101011\\00010110\end{pmatrix},$
$\begin{pmatrix}11000110\\00101001\\00010111\end{pmatrix},\begin{pmatrix}11001101\\00101001\\00010111\end{pmatrix},\begin{pmatrix}11000001\\00101110\\00011001\end{pmatrix},\begin{pmatrix}11000011\\00101110\\00011001\end{pmatrix},\begin{pmatrix}11000010\\00101110\\00011001\end{pmatrix},\begin{pmatrix}11001001\\00100100\\00011100\end{pmatrix},\begin{pmatrix}11001011\\00101001\\00010111\end{pmatrix},\begin{pmatrix}11100001\\00001000\\00000100\end{pmatrix},\begin{pmatrix}11100000\\00001010\\00000101\end{pmatrix},\begin{pmatrix}01000000\\00000010\\00000001\end{pmatrix},$
$\begin{pmatrix}01000000\\00100001\\00010011\end{pmatrix},\begin{pmatrix}10000000\\00001011\\00000110\end{pmatrix},\begin{pmatrix}11000000\\00000010\\00000001\end{pmatrix},\begin{pmatrix}11000000\\00001001\\00000111\end{pmatrix},\begin{pmatrix}10010101\\01011101\\00111111\end{pmatrix},\begin{pmatrix}10011111\\01010000\\00111010\end{pmatrix},\begin{pmatrix}10010101\\01010001\\00101010\end{pmatrix},\begin{pmatrix}10101010\\01100110\\00010101\end{pmatrix},\begin{pmatrix}10011111\\01010101\\00100101\end{pmatrix},\begin{pmatrix}10100101\\01100000\\00011111\end{pmatrix},$
$\begin{pmatrix}10010101\\01011101\\00111111\end{pmatrix},\begin{pmatrix}10011111\\01010000\\00111010\end{pmatrix},\begin{pmatrix}10010101\\01010001\\00101010\end{pmatrix},\begin{pmatrix}10101010\\01100110\\00010101\end{pmatrix},\begin{pmatrix}10011111\\01010101\\00100101\end{pmatrix},\begin{pmatrix}10100101\\01100000\\00011111\end{pmatrix}.$

$n_2(8,3;22) \geq 180$: $\begin{pmatrix}00000100\\00000010\\00000001\end{pmatrix},\begin{pmatrix}00001100\\00000010\\00000001\end{pmatrix},\begin{pmatrix}00001000\\00000100\\00000010\end{pmatrix},\begin{pmatrix}00001000\\00000100\\00000011\end{pmatrix},\begin{pmatrix}00001000\\00000110\\00000001\end{pmatrix},\begin{pmatrix}00100001\\00011000\\00000101\end{pmatrix},\begin{pmatrix}11100110\\00010010\\00001101\end{pmatrix},\begin{pmatrix}00101010\\00011011\\00000101\end{pmatrix},$
$\begin{pmatrix}11100010\\00010101\\00001101\end{pmatrix},\begin{pmatrix}01000001\\00100011\\00001010\end{pmatrix},\begin{pmatrix}01000010\\00100101\\00001010\end{pmatrix},\begin{pmatrix}01000000\\00100110\\00010101\end{pmatrix},\begin{pmatrix}11001110\\00100000\\00010011\end{pmatrix},\begin{pmatrix}01000011\\00100110\\00011110\end{pmatrix},\begin{pmatrix}11001010\\00101110\\00010011\end{pmatrix},\begin{pmatrix}01000101\\00100100\\00011010\end{pmatrix},\begin{pmatrix}11001010\\00101101\\00010010\end{pmatrix},\begin{pmatrix}01000100\\00101110\\00010011\end{pmatrix},$
$\begin{pmatrix}11001011\\00100010\\00010111\end{pmatrix},\begin{pmatrix}01000100\\00101111\\00010110\end{pmatrix},\begin{pmatrix}11001001\\00100010\\00011000\end{pmatrix},\begin{pmatrix}01001111\\00100001\\00011110\end{pmatrix},\begin{pmatrix}11000000\\00101000\\00011111\end{pmatrix},\begin{pmatrix}01001101\\00100101\\00011001\end{pmatrix},\begin{pmatrix}11001111\\00101001\\00011101\end{pmatrix},\begin{pmatrix}01001000\\00101011\\00010000\end{pmatrix},\begin{pmatrix}11001110\\00100100\\00011010\end{pmatrix},\begin{pmatrix}01001010\\00101011\\00010101\end{pmatrix},$
