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General birationality and hyperelliptic theta divisors

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Abstract

We first state a condition ensuring that having a birational map onto the image is an open property for families of irreducible normal non uniruled varieties. We give then some criteria to ensure general birationality for a family of rational maps, via specializations. Among the applications is a new proof of the main result of Catanese and Cesarano (Electron Res Arch 29(6):4315–4325, 2021) that, for a general pair (A, X) of an (ample) Hypersurface X in an Abelian Variety A, the canonical map Φ_X of X is birational onto its image if the polarization given by X is not principal. The proof is also based on a careful study of the Theta divisors of the Jacobians of Hyperelliptic curves, and some related geometrical constructions. We investigate these here also in view of their beauty and of their independent interest, as they lead to a description of the rings of Hyperelliptic theta functions.

Keywords Birational maps · Hypersurfaces in Abelian varieties · Canonical maps · Gauss maps · Theta divisors · Hyperelliptic curves · Graded rings of theta functions

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1 Introduction

One of the main general problems in algebraic geometry is the study of the canonical and pluricanonical maps of varieties of general type, especially the problem of establishing their birationality, see for instance [3, 11–13], just to name a few items.

We describe here a simple and relatively general method for establishing birationality of a rational map for the general variety in a family, via specializations, see Theorems 2.4 and 2.7.

The applications can be many (see [6] in the case of surfaces), but we focus here on the use of this method for the problem which was our original motivation, and we exhibit a new self-contained proof (Theorem 6.2) of the main result of [7]. The present proof couples this method with an interesting study, of the geometry of Hyperelliptic Jacobians and of some of their unramified cyclic coverings.

In the course of doing this we establish some general results on the graded rings of Theta Functions on Hyperelliptic Jacobians, see Theorems 4.3 and 4.6.

2 Openness of birationality and general birationality

As already mentioned, our present main problem is: given a family of varieties $\{X_t\}_{t\in T}$, and a family of morphisms $f_t : X_t \to Y_t$ (respectively, rational maps), when can we conclude, from the fact that f_0 is birational onto its image Y_0, that, for general t, f_t is birational onto Y_t?

Let us start with a negative example: let X be a hypersurface in $\mathbb{P}^N$ of degree d, let P be a point, $P \in \mathbb{P}^N$, and consider the projection with centre P, $\pi_P : \mathbb{P}^N \setminus P \to \mathbb{P}^{N-1}$.

If the hypersurface X_0 has multiplicity $d-1$ at the point P, then π_P induces a birational map between X_0 and $\mathbb{P}^{N-1}$, but for a general X the projection is not birational, having degree equal to $d - mult_P(X)$, which is ≥ 2 as soon as $mult_P(X) < d-1$.

The important feature of this example, which motivates the assumption in the following theorems, is that X_0 is a uniruled variety, indeed it is a rational variety: and this must be avoided.

The next example, instead, clarifies the hypotheses needed for the validity of an assertion made in the first version of this paper (see for instance the next Proposition 2.2).

Example 2.1 Consider in $\mathbb{P}^N \times \mathbb{P}^1$ the following family

$$\mathcal{X} := \{(x, (\lambda_0, \lambda_1)) | \lambda_0^m f(x) + \lambda_1^m g(x) = 0\},$$

where the Hypersurfaces $X_0 := \{f(x) = 0\}$ and $X_\infty := \{g(x) = 0\}$ intersect transversally, X_∞ is smooth, while X_0 has only one isolated singular point P of multiplicity m, and is of general type if $d := \deg(f) = \deg(g) \geq N + 2 + m$.

An elementary calculation shows that $Sing(\mathcal{X}) = \{(P, (1, 0)\}$, a point of multiplicity equal to m.

Hence X_0, $\mathcal{X}$ are normal (being hypersurfaces in a smooth manifold).

Blowing up the only singular point, we get

$$\mathcal{Z} \to \mathcal{X} \subset \mathbb{P}^N \times \mathbb{P}^1, \; p : \mathcal{Z} \to \mathbb{P}^1,$$

and the fibre Z_0 consists of the union of the blow up X_0' of X_0 in P, together with the hypersurface Z_0' in the exceptional $\mathbb{P}^N$,

$$Z_0' := \{\phi_m(x_1, \dots, x_n) + \lambda^m g(P) = 0\},$$

where we assume that $P = (1, 0, \dots, 0)$ and that ϕ is the leading term of the Taylor development of f at P.

For $m \geq N + 2$ and ϕ general, Z_0' is a smooth variety with ample canonical system, and $X_0' \cap Z_0' = \{\phi_m(x_1, \dots, x_n) = 0\}$, the exceptional divisor of $\pi : X_0' \to X_0$.

The following Proposition is a direct consequence of Hironaka's II Main Theorem in [15]

Proposition 2.2 *Assume that we have a 1-dimensional projective family $p : \mathcal{S} \to T$ where $\mathcal{S}$ is smooth of dimension $n + 1$, T is a smooth connected curve, $0 \in T$, and we have a rational map*

$$f : \mathcal{S} \dashrightarrow \mathbb{P}^N.$$

Then there exists a modification $\pi : \mathcal{Z} \to \mathcal{S}$ such that, setting

$$F := f \times p : \mathcal{S} \dashrightarrow \mathbb{P}^N \times T,$$

and denoting by $\mathcal{Y}$ the closure of the image of F,

(i) $\mathcal{Z}$ is smooth,
(ii) $F' := F \circ \pi$ becomes a morphism $F' : \mathcal{Z} \to \mathcal{Y}$,
(iii) all the fibres of $p' := p \circ \pi : \mathcal{Z} \to T$ consist of the union of the strict transform S_t' of $S_t := p^{-1}(t)$ with other ruled components.
(iv) In particular, if the indeterminacy locus of f is contained in S_0, then $S_t' = S_t$ for $t \neq 0$.
(v) It follows that, if $\Gamma \to \mathcal{S} \times \mathbb{P}^N \times T$ is the normalization of the graph of F, then the fibres Γ_t consist of the strict transform of S_t plus some uniruled components.
(vi) Shrinking T, we may assume that in (iii) and (v) other uniruled components only occur for $t = 0$.

Proof A preliminary observation is that, since $\dim(\mathcal{S}) = n + 1$, the indeterminacy locus of F does not contain any fibre S_t.

Assertions (i) and (ii) follow from Hironaka's II Main Theorem of [15] (see especially page 140, and the affirmative answer to Question (F), (iii), the assertion that f_r is a morphism, and that the centres D_i are smooth and contained in the indeterminacy locus of f_i) ensuring that, via a sequence of blow ups with smooth centres, we get $\pi : \mathcal{Z} \to \mathcal{S}$ such that the rational map $F' := F \circ \pi$ becomes a morphism on $\mathcal{Z}$.

For (iii) we just need to observe that, if we blow up a submanifold W of a manifold M, then the blow up $\tilde{M}$ contains as exceptional divisor the ruled manifold $\mathbb{P}(N_{W|M})$. Hence the exceptional divisors are all ruled, hence so are the new irreducible components of the fibres of p' (as they are divisors in $\mathcal{Z}$ by our assumptions).

(iv) follows since the centres of the blow up are contained in the inverse image of the indeterminacy locus in $\mathcal{S}$.

(v): since $\mathcal{Z}_t$ surjects onto Γ_t, the other components of Γ_t are images of a ruled manifold, hence they are uniruled.

(vi) first of all, the set of $t \in T$ such that S_t is not irreducible is closed; furthermore, since there is only a finite number of exceptional divisors, there is only a finite number of t such that the fibres $\mathcal{Z}_t$ and Γ_t are not irreducible. So we omit these two finite subsets of T. □

Remark 2.3 In view of Hironaka's extension [16] of the resolution results to complex spaces, one can replace the hypothesis that we have a projective family by the hypothesis that we have a proper family.

