CONSTRUCTIONS AND MODULI OF SURFACES OF GENERAL TYPE AND RELATED TOPICS

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Bayreuth, 11.08.2023

Massimiliano Alessandro

A mio padre.

Abstract

In this thesis we treat two topics: the construction of minimal complex surfaces of general type with $p_g = q = 2, 3$ and an extension of Schur's concept of a representation group for projective representations to the setting of semi-projective representations. These are the contents of the two articles [AC22] and [AGK23], which are two joint works: the former with Fabrizio Catanese, the latter with Christian Gleissner and Julia Kotonski.

The first part of the thesis is devoted to the treatment of the construction method for minimal surfaces of general type with $p_g = q$ developed together with Fabrizio Catanese in [AC22].

We give first a construction of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 5$ and Albanese map of degree 3, describing a unirational irreducible connected component of the Gieseker moduli space, which we show to be the only one with these invariants fulfilling a mild technical assumption (*Gorenstein Assumption*, see Assumption 2.6) and whose general element S has Albanese surface $\mathrm{Alb}(S)$ containing no elliptic curve. We call it the component of *CHPP surfaces*, since it contains the family constructed by Chen and Hacon in [CH06], and coincides with the one constructed by Penegini and Polizzi in [PePo13a].

Similarly, we construct a unirational irreducible connected component of the moduli space of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree 4, which we call the component of *PP4 surfaces* since it coincides with the irreducible one constructed by Penegini and Polizzi in [PePo14].

Furthermore, we answer a question posed by Chen and Hacon [CH06] by constructing three families of surfaces with $p_g = q$ whose Tschirnhaus module has a kernel realization with quotient a nontrivial homogeneous bundle. Two families have $p_g = q = 3$ (one of them is just a potential example since a computer script showing the existence is still missing), while the third one is a new family of surfaces with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree 3. The latter, whose existence is showed in [CS22], yields a new irreducible component of the Gieseker moduli space, which we call the component of *AC3 surfaces*. This is the first known component with these invariants, and moreover we show that it is unirational.

We point out that we provide explicit and global equations for all the five families of surfaces we mentioned above.

Finally, in the second and last part of the thesis we treat the content of the joint work [AGK23] with Christian Gleissner and Julia Kotonski.

Here we study *semi-projective representations*, i.e., homomorphisms of finite groups to the group of semi-projective transformations of finite dimensional vector spaces over an arbitrary field K. The main tool we use is *group cohomology*, more precisely explicit computations involving cocycles.

As our main result, we extend Schur's concept of *projective representation groups* [Sch04] to the semi-projective case under the assumption that K is algebraically closed.

Furthermore, a computer algorithm is given: it produces, for a given finite group, all *twisted representation groups* under trivial or conjugation actions on the field of complex numbers.

In order to stress the relevance of the theory, we discuss two important applications, where semi-projective representations occur naturally.

The first one reviews Isaacs' treatment in *Clifford theory for characters* [Isa81], namely the extension problem of invariant characters (over arbitrary fields) defined on normal subgroups.

The second one is our original algebro-geometric motivation and deals with the problem to find linear parts of homeomorphisms and biholomorphisms between complex torus quotients.

Zusammenfassung

In dieser Arbeit werden zwei Themen behandelt: die Konstruktion minimaler komplexer Flächen allgemeinen Typs mit $p_g = q = 2, 3$ und eine Erweiterung des Schur'schen Konzepts einer Darstellungsgruppe für projektive Darstellungen auf den Fall semiprojektiver Darstellungen. Dies sind die Inhalte der beiden Artikel [AC22] und [AGK23], die zwei gemeinsame Arbeiten sind: die erste mit Fabrizio Catanese, die zweite mit Christian Gleißner und Julia Kotonski.

Der erste Teil der Arbeit behandelt die gemeinsam mit Fabrizio Catanese in [AC22] entwickelte Konstruktionsmethode für minimale Flächen allgemeinen Typs mit $p_g = q$.

Wir geben zunächst eine Konstruktion von minimalen Flächen allgemeinen Typs mit $p_g = q = 2$, $K^2 = 5$ und Albanese-Abbildung vom Grad 3 an. Diese beschreibt eine unirationale irreduzible zusammenhängende Komponente des Gieseker-Modulraums. Unter einer schwachen technischen Voraussetzung (*Gorenstein Assumption*, siehe Assumption 2.6) zeigen wir, dass diese Komponente durch diese Invarianten und die Tatsache, dass die Albanese-Fläche Alb(S) eines allgemeinen Elements S keine elliptische Kurve enthält, eindeutig bestimmt ist. Wir nennen sie die Komponente von *CHPP-Flächen*, da sie die von Chen und Hacon in [CH06] konstruierte Familie enthält und mit der von Penegini und Polizzi in [PePo13a] konstruierten Familie übereinstimmt.

In ähnlicher Weise konstruieren wir eine unirationale irreduzible zusammenhängende Komponente des Modul-Raums minimaler Flächen allgemeinen Typs mit $p_g = q = 2$, $K^2 = 6$ und Albanese-Abbildung vom Grad 4, die wir die Komponente von *PP4-Flächen* nennen, da sie mit der von Penegini und Polizzi in [PePo14] konstruierten irreduziblen Komponente übereinstimmt.

Außerdem beantworten wir eine Frage von Chen und Hacon [CH06], indem wir drei Familien von Flächen mit $p_g = q$ konstruieren, deren Tschirnhaus-Modul eine Realisierung als Kern besitzt dessen Quotient ein nichttriviales homogenes Bündel ist. Zwei Familien haben $p_g = q = 3$ (eine davon ist nur ein potentielles Beispiel, da ein Computerskript, das die Existenz zeigt, noch fehlt), während die dritte eine neue Familie von Flächen mit $p_g = q = 2$, $K^2 = 6$ und Albanese-Abbildung vom Grad 3 ist. Letztere, deren Existenz in [CS22] gezeigt wird, führt zu einer neuen irreduziblen Komponente des Gieseker-Modulraums, die wir die Komponente von *AC3-Flächen* nennen. Es ist die erste bekannte Komponente mit diesen Invarianten und darüber hinaus zeigen wir, dass sie unirational ist.

Wir weisen darauf hin, dass wir explizite und globale Gleichungen für alle fünf oben erwähnten Familien von Flächen angeben.

Im zweiten und letzten Teil dieser Dissertation wird der Inhalt der gemeinsamen Arbeit [AGK23] mit Christian Gleißner und Julia Kotonski behandelt.

Hier untersuchen wir *semiprojektive Darstellungen*, d.h. Homomorphismen endlicher

Gruppen in die Gruppe der semiprojektiven Transformationen endlich dimensionaler Vektorräume über einem beliebigen Körper K. Das Hauptwerkzeug, das wir verwenden, ist die *Gruppenkohomologie*, genauer gesagt, explizite Berechnungen mit Kozyklen.

Unser Hauptergebnis ist die Erweiterung des Schur'schen Konzepts der *projektiven Darstellungsgruppen* [Sch04] auf den semiprojektiven Fall unter der Annahme, dass K algebraisch geschlossen ist.

Außerdem wird ein Algorithmus angegeben, mit dessen Hilfe für jede endliche Gruppe alle sogenannten "getwisteten Darstellungsgruppen"bezüglich der trivialen oder der Konjugationswirkung auf dem Körper der komplexen Zahlen bestimmt werden können.

Um die Relevanz der Theorie zu unterstreichen, diskutieren wir zwei wichtige Anwendungen, in denen semiprojektive Darstellungen in natürlicher Art und Weise auftreten.

In der ersten wird Isaacs' Anwendung in der *Clifford-Theorie für Charaktere* [Isa81] behandelt, nämlich das Erweiterungsproblem von invarianten Charakteren (über beliebigen Körpern), die auf normalen Untergruppen definiert sind.

Die zweite Anwendung ist unsere ursprüngliche algebro-geometrische Motivation und befasst sich mit dem Problem, Linearteile von Homöomorphismen und Biholomorphismen zwischen komplexen Torusquotienten zu finden.

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I strongly believe that every day we live is a new occasion to learn and improve ourselves. Life is a wonderful journey towards our dreams and desires. I really hope this is just the beginning of a fruitful professional career.

Ad Maiora Semper!

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Massimiliano Alessandro

Table of Contents

Introduction

In this thesis we mostly treat the contents of the two articles [AC22], [AGK23].

The first part of the thesis is devoted to the treatment of a construction method for minimal surfaces of general type with $p_g = q$ developed together with Fabrizio Catanese in [AC22].

First, we give a construction of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 5$ and Albanese map of degree 3, describing a unirational irreducible connected component of the Gieseker moduli space, which we show to be the only one with these invariants fulfilling a mild technical assumption (*Gorenstein Assumption*, see Assumption 0.7) and whose general element S has Albanese surface $\mathrm{Alb}(S)$ containing no elliptic curve. We call it the component of *CHPP surfaces*, since it contains the family constructed by Chen and Hacon in [CH06], and coincides with the one constructed by Penegini and Polizzi in [PePo13a].

Secondly, we construct a unirational irreducible connected component of the moduli space of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree 4, which we call the component of *PP4 surfaces* since it coincides with the irreducible one constructed by Penegini and Polizzi in [PePo14].

Finally, we answer a question posed by Chen and Hacon [CH06] by constructing three families of surfaces with $p_g = q$ whose Tschirnhaus module has a kernel realization with quotient a nontrivial homogeneous bundle. Two families have $p_g = q = 3$ (one of them is just a potential example since a computer script showing the existence is still missing), while the third one is a new family of surfaces with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree 3. The latter, whose existence is showed in [CS22], yields a new irreducible component of the Gieseker moduli space, which we call the component of *AC3 surfaces*. This is the first known component with these invariants, and moreover we show that it is unirational.

We point out right away that we provide explicit and global equations for all the five families of surfaces we mentioned above.

Later on we will describe the state of the art in the classification of surfaces of general type, and moreover we will highlight the novelty of our construction method in Remark 0.12.

In the second and last part of the thesis we treat the content of the joint work [AGK23].

Recalling that a projective representation of a finite group G is a homomorphism

$$f \colon G \to \mathrm{PGL}(V),$$

where $\mathrm{PGL}(V)$ is the group of projective transformations of a finite dimensional K-vector space V, in [AGK23] we consider more in general homomorphisms

$$f\colon G \to \mathrm{P\Gamma L}(V),$$

where $\mathrm{P\Gamma L}(V) \cong \mathrm{PGL}(V) \rtimes \mathrm{Aut}(K)$ is the group of semi-projective transformations of V. We call such homomorphisms *semi-projective representations* and study them by using as a main tool *group cohomology*, more precisely explicit computations involving cocycles.

The novelty of our approach mainly relies on the fact that we allow nontrivial actions of the group G on the field K (see Remark 0.19).

As our main result, we extend Schur's concept of *projective representation groups* [Sch04] to the semi-projective case under the assumption that K is algebraically closed.

Furthermore, a computer algorithm is given: it produces, for a given finite group, all *twisted representation groups* under trivial or conjugation actions on the field of complex numbers.

In order to stress the relevance of the theory, we discuss two important applications, where semi-projective representations occur naturally.

The first one reviews Isaacs' treatment in *Clifford theory for characters* [Isa81], namely the extension problem of invariant characters (over arbitrary fields) defined on normal subgroups.

The second one is our original algebro-geometric motivation and deals with the problem to find linear parts of homeomorphisms and biholomorphisms between complex torus quotients.

Later on, on pages 13–16, we will describe our working setup and results more in detail. In particular, there we will explain the above-mentioned concept of a projective representation group by introducing the so-called *lifting problem* (see diagram (0.16)), highlighting on page 13 the connection between the two articles [AC22] and [AGK23] (see diagram (0.14)).

Let us now explain with more details the content of the thesis.

The classification of surfaces of general type is a classical and long-standing research topic. Recall that such surfaces have a unique minimal model, hence their birational classification amounts to the classification of minimal surfaces of general type.

In this context, given a minimal surface of general type S, classical inequalities are known:

- $K_S^2 \geq 1$, $\chi(S) \geq 1$ (the second one due to Castelnuovo, [Bea96, Proposition X.1, Theorem X.4]);
- $K_S^2 \geq 2\chi(S) - 6$ (Noether's inequality, [BHPV04, Theorem 3.1]);
- $K_S^2 \leq 9\chi(S)$ (Bogomolov-Miyaoka-Yau inequality, [Miy77], [Yau77, Yau78]);
- $K_S^2 \geq 2p_g$ if $q > 0$ (Debarre's inequality, [Deb82]).

It turns out that isomorphism classes of minimal surfaces of general type can be parametrized by countably many quasi-projective families. More precisely, Gieseker proved ([Gie77, Theorem 1.3]) the following.

Theorem 0.1. *There exists a quasi-projective coarse moduli scheme $\mathcal{M}_{K_S^2,\chi(S)}$ for minimal surfaces of general type S with fixed invariants $K_S^2, \chi(S)$.*

Then, for fixed values of $K_S^2, \chi(S)$, we can consider the Gieseker moduli space $\mathcal{M}_{K_S^2,\chi(S)}$ and its subschemes $\mathcal{M}_{K_S^2,p_g,q}$ corresponding to minimal surfaces of general type with given invariants K_S^2, p_g, q, which are quasi-projective schemes and so they have finitely many irreducible components.

Despite the importance of Gieseker's Theorem, nothing is said about the structure of $\mathcal{M}_{K_S^2,\chi(S)}$ and describing even its subschemes $\mathcal{M}_{K_S^2,p_g,q}$ is a very challenging task. Indeed, it turns out that even constructing minimal surfaces of general type with small invariants is very hard (see for instance [BCP06], [BCP11]). Therefore, one first tries to understand and classify surfaces with particularly small invariants (e.g. those fulfilling the equality in some of the above-mentioned classical inequalities).

In this thesis we focus on surfaces with $p_g = q$ which are those with the lowest value $\chi(S) = 1$ of the invariant $\chi(S) = 1 - q + p_g$. Then from the above inequalities it follows in particular that

$$0 \leq p_g = q \leq 4.$$

The case $p_g = q = 4$ have been classified by Beauville in the appendix to [Deb82], where he shows as a byproduct of his main theorem that such a surface S is isomorphic to the product of two curves of genus $g = 2$. In particular, $K_S^2 = 8$ and the Gieseker moduli space $\mathcal{M}_{8,4,4}$ consists exactly of one connected component of dimension 6.

The case $p_g = q = 3$ have been understood through the work of several authors, see [CCML98], [HP02], [Pir02]. Each minimal surface of general type S with such invariants has either $K_S^2 = 6$ and is the symmetric square of a genus three curve, or $K_S^2 = 8$ and is of the form $(C_2 \times C_3)/\nu$, where C_g is a curve of genus g and ν is an involution acting on C_2 as an elliptic involution (i.e., C_2/ν is an elliptic curve) and freely on C_3. In particular, the Gieseker moduli space of minimal surfaces of general type with $p_g = q = 3$ is the disjoint union of $\mathcal{M}_{6,3,3}$ and $\mathcal{M}_{8,3,3}$, which are both irreducible of dimension 6 and 5 respectively.

The case $p_g = q = 2$ is still widely open despite many contributions, [Zuc03], [Man03], [CML02], [CH06], [Pen11], [PePo13a], [PePo13b], [PePo14], [CMLP14], [Pen13], [PePo17], [BCF15], [Rit18], [CanFrap18], [PiPo17], [PRR20], [PePi22].

It seems that the classification becomes more and more complicated as the value of p_g decreases.

In particular, given a minimal surface of general type S with $p_g = q = 2$, the Albanese variety $\mathrm{Alb}(S)$ is an abelian surface and for the Albanese map $alb_S \colon S \to \mathrm{Alb}(S)$ there are two possibilities:

(1) either the image $alb_S(S)$ is a curve of genus 2, or

(2) alb_S is surjective.

Case (1) was fully understood through the work of Zucconi [Zuc03] and Penegini [Pen11].

Recall that a surface S is said to be *isogeneous to a higher product of curves* if

$$S = (C_1 \times C_2)/G,$$

where C_i is a smooth curve of genus $g_i \geq 2$ and G is a finite group acting freely on $C_1 \times C_2$. The action of G is said to be *of unmixed type* if G does not exchange the two factors, and hence it acts diagonally. Moreover, a surface $S = (C_1 \times C_2)/G$ isogenous to a higher product of curves which is of umixed type is said to be *of generalized hyperelliptic type* if

- the Galois cover $C_1 \to C_1/G$ is unramified, and
- the quotient C_2/G is isomorphic to $\mathbb{P}^1$.

Indeed, Catanese [Cat00] and Zucconi [Zuc03] proved that all minimal surfaces of general type with $p_g = q = 2$ and $\dim alb_S(S) = 1$ are of generalized hyperelliptic type.

Recalling that an *isotrivial fibration* of a surface S is a fibration $f \colon S \to B$ from S onto a smooth curve B such that all the smooth fibres are isomorphic to each other, since Penegini classified in [Pen11] all minimal surfaces of general type S with $p_g = q = 2$ which either are isogenous to a higher product of curves of mixed type or admit an isotrivial fibration, he completed as a byproduct the classification of those surfaces S with Albanese dimension equal to 1.

This is the reason why we focus on minimal surfaces of general type S with $p_g = q = 2$ and surjective Albanese map $alb_S \colon S \to \mathrm{Alb}(S)$. In this context, the degree d of the Albanese map is a topological invariant (see [Cat91]), hence in particular it is a numerical invariant for a connected component of the moduli space. We observe that no explicit upper bound is known for d, even though up to now we have only examples with $d = 2, 3, 4, 6$: in particular, only one family for $d = 6$ ([Pen11]), but already three for $d = 4$ (two in [Pen11] and one in [PePo14]; regarding the latter see also [AC22]). We refer the reader to Appendix A for more details on all the known families with $d = 2, 3, 4, 6$.

In [Pen11], there are several examples of such surfaces, which are

- either isogenous to a higher product of curves (see Table 1 of [Pen11]), or
- the minimal resolution of singularities of a quotient $(C_1 \times C_2)/G$ where C_i is a smooth curve of genus $g_i \geq 2$ and G is a finite group acting faithfully on C_i and diagonally, but not freely, on $C_1 \times C_2$ (see Table 2 of [Pen11]).

However, not all minimal surfaces of general type S with $p_g = q = 2$ and maximal Albanese dimension are of this kind. Indeed, several different examples were found in the last two decades, see for instance [CML02], [CH06], [PePo13a], [PePo13b], [PePo14], [CMLP14], [PePo17], [BCF15], [Rit18], [CanFrap18], [PiPo17], [PRR20], [PePi22].

We observe here that for such surfaces S we have $K_S^2 \geq 4$ by Debarre's inequality or [Par05], and moreover Bogomolov-Miyaoka-Yau inequality implies $4 \leq K_S^2 \leq 9$.

Note that all the values $K_S^2 = 4, 5, 6, 7, 8$ occur, while the case $K_S^2 = 9$ is believed not to occur. Indeed, there are several papers by Sai-Kee Yeung claiming that, though his proofs contain some gaps.

In particular, the case $K_S^2 = 4$, first studied by Catanese (see [Cil97], Example (c) on page 70 and Remark 3.15 on page 72), was fully classified, see [Man03], [CML02], [CMLP14]. In fact, the generically finite double covers S of a principally polarized abelian surface (A, Θ) branched on a divisor $\mathcal{B} \in |2\Theta|$ with simple singularities have $p_g = q = 2$ and $K_S^2 = 4$, and, conversely, it turns out that each surface S with $p_g = q = 2$

and $K_S^2 = 4$ belongs to this family, which is called the family of *Catanese surfaces* (see [Pen13]) or *STANDARD surfaces* (see [AC22]).

Indeed, every surface S with $p_g = q = 2$ and maximal Albanese dimension arises as a generically finite cover $\alpha\colon S \to A$ of a polarized abelian surface A. Considering the Stein factorization $S \to Y \to A$ of α, we get then a finite cover $Y \to A$ where Y is normal. This is the reason why one tries to construct such a surface S by using a bottom-up approach: one can construct a finite cover $\pi\colon Y \to A$, where A is a given abelian surface and Y is normal, by assigning some cover data on A, and then consider the minimal resolution of singularities $\widetilde{S} \to Y$ of Y. Eventually, after contracting all (-1)-curves on $\widetilde{S}$ (if there are any), one gets the desired minimal surface S, and $alb_S\colon S \to \mathrm{Alb}(S)$ is induced by the composition of the resolution $\widetilde{S} \to Y$ and $\pi\colon Y \to A$.

Following this bottom-up strategy, some examples of surfaces of general type S with $p_g = q = 2$ and degree of the Albanese map $d = 2, 3, 4$ have been constructed, see for instance [CH06], [PePo13a], [PePo13b], [PePo14]. All the latter examples have small degree $d \leq 4$: this is due to the fact that structure theorems for covers of degree d are known only for $d = 2, 3, 4$ (and partially for $d = 5$).

Recalling that every degree d cover $\pi\colon Y \to A$ is given as $Y \cong \mathbf{Spec}_{\mathcal{O}_A}(\mathcal{O}_A \oplus \mathcal{E}^\vee)$, where $\mathcal{E}^\vee$ is a rank $d-1$ locally free $\mathcal{O}_A$-module called the *Tschirnhaus bundle of* π, we point out here that the main difficulty of the bottom-up approach described above relies on the fact that constructing covers with a non-split Tschirnhaus bundle is in general very hard.

From this viewpoint, a result by Chen and Hacon ([CH06], Theorem 3.5) has been helpful. Namely, they proved the following.

Theorem 0.2 (Theorem 2.10)**.** *Let S be a minimal surface of general type with $p_g = q = 2$ without any irrational pencil. Denote by $\alpha\colon S \to A$ the Albanese map of S and by $\mathfrak{F}$ the coherent sheaf defined as the cokernel of the map $\omega_A \to \alpha_*\omega_S$. Then there exist a homogeneous vector bundle $\mathfrak{H}$ on A, a negative definite line bundle $\mathfrak{L}$ on $\widehat{A} = \mathrm{Pic}^0(A)$ and a short exact sequence as follows*

$$0 \to \mathfrak{H} \to \widehat{\mathfrak{L}} \to (-1_A)^*\mathfrak{F} \to 0, \tag{0.1}$$

where $\widehat{\mathfrak{L}}$ denotes the Fourier-Mukai transform of $\mathfrak{L}$.

We recall that a surface S is said to have an *irrational pencil of genus b* if there exists a surjective rational map $f\colon S \dashrightarrow B$ onto a smooth curve B of genus $b \geq 1$ with connected fibres ([CCML98], page 278).

Remark 0.3. Given a minimal surface of general type S with $p_g = q = 2$, it turns out that S has no irrational pencil if and only if S has a surjective Albanese map $alb_S\colon S \to \mathrm{Alb}(S)$ and Albanese surface $\mathrm{Alb}(S)$ containing no elliptic curve.

Let us come back to the setting of Theorem 0.2. Considering the dual abelian surface $A' := \widehat{A}$ and the isogeny associated with the polarization $\mathcal{L} := \mathcal{O}_{A'}(D) := \mathfrak{L}^{-1}$ of type (δ_1, δ_2) (hence, with Pfaffian $\delta := \delta_1\delta_2$), namely $\Phi_D : A' \to A \cong A'/\mathcal{K}(D)$ (see Chapter 1, Subsection 1.4.1 for the definition of Φ_D), one main result of the theory of Fourier-Mukai transforms ensures that (see Proposition 1.85)

$$(-\Phi_D)^*(\,\widehat{\mathfrak{L}}\,) \cong \mathcal{L} \otimes V^\vee, \tag{0.2}$$

where $V := H^0(A', \mathcal{O}_{A'}(D))$ is a δ-dimensional vector space.

Hence, if in sequence (0.1) we have $\mathfrak{H} = 0$, then the sheaf $\mathfrak{F}$ is locally free and its pullback $\mathfrak{F}' := (\Phi_D)^*\mathfrak{F}$ is a split locally free $\mathcal{O}_{A'}$-module $\mathfrak{F}' \cong \mathcal{L} \otimes V^\vee$.

From our viewpoint, given an abelian surface A and setting $A' := \widehat{A}$ for its dual, the latter fact suggests that, in order to construct a cover $\pi\colon Y \to A$ with a non-split Tschirnhaus bundle, we can construct a cover $\pi'\colon Y' \to A'$ with a split Tschirnhaus bundle, and then we take the étale quotient $Y := Y'/\mathcal{K}(D)$. In this way, it is possible to bypass the difficulty of dealing with cohomological computations involving a non-split locally free sheaf.

This is exactly the approach followed in [CH06], [PePo13a] and [PePo14]. Here two families of surfaces of general type S with $p_g = q = 2$ and degree of the Albanese map $d = 3, 4$ have been constructed by exploiting the theory of Miranda [Mir85] for $d = 3$ in [CH06] and [PePo13a], respectively the theory of Hahn-Miranda [HM99] for $d = 4$ in [PePo14]. In these constructions we have a diagram as follows

$$\begin{tikzcd}
Y' := Y \times_A \widehat{A} \arrow[r, "/\mathcal{K}(D)"] \arrow[d, "\pi'"'] & Y \arrow[d, "\pi"] \\
\widehat{A} \arrow[r, "\Phi_D"] & A
\end{tikzcd} \tag{0.3}$$

where $S \to Y \to A$ is the Stein factorization of the Albanese map $\alpha\colon S \to A$ and $\pi'\colon Y' \to \widehat{A}$ is a cover with a split Tschirnhaus bundle $\mathfrak{F}' \cong \mathcal{L} \otimes V^\vee$ (here we are using the notation of Theorem 0.2, which applies with $\mathfrak{H} = 0$: $\mathcal{L} := \mathcal{O}_{\widehat{A}}(D) := \mathfrak{L}^{-1}$, $\mathfrak{F} := \alpha_*\omega_S/\omega_A$ and $\mathfrak{F}' := (\Phi_D)^*\mathfrak{F}$).

Inspired by the work of Jungkai Alfred Chen, Christopher Derek Hacon, Matteo Penegini and Francesco Polizzi, namely [CH06], [PePo13a], [PePo14], Fabrizio Catanese and I developed in a joint work [AC22] a new construction method for minimal surfaces of general type S with $p_g = q$. Let us show the main feature of our construction.

Let A' be an abelian surface with a divisor D yielding a polarization of type (δ_1, δ_2) (hence with Pfaffian $\delta := \delta_1\delta_2$).

Then $V := H^0(A', \mathcal{O}_{A'}(D))$ is a δ-dimensional vector space, and we consider the group of translations $G := \mathcal{K}(D)$ leaving the isomorphism class of $\mathcal{O}_{A'}(D)$ invariant, namely the kernel of $\Phi_D : A' \to \widehat{A'}$.

Setting $H_D := (\mathbb{Z}/\delta_1) \times (\mathbb{Z}/\delta_2)$ and $A := \widehat{A'} = A'/G$ for the dual abelian surface of A', we have that $G \cong H_D^2$ and V is an irreducible representation (called the *Schrödinger representation*) of the *finite Heisenberg group* $\mathcal{H}_D := \mathcal{H}(H_D)$ (see Chapter 1, Section 1.3 for the definition of the Heisenberg group $\mathcal{H}(H)$ of a given finite abelian group H) fitting into the following exact sequence

$$1 \to \mu_{\delta_2} \to \mathcal{H}_D \to H_D^2 \to 0, \tag{0.4}$$

where $\mu_{\delta_2} \subset \mathbb{C}^*$ is the group of δ_2-th roots of 1.

This representation has the property that the centre (which is also the commutator subgroup) $\mu_{\delta_2} \subset \mathbb{C}^*$ of $\mathcal{H}_D$ acts by scalar multiplication in a natural way. We observe moreover that $\mathcal{H}_D/\mu_{\delta_2} \cong G$.

Our method consists in describing a surface

$$S' \subset \mathbb{P}^{\delta-1} \times A' = \mathbb{P}(V) \times A',$$

which is G-invariant for the G-action of product type on $\mathbb{P}(V) \times A'$ (the action of G on $\mathbb{P}(V)$ being induced by the action of the Heisenberg group $\mathcal{H}_D$ on V).

Then we obtain the desired surface S with $p_g = q$ as the free quotient $S := S'/G$.

In order to get a full component of the moduli space, we must also consider those normal varieties $X' \subset \mathbb{P}(V) \times A'$ which have at most Rational Double Points as singularities, and then we let S' be the minimal resolution of X' ($S' = X'$ if X' is smooth).

Focusing on the case $p_g = q = 2$, since our method is based on Theorem 0.2, we consider components of the Gieseker moduli space where the Albanese map $alb_S : S \to \mathrm{Alb}(S)$ is surjective and the Albanese surface $\mathrm{Alb}(S)$, for a general S, does not contain any elliptic curve.

More generally, we give the following.

Definition 0.4 (Definition 2.14)**.** A component $\mathcal{M}$ of the moduli space of minimal surfaces of general type with $p_g = q = 2$ is said to be of the **Main Stream** if

(1) the Albanese map is surjective and

(2)
$$\{\mathrm{Alb}(S) | \; [S] \in \mathcal{M}\}$$
contains an open set in a moduli space of polarized abelian surfaces.

Remark 0.5. The hypothesis on S (which is however not necessarily deformation invariant) that the Albanese surface $\mathrm{Alb}(S)$ does not contain any elliptic curve is generically verified if we deal with a component of the Main Stream.

Example 0.6. The simplest example of a component of the Main Stream is given by the component of the above-mentioned STANDARD surfaces having $K_S^2 = 4$.

More generally, we consider a surface S with $p_g = q$ and a surjective morphism $\alpha : S \to A$ of degree d onto an abelian surface A such that α does not factor through any other abelian surface: we call such a surface S *"surface with AP"*, where AP stands for *Albanese Property* (see Chapter 2, Definition 2.1).

One defines the Tschirnhaus bundle $\mathcal{E}^\vee$ of $\alpha\colon S \to A$ via the split exact sequence

$$0 \to \mathcal{O}_A \to \alpha_* \mathcal{O}_S \to \mathcal{E}^\vee \to 0. \tag{0.5}$$

By relative duality, we have then the split exact sequence

$$0 \to \omega_A \cong \mathcal{O}_A \to \alpha_* \omega_S \to \mathfrak{F} \to 0, \tag{0.6}$$

where $\mathfrak{F}$ is a subsheaf of $\mathcal{E}$ and $\mathfrak{F}$ is locally free if and only if $\mathfrak{F} = \mathcal{E}$ (see Chapter 2, Section 2.1).

If such a surface S has $p_g = q = 2$ and its Albanese surface A does not contain any elliptic curve, then S fulfills the hypothesis of Theorem 0.2 and there is a sequence like (0.1).

Hence, considering the isogeny $\Phi_D : \widehat{A} \to A \cong \widehat{A}/\mathcal{K}(D)$ associated with the polarization $\mathcal{L} := \mathcal{O}_{\widehat{A}}(D) := \mathfrak{L}^{-1}$, if we pull back sequence (0.1) via $(-\Phi_D)$, we get on $\widehat{A}$ the Heisenberg-equivariant (and indeed $\mathcal{K}(D)$-equivariant) exact sequence

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathfrak{F}' \to 0, \tag{0.7}$$

where $\mathfrak{H}' := (-\Phi_D)^*\mathfrak{H}$ and $\mathfrak{F}' := (\Phi_D)^*\mathfrak{F}$.

We notice that in general the sheaf $\mathfrak{F}'$ might not be locally free (see Chapter 2, Remark 2.5). Assuming that $\mathfrak{F}'$ is locally free, we get $\mathfrak{F}' = \mathcal{E}' := (\Phi_D)^*\mathcal{E}$, hence a sequence as follows

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathcal{E}' \to 0, \tag{0.8}$$

which is Heisenberg-equivariant.

Another main ingredient in our construction method is the use of the theory by Casnati and Ekedahl of Gorenstein covers of degree $d \geq 3$, [CE96]. The choice to use this theory forces us to make a slightly restrictive assumption, which we now describe.

We have a surjective morphism $\alpha : S \to A$, where A is an abelian surface, and S is the minimal model of a surface of general type. Then α is generically finite of degree $d \geq 2$, and any rational curve C in S is mapped to a point in A. Hence, α factors through a morphism $a : X \to A$ of the canonical model X of S, which is a Gorenstein normal variety.

If $a\colon X \to A$ is a finite morphism and $d \geq 3$, then we can directly apply the factorization theorem by Casnati and Ekedahl (Chapter 1, Theorem 1.37), implying that X embeds into $\mathbb{P}(\mathcal{E}^\vee) := \mathbf{Proj}_{\mathcal{O}_A}\mathbf{Sym}(\mathcal{E})$, where $\mathcal{E}^\vee$ is the Tschirnhaus bundle of α. In particular, we can use the structure theorems of [CE96] for degree $d = 3, 4$.

In general, we can consider the Stein factorization $S \to X \to Y \xrightarrow{\pi} A$, where the last morphism $\pi : Y \to A$ is finite of degree d, but Y need not be Gorenstein. For this reason, one usually uses the theory by Miranda for $d = 3$ ([Mir85]) and Hahn-Miranda for $d = 4$ ([HM99]), describing Y as $\mathbf{Spec}_{\mathcal{O}_A}(\mathcal{O}_A \oplus \mathcal{E}^\vee)^1$.

Still, restricting our attention to the open set

$$A^0 := A \setminus \{z \mid \dim(a^{-1}(z)) = 1\},$$

we have a finite morphism $X^0 \to A^0$, hence a rational map

$$\psi : X \dashrightarrow \mathbb{P}(\mathcal{E}^\vee),$$

with image Z which is birational to S. The natural question is: when is ψ a morphism? For instance, is it so when Z is normal?

At any rate, we propose the following assumption.

Assumption 0.7 (Gorenstein Assumption 2.6). (I) We are given a surjective morphism of degree $d \geq 3$, $\alpha : S \to A$, where A is an abelian surface, S is the minimal model of a surface of general type with $p_g = q$, and α enjoys the property of the Albanese map, that it does not factor through a morphism of S to another abelian surface.

(II) We make the assumption that $\alpha\colon S \to A$ induces an embedding $\psi : X \to \mathbb{P}(\mathcal{E}^\vee)$ of the canonical model X of S.

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[1] In [Mir85] and [HM99] $\mathcal{E}^\vee$ is called $\mathcal{E}$.

Remark 0.8. The Gorenstein Assumption holds true if $a : X \to A$ is finite, but the example of CHPP surfaces shows that it holds more generally without the morphism a being finite (see Chapter 2, Section 2.5, Remark 2.28).

Remark 0.9. If S is a surface with $p_g = q$ fulfilling the Gorenstein Assumption, then $\mathfrak{F} = \mathcal{E}$, where $\mathfrak{F}$ is the sheaf defined via sequence (0.6) (see Chapter 2, Proposition 2.9).

In light of the previous remark, if S is a surface satisfying the hypothesis of Theorem 0.2 and also the Gorenstein Assumption, then there is a sequence like (0.8).

Hence, an alternative to the hypothesis of having a component of the Main Stream fulfilling the Gorenstein Assumption is the following.

Assumption 0.10. (Generality Assumption 2.17) We make here the same assumptions (I), (II) as in Assumption 0.7, and we require moreover that:

(III) there exists an ample line bundle $\mathcal{L} = \mathcal{O}_{\widehat{A}}(D)$ yielding a polarization of type (δ_1, δ_2) on $\widehat{A} = \mathrm{Pic}^0(A)$ such that the pull-back $\mathcal{E}'$ of $\mathcal{E}$ via the isogeny $\Phi_D \colon \widehat{A} \to A$ is a locally free $\mathcal{O}_{\widehat{A}}$ -module fitting into a $\mathcal{H}_D$-equivariant exact sequence

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathcal{E}' \to 0, \tag{0.9}$$

where $\mathfrak{H}'$ is a homogeneous vector bundle and $V := H^0(\widehat{A}, \mathcal{O}_{\widehat{A}}(D))$ is the Schrödinger representation of the Heisenberg group $\mathcal{H}_D := \mathcal{H}(\mathbb{Z}/\delta_1 \times \mathbb{Z}/\delta_2)$.

Moreover, we consider the abelian surface A endowed with the dual polarization corresponding to $\mathcal{L} = \mathcal{O}_{\widehat{A}}(D)$, which is still of type (δ_1, δ_2) (see for instance [BL04, Sec. 14.4] for the notion of dual polarization).

Remark 0.11. We consider the case $d \geq 3$ as we want to use the theory by Casnati-Ekedahl. Concerning the case $d = 2$, denote by $\alpha \colon S \to A$ the (surjective) Albanese map of a minimal surface of general type S with $p_g = q = 2$; even if A does not contain any elliptic curve, the remark made on page 226 of [CH06] is wrong (observe moreover that in this remark there is an error of sign: it should be $\mathcal{O}_A(\Theta)$ instead of $\mathcal{O}_A(-\Theta)$).

Indeed, the hypothesis $d = 2$ does not imply that $\mathfrak{F} := \alpha_* \omega_S / \omega_A$ is a line bundle (yielding a principal polarization), as showed by the existence of the families with $p_g = q = 2$, $K^2 = 8$, $d = 2$ and $p_g = q = 2$, $K^2 = 6$, $d = 2$, constructed respectively in [Pen11] and [PePo13b] (see Appendix A for more details on how $\mathfrak{F}$ looks like in these cases).

Remark 0.12. Let us come back to the constructions given in [CH06], [PePo13a] and [PePo14], where we have a diagram like (0.3).

From the description of the main features of our construction method [AC22], it is clear that the novelty of our approach is given by

(1) Assumption 0.7 (Gorenstein Assumption 2.6), which here corresponds to the assumption that the canonical model X' of the resolution of singularities $S' \to Y'$ of the normal variety $Y' = Y \times_A \widehat{A}$ embeds as follows

$$X' \subset \mathbb{P}(\mathcal{E}'^\vee) = \mathbb{P}^{\delta - 1} \times \widehat{A}, \tag{0.10}$$

where $\mathcal{E}'$ is the dual of the Tschirnhaus bundle of the cover $\pi' \colon Y' \to \widehat{A}$ and $\delta = \delta_1 \delta_2$ is the Pfaffian of the polarization D provided by Theorem 0.2;

(2) the geometric interpretation of the exact sequence (0.9) as the Heisenberg-equivariant embedding of projective bundles

$$\mathbb{P}(\mathcal{E}'^{\vee}) \subset \mathbb{P}^{\delta-1} \times \widehat{A}, \tag{0.11}$$

where $\delta := \delta_1\delta_2$ is the Pfaffian of the polarization D provided by Assumption 0.10 (Generality Assumption 2.17);

(3) the use of the theory by Casnati-Ekedahl for Gorenstein covers of small degree $d = 3, 4$ [CE96].

Indeed, (1), (2) and (3) from the previous remark allowed us to construct some families of surfaces providing for them explicit and global equations (inside a trivial projective bundle).

More in detail, by using our construction method we could find global equations for the two families of surfaces with $p_g = q = 2$ constructed and studied in [CH06] and [PePo13a], respectively in [PePo14]. We named *"CHPP family"* after Chen, Hacon, Penegini and Polizzi, [CH06], [PePo13a], the family with degree of the Albanese map $d = 3$ described in Chapter 2, Sections 2.5–2.6, and similarly we did for the family presented in Chapter 2, Sections 2.7–2.8, which we named *"PP4 family"* after Penegini and Polizzi, [PePo14].

Here are their equations:

(*I*) *CHPP surfaces*: $p_g = q = 2$, $K_S^2 = 5$, $d = 3$, $\delta = 2$,

$$S' := S'(\lambda) := \{x_1(y_1^3 + \lambda y_1 y_2^2) + x_2(y_2^3 + \lambda y_2 y_1^2) = 0\} \subset \mathbb{P}^1 \times A',$$

where $\lambda \in \mathbb{C}$, $\{x_1, x_2\}$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$ and y_1, y_2 are homogeneous coordinates of $\mathbb{P}^1 = \mathbb{P}(V)$ (dual basis of $\{x_1, x_2\}$).

(*II*) *PP4 surfaces*: $p_g = q = 2$, $K_S^2 = 6$, $d = 4$, $\delta = 3$,

$$S' := S'(\mu) := \{\mathrm{rank}(M) \leq 1\} \subset \mathbb{P}^2 \times A',$$

$$M = \begin{pmatrix} x_1 & x_3 & x_2 \\ y_1^2 + \mu y_2 y_3 & y_3^2 + \mu y_1 y_2 & y_2^2 + \mu y_1 y_3 \end{pmatrix},$$

where $\mu \in \mathbb{C}$, $\{x_1, x_2, x_3\}$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$ and y_1, y_2, y_3 are homogeneous coordinates of $\mathbb{P}^2 = \mathbb{P}(V)$ (dual basis of $\{x_1, x_2, x_3\}$).

The examples $(I), (II)$ we have described above yield two components of the Main Stream, and then, considering sequence (0.7), these are just cases where the homogeneous bundle $\mathfrak{H}' = 0$ (equivalently, $\mathfrak{H} = 0$). Under this assumption, the sheaf $\mathfrak{F}$ defined via sequence (0.6) is a locally free $\mathcal{O}_A$-module, and then $\mathfrak{F} = \mathcal{E}$.

On the other hand, given a minimal surface S with $p_g = q = 2$ and surjective Albanese map $\alpha\colon S \to A$ where A contains no elliptic curve, we have sequences like

(0.1) and (0.7), and on page 227 of [CH06] it is asked whether the case $\mathfrak{H} \neq 0$ can occur (equivalently, $\mathfrak{H}' \neq 0$).

We give a positive answer, constructing under Assumption 0.10 (Generality Assumption 2.17) two families of examples with $p_g = q = 3$ (see $(III), (V)$ below) and one family with $p_g = q = 2$ (see (IV) below).

(III) $p_g = q = 3$, $K_S^2 = 6$, $d = \delta = 3$ (see Chapter 2, Subsection 2.9.1.a, Proposition 2.58),

$$S' := S'(\lambda) := \{(y,z) \mid \sum_j y_j x_j(z) = \sum_j y_j^3 + \lambda y_1 y_2 y_3 = 0\} \subset \mathbb{P}^2 \times A',$$

where $\{x_1, x_2, x_3\}$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$, $y := (y_1, y_2, y_3) \in \mathbb{P}^2 = \mathbb{P}(V)$, and $\lambda \in \mathbb{C}$ is such that $F(y) := \sum_j y_j^3 + \lambda y_1 y_2 y_3 = 0$ defines a smooth elliptic curve C; then, for a general λ, $S'(\lambda)$ is smooth.

Hence, $S' \subset C \times A'$ and $S := S'/\mathcal{K}(D)$ has irregularity $q = 3$ since $\mathcal{K}(D) \cong (\mathbb{Z}/3)^2$ acts by translations on C.

(IV) *AC3 surfaces*: $p_g = q = 2$, $K_S^2 = 6$, $d = \delta = 3$ (see Chapter 2, Subsection 2.9.1.b),

$$S' := \{(y,z) \mid \sum_j y_j x_j(z) = 0, \quad \sum_i y_i^2 y_{i+1} = 0\} \subset \mathbb{P}^2 \times A',$$

where $y := (y_1, y_2, y_3) \in \mathbb{P}^2 = \mathbb{P}(V)$, $\{x_1, x_2, x_3\}$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$, $C = \{y \mid \sum_i y_i^2 y_{i+1} = 0\} \subset \mathbb{P}^2$.

Here, $S' \subset C \times A'$ and $S := S'/\mathcal{K}(D)$ has irregularity $q = 2$ since $\mathcal{K}(D) \cong (\mathbb{Z}/3)^2$ does not act by translations on C.

(V) $p_g = q = 3$, $K_S^2 = 6$, $d = \delta = 4$, with a polarization D of type $(1,4)$ [2] (see Chapter 2, Section 2.10),

$$S' := S'(\lambda) := \{(y,z) \mid \sum_j y_j x_j(z) = Q_1(y) = Q_2(y) = 0\} \subset \mathbb{P}^3 \times A',$$

where $y \in \mathbb{P}^3 = \mathbb{P}(V)$, $\{x_1, x_2, x_3, x_4\}$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$ and

$$Q_1(y) := y_1^2 + y_3^2 + 2\lambda y_2 y_4, \qquad Q_2(y) := y_2^2 + y_4^2 + 2\lambda y_1 y_3, \qquad \lambda \neq 0, \pm 1, \pm i.$$

The intersection of the two quadrics defines an elliptic curve C of degree 4,

$$C := \{y \mid Q_1(y) = Q_2(y) = 0\} \subset \mathbb{P}^3,$$

on which $\mathcal{K}(D) \cong (\mathbb{Z}/4)^2$ acts by translations.

Here, $S' \subset C \times A'$ and $S := S'/\mathcal{K}(D)$ has irregularity $q = 3$ since $\mathcal{K}(D)$ acts by translations on C.

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[2] The case of a polarization of type $(2,2)$ cannot occur since $\mathcal{K}(D) \cong (\mathbb{Z}/2)^4$ cannot act faithfully on an elliptic curve.

> **Remark 0.13.** The family of surfaces whose equations are displayed in (V) is just a potential example since here a computer script showing the existence is still missing.

In the examples (III)–(V) above, the Albanese variety $\mathrm{Alb}(S)$ of S admits a surjection onto an abelian surface A, and the composition of the Albanese map alb_S of S with this surjection yields $\alpha : S \to A$ of degree $d = \delta = 3$ in (III) and (IV), respectively $d = \delta = 4$ in (V).

Furthermore, the equations of S' in the above-mentioned cases (III)–(V) explicitly show one of the main features of our construction method, namely the item labelled with (2) above: $S' \subset \mathbb{P}(V) \times A'$ is contained here in the projective subbundle given by $\{\sum_j y_j x_j(z) = 0\} \subset \mathbb{P}(V) \times A'$, and this is indeed the manifestation of the geometric interpretation of the exact sequence (0.9) as the embedding of projective bundles (0.11) which generalizes the equality in (0.10).

Still, as the reader might have observed, the equations that we have shown in all the five examples (I)–(V) are either a cubic equation in the variables (y_j), or some quadratic equations: this is due to the use of the theory by Casnati-Ekedahl of Gorenstein covers of small degree $d = 3, 4$ [CE96].

Remark 0.14. Note that the above example labelled with (IV) provides a new irreducible component of the moduli space of minimal surfaces of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree $d = 3$, as showed in [CS22] (see Theorem 2.61). This is actually the first known component with these invariants. Note also that this component is unirational (see Theorem 2.63).

Remark 0.15. In [PiPo17] the authors provide a component of the Main Stream, and hence for the general surface S of this component Theorem 0.2 applies. We point out here that their construction implicitly provides another example where $\mathfrak{H} \neq 0$. Here we have $p_g = q = 2$, $K_S^2 = 7$ and Albanese map of degree $d = 3$ (and we believe $\delta_1 = \delta_2 = 2$, hence $\delta = 4$), but in this case the construction is quite different and not directly related to our method since our Gorenstein Assumption 2.6 (Assumption 0.7) is not verified.

These are the main results presented in Chapter 2 ([AC22]).

Theorem 0.16 (Theorem 2.35)**.** *The CHPP surfaces yield a unirational irreducible connected component of the moduli space of surfaces of general type, which is the unique component of the Main Stream such that there is a surface in this component which fulfills the Gorenstein Assumption 0.7 and has* $K_S^2 = 5$, $p_g(S) = q(S) = 2$ *and Albanese map* $\alpha\colon S \to A := \mathrm{Alb}(S)$ *of degree* $d = 3$*. In particular, this component coincides with the one constructed in* [PePo13a].

Theorem 0.17 (Theorem 2.47, Theorem 2.49, Theorem 2.51, Subsection 2.8.2)**.** *The four dimensional family of PP4 surfaces of general type yields a unirational irreducible connected component of the moduli space of surfaces of general type with* $p_g = q = 2$, $K_S^2 = 6$, $d = 4$ *and* $\delta = 3$*. This component coincides with the one found by Penegini and Polizzi in* [PePo14].

Theorem 0.18 (Theorem 2.63, Theorem 2.64)**.** *All the minimal surfaces S of general type with $p_g = q = 2$, $K_S^2 = 6$, with Albanese map of degree $d = 3$ and satisfying the Generality Assumption 2.17 (Assumption 0.10) with Pfaffian $\delta = 3$ belong to the family described in Subsection 2.9.1.b, whose existence is proved in* [CS22]. *This family yields an irreducible component of the moduli space which is in particular unirational.*

Moreover, under the Generality Assumption with Pfaffian $\delta = 3$, the only other minimal surfaces S of general type with $p_g = q$, $K_S^2 = 6$, having a surjective morphism $\alpha : S \to A$ of degree $d = 3$ onto an abelian surface A, are the surfaces with $p_g = q = 3$ described in Subsection 2.9.1.a.

We have a similar example with $p_g = q = 3$, $K_S^2 = 6$ and $\alpha : S \to A$ a surjective morphism of degree $d = \delta = 4$ onto an abelian surface A, see Section 2.10 (here there is a computer script still missing, see Remark 0.13).

One of the main ingredients of the construction method developed in the joint work [AC22] and carefully described in Chapter 2 is the equivariance of sequence (0.9) with respect to the action of the finite Heisenberg group $\mathcal{H}_D$ associated to a divisor D yielding a polarization of type (δ_1, δ_2) (hence, with Pfaffian $\delta := \delta_1\delta_2$) on an abelian surface A'. As already said before, this group is a central extension by the group of δ_2-th roots of unity $\mu_{\delta_2} \subset \mathbb{C}^*$ of the group of translations $\mathcal{K}(D) \cong (\mathbb{Z}/\delta_1\mathbb{Z} \times \mathbb{Z}/\delta_2\mathbb{Z})^2$ leaving invariant the isomorphism class of the line bundle $\mathcal{O}_{A'}(D)$. Namely, there is a sequence as follows

$$1 \to \mu_{\delta_2} \to \mathcal{H}_D \to \mathcal{K}(D) \to 0. \tag{0.12}$$

By exploiting the Schrödinger representation $V := H^0(A', \mathcal{O}_{A'}(D))$ of the group $\mathcal{H}_D$ we could provide explicit and global equations for some families of surfaces inside the projective bundle $\mathbb{P}(V) \times A'$.

Assuming that D is a very ample divisor, the associated embedding

$$\varphi_D \ : \ A' \hookrightarrow \mathbb{P}^{\delta-1} = \mathbb{P}(V^\vee) \tag{0.13}$$

has the property that the action of $\mathcal{K}(D)$ on A' extends to an action $\tilde{\rho} \colon \mathcal{K}(D) \to \mathrm{PGL}(V^\vee)$ on the projective space $\mathbb{P}(V^\vee)$ with respect to which the embedding φ_D is equivariant.

Even though it is not possible to lift the projective representation $\tilde{\rho}$ to an ordinary representation of $\mathcal{K}(D)$, it is possible to lift it to an ordinary representation of $\mathcal{H}_D$, namely to the dual of the Schrödinger representation $\rho \colon \mathcal{H}_D \to \mathrm{GL}(V)$.

This particular representation V has the property that it is the unique irreducible representation of $\mathcal{H}_D$ such that its center $\mu_{\delta_2} \subset \mathbb{C}^*$ acts via scalar multiplication in the natural way (*Stone-von Neumann Theorem*, see [Mackey49] or [Igu72], Ch. I, Sec. 5, Proposition 2).

Furthermore, if D is of type $(1, \delta)$, the Heisenberg group $\mathcal{H}_\delta := \mathcal{H}_D$ turns out to be a *representation group* for $\mathcal{K}(D) \cong (\mathbb{Z}/\delta\mathbb{Z})^2$: this means that every projective representation $f \colon \mathcal{K}(D) \to \mathrm{PGL}(n, \mathbb{C})$ lifts to an ordinary representation $F \colon \mathcal{H}_\delta \to \mathrm{GL}(n+1, \mathbb{C})$, namely the following diagram commutes

$$\begin{tikzcd}
1 \arrow[r] & \mu_\delta \arrow[r] \arrow[d] & \mathcal{H}_\delta \arrow[r] \arrow[d, "F"] & \mathcal{K}(D) \arrow[r] \arrow[d, "f"] & 0 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathrm{GL}(n+1, \mathbb{C}) \arrow[r] & \mathrm{PGL}(n, \mathbb{C}) \arrow[r] & 1
\end{tikzcd} \tag{0.14}$$

The notion of a representation group was first introduced by Schur [Sch04] in order to study, over an arbitrary field K, projective representations by means of ordinary representations.

He showed that, given a finite group G and denoting by V an arbitrary finite dimensional K-vector space, there exists a *stem extension*

$$1 \to A \to \Gamma \to G \to 1 \quad \text{with} \quad A \cong H^2(G, K^*) \tag{0.15}$$

such that every projective representation $f\colon G \to \mathrm{PGL}(V)$ lifts to an ordinary representation $F\colon \Gamma \to \mathrm{GL}(V)$ fitting into the following diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d, "F"] & G \arrow[r] \arrow[d, "f"] & 1 \\
1 \arrow[r] & K^* \arrow[r] & \mathrm{GL}(V) \arrow[r] & \mathrm{PGL}(V) \arrow[r] & 1
\end{tikzcd} \tag{0.16}$$

Recall that, considering K^* as a G-module with the trivial action, $H^2(G, K^*)$ denotes *the second cohomology group of the group G with coefficients in K^** (see Chapter 1, Section 1.5) and *stem* means that A is central and contained in the commutator subgroup $[\Gamma, \Gamma]$.

In a joint work with Christian Gleissner and Julia Kotonski [AGK23], we extended (under the assumption that K is an algebraically closed field) Schur's concept of a representation group to *semi-projective representations*, which are homomorphisms $f\colon G \to \mathrm{P\Gamma L}(V)$ from a finite group G to the group of semi-projective transformations $\mathrm{P\Gamma L}(V)$ defined as the quotient of the group of semi-linearities $\Gamma\mathrm{L}(V)$ modulo K^*, V being an arbitrary finite dimensional K-vector space.

Remark 0.19. We observe right away that, given a semi-projective representation $f : G \to \mathrm{P\Gamma L}(V) \cong \Gamma\mathrm{L}(V) \rtimes \mathrm{Aut}(K)$, there is an induced action $\varphi\colon G \to \mathrm{Aut}(K)$, $g \mapsto \varphi_g$, which endows K^* with a structure of G-module. Note that in general φ is nontrivial (φ being trivial means that we are indeed in the projective setting).

In Chapter 3 we treat this topic from both a group-theoretic and an algebro-geometric viewpoint.

First, we explain the interplay between semi-projective representations and group cohomology, showing that to every semi-projective representation $f\colon G \to \mathrm{P\Gamma L}(V)$ we can attach a cohomology class $[\alpha] \in H^2(G, K^*)$; namely we have the following.

Proposition 0.20 (Proposition 3.6)**.** *Let $f\colon G \to \mathrm{P\Gamma L}(V)$ be a semi-projective representation and f_g be a representative of the class $f(g)$ for each $g \in G$. Then there exists a map*

$$\alpha\colon G \times G \to K^* \qquad \textit{such that} \qquad f_{gh} = \alpha(g,h) \cdot (f_g \circ f_h)$$

for all $g, h \in G$. The map α is a 2-cocycle, i.e.,

$$\varphi_g\big(\alpha(h,k)\big) \cdot \alpha(gh,k)^{-1} \cdot \alpha(g,hk) \cdot \alpha(g,h)^{-1} = 1.$$

The cohomology class $[\alpha] \in H^2(G, K^*)$ is independent of the chosen representatives f_g.

Then we phrase the above-mentioned lifting problem (see diagram (0.16)) in terms of semi-projective representations, giving a cohomological criterion for a semi-projective representation of a finite group G to lift to a semi-linear representation of an extension Γ of G by a finite abelian group A.

Theorem 0.21 (Theorem 3.11). *Let* $1 \to A \to \Gamma \to G \to 1$ *be an extension of* G *by a finite abelian group* A *with associated cohomology class* $[\beta] \in H^2(G, A)$. *A semi-projective representation* $f \colon G \to P\Gamma L(V)$ *with class* $[\alpha] \in H^2(G, K^*)$ *is induced by a semi-linear representation* $F \colon \Gamma \to \Gamma L(V)$ *if and only if* $[\alpha]$ *belongs to the image of the transgression map*

$$\mathrm{tra}\colon \mathrm{Hom}_G(A, K^*) \to H^2(G, K^*), \quad \lambda \mapsto [\lambda \circ \beta].$$

Finally, we construct for any given finite group G, together with an action φ on an algebraically closed field K, a *φ-twisted representation group.*

This is our main theorem.

Theorem 0.22 (Theorem 3.18). *Let* G *be a finite group and* K *an algebraically closed field. Let* $\varphi \colon G \to \mathrm{Aut}(K)$ *be a fixed action. Then there exists an extension of* G

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

with A *finite and abelian such that the transgression map*

$$\mathrm{tra}\colon \mathrm{Hom}_G(A, K^*) \to H^2(G, K^*), \qquad \lambda \mapsto [\lambda \circ \beta]$$

is an isomorphism.

Therefore, we give the formal definition of a φ-twisted representation group.

Definition 0.23 (Definition 3.20). Let $\varphi \colon G \to \mathrm{Aut}(K)$ be an action of a finite group G on an algebraically closed field K. A group Γ is called a *φ-twisted representation group* of G if there exists an extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \text{with } A \text{ finite and abelian}$$

such that the following conditions hold:

1. $\mathrm{char}(K) \nmid |A|$,
2. $\mathrm{Hom}_G(A, K^*) = \mathrm{Hom}(A, K^*)$,
3. the transgression map $\mathrm{tra}\colon \mathrm{Hom}_G(A, K^*) \to H^2(G, K^*)$ is an isomorphim.

The definition of a φ-twisted representation group is indeed a generalization of Schur's concept of a representation group.

Proposition 0.24 (Proposition 3.22). *In the projective case, i.e., when the* G*-action on* K *is trivial, Definition 0.23 (Definition 3.20) reduces exactly to the classical notion of a representation group (cf.* [Isa94, Corollary 11.20]*), i.e.,*

1. *the extension* $1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ *is stem,*
2. $|A| = |H^2(G, K^*)|$.

Using the previous proposition, we give a proof of the following well-known fact.

Proposition 0.25 (Proposition 3.25)**.** *The Heisenberg group $\mathcal{H}_r$ of the cyclic group $\mathbb{Z}/r$ is a representation group for the group $(\mathbb{Z}/r)^2$.*

Furthermore, we provide a numerical criterion to decide whether a given extension Γ of G is a φ-twisted representation group of G or not.

Proposition 0.26 (Proposition 3.24)**.** *Let $\varphi\colon G \to \operatorname{Aut}(K)$ be a nontrivial action of a finite group G on an algebraically closed field K. Let*

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be an extension by a finite abelian group A. Then Γ is a φ-twisted representation group if and only if the following conditions are satisfied:

1. $|A| = |H^2(G, K^*)|$,
2. $|\operatorname{Hom}_G(A, K^*)| = |\operatorname{Hom}(A, K^*)|$ *and*
3. $|H^1(G, K^*)| = |H^1(\Gamma, K^*)|$.

After providing some basic examples of semi-projective representatios and twisted representation groups, we also give an algorithm for the case $K = \mathbb{C}$ (see *Algorithm 1* in Chapter 3, Subsection 3.5.1), which takes as inputs a finite group G and an action $\varphi\colon G \to \operatorname{Aut}(\mathbb{C})$, and returns all the φ-twisted representation groups of G. Moreover, we provide a MAGMA implementation of this algorithm (which is presented in Appendix B), running it to determine the φ-twisted representation groups of the dihedral group D_4 for all possible actions φ.

Finally, we discuss two interesting situations where semi-projective representations occur naturally:

(1) the problem of extending G-invariant irreducible L-representations defined on a normal subgroup $N \trianglelefteq G$ to the ambient group G for arbitrary fields L (see Chapter 3, Subsection 3.5.2);

(2) the study of homeomorphisms and biholomorphisms of certain quotients of complex tori (see Chapter 3, Subsection 3.5.3).

Note that (2) is indeed our original geometric motivation for studying semi-projective representations.

This thesis is organized in three chapters, which are subdivided in several sections. We give now a brief explanation of the content of each chapter and section.

Chapter 1 treats some of the tools used in Chapter 2 and Chapter 3 which we believe are relevant for the discussion.

In Section 1.1 we recall the notion of a cover in algebraic geometry. In particular, in Subsection 1.1.2, we define for a given degree d cover $\pi\colon X \to Y$ the so-called *Tschirnhaus bundle* $\mathcal{E}^\vee$, which is a locally free $\mathcal{O}_A$-module of rank $d-1$. Then, after briefly explaining the *relative spectrum construction of a cover* (Subsection 1.1.3), we focus on covers of small degree $d = 3, 4$. More precisely, in Subsection 1.1.4 we briefly recall the theory of triple and quadruple covers developed respectively by Miranda [Mir85] and Hahn-Miranda [HM99], and then the theory of Casnati-Ekedahl for the Gorenstein case [CE96]. Here, given a Gorenstein cover $\pi\colon X \to Y$ of degree $d \geq 3$, the factorization theorem of Casnati-Ekedahl (see Theorem 1.37) implies that the total space X embeds into the projective bundle $\mathbb{P}(\mathcal{E}^\vee)$ associated with the Tschirnhaus bundle $\mathcal{E}^\vee$ of π: this is the reason why in Section 1.2 we discuss some important features of projective bundles, recalling also some formulae which turn out to be useful in Chapter 2.