$\begin{pmatrix}11000010\\00100101\\00011010\end{pmatrix},\begin{pmatrix}01001010\\00101000\\00011101\end{pmatrix},\begin{pmatrix}11001000\\00100101\\00010100\end{pmatrix},\begin{pmatrix}01010110\\00100001\\00001011\end{pmatrix},\begin{pmatrix}11000110\\00010101\\00001010\end{pmatrix},\begin{pmatrix}01010100\\00110111\\00001101\end{pmatrix},\begin{pmatrix}11010010\\00110111\\00001001\end{pmatrix},\begin{pmatrix}01011110\\00110010\\00000001\end{pmatrix},\begin{pmatrix}11010111\\00110110\\00001111\end{pmatrix},\begin{pmatrix}10000010\\00100100\\00010111\end{pmatrix},$
$\begin{pmatrix}10101111\\01000001\\00010010\end{pmatrix},\begin{pmatrix}10000100\\00100011\\00010110\end{pmatrix},\begin{pmatrix}10101111\\01000010\\00011110\end{pmatrix},\begin{pmatrix}10000010\\00100001\\00011001\end{pmatrix},\begin{pmatrix}10100101\\01000001\\00011111\end{pmatrix},\begin{pmatrix}10000011\\00100111\\00011100\end{pmatrix},\begin{pmatrix}10100100\\01001110\\00011100\end{pmatrix},\begin{pmatrix}10000111\\00101001\\00010001\end{pmatrix},\begin{pmatrix}10101000\\01001100\\00011011\end{pmatrix},\begin{pmatrix}10000111\\00101000\\00010100\end{pmatrix},$
$\begin{pmatrix}10101010\\01001100\\00010100\end{pmatrix},\begin{pmatrix}10000110\\00101110\\00010101\end{pmatrix},\begin{pmatrix}10101001\\01000011\\00010111\end{pmatrix},\begin{pmatrix}10001101\\00100000\\00010000\end{pmatrix},\begin{pmatrix}10101001\\01001001\\00010000\end{pmatrix},\begin{pmatrix}10001110\\00101100\\00010110\end{pmatrix},\begin{pmatrix}10100010\\01000111\\00010110\end{pmatrix},\begin{pmatrix}10000100\\00110011\\00001101\end{pmatrix},\begin{pmatrix}10100101\\01000010\\00001001\end{pmatrix},\begin{pmatrix}10010100\\00100011\\00001110\end{pmatrix},$
$\begin{pmatrix}10101011\\00011001\\00000111\end{pmatrix},\begin{pmatrix}10010101\\00100111\\00001111\end{pmatrix},\begin{pmatrix}10100001\\00010100\\00001000\end{pmatrix},\begin{pmatrix}10010111\\00110111\\00001110\end{pmatrix},\begin{pmatrix}10111011\\01010000\\00000111\end{pmatrix},\begin{pmatrix}10101001\\01011011\\00000100\end{pmatrix},\begin{pmatrix}10100001\\01010110\\00001100\end{pmatrix},\begin{pmatrix}10000010\\01000110\\00001011\end{pmatrix},\begin{pmatrix}10000100\\01000001\\00001011\end{pmatrix},\begin{pmatrix}10000101\\01000100\\00010011\end{pmatrix},$
$\begin{pmatrix}10010001\\01001101\\00100010\end{pmatrix},\begin{pmatrix}10000000\\01000000\\00011010\end{pmatrix},\begin{pmatrix}10010101\\01000000\\00100000\end{pmatrix},\begin{pmatrix}10000100\\01000101\\00011000\end{pmatrix},\begin{pmatrix}10011011\\01000010\\00101101\end{pmatrix},\begin{pmatrix}10000001\\01001111\\00010011\end{pmatrix},\begin{pmatrix}10011001\\01001111\\00101000\end{pmatrix},\begin{pmatrix}10000110\\01001011\\00010111\end{pmatrix},\begin{pmatrix}10010101\\01000011\\00101010\end{pmatrix},\begin{pmatrix}10000011\\01001110\\00011100\end{pmatrix},$
$\begin{pmatrix}10011111\\01001110\\00100111\end{pmatrix},\begin{pmatrix}10001110\\01000111\\00010110\end{pmatrix},\begin{pmatrix}10011000\\01000111\\00101100\end{pmatrix},\begin{pmatrix}10001111\\01000011\\00011000\end{pmatrix},\begin{pmatrix}10010010\\01001000\\00101110\end{pmatrix},\begin{pmatrix}10001100\\01000111\\00011111\end{pmatrix},\begin{pmatrix}10010010\\01000110\\00101100\end{pmatrix},\begin{pmatrix}10000001\\01011001\\00000010\end{pmatrix},\begin{pmatrix}10010100\\01001110\\00000001\end{pmatrix},\begin{pmatrix}10001010\\01011001\\00000110\end{pmatrix},$
