Representation of practical nonsmooth control Lyapunov functions by piecewise affine functions and neural networks

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ABSTRACT

In this paper we give conditions under which control Lyapunov functions exist that can be represented by either piecewise affine functions or by neural networks with a suitable number of ReLU layers. The results provide a theoretical foundation for recent computational approaches for computing control Lyapunov functions with optimization-based and machine-learning techniques.

1. Introduction

A control Lyapunov function (CLF) is a powerful device in control theory that provides a systematic method for the design of stabilizing controllers. A CLF is a real-valued function defined on the state space of a (potentially nonlinear) control system that encodes the property that there exists a feedback to stabilize the system to an equilibrium point. This device finds employment in nonlinear and adaptive control, where one finds systematic methods for the design of stabilizing controllers even in the presence of input constraints. The early articles Artstein (1983) and Sontag (1983) on the topic of CLFs were considerably influential and spurred a range of developments, including Sontag (1989); Lin and Sontag (1991), across wide areas of theoretical and applied constructive nonlinear control.

Let us take a brief look at the question of existence of CLFs. For continuous (nonlinear) control systems satisfying mild regularity (Lipschitz growth) conditions, having compact admissible action sets and convex velocity sets, CLFs are known to be intimately connected to the property of their *null controllability* Sontag and Sussmann (1996). The indicated property concerns the existence of controllers that guarantee steering of initial states arbitrarily picked from a domain to the origin over a finite time interval. It is known (Sontag and Sussmann, 1996, Theorem 4.1) (see also (Clarke, 2013, pp. 558-560)) that a control system of the aforementioned kind admits a CLF if and only if it is null controllable, and the minimal-time function (of the initial states) serves as a CLF if the null controllability property holds; moreover, the minimal-time function is continuous if the origin lies in the velocity set at the origin.

While from an engineering viewpoint, continuity of CLFs is a desirable property and numerical methods for the synthesis of CLFs certainly stand to benefit from stronger regularity properties (such as continuous differentiability or smoothness) of candidate CLFs, the points raised in the preceding paragraph indicate that CLFs for continuous control systems could well be nonsmooth. Indeed, as shown in Sontag (1998), nonsmoothness of CLFs may be unavoidable and is linked to topological obstructions that require binary decisions such as whether to move clockwise or counterclockwise. On the other hand, positive results in the direction of structural regularity of CLFs, appearing e.g., in Rifford (2000), provide sufficient conditions for the existence of Lipschitz continuous and semiconcave CLFs. In view of the above facts, algorithmic synthesis of CLFs for nonlinear systems continues to remain a challenging problem from both theoretical and numerical standpoints.

This article makes inroads into the challenging domain of CLF synthesis on both the analysis and synthesis fronts. The following are our key contributions:

(1) We begin by addressing the case of nonsmooth CLFs that are representable as the pointwise minimum of finitely many Lipschitz continuous CLFs. Proposition 9 shows that such a pointwise minimum is itself a CLF on a neighborhood of the origin.

(2) The possibility of approximating CLFs via continuous and piecewise affine (CPwA) functions is investigated next. It is demonstrated that if there exists a semiconcave CLF away from the origin and realized as the pointwise minimum over finitely many C^2 functions, then one can find a continuous piecewise affine CLF away from the origin; this is the content of Theorem 12. The CLF property away from the origin is expressed formally via practical CLFs — they encode the standard properties of CLFs, but only outside a small ball centered at the origin. The omission of a small ball around the origin is part of the approximation.

(3) From a computational standpoint, attention to CPwA functions is desirable because this class of functions

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possesses good numerical properties such as ease of representation and quick computation; we refer the reader to Baier, Braun, Grüne and Kellett (2019) and the references therein for optimization-based computational techniques for control Lyapunov functions using CPWA functions as approximators. Moreover, since each ReLU neural network produces a CPWA function, the inclusion of neural networks as candidate CLFs also benefits from the wide availability of efficient and contemporary computational packages for function approximation using ReLU neural networks; we refer the reader to Grüne (2021); Grüne and Sperl (2023); Sperl, Mysliwitz and Grüne (2024); Gaby, Zhang and Ye (2022); Liu, Meng, Fitzsimmons and Zhou (2023, 2024) for recent work on the synthesis of (control) Lyapunov functions via neural networks. In this connection, Theorem 15 provides structural details of neural networks combining both smooth and ReLU activation functions, that guarantee the existence of a practical CLF under the assumptions of Theorem 12. We draw attention to the at most logarithmic increase in the number of layers of the neural network with the number of C^2 functions in the original CLF.

In summary, the results in this paper justify recent computational approaches for CLFs, because they give conditions under which the approximators used in these approaches can indeed represent CLFs.

This article exposes as follows: Section 2 sets down the setting of CLFs in the context of continuous nonlinear control systems, and Section 3 reviews known and preliminary results. The main results on representation of CLFs by CPWA functions are presented in Section 4 and Section 5 contains the results on representation of CLFs by neural networks. A numerical experiment with the benchmark control system known as Artstein's circles is carried out in Section 6, and we conclude in Section 7 with a discussion of future directions.