Theorem 2.4 **(Openness of birationality)** *Let $p : \mathcal{X} \to T$ be a 1-dimensional (flat) family of reduced projective subschemes of dimension n (i.e., $\mathcal{X}$ is irreducible and T is a smooth connected curve, $0 \in T$) such that $X_0 = p^{-1}(0)$ contains a unique irreducible component X_0'' which is not uniruled.*

Let $f : \mathcal{X} \dashrightarrow \mathbb{P}^N$ be a rational map such that $f_0 : X_0'' \dashrightarrow Y_0'$ is birational to its image. Assume moreover

(**) setting $F := f \times p : \mathcal{X} \dashrightarrow \mathbb{P}^N \times T$, letting $\mathcal{Y}$ be the closure of the image of F, and letting Γ be the normalization of the graph of F, then the fibres Γ_t are irreducible for $t \neq 0$, while Γ_0 consists of the strict transform of X_0 plus some uniruled components.

Then $f_t : X_t \to Y_t$ is birational to its image for all t in a neighbourhood of $0 \in T$.

Proof Clearly $\mathcal{Y}$ is irreducible and it has dimension $n+1$ since its fibre Y_0 over 0 contains Y_0' which has dimension n; the fibre Y_t over $t \neq 0$ contains the image Y_t' of X_t which by assumption is irreducible.

The rational map F induces a surjective morphism $F' : \Gamma \to \mathcal{Y}$.

Γ is irreducible of dimension $n+1$, and the central image Y_0 is the image of Γ_0 under a proper map, and contains Y_0' as a component, since the strict transform X_0' of X_0'' is a component of Γ_0.

The other components of Γ_0 are uniruled, hence they cannot dominate the component Y_0', which is not uniruled.

Hence the general point $y \in Y_0'$ is in the image of only one point x, this point x lies in X_0', and the map F' is of maximal rank in x, hence a formal isomorphism with its image: because f_0 is a local isomorphism and p is a submersion at x (in particular there is no ramification of F' at x).

Consider now a local holomorphic section Σ of $\mathcal{Y} \to T$ passing through y (which is a smooth point of $\mathcal{Y}$ and of the fibre Y_0, since F' and p are local submersions at x).

If the map f_t were non birational for all t, then $f_t' : X_t' \to Y_t'$ would have positive degree, and would be étale outside of a branch locus $\mathcal{B}_t \subset Y_t'$.

We have seen that if $y \in Y_0'$ is chosen general, it is not contained in the closure $\mathcal{B}$ of the branch loci: since there is no ramification at x.

Therefore the inverse image of Σ consists of holomorphic arcs, in a number strictly greater than one, of which only one contains x in its closure, while the other arcs tend to a point z in Γ_0 different from x.

The conclusion is that $F'(z) = y, z \neq x$, and we have reached the desired contradiction: hence we have proven that f_t is birational. □

Remark 2.5 The above Theorem and the following ones can be stated in more general situations.

(i) We can consider more generally[1] a family $\mathcal{X} \to T$ of compact complex spaces, and a meromorphic map $f : \mathcal{X} \dashrightarrow M$, where M is a complex manifold: the above proof works without any change.

(ii) The same theorem is true for a projective family over an algebraically closed field of any characteristic, if we assume that f_0 is separable and birational on X_0''.

We have in fact that $F' : \Gamma \to \mathcal{Y}$ proper, hence there is a closed set $B \subset \mathcal{Y}$ with nontrivial complement

$\mathcal{Y} \setminus B$, such that, over $\mathcal{Y} \setminus B$, F' is finite with all the fibres of cardinality d. If f_t is not birational, then $d \geq 2$. Since we have shown that $y \notin B$, and that $F'^{-1}(y)$ is a single point with multiplicity 1, it follows then that $d = 1$, a contradiction.

[1] Thanks to Thomas Peternell for asking this question.

(ii) The theorem was applied as self evident in the case of canonical maps of algebraic surfaces in [6], but its use was criticized as non self evident in [18]. All details of the proof are now following from Proposition 2.2, (v), applied to the family of smooth minimal models of surfaces of general type, and from Theorem 2.4.

Before moving to a more general Theorem, we need to state a simple group theoretical result.

Lemma 2.6 *Given finite groups $\Pi_X < \Pi' < M_0$, where the maximal normal subgroup of M_0 contained in Π_X is the identity, let us set:*

(1) $M_0^\nu := M_0/K$, where K is the maximal normal subgroup contained in Π', so that
(2) M_0 acts faithfully on the coset space $\mathcal{F}_X := M_0/\Pi_X$, whose cardinality will be denoted by d,
(3) M_0^ν acts on the coset space $\mathcal{F}^\nu := M_0/\Pi'$, whose cardinality will be denoted by m,
(4) $M_t := \Pi'/K_t$ acts on $\mathcal{F}_t := \Pi'/\Pi_X$, where K_t is the largest normal subgroup of Π' contained in Π_X.
Then $d = \delta m$, where δ is the cardinality of the set $\mathcal{F}_t = \Pi'/\Pi_X$.
And the action of M_0 preserves the blocks corresponding to the m elements of $\mathcal{F}^\nu$,

$$\mathcal{F}_X = \cup_{[c]\in\mathcal{F}^\nu} c\Pi'/\Pi_X.$$

Hence we have exact sequences

$$1 \to K \to M_0 \to M_0^\nu \to 1,$$

$$1 \to K_t \to \Pi' \to M_t \to 1,$$

and, setting $G := \Pi'/K$,

$$G < M_0^\nu.$$

.

With a similar proof to Theorem 2.4, we obtain the following more general result which is useful for applications.

Theorem 2.7 *Let $p : \mathcal{X} \to T$ be a 1-dimensional (flat) family of projective varieties of dimension n, with $\mathcal{X}$ irreducible, T a smooth connected curve, $0 \in T$, such that $X_0 = p^{-1}(0)$ is irreducible normal.*

Let $f : \mathcal{X} \dashrightarrow \mathbb{P}^N$ be a rational map such that $f_0 : X_0 \dashrightarrow Y_0'$ is of degree d to its image Y_0', which is not uniruled.

Assume moreover

(**) setting $F := f \times p : \mathcal{X} \dashrightarrow \mathbb{P}^N \times T$, letting $\mathcal{Y}$ be the closure of the image of F, and letting Γ be the normalization of the graph of F, then the fibres Γ_t are irreducible for $t \neq 0$, while Γ_0 consists of the strict transform of X_0 plus some uniruled components.

Then

(i) $f_t : X_t \to Y_t$ has degree δ onto its image for all t in a neighbourhood of $0 \in T$, with δ dividing d, so that we may write $d = m\delta$.
More precisely, $f_0 : X_0 \dashrightarrow Y_0'$ admits a factorization as $\nu_0 \circ F_0''$, where ν_0 has degree m, and the monodromy group $M_0 \subset \mathfrak{S}_d$ of f_0 is thus related to the monodromy group of f_t, $M_t \subset \mathfrak{S}_\delta$, and the monodromy group $M_0^\nu \subset \mathfrak{S}_m$ of ν_0, as in the statement of Lemma 2.6.

(ii) *In particular, if the monodromy group $M_0 \subset \mathfrak{S}_d$ is primitive (that is, there is no nontrivial partition of $\{1, \dots, d\}$ which is M_0-invariant) then either the general f_t is birational ($m = d$) or it has degree $\delta = d$ ($m = 1$).*

Proof Using the same notation as in the proof of Theorem 2.4, we are then in a similar situation.

The general point $y \in Y_0$ is in the image of exactly d smooth points $x_1, \dots, x_d$ of Γ which lie in X'_0, and the map F' is of maximal rank in each x_i.

What may now change is that y could be contained in the singular locus of $\mathcal{Y}$, and there may be m smooth branches of $\mathcal{Y}$ passing through y.