$\begin{pmatrix}10011101\\01000101\\00000011\end{pmatrix},\begin{pmatrix}10010000\\01010011\\00000110\end{pmatrix},\begin{pmatrix}10110000\\01101101\\00000011\end{pmatrix},\begin{pmatrix}10000001\\01000010\\00100011\end{pmatrix},\begin{pmatrix}10000110\\01001100\\00100011\end{pmatrix},\begin{pmatrix}10000000\\01001100\\00101010\end{pmatrix},\begin{pmatrix}10000101\\01001010\\00101101\end{pmatrix},\begin{pmatrix}10001000\\01000011\\00100110\end{pmatrix},\begin{pmatrix}10001000\\01000001\\00101100\end{pmatrix},\begin{pmatrix}10001011\\01001111\\00100101\end{pmatrix},$
$\begin{pmatrix}10001111\\01001001\\00100000\end{pmatrix},\begin{pmatrix}10001010\\01001000\\00101011\end{pmatrix},\begin{pmatrix}10001011\\01001001\\00101010\end{pmatrix},\begin{pmatrix}10000100\\01000110\\00110000\end{pmatrix},\begin{pmatrix}10000010\\01000111\\00111010\end{pmatrix},\begin{pmatrix}10000111\\01000110\\00111101\end{pmatrix},\begin{pmatrix}10000001\\01001010\\00110010\end{pmatrix},\begin{pmatrix}10000110\\01001100\\00110101\end{pmatrix},\begin{pmatrix}10000111\\01001011\\00111100\end{pmatrix},\begin{pmatrix}10001011\\01000101\\00110100\end{pmatrix},$
$\begin{pmatrix}10001101\\01000000\\00111100\end{pmatrix},\begin{pmatrix}10001101\\01001011\\00111111\end{pmatrix},\begin{pmatrix}10001111\\01001001\\00111101\end{pmatrix},\begin{pmatrix}10000011\\01010010\\00100111\end{pmatrix},\begin{pmatrix}10000001\\01010101\\00100110\end{pmatrix},\begin{pmatrix}10000000\\01011011\\00101011\end{pmatrix},\begin{pmatrix}10000110\\01011000\\00101111\end{pmatrix},\begin{pmatrix}10001000\\01010100\\00100110\end{pmatrix},\begin{pmatrix}10001110\\01010001\\00101100\end{pmatrix},\begin{pmatrix}10001110\\01011000\\00100001\end{pmatrix},$
$\begin{pmatrix}10001001\\01011011\\00101110\end{pmatrix},\begin{pmatrix}10001010\\01011101\\00101111\end{pmatrix},\begin{pmatrix}10000000\\01010000\\00110000\end{pmatrix},\begin{pmatrix}10010000\\01000000\\00110000\end{pmatrix},\begin{pmatrix}10000011\\01010100\\00110101\end{pmatrix},\begin{pmatrix}10011100\\01001110\\00111111\end{pmatrix},\begin{pmatrix}10000101\\01010110\\00110010\end{pmatrix},\begin{pmatrix}10011110\\01001101\\00110010\end{pmatrix},\begin{pmatrix}10000101\\01010010\\00111011\end{pmatrix},\begin{pmatrix}10011100\\01001101\\00111011\end{pmatrix},$
$\begin{pmatrix}10001101\\01010011\\00110001\end{pmatrix},\begin{pmatrix}10010111\\01001001\\00110001\end{pmatrix},\begin{pmatrix}10001000\\01010001\\00111101\end{pmatrix},\begin{pmatrix}10011011\\01000100\\00110110\end{pmatrix},\begin{pmatrix}10001000\\01010101\\00111000\end{pmatrix},\begin{pmatrix}10011001\\01000100\\00111001\end{pmatrix},\begin{pmatrix}10001111\\01011010\\00110110\end{pmatrix},\begin{pmatrix}10011101\\01001000\\00110110\end{pmatrix},\begin{pmatrix}10010000\\01010000\\00100000\end{pmatrix},\begin{pmatrix}10010111\\01010011\\00101011\end{pmatrix},$
$\begin{pmatrix}10011000\\01011001\\00100010\end{pmatrix},\begin{pmatrix}10011001\\01010000\\00100010\end{pmatrix},\begin{pmatrix}10011101\\01011010\\00100100\end{pmatrix},\begin{pmatrix}10011110\\01010110\\00101001\end{pmatrix},\begin{pmatrix}10011111\\01011100\\00101001\end{pmatrix},\begin{pmatrix}10010001\\01011110\\00100111\end{pmatrix},\begin{pmatrix}10100000\\01100000\\00010000\end{pmatrix},\begin{pmatrix}10100110\\01100010\\00011010\end{pmatrix},\begin{pmatrix}10100011\\01101000\\00011100\end{pmatrix},\begin{pmatrix}10101010\\01101011\\00010001\end{pmatrix},$