2. Setting and preliminaries

We consider control systems of the form

$$\dot{x}(t) = f(x(t), u(t))$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in U \subset \mathbb{R}^m$. We assume that f is continuous and Lipschitz in x uniformly in u. By Carathèodory's Theorem this implies existence and uniqueness of the solutions for each initial condition $x(t_0) = x_0$ and each control input $u \in L^\infty(\mathbb{R}, U)$, see e.g. (Sontag, 1998, Appendix C).

It is well known that smooth control Lyapunov functions do not exist in general. Hence, for their definition we need a weak definition of a directional derivative. The following definition provides the appropriate concept for Lipschitz functions, which is a sufficiently rich class of functions for our purpose in this paper.

Definition 1. *Let $V : O \to \mathbb{R}$ for an open set $O \subset \mathbb{R}^n$ be a Lipschitz function. The lower right Dini derivative of V at a point $x \in O$ in the direction of $w \in \mathbb{R}^n$ is defined as*

$$DV(x; w) := \liminf_{t \searrow 0} \frac{V(x + tw) - V(x)}{t}.$$

The next definition specifies the notion of a control Lyapunov function, going back to Sontag (1983), which we present here in the by now standard form using $\mathcal{K}_\infty$ functions. A control Lyapunov function is always defined with respect to an equilibrium $x^* \in \mathbb{R}^n$ of the system, which we here assume to be the origin, i.e., $x^* = 0$.

Definition 2. *A Lipschitz function $V : O \to \mathbb{R}$ for an open set $O \subset \mathbb{R}^n$ containing 0 is a* control Lyapunov function (CLF)*, if there exist three $\mathcal{K}_\infty$ functions α_i, $i = 1, 2, 3$, such that the inequalities*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \tag{1}$$

$$\inf_{u \in U} DV(x; f(x, u)) \le -\alpha_3(\|x\|) \tag{2}$$

hold for all $x \in O$. In the case of $O = \mathbb{R}^n$, V is called a global CLF.

A global CLF exists if and only if the system can be globally asymptotically controlled to $x^* = 0$, which in turn holds if and only if the system can be globally asymptotically stabilized at $x^* = 0$ in the sample-and-hold sense (for precise statements and definitions of these properties see, e.g., Clarke, Ledyaev, Sontag and Subbotin (1997)).

If either of these properties is not global, one can restrict the definition of a CLF onto a subset of the system's domain of asymptotic controllability. One can even define CLFs on the entire domain of asymptotic controllability D, cf. Camilli, Grüne and Wirth (2008), but then at least one of the inequalities in (1) and (2) must be modified near the boundary of D.

When we want to find CLFs within a specific class of "simple" functions, we may have to exclude those x for which the α_i are close to 0, as in these points even small errors induced by the restriction to a limited class of functions may lead to a violation of the inequalities (1) and (2).

Definition 3. *Given $\varepsilon > 0$, a Lipschitz function $V : O \to \mathbb{R}$ for an open set $O \subset \mathbb{R}^n$ containing 0 is an ε-*practical control Lyapunov function (ε-PCLF)*, if there exist three $\mathcal{K}_\infty$ functions α_i, $i = 1, 2, 3$, such that the inequalities*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \tag{3}$$

$$\inf_{u \in U} DV(x; f(x, u)) \le -\alpha_3(\|x\|) \tag{4}$$

holds for all $x \in O$ with $\|x\| \ge \varepsilon$.

The existence of an ε-PCLF still implies asymptotic controllability and sample-and-hold stabilizability of a neighborhood of 0 whose size is determined by ε, see Kellett and Teel (2004). More precisely, this neighborhood is at most as large as the smallest sublevel set of V that contains the ball $B_\varepsilon(0)$ with radius ε around 0. The radius of this neighborhood can be conservatively estimated by $\alpha_1^{-1}(\alpha_2(\varepsilon))$.

3. Known and preliminary results

Our construction is motivated by the notion of semiconcavity. Here we provide the definition used in Rifford (2000).

Definition 4. *A function $V : O \to \mathbb{R}$ defined on an open set $O \subset \mathbb{R}^n$ is called* semiconcave *if for any point $x_0 \in O$ there exist $\rho, C > 0$ such that*

$$g(x) + g(y) - 2g\left(\frac{x+y}{2}\right) \le C\|x-y\|^2$$

for all $x, y \in B_\rho(x_0) \subset O$.

In the terminology of Cannarsa and Sinestrari (2004) this is a semiconcave function *with linear modulus*, but in order not to overload the terminology we will stick here to the name semiconcave function.