Therefore, we take the normalization $\nu : \mathcal{Y}^n \to \mathcal{Y}$, and notice that we have a factorization of F' as $\nu \circ F''$, where $F'' : \Gamma \to \mathcal{Y}^n$. We observe then that the morphism $F''|X'_0$ will have degree δ onto its image, where $d = \delta m$.

Hence the d points are grouped in m subsets, corresponding to the inverse images of the points $y_1, \dots, y_m$ lying over y in $\mathcal{Y}^n$, and the previous argument using the local holomorphic sections Γ_i of $\mathcal{Y}^n \to T$ passing through y_i for $i = 1, \dots, m$ shows that the degree of $f_t : X_t \to Y_t$ equals δ.

Assertion ii) follows right away because, if $m \neq 1, d$, then there is a partition of $\{1, \dots, d\}$ in m subsets which are permuted by M_0.

Now, the monodromy of $f_t : X_t \dashrightarrow Y_t$ will be the same as the one of $F'' : X'_0 \to Y^n_0$, and since $f_0 : X'_0 \dashrightarrow Y_0$ is a composition, it follows that the monodromy of f_0 is as claimed, in view of the previous Lemma 2.6, where we divide the respective fundamental groups by the largest normal subgroup of the fundamental group of the open set of X'_0 where all coverings are unramified, so that M_0 and M_t are the monodromy groups we are talking about. □

Corollary 2.8 *Let $p : \mathcal{X} \to T$ be a family of projective varieties of dimension n, where T is smooth and connected. Assume moreover that we are given a rational map $f : \mathcal{X} \dashrightarrow \mathbb{P}^N$ which is a morphism for $t \in V$, where V is an open set $V \subset T$.*

(I) *Assume that for a general point $t \in T$ there are several 1-parameter specializations, for $j = 1, \dots, r$, with base T_j containing t and $t_j \in T$, of the fibre $X_t = p^{-1}(t)$ to the fibre X_{t_j}. Assume that these are, as in Theorem 2.7, such that X_{t_j} is irreducible and normal with monodromy in $\mathfrak{S}_{d(t_j)}$, and that moreover, writing $d_j := d(t_j)$, we have*

$$GCD\{d_j | j = 1, \dots, r\} = 1.$$

Then, for general t, f_t is birational.

(II) *The same conclusion holds if there are two 1-parameter specializations, one such that the monodromy group $M_0 \subset \mathfrak{S}_{d_1}$ is primitive, the other such that d_1 does not divide d_2.*

Proof We denote as above by δ the degree of f_t for general t.

Our claim(I), in the notation of Theorem 2.7, is that $\delta = 1$, which is obvious since, by (i) of theorem 2.7, $d_j = \delta m(t_j)$ therefore δ divides all the integers d_j, hence their GCD.

To show (II), simply apply (i) and (ii) of Theorem 2.7: in fact the general degree δ is either 1 or d_1 by virtue of (ii), while (i) implies that $\delta | d_2$. □

Remark 2.9 One can obtain other more complicated criteria using the above exact sequences of groups.

But, if M_0 is Abelian, then $\Pi_X = 0$, $K = \Pi'$, $K_t = 0$, hence $M_t = \Pi'$.

If all specializations found yield a group $M_{t_0(j)}$ which is Abelian, then a criterion of triviality of M_t follows from a criterion similar to the above Corollary, analyzing the primary decompositions of all the groups $M_{t_0(j)}$.

If we get one specialization such that one M_{t_0} is Abelian, then M_t is Abelian, and is, for any other specialization, a quotient of the Abelianization of Π' by the image of Π_X.

Remark 2.10 The main conjecture raised in [7] is that the canonical map of a general pair (A, X) of an ample hypersurface in an Abelian variety is an embedding if the Pfaffian of the Polarization given by X is at least $dim(X) + 2$.

Also for this purpose it would be useful to establish in a similar way some criteria guaranteeing 'general embedding', that is, embedding for a general variety in a family.

3 Theta divisors of Hyperelliptic curves

We begin with a quite elementary result in group theory.

Lemma 3.1 *Consider the* **Group** G **of the Hypercube***, namely the natural semidirect product (induced by coordinates permutation)*

$$G := (\mathbb{Z}/2)^n \rtimes \mathfrak{S}_n =: K \rtimes H.$$

Then

(i) the only intermediate subgroups H', with $H < H' < G$, and different from H, G, are just two subgroups H_1, H_2, of respective indices

$$[H_1 : H] = 2, [G : H_2] = 2.$$

(ii) the largest subgroup $H'' < H$ which is normal in G is the identity subgroup.

Proof For $v \in K, \sigma \in H$, we write $\sigma(v) := \sigma v \sigma^{-1}$.

For instance, $\sigma(e_i) = e_{\sigma(i)}$.

If H' is as in (i) and $H' \neq H$, then $H' \cap K =: V$ is then an H-invariant subspace. And conversely, if V is H-invariant, then VH is a subgroup, because

$$v_1 \sigma v_2 \tau = v_1 \sigma(v_2) \sigma \tau.$$

Then assertion (i) follows from the

Claim: *The only $\mathfrak{S}_n$-invariant subspaces of K are:*

$$\{0\}, K, (\mathbb{Z}/2)e, e^{\perp}, \text{ where } e := \sum_1^n e_i.$$

Proof of the claim: it is obvious that the four above subspaces are invariant.

For such an invariant subspace V, assuming that $V \neq 0$, consider a vector v of minimal weight $w(v) := |\{i | v_i \neq 0\}|$. Denote by w the minimal weight: if $w = 1$, then $\exists i$ such that $e_i \in V \Rightarrow V = K$.

Otherwise, we may assume, after a basis change, that $v = e_1 + \cdots + e_w$.

If $w = n$, we get that $V = (\mathbb{Z}/2)e$. If instead $w < n$, then there is a σ such that $\sigma(v) = e_2 + \cdots + e_{w+1}$, hence $v + \sigma(v) = e_1 + e_{w+1}$, hence $w = 2$.

Then $e_1 + e_2, e_2 + e_3, \ldots, e_{n-1} + e_n \in V$, hence V is an invariant hyperplane. Hence V is orthogonal to a projectively invariant vector, and we see that $V = e^{\perp}$.

Passing to (ii),

$$H'' = \cap_{v \in K} v H v^{-1} = \cap_{v \in K} [H \cap v H v^{-1}].$$

Now,

$$\sigma \in H \cap vHv^{-1} \Leftrightarrow \exists \tau \in H, \sigma = v\tau v^{-1} \Leftrightarrow \exists \tau \in H, \sigma\tau^{-1} = v\tau(v^{-1}).$$

Since the last equality is between an element of H and one of K, this means that $\sigma = \tau$ and $v = \sigma(v)$; if this is to hold for each $v \in K$, then σ is the identity. □

We now come to an important geometrical occurrence of the group G of the Hypercube.

Let C be a Hyperelliptic curve of genus g, and let $\psi : C \to \mathbb{P}^1$ be the canonical double cover (such that the canonical map ϕ of C is the composition $\phi = v_{g-1} \circ \psi$, where $v_{g-1} : \mathbb{P}^1 \to \mathbb{P}^{g-1}$ is the Veronese embedding of $\mathbb{P}^1$ as a rational normal curve of degree $g - 1$).

Then, setting $Y := C^n$, the group G acts on $Y = C^n$, and we have the following commutative diagrams:

$$\begin{array}{ccc} C^n = Y & \xrightarrow{\Psi} & Y/K = C^n/(\mathbb{Z}/2)^n = (\mathbb{P}^1)^n \\ \downarrow p & & \downarrow \pi \\ C^{(n)} := C^n/\mathfrak{S}_n = Y/H & \xrightarrow{\nu} & Y/G = (\mathbb{P}^1)^n/\mathfrak{S}_n = \mathbb{P}^n \end{array}$$

It is well known that for $n = g$ we have Jacobi inversion, that is, $C^{(g)}$ has a surjective birational morphism (the Abel Jacobi map) to the Jacobi variety $Jac(C) \cong Pic^g(C)$, while for $n = g - 1$, again via the Abel Jacobi map, $C^{(g-1)}$ has a birational morphism onto the Theta divisor $\Theta_C \subset Jac(C)$. We shall need to make these birational statements more precise.