In Section 1.3 we introduce the so-called Heisenberg group $\mathcal{H}(H)$ of a given finite abelian group H, which turns out to be one of the main tools used by the construction method described in Chapter 2. After constructing the group $\mathcal{H}(H)$, we show its main features, focusing on the case where $H \cong (\mathbb{Z}/r)$ is cyclic with some explicit computations for $r = 2, 3$. It is of interest to recall that in the cyclic case the Heisenberg group $\mathcal{H}_r := \mathcal{H}(\mathbb{Z}/r)$ turns out to be a *representation group* (in the sense of Schur [Sch04]) for $(\mathbb{Z}/r)^2$. This is a well-known fact and we provide a new proof in Chapter 3 by using the theory of semi-projective representations developed in [AGK23].

Section 1.4 is devoted to the discussion on some features of line bundles on complex tori. Given a divisor D on a complex torus, we introduce its associated homomorphism Φ_D in Subsection 1.4.1. Then, given an abelian variety A of dimension g, we introduce the *theta group* $\mathcal{G}(D)$ of a divisor D (Subsection 1.4.2). Under the assumption that D yields a polarization of type $(\delta_1, \ldots, \delta_g)$, we recall in Subsection 1.4.3 that the Heisenberg group $\mathcal{H}_D := \mathcal{H}(\bigoplus_{i=1}^{g} \mathbb{Z}/\delta_i)$ can be considered as a finite subgroup of $\mathcal{G}(D)$ in a sense that we make precise therein. Moreover, in Subsection 1.4.4 we briefly explain how the Heisenberg group acts on sheaves over a given abelian variety. Subsection 1.4.5 concludes our overview on line bundles on complex tori recalling in particular the notion of *Fourier-Mukai transform* for a non-degenerate line bundle.

Finally, Section 1.5 is devoted to the description of the most important features of *Group Cohomology*, which is one of the main tool used in Chapter 3 to develop the theory of semi-projective representations and twisted representation groups.

The main purpose of Chapter 2 is to describe the construction method for minimal surfaces of general type S with $p_g = q$ developed in a joint work with Fabrizio Catanese, [AC22]. The chapter is structured as follows.

In Section 2.1 we introduce the objects we would like to construct. We call them *surfaces with AP* (Definition 2.1) and describe in detail their main features.

In Section 2.2 and Section 2.3 we discuss the technical assumptions introduced before as Assumption 0.7 and Assumption 0.10 (see Gorenstein Assumption 2.6 and Generality Assumption 2.17). In Section 2.4 we provide a detailed description of our construction method.

Then we construct some known families of surfaces with $p_g = q = 2, 3$, providing global and explicit equations inside a projective bundle (see Sections 2.5–2.10). We also sketch the construction of a new irreducible component of the Gieseker moduli space $\mathcal{M}_{6,2,2}$ of surfaces of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree $d = 3$ (Subsection 2.9.1.b). We point out that the existence of this component is proved in [CS22] and that this is the first known component with these invariants. Moreover, we show that this component is unirational (see Theorem 2.63).

More in detail, in Section 2.5 we construct a family of surfaces called as in [AC22] *CHPP surfaces* and named after Jungkai Alfred Chen, Christopher Derek Hacon, Matteo Penegini and Francesco Polizzi. Chen and Hacon constructed in [CH06] a surface of general type with $p_g = q = 2$ and $K_S^2 = 5$, and afterwards Penegini and Polizzi studied in [PePo13a] the family containing such a surface, which is called therein family of *Chen-Hacon surfaces* and provides a four dimensional irreducible connected component of the moduli space of surfaces of general type with $p_g = q = 2$, $K_S^2 = 5$ and Albanese map of degree 3. In Section 2.6 we study the moduli space of CHPP surfaces, showing in particular that they yield a component which is *unirational* and corresponds to the component of Chen-Hacon surfaces constructed in [PePo13a].

Analogously, in Section 2.7 we construct a family of surfaces called as in [AC22] *PP4 surfaces* and named after Matteo Penegini and Francesco Polizzi. Indeed, they constructed in [PePo14] a four dimensional irreducible component of the moduli space of surfaces of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 4. In Section 2.8 we study the moduli space of PP4 surfaces, showing in particular that they yield a component which corresponds to the one constructed in [PePo14]. Moreover, we show that this component is *unirational* and *connected*.

Section 2.9 and Section 2.10 are devoted to the construction of surfaces S with $p_g = q$ fulfilling the above-mentioned Generality Assumption (Assumption 2.17).

More precisely, in Section 2.9 we construct two families of surfaces with $p_g = q$, both having $K_S^2 = 6$: one has $p_g = q = 2$ and is called as in [AC22] *AC3 family* (Subsection 2.9.1.b), while the other one has $p_g = q = 3$ (Subsection 2.9.1.a). As showed in Subsection 2.9.2, these two families contain all the surfaces with AP fulfilling the Generality Assumption 2.17 with $d = \delta = 3$, where d is the degree of the surjective morphism $\alpha\colon S \to A$ given by definition of a surface with AP and δ is the Pfaffian of the polarization D provided by the Generality Assumption.

In Section 2.10 we analyze surfaces with AP fulfilling the Generality Assumption 2.17 with $d = 4$, providing a potential example with $d = \delta = 4$: this should give a family of surfaces with $p_g = q = 3$ and $K_S^2 = 6$. Here we sketch a possible proof of the existence of this family, but a computer script is still missing.

Section 2.11 is devoted to the explicit computation of the degree d of the Albanese map for the three components labelled with *UnMix* in Table 1 of [Pen11] (see also items n. 15, 16, 17 of Table A in Appendix A). Indeed, in [Pen13] the author points out that for these families $d \leq 6$ is an upper bound for the degree d of the Albanese map. We show that the degrees are respectively $d = 4, 6, 4$ (using the order of Table 1 in [Pen11]), confirming a personal communication by Penegini, who had a different and more involved proof. We also describe the Galois closure of the Albanese map.

Finally, in Section 2.12 we briefly summarize what we have done from the perspective of our construction method, pointing out some open questions which potentially constitutes the starting point for a future research program.

Chapter 3 is dedicated to the description of the content of the joint work [AGK23]. We now explain how this chapter is structured.

After briefly introducing our general setting in Section 3.1, we discuss in Section 3.2 the interplay between semi-projective representations and group cohomology.

In Section 3.3 we phrase the lifting problem (see diagram (0.16)) and give a cohomological criterion for a semi-projective representation of a finite group G to lift to a semi-linear representation of an extension Γ of G by a finite abelian group A.

In Section 3.4, we construct for any given finite group G, together with an action φ on an algebraically closed field K, a *φ-twisted representation group*. For this purpose, we adapt Isaacs construction of a representation group in the projective case [Isa94, 11] to our setup. Then we give a cohomological characterization of a φ-twisted representation group and show that it coincides with the classical notion in case that φ is the trivial action. As an immediate application, in Subsection 3.4.1 we give a proof of the well-known fact that the Heisenberg group $\mathcal{H}_r$ of a cyclic group $\mathbb{Z}/r$ is a representation group for $(\mathbb{Z}/r)^2$.

The last part of the chapter, Section 3.5, is devoted to examples and applications. Besides basic examples of semi-projective representations and twisted representation groups, in Subsection 3.5.1 we develop an algorithm which allows us to determine all the φ-twisted representation groups of a given finite group G under the assumption that $K = \mathbb{C}$ and φ maps to $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Running a MAGMA implementation (see Appendix B), we determine the φ-twisted representation groups of the dihedral group D_4 for all possible actions φ. Finally, we explain the relations between semi-projective representations and

- the extendability of L-representations (Subsection 3.5.2),
- the study of homeomorphism and biholomorphism classes of torus quotients (Subsection 3.5.3).

Finally, the reader can find in Appendix A a brief overview on the known irreducible components of the moduli space of minimal surfaces of general type with $p_g = q = 2$ and maximal Albanese dimension. Some relevant information on these components are displayed in Table A.

The content of Appendix B is our MAGMA implementation of the algorithm presented in Chapter 3.

Notation

Throughout this thesis we deal with algebraic varieties over the field $\mathbb{C}$ of complex numbers. For us, an *algebraic variety* X is a quasi-projective integral scheme over $\mathbb{C}$ (in the sense of [Har77]) with structure sheaf $\mathcal{O}_X$.

We recall that there is an equivalence of categories between *locally free* $\mathcal{O}_X$*-modules* and *vector bundles over* X. More precisely, let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $r+1$, $r \geq 0$. Denoting by $\mathcal{E}^\vee$ the *dual sheaf of* $\mathcal{E}$, namely $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$, and by $S(\mathcal{E}) = \bigoplus_m S^m(\mathcal{E})$ or $\mathbf{Sym}(\mathcal{E}) = \bigoplus_m \mathbf{Sym}^m(\mathcal{E})$ the *symmetric algebra of* $\mathcal{E}$, we define

$$\mathbb{V}(\mathcal{E}) := \mathbf{Spec}_{\mathcal{O}_X}(S(\mathcal{E}^\vee)) \to X$$

as *the vector bundle associated with* $\mathcal{E}$, which is a vector bundle of rank $r+1$ whose sheaf of sections is, up to isomorphism, $\mathcal{E}$. Namely,

$$\mathscr{S}(\mathbb{V}(\mathcal{E})/X) = \mathscr{S}(\mathbf{Spec}_{\mathcal{O}_X}(S(\mathcal{E}^\vee))/X) \cong \mathcal{E},$$

where $\mathscr{S}(X/Y)$ is defined to be the *sheaf of (regular) sections* of the morphism $X \to Y$, cf. [Har77, Ch. II, Ex. 5.18].

Furthermore, if $r \geq 1$, following topologists' notation we define

$$\mathbb{P}(\mathcal{E}) := \mathbf{Proj}_{\mathcal{O}_X}(S(\mathcal{E}^\vee)) \to X$$

as the *projective bundle associated with* $\mathcal{E}$. This is a rank r projective bundle whose fibres consist of one-dimensional subspaces of the fibres of $\mathbb{V}(\mathcal{E})$ (see Chapter 1, Section 1.2 for further details on the notion of a projective bundle).

Still, we inform the reader that in the case $r = 0$ we will use the words *line bundle* and *geometric line bundle* referring to a locally free $\mathcal{O}_X$-module of rank 1, respectively to a vector bundle of rank 1. Hence, we will speak of line bundles and their associated geometric line bundles.

Let X, Y be two complex algebraic varieties. We denote by:

Ω_X	the *sheaf of Kähler differentials* on X
ω_X	the *dualizing sheaf* of X
$\Omega_{X\vert Y}$	the *sheaf of relative differentials* of a morphism $f\colon X \to Y$
$\omega_{X\vert Y}$	the *relative dualizing sheaf* of a morphism $f\colon X \to Y$

Let S be a *surface*, i.e., a smooth complex projective variety of dimension 2. We denote by:

Ω^p_S	the *sheaf of holomorphic p-forms* on S
$\omega_S := \Omega^2_S =: \mathcal{O}_S(K_S)$	the *canonical sheaf* of S
$p_g := p_g(S) := h^0(S, \omega_S)$	the *geometric genus* of S
$q := q(S) := h^1(S, \mathcal{O}_S)$	the *irregularity* of S
$K^2 := K^2_S$	the *self-intersection of the canonical divisor* K_S
$\chi(S) := \sum_{i=0}^{2}(-1)^i h^i(S, \mathcal{O}_S)$	the *holomorphic Euler-Poincaré characteristic* of S
$e(S) := \sum_{i=0}^{4}(-1)^i b_i(S)$	the *topological Euler number* of S
$P_n := P_n(S) := h^0(S, \omega_S^{\otimes n})$	the *n-th plurigenus* of S
$alb_S \colon S \to \mathrm{Alb}(S)$	the *Albanese map* of S

Other symbols:

$\mathbb{Z}/r\mathbb{Z},\ \mathbb{Z}/r$	the cyclic group of order r
$\mu_r \subset \mathbb{C}^*$	the group of r-th roots of unity
$[G, G]$	the commutator subgroup of a group G
$Z(G)$	the centre of a group G
$\mathcal{H}(H)$	the Heisenberg group of a finite abelian group H
$\mathcal{H}_r$	the Heisenberg group of the cyclic group $\mathbb{Z}/r$
$\widehat{X} = \mathrm{Pic}^0(X)$	the dual complex torus of the complex torus X
$\widehat{\mathscr{L}}$	the Fourier-Mukai transform of the line bundle $\mathscr{L}$ on an abelian variety
$\Phi_D \colon X \to \widehat{X}$	the homomorphism associated with the line bundle $\mathcal{O}_X(D)$ over the complex torus X
$\mathcal{K}(D)$	the kernel of $\Phi_D \colon X \to \widehat{X}$
$\mathcal{G}(D)$	the theta group of the line bundle $\mathcal{O}_A(D)$ over the abelian variety A
$\mathcal{H}^\infty_D$	the infinite Heisenberg group of the line bundle $\mathcal{O}_A(D)$ over the abelian variety A

$\mathcal{H}_D$	the (finite) Heisenberg group of the line bundle $\mathcal{O}_A(D)$ over the abelian variety A
$\mathrm{Stab}_G(p)$	the stabilizer of the point p with respect to the action of G
$\mathrm{Core}_G(H)$	the normal core of the subgroup $H \leq G$
$\mathfrak{S}_n$	the permutation group of n elements
$\mathbb{C}(H)$	the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions defined on the finite abelian group H
$\mathrm{im}\, f$	the image of the map f
$\mathfrak{Im} z$	the imaginary part of $z \in \mathbb{C}$
$\langle\langle S \rangle\rangle$	the subgroup normally generated by $S \subset G$
$\mathrm{Tr}\colon \pi_*\mathcal{O}_X \to \mathcal{O}_Y$	the trace map of the cover $\pi\colon X \to Y$
$\mathrm{tra}\colon \mathrm{Hom}_G(A, K^*) \to H^2(G, K^*)$	the transgression map with respect to the group extension $1 \to A \to \Gamma \to G \to 1$

Chapter 1

Preliminaries

Throughout this chapter we recall several different tools used in Chapter 2 and Chapter 3 which we believe are relevant for the discussion.

1.1 Covers in Algebraic Geometry

Let X, Y be two algebraic varieties and let $\pi\colon X \to Y$ be a dominant morphism, i.e., $\overline{\pi(X)} = Y$. For the following definition we refer the reader to [Sha13], Definition 1.1 on page 60 and Definition 1.2 on page 62.

Definition 1.1 (Finite morphism)**.** A dominant morphism of algebraic varieties $\pi\colon X \to Y$ is said to be *finite* if every point $y \in Y$ has an affine neighborhood $U \subset Y$ such that $\pi^{-1}(U)$ is affine and the induced dominant morphism $\pi^{-1}(U) \to U$ carries an integral ring extension $\mathbb{C}[U] \hookrightarrow \mathbb{C}[\pi^{-1}(U)]$ (equivalently, $\mathbb{C}[\pi^{-1}(U)]$ is a finitely generated $\mathbb{C}[U]$-module).

Remark 1.2. In the definition above the existence of an open affine covering of Y fulfilling that property is equivalent to require the given property for every affine open subset of Y, see [Har77, Ch. II, Sec. 3, p. 84].

Given a finite morphism $\pi\colon X \to Y$, since X, Y are irreducible and π is dominant, we have a field extension

$$\pi^*(\mathbb{C}(Y)) \subset \mathbb{C}(X)$$

whose degree $[\mathbb{C}(X) : \pi^*(\mathbb{C}(Y))]$ is a finite number (see [Sha13], p. 141).

Definition 1.3 (Degree of a finite morphism)**.** Let $\pi\colon X \to Y$ be a finite morphism between algebraic varieties. We define $\deg(\pi) := [\mathbb{C}(X) : \pi^*(\mathbb{C}(Y))]$ to be the *degree of the finite morphism* $\pi\colon X \to Y$.

Definition 1.4 (Galois finite morphism)**.** Let $\pi\colon X \to Y$ be a finite morphism between algebraic varieties. We say that π is *Galois* if the induced field extension

$$\pi^*(\mathbb{C}(Y)) \subset \mathbb{C}(X)$$

is Galois.

Definition 1.5 (Deck transformation). Given a $\pi\colon X \to Y$ between algebraic varieties, an automorphism $g\colon X \to X$ such that the following diagram

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\
& Y &
\end{tikzcd}$$

commutes is said to be a *deck transformation* of the π.

We denote by $Deck(\pi)$ the group of deck transformations of π.

Remark 1.6. Note that $Deck(\pi)$ can also be defined as the Galois group of the finite field extension

$$\pi^*(\mathbb{C}(Y)) \subset \mathbb{C}(X).$$

This implies in particular that $Deck(\pi)$ is a finite group (see [Gab08], page 232).

Remark 1.7. Given a finite morphism $\pi\colon X \to Y$ of degree d between algebraic varieties and denoting by $G := Deck(\pi)$ the group of deck transformations of π, we have the following commutative diagram

$$\begin{tikzcd}
X \arrow[dd, "\pi"'] \arrow[dr] & \\
& X/G \arrow[dl] \\
Y &
\end{tikzcd}$$

In light of Remark 1.6 we have field extensions as follows

$$\mathbb{C}(Y) \subset \mathbb{C}(X)^G \subset \mathbb{C}(X),$$

and hence it follows that $|G|$ divides d. As a result,

$$\pi\colon X \to Y \quad \text{is Galois} \qquad \Longleftrightarrow \qquad |G| = d \qquad \Longleftrightarrow \qquad Y \cong X/G.$$

If $\pi\colon X \to Y$ is a finite morphism, then any point $y \in Y$ has at most a finite number of preimages, as explained in [Sha13], page 61.

The following proposition points out that such a fibre $\pi^{-1}(y)$ is never empty.

Proposition 1.8 ([Sha13, Theorem 1.12]). *Every finite morphism $\pi\colon X \to Y$ between algebraic varieties is surjective.*

Given a finite morphism $\pi\colon X \to Y$ between algebraic varieties, we expect that its degree d gives un upper bound for the cardinality of each fibre $\pi^{-1}(y)$, namely that $|\pi^{-1}(y)| \leq d$ for any $y \in Y$.

This is in general not true as the following example shows.

Example 1.9 (Nodal cubic curve). Let Y be the *nodal cubic curve* described in $\mathbb{C}^2$ by the equation

$$y^2 = x^2 + x^3$$

and consider its rational parametrization

$$\pi\colon \mathbb{C} \to Y, \qquad t \longmapsto (t^2-1,\ t(t^2-1)).$$

It is easy to see that π is a finite morphism and $\deg(\pi) = 1$ since π is birational. However, the singular point $(0,0) \in \mathbb{C}^2$ has two preimages, namely ± 1.

Indeed, what we need is the assumption that Y is normal. More precisely, we have the following result (see [Sha13], Theorem 2.28).

Proposition 1.10. *Let $\pi\colon X \to Y$ be a finite morphism of degree d between algebraic varieties, and assume that Y is normal. Then for any $y \in Y$*

$$|\pi^{-1}(y)| \leq d.$$

For the next definition we refer the reader to page 254 of [Har77] (Ch. III, Sec. 9).

Definition 1.11 (Flat morphism)**.** A morphism of algebraic varieties $\pi\colon X \to Y$ is said to be *flat* if, for every $x \in X$, $\mathcal{O}_{X,x}$ is a flat module over $\mathcal{O}_{Y,\pi(x)}$, where $\mathcal{O}_{X,x}$ is considered as an $\mathcal{O}_{Y,\pi(x)}$-module via the natural map $\pi^{\#}\colon \mathcal{O}_{Y,\pi(x)} \to \mathcal{O}_{X,x}$.

Now we are finally ready to give the definition of a *cover in algebraic geometry.*

Definition 1.12 (Cover)**.** A *cover* is defined to be a finite and flat morphism $\pi\colon X \to Y$ between algebraic varieties. We say that *X is a cover of Y* referring implicitly to the cover $\pi\colon X \to Y$. By *degree of a cover* we mean the degree as a finite morphism.

Given a cover $\pi\colon X \to Y$, it might be unclear for the reader which is the role played by flatness. The following proposition should clarify ideas (we refer to [Mum99, III, Sec. 10, Prop. 2]).

Proposition 1.13. *Let $\pi\colon X \to Y$ be a finite morphism. Then it holds true that*

$$\pi \textit{ is flat} \qquad \Longleftrightarrow \qquad \pi_*\mathcal{O}_X \textit{ is a locally free } \mathcal{O}_Y\textit{-module.}$$

Corollary 1.14. *Let $\pi\colon X \to Y$ be a cover. Then $\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module of rank* $\deg(\pi)$.

Even though the previous result helps us better understand the meaning of flatness, at this stage it is still not clear why we want to consider finite morphisms which are also flat. One of the reasons relies on the fact that we want to deal with "nice" algebraic varieties, and it turns out that a finite morphism $\pi\colon X \to Y$ is also flat if X, Y are "nice" enough. The following theorem, which can be considered as a weaker version of the so-called *miracle flatness theorem* (see [Har77], Ch. III, Exercise 10.9) since every fibre of a finite morphism $\pi\colon X \to Y$ has dimension $0 = \dim X - \dim Y$, makes our previous statement precise.

Theorem 1.15 (Miracle Flatness)**.** *Let $\pi\colon X \to Y$ be a finite morphism between algebraic varieties. Assume that X is Cohen-Macaulay and Y is smooth. Then π is flat.*

Corollary 1.16. *Let $\pi\colon X \to Y$ be a finite morphism between algebraic varieties of dimension* 2*. Assume that X is normal and Y is smooth. Then π is flat.*

Proof. It is enough to recall that, in dimension 2, every normal algebraic variety is Cohen-Macaulay: this is a consequence of the so-called *Serre's criterion for normality* ([Har77], Ch. II, Theorem 8.22A) which states in particular that an algebraic variety is normal if and only if it satifies conditions R_1 (regularity in codimension 1) and S_2. We refer the reader to Definition 5.7.2 on page 103 of [Gro65] for the definition of *Serre's conditions* S_k, $k \in \mathbb{Z}$, and recall that an algebraic variety of dimension n is Cohen-Macaulay if and only if it satisfies S_n (cf. Remark 5.7.3 (i) of [Gro65]). □

1.1.1 Ramification Locus and Branch Locus

Next we define the notions of *ramification locus* and *branch locus* of a given cover $\pi\colon X \to Y$.

Given a cover $\pi\colon X \to Y$ and denoting by Ω_X the *sheaf of Kähler differentials* of X, we consider the *sheaf of relative differentials* $\Omega_{X|Y}$, which is a coherent $\mathcal{O}_X$-module since π is finite (see [Har77], p. 175). We recall that the sheaf $\Omega_{X|Y}$ measures the difference between Ω_X and the pull-back $\pi^*\Omega_Y$. Namely, we have an exact sequence called the *relative cotangent sequence* (cf. [Har77], Ch. II, Proposition 8.11)

$$\pi^*\Omega_Y \to \Omega_X \to \Omega_{X|Y} \to 0. \tag{1.1}$$

Definition 1.17 (Ramification locus)**.** Given a cover $\pi\colon X \to Y$, we define the *ramification locus* R of π as $\operatorname{Supp}\Omega_{X|Y}$, the support of the sheaf of relative differentials $\Omega_{X|Y}$. If $R = \emptyset$ we say that π is *unramified* or *unbranched* or *étale*.

Definition 1.18 (Branch locus)**.** Given a cover $\pi\colon X \to Y$, we define the *branch locus* $\mathcal{B}$ of π as the image of the ramification locus R, namely $\mathcal{B} := \pi(R)$, if $R \neq \emptyset$, otherwise we set $\mathcal{B} := \emptyset$.

Remark 1.19. Recalling that the support of a coherent sheaf is a closed subset ([Har77], exercise 5.6(c)), the ramification locus R of a cover $\pi\colon X \to Y$ is by definition a closed subset of X. Moreover, we observe that $\Omega_{X|Y} = 0$ on an open set $U \subset X$ (since π is dominant by [Har77], Ch. III, Lemma 10.5), and hence $R \subset X$ is a proper closed subset. Also, since a finite morphism is closed (see [Har77], exercise II.3.5(b)), it follows by definition that the branch locus $\mathcal{B}$ is a proper closed subset of Y.

Indeed, given a cover $\pi\colon X \to Y$, under the assumption that X is normal and Y is smooth we have more information on the ramification locus R (and hence on the branch locus $\mathcal{B}$).

Proposition 1.20 (Purity of the Branch Locus, [Zar58])**.** *Let* $\pi\colon X \to Y$ *be a cover. Assume that* X *is normal and* Y *is smooth. Then the ramification locus* R *is a reduced and effective Weil divisor of* X*.*

Hence, given a cover $\pi\colon X \to Y$, the ramification locus R and the branch locus $\mathcal{B}$ are proper closed subsets of X, respectively of Y. Moreover, under the assumption that X is normal and Y is smooth, they are indeed (Weil) divisors.

Remark 1.21. Let $\pi\colon X \to Y$ be a cover. Under the assumption that Y is normal the branch locus $\mathcal{B}$ can also be defined as the set of points $y \in Y$ such that $|\pi^{-1}(y)| < \deg(\pi)$ (see [Sha13], page 142).

Remark 1.22. Note that in literature there are some inconsistencies about the use of the words *ramification locus* and *branch locus*. For instance, Zariski in [Zar58] refers to the ramification locus in our sense by using the term branch locus.

Remark 1.23. Given a cover $\pi\colon X \to Y$ where X, Y are smooth, consider the first map of the relative cotangent sequence (1.1), namely

$$\pi^*\Omega_Y \to \Omega_X. \tag{1.2}$$

Since X, Y are smooth, this is a map between locally free $\mathcal{O}_X$-modules of the same rank. Hence, taking the top exterior power of it we get

$$\pi^*\omega_Y \to \omega_X,$$

and tensoring the latter by $(\pi^*\omega_Y)^{-1}$ we obtain

$$\mathcal{O}_X \to \omega_X \otimes (\pi^*\omega_Y)^{-1}, \tag{1.3}$$

that is a global section s of the line bundle $\omega_X \otimes (\pi^*\omega_Y)^{-1}$. Denoting by R the zero locus of s, we get then an effective divisor R such that

$$\omega_X = \pi^*\omega_Y \otimes \mathcal{O}_X(R). \tag{1.4}$$

The previous formula is known as *Riemann-Hurwitz formula* (cf. [BHPV04], I.16) and the divisor R coincides with the ramification locus of the cover $\pi\colon X \to Y$.

Indeed, under the assumption that X and Y are smooth and both of dimension n, the ramification locus R of $\pi\colon X \to Y$ has a more elementary description as the set of critical points of the derivative of π. More precisely, recalling that the derivative $D\pi_x$ of π at $x \in X$ is a $\mathbb{C}$-linear map $D\pi_x\colon \Theta_{X,x} \to \Theta_{Y,f(x)}$ between the tangent spaces $\Theta_{X,x} \cong (m_x/m_x^2)^\vee \cong \mathbb{C}^n$ and $\Theta_{Y,f(x)} \cong (m_{f(x)}/m_{f(x)}^2)^\vee \cong \mathbb{C}^n$ (see [Sha13], Chapter 2, Section 1.3), we have that

$$R = \{x \in X \mid \operatorname{rank}(D\pi_x) < n\}.$$

Moreover, in this setting the branch locus $\mathcal{B} = \pi(R)$ can be described as the set of points $y \in Y$ such that $|\pi^{-1}(y)| < \deg(\pi)$ (see Remark 1.21).

1.1.2 Tschirnhaus Bundle

Let $\pi\colon X \to Y$ be a cover of degree $d \geq 2$. We define here the so-called *trace map*

$$\operatorname{Tr}\colon \pi_*\mathcal{O}_X \to \mathcal{O}_Y \tag{1.5}$$

as follows (see [HM99]).

Recall that $\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module with respect to the natural structure given by the pull-back map

$$\begin{aligned} \pi^\#\colon \mathcal{O}_Y &\to \pi_*\mathcal{O}_X \\ f &\longmapsto \pi^\#(f) := f \circ \pi. \end{aligned}$$

Then there exists an affine open covering $\{U_i\}_{i\in I}$ of Y such that for each $i \in I$

$$\pi_*\mathcal{O}_X(U_i) \cong \mathcal{O}_Y(U_i)^{\oplus d}.$$

For every affine open set $U_i \subset Y$, each $\alpha \in \pi_*\mathcal{O}_X(U_i) := \mathcal{O}_X(\pi^{-1}(U_i))$ defines a $\mathcal{O}_Y(U_i)$-linear map $\underline{\alpha}$ given by multiplication with α, namely

$$\begin{aligned} \underline{\alpha}\colon \pi_*\mathcal{O}_X(U_i) &\to \pi_*\mathcal{O}_X(U_i) \\ \beta &\longmapsto \alpha\cdot\beta. \end{aligned} \tag{1.6}$$

If we choose a basis for $\pi_*\mathcal{O}_X(U_i) \cong \mathcal{O}_Y(U_i)^{\oplus d}$ over $\mathcal{O}_Y(U_i)$, then $\underline{\alpha}$ determines a $(d\times d)$ matrix A_α.

Hence, if $\mathrm{tr}(A_\alpha)$ denotes the trace of the matrix A_α, we define a map as follows

$$\begin{aligned} \mathrm{Tr}_{U_i}\colon \pi_*\mathcal{O}_X(U_i) &\to \mathcal{O}_Y(U_i) \\ \alpha &\longmapsto \frac{1}{d}\,\mathrm{tr}(A_\alpha). \end{aligned} \tag{1.7}$$

Since this definition is independent of the choice of a basis (different basis give similar matrices), the map is well-defined.

Moreover, we see right away that Tr_{U_i} is surjective since, by definition of Tr_{U_i}, we have $\mathrm{Tr}_{U_i}\circ\pi^{\#}_{U_i} = \mathrm{Id}_{\mathcal{O}_Y(U_i)}$.

Hence, as Tr_{U_i} glue together, we get a surjective map of $\mathcal{O}_Y$-modules like (1.5) which yields the following exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^\vee \arrow[r] & \pi_*\mathcal{O}_X \arrow[r, "\mathrm{Tr}"] & \mathcal{O}_Y \arrow[r] & 0, \end{tikzcd} \tag{1.8}$$

where $\mathcal{E}^\vee$ is a locally free $\mathcal{O}_Y$-module of rank $d-1$. This is locally the *"trace-zero" module.*

Definition 1.24. We call $\mathcal{E}^\vee$ the *Tschirnhaus bundle* of the cover $\pi : X \to Y$.

Remark 1.25. The symbol $\mathcal{E}^\vee$ we have chosen to denote the Tschirnhaus bundle might seem to the reader weird. However, we will see in Chapter 2 that our choice turns out to be useful since we will actually need to work with the dual sheaf $\mathcal{E}$ of the Tschirnhaus bundle $\mathcal{E}^\vee$. Moreover, the notation $\mathcal{E}^\vee$ reminds the reader that the Tschirnhaus bundle is a locally free sheaf with no section, namely $h^0(\mathcal{E}^\vee) = 0$.

Since it holds true

$$\mathrm{Tr}\circ\pi^{\#} = \mathrm{Id}_{\mathcal{O}_Y},$$

the sequence (1.8) splits via $\pi^{\#}$ and we can write

$$\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}^\vee. \tag{1.9}$$

Remark 1.26. Note that, since sequence (1.8) splits via the pull-back map $\pi^{\#}$, we can equivalently define the *Tschirnhaus bundle* $\mathcal{E}^\vee$ as the cokernel of the pull-back map $\pi^{\#}$, namely via the split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_Y \arrow[r, "\pi^{\#}"] & \pi_*\mathcal{O}_X \arrow[r] & \mathcal{E}^\vee \arrow[r] & 0. \end{tikzcd} \tag{1.10}$$

1.1.3 The *Spec* Construction of a Cover

Let Y be a fixed algebraic variety. It turns out that all covers $\pi\colon X \to Y$ over Y can be constructed by assigning some data on Y. Let us make this statement more precise.

Recall that, given a degree d cover $\pi\colon X \to Y$, the pushforward $\pi_*\mathcal{O}_X$ sits into (1.8) which splits as in (1.9)

Hence, $\pi\colon X \to Y$ yields a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ of rank $d-1$ together with a ring structure on $\mathcal{O}_Y \oplus \mathcal{E}^\vee$ which makes the latter an $\mathcal{O}_Y$-algebra.

Also, recall that for every quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{A}$ we can construct an affine morphism

$$\pi'\colon X' := \mathbf{Spec}_{\mathcal{O}_Y}\,\mathcal{A} \to \mathbf{Spec}_{\mathcal{O}_Y}\,\mathcal{O}_Y \cong Y$$

such that $\pi'_*\mathcal{O}_{X'} = \mathcal{A}$ (see [Har77], exercise II.5.17(c)). Note that $\pi'\colon X' \to Y$ is finite if $\mathcal{A}$ is coherent (see [Mum99], Ch. III., Sec. 2, Definition 2). Hence, if $\mathcal{A}$ is locally free, π' is finite and also flat by Proposition 1.13. This construction is called *relative spectrum construction* of an affine morphism.

Conversely, given an affine morphism $\pi' : X' \to Y$, we observe that $\pi'_*\mathcal{O}_{X'}$ is a quasi-coherent $\mathcal{O}_Y$-algebra. Performing the relative spectrum construction with such a sheaf of algebras $\pi'_*\mathcal{O}_{X'}$, we recover the given morphism $\pi'\colon X' \to Y$ since it holds

$$X' \cong \mathbf{Spec}_{\mathcal{O}_Y}(\pi'_*\mathcal{O}_{X'}),$$

see [Har77], exercise II.5.17(d).

Now let $\pi\colon X \to Y$ be a degree d cover with Tschirnhaus bundle $\mathcal{E}^\vee$. Recalling that a finite morphism is affine, if we perform the relative spectrum construction with $\mathcal{A} = \pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}^\vee$, we get back the given cover $\pi\colon X \to Y$, namely

$$X \cong \mathbf{Spec}_{\mathcal{O}_Y}(\mathcal{O}_Y \oplus \mathcal{E}^\vee). \tag{1.11}$$

Therefore, every cover $\pi\colon X \to Y$ of degree d over a fixed algebraic variety Y is uniquely determined by the assignment of a rank $d-1$ locally free $\mathcal{O}_Y$-module $\mathcal{E}$ together with a structure of $\mathcal{O}_Y$-algebra on $\mathcal{O}_Y \oplus \mathcal{E}^\vee$.

Remark 1.27. Let $d \geq 2$ be an integer and $\mathcal{E}$ a rank $d-1$ locally free $\mathcal{O}_Y$-module. Suppose that a structure of $\mathcal{O}_Y$-algebra is given on $\mathcal{O}_Y \oplus \mathcal{E}^\vee$.

Then if we perform the relative spectrum construction with $\mathcal{A} = \mathcal{O}_Y \oplus \mathcal{E}^\vee$, we get a map

$$\pi\colon X := \mathbf{Spec}_{\mathcal{O}_Y}(\mathcal{O}_Y \oplus \mathcal{E}^\vee) \to \mathbf{Spec}_{\mathcal{O}_Y}(\mathcal{O}_Y) = Y \tag{1.12}$$

which is a finite and flat morphism of degree d.

However, the space $X = \mathbf{Spec}_{\mathcal{O}_Y}(\mathcal{O}_Y \oplus \mathcal{E}^\vee)$ constructed in this way is not even an algebraic variety in general. This is an algebraic set which might be neither irreducible nor reduced.

The previous remark gives us the chance to point out an important fact: if we want to construct a degree d cover $\pi : X \to Y$ starting from a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ of rank $d-1$, we need to require that the ring structure we provide on $\mathcal{O}_Y \oplus \mathcal{E}^\vee$ yields a global space X with the desired features. This is not for granted.

Remark 1.28 (Factorization of a cover)**.** Given a cover $\pi\colon X \to Y$ of degree $d \geq 2$ with Tschirnhaus bundle $\mathcal{E}^\vee$, we note that there is a natural surjection of $\mathcal{O}_Y$-algebras

$$S(\mathcal{E}^\vee) \to \mathcal{O}_Y \oplus \mathcal{E}^\vee = \pi_*\mathcal{O}_X \tag{1.13}$$

which induces an embedding

$$X \cong \mathbf{Spec}_{\mathcal{O}_Y}(\mathcal{O}_Y \oplus \mathcal{E}^\vee) \hookrightarrow \mathbb{V}(\mathcal{E}) := \mathbf{Spec}_{\mathcal{O}_Y}(S(\mathcal{E}^\vee)), \tag{1.14}$$

where $\mathbb{V}(\mathcal{E})$ denotes the vector bundle associated with the locally free $\mathcal{O}_Y$-module $\mathcal{E}$.

Hence, every cover $\pi\colon X \to Y$ factors as follows

$$\begin{tikzcd} & \mathbb{V}(\mathcal{E}) \arrow[dr, "p"] & \\ X \arrow[ur, hook] \arrow[rr, "\pi"] & & Y \end{tikzcd} \tag{1.15}$$

where $\mathcal{E}$ is the dual of the Tschirnhaus bundle of π and $p\colon \mathbb{V}(\mathcal{E}) \to Y$ is the vector bundle projection.

Remark 1.29. Note that the case $d = 2$ is well-known and treated by several authors, see for instance [Per78] and [BHPV04, Ch. I, Sec. 17]. Indeed, all double covers are Galois covers with Galois group $\mathbb{Z}/2\mathbb{Z}$.

1.1.4 Triple and Quadruple Covers

In this subsection we will briefly recall the theory of triple and quadruple covers as follows

- first we introduce the general theory developed by Miranda in [Mir85] about triple covers and by Hahn-Miranda in [HM99] about quadruple covers;
- then we describe the theory of Casnati-Ekedahl [CE96] for Gorenstein covers of degree $d = 3, 4$.

1.1.4.a The Theory of Miranda and Hahn-Miranda

As we observed in Subsection 1.1.3, given a cover $\pi\colon X \to Y$ of degree d with Tschirnhaus bundle $\mathcal{E}^\vee$ it holds true

$$X \cong Spec_{\mathcal{O}_Y}(\mathcal{O}_Y \oplus \mathcal{E}^\vee).$$

In other words, a degree d cover $\pi\colon X \to Y$ is uniquely determined, up to isomorphism, by

1. a rank $d-1$ locally free $\mathcal{O}_Y$-module $\mathcal{E}$,
2. a ring structure on $\mathcal{O}_Y \oplus \mathcal{E}^\vee$ (which turns $\mathcal{O}_Y \oplus \mathcal{E}^\vee$ into a $\mathcal{O}_Y$-algebra) such that $\mathcal{E}^\vee$ is locally the trace-zero module.

We will now try to better understand condition (2).

Let Y be a fixed variety and $d \geq 2$ an integer. Given a rank $d-1$ locally free $\mathcal{O}_Y$-module $\mathcal{E}$, we have to give a multiplication map

$$(\mathcal{O}_Y \oplus \mathcal{E}^\vee) \otimes (\mathcal{O}_Y \oplus \mathcal{E}^\vee) \to (\mathcal{O}_Y \oplus \mathcal{E}^\vee). \tag{1.16}$$

Since

$$(\mathcal{O}_Y \oplus \mathcal{E}^\vee) \otimes (\mathcal{O}_Y \oplus \mathcal{E}^\vee) = (\mathcal{O}_Y \otimes \mathcal{O}_Y) \oplus (\mathcal{O}_Y \otimes \mathcal{E}^\vee) \oplus (\mathcal{E}^\vee \otimes \mathcal{O}_Y) \oplus (\mathcal{E}^\vee \otimes \mathcal{E}^\vee), \tag{1.17}$$

giving (1.16) amounts to providing four maps as follows

$$\begin{aligned} &\mathcal{O}_Y \otimes \mathcal{O}_Y \to \mathcal{O}_Y \oplus \mathcal{E}^\vee, \\ &\mathcal{O}_Y \otimes \mathcal{E}^\vee \to \mathcal{O}_Y \oplus \mathcal{E}^\vee, \\ &\mathcal{E}^\vee \otimes \mathcal{O}_Y \to \mathcal{O}_Y \oplus \mathcal{E}^\vee, \\ &\mathcal{E}^\vee \otimes \mathcal{E}^\vee \to \mathcal{O}_Y \oplus \mathcal{E}^\vee. \end{aligned} \tag{1.18}$$

However, note that the first three maps are already given. More precisely, the first map is the multiplication of $\mathcal{O}_Y$, while the second and the third give the $\mathcal{O}_Y$-module structure of $\mathcal{E}^\vee$. Hence, assigning (1.16) is equivalent to providing a map

$$\mathcal{E}^\vee \otimes \mathcal{E}^\vee \to \mathcal{O}_Y \oplus \mathcal{E}^\vee, \tag{1.19}$$

which has to factor through a map

$$\phi \colon S^2(\mathcal{E}^\vee) \to \mathcal{O}_Y \oplus \mathcal{E}^\vee \tag{1.20}$$

since the multiplication is required to be commutative.

Therefore, giving a map ϕ as above amounts to assigning a commutative multiplication on $\mathcal{O}_Y \oplus \mathcal{E}^\vee$ which is $\mathcal{O}_Y$-linear.

However, there are conditions on ϕ in order that the multiplication (1.16) is associative and the $\mathcal{O}_Y$-submodule $\mathcal{E}^\vee$ of $\mathcal{O}_Y \oplus \mathcal{E}^\vee$ consists locally of zero trace functions.

These conditions are carefully analyzed in the cases $d = 3, 4$ in [Mir85], respectively in [HM99]. In both cases, it turns out that giving the map ϕ with the above-mentioned extra conditions amounts to assigning a section η of a locally free sheaf given in terms of $\mathcal{E}$, $\mathcal{E}^\vee$ and their symmetric and exterior powers.

More precisely, in [Mir85] Miranda proves the following.

Theorem 1.30. *Let Y be an algebraic variety. A triple cover of Y is determined by a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ of rank 2 together with a global section*

$$\eta \in H^0(Y, S^3(\mathcal{E}) \otimes \det(\mathcal{E}^\vee)).$$

In [HM99] the authors give the following structure theorem for quadruple covers.

Theorem 1.31. *Let Y be an algebraic variety. A quadruple cover of Y is determined by a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ of rank 3 and a totally decomposable section*

$$\eta \in H^0(Y, \bigwedge^2 S^2(\mathcal{E}) \otimes \det(\mathcal{E}^\vee)),$$

i.e., a map $\eta \colon \det \mathcal{E} \to \bigwedge^2 S^2(\mathcal{E})$ which for every $y \in Y$ induces a map

$$\eta_y \colon (\bigwedge^3 \mathcal{E})_y \to (\bigwedge^2 S^2(\mathcal{E}))_y$$

whose image consists of totally decomposable tensors.

1.1.4.b Gorenstein Covers: the Theory of Casnati-Ekedahl

Till now we have treated the theory of covers $\pi\colon X \to Y$ without any specific assumption. In particular, in Subsection 1.1.4.a we presented the structure theorems for covers $\pi\colon X \to Y$ of degree $d = 3, 4$ given by Miranda in [Mir85], respectively by Hahn-Miranda in [HM99]. Here we briefly introduce the theory of Gorenstein covers of degree $d \geq 3$ developed by Casnati-Ekedahl in [CE96] and state the structure theorems for $d = 3, 4$ given therein.

Let us start with some definitions and observations.

Definition 1.32. A variety X is said to be *Gorenstein* if for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a Gorenstein ring.

Remark 1.33. A Gorenstein variety X has an invertible dualizing sheaf ω_X ([Har66], Proposition V.9.3).

Definition 1.34 (cf. [Har66], Exercise V.9.7)**.** A cover $\pi\colon X \to Y$ between algebraic varieties is said to be *Gorenstein* if all fibres $X_y := \pi^{-1}(y)$ are Gorenstein (scheme-theoretically).

Remark 1.35. Given a cover $\pi\colon X \to Y$, it holds true ([Har66], Exercise V.9.7)

$$\pi \text{ is Gorenstein} \quad \Longleftrightarrow \quad \omega_{X|Y} \text{ is a line bundle,} \tag{1.21}$$

where here $\omega_{X|Y}$ denotes the relative dualizing complex of π.

Remark 1.36. Given a cover $\pi\colon X \to Y$ where Y is Gorenstein, it holds true that (see [Har66], Proposition V.9.6, cf. [Mat89], Ch. 8, Sec. 23, Theorem 23.4)

$$\pi \quad \text{is Gorenstein} \quad \Longleftrightarrow \quad X \quad \text{is Gorenstein.}$$

Given a cover $\pi\colon X \to Y$ of degree $d \geq 3$ with Tschirnhaus bundle $\mathcal{E}^\vee$, from Remark 1.28 it follows that there exists a factorization

$$\begin{tikzcd} & \mathbb{V}(\mathcal{E}) \arrow[dr] & \\ X \arrow[ur, hook] \arrow[rr, "\pi"] & & Y \end{tikzcd} \tag{1.22}$$

Recalling that a subscheme $Z \subset \mathbb{P}^n$ is said to be *arithmetically Gorenstein* if its homogeneous coordinate ring $S(Z) := \mathbb{C}[x_0, \ldots, x_n]/I(Z)$ is a Gorenstein ring, the novelty of Casnati-Ekedahl approach relies on the fact that if we assume $X_y := \pi^{-1}(y)$ to be Gorenstein, then X_y is an arithmetically Gorenstein subscheme of $\mathbb{P}_y$, where $p\colon \mathbb{P} := \mathbb{P}(\mathcal{E}^\vee) \to Y$ is the $\mathbb{P}^{d-2}$-bundle associated with the Tschirnhaus bundle $\mathcal{E}^\vee$ and $\mathbb{P}_y := p^{-1}(y) \cong \mathbb{P}^{d-2}$.

This yields a global embedding $X \subset \mathbb{P}(\mathcal{E}^\vee)$ such that the given cover $\pi\colon X \to Y$ factors as follows

$$\begin{tikzcd} & \mathbb{P}(\mathcal{E}^\vee) \arrow[dr, "p"] & \\ X \arrow[ur, hook] \arrow[rr, "\pi"] & & Y \end{tikzcd} \tag{1.23}$$

In our setting the factorization theorem for Gorenstein covers of degree $d \geq 3$ (see [CE96, Theorem 2.1] and [CN07, Theorem 2.2] for an amended version) can be stated as follows.

Theorem 1.37. *Let X and Y be two algebraic varieties, and let $\pi : X \to Y$ be a Gorenstein cover of degree $d \geq 3$ with Tschirnhaus bundle $\mathcal{E}^\vee$. Then there exists a unique $\mathbb{P}^{d-2}_{\mathbb{C}}$-bundle $p : \mathbb{P} \to Y$ and an embedding $i : X \hookrightarrow \mathbb{P}$ such that $\pi = p \circ i$ and $X_y := \pi^{-1}(y) \subset \mathbb{P}_y := p^{-1}(y) \cong \mathbb{P}^{d-2}_{\mathbb{C}}$ is a non-degenerate arithmetically Gorenstein subscheme for each $y \in Y$.*

Moreover, the following properties hold.

i. $\mathbb{P} \cong \mathbb{P}(\mathcal{E}^\vee) := \mathbf{Proj}(S(\mathcal{E}))$ *where* $\mathcal{E} \cong (\mathcal{E}^\vee)^\vee$ *is the dual of the Tschirnhaus bundle* $\mathcal{E}^\vee$.

ii. The composition $\varphi\colon \pi^*\mathcal{E} \to \pi^*\pi_*\omega_{X|Y} \to \omega_{X|Y}$ *is surjective and the ramification divisor R satisfies*

$$\mathcal{O}_X(R) \cong \omega_{X|Y} \cong \mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1). \tag{1.24}$$

iii. There exists an exact sequence $\mathcal{N}_*$ of locally free $\mathcal{O}_{\mathbb{P}}$-sheaves

$$0 \to \mathcal{N}_{d-2}(-d) \xrightarrow{\alpha_{d-2}} \mathcal{N}_{d-3}(-d+2) \xrightarrow{\alpha_{d-3}} \dots \xrightarrow{\alpha_2} \mathcal{N}_1(-2) \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0 \tag{1.25}$$

unique up to unique isomorphisms and whose restriction to the fibre $\mathbb{P}_y$ over $y \in Y$ is a minimal free resolution of the structure sheaf of X_y; in particular, $\mathcal{N}_i$ is fibrewise trivial. $\mathcal{N}_{d-2}$ is invertible and for $i = 1, \dots, d-3$ one has

$$\operatorname{rank}\ \mathcal{N}_i = \frac{i(d-2-i)}{d-1}\binom{d}{i+1}$$

Moreover, we have

$$p^*p_*\mathcal{N}_* \cong \mathcal{N}_* \qquad \textit{and} \qquad \mathcal{H}om_{\mathbb{P}}(\mathcal{N}_*, \mathcal{N}_{d-2}(-d)) \cong \mathcal{N}_*.$$

iv. If $\mathbb{P} \cong \mathbb{P}(\mathcal{E}'^\vee)$ *then* $\mathcal{E} \cong \mathcal{E}'$ *if and only if* $\mathcal{N}_{d-2} \cong p^* \det \mathcal{E}'$ *in the resolution* (1.25) *computed with respect to the polarization* $\mathcal{O}_{\mathbb{P}(\mathcal{E}'^\vee)}(1)$.

1.1.4.c Gorenstein Triple Covers

Given a Gorenstein cover $\pi\colon X \to Y$ of degree $d = 3$ with Tschirnhaus bundle $\mathcal{E}^\vee$ Theorem 1.37 applies and the sequence (1.25) reads as

$$0 \to p^*\det(\mathcal{E})(-3) \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0. \tag{1.26}$$

Hence, $X \subset \mathbb{P}(\mathcal{E}^\vee) =: \mathbb{P}$ is the zero locus of the section $\delta \in H^0(\mathbb{P}, p^*\det(\mathcal{E})^{-1}(3))$ which corresponds to a section

$$\eta \in H^0(Y, S^3(\mathcal{E}) \otimes \det(\mathcal{E})^{-1}) \tag{1.27}$$

under the natural isomorphism

$$\Phi_3\colon H^0(Y, S^3(\mathcal{E}) \otimes \det(\mathcal{E})^{-1}) \xrightarrow{\sim} H^0(\mathbb{P}, p^*\det(\mathcal{E})^{-1}(3)), \tag{1.28}$$

namely $\delta = \Phi_3(\eta)$.

Definition 1.38 ([CE96], Definition 3.3)**.** Let Y, $\mathcal{E}$ and η be as above. We say that $\eta \in H^0(Y, S^3\mathcal{E} \otimes \det \mathcal{E}^{-1})$ has the *right codimension at* $y \in Y$ if the zero-locus of

$$\delta_y \in H^0(\mathbb{P}_y, p^* \det(\mathcal{E})^{-1} \otimes \mathcal{O}_{\mathbb{P}_y}(3))$$

has dimension 0.

In our setting the structure theorem for Gorenstein triple covers can be stated as follows.

Theorem 1.39 (cf. [CE96], Theorem 3.4)**.** *Let Y be an algebraic variety. Any Gorenstein triple cover $\pi\colon X \to Y$ with Tschirnhaus bundle $\mathcal{E}^\vee$ determines, up to scalars, a global section $\eta \in H^0(Y, S^3(\mathcal{E}) \otimes \det(\mathcal{E})^{-1})$ having the right codimension at every $y \in Y$.*

Conversely, given a locally free $\mathcal{O}_Y$-sheaf $\mathcal{E}$ and a global section

$$\eta \in H^0(Y, S^3(\mathcal{E}) \otimes \det(\mathcal{E})^{-1})$$

*having the right codimension at every $y \in Y$, let X be the zero-locus of $\delta := \Phi_3(\eta)$ inside the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}^\vee) :=$ **Proj**$(S(\mathcal{E})) \to Y$, namely $X \subset \mathbb{P}(\mathcal{E}^\vee)$. Then the restriction $\pi := p_{|X}\colon X \to Y$ of the bundle projection $p : \mathbb{P}(\mathcal{E}^\vee) \to Y$ is a Gorenstein cover of degree 3 with $\mathcal{E}^\vee$ as Tschirnhaus bundle.*

1.1.4.d Gorenstein Quadruple Covers

Given a Gorenstein cover $\pi\colon X \to Y$ of degree $d = 4$ with Tschirnhaus bundle $\mathcal{E}^\vee$ Theorem 1.37 applies and the sequence (1.25) reads as

$$0 \to p^* \det(\mathcal{E})(-4) \to \mathcal{N}(-2) \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0 \tag{1.29}$$

where $\mathcal{N} \cong p^*\mathcal{F}$ for some locally free $\mathcal{O}_Y$-module $\mathcal{F}$ of rank 2.

Note that the Koszul complex of δ (see [Ful84, Appendix B.3]) is globally exact and then, since (1.29) is unique, it must be

$$p^* \det(\mathcal{E})(-4) \cong \det(\mathcal{N})(-4) \iff p^* \det(\mathcal{E}) \cong \det \mathcal{N}$$

and then, observing that $\det \mathcal{N} \cong p^* \det \mathcal{F}$, by injectivity of the pull-back p^* we get

$$\det(\mathcal{E}) \cong \det(\mathcal{F}). \tag{1.30}$$

The quadruple cover X is given by the zero-locus of the section $\delta \in H^0(\mathbb{P}, \mathcal{N}^\vee(2))$, which corrsponds to a section

$$\eta \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{F}^\vee) \tag{1.31}$$

under the natural isomorphism

$$\Phi_4\colon H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{F}^\vee) \xrightarrow{\sim} H^0(\mathbb{P}, \mathcal{N}^\vee(2)), \tag{1.32}$$

namely $\delta = \Phi_4(\eta)$.

Definition 1.40. Let Y, $\mathcal{E}$, $\mathcal{F}$ and η be as above. We say that a section $\eta \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{F}^\vee)$ has the *right codimension at* $y \in Y$ if the zero-locus of

$$\delta_y \in H^0(\mathbb{P}_y, \mathcal{N}^\vee \otimes \mathcal{O}_{\mathbb{P}_y}(2))$$

has dimension 0.

In our setting the statement of the structure theorem for Gorenstein quadruple cover is as follows.

Theorem 1.41 (cf. [CE96], Theorem 4.4)**.** *Let Y be an algebraic variety. Any Gorenstein quadruple cover $\pi\colon X \to Y$ with Tschirhaus bundle $\mathcal{E}^\vee$ determines a locally free $\mathcal{O}_Y$-sheaf $\mathcal{F}$ of rank 2 with* $\det(\mathcal{E}) \cong \det(\mathcal{F})$ *and, up to scalars, a global section $\eta \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{F}^\vee)$ having the right codimension at every $y \in Y$.*

Conversely, given locally free $\mathcal{O}_Y$-modules $\mathcal{E}$, $\mathcal{F}$ of rank 3 *and* 2 *respectively with* $\det \mathcal{F} = \det \mathcal{E}$ *and $\eta \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{F}^\vee)$ having the right codimension at every $y \in Y$, let X be the zero-locus of $\delta := \Phi_4(\eta)$ inside the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E}^\vee) :=$ **Proj**$(S(\mathcal{E}))$, namely $X \subset \mathbb{P}(\mathcal{E}^\vee)$. Then the restriction $\pi := p_{|X}\colon X \to Y$ of the bundle projection $p\colon \mathbb{P}(\mathcal{E}^\vee) \to Y$ is a Gorenstein cover of degree* 4 *such that $\mathcal{E}^\vee$ is its Tschirnhaus bundle and*

$$\mathcal{F} \cong \ker(S^2(\mathcal{E}) \to \pi_*(\omega_{X|Y}^{\otimes 2})),$$

where $\omega_{X|Y}$ denotes the relative dualizing sheaf of π.

1.2 Projective Bundles: a Brief Overview

Let Y be a fixed algebraic variety and $\mathcal{E}$ a locally free $\mathcal{O}_Y$-module of rank $r \geq 2$. As already declared in Notation, for us the *projective bundle associated with* $\mathcal{E}$, denoted by $\mathbb{P}(\mathcal{E})$, is defined as follows

$$p\colon \mathbb{P}(\mathcal{E}) := \mathbf{Proj}_{\mathcal{O}_Y}\,\mathbf{Sym}(\mathcal{E}^\vee) \to Y. \tag{1.33}$$

To avoid any sort of confusion, we point out that this is the projective bundle of one-dimensional subspaces of the vector bundle $\mathbb{V}(\mathcal{E})$, namely we have that

$$\mathbb{P}(\mathcal{E}) = \big(\mathbb{V}(\mathcal{E}) \setminus s_0(Y)\big)/\mathbb{C}^*$$

where $s_0\colon Y \to \mathbb{V}(\mathcal{E})$ is the zero section of the vector bundle

$$\mathbb{V}(\mathcal{E}) := \mathbf{Spec}_{\mathcal{O}_Y}(\mathbf{Sym}(\mathcal{E}^\vee)) \to Y. \tag{1.34}$$

This is the geometric notation adopted in [Ful84, Appendix B.3, B.5], where it is also pointed out that $\mathbb{V}(\mathcal{E})$ is the vector bundle whose sheaf of sections is, up to isomorphism, $\mathcal{E}$.

1.2.1 Tautological Line Bundle and Useful Formulae

Given a locally free $\mathcal{O}_Y$ module $\mathcal{E}$, set $p\colon \mathbb{P} := \mathbb{P}(\mathcal{E}) \to Y$ and $E := \mathbb{V}(\mathcal{E})$.

Recall that the so-called *tautological line bundle* $\mathscr{L}_E$ on $\mathbb{P}$ is defined by gluing together the tautological line bundles of every fibre. This corresponds to the dual of the *Serre's twisting sheaf* $\mathcal{O}_{\mathbb{P}}(1)$, namely

$$\mathscr{L}_E \cong \mathcal{O}_{\mathbb{P}}(-1) \tag{1.35}$$

and we have a natural embedding

$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \to p^*\mathcal{E}, \tag{1.36}$$

or equivalently a surjection

$$p^*\mathcal{E}^\vee \to \mathcal{O}_{\mathbb{P}}(1) \to 0, \tag{1.37}$$

see [Ful84], Appendix B.5 for (1.36) and [Laz04], Appendix A for (1.37).

Moreover, we have for all $m \geq 0$

$$p_*\mathcal{O}_{\mathbb{P}}(m) = \mathbf{Sym}^m(\mathcal{E}^\vee), \tag{1.38}$$

$$R^{r-1}p_*\mathcal{O}_{\mathbb{P}}(-r-m) = \mathbf{Sym}^m(\mathcal{E}) \otimes \det\mathcal{E}. \tag{1.39}$$

and all the other direct images vanish (see [Laz04], Appendix A).

Remark 1.42. From the previous formulae it follows in particular

$$\begin{aligned} &(i) \quad p_*\mathcal{O}_{\mathbb{P}} = \mathcal{O}_Y,\\ &(ii) \quad p_*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{E}^\vee,\\ &(iii) \quad R^{r-1}p_*\mathcal{O}_{\mathbb{P}}(-r) = \det\mathcal{E}. \end{aligned}$$

Remark 1.43. Note that, for every line bundle $\mathscr{L} \in \operatorname{Pic}(Y)$, there is a canonical isomorphism

$$\varphi\colon \mathbb{P}(\mathcal{E} \otimes \mathscr{L}) \xrightarrow{\sim} \mathbb{P}(\mathcal{E}), \tag{1.40}$$

under which

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}\otimes\mathscr{L})}(1) = \varphi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes (p')^*\mathscr{L}^{-1}, \tag{1.41}$$

where $p\colon \mathbb{P}(\mathcal{E}) \to Y$, $p'\colon \mathbb{P}(\mathcal{E} \otimes \mathscr{L}) \to Y$ are the bundle projections, cf. [Har77], Lemma II.7.9.

If Y is smooth, then also the converse holds. Namely, if $\mathcal{E}$ and $\mathcal{E}'$ are two locally free $\mathcal{O}_Y$-modules such that

$$\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}'),$$

then there exists a line bundle $\mathscr{L}$ on Y such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathscr{L}$, see [Har77], exercise II.7.9(b).