Theorem 5. *Assume that f is locally Lipschitz in x uniformly in u and bounded on $B_r(0) \times U$ for all $r > 0$. Assume furthermore that the system is globally asymptotically controllable to $x^* = 0$. Then there exists a* CLF *that is semiconcave and Lipschitz on $\mathbb{R}^n \setminus \{0\}$.*

This follows from Theorem 1 and 2 in Rifford (2000). Although there the properties of the CLF being Lipschitz and semiconcave are stated separately, the construction in the proof of Theorem 2 in this reference in fact provides a CLF that has both properties at the same time, except possibly at 0.

The crucial property of semiconcave functions that serves as the motivation for our approach is described in the following theorem, which is Theorem 3.4.2 in Cannarsa and Sinestrari (2004) in its version for semiconcave functions with linear modulus.

Theorem 6. *Let $V : O \to \mathbb{R}$ be a semiconcave function on an open set $O \subset \mathbb{R}^n$. Then V can be locally written as the minimum of functions of class C^2. More precisely, for any $K \subset O$ compact, there exists a compact set $S \subset \mathbb{R}^{2n}$ and a continuous function $F : S \times K \to \mathbb{R}$ such that $F(s,\cdot)$ is C^2 for any $s \in S$ with uniformly bounded C^2-norm,*[1] *and*

$$V(x) = \min_{s\in S} F(s,x) \text{ for all } x \in K. \tag{5}$$

Now consider a point $x \in O$ in which two different functions $F(s_1,\cdot) \ne F(s_2,\cdot)$ realize the minimum in any neighborhood $\mathcal{N}$ of x. Then, typically the function V will not be differentiable in x. As discussed in Sontag (1999), such points of nondifferentiability correspond to points in which the stabilizing feedback is discontinuous and a decision between one of two or more possibilities for the directions in which to control the system must be taken. In all examples we were able to find in the literature, the number of points at which this is necessary is limited to a finite number of hypersurfaces, suggesting that V can be written as the minimum over finitely many functions. Hence, even though we are not aware of a theorem that gives rigorous conditions for this fact, it appears that assuming that V can be written as the minimum over finitely many functions captures many if not all cases that are discussed in the literature, including, e.g., nonholonomic systems Sontag (1999).

[1] Recall that the C^2-norm of a twice continuously differentiable function $g : O \to \mathbb{R}$ defined on a nonempty subset $O \subset \mathbb{R}^n$ is $\|g\|_{C^2} := \sup_{x\in O}|g(x)| + \sup_{y\in O}\left\|\frac{\partial g}{\partial x}(y)\right\| + \sup_{z\in O}\left\|\frac{\partial^2 g}{\partial^2 x}(z)\right\|$.

We continue this section by discussing approximations of C^2 functions by piecewise affine functions and by neural networks. The following result is well known, but for convenience of the reader we provide its proof. For its formulation and proof we assume a familiarity with the usual way piecewise affine functions can be expressed as functions over a simplicid grid. Details can be found, e.g., in (Hafstein, 2007, Section 6.1).

Theorem 7. *Let $g : O \to \mathbb{R}$ be a C^2 function on an open set $O \subset \mathbb{R}^n$ with C^2 norm bounded by $C > 0$. Consider a grid of simplices with vertices S_k of maximal diameter $\Delta > 0$, covering a compact set $K \subset O$. Let $C_S > 0$ be such that for each simplex S_k and its vertices $x_{i_1}, \dots, x_{i_{n+1}}$ the matrix*

$$(x_{i_1} - x_{i_2},\ x_{i_2} - x_{i_3},\ \cdots\ x_{i_n} - x_{i_{n+1}}) \in \mathbb{R}^{n\times n} \tag{6}$$

has an inverse with norm bounded by C_S/Δ. Let p be the (unique) continuous and piecewise affine function on the grid with $p(x_i) = g(x_i)$ for all vertices in the grid. Then for all $x \in K$ the inequalities

$$|p(x) - g(x)| \le (C_S + 2)C\Delta^2$$

and

$$\|Dp|_{S_k} - Dg(x)\| \le (C_S + 1)C\Delta$$

hold for all $x \in K$, where in the second inequality S_k is a simplex containing x and $Dp|_{S_k}$ is the derivative of p on S_k.

Proof: Let $K \subset O$ be compact and consider a triangulation of K with simplices of diameter $\le \delta$. Consider the piecewise affine function p uniquely defined by $p(x_i) = g(x_i)$ for all vertices x_i of the grid. Then by the fact that the second derivative of g is bounded, Taylor's theorem yields

$$g(x) = g(y) + Dg(y)(x-y) + R(y)$$

with $|R(y)| \le C\|x-y\|^2$ for a constant C independent of x and y. For p we obtain the same relation with $R(y) = 0$ as long as y and x are contained in the same simplex.