We have the following classical result, due to Andreotti [1].

Theorem 3.2 *If C is a Hyperelliptic curve, then $\nu : C^{(g-1)} \to \mathbb{P}^{g-1}$ is the composition of the birational Abel-Jacobi map $\alpha_{g-1} : C^{(g-1)} \to \Theta_C$ with the Gauss map μ_C of Θ_C, and $p : C^{g-1} \to C^{(g-1)}$ yields the Galois closure of the Gauss map.*

For a non hyperelliptic curve C, letting ϕ be the canonical map $\phi : C \to \mathbb{P}^{g-1}$, the composition $\mu_C \circ \alpha_{g-1}$ is the $g - 1$ secant map of $\phi(C)$, and the branch locus of the Gauss map is the dual variety $\phi(C)^\vee$ of the canonical curve in $(\mathbb{P}^{g-1})^\vee$.

The monodromy group of the Gauss map equals the monodromy group of the canonical curve $\phi(C)$, the symmetric group $\mathfrak{S}_{2g-2}$.

Proof As shown by Andreotti, the map $\pi \circ \Psi$ is given as follows:

$$(P_1, \ldots, P_{g-1}) \mapsto \phi(P_1) \wedge \cdots \wedge \phi(P_{g-1}) \in (\mathbb{P}^{g-1})^\vee,$$

where ϕ is the canonical map $\phi : C \to \mathbb{P}^{g-1}$, which is indeed the projective derivative $D(\phi)$ of the Albanese map= first Abel Jacobi map $\alpha : C \to Jac(C)$.

On the other hand, the Gauss map associates to a point $x \in \Theta_C$,

$$x = \alpha_{g-1}(P_1 + \cdots + P_{g-1}) = \alpha(P_1) + \cdots + \alpha(P_{g-1})$$

the Hyperplane spanned by $\phi(P_1), \ldots, \phi(P_{g-1})$ since ϕ is the projective derivative of α. This shows that ν factors as claimed through the Gauss map μ_C.

The assertion on the Galois closure follows now from Lemma 3.1.

See [1] and [2] page 111 for the last assertions. □

The fact that the degree of μ_C equals to 2^{g-1} follows algebraically since $\pi \circ \Psi = \nu \circ p$, but also geometrically since each hyperplane intersects $\phi(C)$, image of $\mathbb{P}^1$ through the Veronese map of degree $(g - 1)$, in exactly $(g - 1)$ points.

For a non hyperelliptic curve the degree equals $\binom{2g-2}{g-1}$.

For a general Theta divisor in a principally polarized Abelian variety the degree of the Gauss map is instead equal to $g!$.

For more general Jacobians, the Gauss map of the Theta divisor is a rational map whose degree was studied in [14].

Proposition 3.3 *If C is a Hyperelliptic curve the map $\nu : C^{(n)} \to P := \mathbb{P}^{g-1}$ has a branch locus $\mathcal{B}$ which set theoretically equals the union of Δ, the Discriminant Hypersurface for divisors in $\mathbb{P}^1$ of degree n (the dual variety of the rational normal curve $\Gamma = \Gamma_n$), with $2g+2$ Hyperplanes $H_1, \dots, H_{2g+2}$, where, if z_i is a branch point of $C \to \mathbb{P}^1$, then H_i corresponds to the divisors containing z_i. Moreover Δ occurs with multiplicity 2^{n-1} in the branch locus, while the divisors H_j occur with multiplicity 2^{n-2}.*

The map ν factors exactly through two intermediate coverings:

i) $C^{(n)} \to Z_n := C^{(n)}/\iota$, where ι is the hyperelliptic involution;
ii) $Z_n \to \tilde{Z}_n$, where $\tilde{Z}_n \to P$ is the double cover branched on the union $\mathcal{H}$ of the hyperplanes $H_1, \dots, H_{2g+2}$.

Proof In view of Lemma 3.1 the main remaining point to show is that the branch locus is as stated.

The ramification locus of $\Psi : C^n \to (\mathbb{P}^1)^n$ equals the union of the divisors

$$R_C(i) := \{(y_1, \dots, y_n) | y_i \in R_C\},$$

where $R_C = \{p_1, \dots, p_{2g+2}\}$ is the ramification divisor of $\psi : C \to \mathbb{P}^1$. These divisors are permuted by $\mathfrak{S}_n$, and their image in $(\mathbb{P}^1)^n$ equals

$$B_C(i) := \{(x_1, \dots, x_n) | x_i \in B_C\},$$

where $B_C = \{z_1, \dots, z_{2g+2}\}$ is the branch divisor of ψ.

Whereas the ramification of $\pi : (\mathbb{P}^1)^n \to \mathbb{P}^n$ consists of the fixpoints for some nontrivial element of $\mathfrak{S}_n$, and its image is the discriminant hypersurface Δ consisting of the nonreduced divisors on $\mathbb{P}^1$, that is, the divisors $x_1 + \dots + x_n$ where the points x_i are not distinct. Δ is irreducible, being the image of

$$\mathbb{P}^1 \times (\mathbb{P}^1)^{n-2} \cong \{(x_1, x_1, x_3, \dots, x_n)\}.$$

Hence the branch locus of $\nu \circ p = \pi \circ \Psi$ is equal to the union of Δ and of hyperplanes $H_1, \dots, H_{2g+2}$, where H_i consists of the effective divisors in $\mathbb{P}^1$ containing z_i.

H_i intersects Δ in the linear space of codimension 2 consisting of the divisors which are $\geq 2z_i$, and in a smaller discriminant Δ'_i consisting of divisors which are the sum of z_i with a nonreduced divisor.

On the other hand, the branch locus of $p : C^n \to C^{(n)}$ equals the discriminant Δ_C, consisting of nonreduced effective divisors of degree n on C.

Δ_C maps then to Δ with degree 2^{n-1}, since for general x_1 and general $x_3, \dots, x_n$ the inverse image of $2x_1 + x_3 + \dots + x_n$ consists of 2^{n-1} divisors.

While the inverse image of the ramification of π contains the $\mathfrak{S}_n$-orbit of the divisors $y'_1 + y''_1 + y_3 + \dots + y_n$, where $y'_1 + y''_1$ is the inverse image of x_1, and $y_j \mapsto x_j$.

Therefore the branch locus of ν consists of Δ with multiplicity 2^{n-1}, and, since for $j \geq 3$ there are two choices for y_j, of the hyperplanes $H_1, \dots, H_{2g+2}$ with multiplicity 2^{n-2}.

Concerning assertion i), observe that the element $e = \sum e_i \in V$ acts on C^n via the hyperelliptic involution ι acting on each coordinate, hence the intermediate quotient is the quotient of the symmetric product $C^{(n)}$ via the action of ι.

For assertion ii), we notice that the quotient of C^n by the subgroup of K orthogonal to e is the double covering of $(\mathbb{P}^1)^n$ branched on the union of the branch divisors $B_C(i)$, whose image in $\mathbb{P}^n$ is the union of the hyperplanes H_j. □

We can rephrase the previous result in the special case $n = g - 1$:

Proposition 3.4 *If C is a Hyperelliptic curve the Gauss map $\mu_C : \Theta_C \to P := \mathbb{P}^{g-1}$ has a branch locus $\mathcal{B}$ which set theoretically equals the union of Δ, the Discriminant Hypersurface for divisors in $\mathbb{P}^1$ of degree $g-1$ (the dual variety of the rational normal curve $\Gamma = \Gamma_{g-1}$), with $2g+2$ Hyperplanes $H_1, \dots, H_{2g+2}$, where, if z_i is a branch point of $C \to \mathbb{P}^1$, then H_i corresponds to the divisors containing z_i. Moreover Δ occurs with multiplicity 2^{g-2} in the branch locus and the Hyperplanes H_j occur with multiplicity 2^{g-3}.*

The Gauss map μ_C factors exactly through two intermediate coverings:

i) $\Theta_C \to Z := \Theta_C / \pm 1$
ii) $\Theta_C \to \tilde{Z}$, where $\tilde{Z} \to P$ is the double cover branched on the union of the hyperplanes H_i.