$\begin{pmatrix}10101100\\01100100\\00011011\end{pmatrix}$, $\begin{pmatrix}10100101\\01101111\\00010001\end{pmatrix}$, $\begin{pmatrix}10101011\\01101100\\00010010\end{pmatrix}$, $\begin{pmatrix}10100011\\01101110\\00011011\end{pmatrix}$, $\begin{pmatrix}10010000\\01010000\\00110000\end{pmatrix}$, $\begin{pmatrix}10010111\\01010011\\00110001\end{pmatrix}$, $\begin{pmatrix}10010000\\01010101\\00111010\end{pmatrix}$, $\begin{pmatrix}10010011\\01011110\\00110111\end{pmatrix}$, $\begin{pmatrix}10011110\\01010110\\00110010\end{pmatrix}$, $\begin{pmatrix}10011101\\01011010\\00110110\end{pmatrix}$, $\begin{pmatrix}10011100\\01011110\\00110100\end{pmatrix}$, $\begin{pmatrix}10011010\\01011101\\00111000\end{pmatrix}$, $\begin{pmatrix}10000000\\01100111\\00010010\end{pmatrix}$, $\begin{pmatrix}10000111\\01101010\\00010001\end{pmatrix}$, $\begin{pmatrix}10001001\\01100111\\00010000\end{pmatrix}$, $\begin{pmatrix}10001100\\01100000\\00010100\end{pmatrix}$, $\begin{pmatrix}10001100\\01100101\\00010111\end{pmatrix}$, $\begin{pmatrix}10001001\\01100001\\00011101\end{pmatrix}$, $\begin{pmatrix}10001011\\01100100\\00011101\end{pmatrix}$, $\begin{pmatrix}10001101\\01100101\\00011001\end{pmatrix}$, $\begin{pmatrix}10001110\\01101011\\00010001\end{pmatrix}$, $\begin{pmatrix}10001010\\01101000\\00011011\end{pmatrix}$, $\begin{pmatrix}10000010\\01110011\\00001001\end{pmatrix}$, $\begin{pmatrix}10000101\\01110001\\00001001\end{pmatrix}$, $\begin{pmatrix}10011011\\01100011\\00000101\end{pmatrix}$, $\begin{pmatrix}10010001\\01110110\\00001011\end{pmatrix}$, $\begin{pmatrix}10110101\\01110110\\00001010\end{pmatrix}$, $\begin{pmatrix}10010100\\01110111\\00001000\end{pmatrix}$, $\begin{pmatrix}10110010\\01111010\\00000100\end{pmatrix}$, $\begin{pmatrix}01000110\\00100000\\00011110\end{pmatrix}$, $\begin{pmatrix}10100001\\01100100\\00000010\end{pmatrix}$, $\begin{pmatrix}10101000\\01101011\\00000101\end{pmatrix}$.

$n_3(5,2;2) \geq 12$: $\binom{00010}{00001}$, $\binom{01001}{00120}$, $\binom{10001}{01020}$, $\binom{10000}{01102}$, $\binom{10012}{01201}$, $\binom{10112}{01002}$, $\binom{10121}{01112}$, $\binom{10222}{01010}$, $\binom{10201}{01110}$, $\binom{10210}{01212}$, $\binom{11010}{00121}$, $\binom{12002}{00110}$.

$n_3(5,2;4) \geq 33$: $\binom{00010}{00001}$, $\binom{01000}{00100}$, $\binom{01011}{00110}$, $\binom{01021}{00122}$, $\binom{10001}{00101}$, $\binom{10000}{00112}$, $\binom{10020}{01002}$, $\binom{10022}{01002}$, $\binom{10020}{01020}$, $\binom{10002}{01120}$, $\binom{10011}{01101}$, $\binom{10021}{01122}$, $\binom{10010}{01210}$, $\binom{10012}{01220}$, $\binom{10101}{01010}$, $\binom{10122}{01011}$, $\binom{10110}{01102}$, $\binom{10111}{01122}$, $\binom{10120}{01111}$, $\binom{10100}{01212}$, $\binom{10121}{01211}$, $\binom{10212}{01001}$, $\binom{10210}{01012}$, $\binom{10211}{01022}$, $\binom{10201}{01102}$, $\binom{10220}{01100}$, $\binom{10222}{01112}$, $\binom{10202}{01222}$, $\binom{10211}{01221}$, $\binom{11002}{00121}$, $\binom{11011}{00120}$, $\binom{12002}{00102}$, $\binom{12001}{00111}$.