Setting $y = x_i$ and choosing x from a simplex S_k containing x_i as a vertex (which implies $\|x - x_i\| \le \delta$), we obtain

$$\begin{aligned} p(x) - g(x) &= g(x_i) + Dg(x_i)(x-x_i) + R(x_i) \\ &- p(x_i) - Dp|_{S_k}(x-x_i) \\ &= Dg(x_i)(x-x_i) - Dp|_{S_k}(x-x_i) + R(x_i). \end{aligned} \tag{7}$$

If we choose $x = x_j \ne x_i$ to be another vertex in S_k, then this implies

$$0 = p(x_j) - g(x_j) = Dg(x_i)(x_j - x_i) - Dp|_{S_k}(x_j - x_i) + R(x_i).$$

Using (6), we thus obtain

$$\|Dg(x_i) - Dp|_{S_k}\| \le C_S C\Delta,$$

from which, using that Dg is Lipschitz with constant C, we obtain

$$\|Dg(x) - Dp|_{S_k}\| \le (C_S + 1)C\Delta$$

for all $x \in S_k$. Inserting this into (7) we immediately obtain

$$\begin{aligned} &|p(y) - g(y)| \\ \le& \|Dg(x_i)(x - x_i) - Dp|_{S_k}(x - x_i) + R(x_i)\| \\ \le& (C_S + 2)C\Delta^2 \end{aligned}$$

□

The next theorem summarizes universal approximation results for neural networks that are relevant for this paper.

Theorem 8. *Let $g : O \to \mathbb{R}$ be a C^2 function on an open set $O \subset \mathbb{R}^n$ with C^2 norm bounded by $C > 0$. Let $K \subset O$ be compact and $\varepsilon > 0$. Then*

(a) there exists a neural network with ReLU activation functions and at most $\lceil \log_2(n+1) \rceil + 1$ layers such that the function $p : K \to \mathbb{R}$ represented by the neural network satisfies

$$|p(x) - g(x)| \le \varepsilon \quad and \quad \|Dp(x) - Dg(x)\| \le \varepsilon, \quad (8)$$

where the first inequality holds for all $x \in K$ and the second for all $x \in K$ in which p is differentiable;

(b) for any activation function $\sigma \in C^l(\mathbb{R}, \mathbb{R})$, $l \ge 2$ and $0 < \int_{\mathbb{R}} |\sigma^{(l)}(r)| dr < \infty$, there exists a neural network with one hidden layer such that the function $p : K \to \mathbb{R}$ represented by the neural network satisfies (8) for all $x \in K$.

Proof: Statement (a) follows from the fact that Theorem 7 implies the existence of a piecewise affine function satisfying (8). By (Arora, Basu, Mianjy and Mukherjee, 2018, Theorem 2.1) this function can be represented by a deep neural network with ReLU activation functions and at most $\lceil \log_2(n+1) \rceil + 1$ layers. Statement (b) follows from (Hornik, Stinchcombe and White, 1990, Corollary 3.5). □

We end this section with two results on functions satisfying the inequalities (1) and (2) in the CLF definition. The first result shows that a minimum of such functions again satisfies these inequalities.

Proposition 9. *Consider Lipschitz functions $V_i : O_i \to \mathbb{R}$, $i = 1, \dots, q$, with $O_i \subset \mathbb{R}^n$ being open sets. Assume that there are $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such each V_i satisfies (1) and (2) for all $x \in O_i$. Then the function V defined for $x \in O := \bigcup_{i=1}^q O_i$ by*

$$V(x) = \min_{\substack{i=1,\dots,q \\ x \in O_i}} V_i(x)$$

satisfies (1) and (2) for all $x \in O$.

Proof: It is obvious that the inequalities in (1) carry over to the minimum of the V_i. Concerning inequality (2), let $x \in O$ and let V_i be the function at which the minimum in the definition of V is realized in this x, i.e., $V(x) = V_i(x)$. Fix $\varepsilon > 0$ and let $u_\varepsilon \in U$ be a control value that satisfies

$$DV_i(x; f(x, u_\varepsilon)) \le -\alpha_3(\|x\|) + \varepsilon.$$

By the definition of the Dini derivative, this implies that there is a sequence $t_j \searrow 0$ with $x + t_j f(x, u_\varepsilon) \in O_i$ and

$$\lim_{j\to\infty} \frac{V_i(x + t_j f(x, u_\varepsilon)) - V_i(x)}{t_j} \le -\alpha_3(\|x\|) + \varepsilon.$$

From this we conclude that

$$\begin{aligned} DV(x; f(x, u_\varepsilon)) &= \liminf_{t\searrow 0} \frac{V(x + tf(x, u_\varepsilon)) - V(x)}{t} \\ &\le \lim_{j\to\infty} \frac{V(x + t_j f(x, u_\varepsilon)) - V(x)}{t_j} \\ &= \lim_{j\to\infty} \frac{V(x + t_j f(x, u_\varepsilon)) - V_i(x)}{t_j} \\ &\le \lim_{j\to\infty} \frac{V_i(x + t_j f(x, u_\varepsilon)) - V_i(x)}{t_j} \\ &\le -\alpha_3(\|x\|) + \varepsilon. \end{aligned}$$

In turn, this yields

$$\inf_{u\in U} DV(x; f(x, u)) \le -\alpha_3(\|x\|) + \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, the preceding inequality gives us (2). □

The second preparatory result shows that if a function satisfies the inequalities (1) and (2) on a compact set, then it also satisfies these inequalities with an adjusted family of α_i's on a neighborhood of this compact set.