1.2.2 Morphisms factoring through $\mathbb{P}(\mathcal{E})$.

As it follows from Proposition II.7.12 of [Har77], given a morphism of varieties $\pi\colon X \to Y$ and a locally free $\mathcal{O}_Y$-module $\mathcal{E}$, it turns out that π factors through the projective bundle $p\colon \mathbb{P} := \mathbb{P}(\mathcal{E}) \to Y$, namely we have a diagram as follows

$$\begin{tikzcd} & \mathbb{P} \arrow[dr, "p"] & \\ X \arrow[ur, "\psi"] \arrow[rr, "\pi"] & & Y \end{tikzcd}$$

if and only if there is a line bundle $\mathscr{L}$ on X and a surjection as follows

$$\pi^*(\mathcal{E}^\vee) \to \mathscr{L} \to 0. \tag{1.42}$$

Under this hypothesis, we have that

$$\mathscr{L} = \psi^*\mathcal{O}_{\mathbb{P}}(1).$$

and (1.42) is the pull-back via ψ of the tautological surjection (1.37).

1.2.3 Relative Canonical Formula

Assume here that Y is smooth. Given a a projective bundle

$$p\colon \mathbb{P} := \mathbb{P}(\mathcal{E}) \to Y,$$

where $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of rank $r \geq 2$, we observe that under the assumption that Y is smooth, $\mathbb{P}$ is also smooth and the sheaf of relative differentials $\Omega_{\mathbb{P}|Y}$ is locally free of rank $r-1$. Furthermore, the relative cotangent sequence 1.1 is a sequence of locally free $\mathcal{O}_{\mathbb{P}}$-modules which reads as follows

$$0 \to p^*\Omega_Y \to \Omega_{\mathbb{P}} \to \Omega_{\mathbb{P}|Y} \to 0. \tag{1.43}$$

Taking determinants we have then

$$\omega_{\mathbb{P}} = p^*\omega_Y \otimes \omega_{\mathbb{P}|Y}. \tag{1.44}$$

Consider now the so-called *Euler sequence* (see exercise III.8.4(b) of [Har77])

$$0 \to \Omega_{\mathbb{P}|Y} \to p^*(\mathcal{E}^\vee) \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}} \to 0. \tag{1.45}$$

Again by taking determinants, we get from the last sequence

$$\omega_{\mathbb{P}|Y} = p^* \det \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}}(-r). \tag{1.46}$$

Finally, putting together formulae (1.44) and (1.46), we get

$$\omega_{\mathbb{P}} = p^*\omega_Y \otimes p^* \det \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}}(-r), \tag{1.47}$$

which is called *relative canonical formula* for the projective bundle $p\colon \mathbb{P} := \mathbb{P}(\mathcal{E}) \to Y$.

Remark 1.44. Note that if $\omega_Y \cong \mathcal{O}_Y$ is trivial, then

$$\omega_{\mathbb{P}|Y} = \omega_{\mathbb{P}} \tag{1.48}$$

from (1.44), and hence formula (1.46) computes the canonical bundle $\omega_{\mathbb{P}}$ of the projective bundle $\mathbb{P} = \mathbb{P}(\mathcal{E})$, namely

$$\omega_{\mathbb{P}} = p^* \det \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}}(-r). \tag{1.49}$$

This is the case, for instance, when Y is an abelian variety.

1.2.4 Néron-Severi Group and Integral Cohomology

Assume now that Y is smooth and projective and let

$$p\colon \mathbb{P} := \mathbb{P}(\mathcal{E}) \to Y$$

be a projective bundle where $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of rank $r \geq 2$. Then, since Y is smooth we can give precise information on the Picard group $\mathrm{Pic}(\mathbb{P})$. Indeed, it holds (see [Har77], exercise II.7.9)

$$\mathrm{Pic}(\mathbb{P}) \cong \mathrm{Pic}(Y) \times \mathbb{Z}. \tag{1.50}$$

Furthermore, since Y is projective, we have, as pointed out in [Laz04], Appendix A, that the Néron-Severi group $N^1(\mathbb{P})$ of $\mathbb{P}$ is given by

$$N^1(\mathbb{P}) \cong p^*N^1(Y) \oplus \mathbb{Z} \cdot H, \tag{1.51}$$

where H is the class of the Serre's twisting sheaf $\mathcal{O}_{\mathbb{P}}(1)$. Moreover, denoting still by H the class of $\mathcal{O}_{\mathbb{P}}(1)$ in $H^2(\mathbb{P}, \mathbb{Z})$, the integral cohomology ring

$$H^*(\mathbb{P}, \mathbb{Z}) := \bigoplus_n H^n(\mathbb{P}, \mathbb{Z}) \tag{1.52}$$

is a finetely generated $H^*(Y, \mathbb{Z})$-algebra. More precisely, it holds

$$H^*(\mathbb{P}, \mathbb{Z}) \simeq \frac{H^*(Y, \mathbb{Z})[H]}{H^r - c_1H^{r-1} + \ldots + (-1)^{r-1}c_{r-1}H + (-1)^r c_r}, \tag{1.53}$$

where $c_i := p^*c_i(\mathcal{E}^\vee)$ are the pull-back of the Chern classes

$$c_i(\mathcal{E}^\vee) \in H^{2i}(Y, \mathbb{Z})$$

of the locally free $\mathcal{O}_Y$-module $\mathcal{E}^\vee$ and

$$H^r - c_1H^{r-1} + \ldots + (-1)^{r-1}c_{r-1}H + (-1)^r c_r = 0 \tag{1.54}$$

is called *Grothendieck relation*.

1.3 The Heisenberg Group

Let H be a finite abelian group with exponent $r \in \mathbb{N}$, and denote by

$$H^* := \mathrm{Hom}(H, \mathbb{C}^*) \cong H$$

its group of characters.

Denoting by V the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on H, namely

$$V := \mathbb{C}(H),$$

for every $h \in H$ and $\chi \in H^*$ we define linear operators $\underline{h}$ and $\underline{\chi}$ on V as follows

$$\begin{aligned}(\underline{h} \cdot f)(x) &:= f(x+h),\\ (\underline{\chi} \cdot f)(x) &:= \chi(x) f(x).\end{aligned}$$

Definition 1.45. We define the *Heisenberg group* $\mathcal{H}(H)$ as the subgroup of $\mathrm{GL}(V)$ generated by the operators introduced above. Namely,

$$\mathcal{H}(H) := \langle \underline{h}, \underline{\chi} \mid h \in H, \chi \in H^* \rangle \leq GL(V).$$

The product in $\mathcal{H}(H)$ is the composition of linear operators and $\mathcal{H}(H)$ acts by definition on the vector space V, which is called *the Schrödinger representation of* $\mathcal{H}(H)$.

Note that the any two operators in $\{\underline{h}\}_{h \in H}$ commutes, and similarly for those in $\{\underline{\chi}_{\chi \in H^*}\}$. However, given $h \in H$ and $\chi \in H^*$, the associated operators $\underline{h}, \underline{\chi}$ in general does not. We give in the following remark some useful properties describing the way all these operators relate to each other.

Remark 1.46. For every $h, h' \in H$, $\chi, \chi' \in H^*$ it holds true:

(*i*) $\underline{h} \circ \underline{h'} = \underline{h+h'}, \quad \underline{\chi} \circ \underline{\chi'} = \underline{\chi\chi'}$,

(*ii*) $[\underline{h}, \underline{\chi}] := \underline{h} \circ \underline{\chi} \circ (\underline{h})^{-1} \circ (\underline{\chi})^{-1} = \chi(h) \cdot \mathrm{id}_V \in Z(\mathcal{H}(H))$,

(*iii*) $[\underline{h}, \underline{\chi}]^{-1} = [\underline{h}, \underline{\chi^{-1}}], \quad [\underline{\chi}, \underline{h}] = [\underline{-h}, \underline{\chi}]$.

The Heisenberg group $\mathcal{H}(H)$ is not abelian, as (*ii*) of the previous remark shows. Our aim is to prove that

$$[\mathcal{H}(H), \mathcal{H}(H)] = Z(\mathcal{H}(H)) \subset \mathbb{C}^*$$

To do so we need some preliminary lemmas.

Lemma 1.47. *Given a finite abelian group H, for every $h \in H$, $\chi \in H^*$ it holds true*

$$\underline{h} \circ \underline{\chi} \in Z(\mathcal{H}(H)) \quad \Longrightarrow \quad (h, \chi) = (0, 1) \in H \times H^*.$$

Proof. From the hypothesis it follows that

$$(\underline{h} \circ \underline{\chi}) \circ \underline{h'} = \underline{h'} \circ (\underline{h} \circ \underline{\chi}) \qquad \forall\ h' \in H,$$

which implies $\chi(h') = 1$ and hence $\chi = 1$. Therefore,

$$\underline{h} \circ \underline{\chi} = \underline{h} \in Z(\mathcal{H}(H))$$

and since $[\underline{h}, \underline{\chi'}] = \chi'(h)\,\mathrm{id}_V$ we get $\chi'(h) = 1$ for all $\chi' \in H^*$, which implies that $h = 0$.

In fact, suppose by contradiction $h \neq 0$. Since H is a finite abelian group we can write

$$H \cong \mathbb{Z}/d_1 \oplus \ldots \oplus \mathbb{Z}/d_n$$

and then, once we choose primitive d_i-roots of unity ζ_{d_i}, we get isomorphisms

$$\varphi_i \colon \mathbb{Z}/d_i \to (\mathbb{Z}/d_i)^*, \qquad x \longmapsto \{y \mapsto (\zeta_{d_i}^x)^y\}$$

yielding a (non-canonical) isomorphism

$$H^* \cong (\mathbb{Z}/d_1)^* \oplus \ldots \oplus (\mathbb{Z}/d_n)^*.$$

Since $h = (h_1, \ldots, h_n) \neq (0, \ldots, 0)$ there exists $1 \leq j \leq n$ such that $h_j \neq 0$.

There are two possibilities for h_j:

1. h_j is not a zero-divisor in $\mathbb{Z}/d_j$. Then we set
$$\chi' := (1, \ldots, \varphi_j(h_j), \ldots, 1).$$
2. h_j is a zero-divisor in $\mathbb{Z}/d_j$. Then there exists k_j such that $h_j k_j = 0$ and we set
$$\chi' := (1, \ldots, \varphi_j(k_j + 1), \ldots, 1).$$

In both cases we found $\chi' \in H^*$ such that $\chi'(h) \neq 1$, a contradiction.

□

Lemma 1.48. *For each element $T \in \mathcal{H}(H)$ there exist $h_i \in H, \chi_i \in H^*$, $i = 1, \ldots, s$, and unique $h \in H, \chi \in H^*$ such that T can be written as follows*

$$T = [\underline{h_1}, \underline{\chi_1}] \cdots [\underline{h_s}, \underline{\chi_s}] \cdot \underline{h} \circ \underline{\chi}. \tag{1.55}$$

Proof. An element $T \in \mathcal{H}(H)$ is by definition of the form

$$T = (\underline{h_1})^{i_1} \circ (\underline{\chi_1})^{j_1} \circ \cdots \circ (\underline{h_s})^{i_s} \circ (\underline{\chi_s})^{j_s} \qquad \text{for some} \quad h_k \in H, \chi_k \in H^*, i_k, j_k \in \mathbb{Z}.$$

By applying Remark 1.46 (i) and renaming by abuse of notation $i_k \cdot h_k, \chi_k^{j_k}$ as h_k, χ_k, we get

$$T = \underline{h_1} \circ \underline{\chi_1} \circ \cdots \circ \underline{h_s} \circ \underline{\chi_s}.$$

Now, thanks to Remark 1.46 (i),(ii), it is easy to see that

$$T = [\underline{\chi_1}, \underline{h_2}] \cdot [\underline{\chi_1 \chi_2}, \underline{h_3}] \cdot \ \cdots \ \cdot [\underline{\chi_1 \cdots \chi_{s-1}}, \underline{h_s}] \cdot \underline{(h_1 + \ldots + h_s)} \circ \underline{(\chi_1 \cdot \ \cdots \ \cdot \chi_s)}$$

and then, by Remark 1.46 (i) and (iii), we get the desired form (1.55).

Now, suppose

$$T = [\underline{h_1}, \underline{\chi_1}] \cdots [\underline{h_s}, \underline{\chi_s}] \cdot \underline{h} \circ \underline{\chi} = [\underline{h'_1}, \underline{\chi'_1}] \cdots [\underline{h'_t}, \underline{\chi'_t}] \cdot \underline{h'} \circ \underline{\chi'} \tag{1.56}$$

Setting

$$\lambda\,\mathrm{id}_V := [\underline{h_1},\underline{\chi_1}]\cdots[\underline{h_s},\underline{\chi_s}], \qquad \mu\,\mathrm{id}_V := [\underline{h'_1},\underline{\chi'_1}]\cdots[\underline{h'_t},\underline{\chi'_t}],$$

we get

$$\lambda\cdot\underline{h}\circ\underline{\chi} = \mu\cdot\underline{h'}\circ\underline{\chi'} \quad\iff\quad \underline{h-h'}\circ\underline{\chi\chi'^{-1}} = \lambda^{-1}\mu\cdot\mathrm{id}_V\,.$$

Hence, thanks to Lemma 1.47, we obtain

$$h = h', \qquad \chi = \chi'.$$

□

Now we are ready to prove the desired result.

Proposition 1.49. *Given a finite abelian group H, for its associated Heisenberg group $\mathcal{H}(H)$ it holds that*

$$[\mathcal{H}(H),\mathcal{H}(H)] = Z(\mathcal{H}(H)) \subset \mathbb{C}^*$$

Proof. First of all, we recall that

$$\begin{aligned}[\mathcal{H}(H),\mathcal{H}(H)] &= \langle\langle\ [\underline{h},\underline{\chi}],[\underline{\chi},\underline{h}] \mid h\in H,\chi\in H^*\ \rangle\rangle\\ &= \langle\langle\ [\underline{h},\underline{\chi}] \mid h\in H,\chi\in H^*\ \rangle\rangle\\ &:= \langle\ T\circ[\underline{h},\underline{\chi}]\circ T^{-1} \mid h\in H,\chi\in H^*, T\in\mathcal{H}(H)\ \rangle,\end{aligned}$$

where the second equality follows from Remark 1.46 (*iii*).

Then since the commutators $[\underline{h},\underline{\chi}] = \chi(h)\cdot\mathrm{id}_V$ are central elements, we have that

$$[\mathcal{H}(H),\mathcal{H}(H)] \subseteq Z(\mathcal{H}(H)).$$

Now, let $T\in Z(\mathcal{H}(H))$ be a central element. By Lemma 1.48

$$T = [\underline{h_1},\underline{\chi_1}]\cdots[\underline{h_s},\underline{\chi_s}]\cdot\underline{h}\circ\underline{\chi}$$

and thus

$$T\in Z(\mathcal{H}(H)) \quad\iff\quad \underline{h}\circ\underline{\chi}\in Z(\mathcal{H}(H)).$$

By Lemma 1.47 we get $h=0, \chi=1$ and hence

$$T = [\underline{h_1},\underline{\chi_1}]\cdots[\underline{h_s},\underline{\chi_s}] \in [\mathcal{H}(H),\mathcal{H}(H)].$$

Finally, we have showed that

$$Z(\mathcal{H}(H)) = [\mathcal{H}(H),\mathcal{H}(H)] = \langle\ [\underline{h},\underline{\chi}] \mid h\in H,\chi\in H^*\ \rangle \subset \mathbb{C}^*$$

where the inclusion follows from the fact that $[\underline{h},\underline{\chi}]$ are scalar operators.

□

Given a finite abelian group H with exponent $r\in\mathbb{N}$, thanks to Lemma 1.48 we can define a surjective group homomorphism

$$\psi\colon \mathcal{H}(H) \twoheadrightarrow H\times H^*$$

by the following assignment

$$\psi(\underline{h}) := (h, 1), \quad \psi(\underline{\chi}) := (0, \chi), \quad \psi([\underline{h}, \underline{\chi}]) := (0, 1).$$

By definition of ψ and again by Lemma 1.48, it is immediate to see that

$$\ker \psi = [\mathcal{H}(H), \mathcal{H}(H)] = Z(\mathcal{H}(H)).$$

Since the latter is a finite group of scalar operators which has exponent $r \in \mathbb{N}$, we get

$$\ker \psi \cong \mu_r \subset \mathbb{C}^*.$$

Therefore, the Heisenberg group $\mathcal{H}(H)$ of a finite abelian group H with exponent $r \in \mathbb{N}$ is a central extension of $H \times H^*$ via $\mu_r \subset \mathbb{C}^*$, namely

$$\begin{tikzcd} 0 \arrow[r] & \mu_r \arrow[r] & \mathcal{H}(H) \arrow[r] & H \times H^* \arrow[r] & 0 \end{tikzcd} \tag{1.57}$$

where $\mu_r = Z(\mathcal{H}(H)) = [\mathcal{H}(H), \mathcal{H}(H)]$.

Recalling that every such a group extension is given by a 2-cocycle (see Section 1.5 or [MacLane95], Ch. IV, Thm 4.1), it easy to see that (1.57) corresponds to the (normalized) 2-cocycle

$$\begin{gathered} \beta\colon (H \times H^*)^2 \to \mu_r \\ ((h_1, \chi_1),(h_2, \chi_2)) \longmapsto \chi_1^{-1}(h_2) \cdot \mathrm{id}_V = [\underline{h}_2, \underline{\chi_1}^{-1}] \end{gathered} \tag{1.58}$$

We recall moreover that β measures the failure of the section

$$s\colon H \times H^* \to \mathcal{H}(H), \qquad (h, \chi) \longmapsto \underline{h} \circ \underline{\chi}$$

to be a homomorphism. More precisely, the cohomology class of β

$$[\beta] \in H^2(H \times H^*, \mu_r)$$

is trivial if and only if the sequence (1.57) splits.

1.3.1 The Infinite Heisenberg Group

We construct now the *infinite Heisenberg group* $\mathcal{H}^\infty(H)$ with $\mathbb{C}^*$ as centre. The idea is to add all the scalar operators to the finite Heisenberg group $\mathcal{H}(H)$, namely $\{\lambda \cdot \mathrm{id}_V\}_{\lambda \in \mathbb{C}^*}$.

Therefore, given a finite abelian group H with exponent $r \in \mathbb{N}$ and keeping the notation $V = \mathbb{C}(H)$, we define

$$\mathcal{H}^\infty(H) := \langle \{\underline{h}, \underline{\chi}, \lambda \cdot \mathrm{id}_V \mid h \in H, \chi \in H^*, \lambda \in \mathbb{C}^*\} \rangle \leq GL(V),$$

getting the following commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mu_r \arrow[r] \arrow[d, hook] & \mathcal{H}(H) \arrow[r] \arrow[d, hook] & H \times H^* \arrow[r] \arrow[d, equal] & 0 \\ 1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathcal{H}^\infty(H) \arrow[r] & H \times H^* \arrow[r] & 0 \end{tikzcd} \tag{1.59}$$

with

$$\mathbb{C}^* = Z(\mathcal{H}^\infty(H)) = [\mathcal{H}^\infty(H), \mathcal{H}^\infty(H)].$$

By construction $\mathcal{H}(H) \leq \mathcal{H}^\infty(H)$ and obviously the Schrödinger representation $\rho\colon \mathcal{H}^\infty(H) \hookrightarrow \mathrm{GL}(V)$ restricted to $\mathcal{H}(H)$ gives the Schrödinger representation of $\mathcal{H}(H)$.

Remark 1.50. (*i*) The *Stone-von Neumann Theorem* (known also as *Mackey's Theorem*, see [Mackey49] or [Igu72], Ch. I, Sec. 5, Propositon 2) ensures that $V = \mathbb{C}(H)$ is the unique irreducible (faithful) representation of the Heisenberg group $\mathcal{H}(H)$ such that its centre $\mu_r \subset \mathbb{C}^*$ acts via scalar multiplication in a natural way.

(*ii*) Considering the canonical basis of $V = \mathbb{C}(H)$, namely the characteristic functions $\{\mathbb{1}_h\}_{h\in H}$ defined as follows

$$\mathbb{1}_h(x) := \begin{cases} 0 & x \neq h \\ 1 & x = h \end{cases} \quad , \tag{1.60}$$

we get the *canonical Schrödinger matrix representation.*

(*iii*) Since the centre of both Heisenberg groups $\mathcal{H}(H)$, $\mathcal{H}^\infty(H)$ act via scalar multiplication on $V = \mathbb{C}(H)$, there is an induced action of

$$H \times H^* \cong \mathcal{H}^\infty(H)/\mathbb{C}^* \cong \mathcal{H}(H)/\mu_r$$

on the projective space $\mathbb{P}(V)$. Namely, denoting by $\widetilde{T}$ the class in $\mathrm{PGL}(V)$ of an operator $T \in \mathrm{GL}(V)$, there is a projective representation

$$\begin{aligned} \bar{\rho}\colon H \times H^* &\to PGL(V) \\ (h,\chi) &\longmapsto \widetilde{\underline{h} \circ \underline{\chi}} \end{aligned} \tag{1.61}$$

which makes the following diagram commute

$$\begin{tikzcd}
1 \arrow[r] & \mu_r \arrow[r] \arrow[d, hook] & \mathcal{H}(H) \arrow[r] \arrow[d, hook] & H \times H^* \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d, equal] & \mathcal{H}(H)_\infty \arrow[r] \arrow[d, hook, "\rho"] & H \times H^* \arrow[r] \arrow[d, "\bar{\rho}"] & 0 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathrm{GL}(V) \arrow[r] & PGL(V) \arrow[r] & 0
\end{tikzcd} \tag{1.62}$$

In other words, $\bar{\rho}$ is a projective representation of $H \times H^*$ which lifts to a linear representation of the Heisenberg group $\mathcal{H}(H)$. We will see in Chapter 3 that for a cyclic group $H = \mathbb{Z}/r$ every projective representation of $H \times H^* \cong (\mathbb{Z}/r)^2$ has this property and the Heisenberg group $\mathcal{H}(\mathbb{Z}/r)$ is called *a representation group of* $(\mathbb{Z}/r)^2$.

1.3.2 The Representation $V \otimes V^\vee$

Given a finite abelian group H, we consider its Heisenberg group $\mathcal{H}(H)$ fitting into the following exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mu_r \arrow[r] & \mathcal{H}(H) \arrow[r] & H \times H^* \arrow[r] & 0
\end{tikzcd} \tag{1.63}$$

where $\mu_r = Z(\mathcal{H}(H)) = [\mathcal{H}(H), \mathcal{H}(H)]$.

Denoting by V the Schrödinger representation, we show here that

$$V \otimes V^\vee = \bigoplus_{\eta \in H^* \times H} \mathbb{C}_\eta, \tag{1.64}$$

where $\eta \in H^* \times H = (H \times H^*)^*$ are the 1-dimensional characters of the abelian group $H \times H^*$.

Indeed, denoting by χ the character of the Schrödinger representation $\rho\colon \mathcal{H}(H) \to \mathrm{GL}(V)$ and by $\overline{\chi}$ its complex conjugate, from character theory (see for instance [Isa94], Chapter 2) it follows that it is enough to show

$$\langle\, \chi \cdot \overline{\chi}, \eta \,\rangle = 1 \qquad \text{for all} \quad \eta \in (H \times H^*)^*. \tag{1.65}$$

Recall that, given a 1-dimensional character η of the Heisenberg group $\mathcal{H}(H)$, η comes from the abelianization of $\mathcal{H}(H)$, namely

$$\mathcal{H}(H)/\mu_r \cong H \times H^*.$$

Moreover, we have that

$$\langle\, \chi \cdot \overline{\chi}, \eta \,\rangle = \langle\, \chi, \chi \cdot \eta \,\rangle \qquad \text{for all} \quad \eta \in (H \times H^*)^*. \tag{1.66}$$

Note that, given $\eta \in (H \times H^*)^*$, $\chi \cdot \eta$ is an irreducible character of $\mathcal{H}(H)$ since χ is irreducible. Moreover, as η comes from the abelianization of $\mathcal{H}(H)$, it holds true

$$\chi = \chi \cdot \eta \qquad \text{on} \quad \mu_r.$$

Hence, by Stone-von Neumann Theorem we get

$$\chi = \chi \cdot \eta.$$

Finally, we have

$$\langle\, \chi \cdot \overline{\chi}, \eta \,\rangle = \langle\, \chi, \chi \cdot \eta \,\rangle = \langle\, \chi, \chi \,\rangle = 1 \qquad \text{for all} \quad \eta \in (H \times H^*)^*, \tag{1.67}$$

proving the decomposition 1.64.

1.3.3 The Cyclic Case

Given a cyclic group $H = \mathbb{Z}/r\mathbb{Z}$, we set

$$\mathcal{H}_r := \mathcal{H}(\mathbb{Z}/r\mathbb{Z}).$$

Then the sequence (1.57) reads as

$$1 \to \mu_r \to \mathcal{H}_r \to (\mathbb{Z}/r)^2 \to 0 \tag{1.68}$$

and a presentation of $\mathcal{H}_r$, which is a group of order r^3, is given by

$$\mathcal{H}_r = \langle a, b, c \mid a^r = b^r = c^r = 1,\ c = [a,b],\ aca^{-1} = c,\ bcb^{-1} = c\rangle. \tag{1.69}$$

Note that the 2-cocycle $\beta \in Z^2((\mathbb{Z}/r)^2, \mathbb{Z}/r)$ giving the extension (1.68) is

$$\begin{aligned} \beta\colon (\mathbb{Z}/r)^2 \times (\mathbb{Z}/r)^2 &\to \mu_r \cong \mathbb{Z}/r \\ ((i,j),(k,l)) &\longmapsto -jk. \end{aligned}$$

Moreover, the Schrödinger representation has in this case dimension r. Fixing the canonical basis of $\mathbb{C}(\mathbb{Z}/r)$ and a primitive r-th root of unity ζ_r, the generators a, b, c act as the following matrices

$$a \longmapsto \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b \longmapsto \begin{pmatrix} 1 & & & & & \\ & \zeta_r & & & & \\ & & \zeta_r^2 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \zeta_r^{r-1} \end{pmatrix},$$

$$c \longmapsto \begin{pmatrix} \zeta_r & & & & & \\ & \zeta_r & & & & \\ & & \zeta_r & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \zeta_r \end{pmatrix}.$$

and they give the canonical Schrödinger matrix representation of $\mathcal{H}_r$.

1.3.3.a The Case $r = 2$

The Heisenberg group $\mathcal{H}_2$ is a group of order 8 with a presentation as follows

$$\mathcal{H}_2 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ c = [a, b],\ aca = c,\ bcb = c \rangle.$$

Since the dihedral group D_4 can be presented by

$$D_4 = \langle r, s \quad \mid \quad s^2 = r^4 = 1, \quad srs = r^{-1} \rangle,$$

we easily see that the following map

$$\begin{aligned} \varphi \colon \mathcal{H}_2 &\to D_4 \\ a &\longmapsto s \\ b &\longmapsto sr \\ c &\longmapsto r^2 \end{aligned} \tag{1.70}$$

is an isomorphism of groups.

The Schrödinger representation V has dimension 2 and the generators a, b, c act as follows

$$a \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, since the dual representation $V^\vee$ is obtained by transposing and inverting, we have in this case that

$$V \cong V^\vee.$$

On the other hand, using the standard formula from representation theory (see [Ser77], Ch. II, Corollary 2(a))

$$|G| = \sum_{i=1}^{s} d_i^2 \tag{1.71}$$

where G is a finite group and d_i are the degrees of its irreducible representations, we get that, up to isomorphism of representations, $\mathcal{H}_2$ has:

- 1 irreducible representations of dimension 2 (the Schrödinger representation V),
- 4 irreducible representations of dimension 1 (i.e., the four characters coming from the abelianization $(\mathbb{Z}/2)^2 \cong \mathcal{H}_2/\mu_2$).

1.3.3.b The Case $r = 3$

The Heisenberg group $\mathcal{H}_3$ is a group with 27 elements given by the following presentation

$$\mathcal{H}_3 = \langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ [a, b] = c,\ aca^2 = c,\ bcb^2 = c \rangle. \tag{1.72}$$

The canonical Schrödinger matrix representation is given in this case by

$$a \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}, \quad c \longmapsto \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}.$$

We immediately see that the dual representation $V^\vee$ is then given by

$$a \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \quad c \longmapsto \begin{pmatrix} \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}.$$

Since the traces of the two matrices corresponding to c are not equal, it follows that the Schrödinger representation V and its dual $V^\vee$ are not isomorphic.

Hence, formula (1.71) implies in this case that, up to isomorphism of representations, $\mathcal{H}_3$ has:

- 2 irreducible rep. of dimension 3 ($V, V^\vee$),
- 9 irreducible rep. of dimension 1 (i.e., the 9 characters coming from the abelianization $(\mathbb{Z}/3)^2 \cong \mathcal{H}_3/\mu_3$).

1.4 Line Bundles on Complex Tori

Let $X = V/\Lambda$ be a complex torus of dimension g, where V is a $\mathbb{C}$-vector space of dimension g and Λ is a lattice inside V.

Recall that the Néron-Severi group $NS(X)$, which is defined as the image of the homomorphism

$$c_1\colon \operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}), \tag{1.73}$$

can be identified with the group of hermitian forms

$$H\colon V \times V \to \mathbb{C} \qquad \text{with} \qquad \operatorname{im} H(\Lambda, \Lambda) \subset \mathbb{Z}, \tag{1.74}$$

see [BL04], Ch. 2, Sec. 2.1).

We collect here, under the above-mentioned identification, some fundamental notions and definitions, as for which we refer the reader to [BL04].

Given on X a holomorphic line bundle $\mathcal{L}$ with first Chern class $H = c_1(\mathcal{L})$, we recall that $E = \mathfrak{Im} H$ is an alternating form $E\colon V \times V \to \mathbb{R}$ which is $\mathbb{Z}$-valued on the lattice Λ, see [BL04], Lemma 2.1.7. Hence, by the *elementary divisor theorem* ([Bou07], Theorem 1 of Sec. 5.1) there exists a basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ of Λ with respect to which E is represented by a matrix of the form

$$\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}, \tag{1.75}$$

where $\Delta = \operatorname{diag}(\delta_1, \dots, \delta_g)$ is a $g \times g$ diagonal matrix with integers $\delta_i \geq 0$ such that $\delta_i \mid \delta_{i+1}$ for $i = 1, \dots, g-1$.

Definition 1.51 (Type of $\mathcal{L}$)**.** The integers δ_i are called *elementary divisors* of $\mathcal{L}$ and the vector $(\delta_1, \dots, \delta_g)$ is defined as the *type of* $\mathcal{L}$.

Definition 1.52 (Pfaffian of $\mathcal{L}$)**.** We call the Pfaffian of the skew-symmetric matrix (1.75) the *Pfaffian of* $\mathcal{L}$, namely

$$\operatorname{Pf}(\mathcal{L}) := \det(\Delta) = \delta_1 \cdots \delta_g.$$

Definition 1.53. A holomorphic line bundle $\mathcal{L}$ on a complex torus X is said to be *non-degenerate* (resp. *positive definite/negative definite*) if its first Chern class $c_1(\mathcal{L})$ is a non-degenerate (resp. positive definite/negative definite) hermitian form.

Definition 1.54. A *polarization* H on a complex torus X is defined as the first Chern class $H = c_1(\mathcal{L})$ of a positive definite holomorphic line bundle $\mathcal{L} = \mathcal{O}_X(D)$. We often say by abuse of notation that $\mathcal{L}$ (or D) itself is a polarization.

Definition 1.55. An *abelian variety* A is a complex torus endowed with a polarization $H = c_1(\mathcal{O}_A(D))$. We say that (A, H) is a *polarized abelian variety* and by abuse of notation we often write $(A, \mathcal{O}_A(D))$ or (A, D) instead of (A, H).

Remark 1.56. From the work of Lefschetz [Lef21a, Lef21b] (see also [Mum70], pp. 29–33, Theorem of Lefschetz) it follows that on a complex torus a holomorphic line bundle is positive definite if and only if it is ample. This means that an abelian variety can be defined as a complex torus admitting a projective embedding (and hence, by Chow's Theorem, it is a projective variety).

1.4.1 The Associated Homomorphism Φ_D

Given a complex torus X of dimension g, we will use the canonical identification between the dual complex torus $\widehat{X}$ and $\mathrm{Pic}^0(X)$ (see [BL04],Proposition 2.4.1), where the latter consists by definition of those line bundles with vanishing first Chern class.

For any given holomorphic line bundle $\mathcal{L} = \mathcal{O}_X(D)$ on X, it is defined a map (see [BL04], p. 36)

$$\begin{aligned} \Phi_D \colon X \to \widehat{X} &= \mathrm{Pic}^0(X) \\ x \longmapsto t_x^*&\mathcal{L} \otimes \mathcal{L}^{-1}, \end{aligned} \tag{1.76}$$

where $t_x \colon X \to X$ denotes the translation by $x \in X$.

Remark 1.57. By definition it follows immediately that

$$\Phi_{(-D)} = (-1_{\widehat{X}}) \circ \Phi_D,$$

where $-1_{\widehat{X}} : \widehat{X} \to \widehat{X}$ denotes the multiplication by -1.

The above-defined map Φ_D is indeed a homomorphism between complex tori by the *Theorem of the Square* (see [BL04], Theorem 2.3.3, or [Mum70], pp. 59–60).

Hence, setting

$$\mathcal{K}(D) := \ker \Phi_D \tag{1.77}$$

for its kernel, we have the following propositions.

Proposition 1.58 ([BL04], Lemma 2.4.7(b))**.**

$$\mathcal{K}(D) = X \qquad \textit{if and only if} \qquad \mathcal{O}_X(D) \in \mathrm{Pic}^0(X).$$

Proposition 1.59 ([BL04], Proposition 2.4.8)**.** *$\mathcal{K}(D)$ is a finite group if and only if $\mathcal{O}_X(D)$ is a non-degenerate line bundle.*

Hence, under the assumption that $\mathcal{O}_X(D)$ is a non-degenerate line bundle of type $(\delta_1, \dots, \delta_g)$, the homomorphism Φ_D is indeed an isogeny between complex tori, whose degree is computed by the following.

Proposition 1.60 ([BL04], Proposition 2.4.9)**.** *For any non-degenerate line bundle $\mathcal{O}_X(D)$ of type $(\delta_1, \dots, \delta_g)$ with Pfaffian $\delta := \prod_{i=1}^g \delta_i$, it holds true*

$$\deg \Phi_D = \delta^2 = \prod_{i=1}^{g} \delta_i^2.$$

1.4.2 The Theta Group $\mathcal{G}(D)$ and its Canonical Representation

Let A be an abelian variety. Given a holomorphic line bundle $\mathcal{L} = \mathcal{O}_A(D)$, we consider the group $\mathcal{G}(D)$ of all automorphisms of its associated geometric line bundle $\mathbb{V}(\mathcal{L})$ which are lifts of some translations $t_x \colon A \to A$, where $x \in A$.

More formally, we give the following definitions, as for which we refer the reader to [BL04], Ch. 6.

Definition 1.61. Suppose $x \in A$. Recalling that $\mathbb{V}(\mathcal{L})$ denotes the geometric line bundle whose sheaf of section is $\mathcal{L}$, a biholomorphic map $\varphi_x \colon \mathbb{V}(\mathcal{L}) \to \mathbb{V}(\mathcal{L})$ is called *an automorphism of $\mathcal{L}$ over x* if the following diagram commutes

$$\begin{tikzcd}
\mathbb{V}(\mathcal{L}) \arrow[r, "\varphi_x"] \arrow[d] & \mathbb{V}(\mathcal{L}) \arrow[d] \\
A \arrow[r, "t_x"] & A
\end{tikzcd}$$

and, for every $y \in A$, the induced map on the fibres

$$\varphi(y) \colon \mathbb{V}(\mathcal{L})_y \to \mathbb{V}(\mathcal{L})_{y+x}$$

is $\mathbb{C}$-linear.

Definition 1.62 (Theta group)**.** Given a line bundle $\mathcal{L} = \mathcal{O}_A(D)$ on an abelian variety A, we define the *theta group* $\mathcal{G}(D)$ as follows

$$\mathcal{G}(D) := \{(\varphi_x, x) \mid x \in A,\ \varphi_x \text{ is an automorphism of } \mathcal{L} \text{ over } x\}.$$

Recalling that $\mathcal{K}(D)$ is the group of translations $t_x \colon A \to A$ such that $t_x^*\mathcal{L} \cong \mathcal{L}$, the image of the map

$$\begin{aligned} \mathcal{G}(D) &\to A \\ (\varphi_x, x) &\mapsto x. \end{aligned} \tag{1.78}$$

is $\mathcal{K}(D)$ by the universal property of the fibre product. Indeed, we can be more precise.

Proposition 1.63 ([BL04], Prop. 6.1.1)**.** *Given a line bundle $\mathcal{L} = \mathcal{O}_A(D)$ on an abelian variety A, the theta group $\mathcal{G}(D)$ is a central extension of $\mathcal{K}(D)$ via $\mathbb{C}^*$, namely it fits into the following exact sequence*

$$1 \to \mathbb{C}^* \to \mathcal{G}(D) \to \mathcal{K}(D) \to 0. \tag{1.79}$$

Given a line bundle $\mathcal{L} = \mathcal{O}_A(D)$ on an abelian variety A, the theta group $\mathcal{G}(D)$ acts in a natural way on the vector space of global sections $H^0(A, \mathcal{O}_A(D))$.

Suppose s is a global section of $\mathcal{L} = \mathcal{O}_A(D)$ and $(\varphi_x, x) \in \mathcal{G}(D)$. Since the following diagram

$$\begin{tikzcd}
\mathbb{V}(\mathcal{L}) \arrow[r, "\varphi_x"] & \mathbb{V}(\mathcal{L}) \\
A \arrow[u, "s"] \arrow[r, "t_x"] & A \arrow[u, "\varphi_x \circ s \circ t_{-x}"']
\end{tikzcd} \tag{1.80}$$

commutes, $\varphi s t_{-x}$ is also a global section of $\mathcal{L}$.

It is immediate to see that the assignment

$$((\varphi_x, x), s) \longmapsto \varphi_x \circ s \circ t_{-x}$$

defines in a canonical way an action

$$\rho \colon \mathcal{G}(D) \to \mathrm{GL}(H^0(A, \mathcal{L}))$$

which is called the *canonical representation of the Theta group $\mathcal{G}(D)$*.

Remark 1.64. Note that if $\mathcal{L} = \mathcal{O}_A(D)$ is a positive definite line bundle, there exists an explicit basis for $H^0(A, \mathcal{O}_A(D))$, namely the set of *canonical theta functions with fixed characteristic*, see [BL04], Theorem 3.2.7. Then, fixing this basis, we obtain the *canonical matrix representation of $\mathcal{G}(D)$*.

1.4.3 The Isomorphism between the Theta Group and the Heisenberg Group

Given on an abelian variety A of dimension g a positive definite line bundle $\mathcal{L} = \mathcal{O}_A(D)$ of type $(\delta_1, \dots, \delta_g)$, we define *the Heisenberg group associated with* D (resp. *the infinite Heisenberg group associated with* D) as the Heisenberg group (resp. the infinite Heisenberg group) of the abelian group $\bigoplus_{i=1}^{g} \mathbb{Z}/\delta_i$, namely

$$\mathcal{H}_D := \mathcal{H}\Big(\bigoplus_{i=1}^{g} \mathbb{Z}/\delta_i\Big), \qquad \left(\text{resp. } \mathcal{H}_D^{\infty} := \mathcal{H}^{\infty}\Big(\bigoplus_{i=1}^{g} \mathbb{Z}/\delta_i\Big)\right). \tag{1.81}$$

Remark 1.65. Note that $\mathcal{H}_D^{\infty}$ coincides with the Heisenberg group defined in Sec. 6.6 of [BL04].

Set $H_D := \bigoplus_{i=1}^{g} \mathbb{Z}/\delta_i$ and recall that, since $\mathcal{O}_A(D)$ is non-degenerate, the group $\mathcal{K}(D)$ is finite by Proposition 1.59. Indeed, in this case we can be more precise as $\mathcal{O}_A(D)$ is positive definite and hence, from Lemma 6.6.5 of [BL04], it follows that there exist group isomorphisms b, b' such that the following diagram commutes

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d, equal] & \mathcal{G}(D) \arrow[r] \arrow[d, "b", "\wr"'] & \mathcal{K}(D) \arrow[r] \arrow[d, "b'", "\wr"'] & 0 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathcal{H}_D^{\infty} \arrow[r] & H_D \times H_D^* \arrow[r] & 0
\end{tikzcd} \tag{1.82}$$

Furthermore, these isomorphisms b, b' induce an isomorphism

$$\beta \colon H^0(A, \mathcal{O}_A(D)) \to \mathbb{C}(H_D) \tag{1.83}$$

sending the basis of canonical theta functions of Remark 1.64 to the canonical basis $\{\mathbb{1}_h\}_{h \in H_D}$ of $\mathbb{C}(H_D)$, and the following diagram commutes (see [BL04, Proposition 6.7.1])

$$\begin{tikzcd}
\mathcal{G}(D) \times H^0(A, \mathcal{O}_A(D)) \arrow[r] \arrow[d, "(b,\beta)"] & H^0(A, \mathcal{O}_A(D)) \arrow[d, "\beta"] \\
\mathcal{H}_D^{\infty} \times \mathbb{C}(H_D) \arrow[r] & \mathbb{C}(H_D)
\end{tikzcd} \tag{1.84}$$

where the first and second row are given by the canonical representation of $\mathcal{G}(D)$, respectively by the Schrödinger representation of $\mathcal{H}_D^{\infty}$.

In light of diagrams (1.82) and (1.84), we see that the Heisenberg group $\mathcal{H}_D^{\infty}$ is an abstract version of the theta group $\mathcal{G}(D)$. Therefore, we will identify $\mathcal{G}(D)$ with $\mathcal{H}_D^{\infty}$ and the canonical representation of $\mathcal{G}(D)$ with the Schrödinger representation of $\mathcal{H}_D^{\infty}$, calling the elements of the canonical basis $\{\mathbb{1}_h\}_{h \in H_D}$ *finite theta functions* (see Remark 6.7.2 of [BL04]).

Still, we will say that the infinite Heisenberg group $\mathcal{H}_D^{\infty}$ acts on $H^0(A, \mathcal{O}_A(D))$ and write a sequence as follows

$$0 \to \mathbb{C}^* \to \mathcal{H}_D^{\infty} \to \mathcal{K}(D) \to 0. \tag{1.85}$$

Note that the Heisenberg group $\mathcal{H}_D \subset \mathcal{H}_D^\infty$ is by definition a finite subgroup of $\mathcal{H}_D^\infty$ of order

$$|\mathcal{H}_D| = \delta_g \cdot \prod_{i=1}^{g} \delta_i^2$$

fitting into a diagram like (1.62), namely

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathcal{H}_D^\infty \arrow[r] & H_D \times H_D^* \arrow[r] & 0 \\
1 \arrow[r] & \mu_{\delta_g} \arrow[r] \arrow[u, hook] & \mathcal{H}_D \arrow[r] \arrow[u, hook] & H_D \times H_D^* \arrow[r] \arrow[u, equal] & 0
\end{tikzcd} \tag{1.86}$$

where $\mu_{\delta_g} \subset \mathbb{C}^*$ is the group of δ_g-th roots of unity.

Hence, under the above-mentioned identification between $\mathcal{G}(D)$ and $\mathcal{H}_D^\infty$, we will write

$$0 \to \mu_{\delta_g} \to \mathcal{H}_D \to \mathcal{K}(D) \to 0, \tag{1.87}$$

considering $\mathcal{H}_D \subset \mathcal{G}(D)$ as a subgroup with the Schrödinger representation of $\mathcal{H}_D$ being induced by restriction of the canonical representation of $\mathcal{G}(D)$ (see Section 1.3).

Remark 1.66. The inclusion $\mathcal{H}_D \subset \mathcal{G}(D)$ means that we can interpret the elements of $\mathcal{H}_D$ as automorphisms over points $x \in \mathcal{K}(D)$. Note that the set $\mathcal{G}(D) \setminus \mathcal{H}_D$ consists exactly of those automorphisms given by multiplication with a non-zero constant $\lambda \in \mathbb{C}^*, \lambda \notin \mu_{\delta_g}$.

Remark 1.67. It is worth pointing out that throughout Chapter 2 we will deal with $\mathcal{H}_D$ and never with its infinite version $\mathcal{H}_D^\infty$.

1.4.4 Heisenberg Action on Sheaves

Let A be an abelian variety of dimension g. Given a positive definite line bundle $\mathcal{O}_A(D)$, we observe that its associated isogeny $\Phi_D \colon A \to \widehat{A}$ is indeed an unramified Galois cover with Galois group $\mathcal{K}(D)$ and degree equal to $\mathrm{Pf}(\mathcal{O}_A(D))^2 = |\mathcal{K}(D)|$ (see Proposition 1.60), namely we can write

$$\begin{aligned} \Phi_D \colon A &\to \mathrm{Pic}^0(A) = A/\mathcal{K}(D) \\ x &\longmapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}, \end{aligned}$$

where $t_x \colon A \to A$ denotes the translation by $x \in A$.

We now recall the notion of G-sheaf (see [Mum70], II.7) which turns out to be useful for our purposes.

Let X be an algebraic variety, $G \subset \mathrm{Aut}(X)$ a finite group of automorphisms, and denotes by $\pi \colon X \to Y = X/G$ the canonical projection.

Definition 1.68 (G-sheaf)**.** A coherent $\mathcal{O}_X$-module $\mathscr{F}$ is said to be a G-sheaf if G acts on $\mathscr{F}$ in a way compatible with its action on X. In other words, for every $g \in G$ there must be an automorphism $g^* \colon \mathscr{F} \to \mathscr{F}$ inducing an isomorphism between the stalks $\mathscr{F}_x \to \mathscr{F}_{gx}$ for every $x \in X$.

Example 1.69. Consider the pull-back $\pi^*\mathscr{G}$ of a coherent $\mathcal{O}_Y$-module $\mathscr{G}$. This is a coherent $\mathcal{O}_X$-module ([Har77], Ch. II, Prop. 5.8(b)) which is a G-sheaf in a natural way. More precisely, for every $g \in G$ there is a commutative diagram as follows

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\
& Y &
\end{tikzcd} \tag{1.88}$$

providing for any open set $U \subset X$

$$\begin{tikzcd}
U \arrow[rr, "g"] \arrow[dr, "\pi"'] & & g(U) \arrow[dl, "\pi"] \\
& \pi(U) &
\end{tikzcd} \tag{1.89}$$

Since $\pi \circ g = \pi$, the pull-back map

$$\begin{aligned} g_U^* \colon \pi^*\mathscr{G}(U) &\to \pi^*\mathscr{G}(g(U)) \\ s &\longmapsto g^*(s) := s \circ g^{-1} \end{aligned} \tag{1.90}$$

yields an automorphism

$$g^* \colon \pi^*\mathscr{G} \to \pi^*\mathscr{G}.$$

Note moreover that g^* induces on the level of stalks isomorphisms as follows

$$(\pi^*\mathcal{G})_x \cong (\pi^*\mathcal{G})_{gx}$$

since sections comes from $Y = X/G$ and locally we have a diagram like (1.89).

Remark 1.70. If in the previous example $\mathscr{G}$ is a locally free $\mathcal{O}_Y$-module, it becomes easier to describe the G-sheaf structure of $\pi^*\mathscr{G}$ since we can do it in terms of vector bundles. Namely, considering the associated vector bundle $E := \mathbb{V}(\mathscr{G})$, for every $g \in G$ we have a diagram as follows

$$\begin{tikzcd}
E \times_Y X \arrow[r] \arrow[d] & E \times_Y X \arrow[d] & (e,x) \arrow[r, mapsto] \arrow[d, mapsto] & (e,gx) \arrow[d, mapsto] \\
X \arrow[r, "g"] & X & x \arrow[r, mapsto] & gx
\end{tikzcd}$$

which shows that G acts on E in a way compatible with its action on X.

Indeed, the following result shows that, if G acts freely, G-sheaves on X are essentially pull-backs of sheaves on $Y = X/G$.

Proposition 1.71. *Let X be an algebraic variety and $G \subset \mathrm{Aut}(X)$ be a finite group of automorphisms of X, acting freely on X. Let $\pi \colon X \to Y := X/G$ be the canonical projection. Then the functor $\mathcal{F} \mapsto \pi^*\mathcal{F}$ is an equivalence between the category of coherent $\mathcal{O}_Y$-modules and that of coherent G-sheaves on X, whose inverse is given by $\mathscr{F} \mapsto \pi_*(\mathscr{F})^G$. Locally free sheaves correspond to locally free sheaves of the same rank.*

Applying the previous proposition to the case of the isogeny $\Phi_D\colon A \to \widehat{A}$ where A is an abelian variety and D a polarization, we see that for every coherent $\mathcal{O}_{\widehat{A}}$-module $\mathscr{F}$, the pull-back $\Phi_D^*\mathscr{F}$ is a $\mathcal{K}(D)$-sheaf. Moreover, given a sequence of coherent $\mathcal{O}_{\widehat{A}}$-module, if we take pull-backs via Φ_D we get a sequence of $\mathcal{K}(D)$-equivariant sheaves.

Notice that the notion of a G-sheaf can be generalized to the case where the action of the group G on the algebraic variety X is not faithful. Namely, in Definition 1.68 we replace the hypothesis $G \subset \mathrm{Aut}(X)$ with $\rho\colon G \to \mathrm{Aut}(X)$ an action.

Example 1.72. Given on an abelian variety A a polarization D, every $\mathcal{K}(D)$-sheaf is also a Heisenberg $\mathcal{H}_D$-sheaf by using the surjection

$$\mathcal{H}_D \twoheadrightarrow \mathcal{K}(D).$$

Hence, since the action of $\mathcal{K}(D)$ on A is free, the pull-back $\Phi_D^*\mathscr{F}$ of a coherent $\mathcal{O}_{\widehat{A}}$-module $\mathscr{F}$ is both a $\mathcal{K}(D)$-sheaf and a $\mathcal{H}_D$-sheaf.

Example 1.73. Given on an abelian variety A a polarization D of type $(\delta_1, \dots, \delta_g)$, the line bundle $\mathcal{O}_A(D)$ is a $\mathcal{H}_D$-sheaf since we consider $\mathcal{H}_D \subset \mathcal{G}(D)$.

1.4.5 The Fourier-Mukai Transform

Given a line bundle $\mathcal{O}_A(D)$ an abelian variety A, we would like to have information on its cohomology. From this viewpoint the so-called *Riemann-Roch Theorem for abelian varieties* provides us with some help. Let us recall it.

Theorem 1.74 ([Mum70], page 150)**.** *Let A be an abelian variety of dimension g and $\mathcal{O}_A(D)$ a line bundle. Then it holds*

$$(i) \quad \chi(D) = \frac{D^g}{g!}$$
$$(ii) \quad \chi(D)^2 = \begin{cases} \deg \Phi_D & \text{if } \;|\mathcal{K}(D)| < \infty \\ 0 & \text{else} \end{cases}$$

where D^g denotes the g-fold self-intersection number of D.

Thanks to the previous theorem, it becomes easy to compute the cohomology of a positive definite line bundle on an abelian variety.

Corollary 1.75. *Let A be an abelian variety of dimension g and $\mathcal{O}_A(D)$ a positive definite line bundle of type $(\delta_1, \dots, \delta_g)$ with Pfaffian $\delta := \prod_{j=1}^{g} \delta_j$. Then*

$$h^j(D) = 0 \quad \text{for } j \neq 0, \qquad h^0(D) = \delta, \qquad D^g = g! \cdot \delta. \tag{1.91}$$

Proof. By Kodaira vanishing we have that $h^j(D) = 0$ for all $j \neq 0$, and hence $\chi(D) = h^0(D)$. Moreover, since D is non-degenerate, we have $\mathcal{K}(D)$ is finite. Thus, applying (ii) of Theorem 1.74 we get

$$h^0(D)^2 = \deg \Phi_D = \delta^2,$$

where the last equality follows from Proposition 1.60. Hence, we get $h^0(D) = \delta$ and

$$D^g = g! \cdot h^0(D) = g! \cdot \delta,$$

where the first equality follows from (i) of Theorem 1.74. □

More generally, we are interested in the case where the line bundle $\mathcal{O}_A(D)$ is non-degenerate (or equivalently by Proposition 1.59, $|\mathcal{K}(D)| < \infty$).

The cohomology of such a line bundle D turns out to be really interesting as the so-called *Mumford's Index Theorem* points out (see [Mum70], page 150).

Theorem 1.76 (Mumford's Index Theorem)**.** *Let A be an abelian variety of dimension g and $\mathcal{O}_A(D)$ a non-degenerate line bundle. Then there exists a unique integer $i(D)$, $0 \leq i(D) \leq g$, such that*

$$H^j(A, \mathcal{O}_A(D)) = 0 \quad \textit{for } j \neq i(D), \qquad H^{i(D)}(A, \mathcal{O}_A(D)) \neq 0. \tag{1.92}$$

Moreover, $i(-D) = g - i(D)$.

Definition 1.77. Let A be an abelian variety of dimension g and $\mathcal{O}_A(D)$ a non-degenerate line bundle. We call the integer $i(D)$ given by Mumford's Index Theorem the *"index of D"*.

Example 1.78. Given a negative definite line bundle $\mathcal{O}_A(D)$ on an abelian variety A, $\dim A = g$, we see right away that its index $i(D) = g$. Indeed, $i(D) = g - i(-D)$ by Serre duality and since $-D$ is ample we get immediately $i(-D) = 0$ applying Corollary 1.75.

A natural question then arises: given a non-degenerate line bundle $\mathcal{O}_A(D)$ on an abelian variety A, $\dim A = g$, how can we compute its index $i(D)$?

The following proposition provides an answer.

Proposition 1.79 ([Mum70], Corollary on page 62)**.** *Let $A = V/\Lambda$ be an abelian variety of dimension g and $\mathcal{O}_A(D)$ a non-degenerate line bundle with first Chern class H, which is a non-degenerate hermitian form defined on V. Then the index $i(D)$ equals the number s of negative eigenvalues of H.*

Corollary 1.80. *Let A be an abelian variety of dimension g and $\mathcal{O}_A(D)$ a non-degenerate line bundle. For any line bundle $\mathcal{O}_A(M) \in \mathrm{Pic}^0(A)$, we have that $i(D + M) = i(D)$.*

Proof. It is enough to observe that a line bundle $\mathcal{O}_A(M) \in \mathrm{Pic}^0(A)$ has vanishing first Chern class. □

Hence, summing up, for a non-degenerate line bundle $\mathcal{O}_A(D)$ on an abelian variety A, $\dim A = g$, there exists an integer $i(D)$, $1 \leq i(D) \leq g$, such that

$$H^j(A, \mathcal{O}_A(D + M)) = 0 \quad \text{for } j \neq i(D), \qquad H^{i(D)}(A, \mathcal{O}_A(D + M)) \neq 0. \tag{1.93}$$

for every line bundle $\mathcal{O}_A(M) \in \mathrm{Pic}^0(A)$.

Consider the *normalized Poincaré bundle* $\mathcal{P}$ on $A \times \widehat{A}$ (see [BL04], Ch. 2, Sec. 5 for its definition and properties) and denote by p_A and $p_{\widehat{A}}$ the two projections from $A \times \widehat{A}$

to A, respectively $\widehat{A}$. Given a line bundle $\mathscr{L} = \mathcal{O}_A(D)$ on A, according to the following picture

$$\begin{tikzcd}
& A \times \widehat{A} \arrow[dl, "p_A"'] \arrow[dr, "p_{\widehat{A}}"] & & & \mathcal{P} \otimes p_A^*\mathscr{L} \arrow[dr, mapsto] & \\
A & & \widehat{A} & \mathscr{L} \arrow[ur, mapsto] & & R^i p_{\widehat{A}*}\big(\mathcal{P} \otimes p_A^*\mathscr{L}\big)
\end{tikzcd} \tag{1.94}$$

we consider all higher direct images

$$R^i p_{\widehat{A}*}\big(\mathcal{P} \otimes p_A^*\mathscr{L}\big), \quad i \geq 0 \tag{1.95}$$

Next proposition gives information on them under the hypothesis that $\mathscr{L}$ is non-degenerate.

Proposition 1.81 (cf. [BL04], Lemma 14.2.1)**.** *Let A be an abelian variety and $\mathscr{L} = \mathcal{O}_A(D)$ a non-degenerate line bundle. Then it holds*

(i) $R^j p_{\widehat{A}*}\big(\mathcal{P} \otimes p_A^*\mathscr{L}\big) = 0$ *for* $j \neq i(D)$

(ii) $R^{i(D)} p_{\widehat{A}*}\big(\mathcal{P} \otimes p_A^*\mathscr{L}\big)$ *is locally free of rank* $h^{i(D)}(A, D)$ *on* $\widehat{A}$.

Thus, it makes sense to give the following definition.

Definition 1.82 (Fourier-Mukai Transform)**.** Let A be an abelian variety and $\mathscr{L} = \mathcal{O}_A(D)$ a non-degenerate line bundle. The locally free $\mathcal{O}_{\widehat{A}}$ -module of rank $h^{i(D)}(A, D)$

$$\widehat{\mathscr{L}} := R^{i(D)} p_{\widehat{A}*}\big(\mathcal{P} \otimes p_A^*\mathscr{L}\big)$$

is called *Fourier-Mukai transform of $\mathscr{L}$*.

Example 1.83. Let A be an abelian variety, $\dim A = g$, and $\mathscr{L} = \mathcal{O}_A(D)$ an ample line bundle yielding a polarization of type $(\delta_1, \dots, \delta_g)$ with Pfaffian $\delta := \prod_{i=1}^{g} \delta_i$. Hence, $\mathscr{L}^{-1}$ is a negative definite line bundle with index g and it holds that

$$h^g(-D) = h^0(D) = \delta,$$

where the first equality follows from Serre, while the second from Corollary 1.75. Therefore, the Fourier-Mukai transform $\widehat{\mathscr{L}^{-1}}$ is a locally free $\mathcal{O}_{\widehat{A}}$ -module of rank δ.

Remark 1.84. We have defined here the Fourier-Mukai transform for a non-degenerate line bundle . Indeed, this is just the very first instance of a more general construction which yields an equivalence of categories between the two derived categories $D(A)$ and $D(\widehat{A})$ of an abelian variety A and its dual $\widehat{A}$. We refer the interested reader to [Muk81] or [BL04], Ch. 14. Sec. 2, for a more detailed account on the topic.

Finally, we present here one main result of the theory of Fourier-Mukai transforms, which actually turns out to be central for the construction method described in Chapter 2.

Proposition 1.85 (cf. [Muk81, formula (3.10)], [Polish03, Prop. 11.9]). *Let $\mathfrak{L}$ be a negative definite line bundle on an abelian surface A. Set $\mathcal{L} := \mathcal{O}_A(D) := \mathfrak{L}^{-1}$ and $V := H^0(A, \mathcal{O}_A(D))$. Then, recalling that $\Phi_D\colon A \to \widehat{A}$ denotes the isogeny associated with the ample line bundle $\mathcal{L} = \mathcal{O}_A(D)$, it holds true*

$$(-\Phi_D)^*(\,\widehat{\mathfrak{L}}\,) \cong \mathcal{L} \otimes V^\vee.$$

Proof. Since $\mathfrak{L}$ is a negative definite line bundle on the abelian surface A, its index equals 2 (see Example 1.78). Hence, applying Corollary 14.3.6 a) of [BL04], we obtain

$$(\Phi_{-D})^*\widehat{\mathfrak{L}} \cong H^2(A, \mathfrak{L}) \otimes \mathfrak{L}^{-1}.$$

Recalling that by definition $\mathcal{L} = \mathcal{O}_A(D) = \mathfrak{L}^{-1}$ and that $\Phi_{(-D)} = -\Phi_D$, by Serre duality we get our formula. □

1.4.6 Polarizations on Abelian Surfaces

Here we recall some well-known facts on polarizations on abelian surfaces which will turn out to be useful in Chapter 2.

Given on an abelian surface A an ample divisor D yielding a polarization of type (δ_1, δ_2) (hence with Pfaffian $\delta := \delta_1\delta_2$), the linear system $|D|$ has no base points if $\delta_1 \geq 2$ (by Proposition 4.1.5 of [BL04]).

1.4.6.a The Case $(1, \delta)$, $\delta \geq 3$

If $\delta_1 = 1$ and $\delta = \delta_2 \geq 3$ we first show that it has no base points if it has no fixed part; since the base-point locus Σ is $\mathcal{K}(D)$-invariant, it has cardinality a multiple of $|\mathcal{K}(D)| = \delta^2$, while $D^2 = 2\delta$, a contradiction.

Note that the system $|D|$ has no fixed part (see [BL04, Lemma 10.1.1]) unless the pair $(A, \mathcal{O}_A(D))$ is isomorphic to a polarized product of two elliptic curves, namely

$$(A, \mathcal{O}_A(D)) \cong (E_1, \mathcal{O}_{E_1}(D_1)) \times (E_2, \mathcal{O}_{E_2}(D_2)), \tag{$*$}$$

where $\deg(D_1) = 1$, $\deg(D_2) = \delta_2$.