$n_3(5,2;5) \geq 41$: $\binom{00010}{00001}$, $\binom{01000}{00100}$, $\binom{01011}{00120}$, $\binom{01012}{00122}$, $\binom{01201}{00010}$, $\binom{10002}{00112}$, $\binom{10021}{00100}$, $\binom{10021}{00121}$, $\binom{10101}{00011}$, $\binom{10000}{01012}$, $\binom{10011}{01011}$, $\binom{10020}{01001}$, $\binom{10000}{01120}$, $\binom{10012}{01110}$, $\binom{10011}{01121}$, $\binom{10022}{01111}$, $\binom{10001}{01222}$, $\binom{10010}{01222}$, $\binom{10101}{01020}$, $\binom{10100}{01102}$, $\binom{10102}{01122}$, $\binom{10122}{01102}$, $\binom{10110}{01201}$, $\binom{10122}{01210}$, $\binom{10210}{01002}$, $\binom{10212}{01021}$, $\binom{10212}{01022}$, $\binom{10221}{01002}$, $\binom{10202}{01112}$, $\binom{10201}{01120}$, $\binom{10211}{01122}$, $\binom{10202}{01200}$, $\binom{10222}{01212}$, $\binom{11000}{00111}$, $\binom{11001}{00122}$, $\binom{11010}{00102}$, $\binom{11012}{00102}$, $\binom{12001}{00101}$, $\binom{12011}{00110}$, $\binom{12012}{00110}$, $\binom{12100}{00001}$.

$n_3(5,2;7) \geq 61$: $\binom{01000}{00110}$, $\binom{01001}{00112}$, $\binom{01010}{00102}$, $\binom{01011}{00112}$, $\binom{01022}{00100}$, $\binom{01210}{00001}$, $\binom{10000}{00101}$, $\binom{10002}{00122}$, $\binom{10011}{00120}$, $\binom{10012}{00120}$, $\binom{10020}{00122}$, $\binom{10200}{00010}$, $\binom{10200}{00012}$, $\binom{10001}{01002}$, $\binom{10002}{01021}$, $\binom{10010}{01011}$, $\binom{10022}{01001}$, $\binom{10020}{01012}$, $\binom{10022}{01021}$, $\binom{10001}{01101}$, $\binom{10010}{01102}$, $\binom{10012}{01102}$, $\binom{10011}{01112}$, $\binom{10000}{01212}$, $\binom{10021}{01201}$, $\binom{10021}{01220}$, $\binom{10100}{01020}$, $\binom{10112}{01020}$, $\binom{10122}{01000}$, $\binom{10110}{01100}$, $\binom{10110}{01121}$, $\binom{10120}{01101}$, $\binom{10120}{01111}$, $\binom{10121}{01111}$, $\binom{10100}{01221}$, $\binom{10111}{01200}$, $\binom{10111}{01201}$, $\binom{10122}{01200}$, $\binom{10211}{01010}$, $\binom{10221}{01012}$, $\binom{10220}{01022}$, $\binom{10201}{01100}$, $\binom{10211}{01120}$, $\binom{10222}{01122}$, $\binom{10202}{01202}$, $\binom{10200}{01210}$, $\binom{10212}{01221}$, $\binom{11000}{00101}$, $\binom{11002}{00110}$, $\binom{11012}{00101}$, $\binom{11010}{00111}$, $\binom{11022}{00102}$, $\binom{11100}{00011}$, $\binom{11200}{00010}$, $\binom{12000}{00100}$, $\binom{12001}{00111}$, $\binom{12012}{00121}$, $\binom{12020}{00121}$, $\binom{12102}{00011}$, $\binom{12102}{00012}$, $\binom{12220}{00001}$.