Lemma 10. *Assume that f is locally Lipschitz in x uniformly in u and bounded on $B_r(0) \times U$ for all $r > 0$. Let $i \in \{1, \dots, q\}$, $O_i \subset \mathbb{R}^n$ be open, and consider a C^2 function $V_i : O_i \to \mathbb{R}$ with C^2 norm of V_i on O_i bounded by some $C > 0$. Assume there are $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, Lipschitz with constant L_α, such that V_i satisfies (1) and (2) for all $x \in \operatorname{int} K_i$ for a compact set $K_i \subset O_i$ with $K_i = \operatorname{cl}\operatorname{int} K_i$. Then, given $\varepsilon > 0$, there exists $\delta > 0$, depending only on the bounds and Lipschitz constants of the involved functions, such that V_i satisfies (1) and (2) with $\tilde{\alpha}_1 = \alpha_1/2$, $\tilde{\alpha}_2 = 2\alpha_1$ and $\tilde{\alpha}_3 = \alpha_3/2$ instead of $\alpha_1, \alpha_2, \alpha_3$ on $(\overline{B}_\delta(K_i) \cap O_i) \setminus B_\varepsilon(0)$.*

Proof: By definition of the C^2 norm, we know that V_i and DV_i are bounded and Lipschitz with constant C on O_i. Let L and M be the Lipschitz constant and bound of f on $\overline{B}_1(K_i) \times U$, respectively, and let $\eta := \min\{\alpha_i(r) \mid i = 1, 2, 3, r \ge \varepsilon\}$.

Now consider a point $x \in O_i$ with $x \notin K_i$ and $x \notin B_\varepsilon(0)$. Let $y \in K_i$ be a closest point in K_i and let $d = \|x - y\|$ be the distance of x to y (and hence to K_i). Then since $K_i = \operatorname{cl}\operatorname{int} K_i$ we can estimate

$$V_i(x) \ge V_i(y) - C\|x - y\| \ge \alpha_1(\|y\|) - Cd$$

$$\geq \alpha_1(\|x\|) - (C + L_\alpha)d$$

and

$$V_i(x) \leq V_i(y) + C\|x - y\| \leq \alpha_2(\|y\|) + Cd$$
$$\leq \alpha_2(\|x\|) + (C + L_\alpha)d,$$

which leads to

$$\inf_{u\in U} DV_i(x; f(x,u)) = \inf_{u\in U} DV(x)f(x,u)$$
$$\leq \inf_{u\in U} DV(y)f(y,u) + CMd + CLd$$
$$\leq -\alpha_3(\|y\|) + (CM + CL)d$$
$$\leq -\alpha_3(\|x\|) + (CM + CL + L\alpha)d.$$

Now if we choose δ such that $(C + L_\alpha)\delta \leq \eta/2$ and $(CM + CL + L_\alpha)\delta \leq \eta/2$, then the assertion follows. □

4. Representation by piecewise affine functions

Now we turn to our main result on the representation of CLFs by piecewise affine functions. Before stating our main result, we first show an approximation result for each component V_i of V.

Lemma 11. *Let the assumptions of Lemma 10 hold and assume that $g = V_i$ satisfies the assumptions of Theorem 7 on $O = O_i$. Then, for any compact set $K \subset O$ and all $\nu_1, \nu_2 > 0$ there is a piecewise affine function V_i^p satisfying*

$$\alpha_1(\|x\|) - \nu_1 \leq V_i^p(x) \leq \alpha_2(\|x\|) + \nu_1 \quad (9)$$
$$\inf_{u\in U} DV_i^p(x; f(x,u)) \leq -\alpha_3(\|x\|) + \nu_2 \quad (10)$$

for all $x \in K$.

Proof: First observe that by standard constructions of simplicid grids for any $\Delta > 0$ we can find a grid covering K and satisfying the requirements of Theorem 7. Choosing $V_i^p = p$ from Theorem 7, this implies the first two inequalities with $\nu_1 = (C_s + 2)C\Delta^2$, which can be made arbitrarily small. The Dini derivative of a piecewise affine and continuous function satisfies

$$DV_i^p(x; f(x,u)) = DV_i^p|_{S_k} f(x,u),$$

where $DV_i^p|_{S_k}$ is the derivative of V_i^p on one of the simplices S_k containing x. More precisely, the relevant simplex S_k is the one that also contains $x + hf(x,u)$ for sufficiently small $h > 0$, but this is not relevant here, as the error estimate for the derivative in Theorem 7 holds for all simplices containing x. From this error estimates and denoting by M a bound on $\|f(x,u)\|$, we obtain

$$|DV_i^p(x; f(x,u)) - DV_i(x; f(x,u))|$$
$$= \left|DV_i^p|_{S_k} f(x,u) - DV_i(x)f(x,u)\right|$$
$$\leq M(C_S + 1)C\Delta$$

for all $x \in K$, $u \in U$. This carries over to the minimum over u and thus shows the claim with $\nu_2 = M(C_S + 1)C\Delta$, which can again be made arbitrarily small. □

As discussed after Theorem 5, we now assume that, at least away from the origin, the minimum in Theorem 5 can be realized as a minimum over finitely many functions V_i. For systems admitting such a CLF, the next theorem shows that for each $\varepsilon > 0$ there exists a practical CLF that can be written as the minimum of finitely many piecewise affine functions.