Hence, we conclude in particular that for $\delta_1 = 1$, $\delta \geq 3$, $|D|$ has no base points if A does not contain any elliptic curve.

1.4.6.b The Case $(1, 2)$

If $\delta_1 = 1$ and $\delta = 2$, D has no fixed part unless (see [Bar87]) A is the polarized product of two elliptic curves,

$$(A, \mathcal{O}_A(D)) = (E_1, \mathcal{O}_{E_1}(P_1)) \times (E_2, \mathcal{O}_{E_2}(2P_2)), \tag{$**$}$$

where P_1, P_2 are points; in this case the base locus equals the curve $\{P_1\} \times E_2$.

If there is no curve in the base locus, by $\mathcal{K}(D)$-invariance, the base locus consists of 4 distinct points.

Hence, in all cases, given a basis x_1, x_2 of $H^0(A, \mathcal{O}_A(D))$, at each base point either x_1 or x_2 is a local parameter.

1.5 Group Cohomology: a Brief Overview

In this section we will give a brief introduction to group cohomology, referring the reader to [Bro94] as a classical textbook on the topic. More precisely, our main goal is to introduce the groups $H^i(G, M)$ which we will use throughout Chapter 3.

We start with the definition of a G-module, where G is a finite group.

Definition 1.86 (G-module)**.** A G-module is an abelian group M equipped with a left action $G \times M \to M$ which is compatible with the abelian group structure on M, namely

$$g * (x \cdot y) = (g * x) \cdot (g * y), \qquad g * 1 = 1.$$

Example 1.87. (1) Every abelian group M where G acts trivially is a G-module.

(2) Every abelian group M is a $\mathbb{Z}$-module with the natural action given, for any $n \in \mathbb{Z}$, $m \in M$, by

$$(n, m) \longmapsto n * m := \begin{cases} \underbrace{m \cdot \ldots \cdot m}_{n \text{ times}} & n \geq 0 \\ \underbrace{m^{-1} \cdot \ldots \cdot m^{-1}}_{-n \text{ times}} & n < 0 \end{cases}$$

Given a G-module M, we define the so-called G-invariant part M^G as follows

$$M^G := \{m \in M \mid g * m = m \text{ for all } g \in G\}$$

It is easy to see that M^G is an abelian subgroup of M where by definition G acts trivially.

Note that $\cdot^G$ gives a functor from the category of G-modules $\mathbf{Mod_G}$ to the category of abelian groups $\mathbf{Ab}$, namely

$$\begin{aligned} \cdot^G \colon \mathbf{Mod_G} &\to \mathbf{Ab} \\ M &\to M^G. \end{aligned}$$

and given a G-equivariant group homomorphism $f \colon M \to N$, the corresponding map $f^G \colon M^G \to N^G$ is simply defined as the restriction

$$f^G := f_{|M^G}.$$

Consider now a short exact sequence of G-modules, that is a short exact sequence of abelian groups

$$1 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 1, \tag{1.96}$$

where the group homomorphisms are G-equivariant. Applying the functor $\cdot^G$, we get an exact sequence

$$1 \longrightarrow M^G \xrightarrow{\varphi^G} N^G \xrightarrow{\psi^G} P^G, \tag{1.97}$$

where ψ^G is in general not surjective.

In other words, $\cdot^G$ is a left exact functor, and therefore, as usual, we would like to construct its right derived functors to get a long exact sequence in cohomology. Given a G-module M, these right derived functors give as values abelian groups denoted by $H^i(G, M)$ and called *the i-th cohomology group of G with coefficients in M.*

Indeed, we can give a more down-to-earth description of these cohomology groups, which turns out to be really useful in order to perform computations in Chapter 3.

1.5.1 Cocycles and Coboundaries

Given a G-module M, we define for $n \geq 0$ the group $C^n(G,M)$ of *n-cochains* as the abelian group of all M-valued functions defined on $\underbrace{G \times \ldots \times G}_{n \text{ times}}$, namely

$$C^n(G,M) := \{f \colon G^n \to M\},$$

where by definition $G^0 := \{1\}$ and the product is naturally given by

$$(\alpha \cdot \beta)(g_1, \ldots, g_n) := \alpha(g_1, \ldots, g_n) \cdot \beta(g_1, \ldots, g_n), \qquad \forall \alpha, \beta \in C^n(G,M).$$

We observe immediately that the group of 0-cochains is indeed isomorphic to M, namely

$$\begin{aligned} C^0(G,M) = \{f \colon \{1\} \to M\} &\to M \\ f &\longmapsto f(1), \end{aligned} \tag{1.98}$$

and we define the *first coboundary operator* as follows

$$\begin{aligned} \partial^1 \colon C^0(G,M) \cong M &\to C^1(G,M) \\ m &\to \{g \mapsto g * m \cdot m^{-1}\}. \end{aligned} \tag{1.99}$$

Moreover, given an n-cochain τ, $n \geq 1$, we define the *$(n+1)$-th coboundary operator*

$$\partial^{n+1} \colon C^n(G,M) \to C^{n+1}(G,M)$$

via the following formula

$$\partial^{n+1}\tau(g_1, \ldots, g_{n+1}) :=$$

$$g_1 * \big(\tau(g_2, \ldots, g_{n+1})\big) \cdot \left(\prod_{j=2}^{n+1} \tau(g_1, \ldots, g_{j-2},\ g_{j-1}g_j,\ g_{j+1}, \ldots, g_{n+1})^{(-1)^{j-1}}\right) \cdot \tau(g_1, \ldots, g_n)^{(-1)^{n+1}}. \tag{1.100}$$

Since for all $n \geq 0$ it holds true

$$\partial^{n+2} \circ \partial^{n+1} = 1,$$

we get a cochain complex

$$0 \to C^0(G,M) \cong M \xrightarrow{\partial^1} C^1(G,M) \xrightarrow{\partial^2} C^2(G,M) \xrightarrow{\partial^3} \ldots \xrightarrow{\partial^n} C^n(G,M) \xrightarrow{\partial^{n+1}} \ldots \tag{1.101}$$

Definition 1.88 (n-cocycles)**.** The group $Z^n(G,M)$ of *n-cocycles* is defined as the kernel of the homomorphism $\partial^{n+1} \colon C^n(G,M) \to C^{n+1}(G,M)$, namely

$$Z^n(G,M) := \ker \partial^{n+1} \subset C^n(G,M).$$

Definition 1.89 (n-coboundaries)**.** The group $B^n(G,M)$ of *n-coboundaries* is defined as the image of the homomorphism $\partial^n \colon C^{n-1}(G,M) \to C^n(G,M)$, namely

$$B^n(G,M) := \operatorname{im} \partial^n \subset C^n(G,M).$$

Note that, since $\partial^{n+2} \circ \partial^{n+1} = 1$, we have indeed

$$B^n(G, M) \subset Z^n(G, M).$$

Definition 1.90 (the n-th cohomology group)**.** We define the *n-th cohomology group of G with coefficients in M* as following quotient

$$H^n(G, M) := Z^n(G, M)/B^n(G, M).$$

Note in particular that

$$H^0(G, M) = \ker \partial^1 = \{m \in M \mid g * m = m \quad \forall g \in G\} = M^G \tag{1.102}$$

$$H^1(G, M) = \frac{\ker \partial^2}{\operatorname{im} \partial^1} = \frac{\{\tau \colon G \to M \mid g * \tau(h) \cdot \tau(gh)^{-1} \cdot \tau(g) = 1\}}{\{\tau \colon G \to M \mid \exists m \in M \text{ s.t. } \tau(g) = g * m \cdot m^{-1}\}} \tag{1.103}$$

$$H^2(G, M) = \frac{\{\tau \colon G \times G \to M \mid g * \tau(h, k) \cdot \tau(gh, k)^{-1} \cdot \tau(g, hk) \cdot \tau(g, h)^{-1} = 1\}}{\{\tau \colon G \times G \to M \mid \exists \gamma \colon G \to M \text{ s.t. } \tau(g, h) = g * \gamma(h) \cdot \gamma(gh)^{-1} \cdot \gamma(g)\}} \tag{1.104}$$

Hence, for a given function $\tau \colon G \to M$, we have the so-called *1-cocycle relation*

$$g * \tau(h) \cdot \tau(gh)^{-1} \cdot \tau(g) = 1, \qquad \forall g, h \in G. \tag{1.105}$$

Similarly, for a given function $\tau \colon G \times G \to M$, we have the so-called *2-cocycle relation*

$$g * \tau(h, k) \cdot \tau(gh, k)^{-1} \cdot \tau(g, hk) \cdot \tau(g, h)^{-1} = 1, \qquad \forall g, h, k \in G. \tag{1.106}$$

Remark 1.91. When dealing with cocycles and coboundaries one often omits the index n for the n-th coboundary operator ∂^n if no confusion arises. This is what we do in Chapter 3.

1.5.2 Group Extensions with Abelian Kernels

Given a finite group G, we consider a group extension

$$1 \to A \to \Gamma \to G \to 1, \qquad \text{with } A \text{ finite and abelian.} \tag{1.107}$$

There is a natural action of G on the kernel A as follows

$$g * a := s(g) \cdot a \cdot s(g)^{-1}, \tag{1.108}$$

where $s \colon G \to \Gamma$ is a set-theoretic section. Note that, since A is abelian, the action is independent of the choice of the section.

Hence, A is a G-module and we observe that for every $g, h \in G$ the two elements

$$s(gh), \qquad s(g) \cdot s(h)$$

must differ by an element $\beta(g,h)$ in the kernel A of the extension, namely

$$s(gh) = \beta(g,h)s(g) \cdot s(h).$$

It is easy to see that the map $\beta\colon G \times G \to A$ is a 2-cocycle since it fulfills the following relation

$$(g * \beta(h,k)) \cdot \beta(gh,k)^{-1} \cdot \beta(g,hk) \cdot \beta(g,h)^{-1} = 1. \tag{1.109}$$

A different choice of a section $s'\colon G \to \Gamma$ yields a cohomologous cocycle $\beta' \in Z^2(G,A)$.

Therefore, we can associate to the given extension a unique cohomology class $[\beta] \in H^2(G,A)$, which is trivial if and only if the extension splits ([MacLane95], Chapter IV, Theorem 4.1).

Chapter 2

Surfaces of General Type with $p_g = q$

As already pointed out in Notation, by *surface* we mean a two dimensional smooth complex projective variety. Using the standard notation from the theory of complex algebraic surfaces (see for instance [Bea96], [Bǎd01],[BHPV04]; see also Notation), throughout this chapter we treat *minimal* surfaces *of general type*. Recall that a surface S is said to be minimal if S does not contain any smooth curve $C \cong \mathbb{P}^1$ such that $C^2 = -1$, and a minimal surface S is of general type if the canonical divisor K_S is big and nef.

In this context, classical inequalities are known:

- $K_S^2 \geq 1$, $\chi(S) \geq 1$ (the second one due to Castelnuovo, [Bea96, Proposition X.1, Theorem X.4]);
- $K_S^2 \geq 2\chi(S) - 6$ (Noether's inequality, [BHPV04, Theorem 3.1]);
- $K_S^2 \leq 9\chi(S)$ (Bogomolov-Miyaoka-Yau inequality, [Miy77], [Yau77, Yau78]);
- $K_S^2 \geq 2p_g$ if $q > 0$ (Debarre's inequality, [Deb82]).

Here we focus on minimal surfaces of general type S with $p_g = q$: they are those with the lowest value $\chi(S) = 1$ of the invariant $\chi(S) = 1 - q + p_g$.

More precisely, the aim of this chapter is to describe the construction method for minimal surfaces of general type with $p_g = q$ developed in [AC22].

We address with particular emphasis the case $p_g = q = 2$ since it is still widely open.

2.1 General Set-up

In this section we define and then analyze those surfaces we want to construct by using our construction method [AC22]. We call them *surfaces with AP (Albanese Property)*.

Let S be a minimal surface of general type with $p_g = q$.

Definition 2.1 (Surface with AP). We say that S is a *surface with AP (Albanese Property)* if there exist an abelian surface A and a surjective morphism $\alpha\colon S \to A$ of degree $d \geq 2$ which enjoys the following property:

if $\widetilde{A}$ is an abelian surface such that $\alpha\colon S \to A$ factors as follows

$$\begin{tikzcd}
 & \widetilde{A} \arrow[dr, "\phi"] & \\
S \arrow[ur, "\widetilde{\alpha}"] \arrow[rr, "\alpha"] & & A
\end{tikzcd} \tag{2.1}$$

then $\phi\colon \widetilde{A} \to A$ is an isomorphism.

Example 2.2. A minimal surface of general type S with $p_g = q = 2$ and Albanese map $alb_S\colon S \to \mathrm{Alb}(S)$ of degree $d \geq 2$ is an example of a surface with AP (take $\alpha = alb_S$).

Remark 2.3. Note that there are examples of surfaces with AP where the surjective morphism $\alpha\colon S \to A$ is not the Albanese map of S, see Proposition 2.58.

Given a surface S with AP, we consider the Stein factorization of $\alpha\colon S \to A$ ([Har77, III, Corollary 11.5])

$$\begin{tikzcd}
 & Y \arrow[dr, "\pi"] & \\
S \arrow[ur, "f"] \arrow[rr, "\alpha"] & & A
\end{tikzcd} \tag{2.2}$$

where

- Y is a normal projective variety of dimension 2,
- $f_*\mathcal{O}_S = \mathcal{O}_Y$ (i.e., f has connected fibres),
- $\pi\colon Y \to A$ is a finite morphism (and hence also flat by Corollary 1.16 since Y is a normal variety of dimension 2 and A is a surface) of degree d, i.e., a cover in the sense of Definition 1.12.

Considering the canonical model X of S, we get a commutative diagram as follows

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[dr, "a"] & Y \arrow[d, "\pi"] \\
S \arrow[u, "f'"] \arrow[r, "\alpha"] & A
\end{tikzcd} \tag{2.3}$$

where f' contracts all the (-2)-curves on S and $f = g \circ f'$.

In particular, from the diagram above it follows that the surjective morphism $\alpha\colon S \to A$ induces a surjective morphism $a\colon X \to A$, which is in general not finite.

More precisely, the following conditions are equivalent:

(1) Y has (at most) Rational Double Points (RDP for short) as singularities,

(2) $X = Y$,

(3) $a\colon X \to A$ is finite.

We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_A \xrightarrow{\pi^{\#}} \pi_*\mathcal{O}_Y \longrightarrow \mathcal{E}^\vee \longrightarrow 0. \tag{2.4}$$

which splits, i.e.,

$$\pi_*\mathcal{O}_Y = \mathcal{O}_A \oplus \mathcal{E}^\vee. \tag{2.5}$$

Since $\pi\colon Y \to A$ is a cover, $\pi_*\mathcal{O}_Y$ is locally free and $\mathcal{E}^\vee$ is the Tschirnhaus bundle of π (see Subsection 1.1.2). Moreover, since $\alpha = \pi \circ f$ and $f_*\mathcal{O}_S = \mathcal{O}_Y$, it holds true that

$$\alpha_*\mathcal{O}_S = \pi_*\mathcal{O}_Y, \tag{2.6}$$

and therefore $\alpha_*\mathcal{O}_S = \mathcal{O}_A \oplus \mathcal{E}^\vee$ is locally free.

Summarizing, we have a split short exact sequence

$$0 \longrightarrow \mathcal{O}_A \xrightarrow{\alpha^{\#}} \alpha_*\mathcal{O}_S \longrightarrow \mathcal{E}^\vee \longrightarrow 0, \tag{2.7}$$

where $\mathcal{E}^\vee$ is a locally free $\mathcal{O}_A$-module of rank $d-1$.

Definition 2.4. Let S be a surface with AP. Using the same notation as in Definition 2.1, the surjective morphism $\alpha\colon S \to A$ of degree $d \geq 2$ yields

$$\alpha_*\mathcal{O}_S = \mathcal{O}_A \oplus \mathcal{E}^\vee, \tag{2.8}$$

where $\mathcal{E}^\vee$, which denotes the cokernel of the pull-back map $\alpha^{\#}\colon \mathcal{O}_A \to \alpha_*\mathcal{O}_S$, is a locally free $\mathcal{O}_A$-module of rank $d-1$. We call $\mathcal{E}^\vee$ the *Tschirnhaus bundle* of $\alpha\colon S \to A$.

Given a surface S with AP, we would like to analyze the cokernel $\mathfrak{F}$ of the injective map $\omega_A \to \alpha_*\omega_S$ given by pull-back of differential 2-forms. Namely, we have a split short exact sequence

$$0 \to \omega_A \cong \mathcal{O}_A \to \alpha_*\omega_S \to \mathfrak{F} \to 0, \tag{2.9}$$

where $\omega_A \cong \mathcal{O}_A$ because A is an abelian surface.

By duality for a finite morphism (see [Har77], Exercises 6.10, page 239, and 7.2, page 249)

$$\mathcal{H}om(\pi_*\mathcal{O}_Y, \omega_A) = \pi_*\omega_Y = \omega_A \oplus (\mathcal{E} \otimes \omega_A)$$

where ω_Y is the dualizing sheaf of Y and clearly $\mathcal{E} \cong (\mathcal{E}^\vee)^\vee$.

In dimension 2, ω_Y equals the sheaf of Zariski's differentials (see [Kni73]); since S is a resolution of singularities of Y, we obtain then that $\alpha_*\omega_S \subset \pi_*\omega_Y$, hence

$$\alpha_*\omega_S = \omega_A \oplus \mathfrak{F}, \qquad \mathfrak{F} \subset \mathcal{E} \otimes \omega_A. \tag{2.10}$$

Since A is an abelian surface, this formula simplifies to

$$\alpha_*\omega_S = \mathcal{O}_A \oplus \mathfrak{F}, \qquad \mathfrak{F} \subset \mathcal{E}. \tag{2.11}$$

In other words, we have that in the split exact sequence (2.9) the cokernel $\mathfrak{F}$ is a subsheaf of $\mathcal{E}$.

Remark 2.5. Some remarks on the coherent sheaf $\mathfrak{F}$ and its relation with the dual sheaf $\mathcal{E}$ of the Tschirnhaus bundle $\mathcal{E}^\vee$.

(*i*) $\mathcal{E}/\mathfrak{F}$ is supported on a finite set contained in the image of the singular points of Y, and of the points where the fibre of $f\colon S \to Y$ is positive dimensional. Hence, since every locally free sheaf on an open set of a surface such that the complement has codimension at least two extends in a unique way to a locally free sheaf on the surface, if $\mathfrak{F}$ is locally free, then $\mathfrak{F} = \mathcal{E}$.

(*ii*) Unfortunately, $\mathfrak{F}$ is in general not locally free as it occurs for the components n. $3, 4, 5$ and 12 in Table A displayed in Appendix A. Since this fact might lead to some technical difficulties, this is one of the reasons why we will make later a nice working assumption, namely the *Gorenstein Assumption* (see Assumption 2.6 and Subsection 2.2.1).

(*iii*) If Y has (at most) Rational Double Points as singularities (i.e., $X = Y$), then $\alpha_*\omega_S = \pi_*\omega_Y$ and we have equality $\mathfrak{F} = \mathcal{E}$.

Indeed, something stronger holds, namely

$$\mathfrak{F} = \mathcal{E} \qquad \Longleftrightarrow \qquad Y \quad \text{has (at most) rational singularities,}$$

see Remark 1.2 of [AC22].

2.2 The Theory of Casnati-Ekedahl

This section is devoted to the discussion of the first main ingredient of our construction method developed in [AC22], namely the structure theorems of Casnati-Ekedahl [CE96] for Gorenstein covers of small degree $d = 3, 4$ and the assumption arisen from them (Gorenstein Assumption 2.6).

Given an abelian surface A, our aim is to construct a minimal surface of general type S together with a surjective morphism $\alpha\colon S \to A$ of degree d.

Using a bottom-up approach, one can construct a degree d cover $\pi\colon Y \to A$, where Y is normal, by assigning some cover data on A (as we have seen in Subsection 1.1.3) and then consider the minimal resolution of singularities $\widetilde{S}$ of Y. Eventually, after contracting all (-1)-curves on $\widetilde{S}$ (if there are any), one gets the desired minimal surface S, and $\alpha\colon S \to A$ is induced by the composition of the resolution $\widetilde{S} \to Y$ and $\pi\colon Y \to A$.

Following this strategy, examples of surfaces of general type S with $p_g = q = 2$ and degree $d = 3, 4$ have been constructed in [PePo13a], [PePo14] by using respectively the theory of Miranda [Mir85] for $d = 3$ and the theory of Hahn-Miranda [HM99] for $d = 4$ (see Subsection 1.1.4.a).

Recalling that on a two-dimensional normal algebraic variety Y a singularity is an RDP if and only if it is a rational Gorenstein singularity (see for instance [Ish18, Theorem 7.5.1]), the reason why most authors have not used the theory of Casnati-Ekedahl for Gorenstein covers (see Subsection 1.1.4.b) relies on the fact that, for a degree d cover $\pi\colon Y \to A$ where A is an abelian surface and Y is normal, the total space Y might have non-Gorenstein singularities, and actually there are examples where it is so (see [PePo13a]). Indeed, if $d \geq 3$ and Y is Gorenstein (equivalently by Remark 1.36 since A is smooth, $\pi\colon Y \to A$ is a Gorenstein cover of degree d), then the factorization theorem of Casnati-Ekedahl for Gorenstein covers of degree $d \geq 3$ (Theorem 1.37) applies, implying that Y embeds into the projective bundle

$$p : \mathbb{P}(\mathcal{E}^\vee) \to A,$$

where $\mathcal{E}^\vee$ is the Tschirnhaus bundle of the cover $\pi\colon Y \to A$, which is given by restriction of p.

More generally, given a surface S with AP whose surjective morphism $\alpha\colon S \to A$ has degree $d \geq 3$, we consider the canonical model X of S and the morphism $a\colon X \to A$ induced by $\alpha\colon S \to A$ (see diagram (2.3)).

If $a\colon X \to A$ is a finite morphism, then it is a Gorenstein cover since X is Gorenstein (by Remark 1.36). Thus, by Theorem 1.37 we have an embeddding

$$\psi\colon X \hookrightarrow \mathbb{P}(\mathcal{E}^\vee),$$

where $\mathcal{E}^\vee$ denotes the Tschirnhaus bundle of $\alpha\colon S \to A$.

However, $a\colon X \to A$ is in general not finite, and then, considering the open set

$$A^0 := A \setminus \{z \mid \dim(a^{-1}(z)) = 1\},$$

we have an induced finite morphism $a^0\colon X^0 \to A^0$, which is a Gorenstein cover. Hence, again by Theorem 1.37, we get a rational map

$$\psi : X \dashrightarrow \mathbb{P}(\mathcal{E}^\vee) \tag{2.12}$$

whose image Z is birational to S.

Since we want to use the structure theorems of [CE96] for Gorenstein covers of degree $d = 3, 4$, namely Theorem 1.39 and Theorem 1.41, to provide a new construction method for surfaces of general type with $p_g = q$, we make a slightly restrictive assumption. Namely, we propose the following.

Assumption 2.6. (Gorenstein Assumption)

(I) We are given a surjective morphism $\alpha : S \to A$ of degree $d \geq 3$, where A is an abelian surface, S is the minimal model of a surface of general type with $p_g = q$, and α enjoys the property of the Albanese map, that it does not factor through a morphism of S to another abelian surface. In other words, we are given a surface S with AP whose surjective morphism $\alpha\colon S \to A$ has degree $d \geq 3$.

(II) We make the assumption that α induces an embedding $\psi : X \hookrightarrow \mathbb{P}(\mathcal{E}^\vee)$ of the canonical model X of S, where $\mathcal{E}^\vee$ denotes the Tschirnhaus bundle of $\alpha\colon S \to A$. Namely, we are given a commutative diagram as follows

$$\begin{tikzcd}
 & & \mathbb{P}(\mathcal{E}^\vee) \arrow[dr, "p"] & \\
S \arrow[r] \arrow[rrr, bend right, "\alpha"'] & X \arrow[ur, hook, "\psi"] \arrow[rr, "a"] & & A
\end{tikzcd} \tag{2.13}$$

Remark 2.7. Given a surface S with AP for which $d \geq 3$, the Gorenstein Assumption holds true if $a : X \to A$ is finite, but the family of CHPP surfaces we will construct in Section 2.5 shows that there exist examples where this assumption holds more generally without a being finite (see Remark 2.28).

Anyhow, there are also examples for which the Gorenstein Assumption does not hold. This is the case for the family of surfaces constructed in [PiPo17] (see item n. 11 in Table A), which are called in [AC22] *PP7 surfaces* and named after Roberto Pignatelli and Francesco Polizzi. Indeed, they construct an irreducible component of the moduli space of surfaces of general type with $p_g = q = 2$, $K_S^2 = 7$ and Albanese map of degree 3. Hence, a PP7 surface S is in particular a surface with AP for which $d \geq 3$, and then it makes sense to ask if the Gorenstein Assumption is fulfilled or not.

From Pignatelli-Polizzi's construction in [PiPo17] one sees that the Albanese map $\alpha \colon S \to \mathrm{Alb}(S)$ of a PP7 surface S contracts only one elliptic curve (Proposition 2.8 of [PiPo17]), yielding a Gorenstein elliptic singularity on the normal variety Y given by the Stein factorization of α. Thus, since Y is a Gorenstein variety, the cover $\pi \colon Y \to \mathrm{Alb}(S)$ induced by the Stein factorization is a Gorenstein cover of degree $d = 3$ (by Remark 1.36), and then the structure theorem of [CE96] for $d = 3$ (Theorem 1.39) applies, yielding in particular an embedding $Y \subset \mathbb{P}(\mathcal{E}^\vee)$, where $\mathcal{E}^\vee$ denotes the Tschirnhaus bundle of the Albanese map α. Denoting by X the canonical model of S, we have that $S = X$ since there are no rational curves inside S, and the rational map (2.12) coincides in this case with the morphism

$$X = S \to Y \subset \mathbb{P}(\mathcal{E}^\vee)$$

induced by the Stein factorization of $\alpha \colon S \to \mathrm{Alb}(S)$ and contracting the aforementioned elliptic curve. Therefore, the Gorenstein Assumption is not fulfilled.

Remark 2.8. It is worth pointing out that in this thesis we use the theory and the structure theorems of Casnati-Ekedahl just as a tool for our construction method. In fact, in general results from [CE96] do not apply directly since we want to deal with a bigger class of morphisms, namely those which are generically finite covers of small degree $d = 3, 4$.

2.2.1 The Gorenstein Assumption implies $\mathfrak{F} = \mathcal{E}$

In (2.8) and in (2.11) of Section 2.1 we have seen that, given a surface S with AP, it holds

$$\alpha_* \mathcal{O}_S = \mathcal{O}_A \oplus \mathcal{E}^\vee,$$

$$\alpha_* \omega_S = \mathcal{O}_A \oplus \mathfrak{F}, \qquad \mathfrak{F} \subset \mathcal{E},$$

where $\mathfrak{F}$ is in general just a subsheaf of $\mathcal{E}$. Moreover, (i) of Remark 2.5 points out that $\mathfrak{F}$ is locally free if and only if $\mathfrak{F} = \mathcal{E}$.

We show here that under the Gorenstein Assumption 2.6 things go well, namely it holds true $\mathfrak{F} = \mathcal{E}$.

Proposition 2.9. *Let S be a surface with AP whose surjective morphism $\alpha \colon S \to A$ has degree $d \geq 3$. Denote by $\mathcal{E}^\vee$, $\mathfrak{F}$ the Tschirnhaus bundle of α, respectively the cokernel of the map $\omega_A \to \alpha_* \omega_S$. If S fulfills the Gorenstein Assumption 2.6, then $\mathfrak{F} = \mathcal{E}$.*

Proof. Denoting by X the canonical model of S, by the Gorenstein Assumption 2.6 we have an embedding $\psi\colon X \hookrightarrow \mathbb{P} := \mathbb{P}(\mathcal{E}^\vee)$, and hence X is a closed subscheme of the $\mathbb{P}^{d-2}$-bundle $p\colon \mathbb{P} \to A$.

Since $\mathbb{P}$ is Cohen-Macaulay at every point (in fact, $\mathbb{P}$ is smooth), Theorem 13.5 of [Lip84] ensures that for the dualizing sheaf ω_X there is an isomorphism as follows

$$\psi_*\omega_X \cong \mathcal{E}xt^{d-2}_{\mathcal{O}_\mathbb{P}}(\psi_*\mathcal{O}_X, \omega_\mathbb{P}). \tag{2.14}$$

Using the same notation as in (2.2) and in (2.3), we recall that for the surjective morphism $a\colon X \to A$ induced by $\alpha\colon S \to A$ via the Stein factorization it holds

$$a = p \circ \psi, \qquad \alpha = a \circ f',$$

where $f'\colon S \to X$ is a morphism with connected fibres contracting all (-2)-curves on S. Moreover, we have that

$$\omega_S = (f')^*\omega_X.$$

Then we apply to both sides of equality (2.14) the direct image p_*. We get on the left-hand side

$$p_*(\psi_*\omega_X) = a_*(\omega_X) = \alpha_*(\omega_S) = \omega_A \oplus \mathfrak{F} = \mathcal{O}_A \oplus \mathfrak{F},$$

where the second equality follows from projection formula since $f'_*\mathcal{O}_S = \mathcal{O}_X$.

On the right-hand side we have

$$p_*\bigg(\mathcal{E}xt^{d-2}_{\mathcal{O}_\mathbb{P}}(\psi_*\mathcal{O}_X, \omega_\mathbb{P})\bigg) = \mathcal{E}xt^{d-2}_p(\psi_*\mathcal{O}_X, \omega_\mathbb{P}), \tag{2.15}$$

where, using the same notation as in [Kle80], page 39, $\mathcal{E}xt^{d-2}_p$ stands for the $(d-2)$-th derived functor of the composition

$$p_*(\cdot) \circ \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\psi_*\mathcal{O}_X, \cdot).$$

Indeed, we have an isomorphism of derived functors (in the derived category setting)

$$R\bigg(p_*(\cdot) \circ \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\psi_*\mathcal{O}_X, \cdot)\bigg) \cong Rp_*(\cdot) \circ R\mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\psi_*\mathcal{O}_X, \cdot)$$

since *Grothendieck's Composition Theorem* (see for instance [GM03, Theorem III.7.1]) applies: in fact, in the category of $\mathcal{O}_\mathbb{P}$-modules, injectives $\mathcal{O}_\mathbb{P}$-sheaves and flabby $\mathcal{O}_\mathbb{P}$-sheaves form two classes of objects *adapted* respectively to the functors $\mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\psi_*\mathcal{O}_X, \cdot)$, $p_*(\cdot)$, see [GM03, Subsection III.6.3] for the notion of *adapted class of objects.*

Hence, since we have the vanishing

$$\mathcal{E}xt^{q}_{\mathcal{O}_\mathbb{P}}(\psi_*\mathcal{O}_X, \omega_\mathbb{P}) = 0 \qquad \text{for} \quad q \neq d-2,$$

formula displayed in line 10 of page 201 of [GM03] holds true for the sheaf $\omega_\mathbb{P}$ with $k = d-2$, and then for $n = d-2$ it reads as follows

$$R^{d-2}(p_*\mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\psi_*\mathcal{O}_X, \omega_\mathbb{P})) \cong R^{d-2-(d-2)}p_*(\mathcal{E}xt^{d-2}_{\mathcal{O}_\mathbb{P}}(\psi_*\mathcal{O}_X, \omega_\mathbb{P})). \tag{2.16}$$

Thus, equality (2.15) follows.

Recalling that $\omega_{\mathbb{P}|A} := \omega_{\mathbb{P}} \otimes p^*\omega_A$ denotes the *relative canonical sheaf* with respect to the morphism $p\colon \mathbb{P} \to A$, in this case $\omega_{\mathbb{P}} = \omega_{\mathbb{P}|A}$ because A is an abelian surface. Then, since $(d-2)$-th order duality holds for $p\colon \mathbb{P} \to A$ (see [Kle80, Definition 10, Example 12]), we have the following isomorphism

$$\mathcal{E}xt_p^{d-2}(\psi_*\mathcal{O}_X, \omega_{\mathbb{P}|A}) \cong \mathcal{H}om_{\mathcal{O}_A}(p_*(\psi_*\mathcal{O}_X), \mathcal{O}_A) = (a_*\mathcal{O}_X)^\vee,$$

and since $\alpha = a \circ f'$, $f'_*\mathcal{O}_S = \mathcal{O}_X$, we have clearly

$$(a_*\mathcal{O}_X)^\vee = (\alpha_*\mathcal{O}_S)^\vee = (\mathcal{O}_A \oplus \mathcal{E}^\vee)^\vee = \mathcal{O}_A \oplus \mathcal{E}.$$

Finally, comparing the two sides of equality (2.14) after applying p_*, we obtain

$$\mathcal{O}_A \oplus \mathfrak{F} \cong \mathcal{O}_A \oplus \mathcal{E},$$

which clearly yields our thesis, i.e., $\mathfrak{F} = \mathcal{E}$, since $\mathfrak{F} \subset \mathcal{E}$.

□

2.3 The Theory of Chen-Hacon

In this section we discuss first the main theorem of [CH06] and then the theory originated from it. This provides us with the second main ingredient of our construction method developed in [AC22] and completes the picture.

2.3.1 The Theorem of Chen-Hacon

Given a minimal surface of general type S with $p_g = q = 2$, for its Albanese map $\alpha\colon S \to A$ there are two possibilities:

(1) $\alpha(S)$ is a smooth projective curve of genus 2 or

(2) α is surjective, i.e., S has maximal Albanese dimension.

The classification of case (1) was started by Zucconi [Zuc03] and completed by Penegini [Pen11]. This is why we are interested just in case (2).

We recall that S is said to have an *irrational pencil of genus* b if there exists a surjective rational map $f\colon S \dashrightarrow B$ onto a smooth projective curve B of genus $b \geq 1$ with connected fibres (cf. [CCML98], page 278).

Thus, since $q = 2$, if S has an irrational pencil of genus b, then it must be $1 \leq b \leq 2$. The case $b = 2$ amounts to the image $\alpha(S)$ of the Albanese map $\alpha\colon S \to A$ being a smooth projective curve of genus 2. Hence, if we assume that S has maximal Albanese dimension, it must occur $b = 1$. Then by the universal property of the Albanese map we get a surjection $A \twoheadrightarrow B$ onto an elliptic curve B and A is isogenous to a product of elliptic curves.

Finally, given a minimal surface of general type S with $p_g = q = 2$, S has no irrational pencil if and only if S has a surjective Albanese map $\alpha\colon S \to A$ and Albanese surface A containing no elliptic curve. In this context the work of Chen and Hacon [CH06] singles out an important property that such a surface S has to fulfill.

Theorem 2.10 ([CH06], Theorem 3.5). *Let S be a minimal surface of general type with $p_g = q = 2$ without any irrational pencil. Denote by $\alpha\colon S \to A$ the Albanese map of S and by $\mathfrak{F}$ the coherent sheaf defined as the cokernel of the map $\omega_A \to \alpha_*\omega_S$. Then there exist a homogeneous vector bundle $\mathfrak{H}$ on A, a negative definite line bundle $\mathfrak{L}$ on $\widehat{A} = \mathrm{Pic}^0(A)$ and a short exact sequence as follows*

$$0 \to \mathfrak{H} \to \widehat{\mathfrak{L}} \to (-1_A)^*\mathfrak{F} \to 0.$$

In other words, given a minimal surface of general type S with $p_g = q = 2$ with surjective Albanese map $\alpha\colon S \to A$ and Albanese surface A containing no elliptic curve, Theorem 2.10 ensures that there exists a short exact sequence as follows

$$0 \to \mathfrak{H} \to \widehat{\mathfrak{L}} \to (-1_A)^*\mathfrak{F} \to 0, \tag{2.17}$$

where $\mathfrak{H}$ is a homogeneous vector bundle on A and $\widehat{\mathfrak{L}}$ is the Fourier-Mukai transform of a negative definite line bundle $\mathfrak{L}$ on the dual abelian surface $\widehat{A}$.

Let us recall here the definition of a homogeneous vector bundle.

Definition 2.11. Let A be an abelian surface. A locally free $\mathcal{O}_A$-module $\mathfrak{H}$ is said to be a *homogeneous vector bundle* if

$$t_x^*\mathfrak{H} \cong \mathfrak{H} \qquad \forall x \in A,$$

where $t_x\colon A \to A$ denotes the translation by $x \in A$.

Example 2.12. Given an abelian surface A, a line bundle $\mathscr{L} \in \mathrm{Pic}^0(A)$ is the very first example of a homogeneous vector bundle since

$$t_x^*\mathscr{L} \cong \mathscr{L} \qquad \forall x \in A,$$

see Proposition 1.58.

Remark 2.13. Note that from the proof of Theorem 2.10 it follows that $\mathfrak{H}$ and $\mathfrak{L}$ are constructed from the coherent sheaf $\mathfrak{F}$ by using the Fourier-Mukai transfom (see Remark 1.84).

Let us come back to the situation where Theorem 2.10 applies. Since $\mathfrak{L}$ is negative definite, the inverse line bundle $\mathcal{L} := \mathcal{O}_{A'}(D) := \mathfrak{L}^{-1}$ is an ample line bundle on the dual abelian surface $A' := \widehat{A}$ of the Albanese surface A yielding a polarization of type (δ_1, δ_2) with Pfaffian $\delta := \delta_1\delta_2$.

We consider the isogeny associated with $\mathcal{L} = \mathcal{O}_{A'}(D)$ (see Subsection 1.4.1), namely

$$\begin{aligned} \Phi_D\colon\ A' &\to \widehat{A'} \cong A \\ x &\longmapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}, \end{aligned} \tag{2.18}$$

whose kernel $\mathcal{K}(D) := \ker \Phi_D$ is a finite group of translations of A' leaving invariant the isomorphism class of $\mathcal{L} = \mathcal{O}_{A'}(D)$. Moreover, it turns out that (see diagram (1.82) in Subsection 1.4.3)

$$\mathcal{K}(D) \cong (\mathbb{Z}/\delta_1 \times \mathbb{Z}/\delta_2)^2.$$

Note also that

$$A \cong A'/\mathcal{K}(D) = \mathrm{Pic}^0(A)/\mathcal{K}(D).$$

As we have seen in Section 1.4.3, the action of the theta group $\mathcal{G}(D)$ on $V := H^0(A', \mathcal{O}_{A'}(D))$ coincides with the action of the finite Heisenberg group $\mathcal{H}_D$ on the $\mathbb{C}$-vector space $\mathbb{C}(H_D)$ of $\mathbb{C}$-valued functions defined on $H_D := \mathbb{Z}/\delta_1 \times \mathbb{Z}/\delta_2$, which is called the *Schrödinger representation* of $\mathcal{H}_D$. Therefore, we shall say that the finite Heisenberg group $\mathcal{H}_D$ acts on the vector space of global sections $V = H^0(A', \mathcal{O}_{A'}(D))$.

Hence, denoting by $\mathfrak{F}'$ and $\mathfrak{H}'$ the pull-back $(\Phi_D)^*\mathfrak{F}$, $(-\Phi_D)^*\mathfrak{H}$ respectively, we pull-back sequence (2.17) by the isogeny $-\Phi_D\colon\ A' \to A'/\mathcal{K}(D) \cong A$, getting as a result by Proposition 1.85 the following exact sequence on A'

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^{\vee} \to \mathfrak{F}' \to 0, \tag{2.19}$$

which is $\mathcal{K}(D)$-equivariant by Proposition 1.71, and hence $\mathcal{H}_D$-equivariant as pointed out in Example 1.72 of Subsection 1.4.4: it is enough to use the surjection

$$\mathcal{K}(D) \twoheadrightarrow \mathcal{H}_D.$$

We want to stress at this stage that sequence (2.19) is in general just a sequence of coherent $\mathcal{O}_{A'}$-modules. In other words, the homogeneous vector bundle $\mathfrak{H}'$ need not be a subbundle of $\mathcal{L} \otimes V^{\vee}$. Anyhow, if $\mathfrak{F}'$ is a locally free $\mathcal{O}_{A'}$-module, then (2.19) becomes a sequence of locally free $\mathcal{O}_{A'}$-modules. What we have just said about sequence (2.19) holds in a similar fashion for sequence (2.17) on A.

Since sequence 2.19 is one of the main ingredients for our construction method, we want to deal mainly with minimal surfaces of general type S with $p_g = q = 2$ without any irrational pencil in order to apply Theorem 2.10. Hence, we give the following definition.

Definition 2.14. A component $\mathcal{M}$ of the moduli space of minimal surfaces of general type with $p_g = q = 2$ is said to be of the **Main Stream** if

(1) the Albanese map is surjective and

(2)
$$\{\mathrm{Alb}(S) |\ [S] \in \mathcal{M}\}$$
contains an open set in a moduli space of polarized abelian surfaces.

Remark 2.15. Note that if $\mathcal{M}$ is a component of the Main Stream, then Theorem 2.10 applies for the general element $[S] \in \mathcal{M}$ since abelian surfaces isogenous to a product of two elliptic curves form a closed subset in the moduli space of polarized abelian surfaces.

Remark 2.16 (The induced polarization on $\widehat{\mathrm{Alb}(S)}$)**.** Given a minimal surface of general type S with $p_g = q = 2$ such that the Albanese map $\alpha\colon S \to A$ is surjective and A does not contain any elliptic curve, Theorem 2.10 applies and from its proof it follows in particular that $\alpha\colon S \to A$ determines an ample line bundle $\mathcal{O}_{A'}(D)$ on the dual abelian surface $A' := \mathrm{Pic}^0(A)$ of the Albanese surface A via the Fourier-Mukai transform of $\mathfrak{F} := \alpha_*\omega_S/\omega_A$ (see Remark 1.84). If we deal with a component of the Main Stream, the general element S satisfies Theorem 2.10, and then we would like to know the value of the Pfaffian δ of the polarization yielded on A' by $\mathcal{O}_{A'}(D)$. However, this strongly depends on the coherent sheaf $\mathfrak{F}$ (which might not be locally free, see (*ii*) of Remark 2.5), involving also its Fourier-Mukai transform. As a result, it is in general not easy to compute δ.

2.3.2 Generality Assumption

Let us sum up which are the consequences of Theorem 2.10 (Theorem 3.5 of [CH06]).

Given a minimal surface of general type S with $p_g = q = 2$ such that the Albanese map $\alpha\colon S \to A$ is surjective and the Albanese surface A does not contain any elliptic curve, Theorem 2.10 ensures that there exist an ample line bundle $\mathcal{O}_{\widehat{A}}(D)$ yielding a polarization of type (δ_1, δ_2) on $\widehat{A}$ and a $\mathcal{H}_D$-equivariant short exact sequence of coherent $\mathcal{O}_{\widehat{A}}$ -modules as follows

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathfrak{F}' \to 0, \tag{2.20}$$

where $\mathcal{L} := \mathcal{O}_{\widehat{A}}(D)$, $\mathcal{H}_D$ denotes the Heisenberg group of $(\mathbb{Z}/\delta_1 \times \mathbb{Z}/\delta_2)$, $\mathfrak{H}'$ is a homogeneous vector bundle, $V := H^0(\widehat{A}, \mathcal{O}_{\widehat{A}}(D))$ is the Schrödinger representation of $\mathcal{H}_D$ and $\mathfrak{F}' = (\Phi_D)^*\mathfrak{F}$, $\mathfrak{F}$ being defined as $\mathfrak{F} := \alpha_*\omega_S/\omega_A$.

If we deal with a component of the Main Stream, the above-mentioned theorem is satisfied by the general surface S of the component, and hence we have a sequence like (2.20). It is important to point out that, denoting by $\alpha\colon S \to A$ the Albanese map of S, in general the coherent $\mathcal{O}_{\widehat{A}}$ -module $\mathfrak{F}'$ (equivalently by Theorem 1.71, $\mathfrak{F}$ on A) is not locally free, see (*ii*) of Remark 2.5.

However, if we deal with a component of the Main Stream fulfilling the Gorenstein Assumption 2.6, for the general surface S of the component it holds by Proposition 2.9 that

$$\mathfrak{F} = \mathcal{E} \quad \text{(equivalently, } \mathfrak{F} \text{ is locally free)},$$

where $\mathcal{E}$ denotes the dual sheaf of the Tschirnhaus bundle of the Albanese map $\alpha\colon S \to A$. In other words, for such a surface S the sequence 2.20 reads as

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathfrak{F}' = \mathcal{E}' \to 0, \tag{2.21}$$

where $\mathcal{E}' := \Phi_D^*(\mathcal{E})$ is a locally free $\mathcal{O}_{\widehat{A}}$ -module.

Till now we have treated minimal surfaces of general type S with $p_g = q = 2$ such that the Albanese map $\alpha\colon S \to A$ is surjective and the Albanese surface A does not contain any elliptic curve, and we have just seen that if such a surface S fulfills the Gorenstein Assumption 2.6, then there exists a sequence like 2.21.

Since we want to consider more generally surfaces with AP fulfilling the Gorenstein Assumption and for which there exists a sequence like (2.21), we propose the following.

Assumption 2.17. (Generality Assumption)

We make here the same assumptions (I), (II) as in the Gorenstein Assumption 2.6, and we require moreover that:

(III) there exists an ample line bundle $\mathcal{L} = \mathcal{O}_{\widehat{A}}(D)$ yielding a polarization of type (δ_1, δ_2) on $\widehat{A} = \mathrm{Pic}^0(A)$ such that the pull-back $\mathcal{E}'$ of $\mathcal{E}$ via the isogeny $\Phi_D\colon \widehat{A} \to A$ is a locally free $\mathcal{O}_{\widehat{A}}$ -module fitting into a $\mathcal{H}_D$-equivariant exact sequence

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathcal{E}' \to 0, \tag{$\diamondsuit$}$$

where $\mathfrak{H}'$ is a homogeneous vector bundle and $V := H^0(\widehat{A}, \mathcal{O}_{\widehat{A}}(D))$ is the Schrödinger representation of the Heisenberg group $\mathcal{H}_D = \mathcal{H}(\mathbb{Z}/\delta_1 \times \mathbb{Z}/\delta_2)$.

Moreover, we consider the abelian surface A endowed with the dual polarization corresponding to $\mathcal{L} = \mathcal{O}_{\widehat{A}}(D)$, which is still of type (δ_1, δ_2) (see for instance [BL04, Sec. 14.4] for the notion of dual polarization).

Remark 2.18. Indeed, in the Generality Assumption the Heisenberg action on $\mathcal{L}$, respectively on $V^\vee$, makes $\mathcal{L} \otimes V^\vee$ a $\mathcal{K}(D)$-sheaf, see Subsection 1.3.2. Hence, the sequence $(\diamondsuit)$ is $\mathcal{K}(D)$-equivariant since $\mathcal{E}'$ is a pull-back via Φ_D (see Subsection 1.4.4).

Remark 2.19. Since $\mathfrak{H}'$ is a successive extension of line bundles in $\mathrm{Pic}^0(\widehat{A})$, from sequence $(\diamondsuit)$ it follows that the total Chern class of $\mathcal{E}'$ equals

$$c(\mathcal{E}') = (1 + D)^\delta, \tag{2.22}$$

and hence in particular

$$c_1(\mathcal{E}') = \delta D, \qquad c_2(\mathcal{E}') = \frac{\delta(\delta - 1)}{2} D^2 = \delta^2(\delta - 1), \tag{2.23}$$

where the last equality follows from $D^2 = 2\delta$.

Remark 2.20. For surfaces S with $p_g = q = 2$ the Generality Assumption can be considered as an alternative to the hypothesis of having a component of the Main Stream fulfilling the Gorenstein Assumption.

Now we are ready to describe in detail the construction method developed in [AC22]. This is what we do in the next section.

2.4 The Construction Method

The goal of our construction method developed in [AC22] is to construct surfaces S with AP fulfilling the Generality Assumption 2.17 (recall that for such surfaces S the surjective morphism $\alpha\colon S \to A$ has degree $d \geq 3$ by definition of Generality Assumption).

More precisely, we construct a two-dimensional normal projective variety X with (at most) RDP as singularities and K_X ample such that there is an embedding

$$X \subset \mathbb{P}(\mathcal{E}^\vee) := \mathbf{Proj}_{\mathcal{O}_A} \mathbf{Sym}(\mathcal{E}),$$

where $\mathcal{E}$ is a locally free sheaf over a given abelian surface A. Then we define S to be the minimal resolution of singularities of X.

However, at this stage it is still not clear to the reader how to do that. In order to figure it out we need to analyze in detail the surfaces we want to construct.

Suppose that a surface S with AP fulfilling the Generality Assumption is given and denote by X its canonical model. Using the same notation as in the Generality Assumption, there is a sequence of locally free $\mathcal{O}_{\widehat{A}}$ -modules as follows

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathcal{E}' \to 0, \tag{2.24}$$

which is $\mathcal{H}_D$-equivariant.

This sequence encodes a geometric interpretation which we are going to explain now. Indeed, it induces in particular a Heisenberg-equivariant surjection

$$\mathbf{Sym}(\mathcal{L} \otimes V^\vee) \to \mathbf{Sym}(\mathcal{E}') \to 0,$$

which yields a Heisenberg-equivariant embedding of projective bundles

$$\mathbb{P}(\mathcal{E}'^\vee) := \mathbf{Proj}_{\mathcal{O}_{\widehat{A}}} \mathbf{Sym}(\mathcal{E}') \hookrightarrow \mathbf{Proj}_{\mathcal{O}_{\widehat{A}}} \mathbf{Sym}(\mathcal{L} \otimes V^\vee) =: \mathbb{P}(\mathcal{L}^{-1} \otimes V).$$

Since there is a natural isomorphism between projective bundles (as described in [Har77, II, Lemma 7.9])

$$\mathbb{P}(\mathcal{O}_{\widehat{A}} \otimes V) \cong \mathbb{P}(\mathcal{L}^{-1} \otimes V),$$

where by definiton $\mathbb{P}(V) \times \widehat{A} := \mathbb{P}(\mathcal{O}_{\widehat{A}} \otimes V)$, we get a Heisenberg-equivariant embedding

$$\mathbb{P}(\mathcal{E}'^\vee) \hookrightarrow \mathbb{P}(V) \times \widehat{A}, \tag{2.25}$$

where the action of the Heisenberg group $\mathcal{H}_D$ on the right-hand side is of product type, induced on $\widehat{A}$ by the action of $\mathcal{K}(D)$ via translations and on $\mathbb{P}(V)$ by the Schrödinger representation V. Indeed, in light of Remark 2.18, the embedding (2.25) is also $\mathcal{K}(D)$-equivariant.

Recall that by the Generality Assumption we have an embedding $\psi\colon X \hookrightarrow \mathbb{P}(\mathcal{E}^\vee)$ of the canonical model X, where $\mathcal{E}^\vee$ is the Tschirnhaus bundle of $\alpha\colon S \to A$.

Considering the fibre product of the morphism $a\colon X \to A$ induced by $\alpha\colon S \to A$ (see diagram 2.3) with the isogeny $\Phi_D\colon \widehat{A} \to A$, we get the following square

$$\begin{tikzcd} X' := X \times_A \widehat{A} \arrow[r, "/\mathcal{K}(D)"] \arrow[d, "a'"'] & X \arrow[d, "a"] \\ \widehat{A} \arrow[r, "\Phi_D"] & A \end{tikzcd} \tag{2.26}$$

where the morphism $X' \to X$ is étale and Galois with Galois group $\mathcal{K}(D)$ since
$\Phi_D\colon \widehat{A} \to A$ is so. Moreover, since $X' \to X$ is étale, X' is a two-dimensional normal projective variety with (at most) RDP as singularities and $K_{X'}$ ample as X is so.

Still, recalling that $\mathcal{E}' := \Phi_D^*(\mathcal{E})$, we have the pull-back square

$$\begin{tikzcd} \mathbb{V}(\mathcal{E}'^\vee) = \mathbb{V}(\mathcal{E}^\vee) \times_A \widehat{A} \arrow[r] \arrow[d] & \mathbb{V}(\mathcal{E}^\vee) \arrow[d] \\ \widehat{A} \arrow[r, "\Phi_D"] & A \end{tikzcd}$$

Since the morphism $\mathbb{V}(\mathcal{E}'^\vee) \to \mathbb{V}(\mathcal{E}^\vee)$ is fibrewise an isomorphism of vector spaces, it induces a map between the respective projectivizations, yielding a diagram as follows

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}'^\vee) = \mathbb{P}(\mathcal{E}^\vee) \times_A \widehat{A} \arrow[r, "/\mathcal{K}(D)"] \arrow[d] & \mathbb{P}(\mathcal{E}^\vee) \arrow[d] \\ \widehat{A} \arrow[r, "\Phi_D"] & A \end{tikzcd}$$

Therefore, we get that the fibre product $X' \subset \mathbb{P}(\mathcal{E}'^{\vee})$ of X is a $\mathcal{K}(D)$-invariant subvariety of the projective bundle $\mathbb{P}(\mathcal{E}'^{\vee})$ and it holds

$$X \cong X'/\mathcal{K}(D).$$

Summarizing, given a surface S with AP fulfilling the Generality Assumption 2.17, its canonical model X fits into a diagram as follows

$$\begin{tikzcd}
\mathbb{P}(V)\times\widehat{A} & \\
\mathbb{P}(\mathcal{E}'^{\vee}) = \mathbb{P}(\mathcal{E}^{\vee}) \times_A \widehat{A} \arrow[u, hook] \arrow[r, "/\mathcal{K}(D)"] & \mathbb{P}(\mathcal{E}^{\vee}) \\
X' := X \times_A \widehat{A} \arrow[u, hook] \arrow[r, "/\mathcal{K}(D)"] \arrow[d, "a'"'] & X \arrow[u, hook] \arrow[d, "a"] \\
\widehat{A} \arrow[r, "\Phi_D"] & A
\end{tikzcd} \tag{2.27}$$

The above picture suggests clearly the strategy we have to follow in order to construct a surface S with AP. In fact, our construction method consists morally speaking in constructing the left-hand side of the above diagram. Then we get the right-hand side of it by taking quotients with respect to the free action of $\mathcal{K}(D)$.

More precisely, we consider an abelian surface A' with an ample line bundle $\mathcal{O}_{A'}(D)$ yielding a polarization of type (δ_1, δ_2) (hence, with Pfaffian $\delta := \delta_1\delta_2$).

Denote by $\mathcal{H}_D$ the Heisenberg group of $H_D := (\mathbb{Z}/\delta_1 \times \mathbb{Z}/\delta_2)$ and recall that (see Chapter 1, Section 1.3)

$$H_D^2 \cong \mathcal{K}(D) \cong \mathcal{H}_D/\mu_D,$$

where μ_D is the centre of $\mathcal{H}_D$.

Assume we are given a homogeneous vector bundle $\mathfrak{H}'$ on A' and a $\mathcal{K}(D)$-equivariant (and hence $\mathcal{H}_D$-equivariant, see Subsection 1.4.4) sequence

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^{\vee} \to \mathcal{E}' \to 0, \tag{2.28}$$

where $\mathcal{L} := \mathcal{O}_{A'}(D)$, $V := H^0(A', \mathcal{O}_{A'}(D))$ is the Schrödinger representation of $\mathcal{H}_D$ and $\mathcal{E}'$ is a locally free $\mathcal{O}_{A'}$-module of rank $d-1 \geq 2$.

Hence, as argued for sequence (2.24), we get a $\mathcal{K}(D)$-equivariant embedding

$$\mathbb{P}(\mathcal{E}'^{\vee}) \hookrightarrow \mathbb{P}(V) \times A'. \tag{2.29}$$

Now the theory of Casnati-Ekedahl comes into the picture. Indeed, our aim is to construct a two-dimensional normal projective variety X' with (at most) RDP as singularities and $K_{X'}$ ample such that

$$X' \subset \mathbb{P}(\mathcal{E}'^{\vee}) \subset \mathbb{P}(V) \times A'$$

and X' is $\mathcal{K}(D)$-invariant.

Using the strategy provided by the structure theorems of Casnati-Ekedahl [CE96] for Gorenstein covers of small degree $d = 3, 4$, we construct X' as a generically finite cover of the abelian surface A'. Namely, we give X' as a closed subvariety of $\mathbb{P}(\mathcal{E}'^{\vee}) \subset \mathbb{P}(V) \times A'$ in such a way that

$$a' := p'_{|X'} \colon X' \to A'$$

is a generically finite cover of small degre $d = 3, 4$, where $p' \colon \mathbb{P}(\mathcal{E}'^{\vee}) \to A'$ denotes the canonical bundle projection.

Remark 2.21. In principle we can try to perform the same construction of X' also for degree $d \geq 5$. Indeed, for $d = 5$ it is possible to use results contained in [Cas96], whereas for $d \geq 6$ the serious drawback relies on the fact that no structure theorems for covers of degree $d \geq 6$ are known.

Finally, we define X as the free quotient

$$X := X'/\mathcal{K}(D),$$

and take its minimal free resolution S.

Since the action of $\mathcal{K}(D)$ on the projective bundle $\mathbb{P}(\mathcal{E}'^{\vee})$ is compatible with the action of $\mathcal{K}(D)$ on A', we have that the bundle projection $p' \colon \mathbb{P}(\mathcal{E}'^{\vee}) \to A'$ descends to a map between the quotients, namely there is diagram as follows

$$\begin{tikzcd}
\mathbb{P}(\mathcal{E}'^{\vee}) \arrow[r, "/\mathcal{K}(D)"] \arrow[d, "p'"'] & \mathbb{P}(\mathcal{E}'^{\vee})/\mathcal{K}(D) \arrow[d, "p"] \\
A' \arrow[r, "\Phi_D"] & A'/\mathcal{K}(D) = \widehat{A'} =: A
\end{tikzcd} \tag{2.30}$$

As the map Φ_D is étale, it is clear that $p \colon \mathbb{P}(\mathcal{E}'^{\vee})/\mathcal{K}(D) \to A$ is a projective bundle over the abelian surface A defined above as the dual abelian surface of A'. Therefore, since every projective bundle over a regular scheme arises from a locally free sheaf (see [Har77], exercise II.7.10(c)), there exists a locally free $\mathcal{O}_A$-module $\mathcal{E}$ such that

$$\mathbb{P}(\mathcal{E}'^{\vee})/\mathcal{K}(D) \cong \mathbb{P}(\mathcal{E}^{\vee}).$$

Finally, we have shown that there is an embedding $X \subset \mathbb{P}(\mathcal{E}^{\vee})$ and a surjective map $a := p_{|X} \colon X \to A$ of degree $d = 3, 4$ which in turn provides a surjective morphism $\alpha \colon S \to A$ of the same degree by composition with the minimal resolution of singularities $S \to X$.

Note that S is a minimal surface since a (-1)-curve on S would yield a (-1)-curve on X: a contradiction since K_X is ample.

Hence, the surface S together with the surjective morphism $\alpha \colon S \to A$ gives us the desired surface with AP.

Remark 2.22. It is worth pointing out that some sanity checks have to be done while constructing a surface S with AP in the way we have just showed. More precisely, it is important to have the invariants of X' under control so that the ones of X (and hence of S) are correct, i.e., $p_g(S) = q(S)$. Also, we have to check that the surjective morphism $\alpha \colon S \to A$ constructed as above does not factor through a morphism of S to another abelian surface.

2.5 Construction of CHPP Surfaces

In this section A' is an abelian surface with a divisor D yielding a polarization of type $(1,2)$.

Then $V := H^0(A', \mathcal{O}_{A'}(D))$ is a two-dimensional vector space, and the kernel $\mathcal{K}(D)$ of the isogeny $\Phi_D : A' \to A := \widehat{A'}$ is here

$$G := \mathcal{K}(D) \cong (\mathbb{Z}/2)^2.$$

Consider the order 8 Heisenberg group $\mathcal{H} := \mathcal{H}_2 \cong D_4$ with centre $\mu_2 \cong \mathbb{Z}/2$, namely

$$1 \to \mu_2 \to \mathcal{H} \to G \cong (\mathbb{Z}/2)^2 \to 0.$$

Recalling that V is the Schrödinger representation of $\mathcal{H}$, there are two generators g_1, g_2 of $\mathcal{H}$ acting on $V := H^0(A', \mathcal{O}_{A'}(D))$ by transforming a suitable basis x_1, x_2 as follows:

$$g_1(x_1) = x_1,\ g_1(x_2) = -x_2,\ g_2(x_1) = x_2,\ g_2(x_2) = x_1.$$

The action of g_1, g_2 has the property that $\gamma := g_1 g_2 g_1 g_2$ acts by multiplication by -1, hence $\langle \gamma \rangle = \mu_2$ and $\mathcal{H}/\langle \gamma \rangle \cong G$.