$n_3(5,2;8) \geq 71$: $\binom{01001}{00111}$, $\binom{01010}{00120}$, $\binom{01021}{00102}$, $\binom{01020}{00112}$, $\binom{01101}{00010}$, $\binom{10000}{00101}$, $\binom{10002}{00102}$, $\binom{10012}{00100}$, $\binom{10010}{00120}$, $\binom{10011}{00122}$, $\binom{10022}{00121}$, $\binom{10102}{00011}$, $\binom{10120}{00001}$, $\binom{10020}{01000}$, $\binom{10020}{01002}$, $\binom{10022}{01000}$, $\binom{10020}{01010}$, $\binom{10000}{01102}$, $\binom{10002}{01101}$, $\binom{10001}{01121}$, $\binom{10011}{01122}$, $\binom{10001}{01212}$, $\binom{10010}{01210}$, $\binom{10012}{01211}$, $\binom{10012}{01221}$, $\binom{10021}{01212}$, $\binom{10022}{01221}$, $\binom{10111}{01011}$, $\binom{10112}{01012}$, $\binom{10120}{01021}$, $\binom{10112}{01100}$, $\binom{10111}{01110}$, $\binom{10120}{01122}$, $\binom{10102}{01211}$, $\binom{10102}{01222}$, $\binom{10110}{01201}$, $\binom{10121}{01210}$, $\binom{10122}{01222}$, $\binom{10201}{01001}$, $\binom{10202}{01000}$, $\binom{10200}{01011}$, $\binom{10202}{01020}$, $\binom{10210}{01022}$, $\binom{10222}{01002}$, $\binom{10200}{01110}$, $\binom{10211}{01100}$, $\binom{10211}{01101}$, $\binom{10210}{01120}$, $\binom{10220}{01112}$, $\binom{10221}{01111}$, $\binom{10201}{01202}$, $\binom{10212}{01202}$, $\binom{10210}{01221}$, $\binom{10221}{01200}$, $\binom{10221}{01210}$, $\binom{11001}{00100}$, $\binom{11012}{00100}$, $\binom{11021}{00101}$, $\binom{11021}{00102}$, $\binom{11020}{00112}$, $\binom{11102}{00011}$, $\binom{11110}{00001}$, $\binom{11201}{00012}$, $\binom{12002}{00012}$, $\binom{12000}{00110}$, $\binom{12001}{00111}$, $\binom{12000}{00121}$, $\binom{12000}{00122}$, $\binom{12012}{00110}$, $\binom{12102}{00012}$, $\binom{12201}{00010}$.

$n_3(5,2;11) \geq 101$: $\binom{01000}{00011}$, $\binom{01001}{00010}$, $\binom{01000}{00112}$, $\binom{01002}{00112}$, $\binom{01010}{00120}$, $\binom{01010}{00122}$, $\binom{01012}{00120}$, $\binom{01012}{00122}$, $\binom{01022}{00100}$, $\binom{01020}{00110}$, $\binom{01020}{00112}$, $\binom{10001}{00012}$, $\binom{10000}{00101}$, $\binom{10001}{00110}$, $\binom{10001}{00121}$, $\binom{10011}{00102}$, $\binom{10012}{00120}$, $\binom{10020}{00120}$, $\binom{10100}{00001}$, $\binom{10102}{00010}$, $\binom{10102}{00012}$, $\binom{10202}{00012}$, $\binom{10002}{01011}$, $\binom{10011}{01000}$, $\binom{10012}{01011}$, $\binom{10010}{01020}$, $\binom{10022}{01002}$, $\binom{10002}{01110}$, $\binom{10000}{01120}$, $\binom{10010}{01111}$, $\binom{10011}{01112}$, $\binom{10010}{01120}$, $\binom{10021}{01101}$, $\binom{10020}{01112}$, $\binom{10021}{01121}$, $\binom{10021}{01122}$, $\binom{10000}{01202}$, $\binom{10002}{01201}$, $\binom{10000}{01221}$, $\binom{10011}{01201}$, $\binom{10021}{01202}$, $\binom{10020}{01211}$, $\binom{10022}{01211}$, $\binom{10100}{01021}$, $\binom{10112}{01001}$, $\binom{10120}{01001}$, $\binom{10121}{01022}$, $\binom{10101}{01101}$, $\binom{10110}{01100}$, $\binom{10111}{01112}$, $\binom{10112}{01122}$, $\binom{10121}{01121}$, $\binom{10100}{01212}$, $\binom{10111}{01201}$, $\binom{10121}{01200}$, $\binom{10122}{01200}$, $\binom{10121}{01212}$, $\binom{10122}{01210}$, $\binom{10200}{01002}$, $\binom{10201}{01012}$, $\binom{10202}{01021}$, $\binom{10212}{01001}$, $\binom{10222}{01012}$, $\binom{10222}{01022}$, $\binom{10202}{01111}$, $\binom{10200}{01122}$, $\binom{10210}{01110}$, $\binom{10211}{01110}$, $\binom{10210}{01120}$, $\binom{10220}{01100}$, $\binom{10220}{01120}$, $\binom{10201}{01200}$, $\binom{10200}{01222}$, $\binom{10210}{01201}$, $\binom{10211}{01211}$, $\binom{10212}{01220}$, $\binom{10221}{01200}$, $\binom{10221}{01210}$, $\binom{10222}{01212}$, $\binom{11002}{00012}$, $\binom{11001}{00101}$, $\binom{11001}{00102}$, $\binom{11001}{00110}$, $\binom{11002}{00110}$, $\binom{11001}{00122}$, $\binom{11011}{00100}$, $\binom{11012}{00100}$, $\binom{11010}{00121}$, $\binom{11021}{00111}$, $\binom{11220}{00001}$, $\binom{12001}{00011}$, $\binom{12002}{00010}$, $\binom{12001}{00101}$, $\binom{12000}{00111}$, $\binom{12011}{00102}$, $\binom{12010}{00112}$, $\binom{12022}{00102}$, $\binom{12021}{00121}$, $\binom{12100}{00001}$, $\binom{12120}{00001}$, $\binom{12201}{00011}$.