Proof We just need to observe that the hyperelliptic involution ι acts on the Jacobian $Jac(C)$ as multiplication by -1, for a suitable choice of the origin as a thetacharacteristic.

The conclusion is that a hyperplane H is in the branch locus if H intersects Γ in a divisor which is the image of a canonical divisor of C which contains a ramification point p_i, or contains a divisor of the form $x' + x''$, the inverse image of a point $x \in \mathbb{P}^1$: this amounts to saying that H intersects Γ in a divisor containing a branch point z_i or containing a point x with multiplicity at least 2. □

For further purposes, we must clarify the different roles played by the discriminant Δ and the union of Hyperplanes $H_1 \cup \dots \cup H_{2g+2}$ in the branch locus.

To quickly get an understanding of this issue, let us consider the case $g = n$: then the double covering $\tilde{Z}_g$ is a variety with trivial canonical divisor, while Z is birational to the Kummer variety of the Jacobian. Hence the map $Z \to \tilde{Z}_g$ is unramifed in codimension 1. The main point is, as we are now going to explain, that Δ contributes to an exceptional divisor on the symmetric product of the curve.

4 Theta functions on Hyperelliptic Jacobians

Let C be a curve of genus g, and let

$$A := Jac(C) = Pic^0(C).$$

Indeed, every divisor of degree g is effective, and, if we fix a point $y_0 \in C$, we have the Abel Jacobi maps

$$\alpha : C \to Jac(C),\ \alpha(y) := \int_{y_0}^{y},\ \text{and}\ \alpha_n : C^n \to Jac(C),\ \alpha_n(y_1, \dots, y_n) := \sum_1^n \int_{y_0}^{y_i}.$$

The Abel-Jacobi maps factor through the symmetric products $C^{(n)} = C^n / \mathfrak{S}_n$, and to simplify notation we shall use the same symbol for all of them. We denote also as usual

$$W_n := \alpha(C^{(n)}), n \leq g,$$

recalling once more that $W_g = A = Jac(C)$.

For many assertions we are going to make, see [2] pages 250 and around it.

By Riemann's singularity Theorem, if $u_0 = \alpha(D)$, $D \in C^{(g-1)}$, then there is a thetacharacteristic $\mathcal{K}$ such that

$$Mult_{u_0}(\Theta_C - \mathcal{K}) = h^0(\mathcal{O}_C(D)).$$

Up to a translation, we may assume

$$(A, \Theta_C) = (A, W_{g-1}),\ W_{g-1} = \alpha(y_0 + C^{(g-1)}) \subset \alpha(C^{(g)}) = A.$$

Hence the classical result that

$$\alpha : C^{(g)} \to A$$

is surjective, birational and locally invertible outside

$$\{\alpha(D')|\ deg(D') = g, h^0(\mathcal{O}_C(D')) \geq 2\} \subset W_{g-1} :$$

in fact for such divisors D' there exists $D'' \in |D'|$ with $D'' \geq y_0$.

Corollary 4.1 *The graded ring of Hyperelliptic Jacobian Theta Functions*

$$\mathcal{R}(A, \Theta_C) := \oplus_{m\geq 0} H^0(A, \mathcal{O}_A(m\Theta_C))$$

equals the graded ring

$$\mathcal{R}(C^{(g)}, \alpha^{-1}(W_{g-1})).$$

Hence in this approach it is necessary to study the divisor $\alpha^{-1}(W_{g-1})$, which contains the divisor $y_0 + C^{(g-1)}$.

Remark 4.2 (1) $\alpha^{-1}(u)$, for $u = \alpha(D)$, and D an effective divisor of degree $g-1$, is the linear system $|D|$, whose dimension is classically denoted by r.

Since D is a special divisor, it follows by Clifford's Theorem that $r \leq \frac{g-1}{2}$, equality holding if and only if C is hyperelliptic and D is a multiple of the hyperelliptic divisor $\mathfrak{H}$.

(2) If C is hyperelliptic, then $|\mathfrak{H}| + C^{(g-2)} \subset C^{(g)}$ is a divisor whose image under α has dimension $g-2$.

Its intersection with W_{g-1} has dimension equal to $g-3$ and is contained in the singular locus $Sing(W_{g-1})$.

Theorem 4.3 *If C is a hyperelliptic curve, then*

$$\alpha^{-1}(W_{g-1}) = (y_0 + C^{(g-1)}) \cup (|\mathfrak{H}| + C^{(g-2)}) =: \tilde{C}^{(g-1)} \cup E \subset C^{(g)},$$

where the divisor E is exceptional for α.

Proof Assume that there is a divisor $\mathcal{D}$ inside C^g which is contracted under the Abel Jacobi map α to a lower dimensional variety.

This means that, for all $(y_1, \ldots, y_g) \in \mathcal{D}$, the canonical images $\phi(y_1), \ldots, \phi(y_g)$ are linearly dependent.

After possibly reordering, $\mathcal{D}$ maps onto C^{g-1}, and for each $y_1, \ldots, y_{g-1}$ there is a point y such that $(y_1, \ldots, y_{g-1}, y) \in \mathcal{D}$.

For a general choice of $(y_1, \ldots, y_{g-1}), \phi(y_1), \ldots, \phi(y_{g-1})$ are linearly independent, span a Hyperplane H, and

$$H \cap \phi(C) = \{\phi(y_1), \ldots, \phi(y_{g-1})\};$$

therefore there exists j such that $\phi(y) = \phi(y_j)$, hence the corresponding divisor on C is in $|\mathfrak{H}| + C^{(g-2)}$. $\square$

Definition 4.4 We let

$$\hat{C}_{(g-1)} := \{(y_1, \dots, y_g) \in C^g | \exists\, 1 \leq j \leq g, y_j = y_0\},$$

$$\hat{E} := \{(y_1, \dots, y_g) \in C^g | \exists\, j < h, y_j = \iota(y_h)\}.$$

Here ι is the hyperelliptic involution; note that $\hat{C}_{(g-1)}$ maps onto $\tilde{C}^{(g-1)}$, $\hat{E}$ maps onto E.

For convenience, we choose now the base point $y_0 \in C$ to be a Weierstrass point, that is, a fixpoint for ι: this means that $2y_0 \in |\mathfrak{H}|$, $2y_0 = \psi^{-1}(x_0)$.

Remark 4.5 (a) The divisor $\hat{C}_{(g-1)}$ is invariant for the Hypercube group G, actually

$$2\hat{C}_{(g-1)} = \Psi^{-1}(H_0') := \Psi^{-1}\{(x_1, \dots, x_g)|\exists j, x_j = x_0\} = (\pi \circ \Psi)^{-1}(H_0),$$

where H_0 is the hyperplane in $\mathbb{P}^g$ of divisors containing x_0.

(b) We observe here that the divisor E maps onto the discriminant $\Delta \subset \mathbb{P}^{g-1}$ under the map ν.

(c) The big diagonal $\Delta' \subset (\mathbb{P}^1)^g$, the inverse image of the Discriminant Hypersurface, has the property that $\Psi^{-1}(\Delta') = \hat{E} \cup \Delta_C'$, where Δ_C' is the big diagonal in C^g. $\hat{E}$ and Δ_C' alone are not G-invariant.