Let us call $W := V^\vee$ the dual representation of V, which actually turns out to be isomorphic to V. Namely, y_1, y_2 being the dual basis of x_1, x_2,

$$g_1(y_1) = y_1,\ g_1(y_2) = -y_2,\ g_2(y_1) = y_2,\ g_2(y_2) = y_1,$$

and W, V are the same representation of the Heisenberg group $\mathcal{H}$.

The basic observation is that on the tensor product $V \otimes W$ we have an action of G, since the centre of $\mathcal{H}$, generated by γ, acts trivially. Also, $V \otimes W$ contains (up to constants) precisely one invariant element, namely $x_1 y_1 + x_2 y_2$.

We define now an action of G on $\mathbb{P}^1 \times A'$, of product type, where G acts on $\mathbb{P}^1 = \mathbb{P}(V)$ via the previous action of $\mathcal{H}$ on V, whereas G acts on A' by translations.

Let H be the hyperplane divisor on $\mathbb{P}^1$. Then we consider the family of divisors X' in $\mathbb{P}^1 \times A'$ which belong to the linear system

$$|3H + D| := |p_1^*(3H) + p_2^*(D)|$$

and which are left invariant by the action of G.

The general equation of such divisors in $|3H + D|$ is of the form

$$X' := \{x_1 P(y_1, y_2) + x_2 Q(y_1, y_2)\},$$

with P, Q homogeneous polynomials of degree 3.

g_2-invariance is equivalent to $Q(y_1, y_2) = \epsilon P(y_2, y_1), \epsilon = \pm 1$: here the choice of ϵ amounts to requiring the equation $f := x_1 P(y_1, y_2) + x_2 Q(y_1, y_2)$ to be an ϵ-eigenvector for the action of g_2.

g_1-invariance is equivalent to

$$x_1 P(y_1, y_2) + x_2 \epsilon P(y_2, y_1) = \epsilon' [x_1 P(y_1, -y_2) - x_2 \epsilon P(-y_2, y_1)], \quad \epsilon' = \pm 1$$

(the choice of ϵ' amounts to requiring the equation to be an ϵ'-eigenvector for the action of g_1).

We can write

$$P(y_1, -y_2) = \epsilon' P(y_1, y_2), \quad \epsilon' = \pm 1,$$

that is, either

$$P(y_1, -y_2) = P(y_1, y_2), \quad \text{or} \quad P(y_1, -y_2) = -P(y_1, y_2).$$

In the first case P is a linear combination of $y_1^3, y_1 y_2^2$, in the second case a linear combination of $y_2^3, y_1^2 y_2$.

Remark 2.23. (i) One may observe in an elementary way that the choice of $\epsilon = -1$ reduces to the case $\epsilon = 1$ by replacing the basis element x_2 with $-x_2$.

(ii) The equation $f \in H^0(\mathcal{O}_{\mathbb{P}^1 \times A'}(3H+D)) = \mathbf{Sym}^3(W) \otimes V$. Since $X' := \{f = 0\}$ is G-invariant, follows that f is an eigenvector for the G-action, with eigenvalue a character $\chi \in G^*$.

We can then take as new equation $(f \otimes \chi) \in \mathbf{Sym}^3(W) \otimes (V \otimes \chi) \cong \mathbf{Sym}^3(W) \otimes V$, where the last isomorphism follows since $\mathcal{H}$ has a unique irreducible representation of dimension 2, and 4 of dimension 1, corresponding to $G^* = \mathcal{H}^* := \mathrm{Hom}(\mathcal{H}, \mathbb{C}^*)$, see Section 1.3.

Hence, by a suitable change of basis in V we may always assume that not only X' is G-invariant, but also its equation f is G-invariant.

Hence, we get for X' the following equation

$$X' := X'(\lambda) := \{x_1(y_1^3 + \lambda y_1 y_2^2) + x_2(y_2^3 + \lambda y_2 y_1^2) = 0\} \subset \mathbb{P}^1 \times A' =: Z,$$

where $\lambda \in \mathbb{C}$. We will denote by $a' \colon X' \to A'$ the restriction of the natural projection onto A'.

Definition 2.24. We define an *extended CHPP surface* X as the quotient $X := X'/G$ of a surface

$$X' := X'(\lambda) := \{x_1(y_1^3 + \lambda y_1 y_2^2) + x_2(y_2^3 + \lambda y_2 y_1^2) = 0\} \subset \mathbb{P}^1 \times A' =: Z,$$

where $\lambda \in \mathbb{C}$.

A *CHPP surface* is defined to be the minimal resolution of singularities of an extended CHPP surface which has (at most) Rational Double Points as singularities.

Remark 2.25. Observe that, for $\lambda = 0$, X' is a Galois cover of A' with group $(\mathbb{Z}/3)$.

Proposition 2.26. *An extended CHPP surface is reducible if we are in the exceptional case* $(**)$ *of Subsection 1.4.6.b, that is, if* (A', D) *is a polarized product of elliptic curves.*

Otherwise, an extended CHPP surface is always normal, and smooth for general λ *and general* (A', D).

G *acts freely on* X'*, and the canonical models* $X := X'/G$ *of CHPP surfaces have ample canonical divisor and invariants*

$$K_{X'}^2 = 20, \; K_X^2 = 5, \; q(X') = q(X) = 2, \; p_g(X') = 5, \; p_g(X) = 2,$$

$$\pi_1(X') \cong \pi_1(X) = \mathbb{Z}^4.$$

Their Albanese map has degree 3.

Moreover, the branch locus Δ *of the Albanese map of* X' *consists of 4 curves in the linear system* $|D|$*, which are generally distinct (hence* Δ *has a 4-uple point at the points* $x_1 = x_2 = 0$*); for* $\lambda = 0$ *instead* Δ *consists of the two curves* $\{x_1 = 0\}$, $\{x_2 = 0\}$ *counted with multiplicity 2.*

Proof. i) If we are in the exceptional case $(**)$, where $x_1 = e_1 s_1$, $x_2 = e_1 s_2$, e_1 is the pull-back of a section defining P_1 on E_1, while s_1, s_2 are pull-backs of a basis of $H^0(\mathcal{O}_{E_2}(2P_2))$, then over the curve $E_2' := \{P_1\} \times E_2$ we have $x_1 = x_2 = 0$, hence $(\mathbb{P}^1 \times E_2') \subset X'$, and X' is reducible.

From now on we assume that we are not in case $(**)$, hence the equations $x_1 = x_2 = 0$ define 4 points and x_1, x_2 are local parameters for A'.

ii) For $\lambda = 0$, we get that the derivatives with respect to y_1, y_2 vanish only when $x_1 y_1 = x_2 y_2 = 0$, which implies that $x_1 x_2 = 0$.

Since (A', D) is not the exception $(**)$ of Subsection 1.4.6.b, for $x_1 = x_2 = 0$ the divisors $x_1 = 0, x_2 = 0$ are smooth and they intersect transversally in 4 points; hence x_1, x_2 are local coordinates, and the partial derivatives with respect to x_1, x_2 vanish only on $y_1 = y_2 = 0$: but these equations define the empty set in $\mathbb{P}^1$.

If only one of x_1, x_2 vanishes, say $x_1 = 0$, then $y_2 = 0$ and we have a smooth point if the divisor $x_1 = 0$ is smooth: this happens for general (A', D).

iii) Identify H, D with their pull back on $Z = \mathbb{P}^1 \times A'$.

Since $K_Z = -2H$, adjunction gives $K_{X'} = (H + D)|_{X'}$, and

$$K_{X'}^2 = (3H + D)(H + D)^2 = 5HD^2 = 20.$$

G acts freely on A', hence also on X', therefore $K_X^2 = 5$.

We have the exact cohomology sequence associated to the exact sequence

$$0 \to \mathcal{O}_Z(-2H) \to \mathcal{O}_Z(H + D) \to \mathcal{O}_{X'}(K_{X'}) \to 0,$$

and since

$$H^0(\mathcal{O}_Z(-2H)) = 0,\ h^1(\mathcal{O}_Z(-2H)) = 1,\ h^2(\mathcal{O}_Z(-2H)) = 2,$$

$$H^1(\mathcal{O}_Z(H + D)) = 0,\ H^2(\mathcal{O}_Z(H + D)) = 0,\ h^0(\mathcal{O}_Z(H + D)) = 4,$$

it follows that

$$p_g(X') = 5, \quad q(X') = h^1(\mathcal{O}_{X'}(K_{X'})) = 2.$$

Since G acts trivially on $H^0(\Omega^1_{A'}) \cong H^0(\Omega^1_{X'})$, it follows that $q(X) = 2$. Finally G acts trivially on $H^1(\mathcal{O}_Z(-2H))$, while, as remarked at the beginning, $H^0(\mathcal{O}_Z(H+D)) = V \otimes W$, hence $H^0(\mathcal{O}_Z(H + D))^G$ has dimension 1 and thus $p_g(X) = 2$.

The isomorphism $\pi_1(X') \cong \pi_1(A')$ follows from Lefschetz hyperplane theorem since X' is an ample divisor on $Z = \mathbb{P}^1 \times A'$.

Finally, $\pi_1(X') \subset \pi_1(X)$ is a normal subgroup of index 4, with quotient group G. Recall that $A = \widehat{A'} \cong A'/G$. Then A is the Albanese variety of X, hence $\pi_1(A)$ is a quotient of $\pi_1(X)$. But $\pi_1(X') \cong \pi_1(A') \subset \pi_1(A)$ has index 4, hence $\pi_1(X) \cong \pi_1(A)$.

iv) In general, we ask when X' has (at most) Rational Double Points as singularities, for $\lambda \neq 0$.

To calculate the singular points we may use the Remark 2.23, and restrict to the equation $f = x_1(y_1^3 + \lambda y_1 y_2^2) + x_2(y_2^3 + \lambda y_2 y_1^2)$.

The partials with respect to y_1, respectively y_2, yield:

$$\frac{\partial f}{\partial y_1} = x_1(3y_1^2 + \lambda y_2^2) + x_2(2\lambda y_1 y_2) = 0, \quad \frac{\partial f}{\partial y_2} = x_1(2\lambda y_1 y_2) + x_2(3y_2^2 + \lambda y_1^2) = 0.$$

If x_1 vanishes, but x_2 does not, we have a singular point only if $y_1 y_2 = 0 = (3y_2^2 + \lambda y_1^2)$, but the two polynomials do not vanish simultaneously, hence we have no singular point. Similarly if x_2 vanishes, but x_1 does not.

If both x_1, x_2 vanish, the two partials with respect to the (local parameters) x_1, x_2 vanish if and only if

$$(y_1^3 + \lambda y_1 y_2^2) = (y_2^3 + \lambda y_2 y_1^2) = 0 \iff (y_1^2 + \lambda y_2^2) = (y_2^2 + \lambda y_1^2) = 0.$$

This may occur only for $\lambda = \pm 1$, and we get then exactly two singular points.

If both x_1, x_2 do not vanish, then a necessary condition for a singular point (or a ramification point for a') is that

$$(3y_1^2 + \lambda y_2^2)(3y_2^2 + \lambda y_1^2) - (2\lambda y_1 y_2)^2 = 0 \iff y_1^4 + y_2^4 + \frac{1}{\lambda}(3 - \lambda^2) y_1^2 y_2^2 = 0.$$

This equation does not vanish for $y_1 = 0$, hence we write $y_1 = 1, y_2 = z$, and we get the equation

$$1 + z^4 + \frac{1}{\lambda}(3 - \lambda^2) z^2 = 0, \tag{$***$}$$

whose roots come in opposite pairs $z, -z$.

At a singular point of X' we have:

$$f := x_1 f_1(\lambda, z) + x_2 f_2(\lambda, z) = 0, \quad \nabla x_1(f_1(\lambda, z)) + \nabla x_2(f_2(\lambda, z)) = 0,$$

whence we get as second coordinate a singular point of the pencil $|D|$, corresponding to the point $(f_1(\lambda, z), f_2(\lambda, z)) \in \mathbb{P}^1$.

Now, since we are not in the exceptional case ($**$), by the Zeuthen-Segre formula it follows that the pencil $|D|$ gives rise to at most such 12 singular points, since the Euler number of the blow up of A' equals 4, and then $4 = -2D^2 + \mu = -8 + \mu$, hence we have $\mu = 12$ singular fibres counted with multiplicity.

For each such value of (u_1, u_2) corresponding to a singular fibre we get the equation $u_2 f_1(\lambda, z) - u_1 f_2(\lambda, z) = 0$, and substituting the four values of z gotten by ($***$), we get equations for the parameter λ for which X' is singular.

v) We want to show that X' has always only finitely many singularities, hence X' is always normal.

In fact, a fibre of $\mathbb{P}^1 \times A' \to A'$ is contained in X' if and only if $x_1 = x_2 = 0$. But x_1, x_2 are local parameters, hence the whole fibre cannot be contained in the singular locus.

The above proof shows that, in the other cases where $x_1 \neq 0$ or $x_2 \neq 0$, we have always a finite number of singular points on X'.

vi) Finally, the discriminant of the projection of X' to A', namely $a' \colon X' \to A'$, equals

$$\Delta := \det \begin{pmatrix} 3x_1 & 2\lambda x_2 & \lambda x_1 & 0 \\ 0 & 3x_1 & 2\lambda x_2 & \lambda x_1 \\ \lambda x_2 & 2\lambda x_1 & 3x_2 & 0 \\ 0 & \lambda x_2 & 2\lambda x_1 & 3x_2 \end{pmatrix}. \tag{2.31}$$

Since Δ is given by the vanishing of a homogeneus polynomial of degree 4 in (x_1, x_2) we get, for each λ, a product of 4 linear factors, hence the discriminant consists of 4 curves in the linear system $|D|$, counted with multiplicity.

For $\lambda = 0$, we get $81x_1^2x_2^2 = 0$, which is of course expected since then we have a Galois cover with cyclic Galois group of order 3.

□

Remark 2.27. The morphism $a : X \to A$ never yields a Galois extension of function fields.

The argument is as follows: if a is Galois, then also the fibre product $X' \to A$ is Galois, hence $X' \to A'$ is Galois and the equation of X' is

$$X' = \{x_1y_1^3 + x_2y_2^3 = 0\}.$$

The group μ_3 of third roots of unity acts by

$$y_1 \mapsto y_1, \quad y_2 \mapsto \epsilon y_2, \quad \epsilon^3 = 1.$$

We claim that G, μ_3 generate a group G' of order 12. Indeed, we see right away that g_1 and ϵ commute, while

$$g_2\epsilon g_2(y_1, y_2) = (\epsilon y_1, y_2) = (y_1, \epsilon^{-1}y_2) \iff g_2\epsilon g_2 = \epsilon^{-1}.$$

Hence, g_2 and μ_3 generate $\mathfrak{S}_3$, and

$$G' = \mathfrak{S}_3 \times \mathbb{Z}/2, \quad \mathbb{Z}/2 = \{0, g_2\}.$$

Since X corresponds to the intermediate subgroup $G < G'$ which is not normal $(G \cong \mathbb{Z}/2 \times \mathbb{Z}/2)$, $a \colon X \to A$ is not Galois, a contradiction.

Remark 2.28. The morphism $a \colon X \to A$ contracts exactly one smooth rational curve $C \cong \mathbb{P}^1$. Indeed, observe that the morphism $a' \colon X' \to A'$ contracts only the 4 rational fibres $\mathbb{P}^1 \times \{z_i\} \subset \mathbb{P}^1 \times A'$ over the base locus $\{z_1, \dots, z_4\}$ of the linear system $|D|$ given by $\{x_1 = x_2 = 0\}$. Since the fibres $\mathbb{P}^1 \times \{z_i\}$ are identified under the action of $G = \mathcal{K}(D) \cong (\mathbb{Z}/2)^2$, we get our conclusion.

2.6 Moduli Space of CHPP Surfaces

In this section we study the family of CHPP surfaces we have constructed in Section 2.5, hence we keep using the same notation and conventions adopted therein.

In particular, we remind the reader that A' is here an abelian surface with a divisor D yielding a polarization of type $(1,2)$, $V := H^0(A', \mathcal{O}_{A'}(D))$ is the two dimensional Schrödinger representation of the order 8 Heisenberg group $\mathcal{H} := \mathcal{H}_2 \cong D_4$, $G := \mathcal{K}(D) \cong (\mathbb{Z}/2)^2$ is the kernel of the isogeny $\Phi_D \colon A' \to A := \widehat{A'}$ and $Z := \mathbb{P}^1 \times A'$.

Moreover, given a CHPP surface S with canonical model $X := X'/G$ and Albanese map $\alpha \colon S \to A$, by abuse of notation we will often call Albanese map the induced morphism $a \colon X \to A$.

We have constructed an irreducible 4-dimensional family (three parameters for the abelian surface A', and λ as fourth parameter) of CHPP surfaces, and we want to see that this yields a component of the moduli space of surfaces of general type.

In order to achieve this goal, it suffices to analyze deformations $\mathcal{X} \to T$ with connected base.

There are two guiding principles, coming from topology:

I) every deformation of X comes together with a deformation of X' preserving the G-action (up to an automorphism of G),

II) every deformation of X, respectively of X', comes together with a deformation of their Albanese maps $a' : X' \to A'$, $a : X \to A$ which are generically finite cover of degree 3; indeed any other surface homotopically equivalent to X, resp. X', has an Albanese map of degree 3.

Taking the Stein factorization of the Albanese maps, we get finite triple covers $Y(t) \to A(t)$, $Y(t) := \mathbf{Spec}(a(t)_*(\mathcal{O}_{X_t}))$, and similarly for the deformations of X'.

We observe that for our surfaces X' we have the exact sequence

$$0 \to \mathcal{O}_Z(-3H - D) \to \mathcal{O}_Z \to \mathcal{O}_{X'} \to 0,$$

whence by direct image the exact sequence

$$0 \to \mathcal{O}_{A'} \to a'_*(\mathcal{O}_{X'}) \to \mathcal{O}_{A'}(-D)^{\oplus 2} \to 0,$$

and the so-called Tschirnhaus bundle $(\mathcal{E}')^\vee$ of the degree 3 map equals $(\mathcal{E}')^\vee = \mathcal{O}_{A'}(-D)^{\oplus 2}$.

Moreover, for small deformations, we shall have a composite morphism

$$X'_t \to Y'(t) \to \mathbb{P}(a'(t)_*(\mathcal{O}_{X'_t})/\mathcal{O}_{A'(t)}),$$

which is a $\mathbb{P}^1$-bundle over $A'(t)$.

The deformations of X' turn out to be more complicated to describe than the ones of X, since the $\mathbb{P}^1$-bundle can admit nontrivial deformations as X' deforms.

However, the situation for X is simpler.

Lemma 2.29. *For every deformation X_t of X, the Albanese map of X'_t factors through a birational morphism into $\mathbb{P}^1 \times A'(t)$.*

Proof. Any deformation of X yields, as we already observed, a deformation of X' which preserves the G-action.

This implies that the Tschirnhaus bundle $(\mathcal{E}')^\vee$ splits according to the two eigensheaves for g_1, and since they have to be exchanged by g_2, we have that $(\mathcal{E}')^\vee$ is always a direct sum of two copies of the same line bundle, which, of course, is a deformation of $\mathcal{O}_{A'}(-D)$. Hence, it is this bundle up to translation on $A'(t)$.

□

Corollary 2.30. *Any small deformation of X yields an embedding $X'_t \subset \mathbb{P}^1 \times A'(t)$. The divisor class of X'_t is the class $3H + D_t$, where D_t is a polarization of type $(1,2)$ on $A'(t)$.*

The previous results allow us to conclude that all small deformations of X are given by deformations of X' as hypersurfaces inside a threefold $\mathbb{P}^1 \times A'(t)$, where $A'(t)$ is a deformation of A', and the action of G is preserved; hence every deformation of X comes from a G-invariant deformation of X', and we conclude that our families are locally complete.

We want to show more.

Theorem 2.31. *Every deformation in the large of a CHPP surface is a CHPP surface.*

Proof. As well known (see for instance [BC18], pages 625–626), it suffices to show that if we have a 1-parameter family $X_t, t \in T$, where T is a smooth curve, which is a deformation in the large of the canonical model X of a CHPP surface, then all the surfaces X_t are canonical models of CHPP surfaces.

Under the above assumption X'_t is a deformation of X', and we have a birational morphism $X'_t \to \mathbb{P}^1 \times A'(t)$, whose image is a divisor Σ_t in a linear system $|3H + D_t|$, where D_t is a polarization of type $(1,2)$ on $A'(t)$.

The dualizing sheaf ω_{Σ_t} is the restriction of the invertible sheaf $\mathcal{O}_{Z(t)}(H + D_t)$, and it has $h^0(\omega_{\Sigma_t}) = 5 = p_g(X'_t)$.

Let S'_t be the minimal model of X'_t. Since $S'_t \to \Sigma_t$ is a resolution of singularities, we see now that there are no conditions of subadjunction, nor of adjunction (see the appendix by Mumford to Chapter III of [Zar71])

Σ_t yields an extended CHPP surface and, since Σ_t is irreducible, by Proposition 2.26 we are not in the exceptional case $(**)$ of Subsection 1.4.6.b and Σ_t is normal.

If Σ_t is normal and does not have Rational Double Points as singularities, then $K_{X'_t}$ is the pull-back of $(H + D_t)$ minus a non zero effective exceptional divisor, hence $K^2_{X'_t} < 20$, a contradiction.

□

Finally, we have shown the following theorem.

Theorem 2.32. *The 4-dimensional family of CHPP surfaces yields an irreducible connected component $\mathcal{M}_{CHPP}$ of the moduli space of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 5$ and Albanese map of degree $d = 3$.*

In the next subsection, which can be seen as a longer digression, we consider the more difficult question of studying the deformations of the surfaces X'.

2.6.1 The Deformations of X'

This subsection is sort of a digression. We want here to look at the deformations of X': hence we look at the cohomology group $H^1(X', \Theta_{X'})$ and the Kodaira-Spencer map.

The first isomorphism that we observe is

$$\Theta_Z \cong \mathcal{O}_Z(2H) \oplus \mathcal{O}_Z^2.$$

Then we consider the exact sequence

$$0 \to \Theta_{X'} \to \mathcal{O}_{X'}(2H) \oplus \mathcal{O}_{X'}^2 \to \mathcal{O}_{X'}(3H + D) \to 0,$$

and, since X' is of general type, $H^0(\Theta_{X'}) = 0$, while $H^0(\mathcal{O}_{X'}(2H) \oplus \mathcal{O}_{X'}^2)$ has dimension 5, and $H^0(\mathcal{O}_{X'}(3H+D))$ has dimension $9 = 8-1+2$, since it fits into the exact sequence

$$0 \to H^0(\mathcal{O}_Z) \to H^0(\mathcal{O}_Z(3H + D)) \to H^0(\mathcal{O}_{X'}(3H + D)) \to H^1(\mathcal{O}_Z) \to 0.$$

Finally, since $\mathcal{O}_{X'}(3H + D) = \mathcal{O}_{X'}(2H + K_{X'})$ has vanishing second cohomology group, and first of dimension 1, we have the exact cohomology sequence

$$0 \to H^0(\mathcal{O}_{X'}(2H) \oplus \mathcal{O}_{X'}^2) \to H^0(\mathcal{O}_{X'}(3H + D)) \to H^1(\Theta_{X'}) \to H^1(\mathcal{O}_{X'}(2H) \oplus \mathcal{O}_{X'}^2) \to$$

$$\to H^1(\mathcal{O}_{X'}(3H + D)) \to H^2(\Theta_{X'}) \to H^2(\mathcal{O}_{X'}(2H) \oplus \mathcal{O}_{X'}^2) \to 0,$$

and since, by the next lemma, $H^i(\mathcal{O}_{X'}(2H))$ has dimension 6 for $i = 1$, respectively 3 for $i = 2$, we get that $H^1(\Theta_{X'})$ has dimension at most 14, while $H^2(\Theta_{X'})$ has dimension either 13 or 14.

Since however $10\chi(X') - 2K_{X'}^2 = 0$, $H^1(\Theta_{X'}), H^2(\Theta_{X'})$ have the same dimension.

Lemma 2.33. *$H^1(\mathcal{O}_{X'}(2H))$ has dimension 6, $H^2(\mathcal{O}_{X'}(2H))$ has dimension 3.*

Proof. We use the exact sequence

$$0 \to \mathcal{O}_Z(-H - D) \to \mathcal{O}_Z(2H) \to \mathcal{O}_{X'}(2H) \to 0,$$

and the fact that by the Künneth formula $\mathcal{O}_Z(-H - D)$ has all cohomology groups vanishing, hence

$$H^1(\mathcal{O}_{X'}(2H)) \cong H^1(\mathcal{O}_Z(2H)) = H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \otimes H^1(\mathcal{O}_{A'})$$

has dimension 6, while

$$H^2(\mathcal{O}_{X'}(2H)) \cong H^2(\mathcal{O}_Z(2H)) = H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \otimes H^2(\mathcal{O}_{A'})$$

has dimension 3.

□

We observe that the image of $H^1(\Theta_{X'})$ inside $H^1(\mathcal{O}_{X'})^2$ corresponds to the deformations of A' as a complex torus, but each deformation of X' yields a deformation of A' as an abelian surface.

The deformations of X' contains a family of dimension $3 + 7 - 3 = 7$ if we keep a hypersurface in $\mathbb{P}^1 \times A'$, but indeed deforming X' we could take a deformation of the trivial rank 2 bundle.

The tangent space to the deformations of the trivial rank 2 bundle on A' is given by the vector space $\mathrm{Ext}^1(\mathcal{O}^2_{A'}, \mathcal{O}^2_{A'}) \cong H^1(\mathcal{O}^4_{A'})$, which has dimension 8, but since we are interested in the deformations of the associated projective bundle, we get a vector space of dimension 6, corresponding to the deformations with trivial determinant, $H^1(End^0(\mathcal{O}^2_{A'}))$ (End^0 denotes as usual the space of trace zero endomorphisms): this is the explanation of the map to the 6-dimensional vector space $H^1(\mathcal{O}_{X'}(2H)) \cong H^1(\mathcal{O}_Z(2H))$.

2.6.2 Unirational Moduli Space for CHPP Surfaces and its Characterization

We now show that the irreducible connected component $\mathcal{M}_{\mathrm{CHPP}}$ of CHPP surfaces is unirational.

Theorem 2.34. *The irreducible connected component $\mathcal{M}_{CHPP}$ corresponding to CHPP surfaces is unirational.*

Proof. Denoting by $\mathcal{A}_2^{(1,2)}$ the moduli space of $(1,2)$-polarized abelian surfaces, it is clear from the construction of the family $\mathcal{M}_{\mathrm{CHPP}}$ of CHPP surfaces that there is a dominant rational map

$$\mathcal{A}_2^{(1,2)} \times \mathbb{P}^1 \dashrightarrow \mathcal{M}_{\mathrm{CHPP}}. \tag{2.32}$$

Since $\mathcal{A}_2^{(1,2)}$ is known to be rational (see [Gri94]), we get right away our conclusion that $\mathcal{M}_{\mathrm{CHPP}}$ is unirational. □

The following result shows in particular that the component $\mathcal{M}_{CHPP}$ of CHPP surfaces coincides with the one constructed by Penegini and Polizzi in [PePo13a].

Theorem 2.35. *The unirational irreducible connected component corresponding to the CHPP surfaces is the unique component of the Main Stream such that there is a surface in this component which fulfills the Gorenstein Assumption 2.6 and has $K_S^2 = 5$, $p_g(S) = q(S) = 2$, and Albanese map $\alpha : S \to A = Alb(S)$ of degree $d = 3$. In particular, this component coincides with the component constructed in* [PePo13a].

Proof. We have the isogeny $\Phi_D : A' \to A'/\mathcal{K}(D) = A'/G$, and if we have a component $\mathfrak{N}$ of the Main Stream, there is some surface S such that $A = \mathrm{Alb}(S)$ contains no elliptic curve.

Under this condition, by Theorem 3.5 of [CH06], we have an exact sequence for the pull-back $\mathfrak{F}'$ of $\mathfrak{F} := \alpha_* \omega_S / \omega_A$, namely

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathfrak{F}' \to 0,$$

where $\mathfrak{H}'$ is a homogeneous bundle and $\mathcal{L}$ a polarization with Pfaffian $\delta = \mathrm{rank}(\mathfrak{H}') + 2$.

We consider now those surfaces S which satisfy the Gorenstein Assumption 2.6: then, denoting by $\mathcal{E}$ the dual of the Tschirnhaus bundle of $\alpha \colon S \to A$, we get $\mathfrak{F} = \mathcal{E}$ (see Proposition 2.9), and we have $K_S^2 = 3 + \delta$ (see Proposition 2.56). This implies immediately that $\delta = 2$ and therefore $\mathfrak{H}' = 0$.

Hence, we get that D yields a polarization of type $(1,2)$.

Taking the fibre-product S' to A' of the surfaces S in $\mathfrak{N}$, we find that on an open set of $\mathfrak{N}$, by the theory by Miranda-Casnati-Ekedahl, we get a section of

$$S^3(\mathcal{E}') \otimes \det(\mathcal{E}')^{-1} = S^3(V^\vee) \otimes \mathcal{O}_{A'}(D),$$

an equation defining a two dimensional variety Z' yielding an extended CHPP surface. For the surfaces S satisfying the Gorenstein Assumption we have $Z' = X'$, where X' denotes the fibre-product of the canonical model X. Hence, the latter surfaces S are CHPP surfaces and we conclude by Theorem 2.31 that $\mathfrak{N}$ is the connected component of the CHPP surfaces.

Finally, thanks to Proposition 6.1 of [PePo13a], we see immediately that this component $\mathfrak{N}$ must coincide with the one constructed by Penegini-Polizzi in [PePo13a].

□

Indeed, we can prove a stronger version of the previous result.

Theorem 2.36. *The unirational irreducible connected component corresponding to the CHPP surfaces is the unique component with* $K_S^2 = 5$, $p_g(S) = q(S) = 2$, *and Albanese map* $\alpha : S \to A = Alb(S)$ *of degree* $d = 3$ *such that there is a surface in this component which fulfills property (III) of the Generality Assumption 2.17 with* $\mathfrak{H}' = 0$.

Proof. If (III) of the Generality Assumption holds with $\mathfrak{H}' = 0$ for some surfaces of the component $\mathfrak{N}$, then their canonical models X have a fibre-product X' with a birational map $\psi : X' \dashrightarrow Z'$, where $Z' \subset \mathbb{P}^1 \times A'$ yields an extended CHPP surface and has dualizing sheaf $\omega_{Z'} = \mathcal{O}_{Z'}(H + D)$. Note that by Proposition 2.26 Z' is normal.

The map ψ is an isomorphism where $X' \to A'$ is finite, that is, outside a finite number of fibres of $\mathbb{P}^1 \times A' \to A'$. Where Z' does not contain such a fibre, $Z' \to A'$ is finite and Z' coincides with the Stein factorization, so ψ is a morphism there.

There remain the fibres which are contained in Z' and for which $X' \to A'$ has a positive dimensional fibre.

Take a blow-up of the minimal resolution of singularities S' of X', say S^*, such that ψ becomes a birational morphism on S^* and use the same notation for a divisor and its pull-back to S^*.

By adjunction we have $K_{S^*} = K_{Z'} - \mathcal{A}$, where $\mathcal{A}$ is the adjoint divisor (an effective divisor, see the appendix by Mumford to Chapter III of [Zar71]).

Similarly, $K_{S^*} = K_{S'} + E$, for some effective exceptional divisor E.

Then the conclusion is that

$$K_{Z'} = K_{S'} + E + \mathcal{A}.$$

Since $K_{S'}^2 = K_{X'}^2 = K_{Z'}^2 = 20$, and $E + \mathcal{A}$ has negative self-intersection (alternatively, use that $K_{Z'}, K_{S'}$ are nef and big, and uniqueness of the Zariski decomposition), we conclude then that $E + \mathcal{A} = 0$, which means that $K_{Z'}$ pulls back to $K_{X'}$. Since $K_{X'}$ is ample, it follows that $X' = Z'$ (else there is a curve C in X' which is contracted to a point in Z', a contradiction), and X is the canonical model of a CHPP surface.

In conclusion, by Theorem 2.31 we see that the component $\mathfrak{N}$ is the component of CHPP surfaces.

□

2.7 Construction of PP4 Surfaces

Let here A' be an abelian surface with a divisor D yielding a polarization of type $(1,3)$ and set $\mathcal{L} := \mathcal{O}_{A'}(D)$. Then $V := H^0(A', \mathcal{O}_{A'}(D))$ is a three dimensional vector space, and the kernel $\mathcal{K}(D)$ of the isogeny $\Phi_D : A' \to A := \widehat{A'}$ is here

$$G := \mathcal{K}(D) \cong (\mathbb{Z}/3)^2.$$

Consider the order 27 Heisenberg group $\mathcal{H} := \mathcal{H}_3$ with centre $\mu_3 \cong \mathbb{Z}/3$, namely

$$1 \to \mu_3 \to \mathcal{H} \to G \cong (\mathbb{Z}/3)^2 \to 0.$$

Recalling that V is the Schrödinger representation of $\mathcal{H}$, we now describe, using the method of Casnati-Ekedahl [CE96], a family of generically finite covers $X' \to A'$ of degree 4 such that X' (which we require to be normal with at most RDP as singularities) is invariant under the action of the group G.

We will call *PP4 surfaces* the minimal resolution of singularities S of the free quotients

$$X := X'/G,$$

since our family coincides generically with the family constructed by Penegini and Polizzi in [PePo14], see Subsection 2.8.2.

Setting

$$\mathcal{E}' := V^\vee \otimes \mathcal{O}_{A'}(D) = V^\vee \otimes \mathcal{L},$$

we need to construct, according to Casnati-Ekedahl [CE96], a rank two locally free sheaf $\mathcal{F}$ with an embedding

$$\mathcal{F} \hookrightarrow S^2(\mathcal{E}') = (\mathcal{L}^{\otimes 2}) \otimes S^2(V^\vee) = 6\mathcal{L}^{\otimes 2} = \bigwedge^2 (3\mathcal{L}) \oplus \bigwedge^2 (3\mathcal{L}). \qquad (2.33)$$

Suppose that such an embedding is given and assume that the corresponding map
$S' \to A'$ is a finite quadruple cover with S' smooth. Then by Casnati-Ekedahl we must have either

$$\det(\mathcal{F}) = \det(\mathcal{E}') = 3D \qquad (2.34)$$

and

$$c_2(\mathcal{F}) = K_{S'}^2 - 2c_1(\mathcal{E}')^2 + 4c_2(\mathcal{E}') = 9\ K_S^2 - 2c_1(\mathcal{E}')^2 + 4c_2(\mathcal{E}') = 9(\ K_S^2 - 4). \qquad (2.35)$$

Hence, for $K_S^2 = 6$ we must have $c_2(\mathcal{F}) = 18$, whereas for $K_S^2 = 5$ we would have $c_2(\mathcal{F}) = 9$.

We observe that

$$h^0(\mathcal{L}) = 3, \quad D^2 = 6,$$

and $|D|$ has no base points if (A', D) is not a product polarization as in $(*)$ of Subsection 1.4.6.a. We make this assumption from now on.

2.7.1 The Definition of the Sheaf $\mathcal{F}$ and its Description

We define the sheaf $\mathcal{F}$ as the cokernel of the natural homomorphism

$$\mathcal{O}_{A'} \to V^\vee \otimes \mathcal{O}_{A'}(D)$$

given by the natural invariant

$$s := \sum_j y_j x_j \in V^\vee \otimes V,$$

where $\{x_1, x_2, x_3\}$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$ and $\{y_1, y_2, y_3\}$ is the dual basis of $V^\vee$. Namely, we have

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{A'} \arrow[r, "s=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}"] & V^\vee \otimes \mathcal{O}_{A'}(D) = 3\mathcal{O}_{A'}(D) \arrow[r] & \mathcal{F} \arrow[r] & 0\ . \end{tikzcd} \tag{2.36}$$

Since $|D|$ has no base points, $\mathcal{F}$ is a rank 2 locally free $\mathcal{O}_{A'}$-module.

The Description of $\mathcal{F}$ via the Koszul Complex. We are going to show that $\mathcal{F}$ is the image of the map $\wedge s$ in the Koszul complex associated with the section

$$s = \sum_j y_j x_j \in V^\vee \otimes H^0(\mathcal{L})$$

(or equivalently, $s = {}^t(x_1, x_2, x_3) \in H^0(\mathcal{E}')$), see [Ful84], Appendix B.3.4.

Namely, defining $Z(s)$ to be the zero subscheme of the section s, we consider the sequence

$$0 \to \bigwedge^3 \mathcal{E}'^\vee \to \bigwedge^2 \mathcal{E}'^\vee \to \bigwedge^1 \mathcal{E}'^\vee = \mathcal{E}'^\vee \to \mathcal{I}_{Z(s)} \to 0. \tag{2.37}$$

which is exact on $A' \setminus Z(s)$.

Note that $Z(s) = \emptyset$ if we assume that x_1, x_2, x_3 have no common zeroes, as does in our case occur for a general choice of (A', D).

Then since $\mathcal{E}' = V^\vee \otimes \mathcal{O}_{A'}(D)$, we get the following exact sequence

$$0 \to \left(\bigwedge^3 V\right) \otimes \mathcal{O}_{A'}(-3D) \to \left(\bigwedge^2 V\right) \otimes \mathcal{O}_{A'}(-2D) \to V \otimes \mathcal{O}_{A'}(-D) = 3\mathcal{O}_{A'}(-D) \to \mathcal{O}_{A'} \to 0. \tag{2.38}$$

Dualizing the previous sequence, we get

$$0 \to \mathcal{O}_{A'} \overset{s}{\to} V^\vee \otimes \mathcal{O}_{A'}(D) = 3\mathcal{O}_{A'}(D) \overset{\wedge s}{\to} \left(\bigwedge^2 V^\vee\right) \otimes \mathcal{O}_{A'}(2D) \to \left(\bigwedge^3 V^\vee\right) \otimes \mathcal{O}_{A'}(3D) \to 0 \tag{2.39}$$

Hence,

$$\mathcal{F} \overset{\wedge s}{\hookrightarrow} \left(\bigwedge^2 V^\vee\right) \otimes \mathcal{O}_{A'}(2D) = 3\mathcal{O}_{A'}(2D). \tag{2.40}$$

2.7.2 The Construction of the Quadruple Covers

Since we constructed a rank two locally free $\mathcal{O}_{A'}$-module $\mathcal{F}$ with an embedding as follows

$$\mathcal{F} \hookrightarrow S^2(\mathcal{E}'),$$

following Casnati-Ekedahl [CE96] we can now define the desired (generically finite) quadruple cover $X' \to A'$ as given by the zero locus of a section

$$\eta \in H^0(A', \mathcal{F}^\vee \otimes S^2(\mathcal{E}')).$$

This means that the fibres of $X' \to A'$ are the intersections of two conics in the $\mathbb{P}^2$-bundle

$$p\colon \mathbb{P}(\mathcal{E}'^\vee) = \mathbf{Proj}_{\mathcal{O}_{A'}} \mathbf{Sym}(\mathcal{E}') \to A',$$

where $\mathbb{P}(\mathcal{E}'^\vee) \cong \mathbb{P}(V) \times A' = \mathbb{P}^2 \times A'$.

More precisely, we will describe a 2-dimensional family of sections

$$\eta_{\lambda,\mu} := \lambda i_1 \oplus \mu i_2$$

depending on two complex parameters λ, μ. Namely,

$$0 \longrightarrow \mathcal{F} \xrightarrow[\lambda i_1 \oplus \mu i_2]{\eta_{\lambda,\mu}} S^2(\mathcal{E}') \cong \left(\bigwedge^2 V^\vee \otimes \mathcal{O}_{A'}(2D)\right) \bigoplus \left(\bigwedge^2 V^\vee \otimes \mathcal{O}_{A'}(2D)\right) \tag{2.41}$$

where i_1, i_2 are the respective inclusions of $\mathcal{F}$ in the respective summands.

We need to give explicitly the isomorphism

$$S^2(\mathcal{E}') \cong \left(\bigwedge^2 V^\vee \otimes \mathcal{O}(2D)\right) \bigoplus \left(\bigwedge^2 V^\vee \otimes \mathcal{O}(2D)\right). \tag{2.42}$$

We take this isomorphism in such a way that the respective summands correspond to the following irreducible subrepresentations of the Heisenberg group $\mathcal{H}$ inside $S^2(V^\vee)$:

$$S^2(V^\vee) = \langle y_1^2, y_2^2, y_3^2 \rangle \bigoplus \langle y_2 y_3, y_1 y_3, y_1 y_2 \rangle \cong V \bigoplus V \cong \bigwedge^2 V^\vee \bigoplus \bigwedge^2 V^\vee. \tag{2.43}$$

Observe in fact that $y_i y_j = \frac{y_1 y_2 y_3}{y_k}$, for i, j, k a permutation of $\{1, 2, 3\}$ of signature 1, and similarly $y_i \wedge y_j = \frac{y_1 \wedge y_2 \wedge y_3}{y_k} \in \mathbb{C} \otimes (V^\vee)^\vee$, where here dividing by y_k stands for the contraction with the corresponding vector in the dual basis.

Recall that the map

$$\wedge s\colon 3\mathcal{O}_{A'}(D) \to \mathcal{F} \subset \left(\bigwedge^2 V^\vee\right) \otimes \mathcal{O}_{A'}(2D) = 3\mathcal{O}_{A'}(2D)$$

is given as follows

$$\sigma = {}^t(\sigma_1, \sigma_2, \sigma_3) \longmapsto s \wedge \sigma = \begin{pmatrix} \sigma_3 x_2 - \sigma_2 x_3 \\ \sigma_1 x_3 - \sigma_3 x_1 \\ \sigma_2 x_1 - \sigma_1 x_2 \end{pmatrix} \tag{2.44}$$

and then

$$\begin{pmatrix} \sigma_3x_2-\sigma_2x_3 \\ \sigma_1x_3-\sigma_3x_1 \\ \sigma_2x_1-\sigma_1x_2 \end{pmatrix} \overset{\eta_{\lambda,\mu}}{\longmapsto} \lambda \begin{pmatrix} \sigma_3x_2-\sigma_2x_3 \\ \sigma_1x_3-\sigma_3x_1 \\ \sigma_2x_1-\sigma_1x_2 \end{pmatrix} \bigoplus \mu \begin{pmatrix} \sigma_3x_2-\sigma_2x_3 \\ \sigma_1x_3-\sigma_3x_1 \\ \sigma_2x_1-\sigma_1x_2 \end{pmatrix} \tag{2.45}$$

For each element $\sigma \in V^\vee \otimes \mathcal{L}$ we get therefore the following equations

$$\sigma \longmapsto \lambda \begin{pmatrix} y_1^2(\sigma_3x_2-\sigma_2x_3)+ \\ +y_2^2(\sigma_1x_3-\sigma_3x_1)+ \\ +y_3^2(\sigma_2x_1-\sigma_1x_2) \end{pmatrix} + \mu \begin{pmatrix} y_2y_3(\sigma_3x_2-\sigma_2x_3)+ \\ +y_1y_3(\sigma_1x_3-\sigma_3x_1)+ \\ +y_1y_2(\sigma_2x_1-\sigma_1x_2) \end{pmatrix}. \tag{2.46}$$

The generators of the ideal sheaf $\mathcal{I}_{X'}$ of X' are given, since $\mathcal{L}$ is generated by global sections, by the image of the space $H^0(V^\vee \otimes \mathcal{O}_{A'}(D))$, hence by the images of the elements

$$\sigma = (\sigma_1, 0, 0),\ (0, \sigma_2, 0),\ (0, 0, \sigma_3). \tag{2.47}$$

Namely, we have

$$\begin{aligned} \sigma_1F_1 &:= \sigma_1(\lambda(y_2^2x_3 - y_3^2x_2) + \mu(x_3y_1y_3 - x_2y_1y_2)) \\ \sigma_2F_2 &:= \sigma_2(\lambda(-x_3y_1^2 + x_1y_3^2) + \mu(-x_3y_2y_3 + x_1y_1y_2)) \\ \sigma_3F_3 &:= \sigma_3(\lambda(x_2y_1^2 - x_1y_2^2) + \mu(x_2y_2y_3 - x_1y_1y_3)) \end{aligned} \tag{2.48}$$

Rearranging them and observing that

$$\sigma_jF_j = 0 \quad \forall \sigma_j \in H^0(\mathcal{L}) \qquad \Longleftrightarrow \qquad F_j = 0,$$

we finally obtain the following equations for X':

$$\begin{aligned} F_1 &= x_3(\lambda y_2^2 + \mu y_1y_3) - x_2(\lambda y_3^2 + \mu y_1y_2) = 0 \\ F_2 &= x_1(\lambda y_3^2 + \mu y_1y_2) - x_3(\lambda y_1^2 + \mu y_2y_3) = 0 \\ F_3 &= x_2(\lambda y_1^2 + \mu y_2y_3) - x_1(\lambda y_2^2 + \mu y_1y_3) = 0. \end{aligned} \tag{2.49}$$

We observe now that X' is a subscheme of $\mathbb{P}^2 \times A' = \mathbb{P}(V) \times A'$, which has an action of $G = \mathcal{K}(D)$ of product type, where G acts on $\mathbb{P}^2 = \mathbb{P}(V)$ via the Schrödinger representation V of $\mathcal{H} = \mathcal{H}_3$, whereas G acts on A' by translations.

Remark 2.37. We set $\mathbb{P} := \mathbb{P}^2 \times A'$ and $\mathbb{P}' := \mathbb{P}(\mathcal{E}'^\vee)$. When using the description $\mathbb{P} = \mathbb{P}^2 \times A'$, we denote by H the hyperplane section on $\mathbb{P}^2$ and use the same notation for a divisor (either on $\mathbb{P}^2$ or on A') and its pull-back. Moreover, when using the description $\mathbb{P}' = \mathbb{P}(\mathcal{E}'^\vee)$, with bundle projection $p\colon \mathbb{P}' \to A'$, we denote by $\mathcal{O}_{\mathbb{P}'}(h)$ the Serre's twisting sheaf $\mathcal{O}_{\mathbb{P}'}(1)$. Hence, in particular we have

$$\mathcal{O}_{\mathbb{P}'}(h) \cong \mathcal{O}_{\mathbb{P}}(H + D),$$

see Remark 1.43 in Section 1.2.

We conclude that X' is G-invariant by the following Lemma

Lemma 2.38. *The algebraic set X' is $G = \mathcal{K}(D)$-invariant.*

Proof. Recalling that $G = \mathcal{K}(D) \cong \mathbb{Z}/3 \times \mu_3$, we observe that the group μ_3 acts, if ϵ is a primitive third root of unity, multiplying x_1, x_2, x_3 by $1, \epsilon, \epsilon^2$, and y_1, y_2, y_3 by $1, \epsilon^2, \epsilon$: hence the equations F_1, F_2, F_3 are respectively multiplied by $1, \epsilon^2, \epsilon$.

The group $\mathbb{Z}/3$ acts by a cyclical permutation of x_1, x_2, x_3, and with the same permutation of y_1, y_2, y_3, hence F_1, F_2, F_3 are also cyclically permuted.

We can also show our assertions using the fact that the inclusion of $\mathcal{F}$ inside $S^2(\mathcal{E}')$ was chosen to be Heisenberg equivariant.

□

The above G-invariant equations on $\mathbb{P}^2 \times A'$ can be written as describing a determinantal variety of Hilbert-Burch type, given by the vanishing of the 2×2 minors of the following matrix (we set $\lambda = 1$)

$$M = \begin{pmatrix} x_1 & x_3 & x_2 \\ y_1^2 + \mu y_2 z_3 & y_3^2 + \mu y_1 y_2 & y_2^2 + \mu y_1 y_3 \end{pmatrix} \tag{2.50}$$

Remark 2.39. The global Hilbert-Burch resolution for the ideal sheaf $\mathcal{I}_{X'}$ is the following

$$0 \to \mathcal{O}_{\mathbb{P}}(-2H - 2D) \oplus \mathcal{O}_{\mathbb{P}}(-4H - D) \overset{{}^tM}{\to} \mathcal{O}_{\mathbb{P}}(-2H - D)^{\oplus 3} \overset{(F_1, -F_3, F_2)}{\to} \mathcal{I}_{X'} \to 0. \tag{2.51}$$

Remark 2.40. The dualizing sheaf $\omega_{X'}$ of a normal projective variety

$$X' \subset \mathbb{P} = \mathbb{P}^2 \times A' \cong \mathbb{P}(\mathcal{E}'^{\vee}) = \mathbb{P}'$$

with (at most) RDP as singularities given by the vanishing of the 2×2 minors of M is an invertible sheaf as follows

$$\omega_{X'} \cong \mathcal{E}xt^2_{\mathbb{P}'}(\mathcal{O}_{X'}, \omega_{\mathbb{P}'}), \tag{2.52}$$

see [Lip84], Theorem 13.5. Moreover, since X' is Gorenstein we have the vanishing

$$\mathcal{E}xt^i_{\mathbb{P}'}(\mathcal{O}_{X'}, \omega_{\mathbb{P}'}) = 0 \qquad \text{for} \quad i \neq 2, \tag{2.53}$$

and there is an exact sequence as follows (cf. [CE96])

$$0 \to \mathcal{O}_{\mathbb{P}'}(-4h + 3D) \to p^*\mathcal{F}(-2h) \to \mathcal{O}_{\mathbb{P}'} \to \mathcal{O}_{X'} \to 0. \tag{2.54}$$

Recall that (see Subsection 1.2.3)

$$\omega_{\mathbb{P}'} = 3D - 3h.$$

Hence, applying the functor $\mathcal{H}om_{\mathbb{P}'}(\cdot, \omega_{\mathbb{P}'})$ to the previous sequence and taking into account the vanishing 2.53, we get in particular a surjection

$$\mathcal{O}_{\mathbb{P}'}(h) \cong \mathcal{H}om_{\mathbb{P}'}(\mathcal{O}_{\mathbb{P}'}(-4h + 3D), \omega_{\mathbb{P}'}) \twoheadrightarrow \mathcal{E}xt^2_{\mathbb{P}'}(\mathcal{O}_{X'}, \omega_{\mathbb{P}'}). \tag{2.55}$$

Then, in view of the isomorphism 2.52, this yields an isomorphism as follows

$$\mathcal{O}_{\mathbb{P}}(H + D)_{|X'} \cong \mathcal{O}_{\mathbb{P}'}(1)_{|X'} \cong \omega_{X'}, \tag{2.56}$$

and hence $\omega_{X'}$ is ample.

In light of the previous remark we see that, given a generically finite quadruple cover $X' \to A'$ as above, if X' is normal and has at most RDP as singularities, then X' is the canonical model of a surface of general type S'. Indeed, we can prove that $S' = X'$ in most cases.

Proposition 2.41. *The generically finite quadruple cover $X' = X'_\mu$ is smooth for a general $\mu \in \mathbb{C}$ and a general pair (A', D).*

Proof. It suffices to show this for $\mu = 0$. Using the symmetry of these equations, and since for a general pair (A', D) x_1, x_2, x_3 do not have common zeros, we may assume that if $p \in X'$ is singular, then $x_1 \neq 0$ (at p).

Then by the criterion of bordering minors we have two equations which locally define X', namely

$$x_1 y_3^2 - x_3 y_1^2 = 0, \; x_1 y_2^2 - x_2 y_1^2 = 0.$$

Requiring that the respective z-gradients are proportional implies the proportionality of the vectors

$$(-x_3 y_1, 0, x_1 y_3), \; (-x_2 y_1, x_1 y_2, 0),$$

and again by symmetry, since we must have $y_2 y_3 = 0$, we may assume $y_3 = 0$, and either $y_2 = 0$ or $x_3 y_1 = 0$.

In the first case we would have $x_2 = x_3 = 0$; looking then at the gradient on A', we would have that $x_2 = 0$ and $x_3 = 0$ do not intersect transversally, a contradiction since we know that the effective divisors $\{x_2 = 0\}$ and $\{x_3 = 0\}$ intersect transversally (see [PePo14, p. 776, Prop. 2.2]).

If $y_3 = y_1 = 0$ we would have the contradiction that $x_1 = 0$. Whereas, if $y_3 = x_3 = 0$, by the remark in the previous line $y_1 \neq 0$; if the gradient of x_3 vanishes, then we get a singular point of $x_3 = 0$. But for a general A' the divisors of the sections x_j are smooth, a contradiction. Hence, the gradient of the first equation on A' is non-zero, and we have a singular point only if $y_2 = x_2 = 0$ and the gradients of x_3, x_2 are proportional. But we have already seen that this is impossible.

□

Proposition 2.42. *A surface S' constructed as a generically finite quadruple cover $S' \to A'$ by the vanishing of the 2×2 minors of the matrix M, see* (2.50)*, has invariants as follows*

$$p_g(S') = 10, \qquad q(S') = 2, \quad K_{S'}^2 = 6 \cdot 9 = 54. \tag{2.57}$$

Moreover, $S' \to A'$ is the Albanese map of S' for a general pair (A', D).

Proof. (i) By Remark 2.39, we get right away the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-2H - 2D) \oplus \mathcal{O}_{\mathbb{P}}(-4H - D) \overset{{}^tM}{\to} \mathcal{O}_{\mathbb{P}}(-2H - D)^{\oplus 3} \overset{(F_1, -F_3, F_2)}{\to} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{S'} \to 0. \tag{2.58}$$

Hence, computing holomorphic Euler-Poincaré characteristics we obtain that

$$\chi(S') = \chi(\mathcal{O}_{\mathbb{P}}) - 3\chi(\mathcal{O}_{\mathbb{P}}(-2H - D)) + \chi(\mathcal{O}_{\mathbb{P}}(-2H - 2D)) + \chi(\mathcal{O}_{\mathbb{P}}(-4H - D)). \tag{2.59}$$

Recalling that $\omega_{\mathbb{P}} = -3H$, by Serre duality we get

$$\chi(S') = \chi(\mathcal{O}_{\mathbb{P}}) - 3\chi(\mathcal{O}_{\mathbb{P}}(D - H)) + \chi(\mathcal{O}_{\mathbb{P}}(2D - H)) + \chi(\mathcal{O}_{\mathbb{P}}(H + D)). \tag{2.60}$$

Thus, by Künneth formula we immediately see that

$$\begin{aligned}&\chi(\mathcal{O}_{\mathbb{P}}) = \chi(\mathcal{O}_{\mathbb{P}}(D-H)) = \chi(\mathcal{O}_{\mathbb{P}}(2D-H)) = 0,\\ &\chi(\mathcal{O}_{\mathbb{P}}(H+D)) = 9,\end{aligned}. \tag{2.61}$$

and hence $\chi(S') = 9$.

(ii) Now we split the sequence 2.58 into two short exact sequences, namely

$$\begin{aligned}&(i)\quad 0 \to \mathcal{O}_{\mathbb{P}}(-2H-2D) \oplus \mathcal{O}_{\mathbb{P}}(-4H-D) \overset{{}^tM}{\to} \mathcal{O}_{\mathbb{P}}(-2H-D)^{\oplus 3} \overset{(F_1,-F_3,F_2)}{\to} \mathcal{I}_{S'} \to 0,\\ &(ii)\quad 0 \to \mathcal{I}_{S'} \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{S'} \to 0.\end{aligned} \tag{2.62}$$

Considering the long exact sequence in cohomology associated with (i) we get in particular that

$$h^1(\mathcal{I}_{S'}) = h^2(\mathcal{I}_{S'}) = 0 \tag{2.63}$$

since $h^i(-2H-2D) = h^i(-4H-D) = h^i(-2H-D) = 0$ for $i = 1, 2, 3$ (by Serre duality and Kodaira vanishing theorem).

Therefore, considering the long exact sequence in cohomology associated with (ii) and taking into account (2.63), we find out that

$$2 = h^1(\mathcal{O}_{\mathbb{P}}) = h^1(\mathcal{O}_{S'}) = q(S'),$$

where the first equality follows from Künneth formula.

Finally, $\chi(S') = 9$ and $q(S') = 2$ immediately imply that $p_g(S') = 10$.

(iii) Since $q(S') = 2$, it follows immediately that the surjective map $S' \to A'$ is the Albanese map of S'. Indeed, for a general pair (A', D), the Picard group of A' has no torsion, namely $\mathrm{Pic}(A') \cong \mathbb{Z}$, and hence in particular there is no degree 2 isogeny $\widetilde{A} \to A'$, where $\widetilde{A}$ is another abelian surface. As a result, by the universal property of the Albanese map we get our conclusion.

(iv) Since for a general $\mu \in \mathbb{C}$ the map $S' \to A'$ is a finite quadruple cover, the self-intersection $K^2_{S'}$ of the canonical divisor of S' can be computed by using Casnati-Ekedahl's formula ([CE96], Proposition 5.3,(ii); cf. (2.35)), namely

$$K^2_{S'} = c_2(\mathcal{F}) + 2c_1(\mathcal{E}')^2 - 4c_2(\mathcal{E}'). \tag{2.64}$$

Recalling that

$$c_1(\mathcal{E}') = 3D, \quad c_2(\mathcal{E}') = c_2(\mathcal{F}) = 3D^2, \quad D^2 = 6,$$

we obtain right away

$$K^2_{S'} = c_2(\mathcal{F}) + 2c_1(\mathcal{E}')^2 - 4c_2(\mathcal{E}') = 18 + 2 \cdot 54 - 4 \cdot 18 = 54.$$

□

Corollary 2.43. *Let $S' \to A'$ be a generically finite quadruple cover constructed as above, where S' is smooth. Then the free quotient*

$$S := S'/G = S'/\mathcal{K}(D)$$

is a minimal surface of general type with $p_g(S) = q(S) = 2$*,* $K_S^2 = 6$ *and Albanese map*

$$\alpha\colon S \to A := \widehat{A'} = A'/G$$

of degree $d = 4$*.*

Proof. Observe that $G = \mathcal{K}(D)$ acts trivially on $H^0(A', \Omega^1_{A'}) \cong H^0(S', \Omega^1_{S'})$, and then

$$q(S) = \dim H^0(S', \Omega^1_{S'})^G = h^0(S', \Omega^1_{S'}) = q(S') = 2.$$

Hence, the thesis follows immediately from the previous proposition recalling that

$$K_S^2 = K_{S'}^2/|G| = 54/9 = 6, \qquad \chi(S) = \chi(S')/|G| = 9/9 = 1.$$

$\square$

Finally, we give the formal definition of a PP4 surface.

Definition 2.44. Given an abelian surface A' with a polarization $\mathcal{O}_{A'}(D)$ of type $(1,3)$, consider the following matrix

$$M = \begin{pmatrix} x_1 & x_3 & x_2 \\ y_1^2 + \mu y_2 z_3 & y_3^2 + \mu y_1 y_2 & y_2^2 + \mu y_1 y_3 \end{pmatrix},$$

where

- $\mu \in \mathbb{C}$,
- $\{x_1, x_2, x_3\}$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$,
- y_1, y_2, y_3 are homogeneous coordinates of $\mathbb{P}^2 = \mathbb{P}(V)$ (the dual basis of $\{x_1, x_2, x_3\}$).

We call *extended PP4 surface* an étale quotient $X := X'/G$, where $X' \subset \mathbb{P}^2 \times A'$ is defined by the vanishing of the 2×2 minors of M and $G := \mathcal{K}(D) \cong (\mathbb{Z}/3)^2$ is the kernel of the isogeny $\Phi_D\colon A' \to A := \widehat{A'}$.

A *PP4 surface* is defined to be the minimal resolution of singularities of an extended PP4 surface which is normal and has at most Rational Double Points as singularities.

2.8 Moduli Space of PP4 Surfaces

In this section we study the family of PP4 surfaces we have constructed in Section 2.7, hence we keep using the same notation and conventions adopted therein. In particular, we remind the reader that A' is here an abelian surface with a divisor D yielding a polarization of type $(1,3)$ (not a polarized product), $V = H^0(A', \mathcal{O}_{A'}(D))$ is the three dimensional Schrödinger representation of the order 27 Heisenberg group $\mathcal{H} = \mathcal{H}_3$ and $G = \mathcal{K}(D) \cong (\mathbb{Z}/3)^2$ is the kernel of the isogeny $\Phi_D \colon A' \to A = \widehat{A'} = A'/G$.

Proposition 2.45. *The family of Heisenberg invariant deformations $\mathcal{F}_t$ of the locally free $\mathcal{O}_{A'}$-module $\mathcal{F}$ with fixed determinant* $\det(\mathcal{F}_t) = 3D$ *is smooth of dimension 2, is parametrized by* $\mathrm{Pic}^0(A')$*, and consists, for $M \in \mathrm{Pic}^0(A')$, of the cokernel bundles*

$$\mathcal{F}_M = \mathrm{coker}(f : \mathcal{O}_{A'}(-3M) \to V^\vee \otimes \mathcal{O}_{A'}(D+M)), \quad f = \sum_j x_j y_j,$$

y_1, y_2, y_3 being the dual basis of $V^\vee$ of a canonical basis x_1, x_2, x_3 of $H^0(\mathcal{O}_{A'}(4M+D))$.