$n_3(6,2;8) \geq 66$: $\binom{100011}{001020}, \binom{110012}{000102}, \binom{100001}{001212}, \binom{100121}{010121}, \binom{100120}{001101}, \binom{100012}{010022}, \binom{100122}{001210}, \binom{120012}{000121}, \binom{100222}{001202}, \binom{100202}{010122},$
$\binom{101011}{000122}, \binom{100110}{010200}, \binom{102002}{012210}, \binom{102000}{011221}, \binom{102011}{012200}, \binom{100020}{011012}, \binom{100002}{012111}, \binom{100111}{011121}, \binom{102202}{011021}, \binom{100100}{011222}, \binom{100010}{012120}, \binom{100200}{011001},$
$\binom{100201}{011201}, \binom{102200}{011002}, \binom{100221}{011212}, \binom{101110}{012121}, \binom{100022}{012001}, \binom{101202}{011112}, \binom{101211}{011120}, \binom{100220}{012112}, \binom{101000}{010101}, \binom{101021}{010202}, \binom{101122}{012222}, \binom{101101}{010222},$
$\binom{101102}{012220}, \binom{101221}{010100}, \binom{110001}{001010}, \binom{101022}{011011}, \binom{101200}{011010}, \binom{101111}{011122}, \binom{101220}{012100}, \binom{102012}{012101}, \binom{110000}{001112}, \binom{101201}{011220}, \binom{110002}{001122}, \binom{120201}{001222},$
$\binom{102220}{012020}, \binom{102010}{010221}, \binom{102112}{010120}, \binom{120002}{001012}, \binom{102121}{010201}, \binom{102222}{011202}, \binom{102210}{010001}, \binom{102202}{010111}, \binom{102221}{010211}, \binom{102201}{011200}, \binom{110102}{001120}, \binom{102102}{012122},$
$\binom{102001}{012000}, \binom{110112}{001200}, \binom{120021}{001221}, \binom{110110}{001220}, \binom{111010}{000101}, \binom{121022}{000112}, \binom{121012}{000120}, \binom{122201}{000012}.$

$n_3(6,2;11) \geq 95$: $\binom{010011}{001010}, \binom{012021}{000101}, \binom{010102}{001001}, \binom{010110}{001011}, \binom{010222}{001101}, \binom{100001}{000110}, \binom{102200}{000011}, \binom{100022}{000111}, \binom{100010}{001100},$
$\binom{100220}{001110}, \binom{100021}{010020}, \binom{102221}{010002}, \binom{100002}{010120}, \binom{101111}{010012}, \binom{100100}{010021}, \binom{100010}{012221}, \binom{100102}{010121}, \binom{101121}{012201}, \binom{100121}{010112}, \binom{102201}{011122}, \binom{100110}{010212},$
$\binom{100011}{011102}, \binom{100212}{010001}, \binom{101102}{012222}, \binom{100201}{010202}, \binom{102212}{011101}, \binom{100210}{010200}, \binom{102220}{011221}, \binom{100020}{011200}, \binom{100120}{011001}, \binom{100012}{012022}, \binom{100101}{011211}, \binom{102202}{012010},$
$\binom{100112}{011212}, \binom{101122}{011202}, \binom{100202}{011102}, \binom{101101}{011221}, \binom{100200}{011222}, \binom{101112}{011010}, \binom{100001}{012111}, \binom{102222}{012100}, \binom{100122}{012012}, \binom{101120}{011012}, \binom{100121}{012211}, \binom{102201}{012110},$
$\binom{100222}{012011}, \binom{101100}{012120}, \binom{101012}{010100}, \binom{101212}{010010}, \binom{101022}{010122}, \binom{101210}{011120}, \binom{101001}{010201}, \binom{102022}{012212}, \binom{101000}{010210}, \binom{101021}{010220}, \binom{102021}{010022}, \binom{102000}{011112},$
$\binom{101202}{010001}, \binom{101201}{012222}, \binom{101222}{010101}, \binom{101200}{012202}, \binom{101210}{010221}, \binom{101010}{011002}, \binom{101020}{011210}, \binom{102002}{012011}, \binom{101220}{011201}, \binom{101211}{011011}, \binom{102010}{012020}, \binom{101221}{011000},$