Theorem 12. *Consider an open set O containing the origin and an $\varepsilon > 0$. Assume that there exists a semiconcave* CLF *on O that on $O \setminus B_\varepsilon(0)$ is given by a minimum over finitely many functions, i.e.,*

$$V(x) = \min_{i=1,\ldots,N} V_i(x) \text{ for all } x \in O_\varepsilon := O \setminus B_\varepsilon(0),$$

*with each V_i being C^2. Then for any compact set $K \subset O$ with $K = \operatorname{cl}\operatorname{int} K$ and $\operatorname{cl} B_\varepsilon(0) \subset \operatorname{int} K$ there exists an ε-*PCLF *V^p on K that can be written as the minimum of finitely many piecewise affine functions V_i^p, i.e.,*

$$V_p(x) = \min_{i=1,\ldots,N} V_i^p(x) \text{ for all } x \in K_\varepsilon := K \setminus B_\varepsilon(0).$$

Proof: Consider the sets

$$C_i := \{x \in K_\varepsilon \mid V(x) = V_i(x)\} \text{ and } O_i := \operatorname{int} C_i.$$

Since the V_i are continuous and K_ε is compact, the sets C_i are compact, hence closed. Moreover, $K_\varepsilon \subset \bigcup_{i=1,\ldots,N} C_i$ holds. We claim that

$$K_\varepsilon \subset \bigcup_{i=1,\ldots,N} \operatorname{cl} O_i \quad (11)$$

holds. In order to prove (11), it is sufficient to show that each $x \in \operatorname{int} K_\varepsilon$ is contained in $\operatorname{cl} O_i$ for some i. Hence, consider an arbitrary $x \in \operatorname{int} K_\varepsilon$ and the closed ball $\overline{B}_\delta(x)$ for a sufficiently small $\delta > 0$ such that $\overline{B}_\delta(x) \subset K_\varepsilon$. Define $C_{\delta,i} := C_i \cap \overline{B}_\delta(x)$. Then

$$\overline{B}_\delta(x) \subset \bigcup_{i=1,\ldots,N} C_{\delta,i}. \quad (12)$$

Now, if all closed sets $C_{\delta,i}$ have empty interior, then it follows from Baire's Category Theorem that their union has empty interior, too, but then the inclusion (12) cannot hold. Hence, at least one of the $C_{\delta,i}$ has nonempty interior.

Now consider a sequence $\delta_k \to 0$. Then the argument above implies that there is a sequence of indices i_k and points $x_k \in \operatorname{int} C_{\delta_k,i_k} \subset \operatorname{int} C_{i_k} = O_{i_k}$. Since $\delta_k \to 0$, it follows that $x_k \to x$ as $k \to \infty$. Since the i_k can only assume finitely many different values, there exists a subsequence i_{k_l}, $k_l \to \infty$, such that $i_{k_l} = i'$ for all $l \in \mathbb{N}$. Hence, $x_{k_l} \to x$ as $l \to \infty$ and $x_{k_l} \in O_{i'}$ for all $l \in \mathbb{N}$. Thus, $x \in \operatorname{cl} O_{i'}$ and (11) follows.

Now, by Lemma 3.2 in Calderón and Zygmund (1961), for each $i = 1, \dots, N$, there exists a C^∞ function $f_i : \mathbb{R}^n \to \mathbb{R}$, such that

$$c_1 d(x, \operatorname{cl} O_i) \le f_i(x) \le c_2 d(x, \operatorname{cl} O_i),$$

for constants $c_1, c_2 > 0$, where $d(x, \operatorname{cl} O_i)$ denotes the distance from x to $\operatorname{cl} O_i$. Replacing each V_i by $V_i + f_i$, the assumptions on V and V_i obviously remain true, but now we have the additional property that $V_j(x) \ge V_i(x) + c_1\delta$ for all $x \in O_i \setminus B_\delta(O_j)$. For these modified V_i we now pick $\delta > 0$ from Lemma 10 for $K_i := \operatorname{cl} O_i$. Next, using Theorem 7 we approximate each V_i by a piecewise affine function V_i^p with error in the function values $\le \nu_1 \le \delta/3$ and error in the derivatives $\le \nu_1$, where ν_1 and ν_2 will be determined below. Then it follows that for all $x \notin B_\delta(K_j)$ we have

$$\begin{aligned} V_j^p(x) &\ge V_j(x) - \delta/3 \ge \min_{i=1,\dots,N} V_i(x) + \delta - \delta/3 \\ &\ge \min_{i=1,\dots,N} V_i^p(x) - \delta/3 + \delta - \delta/3 > \min_{i=1,\dots,N} V_i^p(x). \end{aligned}$$