Theorem 4.6 *The graded ring of Hyperelliptic Jacobian Theta Functions is a subring of invariants as follows:*

$$\mathcal{R} := \mathcal{R}(A, \Theta_C) = \mathcal{R}(C^{(g)}, \tilde{C}^{(g-1)} + E) = \mathcal{R}(C^g, \hat{C}^{(g-1)} + \hat{E})^{\mathfrak{S}_g} \subset$$

$$\subset \mathcal{R}(C^g, \hat{C}^{(g-1)} + \Psi^{-1}(\Delta')) =: \mathcal{A}.$$

Proof The first equality is the same equality stated in Corollary 4.1, in view of Theorem 4.3.

For the second equality we need to observe that $\hat{C}^{(g-1)} + \hat{E}$ is the pull back of the divisor $\tilde{C}^{(g-1)} + E$, and that the ramification divisor of $C^g \to C^{(g)}$ is the big diagonal Δ_C', mapping to the irreducible discriminant divisor Δ_C which is not contained in the divisor $\tilde{C}^{(g-1)} + E$.

Holomorphic sections downstairs (on $C^{(g)}$) clearly lift to invariant (holomorphic) sections upstairs (on C^g); conversely, we claim that invariant sections upstairs descend on the complement of a Zariski closed set of codimension 2 in $C^{(g)}$, and then they extend throughout by virtue of Hartogs' Theorem.

Our claim follows because on an open set of the ramification locus the pull back divisor is trivial, and invariant functions are pull-backs of functions on the quotient.

For the last inclusion, we simply use that $\Psi^{-1}(\Delta') = \hat{E} + \Delta_C'$. $\square$

Remark 4.7 The graded ring $\mathcal{A}$ has the property that its subring $\mathcal{A}^{even}$ is the graded ring associated to the pull-back

$$\Psi^{-1}(2\Delta' + H_0') = (\pi \circ \Psi)^{-1}(H + 2\Delta).$$

The ring $\mathcal{A} = \mathcal{R}(C^g, \hat{C}^{(g-1)} + \hat{E} + \Delta_C')$ is a representation of the group G of the Hypercube, hence $\mathcal{R}$ is a subring of $\mathcal{A}^{\mathfrak{S}_g}$, and it can be detected by considering the subring of sections of degree n vanishing of order n on the Diagonal Δ_C', as done for instance by Canonaco in small genus [4].

The best way to describe $\mathcal{A}^{even}$ is to write its direct image on $(\mathbb{P}^1)^g$, but we do not pursue this further here.

5 Étale double covers of Hyperelliptic curves and Jacobians

Let $\varphi : C' \to C$ be an étale double covering of a Hyperelliptic curve C of genus g, so that

$$\varphi_*(\mathcal{O}_{C'}) = \mathcal{O}_C \oplus \eta.$$

Since the hyperelliptic involution ι acts trivially on $\mathrm{Pic}(C)[2]$, ι lifts to C' and we have an action of $(\mathbb{Z}/2)^2$ on C' with quotient $\mathbb{P}^1$, that is, a bidouble cover of $\mathbb{P}^1$.

Hence (see [5]) there is a factorization of the homogeneous polynomial f of degree $2g+2$ whose equation is the equation for the branch locus of ψ,

$$f(x) = f_1(x) f_2(x) \in \mathbb{C}[x_0, x_1],$$

with factors of respective degrees $2d_1, 2d_2$, with $d_1 + d_2 = g + 1$, and such that

$$C' = \{v_1^2 = f_1(x), v_2^2 = f_2(x)\},\ C = \{v^2 = f(x)\}.$$

Moreover,

$$C = C'/j,\ j(v_1) = -v_1,\ j(v_2) = -v_2,\ \varphi(x, v_1, v_2) = (x, v), \text{with} v = v_1 v_2.$$

$$H^0(C', K_{C'}) = v_2 H^0(\mathcal{O}_{\mathbb{P}^1}(d_1 - 2)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(g - 1)) \oplus v_1 H^0(\mathcal{O}_{\mathbb{P}^1}(d_2 - 2)).$$

This is the Eigenspace decomposition according to the (nontrivial) characters of $(\mathbb{Z}/2)^2$, and we identify $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ to its pull-back under φ. The formula clearly shows that C' is hyperelliptic if and only if some $d_i = 1$.

We run now for C' a similar game to the one we played for C:

$$(C')^n \to C^n = (C')^n/(\mathbb{Z}/2)^n \to (\mathbb{P}^1)^n = (C')^n/((\mathbb{Z}/2)^2)^n.$$

The first quotient is étale, while if we divide by the Symmetric group $\mathfrak{S}_n$, we get

$$(C')^{(n)} \to C^{(n)} \to (\mathbb{P}^1)^{(n)} = \mathbb{P}^n.$$

The map $(C')^{(n)} \to C^{(n)}$ above is no longer étale, since its degree equals 2^n, but the fibre cardinality drops over the discriminant hypersurface (for instance, if $y', y'' \mapsto y$, then only three divisors $2y', 2y'', y' + y''$ map to the divisor $2y$).

Proposition 5.1 *Consider the subgroup $\Lambda \subset (\mathbb{Z}/2)^n \subset Aut((C')^n \to C^n)$, defined as*

$$\Lambda := e^{\perp} = \{(\sigma_i) | \sum_i \sigma_i = 0\}.$$

Then Λ is normalized by $\mathfrak{S}_n$, and defining

$$\hat{X}_n := (C')^n/(\Lambda \rtimes \mathfrak{S}_n),$$

$\hat{X}_n$ dominates $C^{(n)}$ via an étale double covering.

Proof By the factorization

$$(C')^n \to \hat{X}_n \to C^{(n)} = (C')^n/((\mathbb{Z}/2)^n \rtimes \mathfrak{S}_n),$$

$\hat{X}_n \to C^{(n)}$ is étale outside of the discriminant Δ_C, and since $C^{(n)}$ is smooth, it suffices to show that the covering is quasi-étale, that is, étale outside of codimension 2.

Given an effective divisor $\sum_i m_i y_i$, where $y_i', y_i'' \mapsto y_i$, we have as inverse images the effective divisors $\sum_i m_i' y_i' + m_i'' y_i''$ with $m_i = m_i' + m_i''$. For $m_i = 1$, there are two possibilities, for $m_i = 2$, as already observed, we have three possibilities:

$$2y_i', 2y_i'', y_i' + y_i''.$$

Assume that $m_1 = 2$, and all others $m_i = 1$: then we simply observe, that, writing the divisors as images of the n-tuples

$$(y_1', y_1', \dots), (y_1'', y_1'', \dots), (y_1', y_1'', \dots),$$

there is an element of Σ, namely the involution $j_1 \times j_2 \times identity$ which sends the first element to the second, and the third to $(y_1'', y_1', \dots)$, which is equivalent modulo the action of $\mathfrak{S}_n$. Hence, over the set of divisors with $\sum_i (m_i - 1) = 1$ (whose complement has codimension 2), the inverse image consists of two distinct points of $\hat{X}_n$. □

The previous construction is especially useful in two cases: $n = g$, where it provides an étale double covering of $Jac(C)$, which is birational to $C^{(g)}$, and for $n = g - 1$, where it provides the corresponding étale double covering of the Theta divisor Θ_C, which is birational to $C^{(g-1)}$.

In order to simplify the exposition, we recall the following Lemma, whose proof can be found in [1] (Proposition 3, page 806).

Lemma 5.2 *There is a natural isomorphism between the canonical system on the symmetric product of a curve and the exterior product of the canonical system of the curve C*

$$\Lambda^n\big(H^0(\Omega_C^1)\big) \cong H^0\big(\Omega^n_{C^{(n)}}\big) = H^0\big(\Omega^n_{C^n}\big)^{\mathfrak{S}_n},$$

associating to $\eta_1 \wedge \dots \wedge \eta_n$ the symmetrization of $\eta_1(x_1) \wedge \dots \wedge \eta_n(x_n)$.