Proof. The tangent space to the deformations of $\mathcal{F}$ with fixed determinant is the space

$$H^1(End^0(\mathcal{F})),$$

where $End^0(\mathcal{F})$ denotes the subbundle of trace zero endomorphisms; that is, we have the direct sum decomposition $End(\mathcal{F}) = End^0(\mathcal{F}) \oplus \mathcal{O}_{A'}$.

The Heisenberg invariant deformations have as tangent space the subspace

$$H^1(End^0(\mathcal{F}))^{\mathcal{H}}.$$

By the exact sequence

$$(I)\ 0 \to \mathcal{O}_{A'} \to V^\vee \otimes \mathcal{O}_{A'}(D) \to \mathcal{F} \to 0$$

follows the exact sequence

$$(II)\ 0 \to \mathcal{F}^\vee \to V \otimes \mathcal{O}_{A'}(-D) \to \mathcal{O}_{A'} \to 0,$$

hence

$$(III)\ 0 \to V \otimes \mathcal{O}_{A'}(-D) \to V \otimes V^\vee \otimes \mathcal{O}_{A'} \to V \otimes \mathcal{F}(-D) \to 0,$$

and finally the exact sequence

$$(IV)\ 0 \to \mathcal{F}^\vee \otimes \mathcal{F} = End(\mathcal{F}) \to V \otimes \mathcal{F}(-D) \to \mathcal{F} \to 0.$$

The respective long exact cohomology sequences (in cases (I), (III)) yield:

$$0 \to \mathbb{C} \to V \otimes V^\vee \to H^0(\mathcal{F}) \to H^1(\mathcal{O}_{A'}) \to 0,$$

$$H^1(\mathcal{F}) \cong H^2(\mathcal{O}_{A'}) \cong \mathbb{C},$$

$$H^0(V \otimes \mathcal{F}(-D)) \cong V \otimes V^\vee,$$

$$0 \to V \otimes V^\vee \otimes H^1(\mathcal{O}_{A'}) \to H^1(V \otimes \mathcal{F}(-D)) \to V \otimes H^2(\mathcal{O}_{A'}(-D)) = V \otimes V^\vee \to$$

$$\to V \otimes V^\vee \otimes H^2(\mathcal{O}_{A'}) \to H^2(V \otimes \mathcal{F}(-D)) \to 0.$$

Taking Heisenberg invariants we have

$$H^0(\mathcal{F})^{\mathcal{H}} \cong H^1(\mathcal{O}_{A'}), \quad H^1(\mathcal{F})^{\mathcal{H}} \cong \mathbb{C},$$

where $\mathbb{C}$ denotes the trivial representation, and

$$H^0(V \otimes \mathcal{F}(-D))^{\mathcal{H}} \cong \mathbb{C},\ H^1(V \otimes \mathcal{F}(-D))^{\mathcal{H}} \cong H^1(\mathcal{O}_{A'}) \cong \mathbb{C}^2,\ H^2(V \otimes \mathcal{F}(-D))^{\mathcal{H}} = 0.$$

We take now the Heisenberg invariants of the cohomology sequence (IV), observing that by Serre duality

$$H^0(End(\mathcal{F})) = H^2(End(\mathcal{F}))^\vee,\ H^0(End(\mathcal{F}))^{\mathcal{H}} \supset \mathbb{C}.$$

We get then

$$H^0(End(\mathcal{F}))^{\mathcal{H}} \cong \mathbb{C} \cong H^2(End(\mathcal{F}))^{\mathcal{H}},$$

$$0 \to H^1(\mathcal{O}_{A'}) \to H^1(End(\mathcal{F}))^{\mathcal{H}} \to H^1(\mathcal{O}_{A'}) \to 0.$$

Since however $End(\mathcal{F}) = End^0(\mathcal{F}) \oplus \mathcal{O}_{A'}$, we infer that

$$H^1(End^0(\mathcal{F}))^{\mathcal{H}} \cong H^1(\mathcal{O}_{A'}), \quad H^2(End^0(\mathcal{F}))^{\mathcal{H}} = 0.$$

This means that our deformations are unobstructed, with tangent space $H^1(\mathcal{O}_{A'})$, which is the tangent space to $\mathrm{Pic}^0(A')$.

It is easy then to see that the universal family of deformations is our family $\{\mathcal{F}_M\}$. □

Proposition 2.46. *Let the locally free $\mathcal{O}_{A'}$-module $\mathcal{F}_M$ be as in Proposition 2.45, and assume that we have a Heisenberg equivariant injective homomorphism*

$$\mathcal{F}_M \to S^2(\mathcal{E}') = \mathcal{O}_{A'}(2D) \otimes S^2(V^\vee).$$

Then $2M$ is trivial, hence every deformation of $\mathcal{F}$ as a (Heisenberg invariant) subbundle of $S^2(\mathcal{E}')$ is trivial. Moreover, the homomorphism, for M trivial, belongs to the two dimensional vector family described above.

Proof. By composition we obtain an equivariant homomorphism

$$V^\vee \otimes \mathcal{O}_{A'}(D+M) \to \mathcal{F}_M \to (V \bigoplus V) \otimes \mathcal{O}_{A'}(2D),$$

equivalently

$$V^\vee \otimes \mathcal{O}_{A'} \to (V \bigoplus V) \otimes \mathcal{O}_{A'}(D-M),$$

determined by a homomorphism of representations

$$V^\vee \to (V \bigoplus V) \otimes V.$$

To determine this we can use our previous Koszul-type arguments, observing that again x_1, x_2, x_3 is a regular sequence: this implies that the inclusion of $\mathcal{F}_M$ factors through (here $s = (x_1, x_2, x_3)$)

$$\mathcal{F}_M \overset{\wedge s}{\hookrightarrow} \left(\bigwedge^2 V^\vee\right) \otimes \mathcal{O}_{A'}(2D + 2M) \cong V \otimes \mathcal{O}_{A'}(2D + 2M),$$

and hence our proof follows immediately, since

$$\operatorname{Hom}(V \otimes \mathcal{O}_{A'}(2D+2M), (V \bigoplus V) \otimes \mathcal{O}_{A'}(2D))^{\mathcal{H}} = \begin{cases} 0 & \text{for } 2M \neq 0, \\ \operatorname{Hom}(V, V \bigoplus V) & \text{for } 2M = 0. \end{cases}$$

□

Theorem 2.47. *The four dimensional family of PP4 surfaces yields an irreducible component of the moduli space.*

Proof. First of all, notice that the Generality Assumption 2.17 (which holds when the abelian surface A' does not contain an elliptic curve) is an open condition.

Once this is satisfied, by the theorem of Casnati and Ekedahl X is determined by X' which in turn is determined by the Heisenberg invariant inclusion of a locally free $\mathcal{O}_{A'}$-module $\mathcal{F}$ of rank two and with $\det(\mathcal{F}) = 3D$ inside $S^2(\mathcal{E}') = \mathcal{O}_{A'}(2D) \otimes S^2(V^\vee)$.

By Propositions 2.45 and 2.46, $\mathcal{F}$ and the inclusion are determined in an open set containing our family, and this open set is equal to our family.

□

Remark 2.48. 1) Proving that $\mathcal{F}$ is the unique Heisenberg invariant subbundle $\mathcal{F} \subset S^2(\mathcal{E}'^\vee)$ of rank 2 and with $\det(\mathcal{F}) = 3D$ would show that the PP4 family is the only component of the Main Stream of the moduli space of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 6$, $d = 4$, $\delta = 3$.

2) We shall show that the closure of our family yields a connected component of the moduli space. While it is clear, as in the case $K_S^2 = 5$, $d = 3$, that a limit of Tschirnhaus bundles of the form $V^\vee \otimes \mathcal{O}_{A'}(D)$ is again of this form, we show now an analogous statement for the locally free $\mathcal{O}_{A'}$-module $\mathcal{F}$.

Now we show that the irreducible component corresponding to PP4 surfaces is unirational.

Theorem 2.49. *The four dimensional irreducible component of the moduli space corresponding to PP4 surfaces is unirational.*

Proof. The argument is analogous to the one given in the proof of Theorem 2.34.

Denoting by $\mathcal{A}_2^{(1,3)}$ the moduli space of $(1,3)$-polarized abelian surfaces, it is clear from the construction of the component $\mathcal{M}_{\mathrm{PP4}}$ corresponding to PP4 surfaces that there is a dominant rational map

$$\mathcal{A}_2^{(1,3)} \times \mathbb{P}^1 \dashrightarrow \mathcal{M}_{\mathrm{PP4}}. \tag{2.65}$$

Since $\mathcal{A}_2^{(1,3)}$ is known to be unirational (see [Gri94]), we get right away our conclusion that $\mathcal{M}_{\mathrm{PP4}}$ is unirational.

□

In the light of Remark 2.48, especially in the direction of part 2), we establish some characterizations of $\mathcal{F}$ and its Heisenberg equivariant embeddings in $S^2(V^\vee) \otimes \mathcal{O}_{A'}(2D)$.

We recall here that (A', D) is not a polarized product, hence the linear system $|D|$ has no fixed part and no base points.

Set $F := \mathcal{F}(-D)$, so that we have an exact sequence

$$0 \to \mathcal{O}_{A'}(-D) \to V^\vee \otimes \mathcal{O}_{A'} \to F \to 0.$$

Observe that $c_1(F) = D, \ c_2(F) = D^2 = 6$.

Using a non zero element in $V^\vee$, for instance the first element of the canonical basis, we get a bundle inclusion $\mathcal{O}_{A'} \to V^\vee \otimes \mathcal{O}_{A'}$, and, by composition, an exact sequence

$$0 \to \mathcal{O}_{A'} \to F \to \mathcal{I}_Z(D) \to 0,$$

where injectivity follows since $\mathcal{O}_{A'} \cap \mathcal{O}_{A'}(-D) = 0$, and moreover the induced section vanishes only in the 0-dimensional subscheme

$$Z := \{x_2 = x_3 = 0\},$$

since

$$F/\mathcal{O}_{A'} = (V^\vee \otimes \mathcal{O}_{A'})/(\mathcal{O}_{A'} \oplus \mathcal{O}_{A'}(-D)) = \mathcal{O}_{A'}^2/\mathcal{O}_{A'}(-D),$$

and the composed map $\mathcal{O}_{A'}^2 \to F \to \mathcal{O}_{A'}(D)$ is given by $(x_3, -x_2)$.

Let now $\mathcal{F}'$ be a locally free $\mathcal{O}_{A'}$-module with the same Chern classes as $\mathcal{F}$, and again admitting a Heisenberg equivariant embedding in $S^2(V^\vee) \otimes \mathcal{O}_{A'}(2D)$.

Set then $F' := \mathcal{F}'(-D)$ and assume that F' admits a non zero section, leading to an exact sequence

$$0 \to \mathcal{O}_{A'}(C) \to F' \to \mathcal{I}_Z(D - C) \to 0, \tag{2.66}$$

where Z is a zero-dimensional subscheme and C is an effective divisor.

Since F' embeds into $S^2(V^\vee) \otimes \mathcal{O}_{A'}(D)$, the effective divisor C is contained in a divisor in $|D|$.

We observe that $H^0(F')$ is a representation of the Heisenberg group $\mathcal{H}_3$, and if $h^0(F') \leq 2$, the representation is a sum of 1-dimensional representations (se Chapter 1, Subsection 1.3.3.b). Hence, there is a Heisenberg invariant extension and the subscheme Z is $\mathcal{H}_3$-invariant: this implies that 9 divides $|Z|$, but the length $|Z|$ equals $6 - C(D-C)$ and since $6 = D^2 = C^2 + (D-C)^2 + 2C \cdot (D-C)$, we get $3 \leq |Z| \leq 6$, a contradiction.

We can exclude the case $C = D$, since then $H^0(F')$ comes from the subsheaf $\mathcal{O}_{A'}(C)$ (that is, $H^0(F') = H^0(\mathcal{O}_{A'}(C))$), hence the subsheaf is unique and the subscheme Z is $\mathcal{H}_3$-invariant: this implies that 9 divides $|Z|$, again a contradiction.

We can assume therefore $h^0(F') \geq 3$, and we take the sections with a minimal curve C of vanishing.

If $0 < C < D$, then first of all $C \cdot (D - C) \geq 1$, since $C + (D - C)$ is numerically connected (this follows for instance since $H^1(\mathcal{O}_{A'}(-D)) = 0$) and moreover $C \cdot (D-C) \geq 2$ since $C \cdot (D-C) = 1$ implies that its canonical system has a base point (see [CatFran96], [CFHR99]), while $|D|$ is base-point free.

Hence, from $6 = D^2 = C^2 + (D - C)^2 + 2C \cdot (D - C)$ follows that either

1) $C \cdot (D - C) = 3$ and both $D - C$ and C have self-intersection zero, or

2) $C \cdot (D - C) = 2$ and one of them has self-intersection zero.

In case 1) we have that one of the systems $|C|$ or $|D - C|$ has dimension 2, hence $|D|$ has a fixed part, a contradiction.

In case 2), one of the systems has dimension ≤ 1, and the other has dimension 0. Since $h^0(F') \geq 3$ and Z is non trivial, it follows that $\dim |C| + \dim |D - C| = 1$, and Z is in the base locus of $|D - C|$, which has therefore dimension 0 and $(D - C)^2 = 2$. Then $|C|$ consists of curves which are the union of two elliptic curves E_1, E_2. But then $E_1 \cdot (D - C) = 1$, hence $D - C$ maps isomorphically to the elliptic curve A'/E_1 and thus $D - C$ consists of two elliptic curves, of which one is algebraically equivalent to E_1. This implies that $|D|$ has a fixed part, a contradiction.

We have therefore reached the conclusion that, under the assumption of existence of a section, the general section vanishes only on a finite set, and then we have the desired exact sequence

$$0 \to \mathcal{O}_{A'} \to F' \to \mathcal{I}_Z(D) \to 0, \tag{2.67}$$

where we know moreover that $h^0(F') \geq 3$.

Hence, since $|D|$ has dimension 2 and does not have 6 base points, then $h^0(F') = 3$ and our exact sequence is exact on global sections.

Whence, Z is the complete intersection of 2 sections of $\mathcal{O}_{A'}(D)$. Since the base-point scheme of $H^0(F')$ is Heisenberg invariant and is contained in Z, then F' is generated by global sections, and by Heisenberg invariance we have the exact sequence

$$0 \to \mathcal{O}_{A'}(-D) \to V^\vee \otimes \mathcal{O}_{A'} \to F' \to 0,$$

showing that F' is isomorphic to F.

We summarize our conclusion:

Lemma 2.50. *Let (A', D) be a polarization of type $(1, 3)$ which is not a polarized product and let $\mathcal{F}'$ be a rank 2 locally free $\mathcal{O}_{A'}$-module with the same Chern classes as $\mathcal{F}$, with a Heisenberg equivariant embedding in $S^2(V^\vee) \otimes \mathcal{O}_{A'}(2D)$, and moreover with $H^0(\mathcal{F}'(-D)) \neq 0$. Then $\mathcal{F}'$ is isomorphic to $\mathcal{F}$.*

The case where (A', D) is a polarized product cannot occur.

Proof. We have given the proof assuming that $|D|$ is base-point free.

But the only exception is when we have a polarized product of elliptic curves, namely

$$(A', D) = (E_1, P_1) \times (E_2, 3P_2). \tag{2.68}$$

In this case however we can run the same proof; in particular, considering sequence (2.66), we infer as before that $h^0(F') \geq 3$.

We know that the curves of the linear system $|D|$ consist of a fixed elliptic curve E_2' and three elliptic curves E_1'', E_2'', E_3'' which are algebraically equivalent.

Hence, it is possible to have the situation $C \cdot (D - C) = 1$: this happens iff either $C = E_3''$ or $D - C = E_3''$.

Moreover, we can have the situation where $C^2 = (D - C)^2 = 0$, $C \cdot (D - C) = 3$.

If the first case occurs, since $h^0(F') \geq 3$, we must have that Z is in the base locus of $|D - C|$. Hence, the subsheaf $\mathcal{O}_{A'}(C)$ cannot be embedded in $S^2(V^\vee) \otimes \mathcal{O}_{A'}(D)$.

In the second case, we get a similar contradiction if $|D - C|$ has base points.

There remains only the possibility that $C = E'_2$.

Consider now another section of F': it must vanish on the curve E'_2, thus it gives another injective map of $\mathcal{O}_{A'}(E'_2)$ in F', which yields another subsheaf, since otherwise $h^0(F') = 1$. The conclusion is that there is a map of $\mathcal{O}_{A'}(E'_2)^2 \to F'$ with nontrivial determinant, hence $D \geq 2E'_2$, a contradiction.

Therefore, we have a section vanishing only on a finite number of points and then we get a sequence like (2.67), namely

$$0 \to \mathcal{O}_{A'} \to F' \to \mathcal{I}_Z(D) \to 0.$$

We find again a contradiction since $\mathcal{O}_{A'}$ cannot be embedded in $S^2(V^\vee) \otimes \mathcal{O}_{A'}(D)$ with torsion free cokernel: because all sections of $\mathcal{O}_{A'}(D)$ vanish on E'_2.

□

Theorem 2.51. *The family of PP4 surfaces yields a connected component of the moduli space.*

Proof. The first part of our argument runs as in the case of CHPP surfaces.

We consider a 1-parameter limit X'_0 of PP4 surfaces, and we observe that, by virtue of Remark 2.48 and Lemma 2.50, in the limit the locally free sheaves $\mathcal{E}'$ and $\mathcal{F}$ are exactly as for the PP4 surfaces, and that moreover (A', D) is not a product polarization.

The subbundle $\mathcal{F}$ defines a subscheme $\Sigma \subset \mathbb{P}^2 \times A'$ whose equations can be written in Hilbert-Burch form,

$$\Sigma := \{(y, z) \mid \operatorname{rank}(M) \leq 1\}$$

$$M = \begin{pmatrix} x_1 & x_3 & x_2 \\ y_1^2 + \mu y_2 y_3 & y_3^2 + \mu y_1 y_2 & y_2^2 + \mu y_1 y_3 \end{pmatrix}$$

and it would suffice to see that Σ is normal: this however is not always the case, as we shall see later on. Therefore we use another argument.

Since $|D|$ has no base points, Σ is the union of the open sets $\mathcal{U}_j \cap \Sigma$, where

$$\mathcal{U}_j := \{x_j \neq 0\}.$$

By symmetry, we may analyze the open set $\mathcal{U}_1$ where $x_1 \neq 0$, and set

$$s := \frac{x_2}{x_1}, \quad s' := \frac{x_3}{x_1}, \quad q_j := y_j^2 + \mu y_{j+1} y_{j+2}$$

(here the indices have to be understood as elements of $\mathbb{Z}/3$).

Then on this open set we have, by the criterion of bordering minors, the complete intersection

$$\Sigma \cap \mathcal{U}_1 = \{q_2 - s q_1 = q_3 - s' q_1 = 0\}.$$

Hence, Σ is Gorenstein, with dualizing (canonical) sheaf $\omega_\Sigma = \mathcal{O}_{\mathbb{P}(\mathcal{E}'^\vee)}(1)_{|\Sigma}$, and thus the vector space of sections of the canonical system is generated by the pull-back of the 2-form on A' and by the elements $\{y_i x_j\}, i, j \in \{1, 2, 3\}$, which are a basis of $V^\vee \otimes H^0(\mathcal{O}_{A'}(D))$.

Observe that the sub-system generated by $\{y_i x_j\}, i, j \in \{1, 2, 3\}$, is base point free and factors through $\Sigma \to \mathbb{P}^2 \times A' \to \mathbb{P}^2 \times \mathbb{P}^2$, where the last map is the product of the identity with φ_D.

Assume now that Σ is not normal, and let X'' be a resolution of singularities of Σ: then $p_g(X'') < 10$, since the sections of $\mathcal{O}_{X''}(K_{X''})$ correspond to a subspace of the canonical system of Σ contained in the subspace of sections vanishing on the singular curve of Σ.

This is a contradiction, since X'_0 is birational to X'' and $p_g(X'_0) = 10$ (since $\chi(X'_0) = 9$, $q(X'_0) = 2$).

A similar contradiction is found if Σ is normal but its singularities are not Rational Double Points.

□

In order to give a more explicit description of the surfaces in this connected component of the moduli space, it is desirable to describe the singular sets of such two dimensional varieties Σ. This is the main goal of the next subsection.

2.8.1 Singular Sets of Extended PP4 Surfaces

As in Theorem 2.51, we consider a variety

$$\Sigma := \{(y, z) \mid \operatorname{rank}(M) \le 1\} \subset \mathbb{P}^2 \times A',$$

$$M = \begin{pmatrix} x_1 & x_2 & x_3 \\ q_1 & q_2 & q_3 \end{pmatrix},$$

yielding an extended PP4 surface.

To avoid cumbersome calculations, we write (the indices being understood as elements of $\mathbb{Z}/3$)

$$q_j(y) := y_j^2 + 2m y_{j+1} y_{j+2},$$

observing right away that $3q_j(y) = \frac{\partial f_m}{\partial y_j}$, where

$$f_m(y) = \sum_j y_j^3 + 6m y_1 y_2 y_3.$$

Hence, $R_m := \{y \mid f_m(y) = 0\} \subset \mathbb{P}^2$ is a cubic in the Hesse pencil of cubic curves, which is a smooth cubic or the union of 3 lines: the latter situation occurs precisely for $\mu^3 = (2m)^3 = -1$.

The crucial remark is that Σ is a birational fibre product

$$\Sigma = \{(y, z) \mid q(y) = \varphi_D(z)\} = \mathbb{P}^2 \times_{\mathbb{P}^2} A',$$

$$\varphi_D(z) := (x_1(z), x_2(z), x_3(z)), \qquad q(y) := (q_1(y), q_2(y), q_3(y)).$$

Observe that φ_D is always a finite morphism, since we assume that (A', D) is not a polarized product, whereas q is a morphism (hence also finite) if $(2m)^3 \neq -1$.

If instead $(2m)^3 = -1$, R_m consists of three lines, hence q is a standard birational Cremona transformation contracting the three lines and blowing up the three singular points of R_m.

From this remark and a trivial calculation in local coordinates follows that, defining R_m to be the ramification divisor for q and R as the ramification divisor for φ_D, the point (y, z) is a smooth point if either $y \notin R_m$ or $z \notin R$.

Similarly, (y, z) is a smooth point if $y \in R_m$, $z \in R$, but the rank of the derivatives are

$$\operatorname{rank}(Dq_y) = 1, \quad \operatorname{rank}(D(\varphi_D)_z) = 1,$$

and the respective images of Dq_y, $D(\varphi_D)_z$ are not the same tangent line.

A partial conclusion is that, defining B_m as the branch divisor for q and $\mathcal{B}$ as the branch divisor for φ_D, the singular points lie above the points in the plane in $B_m \cap \mathcal{B}$ where the two curves do not intersect transversally.

It follows then:

Proposition 2.52. *An extended PP4 surface yielded by* Σ *is normal unless the two curves* $\mathcal{B}$ *and* B_m *have a common component or* $(2m)^3 = -1$.

Proof. We have seen that Σ is a local complete intersection, whence it is normal if and only if it is smooth in codimension one, that is, $\mathrm{Sing}(\Sigma)$ is a finite set.

If $\mathcal{B}$ and B_m have no common component, their intersection is a finite set. We use then the fact that φ_D is a finite morphism, and also q is a morphism for $(2m)^3 \neq -1$, hence necessarily finite.

□

The reason for assuming $(2m)^3 \neq -1$ is that, for $(2m)^3 = -1$, R_m consists of a triangle, and if y' is a vertex of the triangle, then $q(y') = 0$. Thus, it follows that $\Sigma \supset \{y'\} \times A'$, hence Σ has at least four irreducible components.

For $(2m)^3 \neq -1$ the branch locus B_m consists of the dual sextic curve to the cubic R_m, which has equation (see [Cas99], page 383)

$$\begin{aligned} B_m := \{x \mid\ & \sum_j x_j^6 + 2(-16m^3 - 1)(\sum_{i \neq j} x_i^3 x_j^3) \\ & - 24m^2 x_1 x_2 x_3 (\sum_j x_j^3) + 6m(-8m^3 - 4) x_1^2 x_2^2 x_3^2 = 0\}. \end{aligned} \tag{2.69}$$

Using [Cas99] we have:

Example 2.53. There exist extended PP4 surfaces which are not normal.

It suffices to take (A', D) a bielliptic abelian surface of type $(1, 3)$ with B_m contained in the branch locus $\mathcal{B}$. This exists by [Cas99].

2.8.2 The PP4 Family and the Construction of [PePo14]

We now show that the 4-dimensional family of PP4 surfaces contains the family of surfaces described in [PePo14].

Recalling that $\alpha' : S' \to A'$ is in general a finite quadruple cover, we have

$$\alpha'_* \mathcal{O}_{S'} = \mathcal{O}_{A'} \oplus \mathcal{E}'^{\vee} = \mathcal{O}_{A'} \oplus (V \otimes \mathcal{O}_{A'}(-D)).$$

The multiplication tensor is given by two tensors

$$\tau_0 : \mathcal{E}'^{\vee} \times \mathcal{E}'^{\vee} \to \mathcal{O}_{A'},$$

$$\tau_1 : \mathcal{E}'^{\vee} \times \mathcal{E}'^{\vee} \to \mathcal{E}'^{\vee}.$$

As Hahn and Miranda prove [HM99], τ_0 is determined by

$$\tau_1 \in H^0(A', S^2(\mathcal{E}') \otimes \mathcal{E}'^{\vee}) = H^0(A', S^2(V^{\vee}) \otimes V \otimes \mathcal{O}_{A'}(D)) = S^2(V^{\vee}) \otimes V \otimes V.$$

Indeed, they show that τ_1 is in turn determined by a totally decomposable section (that is, a section which locally is the wedge product of two local sections; see also Theorem 1.31):

$$\begin{aligned}\gamma_{\lambda,\mu} \in H^0(A', \bigwedge^3 \mathcal{E}'^{\vee} \otimes \bigwedge^2 S^2(\mathcal{E}')) &= H^0(A', \bigwedge^3 V \otimes \bigwedge^2 S^2(V^{\vee}) \otimes \mathcal{O}_{A'}(D)) \\ &= \bigwedge^3 V \otimes \bigwedge^2 S^2(V^{\vee}) \otimes V.\end{aligned}$$

An easy calculation yields

$$\begin{aligned}\gamma_{\lambda,\mu} = (x_1 \wedge x_2 \wedge x_3) \otimes \big(&- \lambda\mu x_2 \otimes y_1^2 \wedge y_1y_2 + \lambda\mu x_3 \otimes y_1^2 \wedge y_1y_3 + \lambda^2 x_3 \otimes y_1^2 \wedge y_2^2 \\ &+ 0 \otimes y_1^2 \wedge y_2y_3 - \lambda^2 x_2 \otimes y_1^2 \wedge y_3^2 - \mu^2 x_1 \otimes y_1y_2 \wedge y_1y_3 \\ &- \lambda\mu x_1 \otimes y_1y_2 \wedge y_2^2 + \mu^2 x_2 \otimes y_1y_2 \wedge y_2y_3 + 0 \otimes y_1y_2 \wedge y_3^2 \\ &+ 0 \otimes y_1y_3 \wedge y_2^2 - \mu^2 x_3 \otimes y_1y_3 \wedge y_2y_3 + \lambda\mu x_1 \otimes y_1y_3 \wedge y_3^2 \\ &- \lambda\mu x_3 \otimes y_2^2 \wedge y_2y_3 + \lambda^2 x_1 \otimes y_2^2 \wedge y_3^2 - \lambda\mu x_2 \otimes y_2y_3 \wedge y_3^2 \big).\end{aligned}$$

As pointed out in [HM99, p. 12], recalling that $\bigwedge^2 \mathcal{F} \cong \bigwedge^3 \mathcal{E}'$, $\gamma_{\lambda,\mu}$ corresponds to the Plucker embedding

$$\bigwedge^2 \mathcal{F} \to \bigwedge^2 S^2(\mathcal{E}').$$

Choosing for $S^2(V^{\vee})$ the ordered basis

$$\begin{aligned}\{&y_1^2 \wedge y_1y_2,\ y_1^2 \wedge y_1y_3,\ y_1^2 \wedge y_2^2,\ y_1^2 \wedge y_2y_3,\ y_1^2 \wedge y_3^2, \\ &y_1y_2 \wedge y_1y_3,\ y_1y_2 \wedge y_2^2,\ y_1y_2 \wedge y_2y_3,\ y_1y_2 \wedge y_3^2,\ y_1y_3 \wedge y_2^2, \\ &y_1y_3 \wedge y_2y_3,\ y_1y_3 \wedge y_3^2,\ y_2^2 \wedge y_2y_3,\ y_2^2 \wedge y_3^2,\ y_2y_3 \wedge y_3^2\}\end{aligned}$$

as in [PePo14], under the identification

$$X, Y, Z \longleftrightarrow x_1, x_2, x_3, \qquad \hat{X}, \hat{Y}, \hat{Z} \longleftrightarrow y_1, y_2, y_3,$$

we can write

$$\begin{aligned}\gamma_{\lambda,\mu} = \big(&- \lambda\mu x_2,\ \lambda\mu x_3,\ \lambda^2 x_3,\ 0,\ -\lambda^2 x_2, \\ &- \mu^2 x_1,\ -\lambda\mu x_1, \mu^2 x_2,\ 0,\ 0, \\ &- \mu^2 x_3,\ \lambda\mu x_1,\ -\lambda\mu x_3,\ \lambda^2 x_1,\ -\lambda\mu x_2 \big) \in H^0(A', \mathcal{O}_{A'}(D))^{\oplus 15}.\end{aligned}$$

Then it is easy to see that $\gamma_{\lambda,\mu}$ has the same form provided by Penegini and Polizzi in [PePo14], Proposition 2.3, setting

$$a := -\lambda\mu, \quad b := \lambda\mu, \quad c := \lambda^2,$$
$$d := 0, \quad e := -\mu^2,$$

and it fulfills the properties stated in Proposition 2.4 therein.

Therefore, this shows that the family of PP4 surfaces contains the one in [PePo14].

Remark 2.54. Note that in [PePo14], in the statement of Proposition 2.3, we have to switch Y, Z since in our case the dual of the Heisenberg representation $V^\vee$ is equivalent to the one given in [PePo14] via the matrix

$$C := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Furthermore, we point out that the correspondence between the parameters (λ, μ) and (a, c) giving respectively the family of PP4 surfaces and the family in [PePo14] is as follows

$$(\lambda, \mu) \longmapsto (-\lambda\mu, \lambda^2).$$

2.8.3 Branch Locus

We compute now the branch locus of the (generically finite) degree 4 cover

$$\alpha' \colon S' \to A',$$

induced by the second projection map $\mathbb{P}(V) \times A' \to A'$.

The equations of S' are given by the vanishing of the 2×2 minors of the matrix M, which we have seen to equal

$$\begin{cases} F_1 = x_3(y_2^2 + \mu y_1 y_3) - x_2(y_3^2 + \mu y_1 y_2) = 0 \\ F_2 = x_1(y_3^2 + \mu y_1 y_2) - x_3(y_1^2 + \mu y_2 y_3) = 0 \\ F_3 = x_2(y_1^2 + \mu y_2 y_3) - x_1(y_2^2 + \mu y_1 y_3) = 0 \end{cases} \quad \mu \in \mathbb{C}. \tag{2.70}$$

Recalling that a fixed fibre is singular if and only if it has strictly less than 4 points, we are going to find the conditions which the coefficients of the system (2.70), namely x_1, x_2, x_3, μ, must satisfy in order that this happens.

Recall that the case

$$x_1 = x_2 = x_3 = 0$$

cannot occur since $|D|$ has no base points.

Then one of them is nonzero and by symmetry we may assume that

$$x_1 \neq 0.$$

Then as in Proposition 2.41, the local equations are

$$F_2 = 0, \quad F_3 = 0.$$

The equations describe the intersection of two conics in $\mathbb{P}^2$ and the intersection points are exactly the base points of the pencil generated by them.

Let us denote by $A_{u,v} \in Mat(\mathbb{C}, 3)$ the 3×3 symmetric matrix of a conic in the pencil, namely

$$A_{u,v} = \begin{bmatrix} x_2v - x_3u & \frac{1}{2}\mu x_1 u & -\frac{1}{2}\mu x_1 v \\ \frac{1}{2}\mu x_1 u & -x_1 v & \frac{1}{2}\mu(x_2v - x_3u) \\ -\frac{1}{2}\mu x_1 v & \frac{1}{2}\mu(x_2v - x_3u) & x_1 u \end{bmatrix}, \tag{2.71}$$

and consider its determinant

$$p(u, v) := \det A_{u,v} \in \mathbb{C}[u, v]_3.$$

We point out that the the case $p(u, v) \equiv 0$ does not occur for a general fibre since S' is irreducible, hence the pencil does not have any fixed component.

Then $p(u, v)$ is a nonzero homogeneous polynomial of degree 3 whose roots correspond to the degenerate conics of the pencil.

Recall that the base points of a pencil of conics are less then 4 if and only if the pencil contains at most 2 degenerate conics, that is $p(u, v)$ has at least one multiple root, equivalently the discriminant of $p(u, v)$ vanishes.

Since we have that

$$\begin{aligned} p(u, v) = {} & \frac{1}{4}\mu^2(x_3^3 - x_1^3)u^3 + \\ & + \Big(\frac{1}{4}\mu^3 x_1^2 x_3 - \frac{3}{4}\mu^2 x_2 x_3^2 + x_1^2 x_3\Big)u^2 v + \\ & + \Big(-\frac{1}{4}\mu^3 x_1^2 x_2 + \frac{3}{4}\mu^2 x_2^2 x_3 - x_1^2 x_2\Big)uv^2 + \\ & + \frac{1}{4}\mu^2(x_1^3 - x_2^3)v^3, \end{aligned} \tag{2.72}$$

and we have the well-known formula saying that the discriminant of a polynomial $p = ax^3 + bx^2 + cx + d$ equals

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd,$$

a long but straightforward computation shows that the equation of the discriminant is in our case:

$$\begin{aligned} x_1^6\Big[& -27\mu^8(x_1^6 + x_2^6 + x_3^6) + \mu^2\Big(-4\mu^9 + 6\mu^6 - 192\mu^3 - 256\Big)(x_1^3x_2^3 + x_1x_3 + x_2^3x_3^3) + \\ & + \mu^4\Big(18\mu^6 + 144\mu^3 + 288\Big)x_1x_2x_3(x_1^3 + x_2^3 + x_3^3) + \\ & + \Big(\mu^{12} - 92\mu^9 - 336\mu^6 + 256\mu^3 + 256\Big)x_1^2x_2^2x_3^2\Big] = 0 \end{aligned} \tag{2.73}$$

Since we worked on the open set $x_1 \neq 0$, we finally get the branch locus equation

$$\begin{aligned}
&-27\mu^8(x_1^6+x_2^6+x_3^6)+\mu^2\Bigg(-4\mu^9+6\mu^6-192\mu^3-256\Bigg)(x_1^3x_2^3+x_1x_3+x_2^3x_3^3)+\\
&+\mu^4\Bigg(18\mu^6+144\mu^3+288\Bigg)x_1x_2x_3(x_1^3+x_2^3+x_3^3)+\\
&+\Bigg(\mu^{12}-92\mu^9-336\mu^6+256\mu^3+256\Bigg)x_1^2x_2^2x_3^2=0.
\end{aligned} \tag{2.74}$$

One easily sees by symmetry that the cases $x_2 \neq 0$, $x_3 \neq 0$ lead exactly to the same equation.

Remark 2.55. Note that this is the same branch locus as the one found by Penegini and Polizzi in [PePo14, p. 749, equation (14)]. In fact, by multiplying their equation with -27 and setting $c = 1$, $a = -\mu$, $X = x_1$, $Y = x_2$, $Z = x_3$ one gets (2.74).

2.9 Analysis of the Case $d = 3$ under the Generality Assumption

We want to construct surfaces S with AP fulfilling the Generality Assumption 2.17 and having a surjective morphism $\alpha\colon S \to A$ of degree $d = 3$.

Hence, given such a surface S and setting $A' := \widehat{A}$, there is a polarization $\mathcal{L} = \mathcal{O}_{A'}(D)$ of type (δ_1, δ_2) and hence with Pfaffian $\delta := \delta_1\delta_2$. Considering the associated isogeny $\Phi_D\colon A' \to A$ with kernel $G := \mathcal{K}(D)$, since $d = 3$ we have that the dual $\mathcal{E}$ of the Tschirnhaus bundle of α and its pull-back $\mathcal{E}' = (\Phi_D)^*\mathcal{E}$ have rank

$$\operatorname{rank}(\mathcal{E}) = \operatorname{rank}(\mathcal{E}') = 2,$$

and moreover there is a $\mathcal{H}_D$-equivariant exact sequence like ($\diamondsuit$), namely

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathcal{E}' \to 0. \tag{2.75}$$

Since $\mathcal{L} \otimes V^\vee$ has rank δ, $\mathfrak{H}'$ has rank $\delta - 2$, and by definition it is a successive extension of line bundles in $\operatorname{Pic}^0(A')$.

Since for such a suface S the Gorenstein Assumption 2.6 holds, it follows that $S' := S \times_A A'$ is a subscheme in $\mathbb{P}(V) \times A' = \mathbb{P}^{\delta-1} \times A'$ corresponding to a divisor of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}'^\vee)$ given by a section of

$$\mathbf{Sym}^3(\mathcal{E}') \otimes (\det \mathcal{E}')^{-1}.$$

Notice that

$$c_1(\mathcal{E}') = \delta D, \qquad c_2(\mathcal{E}') = \frac{\delta(\delta-1)}{2} D^2, \tag{2.76}$$

see Remark 2.19. Moreover, letting H be the hyperplane divisor of

$$\mathbb{P} := \mathbb{P}(\mathcal{E}'^\vee) = \mathbf{Proj}\,\mathbf{Sym}(\mathcal{E}'),$$

we have the so-called Grothendieck relation (see Chapter 1, Subsection 1.2.4)

$$H^2 - c_1(\mathcal{E}')H + c_2(\mathcal{E}') = 0. \tag{2.77}$$

We observe that the class of S' equals $3H - c_1(\mathcal{E}')$ and (cf. formula (1.49))

$$K_{\mathbb{P}} = -2H + c_1(\mathcal{E}'),$$

hence $K_{S'} = H_{|S'}$.

Therefore, we have

$$\begin{aligned} K_{S'}^2 &= H^2(3H - c_1(\mathcal{E}')) = (c_1(\mathcal{E}')H - c_2(\mathcal{E}'))(3H - c_1(\mathcal{E}')) = \\ &= 3c_1(\mathcal{E}')H^2 - c_1(\mathcal{E}')^2 - 3c_2(\mathcal{E}') = 2c_1(\mathcal{E}')^2 - 3c_2(\mathcal{E}') = \\ &= 2\delta^2 D^2 - \frac{3}{2}\delta(\delta-1)D^2 = \delta^2(4\delta - 3(\delta-1)) = \delta^2(\delta+3), \end{aligned} \tag{2.78}$$

where the second-last equality follows from $D^2 = 2\delta$.

Since the degree of $S' \to S$ equals $|G| = \delta^2$, we have shown the first assertion of the following.

Proposition 2.56. *$K_S^2 = \delta + 3$ and $\chi(S) = 1$.*

Proof. There remains to show the second assertion. Indeed, it suffices to show that $\chi(S') = \delta^2$. This follows from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}}) = \mathcal{O}_{\mathbb{P}}(-2H + c_1(\mathcal{E}')) \to \mathcal{O}_{\mathbb{P}}(H) \to \mathcal{O}_{S'}(K_{S'}) \to 0.$$

We have $\chi(\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}})) = -\chi(\mathcal{O}_{\mathbb{P}}) = 0$, more precisely, $h^0(\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}})) = 0$, $h^1(\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}})) = h^2(\mathcal{O}_{\mathbb{P}}) = 1$, $h^2(\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}})) = h^1(\mathcal{O}_{\mathbb{P}}) = 2$, $h^3(\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}})) = 1$.

On the other hand, $h^i(\mathcal{O}_{\mathbb{P}}(H)) = h^i(\mathcal{E}')$ yields

$$\chi(S') = \chi(\mathcal{O}_{S'}(K_{S'})) = \chi(\mathcal{E}') = \delta\chi(\mathcal{L}) - \chi(\mathfrak{H}') = \delta^2 - 0 = \delta^2,$$

as we wanted to show.

□

Remark 2.57. In the above proposition we certainly have $q(S') = 2$ provided that $h^1(\mathcal{E}') = 0$, $h^2(\mathcal{E}') = 0$, and this follows if $h^i(\mathfrak{H}') = 0\ \forall i$.

The case $\delta = 2$ is the case of CHPP surfaces, that we have already described in detail, so let us proceed to the next case $\delta = 3$.

2.9.1 The Case $d = \delta = 3$ with Trivial Homogeneous Bundle

Given an abelian surface A' with an ample divisor $\mathcal{L} = \mathcal{O}_{A'}(D)$ yielding a polarization of type $(\delta_1, \delta_2) = (1, 3)$ (hence, with Pfaffian $\delta = 3$), we want to get a Heisenberg-equivariant exact sequence

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^{\vee} \to \mathcal{E}' \to 0 \tag{2.79}$$

where $\mathfrak{H}'$ is a line bundle in $\mathrm{Pic}^0(A')$.

The first guess is to take $\mathfrak{H}' = \mathcal{O}_{A'}$, so that the inclusion $j : \mathcal{O}_{A'} \to \mathcal{L} \otimes V^{\vee}$ is given by a Heisenberg invariant section of $H^0(\mathcal{L} \otimes V^{\vee}) = V \otimes V^{\vee}$.

Since V is an irreducible representation, the only invariant by Schur's lemma corresponds to the identity of V, hence to the element $\sum_j x_j y_j$, where x_1, x_2, x_3 is a natural basis of V and y_1, y_2, y_3 is the dual basis.

In order to get a section of $\mathbf{Sym}^3(\mathcal{E}') \otimes (\det \mathcal{E}')^{-1}$ we use the surjection

$$\mathbf{Sym}^3(V^{\vee}) \otimes \mathcal{O}_{A'} \to \mathbf{Sym}^3(\mathcal{E}') \otimes (\det \mathcal{E}')^{-1},$$

hence the surjection $\mathbf{Sym}^3(V^{\vee}) \to H^0(\mathbf{Sym}^3(\mathcal{E}') \otimes (\det \mathcal{E}')^{-1})$, and we consider a cubic form $F(y) \in \mathbf{Sym}^3(V^{\vee})$.

Then $F(y) = 0$ defines a smooth cubic curve $C \subset \mathbb{P}^2 = \mathbb{P}(V)$, and we define

$$S' := \{(y, z) \in \mathbb{P}(V) \times A' \mid \sum_j y_j x_j(z) = 0,\ F(y) = 0\}. \tag{2.80}$$

The class of S' as a divisor inside the abelian variety $Z := C \times A'$ is $(H+D)_{|Z}$, where H denotes the hyperplane section on $\mathbb{P}(V) = \mathbb{P}^2$ (as usual, we use the same notation for a divisor and its pull-back).

So S' is an ample divisor inside Z, hence, by Lefschetz hyperplane theorem, $q(S') = 3$, and moreover

$$K_{S'}^2 = (H + D)^3(3H) = 9H^2D^2 = 9 \cdot 2\delta = 9 \cdot 6.$$

Thus, setting $G := \mathcal{K}(D)$, this calculation shows that $S = S'/G$ has $K_S^2 = 6$.

On the other hand, the exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z(H + D) \to \mathcal{O}_{S'}(K_{S'}) \to 0$$

shows that $\chi(S') = \chi(\mathcal{O}_Z(H + D)) = 9$, hence $\chi(S) = 1$.

We have taken for granted that S' is G-invariant and smooth, let's now show it.

First of all, C must be G-invariant, and since G is generated by a cyclical permutation $g_1 : y_1 \mapsto y_3 \mapsto y_2 \mapsto y_1$, follows that F is a linear combination of

$$\sum_j y_j^3, \quad y_1y_2y_3, \quad \sum_i y_i^2 y_{i+1}, \quad \sum_i y_i^2 y_{i-1}.$$

Since the other generator g_2 of G acts via the diagonal matrix with entries $1, \epsilon^2, \epsilon$ (ϵ being a primitive cubic root of unity), the above monomials are eigenvectors for respective eigenvalues $1, 1, \epsilon^2, \epsilon$. Hence, either

1. $F = \sum y_j^3 + \lambda y_1y_2y_3$, or
2. $F = \sum_i y_i^2 y_{i+1}$, or
3. $F = \sum_i y_i^2 y_{i-1}$.

Note that the third case is projectively equivalent to the second, via the projectivity ι_2 which exchanges the coordinates y_2, y_3, but the isomorphism does not preserve the action of G.

However, see [LS02], we can use the involution ι coming from the extended Heisenberg group $\mathcal{H}_3 \rtimes \langle\iota\rangle \cong \mathcal{H}_3 \rtimes \mathbb{Z}/2$ and such that $\iota(x_1, x_2, x_3) = (x_1, x_3, x_2)$: it is associated to an automorphism ι of A' which equals to multiplication by -1 for a suitable choice of the origin.

Now, the isomorphism $\iota \times \iota_2$ normalizes the action of the group G (sending each element of G to its inverse), and leaves the equation $\sum_i x_iy_i = 0$ invariant.

Hence, in the sequel we restrict ourselves to consider only the first and second case. In the first case C is a curve of the Hesse pencil of cubics, hence it is either smooth or the product of three linear forms, while in the second case C is smooth, since the three polynomials

$$y_1^2 + 2y_2y_3, \quad y_2^2 + 2y_1y_3, \quad y_3^2 + 2y_1y_2$$

cannot vanish simultaneously (one observes that $y_j \neq 0$ for all j, then $y_1 = 1$ implies $|y_2| = |y_3| = 2$, hence $1 = 8$, a contradiction).

2.9.1.a A Family of Surfaces with $p_g = q = 3,\ K^2 = 6,\ d = 3$

We have to show the smoothness of S' in the first case, that is when C is given by

$$F = \sum_j y_j^3 + \lambda y_1 y_2 y_3 = 0.$$

Here, we have a linear system on $\mathbb{P}(\mathcal{E}'^{\vee})$, which has as base-point set the intersection of $\mathbb{P}(\mathcal{E}'^{\vee})$ with $\mathcal{B} \times A'$, where $\mathcal{B}$ is the base-point set of the Hesse pencil, consisting of the 9-point orbit via the symmetric group $\mathfrak{S}_3$ of the points $(0, 1, -\epsilon^j)$, where ϵ is a primitive cubic root of 1.

By the first Bertini's theorem (see [Ber82], p. 26 , [Sev06], p. 207) it suffices to show smoothness at these points, and by cyclic symmetry it suffices to look at the points $(0, 1, \zeta) := (0, 1, -\epsilon^j)$.

Notice that we have a smooth point of S' if the gradients of the two equations are not proportional in (y, z). This certainly happens if z is a smooth point of the curve $D_y := \{z \mid \sum_j y_j x_j(z) = 0\}$.

Now, for a general A', the curve D_y is always irreducible, hence it has only a finite number of singular points. Making y vary in the finite set of the 9 base points of the Hesse pencil, we get only a finite set of points in the plane $\mathbb{P}^2$ with coordinates (x_1, x_2, x_3), image of A' under the finite morphism φ_D of degree 6 associated to $H^0(A', \mathcal{O}_{A'}(D))$.

These points must then satisfy the equation $x_2 + \zeta x_3 = 0$, coming from the equation $\sum_j x_j y_j = 0$. Moreover, the condition that the gradients are proportional means that the rank of the following matrix

$$N = \begin{pmatrix} \lambda\zeta & 3 & 3\zeta^2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

is at most 1. This leads to further equations

$$3x_1 - \lambda\zeta x_2 = x_3 - \zeta^2 x_2 = -\lambda x_3 + 3\zeta x_1 = 0.$$

We may set $x_1 = 1$, hence we get

$$x_2 = \frac{3}{\lambda\zeta}, \quad x_3 = \frac{3\zeta}{\lambda}.$$

Therefore, the point $(x_1, x_2, x_3) := (\lambda\zeta^2, 3\zeta, -3)$ is the only solution in the plane. Then z must be a singular point for the curve $D_{(0,1,\zeta)}$, and belong to the preimage of $(\lambda\zeta^2, 3\zeta, -3) \in \mathbb{P}^2$ via the degree 6 morphism

$$\varphi_D \colon A' \to \mathbb{P}^2$$

associated to $H^0(A', \mathcal{O}_{A'}(D))$.

This cannot happen for a general pair (A', D) since then the divisor

$$D' := \{x_2(z) + \zeta x_3(z) = 0\}$$

is smooth.

Therefore, we have shown that S' is smooth for a general $\lambda \in \mathbb{C}$ and a general pair (A', D). Furthermore, the action of G on a curve of the Hesse pencil is free, since g_1 has as fixed points only the three points $(1, \epsilon^j, \epsilon^{2j})$, $j = 0, 1, 2$, while g_2 has as fixed points only the coordinate points, and all these points do not belong to the general cubic C.

The conclusion is that G acts by translation via the nine 3-torsion points of C, hence $q(S) = 3$.

Summarizing, we have the following.

Proposition 2.58. *Let A' be an abelian surface with a divisor D yielding a polarization of type $(1, 3)$. Let $V := H^0(A', \mathcal{O}_{A'}(D))$ be the Schrödinger representation of the order 27 Heisenberg group $\mathcal{H}_3$ with a natural basis x_1, x_2, x_3 and denote by y_1, y_2, y_3 its dual basis. Then the equations*

$$S' := \{(y, z) \in \mathbb{P}(V) \times A' \mid \sum_j y_j x_j(z) = 0, \ \sum_j y_j^3 + \lambda y_1 y_2 y_3 = 0\} \subset \mathbb{P}^2 \times A' \quad (2.81)$$

yields, for a general $\lambda \in \mathbb{C}$ and a general pair (A', D), a family of minimal surfaces $S := S'/\mathcal{K}(D)$ of general type with $p_g = q = 3$, $K^2 = 6$ and having a surjective morphism $\alpha\colon S \to A$ of degree $d = 3$, where $A := \widehat{A}'$ and α is induced by the projection $\mathbb{P}^2 \times A' \to A'$ restricted to S'.

2.9.1.b A New Component, consisting of Surfaces with $p_g = q = 2$, $K_S^2 = 6$, $d = 3$.

The main point to establish in the second case, that is when C is the curve

$$C := \{F(y) := \sum_i y_i^2 y_{i+1} = 0\} \subset \mathbb{P}^2,$$

is that the surface S' is smooth (or has only Rational Double Points as singularities).

This is done in Theorem 0.2 of [CS22].

In this case the action is not free on C, since the coordinate points belong to C and are fixed for g_2. Thus, the quotient surface $S = S'/G$ has $q(S) = 2$ and $\chi(S) = 1$, as desired.

Let us therefore discuss the singularities of S', which has equations

$$S' := \{(y, z) \in \mathbb{P}(V) \times A' \mid \sum_j y_j x_j(z) = 0, \sum_i y_i^2 y_{i+1} = 0\}. \quad (2.82)$$

We have already shown that C is smooth.

We notice that a point (y, z) is a singular point of S' if and only if z is a singular point of the curve $D_y := \{z \mid \sum_j y_j x_j(z) = 0\}$ and the rows of the matrix

$$\begin{pmatrix} y_3^2 + 2y_1y_2 & y_1^2 + 2y_2y_3 & y_2^2 + 2y_1y_3 \\ x_1 & x_2 & x_3 \end{pmatrix} \quad (2.83)$$

are proportional. This means that

$$x := (x_1, x_2, x_3) = \nabla F(y), \qquad y := (y_1, y_2, y_3), \quad (2.84)$$

and we view x as a point of $(\mathbb{P}^2)^\vee =: \mathbb{P}'$, while $y \in \mathbb{P} := \mathbb{P}^2$.

Geometrically, this means that $x \in C^\vee$, and x represents a tangent line to C at y, hence y represents a line Λ_y tangent to $C^\vee$ at x.

Moreover, since z is a singular point of D_y, which is the inverse image under

$$\varphi_D \colon A' \to \mathbb{P}'$$

of the line Λ_y corresponding to y, we require that the line Λ_y is tangent at x to the branch curve $\mathcal{B}$ of φ_D. Hence, that $\mathcal{B}$ and $C^\vee$ are tangent.

Therefore, we have reached the conclusion that S' is smooth if $\mathcal{B}$ and $C^\vee$ intersect transversally.

The following is the content of Theorem 0.2 of [CS22]:

Theorem 2.59. *Let $\mathcal{B}$ be the branch curve of $\varphi_D : A' \to \mathbb{P}^2$, where D is a polarization of type $(1, 3)$ and the pair (A', D) is general.*

Then, if C is the plane curve $C := \{\sum_i y_i^2 y_{i+1} = 0\} \subset (\mathbb{P}^2)^\vee$, $\mathcal{B}$ intersects transversally the dual sextic curve $C^\vee$ and C intersects transversally the discriminant curve W of the linear system $|D|$.

Definition 2.60. We call *AC3 surface* a minimal surface S of general type with $p_g = q = 2$, $K^2 = 6$ and degree of the Albanese map $d = 3$, which is the étale quotient $S = S'/G$ of a surface

$$S' := \{(y, z) \in \mathbb{P}(V) \times A' \mid \sum_j y_j x_j(z) = 0, \quad \sum_i y_i^2 y_{i+1} = 0\} \subset \mathbb{P}^2 \times A',$$

where

- A' is a polarized abelian surface with a polarization $\mathcal{O}_{A'}(D)$ of type $(1, 3)$ and the pair (A', D) is general,
- $G := \mathcal{K}(D) \cong (\mathbb{Z}/3)^2$,
- $\{x_1, x_2, x_3\}$ is a natural basis of $V := H^0(A', \mathcal{O}_{A'}(D))$,
- y_1, y_2, y_3 are homogeneous coordinates of $\mathbb{P}^2 = \mathbb{P}(V)$ (dual basis of $\{x_1, x_2, x_3\}$).

Summarizing, in the light of Theorem 0.2 of [CS22] we get the following.

Theorem 2.61. *The three dimensional family of AC3 surfaces yields a new irreducible component of the moduli space of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree $d = 3$.*

Remark 2.62. For the reader's convenience, we point out here that this is the first known irreducible component with these invariants.

Furthermore, we can prove right away the following.

Theorem 2.63. *The irreducible component corresponding to AC3 surfaces is unirational.*

Proof. The argument is analogous to the one given in the proofs of Theorem 2.34 and Theorem 2.49.

Denoting by $\mathcal{A}_2^{(1,3)}$ the moduli space of $(1,3)$-polarized abelian surfaces, it is clear from the construction of the component $\mathcal{M}_{\mathrm{AC3}}$ of AC3 surfaces that there is a dominant rational map

$$\mathcal{A}_2^{(1,3)} \dashrightarrow \mathcal{M}_{\mathrm{AC3}}. \tag{2.85}$$

Since $\mathcal{A}_2^{(1,3)}$ is known to be unirational (see [Gri94]), we get right away our conclusion that $\mathcal{M}_{\mathrm{AC3}}$ is unirational.

□

2.9.2 The Case $d = \delta = 3$ with Nontrivial Homogeneous Bundle

Given an abelian surface A' with an ample divisor $\mathcal{L} = \mathcal{O}_{A'}(D)$ yielding a polarization of type $(\delta_1, \delta_2) = (1,3)$ (hence, with Pfaffian $\delta = 3$), the next option to get a sequence like (2.79) is to take $\mathfrak{H}' = \mathcal{O}_{A'}(M)$, a nontrivial line bundle in $\mathrm{Pic}^0(A')$.

Observe that the inclusion $\mathcal{O}_{A'}(M) \hookrightarrow \mathcal{L} \otimes V^\vee$ comes from a section

$$\xi \in H^0(\mathcal{L}(-M) \otimes V^\vee),$$

and there is a point $z \in A'$ such that, if t_z denotes the translation by z,

$$\xi \in t_z^* H^0(\mathcal{L}) \otimes V^\vee.$$

Then from the exact sequence (2.79), which in this case reads as

$$0 \to \mathcal{O}_{A'}(M) \to \mathcal{L} \otimes V^\vee \to \mathcal{E}' \to 0,$$

we get the following Eagon-Northcott exact sequence for $\mathbf{Sym}^3(\mathcal{E}')$:

$$0 \to \mathbf{Sym}^2(V^\vee) \otimes \mathcal{O}_{A'}(2D + M) \to \mathbf{Sym}^3(V^\vee) \otimes \mathcal{O}_{A'}(3D) \to \mathbf{Sym}^3(\mathcal{E}') \to 0. \tag{$+$}$$

By tensoring with $(\det \mathcal{E}')^{-1} = \mathcal{O}_{A'}(M - 3D)$ we get:

$$0 \to \mathbf{Sym}^2(V^\vee) \otimes \mathcal{O}_{A'}(-D + 2M) \to \mathbf{Sym}^3(V^\vee) \otimes \mathcal{O}_{A'}(M) \to \mathbf{Sym}^3(\mathcal{E}') \otimes \det(\mathcal{E}')^{-1} \to 0. \tag{$++$}$$

Since $\mathcal{O}_{A'}(M) \in Pic^0(A')$ is nontrivial, we know that $H^i(M) = 0$ for all i, and therefore

$$(i) \qquad H^0(\mathbf{Sym}^3(V^\vee) \otimes \mathcal{O}_{A'}(M)) = 0.$$

Furthermore, since D is ample, we have by Kodaira vanishing theorem that

$$(ii) \qquad H^1(\mathbf{Sym}^2(V^\vee) \otimes \mathcal{O}_{A'}(-D + 2M)) = 0.$$

Finally, relations (i) and (ii) together with the long exact cohomology sequence associated with $(++)$ imply that

$$H^0(\mathbf{Sym}^3(\mathcal{E}') \otimes \det(\mathcal{E}')^{-1}) = 0.$$

The conclusion is then the following.

Theorem 2.64. *The case* $d = \delta = 3$, $p_g = q = 2$ *occurs under the Generality Assumption 2.17 exactly for the family of AC3 surfaces.*

That is, all the minimal surfaces S *of general type with* $p_g = q = 2$, $K^2 = 6$, *with Albanese map of degree* 3 *and satisfying the Generality Assumption with Pfaffian* $\delta = 3$ *belong to the family described in Subsection 2.9.1.b, whose existence is proved in* [CS22]. *This family yields an irreducible component of the moduli space which is in particular unirational.*

Moreover, the only other minimal surfaces S *of general type with* $p_g = q$, $K^2 = 6$, *with* $\alpha : S \to A$ *a surjective morphism of degree* $d = 3$ *onto an abelian surface* A *and satisfying the Generality Assumption with Pfaffian* $\delta = 3$ *are the surfaces with* $p_g = q = 3$ *described in Subsection 2.9.1.a.*

2.10 The Case $d = 4$ under the Generality Assumption: an Example with $d = \delta = 4$ and Nonzero Homogeneous Bundle $\mathfrak{H}$.

Here we want to construct surfaces S with AP fulfilling the Generality Assumption 2.17 and having a surjective morphism $\alpha\colon S \to A$ of degree $d = 4$.

Hence, given such a surface S and setting $A' := \widehat{A}$, there is a polarization $\mathcal{L} = \mathcal{O}_{A'}(D)$ of type (δ_1, δ_2) and hence with Pfaffian $\delta := \delta_1\delta_2$. Considering the associated isogeny $\Phi_D\colon A' \to A$ with kernel $G := \mathcal{K}(D)$, since here $d = 4$ we have that the dual $\mathcal{E}$ of the Tschirnhaus bundle of α and its pull-back $\mathcal{E}' = (\Phi_D)^*\mathcal{E}$ have rank

$$\operatorname{rank}(\mathcal{E}) = \operatorname{rank}(\mathcal{E}') = 3,$$

and moreover there is a $\mathcal{H}_D$-equivariant exact sequence like $(\diamondsuit)$, namely

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathcal{E}' \to 0. \tag{2.86}$$

Since $\mathcal{L} \otimes V^\vee$ has rank δ, $\mathfrak{H}'$ has rank $\delta - 3$, and by definition it is a successive extension of line bundles in $\operatorname{Pic}^0(A')$.

Since for such a suface S the Gorenstein Assumption 2.6 holds, it follows that $S' := S \times_A A'$ is a subscheme of the $\mathbb{P}^2$-bundle

$$\mathbb{P}(\mathcal{E}'^\vee) \subset \mathbb{P}(V) \times A' = \mathbb{P}^{\delta-1} \times A'$$

given by Casnati-Ekedahl [CE96] by an embedding

$$\mathcal{F} \hookrightarrow S^2(\mathcal{E}'),$$

where $\mathcal{F}$ is a rank 2 locally free $\mathcal{O}_{A'}$-module with $\det \mathcal{F} = \det \mathcal{E}'$.