This implies that $V_p^j(x)$ can only attain the minimum $\min_{i=1,\dots,N} V_i^p(x)$ for $x \in O_{\delta,i} := B_\delta(K_j)$. This implies that for all $x \in K$ we obtain

$$\min_{i=1,\dots,N} V^p(x) = \min_{\substack{i=1,\dots,q \\ x \in O_{\delta,i}}} V_i^p(x). \tag{13}$$

By choosing ν_1 and ν_2 sufficiently small (depending on ε), using Lemma 11 we can ensure that V_i^p satisfy (1) and (2) on $O_{\delta,i}$ for suitably adapted $\alpha_1, \alpha_2, \alpha_3$. Now the statement follows from Proposition (9) with $O_{\delta,i}$ in place of O_i, because by (13) the minimum in the assertion coincides with the minimum in Proposition (9). □

Corollary 13. *Under the assumption of Theorem 12, for each compact set $K \subset O$ and each $\varepsilon > 0$ there exists a continuous and piecewise affine ε-PCLF on K.*

Proof: This statement follows immediately from Theorem 12, because the minimum of finitely many continuous and piecewise affine functions is again a continuous and piecewise affine function. □

5. Representation by neural networks

We now turn to the representation of ε-PCLFs via neural networks. We first consider the case of ReLU networks, for which the existence proof works similar to the proof of Theorem 8(a).

Corollary 14. *Under the assumption of Theorem 12, for each compact set $K \subset O$ and each $\varepsilon > 0$ there exists a continuous and piecewise affine ε-PCLF on K that can be represented by a neural network with ReLU activation functions and at most $\lceil \log_2(n+1) \rceil + 1$ layers.*

Proof: Theorem 7 implies the existence of a piecewise affine function satisfying (8). By (Arora et al., 2018, Theorem 2.1) this function can be represented by a deep neural network with ReLU activation functions and at most $\lceil \log_2(n+1) \rceil + 1$ layers. □

While this result is theoretically appealing, its practical relevance may be limited. The reason is that, as worked out in detail in Baier et al. (2019), for checking inequality (2) or (4) for a continuous and piecewise affine function, the points at which the function is not differentiable need to be treated differently depending on their local convexity or concavity: While for points of nondifferentiability x at which V is locally concave it is sufficient to know that (2) or (4) are satisfied in all adjacent regions in which V is smooth, if V is locally convex near x then additional conditions need to be checked (see condition (iv) in Algorithm 2 in Baier et al. (2019) for details). This does not only complicate the construction of a loss function for the training of a neural network, but also requires that sampling points are placed on each edge between two simplices defining the piecewise affine function.

It is therefore desirable to avoid points of nondifferentiability in which the function represented by the neural network is not locally convex. Fortunately, this is possible if we design our function such that it is the minimum of a finite number of smooth functions V_i^p, $i = 1, \dots, N$. This is because in this case points of nondifferentiability x can only occur when the minimum is attained in two different V_i^p in any neighborhood of x and in such points the function must be locally convex. The following theorem and the network construction in its proof show how this can be achieved.

Theorem 15. *Under the assumption of Theorem 12, for each compact set $K \subset O$ and each $\varepsilon > 0$ there exists a continuous ε-PCLF on K, which is the minimum over N twice continuously differentiable functions and can be represented by a neural network with at most $\lceil \log_2(N) \rceil + 1$ hidden layers, of which one uses a smooth activation function as specified in Theorem 8(b) and the remaining layers use ReLU activation functions.*

Proof: We first follow the proof of Theorem 12, replacing the piecewise affine approximations V_i^p provided by Theorem 7 with C^2 approximations V_i^s provided by Theorem 8(b). This results in an ε-PCLF of the form $\min_{i=1,\dots,N} V_i^s$, where each of the V_i^s can be represented by a neural network with one hidden layer. We combine the hidden layers of these N neural networks in the first hidden layer of the network we construct.

Now we observe that the minimum $\min\{x, y\}$ of two reals $x, y \in \mathbb{R}$ can be realized in an NN by a ReLU layer with 4 nodes, since

$$\min\{x, y\} = \frac{1}{2}\Big(\rho(x+y) - \rho(-x-y) - \rho(x-y) - \rho(y-x)\Big)$$

, where $\rho(x) = \max\{x, 0\}$ is the ReLU activation function. Hence, by adding another $\lceil \log_2(N) \rceil$ additional ReLU layers (with at most $2N$, N, $N/2$, … 4 nodes), the network represents the desired function $\min_{i=1,\dots,N} V_i^s$. □

6. Numerical example

We illustrate our numerical findings by the following two-dimensional control system known as Artstein's circles Artstein (1983), whose dynamics is given by

$$\dot{x} = f(x, u) = \begin{pmatrix} (-x_1^2 + x_2^2)u \\ -2x_1 x_2 u \end{pmatrix} \tag{14}$$

with $u \in U = [-1, 1]$. The solutions of this control systems evolve on the circles shown in Figure 1, where u determines whether the solutions move clockwise or counterclockwise.