$\binom{102011}{010100}, \binom{101002}{012002}, \binom{101011}{012120}, \binom{102020}{010212}, \binom{101022}{012210}, \binom{102001}{012112}, \binom{102012}{011000}, \binom{102121}{011110}, \binom{102120}{010010}, \binom{102101}{010111}, \binom{102102}{012200}, \binom{102112}{010211},$
$\binom{102111}{011111}, \binom{102122}{011022}, \binom{102100}{012000}, \binom{102112}{011220}, \binom{110202}{001022}, \binom{110221}{001210}, \binom{110212}{001102}, \binom{110211}{001221}, \binom{112002}{000121}, \binom{120101}{001000}, \binom{122002}{000100}, \binom{120202}{001111},$
$\binom{121001}{000010}, \binom{122020}{000001}.$

$n_3(6,2;12) \geq 105$: $\binom{001100}{000012}, \binom{121020}{000110}, \binom{010102}{001012}, \binom{120101}{001212}, \binom{010101}{001102}, \binom{120202}{001120}, \binom{010112}{001122}, \binom{120122}{001202}, \binom{010222}{001020},$
$\binom{122000}{000102}, \binom{010220}{001111}, \binom{120112}{001022}, \binom{011000}{000120}, \binom{011010}{000111}, \binom{120100}{001101}, \binom{012102}{000011}, \binom{120000}{001211}, \binom{100011}{000112}, \binom{101202}{010012}, \binom{100020}{001002}, \binom{101011}{010002},$
$\binom{100001}{001220}, \binom{111001}{000122}, \binom{100022}{001201}, \binom{101211}{010122}, \binom{100110}{001222}, \binom{101012}{010011}, \binom{100200}{001001}, \binom{101102}{010020}, \binom{100212}{001210}, \binom{101021}{000121}, \binom{101020}{010120}, \binom{102010}{000001},$
$\binom{101110}{010201}, \binom{102001}{000100}, \binom{111002}{000010}, \binom{102001}{000112}, \binom{101101}{010211}, \binom{100000}{010111}, \binom{100201}{010000}, \binom{100002}{010221}, \binom{101212}{011022}, \binom{100100}{010001}, \binom{100222}{010010}, \binom{100102}{010021},$
$\binom{101222}{011012}, \binom{100101}{010110}, \binom{110221}{001011}, \binom{100111}{010210}, \binom{110210}{001021}, \binom{100111}{010212}, \binom{100202}{011002}, \binom{110222}{001000}, \binom{100211}{010202}, \binom{100210}{011011}, \binom{100010}{011101}, \binom{100021}{011100},$
$\binom{110100}{001110}, \binom{100012}{011211}, \binom{101210}{011122}, \binom{101202}{012220}, \binom{100201}{011200}, \binom{110102}{001120}, \binom{100112}{012010}, \binom{100121}{012000}, \binom{110000}{001200}, \binom{100101}{012101}, \binom{102102}{012100}, \binom{100122}{012200},$
$\binom{100220}{012212}, \binom{102122}{010022}, \binom{100221}{012222}, \binom{101010}{011102}, \binom{102010}{012122}, \binom{101002}{012122}, \binom{101010}{011020}, \binom{102120}{010101}, \binom{101112}{011022}, \binom{101020}{012012}, \binom{101110}{011102}, \binom{102100}{010111},$
$\binom{101111}{011222}, \binom{101120}{011220}, \binom{101000}{012022}, \binom{101001}{012021}, \binom{102112}{010121}, \binom{102211}{010100}, \binom{102211}{012201}, \binom{102212}{010102}, \binom{101012}{012221}, \binom{101100}{012211}, \binom{101121}{012210}, \binom{110212}{001221},$
$\binom{102022}{010201}, \binom{110011}{001010}, \binom{102220}{010200}, \binom{102000}{011112}, \binom{102002}{011221}, \binom{102101}{011021}, \binom{102221}{012111}, \binom{102210}{011011}, \binom{102200}{011110}, \binom{102012}{012011}, \binom{102020}{012001}, \binom{110012}{000101}.$