Considering sequence 2.86, we must have either

1. $\det(\mathcal{F}) = \det(\mathcal{E}') = \delta D$, or
2. $\det(\mathcal{F}) = \det(\mathcal{E}') = \delta D - M$, for M a nontrivial line bundle in $\operatorname{Pic}^0(A')$.

Again by [CE96] we have the following formula

$$c_2(\mathcal{F}) = K_{S'} - 2c_1(\mathcal{E}')^2 + 4c_2(\mathcal{E}')) = \delta^2\ K_S^2 - 2c_1(\mathcal{E}')^2 + 4c_2(\mathcal{E}')) = \delta^2(\ K_S^2 - 4), \tag{2.87}$$

where the last equality follows from

$$c_1(\mathcal{E}') = \delta D, \qquad c_2(\mathcal{E}') = \frac{\delta(\delta-1)}{2} D^2 = \delta^2(\delta - 1),$$

see Remark 2.19.

The first admissible value for δ is $\delta = 3$, which indeed corresponds to the family of PP4 surfaces, already described in detail in Section 2.7 and Section 2.8.

Hence, let us proceed to the next case, that is $\delta = 4$.

2.10.1 The Case $d = \delta = 4$: a Potential Example

Let A' be an abelian surface with an ample divisor $\mathcal{L} = \mathcal{O}_{A'}(D)$ yielding a polarization of type (δ_1, δ_2) with Pfaffian $\delta = 4$.

Here, since $\delta = 4$, we want to get a Heisenberg-equivariant exact sequence

$$0 \to \mathfrak{H}' \to \mathcal{L} \otimes V^\vee \to \mathcal{E}' \to 0, \tag{2.88}$$

where $\mathfrak{H}'$ is a line bundle in $\mathrm{Pic}^0(A')$.

There are two cases:

1. $\mathfrak{H}' = \mathcal{O}_{A'}$,
2. $\mathfrak{H}' = \mathcal{O}_{A'}(M)$, with M nontrivial in $\mathrm{Pic}^0(A')$.

In the first case, one has to choose a rank 2 locally free $\mathcal{O}_{A'}$-module $\mathcal{F}$ with an embedding

$$\mathcal{F} \hookrightarrow S^2(\mathcal{E}')$$

and such that

$$\det(\mathcal{F}) = \det(\mathcal{E}') = \delta D = 4D.$$

A natural choice is to take $\mathcal{F} := \mathcal{O}_{A'}(2D) \oplus \mathcal{O}_{A'}(2D)$ with the inclusion $\mathcal{F} \subset S^2(\mathcal{E}')$ being induced by a two dimensional subspace of $S^2(V^\vee)$.

Then, setting $G := \mathcal{K}(D)$, suppose that we have already constructed a G-invariant and smooth $S' \subset \mathbb{P}^3 \times A'$ and define $S := S'/G$. Hence, we have

$$K_S^2 = 6,$$

by formula (2.87) since $c_2(\mathcal{F}) = 4D^2 = 8\delta = 32$, and moreover

$$\chi(S) = \chi(S')/\delta^2 = \left(\frac{1}{2}c_1(\mathcal{E}')^2 - c_2(\mathcal{E}')\right)/\delta^2 = 1,$$

see [CE96], Proposition 5.3 *i*).

We show now that this case should occur for a polarization D of type $(1, 4)$, but without yielding $p_g = q = 2$.

The Construction of the Potential Example Consider the Heisenberg-equivariant sequence

$$0 \to \mathcal{O}_{A'} \to \mathcal{L} \otimes V^\vee \to \mathcal{E}' \to 0 \tag{2.89}$$

and the rank 2 locally free $\mathcal{O}_{A'}$-module

$$\mathcal{F} = \mathcal{O}_{A'}(2D) \oplus \mathcal{O}_{A'}(2D).$$

We define $S' \subset \mathbb{P}(V) \times A' = \mathbb{P}^3 \times A'$ as the complete intersection of three divisors of respective classes $2H, 2H, H + D$, where H denotes here the hyperplane section in $\mathbb{P}^3$ (as usual, we use the same notation for a divisor and its pull-back).

More precisely,

$$S' := \{(y, z) \in \mathbb{P}(V) \times A' \mid Q_1(y) = Q_2(y) = \sum_{j=1}^{4} y_j x_j(z) = 0\} \subset \mathbb{P}^3 \times A', \tag{2.90}$$

where $x_1, \dots, x_4$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$, $y_1, \dots, y_4$ denote the dual basis of $V^\vee$ and the subspace generated by the quadrics $Q_1(y)$, $Q_2(y)$ is Heisenberg invariant.

Hence, $S' \subset C \times A'$, where $C := Q_1 \cap Q_2$ is a normal elliptic quartic which is Heisenberg invariant.

We see immediately that the polarization D is of type $(1,4)$ since for type $(2,2)$ we would have $G \cong (\mathbb{Z}/2)^4$ acting faithfully on $\mathbb{P}^3 = \mathbb{P}(V)$, while there is no faithful action of $(\mathbb{Z}/2)^4$ on an elliptic curve C.

By classical formulae, see [Hul86], page 28, we have that by Heisenberg invariance the two quadratic equations are:

$$\begin{aligned} Q_1(y) &:= y_1^2 + y_3^2 + 2\lambda y_2 y_4 = 0, \\ Q_2(y) &:= y_2^2 + y_4^2 + 2\lambda y_1 y_3 = 0, \end{aligned} \qquad \lambda \neq 0, \pm 1, \pm i. \tag{2.91}$$

The group $G \cong (\mathbb{Z}/4)^2$ acts by translations on the normal elliptic curve C of degree 4, namely

$$C = \{y \in \mathbb{P}^3 = \mathbb{P}(V) \mid Q_1(y) = Q_2(y) = 0\},$$

hence the quotient surface $S = S'/G$ has $q(S) = 3$.

Thus, since $\chi(S) = 1$ and $K_S^2 = 6$, equations (2.90) together with (2.91) should yield a family of minimal surfaces of general type S with $p_g = q = 3$, $K^2 = 6$ and having a surjective morphism $\alpha\colon S \to A$ of degree $d = 4$ onto an abelian surface A.

The Smoothness of S': a Sketch of the Proof There remains to show that, for a general choice of $\lambda \in \mathbb{C}$ and a general pair (A', D), S' is smooth. Let's now sketch how to prove the smoothness of S'.

To this purpose we apply the Theorem of Bertini-Sard, and we need only to show that the singular locus of

$$\mathcal{S} := \{(y, z, \lambda) \in \mathbb{P}(V) \times A' \times \mathbb{C} \mid y_1^2 + y_3^2 + 2\lambda y_2 y_4 = y_2^2 + y_4^2 + 2\lambda y_1 y_3 = \sum_{j=1}^{4} y_j x_j(z) = 0\}$$

does not map surjectively onto the complex line $\mathbb{C}$ with coordinate λ.

We define $\mathcal{C} \subset \mathbb{P}^3 \times \mathbb{C}$ as the family of normal elliptic quartics given by the zero set

$$\begin{cases} Q_1(y, \lambda) := y_1^2 + y_3^2 + 2\lambda y_2 y_4 = 0 \\ Q_2(y, \lambda) := y_2^2 + y_4^2 + 2\lambda y_1 y_3 = 0 \end{cases} \tag{2.92}$$

Remark 2.65. $\mathcal{C}$ is birational to a smooth quartic surface $\mathcal{C}'$ in $\mathbb{P}^3$, defined by the following equation

$$(y_1^2 + y_3^2) y_1 y_3 = (y_2^2 + y_4^2) y_2 y_4,$$

since

$$Q_1(y, \lambda) = Q_2(y, \lambda) = 0 \qquad \Longleftrightarrow \qquad -2\lambda = \frac{y_1^2 + y_3^2}{y_2 y_4} = \frac{y_2^2 + y_4^2}{y_1 y_3}.$$

Hence, $\mathcal{S}$ is a hypersurface in the 4-fold $\mathcal{C} \times A'$.

We observe that

$$\operatorname{Sing}(\mathcal{S}) = \{(y, \lambda, z) \in \mathcal{C} \times A' \mid z \in \operatorname{Sing}(D_y) \text{ and } \operatorname{rank} M = 2\},$$

where $D_y := \{z \in A' \mid \sum_j y_j x_j(z) = 0\}$ and

$$M := \begin{pmatrix} 2y_1 & 2\lambda y_4 & 2y_3 & 2\lambda y_2 & 2y_2y_4 \\ 2\lambda y_3 & 2y_2 & 2\lambda y_1 & 2y_4 & 2y_1y_3 \\ x_1 & x_2 & x_3 & x_4 & 0 \end{pmatrix}.$$

Since for $\lambda \neq 0, \pm 1, \pm i$ the curve $\{Q_1(y,\lambda) = Q_2(y,\lambda) = 0\}$ is smooth, the first two rows M_1, M_2 of M are linearly independent and then, on an open set of $\mathcal{C}$, it must hold

$$x = y_1y_3 \cdot M_1 - y_2y_4 \cdot M_2.$$

Hence, writing x as a column, we have

$$x = \begin{pmatrix} y_1^2y_3 - \lambda y_2y_3y_4 \\ -y_2^2y_4 + \lambda y_1y_3y_4 \\ y_1y_3^2 - \lambda y_1y_2y_4 \\ -y_2y_4^2 + \lambda y_1y_2y_3 \end{pmatrix} =: \beta(y, \lambda).$$

That is, we have a rational map

$$\beta\colon \mathcal{C} \dashrightarrow \mathbb{P}^3$$

such that

$$x = \beta(y, \lambda) = \beta_0(y) + \lambda \beta_1(y),$$

where clearly

$$\beta_0(y) := {}^t(y_1^2y_3,\ -y_2^2y_4,\ y_1y_3^2,\ -y_2y_4^2),$$
$$\beta_1(y) := {}^t(-y_2y_3y_4,\ y_1y_3y_4,\ -y_1y_2y_4,\ y_1y_2y_3).$$

Recalling that

$$-2\lambda = \frac{y_1^2 + y_3^2}{y_2y_4} = \frac{y_2^2 + y_4^2}{y_1y_3},$$

we can write

$$x = 2\beta_0(y) - \frac{y_1^2 + y_3^2}{y_2y_4}\beta_1(y) = 2y_2y_4\beta_0(y) - (y_1^2 + y_3^2)\beta_1(y),$$

and this shows that the rational map β is given by homogeneous polynomials of degree 5.

Recall also that, for a general pair (A', D), the image Σ of the map associated to the linear system $|D|$, namely

$$x\colon A' \to \Sigma \subset \mathbb{P}^3, \qquad z \mapsto (x_1(z),\ x_2(z),\ x_3(z),\ x_4(z)),$$

is an octic surface in $\mathbb{P}^3$ whose equation depends on some $c = (c_0, \dots, c_3) \in \mathbb{P}^3$ (see [BLvS89]).

Let $\Delta \subset \mathbb{P}^3$ be the discriminant of the linear system $|D|$, namely

$$\Delta := \{y \mid D_y \text{ is singular}\}.$$

Hence, we define the following divisors in $\mathcal{C}$

$$N_1 := \beta^{-1}(\Sigma), \qquad N_2 := \mathcal{C} \cap \Delta$$

(N_2 is the birational inverse image of $\mathcal{C}' \cap \Delta$).

Moreover, we have the equation (asserting that $x = \beta(y, \lambda)$ belongs to the plane $y^{\perp}$)

$$y \cdot \beta(y, \lambda) = 0,$$

defining

$$N_3 := \{(y, \lambda) \in \mathcal{C} \mid y \cdot \beta(y, \lambda) = 0\} \subset \mathcal{C}.$$

Remark 2.66. A straightforward computation shows that actually $N_3 = \mathcal{C}$. In fact:

$$y \cdot \beta(y, \lambda) = 0 \iff y_1^3 y_3 - y_2^3 y_4 + y_1 y_3^3 - y_2 y_4^3 = 0 \iff (y_1^2 + y_3^2) y_1 y_3 = (y_2^2 + y_4^2) y_2 y_4.$$

Therefore, $\operatorname{Sing}(\mathcal{S})$ does not map onto $\mathbb{C}$ if

$$|N_1 \cap N_2| < \infty.$$

Remark 2.67. (a) Certainly, N_2 is a curve, since the surfaces Σ vary, hence their discriminants, and $\mathcal{C}$, Σ are irreducible. By a similar argument also N_1 is a curve in $\mathcal{C}$ (Σ moves).

(b) We should also impose the condition that $x^{\perp}$ is tangent to Δ at y.

Since Δ is the dual surface to Σ, this condition means that $y^{\perp}$ is tangent to Σ at x.

It suffices in any case to show that $N_1 \cap N_2$ is a finite set for a general A'.

Let $F(c, x) = 0$, for $c \in \mathbb{P}^3, x \in \mathbb{P}^3$, be the equation of the family of octics Σ_c, given in [BLvS89]; let $p(y) = 0,\ y \in \mathbb{P}^3$, be the equation for the surface $\mathcal{C}'$.

Since Δ is the dual surface of Σ, we denote by ∇F the gradient with respect to the variables x, and we set

$$y = \nabla F(c, x).$$

Consider then the three equations

$$\begin{aligned} &F(c, x) = 0, \\ &p' := p(\nabla F(c, x)) = 0, \\ &F' := F(c, \beta(\nabla F(c, x)) = 0, \end{aligned} \tag{$\star\star\star$}$$

which, for a general $c \in \mathbb{P}^3$, describe the set $N_1 \cap N_2$.

Since our aim is to show that, for a general c, we have a finite number of solutions, we view the equations ($\star\star\star$) as equations on $\mathbb{P}^3 \times \mathbb{P}^3$, describing

$$W := \{(c, x) \in \mathbb{P}^3 \times \mathbb{P}^3 | F(c, x) = p'(c, x) = F'(c, x) = 0\}.$$

Hence, our claim is equivalent to showing that the components of W which dominate the $\mathbb{P}^3$ with coordinates c have dimension 3.

We therefore need to calculate the Jacobian matrix of the vector valued function (F, p', F'), and show that this matrix has generically rank equal to three.

This should be done by a computer algebra program. I have been developing a script which is still incomplete at the time I am writing.

2.11 The Degree of the Albanese Map of the *UnMix* Components of [Pen11]

In [Pen13] the author points outs out that the three families of surfaces found in [Pen11] and listed in Table 1 ibidem as "UnMix" (see also Table A, items n. $15, 16, 17$) have Albanese map of degree $d \leq 6$. They yield three irreducible connected components of the moduli space and consist of surfaces isogenous to a product of curves with Albanese surface isogenous to a product of elliptic curves. In particular, these components are not of the Main Stream.

In this section we compute the degree d for each of these families, showing that $d = 4, 6, 4$ (using the order of Table 1 in [Pen11]).

Recall that a surface S is said to be *isogenous to a product of curves* if

$$S = (C_1 \times C_2)/G,$$

where C_i, $i = 1, 2$, is a smooth projective curve of genus $g_i := g(C_i) \geq 1$ and G a finite group acting freely on the product $C_1 \times C_2$. Surfaces isogenous to a product were introduced by Catanese in [Cat00] and they are of general type if and only if $g_i \geq 2$, $i = 1, 2$.

Remark 2.68. A surface $S = (C_1 \times C_2)/G$ isogenous to a product of curves is always minimal since it does not contain any smooth rational curve. Indeed, assuming by contradiction the converse would imply that the product $C_1 \times C_2$ contains a smooth rational curve and this is not possible since C_1, C_2 are irrational by definition.

There are two possibilities for the action of G:

1. there exists an automorphism of G exchanging the two factors (in this case it must be $C \cong C_1 \cong C_2$) and S is said to be of *mixed type*;
2. G acts faithfully on both curves C_i and diagonally on their product $C_1 \times C_2$, i.e., it acts as $\Delta_G \subset G \times G$; in this case S is said to be of *unmixed type.*

The families of surfaces we shall treat in this section are of unmixed type, and hence we tacitly assume that case 2. holds.

Moreover, we denote by Σ_i and Σ the subsets of G consisting of those transformations (different from the identity) having some fixed points on C_i, respectively on $C_1 \times C_2$. Note that by definition $\Sigma = \Sigma_1 \cap \Sigma_2$, and moreover, since $G \cong \Delta_G \subset G \times G$ acts freely on the product $C_1 \times C_2$, it must hold

$$\Sigma = \Sigma_1 \cap \Sigma_2 = \emptyset.$$

Proposition 2.69. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product of unmixed type. Then it holds*

$$q(S) = g(C_1/G) + g(C_2/G).$$

Proof. Setting $p_i\colon C_1 \times C_2 \to C_i$ for the natural projection to C_i, we recall the following fact ([Bea96, Fact III.22])

$$\Omega^1_{C_1 \times C_2} \cong p_1^*(\Omega^1_{C_1}) \oplus p_2^*(\Omega^1_{C_2}),$$

and then observe that

$$\begin{aligned} H^0(\Omega^1_{C_1\times C_2}) &\cong H^0(p_1^*(\Omega^1_{C_1}) \oplus p_2^*(\Omega^1_{C_2})) \cong H^0(p_1^*(\Omega^1_{C_1})) \oplus H^0(p_2^*(\Omega^1_{C_2})) \\ &\cong H^0(p_1^*(\Omega^1_{C_1}) \otimes p_2^*(\mathcal{O}_{C_2})) \oplus H^0(p_1^*(\mathcal{O}_{C_1}) \otimes p_2^*(\Omega^1_{C_2})) \\ &\cong \Big(H^0(C_1, \Omega^1_{C_1}) \otimes H^0(C_2, \mathcal{O}_{C_2})\Big) \oplus \Big(H^0(C_1, \mathcal{O}_{C_1}) \otimes H^0(C_2, \Omega^1_{C_2})\Big), \end{aligned}$$

where the last isomorphism follows by Künneth formula.

Hence, we get

$$\begin{aligned} q(S) = h^0(S, \Omega^1_S) &= \dim H^0(C_1 \times C_2, \Omega^1_{C_1\times C_2})^G \\ &= \dim H^0(C_1, \Omega^1_{C_1})^G + \dim H^0(C_2, \Omega^1_{C_2})^G = g(C_1/G) + g(C_2/G), \end{aligned}$$

where the second and the last equalities follow from [Bea96, Lemma VI.11].

□

Define Γ as the subgroup of $G \times G$ normally generated by the set $\Delta_G \cup \big(\Sigma_1 \times \{1\}\big) \cup \big(\{1\} \times \Sigma_2\big)$, namely

$$\Gamma := \langle\langle \Delta_G \cup \big(\Sigma_1 \times \{1\}\big) \cup \big(\{1\} \times \Sigma_2\big)\rangle\rangle \trianglelefteq G \times G, \tag{2.93}$$

and assume moreover that $S = (C_1 \times C_2)/G$ is of maximal Albanese dimension.

Then it follows from Proposition 2.69 that

$$q(S) = 2 \qquad \Longleftrightarrow \qquad g(C_1/G) = g(C_2/G) = 1.$$

We can now prove the following proposition, which is inspired by the method used in [Pig20].

Proposition 2.70. *Let $S = (C_1 \times C_2)/G$ be a surface of general type isogenous to a product of unmixed type. Assume moreover that $q(S) = 2$ and that the Albanese map $\alpha\colon S \to A$ is surjective. Then the Albanese surface is*

$$A = \mathrm{Alb}(S) = (C_1 \times C_2)/\Gamma,$$

and moreover,

$$d := \deg \alpha = |\Gamma|/|G|.$$

Proof. Let us consider the following commutative diagram

$$\begin{tikzcd} & C_1 \times C_2 \arrow[dl, "f"'] \arrow[dr] & \\ S = (C_1 \times C_2)/\Delta_G \arrow[rr] \arrow[dr, "\alpha"'] & & (C_1 \times C_2)/(G \times G) = E_1 \times E_2 \\ & A \arrow[ur, "\pi"'] & \end{tikzcd} \tag{2.94}$$

where $E_i := C_i/G$ is an elliptic curve by Proposition 2.69.

Note that all maps are finite and the existence of the map $\pi\colon A \to E_1 \times E_2$ is guaranteed by the universal property of the Albanese variety.

Still, $\pi\colon A \to E_1 \times E_2$ is a morphism between abelian surfaces, hence an isogeny with kernel $H := \ker(\pi)$.

The map $C_1 \times C_2 \to E_1 \times E_2$ is also Galois with Galois group $G \times G$, whence the morphism

$$g := \alpha \circ f\colon\ C_1 \times C_2 \to A$$

is Galois with Galois group K such that $(G \times G)/K = H$ (see for instance [Pig20, Lemma 4.1]). In other words,

$$A = (C_1 \times C_2)/K, \qquad \Delta_G \leq K \trianglelefteq G \times G.$$

Moreover, $A = (C_1 \times C_2)/K \to E_1 \times E_2$ is étale, i.e.,

$$\mathrm{Stab}_K(p) = \mathrm{Stab}_{G\times G}(p) \quad \text{for all} \quad p \in C_1 \times C_2.$$

Therefore, since a finite étale cover of an abelian surface is an abelian surface, it follows from the universal property of the Albanese variety that K is the smallest normal subgroup $K \trianglelefteq G \times G$ such that

i. $\Delta_G \leq K$,

ii. $\mathrm{Stab}_K(p) = \mathrm{Stab}_{G\times G}(p) \quad \text{for all} \quad p \in C_1 \times C_2$.

Since K fulfilling condition *ii.* amounts to requiring that

$$K \supset \Sigma_1 \times \Sigma_2,$$

it turns out that K is the subgroup of $G \times G$ normally generated by $\Delta_G \cup \big(\Sigma_1 \times \Sigma_2\big)$, or equivalently by $\Delta_G \cup \big(\Sigma_1 \times \{1\}\big) \cup \big(\{1\} \times \Sigma_2\big)$.

Thus, $K = \Gamma$ and we are done.

□

In light of the previous proposition, the Galois closure of the Albanese map

$$\alpha\colon S = (C_1 \times C_2)/\Delta_G \to A = (C_1 \times C_2)/\Gamma$$

is given by the *normal core of* Δ_G *in* Γ, i.e., the biggest subgroup $\mathrm{Core}_\Gamma(\Delta_G)$ contained in Δ_G which is normal in Γ, namely

$$\mathrm{Core}_\Gamma(\Delta_G) := \bigcap_{(g_1,g_2)\in\Gamma} (g_1,g_2)\Delta_G(g_1,g_2)^{-1}. \tag{2.95}$$

2.11.1 The Case $G = (\mathbb{Z}/2)^2$

In this case (see item n. 15 of Table A in Appendix A) we have the following data (cf. [Pen10, p. 79]):

- $g_1 = g(C_1) = 3$, $g_2 = g(C_2) = 3$,
- $G \cong \langle \sigma_1 \rangle \oplus \langle \sigma_2 \rangle = \{\sigma_1, \sigma_2, \sigma_1\sigma_2, \mathrm{id}\} \cong (\mathbb{Z}/2)^2$.

The action is given as follows: on C_i the automorphism σ_i fixes 4 points, while σ_{i+1}, $\sigma_1\sigma_2$ act freely (here the subscript has to be understood modulo 2). In particular, G acts on C_i yielding two branch points of multiplicity 2.

In other words, we have that

$$\Sigma_i = \{\sigma_i\}, \qquad \Sigma = \emptyset,$$

and hence the action of $G \cong \Delta_G \cong (\mathbb{Z}/2)^2$ is free.

Setting

$$E_i := C_i/G, \quad E_i' := C_i/\langle\sigma_i\rangle, \quad D_i' := C_i/\langle\sigma_{i+1}\rangle, \quad D_i'' := C_i/\langle\sigma_1\sigma_2\rangle,$$

we have a diagram as follows

$$\begin{tikzcd}
 & E_i' = C_i/(\mathbb{Z}/2) \arrow[dr, "\text{étale}"] & \\
C_i \arrow[ur, "\langle\sigma_i\rangle"] \arrow[r, "{\langle\sigma_1\sigma_2\rangle,\ \text{étale}}"] \arrow[dr, "{\langle\sigma_{i+1}\rangle,\ \text{étale}}"'] & D_i' = C_i/(\mathbb{Z}/2) \arrow[r] & E_i \\
 & D_i'' = C_i/(\mathbb{Z}/2) \arrow[ur] &
\end{tikzcd} \tag{2.96}$$

where E_i, E_i' are elliptic curves and D_i', D_i'' have genus 2.

Thus, we have in this case

$$\Gamma \supset \{(\sigma_1, \mathrm{id}),\ (\mathrm{id}, \sigma_2)\} \cup \Delta_G \qquad \Longrightarrow \qquad \Gamma = G \times G.$$

In light of Proposition 2.70, this implies in particular that

$$d = |(\mathbb{Z}/2)^2 \times (\mathbb{Z}/2)^2| / |(\mathbb{Z}/2)^2| = 4.$$

Finally, note that, since

$$G \text{ is abelian} \qquad \Longleftrightarrow \qquad \Delta_G \trianglelefteq G \times G,$$

the Galois closure of the Albanese map is α itself, with Galois group $(\mathbb{Z}/2)^2$.

In conclusion, we summarize here what we have shown in this subsection.

Proposition 2.71. *The irreducible and connected component of the moduli space of surfaces of general type with $p_g = q = 2$ and $K^2 = 8$ corresponding to item n. 15 of Table A in Appendix A has degree of the Albanese map $d = 4$.*

Moreover, for each surface $S = (C_1 \times C_2)/(\mathbb{Z}/2)^2$ in this component, the Albanese map

$$\alpha \colon S \to A = (C_1 \times C_2)/((\mathbb{Z}/2)^2 \times (\mathbb{Z}/2)^2) = E_1 \times E_2$$

is Galois with Galois group $(\mathbb{Z}/2)^2$.

2.11.2 The Case $G = \mathfrak{S}_3$

In this case (see item n. 16 of Table A in Appendix A) we have the following data (cf. [Pen10, p. 79])

- $g_1 = g(C_1) = 3$, $g_2 = g(C_2) = 4$,
- $G \cong \mathfrak{S}_3 = \langle x, y \mid x^2 = y^3 = (xy)^2 = 1 \rangle$.

$G \cong \mathfrak{S}_3$ acts as follows:

- on C_1 yielding one branch point of multiplicity 3 and
$$\Sigma_1 = \{y, y^2\} \cong (\mathbb{Z}/3\mathbb{Z})^*,$$
- on C_2 yielding two branch points of multiplicity 2 and
$$\Sigma_2 = \{x, xy, xy^2\}.$$

Since $\Sigma_1 \cap \Sigma_2 = \emptyset$, the action of $G \cong \Delta_G \cong \mathfrak{S}_3$ on $C_1 \times C_2$ is free.
Moreover, we have the following diagrams

$$\begin{tikzcd}
 & C_1/(\mathbb{Z}/3) \arrow[dr, "\text{étale}"] & \\
C_1 \arrow[ur, "\langle y \rangle"] \arrow[rr] \arrow[dr, "\text{étale}", "{\langle x \rangle, \langle xy \rangle, \langle xy^2 \rangle}"'] & & E_1 \\
 & C_1/(\mathbb{Z}/2) \arrow[ur] &
\end{tikzcd} \tag{2.97}$$

$$\begin{tikzcd}
 & C_2/(\mathbb{Z}/3) \arrow[dr] & \\
C_2 \arrow[ur, "\langle y \rangle", "\text{étale}"'] \arrow[rr] \arrow[dr, "{\langle x \rangle, \langle xy \rangle, \langle xy^2 \rangle}"'] & & E_2 \\
 & C_2/(\mathbb{Z}/2) \arrow[ur, "{\text{étale, non-Galois}}"'] &
\end{tikzcd} \tag{2.98}$$

Since $\{1\} \times \Sigma_2$ generates $\{1\} \times \mathfrak{S}_3$, we have that

$$\Gamma = \langle \langle \Delta_G \cup \big(\Sigma_1 \times \{1\}\big) \cup \big(\{1\} \times \Sigma_2\big) \rangle \rangle = \mathfrak{S}_3 \times \mathfrak{S}_3,$$

and then, by Proposition 2.70,

$$d = |\mathfrak{S}_3 \times \mathfrak{S}_3| / |\mathfrak{S}_3| = 6.$$

Furthermore, we observe that in this case the Albanese map

$$\alpha \colon S = (C_1 \times C_2)/\Delta_G \to A = (C_1 \times C_2)/(G \times G) = E_1 \times E_2$$

is not Galois since $G = \mathfrak{S}_3$ is not abelian (and hence, Δ_G is not normal in $G \times G$).

The Galois closure of the Albanese map α is then given by (2.95), namely

$$\mathrm{Core}_{\mathfrak{S}_3 \times \mathfrak{S}_3}(\Delta_{\mathfrak{S}_3}) := \bigcap_{(g_1, g_2) \in \mathfrak{S}_3 \times \mathfrak{S}_3} (g_1, g_2)\Delta_{\mathfrak{S}_3}(g_1^{-1}, g_2^{-1}) = \Delta_{Z(\mathfrak{S}_3)} = \{(1,1)\},$$

and hence corresponds to the the map

$$C_1 \times C_2 \to (C_1 \times C_2)/(\mathfrak{S}_3 \times \mathfrak{S}_3) = E_1 \times E_2.$$

with Galois group $G \times G = \mathfrak{S}_3 \times \mathfrak{S}_3$.

In conclusion, we summarize here what we have shown in this subsection.

Proposition 2.72. *The irreducible and connected component of the moduli space of surfaces of general type with $p_g = q = 2$ and $K^2 = 8$ corresponding to item n.* 16 *of Table A in Appendix A has Albanese map of degree $d = 6$.*

Moreover, for each surface $S = (C_1 \times C_2)/\mathfrak{S}_3$ in this component, the Albanese map

$$\alpha \colon S \to A = (C_1 \times C_2)/(\mathfrak{S}_3 \times \mathfrak{S}_3) = E_1 \times E_2$$

is not Galois and its Galois closure is the map

$$C_1 \times C_2 \to E_1 \times E_2$$

with Galois group $\mathfrak{S}_3 \times \mathfrak{S}_3$.

2.11.3 The Case $G = D_4$

In this case (see item n. 17 of Table A in Appendix A) we have the following data (cf. [Pen10, p. 79]):

- $g_1 = g(C_1) = 3$, $g_2 = g(C_2) = 5$,
- $G \cong D_4 = \langle x, y, z \mid x^2 = y^2 = z^2 = 1,\ z = [x, y],\ [x, z] = [y, z] = 1 \rangle$.

$G \cong D_4$ acts as follows:

- on C_1 yielding one branch point of multiplicity 2 and

$$\Sigma_1 = \{z\} \cong (\mathbb{Z}/2\mathbb{Z})^*,$$

- on C_2 yielding two branch points of multiplicity 2 and

$$\Sigma_2 = \{y, yz\}.$$

Since $\Sigma_1 \cap \Sigma_2 = \emptyset$, the action of $G \cong \Delta_G \cong D_4$ on $C_1 \times C_2$ is free. More precisely, we have

$$\begin{tikzcd}
C_1 \arrow[r] & E_1' = C_1/\langle z\rangle \arrow[r, "\text{étale}", "(\mathbb{Z}/2)^2"'] & E_1
\end{tikzcd}$$

$$\begin{tikzcd}
C_2 \arrow[r] & E_2' = C_2/\langle y\rangle \arrow[r, "\text{étale}", "(\mathbb{Z}/2)^2"'] & E_2
\end{tikzcd}$$

where E_i' are elliptic curves and z fixes 4 points on C_1, whereas y fixes 8 points on C_2.

Here, the group Γ defined in (2.93)) is generated by

$$\Delta_G \,\cup\, \{(z,1)\} \,\cup\, \{(1,y),(1,yz)\}. \tag{2.99}$$

Observe that, setting $H := \langle y, z\rangle$, the subset $\{(1,y),(1,yz)\}$ generates the group

$$\big(\{1\}\times H\big) \cong (\mathbb{Z}/2)^2.$$

Then we have that

$$\Gamma = \langle\langle \Delta_G \,\cup\, \big(\Sigma_1\times\{1\}\big) \,\cup\, \big(\{1\}\times\Sigma_2\big)\rangle\rangle = \langle\langle \Delta_G \,\cup\, \big(\{1\}\times\Sigma_2\big)\rangle\rangle = \Delta_G\cdot\big(\{1\}\times H\big),$$

where the last equality holds since $\{1\}\times H$ is normal in $D_4\times D_4$.

Therefore,

$$|\Gamma| = |\Delta_G\cdot(\{1\}\times H)| = 32,$$

and by Proposition 2.70 we get

$$d = |\Gamma|/|D_4| = 32/8 = 4.$$

Still, since Δ_G is not normal in $\Gamma = \Delta_G\cdot\big(\{1\}\times H\big)$, the Albanese map α is not Galois and its Galois closure is then given by the normal core

$$\mathrm{Core}_\Gamma(\Delta_{D_4}) = \Delta_H \cong (\mathbb{Z}/2)^2, \tag{2.100}$$

and it corresponds to the map

$$(C_1\times C_2)/\Delta_H \to A = \mathrm{Alb}(S) = (C_1\times C_2)/\Gamma$$

with Galois group Γ/Δ_H.

Since

$$\Gamma/\Delta_H = \{[(1,1)],\ [(1,z)],\ [(1,y)],\ [(1,yz)], [(x,x)],\ [(x,xy)],\ [(x,xz)],\ [(x,yx)]\},$$

it is easy to see that Γ/Δ_H is a non-abelian group of order 8 with 2 elements of order 4, namely

$$[(x,xy)], \qquad [(x,yx)],$$

and 5 elements of order 2, namely

$$[(1,z)],\ [(1,y)],\ [(1,yz)],\ [(x,x)],\ [(x,xz)],$$

and hence

$$\Gamma/\Delta_H \cong D_4.$$

In conclusion, we summarize here what we have shown in this subsection.

Proposition 2.73. *The irreducible and connected component of the moduli space of surfaces of general type with $p_g = q = 2$ and $K^2 = 8$ corresponding to item n.* 17 *of Table A in Appendix A has Albanese map of degree $d = 4$.*

Moreover, for each surface $S = (C_1 \times C_2)/D_4$ in this component, the Albanese surface $A = \mathrm{Alb}(S)$ is isogenous to the product

$$E_1 \times E_2 = (C_1 \times C_2)/(D_4 \times D_4),$$

and the Albanese map $\alpha\colon S \to A$, which is not Galois, has Galois closure with Galois group D_4.

2.12 What is left to do? Some Open Research Questions

As stated at the beginning of this chapter, our aim is to construct surfaces S with AP (see Definition 2.1) fulfilling the Generality Assumption 2.17.

By exploiting our construction method [AC22] we analyzed cases where the surjective morphism $\alpha\colon S \to A$ has degree $d = 3, 4$ and the Pfaffian δ of the polarization D is $\delta = 2, 3, 4$.

More precisely, here are our results:

(I) $d = 3$, $\delta = 2$: CHPP surfaces;

(II) $d = \delta = 3$: AC3 surfaces and the family with $p_g = q = 3$ described in Subsection 2.9.1.a;

(III) $d = 4$, $\delta = 3$: PP4 surfaces;

(IV) $d = \delta = 4$: there is a potential example with $p_g = q = 3$ (Section 2.10).

Note that the previous list suggests the following natural questions.

Question 2.74. (1) Are there surfaces S with AP fulfilling the Generality Assumption 2.17 with $d = 3$ and $\delta \geq 4$? If so, can we classify them?

(2) Is the potential example mentioned in (IV) indeed an example?

(3) Can we thoroughly understand the case $d = \delta = 4$?

(4) Does the case $d = 4$, $\delta \geq 5$ occur?

Still, we recall that our construction method is based on the theory by Casnati-Ekedahl for covers of small degree $d = 3, 4$, and this is the reason why all our examples have such a degree. However, one might ask the following question.

Question 2.75. Are there surfaces S with AP fulfilling the Generality Assumption 2.17 with $d \geq 5$?

Here the main drawback is that no structure theorem for covers of degree $d \geq 6$ is known. Anyhow, for $d = 5$ there are some results contained in [Cas96] which might be helpful towards this direction.

Another interesting question arises from Section 2.2. Considering a surface S with AP whose surjective morphism $\alpha\colon S \to A$ has degree $d \geq 3$, there is always a rational map $\psi\colon X \dashrightarrow \mathbb{P}(\mathcal{E}^\vee)$ as in (2.12), where X is the canonical model of S and $\mathcal{E}$ is the dual of the Tschirnhaus bundle of

Question 2.76. When is ψ a morphism? Is it so when its image Z is normal?

Finally, we showed that the three components of CHPP, PP4 and AC3 surfaces are unirational (see Theorem 2.34, Theorem 2.49 and Theorem 2.63). The natural question is then the following.

Question 2.77. Are the moduli spaces of CHPP surfaces, PP4 surfaces and AC3 surfaces rational?

Chapter 3

Semi-projective Representations and Twisted Representation Groups

In [Sch04], Schur developed the theory of *projective representations*, which are homomorphisms from a group G to the group of projective transformations $\mathrm{PGL}(V)$. Here, G is a finite group, K a field and V a non-trivial finite dimensional K-vector space. It is clear that every ordinary representation induces a projective representation. However, the converse is in general not true, more precisely the obstructions to lift are the elements of the second cohomology group $H^2(G, K^*)$, where K^* is considered as a trivial G-module. In order to study projective representations via ordinary representations in the case $K = \mathbb{C}$, Schur showed the existence of a *representation group* Γ, which is a particular kind of central extension of G having the property that all projective representations of G lift to ordinary representations of Γ.

An example of such a group Γ is provided by the Heisenberg group $\mathcal{H}_r$ of the cyclic group $\mathbb{Z}/r$, defined by a sequence as follows

$$1 \to \mu_r \to \mathcal{H}_r \to (\mathbb{Z}/r)^2 \to 0,$$

see Chapter 1, Section 1.3. Indeed, this is a well-known fact.

Recently, the authors of [DG23] and [GK22]) have constructed certain quotients of complex tori by holomorphic actions of finite groups and investigate their homeomorphism and biholomorphism classes. Under mild assumptions on the fixed loci of the actions, *Bieberbach's theorems* about crystallographic groups (see [Cha86, I]) imply that homeomorphisms and biholomorphisms of such quotients are induced by affine transformations. When determining the linear parts of these transformations, one come across an object similar to a projective representation, namely a homomorphism from a finite group to $\mathrm{PGL}(n, \mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Moreover, they had to determine a particular kind of lift of this map to $\mathrm{GL}(n, \mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

This example served for the authors of [AGK23] as a motivation to extend Schur's theory to *semi-projective representations*, i.e., homomorphisms from a finite group G to the group of semi-projective transformations $\mathrm{P\Gamma L}(V)$. Here, $\mathrm{P\Gamma L}(V)$ is defined as the quotient of the group of semi-linearities

$$\Gamma\mathrm{L}(V) \simeq \mathrm{GL}(V) \rtimes \mathrm{Aut}(K)$$

modulo the action of the multiplicative group K^*. A semi-projective representation yields an action φ of G on K by automorphisms. In this way, K^* becomes a G-module and we can consider the *second cohomology group* $H^2(G, K^*)$ with respect to this action. In analogy to the projective case, this group plays an important role since it is the obstruction space of the *lifting problem* of semi-projective representations to semi-linear representations, i.e., homomorphisms from G to $\Gamma\mathrm{L}(V)$.

As a main result, in [AGK23] it is showed that if K is algebraically closed, then for any given action φ of a finite group G, there exists a finite *φ-twisted representation group* Γ, which has the property that any semi-projective representation inducing the action φ admits a semi-linear lift to Γ. Despite the fact that Γ is in general not unique, it has minimal order among all groups enjoying the lifting property. This allows us to study semi-projective representations of G via semi-linear representations of Γ.

It is also given a *cohomological characterization* of a group Γ to be a φ-twisted representation group, which reduces to the classical description of a representation group in the case where the action φ is trivial.

In general, it seems to be difficult to determine explicitly a φ-twisted representation group, even in the projective case, i.e., where φ is trivial. Indeed, there is a vast amount of literature dedicated to this problem for specific classes of groups, e.g. [Sch11], [Kar85, Section 3.7] or the more recent article [HaSi21]. In [AGK23] the authors approach this problem in the semi-projective case via an algorithm for the case $K = \mathbb{C}$ under the assumption that φ takes values in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. This algorithm produces all φ-twisted representation groups of a given finite group G and a given action φ.

Apart from the *algebro-geometric* application to torus quotients, there are other situations where semi-projective representations arise naturally, for example in *Clifford theory*: in [Isa81], Isaacs developed the concept of crossed-projective representations, which is analogous to our notion of semi-projective representations, in order to study the problem of extending G-invariant irreducible L-representations defined on a normal subgroup $N \trianglelefteq G$ to the ambient group G for arbitrary fields L. In the section dedicated to applications and examples, we briefly review Isaacs' work and rephrase it in our language.

3.1 General Setting

In this section, we introduce semi-linear and semi-projective representations. Throughout this chapter V is a non-trivial finite-dimensional K-vector space and G a finite group.

Definition 3.1. A bijective map $f\colon V \to V$ is called a *semi-linear transformation* if there exists an automorphism $\varphi_f \in \mathrm{Aut}(K)$ such that for all $v, w \in V$ and all $\lambda \in K$, it holds:

$$f(v+w) = f(v) + f(w) \qquad \text{and} \qquad f(\lambda v) = \varphi_f(\lambda) f(v).$$

The set of all semi-linear transformations of V forms a group, which is denoted by $\Gamma\mathrm{L}(V)$.

In the following remark we collect some basic properties describing the structure of $\Gamma\mathrm{L}(V)$.

Remark 3.2. (1) The group $\Gamma\mathrm{L}(V)$ contains $\mathrm{GL}(V)$ as a normal subgroup and sits inside the following short exact sequence

$$1 \longrightarrow \mathrm{GL}(V) \longrightarrow \Gamma\mathrm{L}(V) \longrightarrow \mathrm{Aut}(K) \longrightarrow 1.$$

This sequence splits, i.e., $\Gamma\mathrm{L}(V) \simeq \mathrm{GL}(V) \rtimes \mathrm{Aut}(K)$.

(2) Let $v_1, \ldots, v_n$ be a basis of V. Then we can associate to every $f \in \Gamma\mathrm{L}(V)$ an invertible matrix $A_f := (a_{ij})_{ij}$ by

$$f(v_j) = \sum_{i=1}^{n} a_{ij} v_i.$$

This procedure establishes an isomorphism between $\Gamma\mathrm{L}(V)$ and the semidirect product $\mathrm{GL}(n, K) \rtimes \mathrm{Aut}(K)$ with group structure

$$(A, \varphi) \cdot (B, \psi) := (A\varphi(B), \varphi \circ \psi).$$

Here, $\varphi(B)$ is the matrix obtained by applying the automorphism φ to all of the entries of B.

In analogy to the group of *projective transformations* $\mathrm{PGL}(V)$, the group of *semi-projective transformations* $\mathrm{P}\Gamma\mathrm{L}(V)$ is defined as the quotient of $\Gamma\mathrm{L}(V)$ modulo the equivalence relation

$$f \sim g \quad \text{if and only if there exists } \lambda \in K^*, \text{ such that} \quad f = \lambda g.$$

By construction, we have a short exact sequence

$$1 \longrightarrow K^* \longrightarrow \Gamma\mathrm{L}(V) \longrightarrow \mathrm{P}\Gamma\mathrm{L}(V) \longrightarrow 1.$$

Remark 3.3. The structure of $\mathrm{P}\Gamma\mathrm{L}(V)$ is similar to the one of $\Gamma\mathrm{L}(V)$, namely:

(1) The group $\mathrm{PGL}(V)$ is a normal subgroup of $\mathrm{P}\Gamma\mathrm{L}(V)$, and there is a split exact sequence

$$1 \longrightarrow \mathrm{PGL}(V) \longrightarrow \mathrm{P}\Gamma\mathrm{L}(V) \longrightarrow \mathrm{Aut}(K) \longrightarrow 1.$$

Note that the map $\mathrm{P}\Gamma\mathrm{L}(V) \to \mathrm{Aut}(K)$ is well-defined because all representatives of a given class in $\mathrm{P}\Gamma\mathrm{L}(V)$ have the same automorphism.

(2) After choosing a *projective frame*, we can identify $\mathrm{P}\Gamma\mathrm{L}(V)$ with the semidirect product

$$\mathrm{PGL}(n, K) \rtimes \mathrm{Aut}(K).$$

(3) If $\dim(V) \geq 3$, then *the fundamental theorem of projective geometry* characterizes the *semi-projective transformations* as the bijective self maps of the projective space $\mathbb{P}(V)$ mapping collinear points to collinear points (see [Sam88, Theorem 7]).

We can now introduce our main objects:

Definition 3.4. Let G be a finite group.

(1) A *semi-linear representation* is a homomorphism $F\colon G \to \Gamma\mathrm{L}(V)$.

(2) A *semi-projective representation* is a homomorphism $f\colon G \to \mathrm{P}\Gamma\mathrm{L}(V)$.

Remark 3.5 (The lifting problem)**.** Note that every semi-linear representation $F\colon G \to \Gamma\mathrm{L}(V)$ induces a semi-projective representation $f\colon G \to \mathrm{P}\Gamma\mathrm{L}(V)$ by composition with the quotient map:

$$\begin{tikzcd}
\Gamma\mathrm{L}(V) \arrow[r] & \mathrm{P}\Gamma\mathrm{L}(V) \\
G \arrow[u, "F"] \arrow[ur, "f"'] &
\end{tikzcd}$$

However, it is not true that every semi-projective representations can be obtained in this way. The obstruction to the existence of a lift to $\Gamma\mathrm{L}(V)$, or more generally, the interplay between semi-linear and semi-projective representations can be described by using *group cohomology* in analogy to the classical theory of projective representations.

3.2 Cohomological Description of Semi-projective Representations

Given a semi-linear or semi-projective representation of G, we obtain an action

$$\varphi\colon G \to \mathrm{Aut}(K), \quad g \mapsto \varphi_g,$$

by composition with the projection from $\Gamma\mathrm{L}(V)$ or $\mathrm{P}\Gamma\mathrm{L}(V)$ to $\mathrm{Aut}(K)$, respectively. Via this action, the abelian group K^* obtains the structure of a G-module. In particular, we can define cocycles $Z^i(G, K^*)$, coboundaries $B^i(G, K^*)$ and the cohomology groups

$$H^i(G, K^*) = Z^i(G, K^*)/B^i(G, K^*).$$

For details on group cohomology, we refer the reader to the textbook [Bro94] (see also Section 1.5 for a brief overview). The basic observation is that we can associate to every semi-projective representation a well-defined class in the second cohomology group.

Proposition 3.6. *Let $f\colon G \to \mathrm{P}\Gamma\mathrm{L}(V)$ be a semi-projective representation and f_g be a representative of the class $f(g)$ for each $g \in G$. Then there exists a map*

$$\alpha\colon G \times G \to K^* \qquad \textit{such that} \qquad f_{gh} = \alpha(g,h) \cdot (f_g \circ f_h)$$

for all $g, h \in G$. The map α is a 2-cocycle, i.e.,

$$\varphi_g\big(\alpha(h,k)\big) \cdot \alpha(gh,k)^{-1} \cdot \alpha(g,hk) \cdot \alpha(g,h)^{-1} = 1.$$

The cohomology class $[\alpha] \in H^2(G, K^*)$ is independent of the chosen representatives f_g.

Proof. Since f is a homomorphism, it holds $[f_{gh}] = [f_g] \circ [f_h]$, which implies that f_{gh} and $f_g \circ f_h$ differ by an element $\alpha(g,h) \in K^*$. To show that α is a cocycle, we use the associativity of the multiplication in G to compute f_{ghk} in two different ways. On the one hand, we have

$$\begin{aligned} f_{g(hk)} = \alpha(g,hk) \cdot (f_g \circ f_{hk}) &= \alpha(g,hk) \cdot (f_g \circ \alpha(h,k) \cdot (f_h \circ f_k)) \\ &= \alpha(g,hk) \cdot \varphi_g(\alpha(h,k)) \cdot (f_g \circ f_h \circ f_k). \end{aligned}$$

On the other hand,

$$f_{(gh)k} = \alpha(gh,k) \cdot (f_{gh} \circ f_k) = \alpha(gh,k) \cdot \alpha(g,h) \cdot (f_g \circ f_h \circ f_k).$$

Comparing the two expressions yields

$$\alpha(g,hk) \cdot \varphi_g(\alpha(h,k)) = \alpha(gh,k) \cdot \alpha(g,h).$$

Let f'_g be another representative for $f(g)$, then there exists $\tau(g) \in K^*$ such that $f_g = \tau(g) f'_g$. Let α' be the 2-cocycle associated to the collection of the f'_g, i.e.,

$$f'_{gh} = \alpha'(g,h) \cdot (f'_g \circ f'_h) \qquad \text{for all} \quad g,h \in G.$$

A computation as above shows that

$$\alpha'(g,h) = \varphi_g(\tau(h)) \cdot \tau(gh)^{-1} \cdot \tau(g) \cdot \alpha(g,h).$$

Thus, α and α' differ by the 2-coboundary $\partial\tau(g,h) = \varphi_g(\tau(h)) \cdot \tau(gh)^{-1} \cdot \tau(g)$. ☐

Remark 3.7. Let $f\colon G \to \mathrm{P\Gamma L}(V)$ be a semi-projective representation.

(1) If we choose id_V as a representative for $f(1)$, then the 2-cocycle α is normalized, i.e.,

$$\alpha(1,g) = \alpha(g,1) = 1.$$

(2) If f is induced by a semi-linear representation F, then the attached cohomology class is trivial. Conversely, assume that α is a coboundary, that is

$$\alpha(g,h) = \varphi_g(\tau(h)) \cdot \tau(gh)^{-1} \cdot \tau(g) \qquad \text{for some} \quad \tau\colon G \to K^*.$$

Then the map

$$F\colon G \to \mathrm{\Gamma L}(V), \quad g \mapsto F_g := \tau(g) f_g$$

is a semi-linear representation inducing f. Indeed, F is a homomorphism, as the following computation shows:

$$\begin{aligned} F_g \circ F_h = (\tau(g) \cdot f_g) \circ (\tau(h) \cdot f_h) &= \tau(g) \cdot \varphi_g(\tau(h)) \cdot (f_g \circ f_h) \\ &= \tau(gh) \cdot \alpha(g,h) \cdot (f_g \circ f_h) \\ &= \tau(gh) \cdot f_{gh} = F_{gh}. \end{aligned}$$

In the theory of projective representations, the action $\varphi\colon G \to \mathrm{Aut}(K)$ is trivial and $H^2(G,K^*)$ is called the *Schur multiplier*. In the semi-projective setting φ is in general non-trivial. This motivates the next definition.

Definition 3.8. Let $\varphi\colon G \to \mathrm{Aut}(K)$ be an action and consider the induced G-module structure on K^*. Then we call $H^2(G,K^*)$ the *φ-twisted Schur multiplier* of G.

Up to now, we assigned to every semi-projective representation an element in $H^2(G,K^*)$. The next remark shows that all cohomology classes arise in this way.

Remark 3.9. Let $\varphi\colon G \to \mathrm{Aut}(K)$ be an action of a finite group G of order n on the field K and $\alpha \in Z^2(G, K^*)$ be a 2-cocycle. In analogy to the *regular representation*, we consider the vector space V with basis $\{e_h \mid h \in G\}$ and define for every $g \in G$ an element $R_g \in \mathrm{GL}(V)$ via

$$R_g(e_h) := \alpha(g,h)^{-1} e_{gh}.$$

Then the map

$$f\colon G \to \mathrm{PGL}(V) \rtimes \mathrm{Aut}(K), \quad g \mapsto ([R_g], \varphi_g),$$

is a semi-projective representation with assigned cohomology class $[\alpha] \in H^2(G, K^*)$.

3.3 Schur's Lifting Problem

Remark 3.7(2) and Remark 3.9 show that if $H^2(G, K^*) \neq 0$, there are semi-projective representations without a semi-linear lift. In the projective case, this problem was first noticed and investigated by Schur [Sch04]. In order to study projective representations by means of ordinary linear representations, he constructed a so-called *representation group* Γ of G: in modern terminology, a *stem extension*

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1, \qquad \text{with} \qquad A \simeq H^2(G, K^*).$$

Such an extension has the property that for every projective representation $f\colon G \to \mathrm{PGL}(V)$ there exists an ordinary linear representation $F\colon \Gamma \to \mathrm{GL}(V)$ fitting inside the following diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d, "F"] & G \arrow[r] \arrow[d, "f"] & 1 \\
1 \arrow[r] & K^* \arrow[r] & \mathrm{GL}(V) \arrow[r] & \mathrm{PGL}(V) \arrow[r] & 1
\end{tikzcd}$$

Recall that *stem* means that A is central and contained in the commutator group $[\Gamma, \Gamma]$.

If we want to generalize Schur's construction to the semi-projective case, we have to deal with more general finite extensions. Let us recall some facts about group extensions.

Remark 3.10. Let $1 \to A \to \Gamma \to G \to 1$ be an extension of G by a finite abelian group A and $s\colon G \to \Gamma$ a set-theoretic section.

(1) There is an action of G on A defined by $g * a := s(g) \cdot a \cdot s(g)^{-1}$. Since A is abelian, the action is independent of the choice of the section.

(2) In general, s is not a homomorphism, but, as already recalled in Subsection 1.5.2, we may write

$$s(gh) = \beta(g,h) s(g) s(h) \qquad \text{for some} \quad \beta(g,h) \in A.$$

In this way, we obtain a 2-cocycle $\beta\colon G \times G \to A$ whose cohomology class $[\beta] \in H^2(G, A)$ uniquely determines the given extension, see [MacLane95], Chapter IV, Theorem 4.1.

(3) Assume that we have an action $\varphi\colon G \to \operatorname{Aut}(K)$ on the field K. Then by composition with the projection $\Gamma \to G$, we also obtain an action of Γ on K with kernel containing A. In this situation, the *inflation-restriction exact sequence* of Hochschild and Serre [HoSe53, Theorem 2, p. 129] reads:

$$1 \longrightarrow H^1(G,K^*) \xrightarrow{\text{inf}} H^1(\Gamma,K^*) \xrightarrow{\text{res}} \operatorname{Hom}_G(A,K^*) \xrightarrow{\text{tra}} H^2(G,K^*) \xrightarrow{\text{inf}}$$

$$\xrightarrow{\text{inf}} H^2(\Gamma,K^*).$$

Here, *inf* and *res* are induced by inflation and restriction of cocycles and the *transgression* map *tra* is defined as

$$\text{tra}\colon \operatorname{Hom}_G(A,K^*) \to H^2(G,K^*), \quad \lambda \mapsto [\lambda \circ \beta].$$

Clearly, this map depends only on the cohomology class of β.

By using the terminology of the previous remark, we get a far-reaching generalization of Remark 3.7 (2); see [Isa94, Theorem 11.13] for the corresponding statement in the projective setting.

Theorem 3.11. *Let* $1 \to A \to \Gamma \xrightarrow{\pi} G \to 1$ *be an extension of* G *by a finite abelian group* A *with associated cohomology class* $[\beta] \in H^2(G,A)$. *A semi-projective representation* $f\colon G \to P\Gamma L(V)$ *with class* $[\alpha] \in H^2(G,K^*)$ *is induced by a semi-linear representation*

$$F\colon \Gamma \to \Gamma L(V), \quad \gamma \mapsto F_\gamma$$

if and only if $[\alpha]$ *belongs to the image of the transgression map.*

Proof. Assume that f is induced by a semi-linear representation F. By assumption, there exists a function $\lambda\colon \Gamma \to K^*$ such that $F_\gamma = \lambda(\gamma) f_{\pi(\gamma)}$ for all $\gamma \in \Gamma$. Since we assume that $f_1 = \text{id}$, it follows that

$$F_a = \lambda(a) f_{\pi(a)} = \lambda(a)\,\text{id} \qquad \text{for all} \quad a \in A.$$

As a result, λ restricted to A is a homomorphism. We claim that $\lambda \in \operatorname{Hom}_G(A,K^*)$, i.e.,

$$\lambda(g * a) = \varphi_g(\lambda(a))$$

for all $g \in G$ and $a \in A$. Indeed, we get

$$\varphi_g(\lambda(a))\,\text{id} = F_{s(g)} \circ (\lambda(a)\,\text{id}) \circ F_{s(g)^{-1}} = F_{s(g)} \circ F_a \circ F_{s(g)^{-1}} = F_{s(g)as(g)^{-1}} = \lambda(g * a)\,\text{id}\,.$$

By using the definition of β, we compute

$$\begin{aligned} F_{s(gh)} = F_{\beta(g,h)s(g)s(h)} &= F_{\beta(g,h)} \circ F_{s(g)} \circ F_{s(h)} \\ &= \lambda(\beta(g,h)) \cdot (\lambda(s(g)) f_g) \circ (\lambda(s(h)) f_h) \\ &= \lambda(\beta(g,h)) \cdot \lambda(s(g)) \cdot \varphi_g(\lambda(s(h)) \cdot (f_g \circ f_h). \end{aligned}$$

On the other hand,

$$F_{s(gh)} = \lambda(s(gh)) f_{gh} = \lambda(s(gh)) \cdot \alpha(g,h) \cdot (f_g \circ f_h).$$

Comparing the results, we obtain $\alpha(g,h) = \lambda(\beta(g,h)) \cdot \partial(\lambda \circ s)(g,h)$, which means that

$$[\lambda \circ \beta] = [\alpha] \in H^2(G, K^*).$$

Conversely, assume there is a function $\tau\colon G \to K^*$ and $\lambda \in \mathrm{Hom}_G(A, K^*)$ such that

$$\alpha(g,h) = \lambda(\beta(g,h)) \cdot \varphi_g(\tau(h)) \cdot \tau(gh)^{-1} \cdot \tau(g).$$

We define the following map

$$F\colon \Gamma \to \Gamma L(V), \quad a \cdot s(g) \mapsto \lambda(a)\tau(g)f_g.$$

As in Remark 3.7, one can show that F is a homomorphism inducing f. □

A natural question arises:

Question 3.12. Is it possible to find for every finite group G together with a fixed action $\varphi\colon G \to \mathrm{Aut}(K)$ an extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \text{with } A \text{ finite and abelian}$$

such that every semi-projective representation $f\colon G \to \mathrm{P\Gamma L}(V)$ with action φ is induced by a semi-linear representation $F\colon \Gamma \to \Gamma\mathrm{L}(V)$?

By virtue of Remark 3.9 and Theorem 3.11, answering this question amounts to constructing an extension with surjective trangression map

$$\mathrm{tra}\colon \mathrm{Hom}_G(A, K^*) \twoheadrightarrow H^2(G, K^*), \quad \lambda \mapsto [\lambda \circ \beta].$$

Clearly, this may only be possible if the twisted Schur multiplier is finite. In case such an extension Γ exists, its order is bounded from below:

$$|G| \cdot |H^2(G, K^*)| \leq |G| \cdot |\operatorname{Hom}(A, K^*)| \leq |G| \cdot |A| = |\Gamma|.$$

Remark 3.13. Note that $H^2(G, K^*)$ is in general not finite. As an example, consider $K = \mathbb{Q}(i)$ and $G = \mathrm{Gal}(K/\mathbb{Q})$ acting naturally on K. Then the cohomology group

$$H^2(G, K^*) \simeq \mathbb{Q}^*/N_{K/\mathbb{Q}}(K^*)$$

is infinite. Indeed, an application of the *sum of two squares theorem* shows that all primes p with $p \equiv 3 \mod 4$ yield non-trivial distinct elements. Nevertheless, in many important situations $H^2(G, K^*)$ is finite: e.g., if K is a finite Galois extension of $\mathbb{Q}_p$ and $G = \mathrm{Gal}(K/\mathbb{Q}_p)$ is acting naturally (cf. [Neu13, II, Lemma 5.1]), or, as we shall see in the next section, if K is algebraically closed and $\varphi\colon G \to \mathrm{Aut}(K)$ is an arbitrary action.

3.4 Twisted Representation Groups: the Algebraically Closed Case

Throughout this section, K is an algebraically closed field and G a finite group together with a given action

$$\varphi\colon G \to \operatorname{Aut}(K).$$

We want to provide an answer to Question 3.12 under the above assumptions. Indeed, we will construct an extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \text{with } A \text{ finite and abelian}$$

such that the transgression map

$$\operatorname{tra}\colon \operatorname{Hom}_G(A, K^*) \to H^2(G, K^*)$$

is an isomorphism and Γ has minimal order, namely

$$|\Gamma| = |G| \cdot |H^2(G, K^*)|.$$

Remark 3.14. Note that, under our assumptions, we mainly deal with a case similar to $K = \mathbb{C}$, where φ acts just by the identity and/or complex conjugation. Indeed, $H := \varphi(G)$ is a finite group and $F := K^H \subset K$ is a Galois extension with Galois group H. Since we assume K to be algebraically closed, the Artin-Schreier Theorem [AS27] implies that if H is non-trivial, then it is isomorphic to $\mathbb{Z}/2$, $K = F(i)$ with $i^2 = -1$ and $\operatorname{char}(K) = 0$. In particular, if $\operatorname{char}(K) \neq 0$, then the action is necessarily trivial and we are in the projective setting.