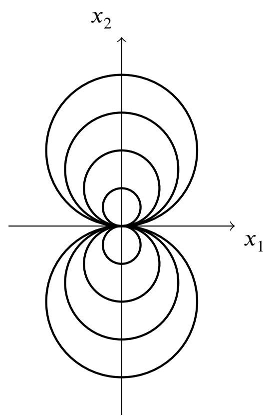

Figure 1: Invariant sets for the solutions of (14)

It is known that this system admits a CLF, but not a smooth one. In fact, in order to asymptotically stabilize the system at 0, at some point on each circle a discontinuous decision to move clockwise or counterclockwise must be taken. A natural choice for the points where the direction of movement changes is the x_2-axis. A known CLF for this system is

$$V(x) = \sqrt{4x_1^2 + 3x_2^2} - |x_1|, \tag{15}$$

in which the nondifferentiability, which corresponds to the change of direction, occurs precisely at $x_1 = 0$, i.e., on the x_2-axis.

Figure 2 shows a piecewise affine PCLF computed with the mixed integer programming based technique proposed in Baier et al. (2019). One clearly sees that also in this CLF the concave "ridge" at which the direction of movement changes (approximately) lies on the x_2-axis.

Figure 3 shows a piecewise smooth PCLF represented by a neural network of the form discussed in Theorem 15 and its proof. The wireframe in the lower part of the figure shows the expression on the left hand side in (4). The nondifferentiability of the function is clearly visible and again lies along the x_2-axis.

The corresponding neural network architecture is depicted in Figure 4. It consists of two shallow subnetworks, each containing $M = 512$ neurons using the sigmoid activation function. These subnetworks are trained in a supervised way such that their outputs W_1 and W_2 satisfy $W_1(x) \approx \sqrt{4x_1^2 + 3x_2^2} - x_1$ and $W_2(x) \approx \sqrt{4x_1^2 + 3x_2^2} + x_1$, respectively. To compute the minimum of the two functions, the neurons $z_1, \ldots, z_4$ use the ReLU activation function together with fixed weights as described in the proof of Theorem 15.

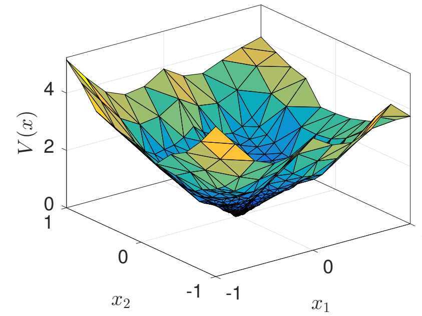

Figure 2: Piecewise affine PCLF for Artstein's circles

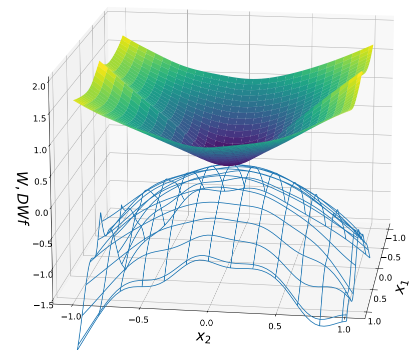

Figure 3: Neural network PCLF for Artstein's circles

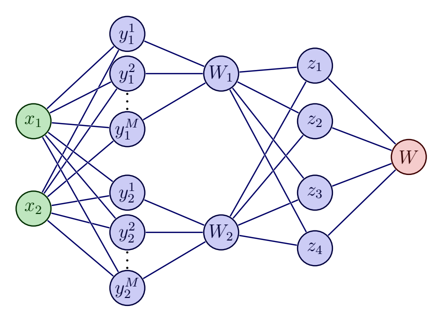

Figure 4: Neural network architecture for representing the PCLF from Figure 3

7. Conclusion and future work

We have shown that nonsmooth CLFs can be approximated by piecewise affine functions and by suitably designed neural networks, provided they can be expressed as the minimum over finitely many smooth functions. Approximation here is to be understood in an ε-practical sense on compact subsets of the state space. These results on the one hand justify the algorithmic approach using piecewise affine functions presented in Baier et al. (2019), because they show that the piecewise affine functions PCLFs constructed in this reference exist. On the other hand, the results yield a neural network architecture that is able to express nonsmooth CLFs. This motivates the development of unsupervised training algorithms that are able to learn nonsmooth CLFs without a priori information on their functional form, which will be an important topic of future research.

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