Proposition 5.3 *Let $\hat{X} := \hat{X}_{g-1}$ be as in Proposition 5.1 the étale double covering of $C^{(g-1)}$: then the canonical image of $\hat{X} \subset \mathbb{P}^g$ is a finite covering $\hat{W}$ of $\mathbb{P}^{g-1}$ via a linear projection $\mathbb{P}^g \dashrightarrow \mathbb{P}^{g-1}$.*

In terms of the two integers $d_1, d_2 \geq 1$ such that $d_1 + d_2 = g + 1$, if $d_1 = 1, d_2 = g$, then the canonical image $\hat{W}$ of $\hat{X}$ is birational to the double covering of $\mathbb{P}^{g-1}$ branched on the union of two hyperplanes $\mathcal{H} := H_1 + H_2$.

When $d_1, d_2 \geq 2$, $\hat{W}$ is not a double covering of $\mathbb{P}^{g-1}$.

Proof The canonical system of $\hat{X}_n$ pulls back to the $\Lambda \rtimes \mathfrak{S}_n$-invariant part of the canonical system of $(C')^n$, which is

$$H^0(\mathcal{O}_{(C')^n}(K)) = \otimes_1^n H^0(\mathcal{O}_{C'}(K_{C'})).$$

By Andreotti's Lemma 5.2 the $\mathfrak{S}_n$-invariance determines a subspace isomorphic to $\Lambda^n(H^0(\Omega^1_{C'}))$.

We use now the formula

$$H^0(C', K_{C'}) = v_2 H^0(\mathcal{O}_{\mathbb{P}^1}(d_1 - 2)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(g-1)) \oplus v_1 H^0(\mathcal{O}_{\mathbb{P}^1}(d_2 - 2)),$$

and replace v_1 by u, v_2 by w, so that $uw = v$, and denoting $u(i)$ for the section u on the i-th copy of C', and similarly for the other variables,

$$H^0(\mathcal{O}_{(C')^n}(K)) = \otimes_1^n \{w(i) Q_i(x(i)) + P_i(x(i)) + u(i) M_i(x(i))\}.$$

Set now $n = g-1$, and observe that, taking $Q_i = M_i = 0$, that is, taking the invariants for $(\mathbb{Z}/2)^{g-1}$, we get the canonical system of C^n. Taking the further subring of $\mathfrak{S}_{g-1}$-invariants, we get

$$\Lambda^{g-1}\big(H^0(\Omega^1_C)\big) \cong H^0\big(\Omega^{g-1}_{C^{(g-1)}}\big)$$

and this linear system, by Theorem 3.2 corresponds to the morphism $C^{(g-1)} \to \mathbb{P}^{g-1}$.

The other sections s for which we are looking for must be eigenvectors for the group $\mathbb{Z}/2 = (\mathbb{Z}/2)^{g-1}/\Lambda$, and with nontrivial eigenvalue, hence they must be left invariant by $\mathfrak{S}_{g-1}$ and each σ_i should send them to $-s$.

The second property implies that for them $P_i \equiv 0$, for all i.

Hence we get exactly one new element v_*, corresponding to the symmetrization of

$$w_1 \dots w_{d_1-1}\Lambda^{d_1-1}H^0(\mathcal{O}_{\mathbb{P}^1}(d_1-2))u_{d_1}\dots u_{g-1}\Lambda^{d_2-1}H^0(\mathcal{O}_{\mathbb{P}^1}(d_2-2)),$$

where we let $w_i := w(i) \dots$.

Observe now that, if $d_1, d_2 \geq 2$, then $Q_i, M_i \not\equiv 0$, and it is complicated to calculate v_*^2.

We can however say that v_* is not an eigenvector for $((\mathbb{Z}/2)^2)^{g-1}$, and v_*^2 as well, hence v_*^2 is not a section of a line bundle on $\mathbb{P}^{g-1}$.

If instead $d_1 = 1$, then $Q_i \equiv 0$, then $d_2 = g$, and v_* equals the symmetrization of

$$u_1 \dots u_{g-1}\Lambda^{g-1}H^0(\mathcal{O}_{\mathbb{P}^1}(g-2)),$$

and is a multiple of $u_* := u_1 \dots u_{g-1}$.

Hence the canonical map of $\hat{X}$ factors through the double covering given by

$$u_*^2 = f_1(x(1)) \cdot \dots \cdot f_1(x(g-1)).$$

Then we see that, setting z_1, z_2 to be the roots of f_1, and $z_3, \dots z_{2g+2}$ to be the roots of f_2, then

$$u_*^2 = h_1 \cdot h_2,$$

where h_i is the linear form on $\mathbb{P}^{g-1}$ whose zero set is the hyperplane H_i corresponding to the symmetrization of the divisor $\{z_i\} \times (\mathbb{P}^1)^{g-2}$. ◻

6 Application to canonical maps of hypersurfaces in Abelian Varieties

Let A be an Abelian variety of dimension g, and let $X \subset A$ be a smooth ample hypersurface in A such that the Chern class $c_1(X)$ of the divisor X is a polarization of type $\overline{d} := (d_1, d_2, \dots, d_g)$, so that the vector space $H^0(A, \mathcal{O}_A(X))$ has dimension equal to the Pfaffian $d := d_1 \cdot \dots \cdot d_g$ of $c_1(X)$.

The classical results of Lefschetz [17] say that the rational map associated to $H^0(A, \mathcal{O}_A(X))$ is a morphism if $d_1 \geq 2$, and is an embedding of A if $d_1 \geq 3$.

By adjunction, the canonical sheaf of X is the restriction $\mathcal{O}_X(X)$, so a natural generalization of Lefschetz' theorems is to ask about the behaviour of the canonical systems of such hypersurfaces X. This behaviour depends on the hypersurface X and not just on the polarization type only, as shown in [8]: if we have a polarization of type $(1, 1, 2)$ then the image Σ of the canonical map Φ_X is in general a surface of degree 12 in $\mathbb{P}^3$, birational to X, while for the special case where X is the pull-back of the Theta divisor of a curve of genus 3, then the canonical map has degree 2, and Σ has degree 6.

The canonical map of such a hypersurface X is, via the following folklore Lemma, a mixture of the restriction of the Lefschetz map with the Gauss map of X, which is a morphism for X smooth by a theorem of Ziv Ran [21].

Lemma 6.1 *Let X be an ample hypersurface of dimension n in an Abelian variety A, such that the class of X is a polarization of type $\overline{d} := (d_1, d_2, \dots, d_{n+1})$.*

Let $\theta_1, \dots, \theta_d$ be a basis of $H^0(A, \mathcal{O}_A(X))$ such that $X = \{\theta_1 = 0\}$.

Then, if $z_1, \dots, z_g$ are linear coordinates on the complex vector space V such that A is the quotient of V by a lattice Λ, $A = V/\Lambda$, then the canonical map Φ_X is given by

$$\left(\theta_2, \dots, \theta_d, \frac{\partial\theta_1}{\partial z_1}, \dots, \frac{\partial\theta_1}{\partial z_g}\right).$$

Hence first of all the canonical map is an embedding if $H^0(A, \mathcal{O}_A(X))$ yields an embedding of A; secondly, since a projection of Φ_X is the Gauss map of X, given by $(\frac{\partial\theta_1}{\partial z_1}, \dots, \frac{\partial\theta_1}{\partial z_g})$, follows that the canonical system $|K_X|$ is base-point-free and Φ_X a finite morphism.

This is the main Theorem of [7]:

Theorem 6.2 *Let (A, X) be a general pair, consisting of a hypersurface X of dimension $n = g - 1$ in an Abelian variety A, such that the class of X is a polarization of type $\overline{d} := (d_1, d_2, \dots, d_g)$ with Pfaffian $d = d_1 \dots d_g > 1$.*

Then the canonical map Φ_X of X is birational onto its image Σ.

6.1 A new proof of theorem 6.2

For the reader's convenience we borrow now a simple argument contained in [7], yielding first a reduction step:

Step I: It suffices to prove the Theorem in the case of a polarization of type $(1, \dots, 1, p)$, **with** p **a prime number, and assuming** $g \geq 2$.

We deal then with the following specializations:

Step II: Consider the cases where X **is an étale pull-back of a Theta divisor** Θ.

Here, we shall assume that X is a polarization of type $(1, \dots, 1, p)$, and that X is the pull-back of a Theta divisor $\Theta \subset A'$ (that is, Θ yields a principal polarization) via an isogeny $\beta : A \to A'$ with kernel $\cong \mathbb{Z}/p$.

We define

$$Z := \Theta / \pm 1, \tag{6.1}$$

and observe that Z is a dihedral quotient of X, $Z = X/D_p$.

In this situation, 4.7 of [7] uses that the canonical system is a representation of the group D_p to show that the canonical map of X separates the general fibres of $X \to Z$ for $p > 2$, and that we may assume this also for $p = 2$ after a deformation of X (this argument shall be recalled in the final step).

Step III: First we shall assume that $\Theta = \Theta_C$ is the Theta divisor of a hyperelliptic curve, hence we have $\beta : A \to Jac(C) =: A'$.

Here, there is a dihedral covering of $\mathbb{P}^1$ with group D_p yielding an unramified $\mathbb{Z}/p$ covering $C' \to C$, and $D_p^{g-1} \rtimes \mathfrak{S}_{g-1}$ acts on $(C')^{g-1}$.

Observe that $\mathbb{Z}/p$ acts also on the canonical image, non trivially as we saw.

Assume that the canonical map $\Phi_X : X \dashrightarrow \Sigma$ is not birational, and that it factors through a normal variety W which is birational to Σ; set then

$$\tilde{W} := W/(\mathbb{Z}/p). \tag{6.2}$$

We have a factorization of

$$f \circ \Phi_X : X \to \Theta \to \tilde{W} \to \mathbb{P}^{g-1},$$

where $f : \Sigma \dashrightarrow \mathbb{P}^{g-1}$ is the projection corresponding to the Gauss map of X, which equals the Gauss map of Θ_C.

By Lemma 3.1 there are four cases possible:

(1) $\tilde{W} = \Theta$;
(2) $\Theta \to \tilde{W}$ has degree 2, and $\tilde{W} = Z = \Theta / \pm 1$;
(3) $\tilde{W} = P := \mathbb{P}^{g-1}$.
(1) $\tilde{W} \to P$ has degree 2 and $\tilde{W} = \tilde{Z}$.

Cases (1) and (2) are eliminated by virtue of Step II, as follows.

In case (1) we would have either $W = X$, hence birationality holds, or $W = \Theta$, contradicting Step II.

In case (2) the general fibres of $X \to W$ would be contained in the fibres of $X \to Z = X/D_p$, again contradicting Step II for $p \neq 2$.

For $p = 2$, either $W = Z$, and we are done by Step II, or we have a double covering, and by Step II a general deformation becomes birational.

In cases (3) and (4) $\tilde{W} \to P := \mathbb{P}^{g-1}$ is either the identity or a double covering. But, in any case, since $\mathbb{Z}/p$ acts faithfully on the fibres of $\Sigma \dashrightarrow \tilde{W}$, it follows that the degree m of the covering $\Sigma \dashrightarrow P$ (hence of $W \to P$) is either $2p$ or p.

We have two factorizations of the Gauss map f:

$$X \to Z \to P, \; X \to W \to P.$$

Consider now the respective ramification divisors $\mathcal{R}_f, \mathcal{R} = \mathcal{R}_Z, \mathcal{R}_W$ of the respective maps $f : X \to P, \Psi : Z \to P, W \to P$.

Since $X \to Z$ is quasi-étale (unramified in codimension 1), $\mathcal{R}_f$ is the inverse image of $\mathcal{R}$, hence $\mathcal{R}_f$ maps to $\mathcal{R}$ with mapping degree $2p$.

We use now the notation and the results of Proposition 3.3. It turns out that all the components of $\mathcal{R}_f$ have multiplicity 1, since the same happens for the components of $\mathcal{R}$.

This excludes right away the case $p \geq 3$, since a cyclic covering of degree p has a ramification divisor occurring with multiplicity $(p-1)$, and moreover P and $\tilde{Z}$ do not admit unramified coverings.

Moreover, by Step II, we may assume that $Z \neq \tilde{Z}$, hence that $g \geq 4$.

We are then left with the case where $p = 2$.

Here we can use Proposition 5.1, first under the assumption that we choose $d_1 = 1, d_2 = g$.

The image of X, which equals the one of $\hat{X}$, is the double cover $\hat{W}$ of P with branch locus $H_1 + H_2$, the union of 2 Hyperplanes.

Since $W = \hat{W}$ case (4) is clearly excluded, since $\hat{W}$ is not a double covering of $\tilde{Z}$.

Use now Proposition 5.1 under the assumption that $d_1, d_2 \geq 2$. Then case (3), where W would be the double covering of $P = \mathbb{P}^{g-1}$ branched on a branch divisor $\mathcal{B}' \subset \Delta \cup \mathcal{H}$, is also excluded.

Step IV: To finish the proof, consider the more general case where X is a double étale covering of a smooth Theta divisor Θ.

As observed in [8], we have a basis θ_1, θ_2 of even functions, i.e., such that $\theta_i(-z) = \theta_i(z)$, and $Z := \Theta / \pm 1 = X/(\mathbb{Z}/2)^2$, where $(\mathbb{Z}/2)^2$ acts sending $z \mapsto \pm z + \eta$, where η is a 2-torsion point on A. Then the canonical map Φ_X, since the partial derivatives of θ_1 are invariant for $z \mapsto z + \eta$, while $\theta_2(z+\eta) = -\theta_2(z)$, factors through the involution $\iota : X \to X$

such that

$$\iota(z) = -z + \eta.$$

Assume that we have a further factorization $X \to X/\iota \to \Sigma$ of the canonical map, and recall that X/ι is a double covering of Z.

Then, specializing to the case where we have the double étale covering X_0 of the Theta divisor of a hyperelliptic curve, we see by Theorem 2.7 that we have a further factorization of the canonical map of X_0, $X_0 \to \Sigma_0 \to \hat{W}$.

Indeed, Hypothesis (**) can be seen to hold using Theorem 4.3 and assertion (v) of Proposition 2.2.

As we argued before, Σ_0 is a double cover of $\Sigma_0/(\mathbb{Z}/2)$, which is therefore either Z or $\tilde{Z}$.

Accordingly, either $\Sigma = X/\iota$ or $\Sigma_0/(\mathbb{Z}/2) = \tilde{Z}$, hence Σ_0 is a degree four covering of P.

In the latter case, if we take the degrees $d_1, d_2 \geq 2$, it would follow that $\Sigma_0 = \hat{W}$.

Hence the monodromy of $\Sigma \to P$ would land in $\mathfrak{S}_4$.

However, if we specialize to the Theta divisor of a non hyperelliptic curve, the Monodromy group of the covering $C^{(g-1)} \sim \Theta_C \to P$ is equal to $\mathfrak{S}_{2g-2}$, acting on the subsets of cardinality $(g-1)$.

Since $g \geq 4$, $2g - 2 \geq 6$ and the group $\mathfrak{A}_{2g-2}$ is simple: hence the monodromy image in $\mathfrak{S}_4$ has order 2, and cannot be transitive, whence a contradiction.

6.2 Final step

We repeat here the final argument which takes care of the case $\Sigma = X/\iota$, as in [8]: if for a general deformation of X as a symmetric divisor the canonical map would factor through ι, then X would be ι-invariant; being symmetric, it would be $(\mathbb{Z}/2)^2$-invariant, hence for all deformations X would remain the pull-back of a Theta divisor. This is a contradiction, since the Kuranishi family of X has higher dimension than the Kuranishi family of a Theta divisor Θ (see [8]).

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