The first step towards our goal is to prove the finiteness of the twisted Schur multiplier, or more generally, of all higher cohomology groups $H^i(G, K^*)$. In order to show this, we adapt the proof of the finiteness of the Schur multiplier given in [Isa94].

Lemma 3.15 ([Isa94], Lemma 11.14)**.** *Let A be an abelian group (not necessarily finite) and $Q \leq A$ with Q divisible, i.e., for all positive integers n, the maps*

$$Q \to Q, \quad \alpha \mapsto \alpha^n$$

are surjective. Assume $|A : Q| < \infty$. Then Q is complemented in A.

Lemma 3.16. *Under our assumptions, the groups $B^i(G, K^*)$ are divisible.*

Proof. Let n be a positive integer and $\beta \in B^i(G, K^*)$ a coboundary. Then there is a function $\tau\colon G^{i-1} \to K^*$ such that $\beta = \partial\tau$, where

$$\partial\tau(g_1, \ldots, g_i) :=$$

$$\varphi_{g_1}\big(\tau(g_2, \ldots, g_i)\big) \cdot \left(\prod_{j=2}^{i} \tau(g_1, \ldots, g_{j-2}, g_{j-1}g_j, g_{j+1}, \ldots, g_i)^{(-1)^{j-1}}\right) \cdot \tau(g_1, \ldots, g_{i-1})^{(-1)^i}. \tag{3.1}$$

Since we assume K to be algebraically closed, for all $(g_1, \ldots, g_{i-1}) \in G^{i-1}$, there is an element $\nu(g_1, \ldots, g_{i-1}) \in K^*$ such that $\nu(g_1, \ldots, g_{i-1})^n = \tau(g_1, \ldots, g_{i-1})$. As φ_{g_1} is a field automorphism, it holds

$$\beta(g_1, \ldots, g_i) = \partial\tau(g_1, \ldots, g_i) = \partial\nu^n(g_1, \ldots, g_i) = \Big(\partial\nu(g_1, \ldots, g_i)\Big)^n. \qquad \square$$

Now, we are ready to prove the finiteness of the higher cohomology groups $H^i(G, K^*)$.

Proposition 3.17. *For each $i \geq 1$, the cohomology groups $H^i(G, K^*)$ are finite with exponent dividing the order of G. Moreover, $B^i(G, K^*)$ has a complement in $Z^i(G, K^*)$.*

Proof. It is well known that $\alpha^{|G|} \in B^i(G, K^*)$ for every cocycle $\alpha \in Z^i(G, K^*)$, see [Bro94, III, Corollary 10.2]. In other words, the exponent of $H^i(G, K^*)$ divides the order of G. Take a cocycle $\alpha \in Z^i(G, K^*)$ and consider the group $A := \langle B^i(G, K^*), \alpha \rangle$. By construction, $A/B^i(G, K^*) = \langle [\alpha] \rangle$, which implies that the order of the quotient divides the order of G. Since $B^i(G, K^*)$ is divisible, it is complemented in A thanks to Lemma 3.15. Thus, there exists a subgroup $W \leq A$ such that

$$W \cap B^i(G, K^*) = \{1\} \qquad \text{and} \qquad WB^i(G, K^*) = A.$$

Note that, for all $\gamma \in W$, it holds

$$\gamma^{|G|} \in W \cap B^i(G, K^*) = \{1\}.$$

This shows that W is contained in the group

$$U := \{\eta \in Z^i(G, K^*) \mid \eta^{|G|} = 1\}.$$

In particular,

$$\alpha \in A = WB^i(G, K^*) \leq UB^i(G, K^*).$$

Since $\alpha \in Z^i(G, K^*)$ is arbitrary, the above relation implies

$$Z^i(G, K^*) = UB^i(G, K^*).$$

The group U is finite because it consists of functions $G^i \to K^*$ with image contained in the group of $|G|$-th roots of unity. It follows that

$$|H^i(G, K^*)| = |Z^i(G, K^*)/B^i(G, K^*)| \leq |U| < \infty,$$

and Lemma 3.15 implies that $B^i(G, K^*)$ has a complement in $Z^i(G, K^*)$. □

The main result of this section is the following.

Theorem 3.18. *Let G be a finite group and K an algebraically closed field. Let $\varphi\colon G \to \mathrm{Aut}(K)$ be a fixed action. Then there exists an extension of G*

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

with A finite and abelian such that the transgression map

$$\mathrm{tra}\colon \mathrm{Hom}_G(A, K^*) \to H^2(G, K^*), \qquad \lambda \mapsto [\lambda \circ \beta]$$

is an isomorphism.

Proof. Take a complement M of $B^2(G,K^*)$ in $Z^2(G,K^*)$. Such a group M exists and is finite thanks to Proposition 3.17. Consider $A := \mathrm{Hom}(M,K^*)$ and define via φ an action on it:

$$(g * a)(m) := \varphi_g(a(m)).$$

We define a map $\beta\colon G \times G \to A$ by

$$\beta(g,h)(m) := m(g,h) \qquad \text{for} \qquad m \in M.$$

A straightforward computation shows that β is a 2-cocycle:

$$\begin{aligned}\partial\beta(g,h,k)(m) &= \Big(g * \beta(h,k) \cdot \beta(gh,k)^{-1} \cdot \beta(g,hk) \cdot \beta(g,h)^{-1}\Big)(m)\\ &= \varphi_g(\beta(h,k)(m)) \cdot \beta(gh,k)^{-1}(m) \cdot \beta(g,hk)(m) \cdot \beta(g,h)^{-1}(m)\\ &= \varphi_g(m(h,k)) \cdot m(gh,k)^{-1} \cdot m(g,hk) \cdot m(g,h)^{-1}\\ &= \partial m(g,h,k) = 1.\end{aligned}$$

Despite the fact that the cocycle β is in general not normalized, we can consider a normalized cocycle β' in its cohomology class. Then it is clear from literature (see [Bro94, IV]) that β' defines an extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

where $\Gamma := A \times G$ with product structure

$$(a,g) \cdot (b,h) := (a(g * b)\beta'(g,h), gh).$$

We point out that the conjugation action of G on A is given by $g * a$.
Now, we claim that the transgression map

$$\mathrm{tra}\colon \mathrm{Hom}_G(A,K^*) \to H^2(G,K^*), \quad \lambda \mapsto [\lambda \circ \beta] = [\lambda \circ \beta'],$$

is surjective. Any class in $H^2(G,K^*)$ is represented by a (unique) element $m_0 \in M \leq Z^2(G,K^*)$. Consider the evaluation homomorphism at m_0, that is

$$\lambda\colon A \to K^*, \quad a \mapsto a(m_0).$$

Note that λ is G-equivariant, in fact

$$\lambda(g * a) = (g * a)(m_0) = \varphi_g(a(m_0)) = \varphi_g(\lambda(a)) \qquad \text{for all} \quad g \in G.$$

Furthermore, we have

$$(\lambda \circ \beta)(g,h) = \lambda(\beta(g,h)) = \beta(g,h)(m_0) = m_0(g,h).$$

This shows that $\mathrm{tra}(\lambda) = [m_0]$ and thus the desired surjectivity. Finally, the injectivity follows from

$$|M| = |H^2(G,K^*)| \leq |\mathrm{Hom}_G(A,K^*)| \leq |\mathrm{Hom}(A,K^*)| \leq |A| \leq |M|. \qquad \square$$

Remark 3.19. (1) From the above chain of inequalities, it follows that

  (a) all characters of A are G-equivariant, namely $\mathrm{Hom}_G(A, K^*) = \mathrm{Hom}(A, K^*)$,
  (b) $A \simeq \mathrm{Hom}(A, K^*)$,
  (c) $A \simeq H^2(G, K^*)$,
  (d) the group Γ has minimal order $|\Gamma| = |G| \cdot |H^2(G, K^*)|$,
  (e) $H^1(G, K^*) \simeq H^1(\Gamma, K^*)$ by the inflation-restriction sequence

$$0 \longrightarrow H^1(G, K^*) \longrightarrow H^1(\Gamma, K^*) \longrightarrow \mathrm{Hom}_G(A, K^*) \xrightarrow{\sim} H^2(G, K^*).$$

(2) If $\mathrm{char}(K) \neq 0$, the action φ is trivial, see Remark 3.14. Moreover, property (1b) amounts to saying that $\mathrm{char}(K) \nmid |A|$. Thus, as a byproduct, we found a general property of the Schur multiplier, namely

$$\mathrm{char}(K) \nmid |H^2(G, K^*)|,$$

whenever G is a finite group and K is algebraically closed.

Remark 3.19 motivates the following definition:

Definition 3.20. Let $\varphi\colon G \to \mathrm{Aut}(K)$ be an action of a finite group G on an algebraically closed field K. A group Γ is called a *φ-twisted representation group* of G if there exists an extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \text{with } A \text{ finite and abelian}$$

such that the following conditions hold:

(1) $\mathrm{char}(K) \nmid |A|$,

(2) $\mathrm{Hom}_G(A, K^*) = \mathrm{Hom}(A, K^*)$,

(3) the transgression map

$$\mathrm{tra}\colon \mathrm{Hom}_G(A, K^*) \to H^2(G, K^*)$$

is an isomorphim.

Hence, given a finite group G together with an action $\varphi\colon G \to \mathrm{Aut}(K^*)$, we see right away thanks to Proposition 3.18 and Remark 3.19 that a φ-twisted representation group of G always exists.

Proposition 3.21. *If $\varphi\colon G \to \mathrm{Aut}(K)$ is the trivial action, then an extension as in Definition 3.20 is a stem extension.*

Proof. Since φ is trivial, the restriction-inflation sequence reads

$$1 \longrightarrow \mathrm{Hom}(G, K^*) \longrightarrow \mathrm{Hom}(\Gamma, K^*) \longrightarrow \mathrm{Hom}_G(A, K^*) \longrightarrow H^2(G, K^*).$$

As the transgression map is an isomorphism, the restriction $\mathrm{Hom}(\Gamma, K^*) \to \mathrm{Hom}_G(A, K^*)$ has to be trivial, which implies $A \leq [\Gamma, \Gamma]$. Suppose it does not, then the map from A to the abelianization Γ^{ab} is non-trivial. We write $\Gamma^{ab} \simeq \mathbb{Z}/d_1 \times \ldots \times \mathbb{Z}/d_m$ and, w.l.o.g., we can assume that the induced map $A \to \mathbb{Z}/d_1$ is not the zero-map. If $p := \mathrm{char}(K) \mid d_1$, then we write $d_1 = p^k l_1$ with $p \nmid l_1 \neq 1$ and obtain a non-trivial map $A \to \mathbb{Z}/d_1 \to \mathbb{Z}/l_1$ since $p \nmid |A|$. Replacing d_1 by l_1, if necessary, we may assume that there exists a primitive d_1-th root of unity. This yields a character $\lambda \in \mathrm{Hom}(\Gamma, K^*)$ such that the restriction $\lambda_A \colon A \to K^*$ is non-trivial. Thus, we get a contradiction.

Assume now that A is not contained in the center of Γ. Then there exist $a \in A$ and $\gamma \in \Gamma$ such that

$$\gamma a \gamma^{-1} a^{-1} \neq 1.$$

Since $\mathrm{char}(K) \nmid |A|$, a similar argument as before shows that there exists a character $\lambda \in \mathrm{Hom}(A, K^*)$ such that $\lambda(\gamma a \gamma^{-1} a^{-1}) \neq 1$. As φ is the trivial action, this means that $\lambda \notin \mathrm{Hom}_G(A, K^*)$, which contradicts the assumption $\mathrm{Hom}_G(A, K^*) = \mathrm{Hom}(A, K^*)$. □

The next proposition shows that Definition 3.20 is well-posed.

Proposition 3.22. *In the projective case, i.e., when the G-action on K is trivial, Definition 3.20 reduces exactly to the classical notion of a representation group (cf.* [Isa94, Corollary 11.20]*), i.e.,*

(1) the extension $1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ *is stem,*

(2) $|A| = |H^2(G, K^*)|$.

Proof. If the extension fulfills the conditions of Definition 3.20, then Proposition 3.21 implies that it is stem. Since $\mathrm{char}(K) \nmid |A|$, we have that $A \simeq \mathrm{Hom}(A, K^*)$ and then (2) follows from the fact that the transgression map is an isomorphism.
Conversely, suppose we have a stem extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

such that $|A| = |H^2(G, K^*)|$. First of all, Remark 3.19 (2) implies that $\mathrm{char}(K) \nmid |A|$. Since the extension is stem, $A \leq Z(\Gamma)$ and therefore the action of G on A is trivial implying $\mathrm{Hom}_G(A, K^*) = \mathrm{Hom}(A, K^*)$. Furthermore, the inflation-restriction sequence says that, for a stem extension, the transgression map is injective because $A \leq [\Gamma, \Gamma]$. Since

$$|\mathrm{Hom}(A, K^*)| = |A| = |H^2(G, K^*)|,$$

we conclude that the transgression map is also surjective and hence an isomorphism. □

Remark 3.23. We want to point out that only the order of a φ-twisted representation group Γ is unique, whereas the group itself is in general not (see examples in Section 3.5), even in the projective case. Here, it is known that the group Γ is unique up to isomorphism if $|G^{ab}|$ and $|H^2(G, K^*)|$ are coprime [BT82, p. 92]. Note that the latter condition is fulfilled if for instance G is *perfect*. However, there are groups with a unique representation group, even though $|G^{ab}|$ and $|H^2(G, K^*)|$ are not coprime. An example is the metacyclic group

$$G := \langle a, b \mid a^8 = b^4 = 1, bab^{-1} = a^5 \rangle,$$

which has abelianization $(\mathbb{Z}/4)^2$, Schur multiplier $\mathbb{Z}/2$ and

$$\Gamma := \langle a, b \mid a^{16} = b^4 = 1, bab^{-1} = a^5 \rangle$$

as the unique representation group.

Now, we want to give a numerical criterion to decide whether a given extension is a φ-twisted representation group or not.

Proposition 3.24 (Numerical criterion)**.** *Let* $\varphi\colon G \to \mathrm{Aut}(K)$ *be a non-trivial action of a finite group* G *on an algebraically closed field* K*. Let*

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be an extension by a finite abelian group A*. Then* Γ *is a* φ*-twisted representation group if and only if the following conditions are satisfied:*

(1) $|A| = |H^2(G, K^*)|$,

(2) $|\operatorname{Hom}_G(A, K^*)| = |\operatorname{Hom}(A, K^*)|$ *and*

(3) $|H^1(G, K^*)| = |H^1(\Gamma, K^*)|$.

Proof. Clearly, every φ-twisted representation group fulfills the three conditions. Conversely, if they hold, then the inflation-restriction sequence together with (3) implies that the transgression map is injective. Condition (1), together with Remark 3.19(2), implies $\mathrm{char}(K) \nmid |A|$. Therefore, by using condition (2), we have

$$|\operatorname{Hom}_G(A, K^*)| = |\operatorname{Hom}(A, K^*)| = |A|.$$

Thus, the transgression map is also surjective and hence an isomorphism. □

3.4.1 The Heisenberg Group as a Representation Group

Let $K = \mathbb{C}$ be the field of complex numbers. We give here a proof of the well-known fact that the Heienberg group $\mathcal{H}_r$ of a cyclic group $\mathbb{Z}/r$ is a representation group for $G_r := (\mathbb{Z}/r)^2$.

We recall that by definition of Heisenberg group (see Chapter 1, Section 1.3) there is a stem extension

$$1 \to \mu_r \to \mathcal{H}_r \to G_r \to 0, \tag{3.2}$$

such that

$$\mu_r = Z(\mathcal{H}_r) = [\mathcal{H}_r, \mathcal{H}_r].$$

Since we are in the classical setting of projective representations, the action $\varphi\colon G_r \to \mathrm{Aut}(\mathbb{C}^*)$ is trivial, and hence, according to Proposition 3.22, it remains to show just that

$$|H^2(G_r, \mathbb{C}^*)| = |\mu_r| = r. \tag{3.3}$$

This follows from the fact that

$$H^2(G_r, \mathbb{C}^*) \cong \mathbb{Z}/r, \tag{3.4}$$

which we will show next.

To do so we can use some well-known formulae. Indeed, given two finite groups H_1 and H_2, we recall that, defining the tensor product $H_1 \otimes H_2$ of H_1 and H_2 as follows

$$H_1 \otimes H_2 := \Big(H_1/[H_1, H_1]\Big) \otimes_{\mathbb{Z}} \Big(H_2/[H_2, H_2]\Big), \tag{3.5}$$

it holds true (see [Kar87], Theorem 2.2.10)

$$H^2(H_1 \times H_2, \mathbb{C}^*) \cong H^2(H_1, \mathbb{C}^*) \times H^2(H_2, \mathbb{C}^*) \times H_1 \otimes H_2. \tag{3.6}$$

Moreover, Proposition 2.1.1 of [Kar87] states in particular that

$$H^2(\mathbb{Z}/r, \mathbb{C}^*) = 1 \qquad \text{for all} \quad r \in \mathbb{N}. \tag{3.7}$$

Hence, applying (3.6) and (3.7), we get right away that

$$H^2(G_r, \mathbb{C}^*) \cong \mathbb{Z}/r \otimes_{\mathbb{Z}} \mathbb{Z}/r \cong \mathbb{Z}/r, \tag{3.8}$$

where the last isomorphism follows from the well-known formula

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/\mathrm{gcd}(m, n).$$

Therefore, we have showed the following.

Proposition 3.25. *The Heisenberg group $\mathcal{H}_r$ of the cyclic group $\mathbb{Z}/r$ is a representation group for the group $G_r = (\mathbb{Z}/r)^2$.*

3.5 Examples and Applications

In this section, we present basic examples of semi-projective representations. Furthermore, we develop an algorithm to compute all φ-twisted representation groups for a given finite group G and a given action φ under the assumption $K = \mathbb{C}$ and that φ maps to $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Finally, as we have announced in the introduction to this chapter, we discuss two more involved situations, where semi-projective representations arise naturally.

The first one deals with a purely representation theoretic question from *Clifford theory*, namely the extendability of G-invariant irreducible L-representations defined on a normal subgroup $N \trianglelefteq G$ to the ambient group G, where L is an arbitrary field. Isaacs investigated this problem in [Isa81] by using the concept of crossed-projective representations, which is analogous to our notion of a semi-projective representation.

The second one is the original geometric motivation which led the authors of [AGK23] to the concept of a twisted representation group. It deals with the problem to find linear parts of homeomorphisms and biholomorphisms of complex torus quotients, cf. [DG23], [GK22] and [HL19]. We show that this problem reduces, in some occasions, to a lifting problem of a certain semi-projective representation.

3.5.1 Basic Examples of Semi-Projective Representations and Twisted Representation Groups

Example 3.26. Consider $K = \mathbb{C}$ as a $G = \mathbb{Z}/2$-module, where $1 \in \mathbb{Z}/2$ acts via complex conjugation $\mathrm{conj}(z) = \overline{z}$. In this example, a twisted representation group Γ is of order 4 because

$$H^2(\mathbb{Z}/2, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\mathbb{Z}/2}/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*) = \mathbb{Z}/2.$$

It is easy to see that Γ must be isomorphic to $\mathbb{Z}/4$. Indeed, since the transgression map is required to be an isomorphism, the extension

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \Gamma \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

has to be non-split, which implies $\Gamma \simeq \mathbb{Z}/4$. Consider the semi-projective representation

$$f \colon \mathbb{Z}/2 \to \mathrm{PGL}(2, \mathbb{C}) \rtimes \mathbb{Z}/2, \quad 1 \mapsto \left(\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right], \mathrm{conj} \right).$$

Its cohomology class in $H^2(\mathbb{Z}/2, \mathbb{C}^*)$ is represented by the normalised 2-cocycle α with $\alpha(1,1) = -1$, see Remark 3.9. It has no lift to a semi-linear representation of $\mathbb{Z}/2$. A semi-linear lift to Γ is given by

$$F \colon \mathbb{Z}/4 \to \mathrm{GL}(2, \mathbb{C}) \rtimes \mathbb{Z}/2, \quad 1 \mapsto \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathrm{conj} \right).$$

In the following, we explain how to use a computer algebra system, such as MAGMA [BoCaPlay97], to produce all twisted representation groups of a given finite group G in the case $K = \mathbb{C}$. We assume that $\varphi \colon G \to \mathrm{Aut}(\mathbb{C})$ takes values in $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \{\mathrm{id}, \mathrm{conj}\}$, cf. Remark 3.14.

Recall that Proposition 3.24 provides necessary and sufficient numerical conditions for an extension Γ of G by a finite abelian group A to be a φ-twisted representation group. The results from the previous section say that A must be isomorphic to $H^2(G, \mathbb{C}^*)$. Furthermore, condition (3) of the proposition requires $H^1(G, \mathbb{C}^*)$ and $H^1(\Gamma, \mathbb{C}^*)$ to be of the same size. In order to check this, we determine the above cohomology groups. Since we want to use a computer, it is necessary to replace the module $\mathbb{C}^*$ by a discrete module. Identifying complex conjugation with multiplication by -1, the homomorphism φ induces an action of G on $\mathbb{Z}$ that is also denoted by φ. In this way, we can consider φ as a complex character of G of degree 1 with values in $\{\pm 1\}$. Furthermore, the exponential sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2\pi i"] & \mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^* \arrow[r] & 1 \end{tikzcd}$$

becomes a sequence of G-modules. Since the cohomology groups $H^n(G, \mathbb{C})$ vanish for $n \geq 1$, see [Bro94, III, Corollary 10.2], the corresponding long exact sequence induces isomorphisms

$$H^n(G, \mathbb{C}^*) \simeq H^{n+1}(G, \mathbb{Z}) \qquad \text{for all} \quad n \geq 1.$$

Similarly, we have these isomorphisms for the cohomology groups of Γ. In order to check the second condition of the proposition, we make use of the identity $\mathrm{Hom}(A, \mathbb{C}^*) = \mathrm{Irr}(A)$, which holds since A is abelian.

These considerations lead to *Algorithm 1*. It takes as inputs a finite group G and an action φ, which is given as a character with values in $\{\pm 1\}$, and it returns all φ-twisted representation groups of G.

Algorithm 1 φ-twisted representation groups

```
function TwistedRepresentationGroups($G, \varphi$)
input: Finite group $G$, $\varphi \in \mathrm{Irr}(G)$ of degree one with values in $\{\pm 1\}$
output: List of all $\varphi$-twisted representation groups of $G$

$A \leftarrow H^3(G, \mathbb{Z})$
$(\Gamma_1, \ldots, \Gamma_k) \leftarrow$ extensions of $G$ by $A$
$L \leftarrow$ empty list
for $j = 1, \ldots, k$ do
    test $\leftarrow$ true
    for $\chi \in \mathrm{Irr}(A)$ do
        if $\chi$ is not $G$-invariant then
            test $\leftarrow$ false
        end if
    end for
    if test = true and $\#H^2(G, \mathbb{Z}) = \#H^2(\Gamma_j, \mathbb{Z})$ then
        $L \leftarrow$ append$(L, \Gamma_j)$                ▷ add $\Gamma_j$ to the list $L$
    end if
end for
return $L$
```

The reader can find a MAGMA implementation on the webpage `http://www.staff.uni-bayreuth.de/~bt300503/publi.html`,

see also Appendix B.

Example 3.27. Running our code, we compute the φ-twisted representation groups of the *dihedral group*

$$D_4 = \langle s, t \mid s^2 = t^4 = 1,\ sts^{-1} = t^3 \rangle$$

for all possible actions $\varphi\colon D_4 \to \mathrm{Aut}(\mathbb{C})$ given as characters with values in $\{\pm 1\}$:

$\varphi(s)$	$\varphi(t)$	$A = H^2(D_4, \mathbb{C}^*)$	φ-twisted representation groups
1	1	$\mathbb{Z}/2$	$\langle 16, 7\rangle$, $\langle 16, 8\rangle$, $\langle 16, 9\rangle$
-1	-1	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\langle 32, 14\rangle$, $\langle 32, 13\rangle$
1	-1	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\langle 32, 9\rangle$, $\langle 32, 10\rangle$, $\langle 32, 14\rangle$, $\langle 32, 13\rangle$
-1	1	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\langle 32, 2\rangle$, $\langle 32, 10\rangle$, $\langle 32, 13\rangle$

Here, the symbol $\langle n, d\rangle$ denotes the d-th group of order n in MAGMA's *Database of Small Groups.*

3.5.2 Extendability of L-Representations

Let L be a field and $\chi \in \mathrm{Irr}_L(N)$ an irreducible character defined on a normal subgroup $N \trianglelefteq G$. Assume that χ is G-invariant, i.e.,

$$\chi(gng^{-1}) = \chi(n) \qquad \text{for all} \quad g \in G, n \in N.$$

Then we can ask the question if χ can be extended to an irreducible character of the ambient group G. Clearly, the G-invariance is a necessary condition for the extendibility, but in general not sufficient. In the following, we will describe how this problem relates to the theory of semi-projective representations.

Remark 3.28. Let K be an algebraically closed field containing L. Then the character χ splits as follows

$$\chi = m(\eta_1 + \ldots + \eta_r), \qquad \text{where} \quad \eta_i \in \mathrm{Irr}_K(N).$$

The irreducible characters $\eta_1, \ldots, \eta_r$ form a single orbit under the action of $\mathrm{Gal}(K/L)$. The common multiplicity m of the constituents η_i is called the *Schur index* of χ.

Let us call $\eta := \eta_1$ and F the subfield of K generated by L and the values of η. The extension $L \subset F$ is Galois of degree r with abelian Galois group. By [Isa81, Lemma 2.1], the $\mathrm{Gal}(F/L)$-orbit of η consists of all constituents η_i of χ. We now make the *crucial assumption* that $m = 1$, which by [Isa94, Theorem 9.21]) is automatically fulfilled in the case $\mathrm{char}(L) \neq 0$. Under this assumption, the character η is afforded by an irreducible F-representation

$$\rho\colon N \to \mathrm{GL}\left(n, F\right),$$

cf. [Isa94, Corollary 10.2]. By the G-invariance of χ, there exists for all $g \in G$ an element $\varphi_g \in \mathrm{Gal}(F/L)$ such that $\eta^g = \varphi_g \circ \eta$, where $\eta^g(n) := \eta(gng^{-1})$. Clearly, φ_g is unique and $\varphi_n = \mathrm{id}$ for all $n \in N$. Thus, we obtain an action $\varphi\colon G \to \mathrm{Gal}(F/L)$ which factors through the quotient map $\pi\colon G \to G/N$. Since the F-representations ρ^g and $\varphi_g(\rho)$ are irreducible and their characters η^g and $\varphi_g \circ \eta$ agree, they are equivalent according to [Isa94, Corollary 9.22]. Thus, for all $g \in G$, there exists a matrix $A_g \in \mathrm{GL}(n, F)$ such that

$$A_g \cdot \varphi_g(\rho) \cdot A_g^{-1} = \rho^g. \tag{$*$}$$

Since ρ is irreducible over K, the matrix A_g is only unique up to a scalar. Clearly, this scalar belongs to F^* because $A_g \in \mathrm{GL}(n, F)$. Let $s\colon G/N \to G$ be a set-theoretic section. Then we define the following map

$$f\colon G \to \mathrm{PGL}(n, F) \rtimes \mathrm{Gal}(F/L), \quad n \cdot s(\gamma) \mapsto \big([\rho(n)A_{s(\gamma)}], \varphi_\gamma\big).$$

Proposition 3.29. *The map $f\colon G \to \mathrm{PGL}(n, F) \rtimes \mathrm{Gal}(F/L)$ from above is a semi-projective representation.*

Proof. We need to show that f is a homomorphism, i.e.,

$$f(n_1 s(\gamma_1) n_2 s(\gamma_2)) = f(n_1 s(\gamma_1)) \circ f(n_2 s(\gamma_2)).$$

For this purpose, we rewrite the left-hand side as

$$\begin{aligned} f(n_1 s(\gamma_1) n_2 s(\gamma_2)) &= f(n_1 s(\gamma_1) n_2 s(\gamma_2) s(\gamma_1\gamma_2)^{-1} s(\gamma_1\gamma_2)) \\ &= \big(\big[\rho(n_1 s(\gamma_1) n_2 s(\gamma_2) s(\gamma_1\gamma_2)^{-1}) A_{s(\gamma_1\gamma_2)}\big], \varphi_{\gamma_1\gamma_2}\big) \\ &= \big(\big[\rho\big(n_1 s(\gamma_1) n_2 s(\gamma_1)^{-1}\big) \cdot \rho\big(s(\gamma_1) s(\gamma_2) s(\gamma_1\gamma_2)^{-1}\big) \cdot A_{s(\gamma_1\gamma_2)}\big], \varphi_{\gamma_1\gamma_2}\big). \end{aligned}$$

Similarly, the right-hand side becomes

$$\begin{aligned} f(n_1 s(\gamma_1)) \circ f(n_2 s(\gamma_2)) &= \big(\big[\rho(n_1) A_{s(\gamma_1)}\big], \varphi_{\gamma_1}\big) \circ \big(\big[\rho(n_2) A_{s(\gamma_2)}\big], \varphi_{\gamma_2}\big) \\ &= \big(\big[\rho(n_1) A_{s(\gamma_1)} \varphi_{\gamma_1}\big(\rho(n_2) A_{s(\gamma_2)}\big)\big], \varphi_{\gamma_1\gamma_2}\big) \\ &= \big(\big[\rho\big(n_1 s(\gamma_1) n_2 s(\gamma_1)^{-1}\big) \cdot A_{s(\gamma_1)} \cdot \varphi_{\gamma_1}\big(A_{s(\gamma_2)}\big)\big], \varphi_{\gamma_1\gamma_2}\big). \end{aligned}$$

In order to show that they are equal, it suffices to prove that the following two matrices

$$C_{(\gamma_1,\gamma_2)} := \rho\big(s(\gamma_1) s(\gamma_2) s(\gamma_1\gamma_2)^{-1}\big) \cdot A_{s(\gamma_1\gamma_2)} \qquad \text{and} \qquad D_{(\gamma_1,\gamma_2)} := A_{s(\gamma_1)} \cdot \varphi_{\gamma_1}\big(A_{s(\gamma_2)}\big)$$

differ by a constant $\bar{\alpha}(\gamma_1, \gamma_2)$ in F^*, namely

$$\bar{\alpha}(\gamma_1, \gamma_2) \cdot D_{(\gamma_1,\gamma_2)} = C_{(\gamma_1,\gamma_2)}.$$

This is an immediate consequence of Schur's lemma and the identity

$$C_{(\gamma_1,\gamma_2)} \cdot \varphi_{\gamma_1\gamma_2}(\rho) \cdot C_{(\gamma_1,\gamma_2)}^{-1} = D_{(\gamma_1,\gamma_2)} \cdot \varphi_{\gamma_1\gamma_2}(\rho) \cdot D_{(\gamma_1,\gamma_2)}^{-1},$$

which we leave to the reader. □

Remark 3.30. We observe from the proof of Proposition 3.29 that the cohomology class of f is represented by a cocycle $\alpha\colon G \times G \to F^*$, which is constant on N. For this reason, it induces a cocycle $\bar{\alpha}\colon G/N \times G/N \to F^*$ whose class in $H^2(G/N, F^*)$ is independent of the chosen section $s\colon G/N \to G$ and of the chosen $A_{s(\gamma)}$, which we recall to be unique only up to a scalar in F^*.

It is clear from Remark 3.7 (2) that f lifts to a semi-linear representation of G if and only if $[\alpha]$ is trivial in $H^2(G, F^*)$. However, this semi-linear representation might not be an extension of ρ, cf. [Isa94, p. 179].

Theorem 3.31 ([Isa81], Theorem 4.3)**.** *The representation* $\rho\colon N \to \mathrm{GL}\big(n, F\big)$ *extends to a semi-linear representation*

$$\hat{\rho}\colon G \to \mathrm{GL}(n, F) \rtimes \mathrm{Gal}(F/L)$$

if and only if $[\bar{\alpha}]$ *is trivial in* $H^2(G/N, F^*)$.

Proof. Given a semi-linear extension

$$\hat{\rho}\colon G \to \mathrm{GL}(n, F) \rtimes \mathrm{Gal}(F/L), \quad g \mapsto (B_g, \varphi_g),$$

the matrices B_g fulfill the conjugation equation $(*)$. Thus, setting $A_g := B_g$, one can see that $\hat{\rho}$ is a lift of the semi-projective representation f and then it is clear that $\bar{\alpha} = 1$ as a cocycle.

Assume now that $[\bar{\alpha}]$ is trivial, where the representative $\bar{\alpha}$ is constructed as above choosing the matrices A_g and the section $s\colon G/N \to G$ such that $s(1) = 1$ and A_1 is the identity matrix E_n. Then there exists a function $\tau\colon G/N \to F^*$ such that

$$\bar{\alpha}(\gamma_1, \gamma_2) = \varphi_{\gamma_1}(\tau(\gamma_2))\tau(\gamma_1\gamma_2)^{-1}\tau(\gamma_1).$$

Define the following map:

$$\hat{\rho}\colon G \to \mathrm{GL}(n, F) \rtimes \mathrm{Gal}(F/L), \quad n \cdot s(\gamma) \mapsto \big(\tau(\gamma)\rho(n)A_{s(\gamma)}, \varphi_\gamma\big).$$

Clearly, by our choice of s and A_g, the map $\hat{\rho}$ is an extension of ρ. Indeed, since $\bar{\alpha}(1, 1) = 1$, it follows that $\tau(1) = 1$ and we obtain

$$\hat{\rho}(n) = \big(\tau(1)\rho(n)A_{s(1)}, \varphi_1\big) = \big(\rho(n), \mathrm{id}\big).$$

It remains to show that $\hat{\rho}$ is a homomorphism. In order to have a compact notation, we use the matrices $C_{(\gamma_1,\gamma_2)}$ and $D_{(\gamma_1,\gamma_2)}$, as defined in the proof of Proposition 3.29, and compute

$$\begin{aligned}
\hat{\rho}(n_1 s(\gamma_1)) \circ \hat{\rho}(n_2 s(\gamma_2)) &= \big(\tau(\gamma_1)\rho(n_1)A_{s(\gamma_1)}, \varphi_{\gamma_1}\big) \circ \big(\tau(\gamma_2)\rho(n_2)A_{s(\gamma_2)}, \varphi_{\gamma_2}\big) \\
&= \big(\tau(\gamma_1)\varphi_{\gamma_1}(\tau(\gamma_2)) \cdot \rho(n_1 s(\gamma_1) n_2 s(\gamma_1)^{-1}) \cdot D_{(\gamma_1,\gamma_2)}, \varphi_{\gamma_1\gamma_2}\big) \\
&= \big(\tau(\gamma_1\gamma_2) \cdot \rho(n_1 s(\gamma_1) n_2 s(\gamma_1)^{-1}) \cdot \bar{\alpha}(\gamma_1, \gamma_2) \cdot D_{(\gamma_1,\gamma_2)}, \varphi_{\gamma_1\gamma_2}\big) \\
&= \big(\tau(\gamma_1\gamma_2) \cdot \rho(n_1 s(\gamma_1) n_2 s(\gamma_1)^{-1}) \cdot C_{(\gamma_1,\gamma_2)}, \varphi_{\gamma_1\gamma_2}\big) \\
&= \hat{\rho}(n_1 s(\gamma_1) n_2 s(\gamma_2)).
\end{aligned}$$

□

Remark 3.32. The extension $\hat{\rho}$ can be considered as an ordinary representation over the field L. Its character $\chi_{\hat{\rho}}$ is an extension of $\chi \in \mathrm{Irr}_L(N)$, see [Isa81, Theorem 3.1].

3.5.3 Homeomorphisms and Biholomorphisms of Torus Quotients

In order to describe the representation theoretic problem, we will briefly sketch the geometric setup. For details, we refer to the articles [DG23] and [GK22].

Let G be a finite group acting holomorphically and faithfully on a compact complex torus $T = \mathbb{C}^n/\Lambda$. Such an action is always affine-linear, i.e., of the form

$$\phi(g)z = \rho(g)z + t(g),$$

where the linear part $\rho\colon G \to \mathrm{GL}(n,\mathbb{C})$ is a representation such that $\rho(g) \cdot \Lambda = \Lambda$ and the translation part $t\colon G \to T$ is a 1-cocycle

$$\rho(g)t(h) - t(gh) + t(g) = 0.$$

Here, we view the torus T as a G-module via ρ. Since a quotient of a complex torus by a finite group of translations is again a complex torus, we may assume that ρ is faithful, or equivalently, ϕ is translation-free. Suppose that ϕ' is another action with the same linear part ρ, but a different translation part t'. If these actions are free, or at least *free in codimension one*, then *Bieberbach's theorems* from *crystallographic group theory* (see [Cha86, I]) allow us to decide if the quotients X and X' of T with respect to these actions are homeomorphic or not. It turns out that X and X' are homeomorphic if and only if there exist a matrix $C \in \mathrm{GL}(2n,\mathbb{R})$ with $C \cdot \Lambda = \Lambda$, an automorphism ψ of the group G and an element $d \in T$, such that

(1) $C \cdot \rho_{\mathbb{R}} \cdot C^{-1} = \rho_{\mathbb{R}} \circ \psi$,

(2) $(\rho'_{\mathbb{R}}(g) - \mathrm{id})d = Ct(\psi^{-1}(g)) - t'(g)$ for all $g \in G$.

Here, the representation $\rho_{\mathbb{R}}\colon G \to \mathrm{GL}(2n,\mathbb{R})$ is the *decomplexification* of ρ. If such C and d exist, then a homeomorphism is given by

$$\Xi\colon X \to X', \qquad x \mapsto Cx + d.$$

The quotients X and X' are biholomorphic if and only if C can be chosen as a $\mathbb{C}$-linear matrix, see [GK22, Remark 3.7], [DG23, Remark 4.6] or [HL19, Section 3].

Note that condition (1) says that the representations $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{R}} \circ \psi$ are equivalent. In particular,

$$\psi \in \mathrm{Stab}(\chi_{\mathbb{R}}) := \{\psi \in \mathrm{Aut}(G) \mid \chi_{\mathbb{R}} = \chi_{\mathbb{R}} \circ \psi\}, \qquad \text{where} \quad \chi_{\mathbb{R}} := \mathrm{tr}(\rho_{\mathbb{R}}).$$

Condition (2) says that the cocycles t' and $C \cdot (t \circ \psi^{-1})$ differ by a coboundary, i.e., they are equal in the cohomology group $H^1(G,T)$.

Concretely, if the torus T and the two actions ϕ and ϕ' are explicitly given, one can easily check the second condition, for example by a computer, provided that the full list of candidates for C is known.

The problem to determine the solutions C of the conjugation equation in condition (1) relates to semi-projective representations, in analogy to the extension problem discussed in Subsection 3.5.2, where we had to solve a similiar conjugation equation, see $(*)$. Note that for each $\psi \in \mathrm{Stab}(\chi_{\mathbb{R}})$ the representations $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{R}} \circ \psi$ are equivalent because they have the same character. Thus, there exists a matrix $C_\psi \in \mathrm{GL}(2n, \mathbb{R})$ fulfilling condition (1).

Assume now that ρ is irreducible and of complex type, i.e., the Schur index $m(\chi) = 1$, where $\chi = \mathrm{tr}(\rho)$. Then the matrix C_ψ is unique up to an element in the endomorphism algebra $\mathrm{End}_G(\rho_\mathbb{R}) \simeq \mathbb{C}$. Since $\chi_\mathbb{R} = \chi + \overline{\chi}$, the automorphism ψ either stabilizes χ or maps χ to $\overline{\chi}$. In the first case the matrix C_ψ is $\mathbb{C}$-linear, whereas in the second case $\mathbb{C}$-antilinear. This yields a semi-projective representation

$$f\colon \mathrm{Stab}(\chi_\mathbb{R}) \to \mathrm{PGL}(n, \mathbb{C}) \rtimes \mathbb{Z}/2.$$

Since ρ is faithful, the representation f is also faithful. The candidates for the linear part C of potential homeomorphisms are the elements in the group

$$\mathcal{N} := \{C \in \mathrm{GL}(n, \mathbb{C}) \rtimes \mathbb{Z}/2 \mid [C] \in \mathrm{im}(f),\ C \cdot \Lambda = \Lambda\}.$$

By construction, the group $\mathcal{N}$ sits inside the short exact sequence

$$1 \longrightarrow A \longrightarrow \mathcal{N} \longrightarrow S \longrightarrow 1,$$

where $A := \{\mu \in \mathbb{C}^* \mid \mu\Lambda = \Lambda\}$ and $S \leq \mathrm{Stab}(\chi_\mathbb{R})$ is the subgroup of automorphisms ψ such that $f(\psi)$ has a representative C_ψ with $C_\psi \cdot \Lambda = \Lambda$.

Proposition 3.33. *The group A is a finite cyclic group. In particular, $\mathcal{N}$ is finite.*

Proof. We claim that $|\mu| = 1$ for all $\mu \in A$. Suppose there exists an element $\mu \in A$ with modulus different from 1; note that we can always assume $|\mu| < 1$, otherwise we replace μ by its inverse. Let $v \in \Lambda$ be a non-zero element of minimal norm. Then $w := \mu v \in \Lambda$ has norm strictly less then v, which contradicts the minimality of v. Thus, $|\mu| = 1$ and the map defined by multiplication with μ restricts to closed balls $\overline{B}_r$ of any radius r. If r is chosen large enough so that $\overline{B}_r$ contains a non-zero element of Λ, then the multiplication-homomorphism

$$A \to \mathrm{SYM}\big(\overline{B}_r \cap \Lambda\big), \quad \mu \mapsto (v \mapsto \mu v)$$

is injective (SYM denotes the permutation group of the set $\overline{B}_r \cap \Lambda$). Since Λ is discrete, the intersection $\overline{B}_r \cap \Lambda$ is finite and it follows that A is a finite cyclic group. □

Remark 3.34. The inclusion $i\colon \mathcal{N} \to \mathrm{GL}(n, \mathbb{C}) \rtimes \mathbb{Z}/2$ is by construction a semi-linear lift of the semi-projective representation $f_{|S}\colon S \to \mathrm{PGL}(n, \mathbb{C}) \rtimes \mathbb{Z}/2$.

Example 3.35. We discuss the example from [GK22], the one from [DG23] is similar. Here, the dimension is three and the lattice of the torus $T = \mathbb{C}^3/\Lambda$ is one of the following

$$\Lambda_1 := \mathbb{Z}[\zeta_3]^3 + \langle (u,\ u,\ u) \rangle \quad \text{or} \quad \Lambda_2 := \Lambda_1 + \langle (u,\ -u,\ 0) \rangle,$$

$$\text{where} \qquad u := \frac{1}{3}(1 + 2\zeta_3), \qquad \zeta_r := \exp(2\pi i/r).$$

The group G is here the Heisenberg group of order 27, namely $G = \mathcal{H}_3$. Recall that it can be presented as follows (see Chapter 1, Sec. 1.3)

$$\mathcal{H}_3 = \langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ [a, b] = c,\ aca^2 = c,\ bcb^2 = c \rangle.$$

and that it has two irreducible complex three-dimensional representations: the first one is the *Schrödinger representation* $\rho\colon \mathcal{H}_3 \to \mathrm{GL}(V)$ and the second one is its complex conjugate $\overline{\rho}$. Note that they both have Schur index one. Furthermore, the decomplexification $\rho_{\mathbb{R}}$ of ρ is the unique irreducible 6-dimensional representation of $\mathcal{H}_3$. Hence, $\mathrm{Stab}(\chi_{\mathbb{R}})$ is the full automorphism group $\mathrm{Aut}(\mathcal{H}_3) \simeq \mathrm{AGL}(2,3)$.

In this example, $A = \langle \zeta_6 \rangle \simeq \mathbb{Z}/6$ and, for both lattices Λ_1 and Λ_2, the group $\mathcal{N}$ contains the $\mathbb{C}$-linear maps

$$C_1 := \begin{pmatrix} \zeta_3 & & \\ & \zeta_3^2 & \\ & & 1 \end{pmatrix}, \qquad C_2 := -u \cdot \begin{pmatrix} 1 & \zeta_3^2 & \zeta_3^2 \\ \zeta_3^2 & 1 & \zeta_3^2 \\ \zeta_3^2 & \zeta_3^2 & 1 \end{pmatrix}, \qquad C_3 := u \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{pmatrix}$$

and the $\mathbb{C}$-antilinear map $C_4(z_1, z_2, z_3) = (\bar{z}_1, \bar{z}_2, \bar{z}_3)$. A MAGMA computation shows that the elements $C_1, \dots, C_4$ generate a subgroup of $\mathcal{N}$ of order $2592 = |A| \cdot |\operatorname{Stab}(\chi_{\mathbb{R}})|$. Hence, this subgroup is actually equal to $\mathcal{N}$ and every class in the image of

$$f\colon \mathrm{Stab}(\chi_{\mathbb{R}}) \to \mathrm{PGL}(n, \mathbb{C}) \rtimes \mathbb{Z}/2$$

is represented by an element in $\mathcal{N}$. However, even if the semi-projective representation f lifts to $\mathcal{N}$, this group is not a φ-twisted representation group for the action $\varphi\colon \mathrm{Stab}(\chi_{\mathbb{R}}) \to \mathrm{Aut}(\mathbb{C})$ induced by f. Indeed, a MAGMA computation yields $H^1(\mathrm{Stab}(\chi_{\mathbb{R}}), \mathbb{C}^*) \simeq \mathbb{Z}/3$ and $H^1(\mathcal{N}, \mathbb{C}^*) \simeq \mathbb{Z}/6$, which violates the third condition of Proposition 3.24.

Appendix A. Components of the Moduli Space ($p_g = q = 2$)

For the benefit of the reader, we summarize here the situation concerning the known irreducible components of the moduli space of minimal surfaces of general type with $p_g = q = 2$ and maximal Albanese dimension.

Here are the components of the Main Stream:

- $K^2 = 4$: there is a unique irreducible connected component, of STANDARD surfaces, with $d = 2, \delta = 1$, and branch curve in $|2\Theta|$;
- $K^2 = 5$, $d = 3$: there is a unique irreducible connected component fulfilling the Gorenstein Assumption 0.7, the component of CHPP surfaces;
- $K^2 = 6$, $d = 2$: there are only three irreducible connected components, see [PePo13b] (note that their construction works, in spite of the incorrect assertion that the elliptic singularity maps to a base point of the linear system $|D'|$, where $\mathfrak{F} = \mathfrak{M}_p(D')$: indeed p is a base point of $|D' + Q_i|$, where Q_i is a 2-torsion line bundle);
- $K^2 = 6$, $d = 3$: there is the new component of AC3 surfaces (see Subsection 2.9.1.b and [CS22]);
- $K^2 = 6$, $d = 4$: there is the irreducible connected component of PP4 surfaces, which equals the irreducible one constructed in [PePo14];
- $K^2 = 7, d = 3$: there is the irreducible component of PP7 surfaces, see [PiPo17].
- $K^2 = 8, d = 2$: there is an irreducible connected component of dimension 3 constructed by Penegini in [Pen11] as follows. Let $f : D \to C$ be an étale double cover of a curve C of genus 2, and let $g : C \to C$ be the covering involution. Then $S = (D \times D)/\mathbb{Z}/4$ where the action is free and generated by $(x, y) \mapsto (y, g(x))$.

  The Albanese surface is the Jacobian $Jac(C)$ and the Albanese map factors through $[(x, y)] \mapsto f(x) + f(y) \in C^{(2)}$, and then via the birational morphism $C^{(2)} \to Jac(C)$. In this way the branch locus $\mathcal{B}$ of the Albanese map is a divisor $\mathcal{B} \in |4\Theta|$ with a point O of multiplicity 6, and the sheaf $\mathfrak{F} = \mathfrak{M}_O^2(2\Theta)$ (part of the facts we state here can also be found in [PRR20]).

Still, here are the other known irreducible components of the moduli space of minimal surfaces of general type with $p_g = q = 2$ and maximal Albanese dimension, which are not of the Main Stream:

- $K^2 = 4$: none, there is only the component of STANDARD surfaces;
- $K^2 = 5$: none;
- $K^2 = 6$: none;
- $K^2 = 7$, $d = 2$: there are 3 irreducible components, all of dimension 2, see [PePi22]. For every surface in them, the Albanese surface has a non-simple polarization of type $(1, 2)$ and the branch curve $\mathcal{B} \in |2D|$ has a singularity of type $(3, 3)$;
- $K^2 = 8$, $d = 2$: there are two complex-conjugate rigid minimal surfaces whose universal cover is not biholomorphic to the bidisk $\mathbb{H} \times \mathbb{H}$, [PRR20].
- $K^2 = 8$, $d = 4, 6, 4$: there are here 3 connected components with $K^2 = 8$, two of them of dimension 3 and one of dimension 4, constructed by Penegini in [Pen11] and listed in Table A as items n. $15, 16, 17$; these are surfaces isogenous to a product of unmixed type and not of the Main Stream (their Albanese surface is isogenous to a product of elliptic curves).

  In [Pen13] the author points out that for these families $d \leq 6$ is an upper bound for the degree d of the Albanese map.

  Indeed, as we calculated by hand, confirming a personal communication by Penegini, the respective degrees are (using the order of Table 1 of [Pen11]) $d = 4, 6, 4$ (see Section 2.11). Moreover, the respective monodromy groups of the Albanese covering are
  $$(\mathbb{Z}/2)^2, \mathfrak{S}_3 \times \mathfrak{S}_3, D_4.$$

Remark A.36. (1) The surfaces with $p_g = q = 2$ constructed in [BCF15], as stated in Proposition 4.11 ibidem, lie in the components described in [PePo13b].

(2) Note that the first examples of PP7 surfaces were given in [CanFrap18]. Indeed, in [PiPo17] the authors studied the family of surfaces containing those examples.

Question A.37. Does the case $K_S^2 = 5$, $d = 2$ occur?

We have collected and displayed in Table A some relevant information concerning the known irreducible components of the moduli space of minimal surfaces of general type with $p_g = q = 2$ and maximal Albanese dimension.

Here, items are ordered according to column "n." and each of them provides an irreducible component of the moduli space of surfaces of general type with $p_g = q = 2$ and surjective Albanese map, whose dimension is listed in the column "dim".

The columns labelled with " K_S^2 " and " d " display the self-intersection K_S^2 of the canonical divisor K_S, respectively the degree d of the Albanese map.

Moreover, the column labelled with "Conn." indicates whether the irreducible component is also a connected component, while in the column "Name & References" one can find the references where the component was discovered and/or described, together with its name (either we gave or used in the original reference).

Finally, the column "M. S." specifies whether the component is of the main stream or not.

n.	K_S^2	d	Conn.	M. S.	dim	Name & References
1	4	2	✓	✓	4	STANDARD, [CMLP14]
2	5	3	✓	✓	4	CHPP, [PePo13a], [AC22]
3	6	2	✓	✓	4	$\mathcal{M}_{Ia}$, [PePo13b]
4	6	2	✓	✓	4	$\mathcal{M}_{Ib}$, [PePo13b]
5	6	2	✓	✓	3	$\mathcal{M}_{II}$, [PePo13b]
6	6	3	?	✓	3	AC3, [AC22], [CS22]
7	6	4	✓	✓	4	PP4, [PePo14], [AC22]
8	7	2	?	✗	2	$\overline{\mathcal{M}_1}$, [PePi22]
9	7	2	?	✗	2	$\overline{\mathcal{M}_2}$, [PePi22]
10	7	2	?	✗	2	$\overline{\mathcal{M}_4}$, [PePi22]
11	7	3	?	✓	3	PP7, [PiPo17]
12	8	2	✓	✓	3	[Pen11, Table 1, Mix]
13	8	2	✓	✗	0	[PRR20]
14	8	2	✓	✗	0	[PRR20]
15	8	4	✓	✗	4	[Pen11, Table 1, UnMix, $G = (\mathbb{Z}/2)^2$]
16	8	6	✓	✗	3	[Pen11, Table 1, UnMix, $G = \mathfrak{S}_3$]
17	8	4	✓	✗	3	[Pen11, Table 1, UnMix, $G = D_4$]

Table A: Known irreducible components of the moduli space of minimal surfaces of general type with $p_g = q = 2$ and maximal Albanese dimension.

Remark A.38. (1) From Table A one immediately sees that up to now 17 irreducible components are known, among which 9 are of the Main Stream.

(2) Besides the first examples in the PP7 family, several other surfaces with $p_g = q = 2$ have been constructed in [CanFrap18], and the last section of [Pig20] shows which row of Table A each of these surfaces belongs to. Indeed, this was the main motivation for [Pig20].

Appendix B. MAGMA Code (Twisted Representation Groups)

```
// This is the MAGMA implementation of our algorithm to determine phi-twisted
// representation groups (cf. Subsection 3.6.1)

/*
For a given finite group G and action phi: G -> Aut(C), we want to determine all
phi-twisted representation groups Gamma of G, i.e. we have to determine all extensions
0 -> A -> Gamma -> G -> 1,
where A=H^2(G,C^*), such that
(1) |H^1(G,C^*)| = |H^1(Gamma,C^*)| and
(2) Hom_G(A,C^*)=Hom(A,C^*).

for this, we identify H^j(G,C^*)=H^{j+1}(G,Z), for j=1,2, where G acts on Z via phi
and sending conj to [-1], which gives a character X of G with values in {1,-1}.

Notice that Gamma is a C^*-module via phi$\circ$pi, where pi: Ga -> G is the quotient map.

The main function will therefore has as input the group G and the character X.
We start with two help functions.
*/

/*
The function "Phi" has as input "x=X(g)", for an element g in G, and an element "v" in C,
and determines the value phi(g)(v), which is
v,   if x=[1],
ComplexConjugate(v), if x=[-1].
*/

function Phi(x,v)
Id1:=DiagonalMatrix([1]);
if x eq Id1 then
return v;
else
return ComplexConjugate(v);
end if;
end function;

/*
The function "TestInvariance" has as input the group "A" with "a" generators
and the group "Ga" with "m" generators. The group A is embedded in Ga such that
the generators of A equal the last a generators of Ga. The action of Ga on Z is
encoded in "actGa", which is a list where the i-th entry is the action (as 1x1-matrix)
of the i-th generator of Ga on Z.
The function checks condition (2). For this, we use that Hom(A,C^*)
equals the set of irreducible characters of A. We need to check, whether all of them are
G-invariant, where G acts on A via
g*a:=s(g)as(g^-1), where s:G -> Ga is a section.
We use that the first m-a generators of Ga define preimages of the generators of G
under pi: Ga -> G.
The function returns "true" if condition (2) is fulfilled, "false" otherwise.
```

```
*/

function TestInvariance(A,Ga,actGa,m,a)
CT:=CharacterTable(A);
for x in CT do
for i in [1..m-a] do
for j in [m-a+1..m] do
if not x(Ga.i*Ga.j*Ga.i^-1) eq Phi(actGa[i],x(Ga.j)) then
return false;
end if;
end for;
end for;
end for;
return true;
end function;

/*
The function "KernelCokernelExtension" has as input an extension "Ga" (of G by A),
its image "GaRef" under the Cayley-embedding "f", the number "m" of generators of G and "a"=#A.
It returns the kernel "APer" as subgroup of GaRef and the quotient "Quot"=Ga/APer
and the quotient map "pi":Ga -> Quot.
Note that the kernel A is generated by the last generators of GaRef, the problem is that
we don't know how many generators we have to take (the number can differ from #Generators(A).
Therefore, the last output "i" gives this number of generators of APer.
*/


function KernelCokernelExtension(Ga,GaRef,f,m,a)
for i in [1..m] do
APer:=sub<Ga | [f(GaRef.j): j in [(m-i+1)..m]]>;
if #APer eq a then
Quot, pi:= quo<Ga|APer>;
return APer, Quot, pi, i;
end if;
end for;
end function;

/*************** MAIN FUNCTION **************************************************************/

/*
INPUT: finite, solvable Group G of type GrpPerm, character X of G of degree 1 with values
 in {1,-1} representing an action phi of G on C
OUPUT: A=H^2(G,C^*)(in terms of invariants) and a list of all phi-twisted representation
groups of G

Explanation: the invariants [n_1,...n_k] correspond to the abelian group
Z/{n_1} x ... x Z/{n_k}
*/


function RepGroups(G,X)
g:=#Generators(G);
Id1:=DiagonalMatrix([1]);
act:=[X(G.i)*Id1: i in [1..g]];  // The i-th element of act gives the action of
                                //the i-th generator of G on Z as a 1x1-matrix.
CMG:= CohomologyModule(G,[0],act);
TwistedSchurG:=CohomologyGroup(CMG,3); // TwistedSchurG=H^3(G,Z)=H^2(G,C^*)

invarA:=Moduli(TwistedSchurG); // #invariants of the abelian Group A = #generators of A
if invarA eq [] then // in this case, the twisted Schur multiplier is trivial.
return invarA, G;
end if;
A:=AbelianGroup(GrpPerm,invarA); // A = H^2(G,C^*), of type GrpPerm
a:=#A;

E:=ExtensionsOfSolubleGroup(A,G);
// all candidates for the phi-twisted representation groups, each group in the list is given
```

```
// as GrpFP, the last generators correspond to A


ListRepGroups:=[];
h1G:=#CohomologyGroup(CMG,2);

for k in [1..#E] do
GaRef:=E[k];
f,Ga:= CosetAction(GaRef,sub<GaRef|>); //Transform the extension GaRef into GrpPerm using
the Cayley-embedding f
m:=#Generators(GaRef);
APer, Quot, pi, genA:=KernelCokernelExtension(Ga,GaRef,f,m,a);
test, psi:=IsIsomorphic(Quot,G); // psi: Quot -> G defines an isomorphism
actGa:=[X(psi(pi(Ga.i)))*Id1 : i in [1..m]]; // the action of Ga is given by composing the
action of G with psi and pi.
CMGa:=CohomologyModule(Ga,[0],actGa);
if h1G eq #CohomologyGroup(CMGa,2) and TestInvariance(APer,Ga,actGa,m,genA) then
Append(~ListRepGroups,Ga);
end if;
end for;
return invarA, ListRepGroups;
end function;

// Here we compute the representation groups in Example 3.27

G:=DihedralGroup(4);
CT:=CharacterTable(G);
X:=CT[1];
RepGroups(G,X);

// *****************************************************************************************

// With the MAGMA code from below, we show that the group "N" in Example 3.35 is not a
//covering group for the given action.

F:=CyclotomicField(12);
ze:=F.1^4;
i:=F.1^3;
t:=(1+2*ze)/3;

// The function RI returns real and imaginary part of a complex number "c". */

RI:=function(c)
re:=(c+ComplexConjugate(c))/2;
im:=-i*(c-re);
return [re, im];
end function;

// The function "RealMat" turns a complex 3x3 matrix "D" into a real 6x6 matrix under
// the canonical embedding

RealMat:=function(D)
return Matrix(F, 6, 6,
[RI(D[1][1])[1],-RI(D[1][1])[2],RI(D[1][2])[1],-RI(D[1][2])[2],RI(D[1][3])[1],-RI(D[1][3])[2],
RI(D[1][1])[2],RI(D[1][1])[1],RI(D[1][2])[2],RI(D[1][2])[1],RI(D[1][3])[2],RI(D[1][3])[1],
RI(D[2][1])[1],-RI(D[2][1])[2],RI(D[2][2])[1],-RI(D[2][2])[2],RI(D[2][3])[1],-RI(D[2][3])[2],
RI(D[2][1])[2],RI(D[2][1])[1],RI(D[2][2])[2],RI(D[2][2])[1],RI(D[2][3])[2],RI(D[2][3])[1],
RI(D[3][1])[1],-RI(D[3][1])[2],RI(D[3][2])[1],-RI(D[3][2])[2],RI(D[3][3])[1],-RI(D[3][3])[2],
RI(D[3][1])[2],RI(D[3][1])[1],RI(D[3][2])[2],RI(D[3][2])[1],RI(D[3][3])[2],RI(D[3][3])[1]]);
end function;

// These are the three C-linear matrices C1,..,C3, which generate N.

C1:=DiagonalMatrix([ze,ze^2,1]);
C2:=-t*Matrix([[1,ze^2,ze^2],[ze^2,1,ze^2],[ze^2,ze^2,1]]);
C3:=t*Matrix([[1,1,1],[1,ze^2,ze],[1,ze,ze^2]]);
C4:=Matrix(F,6,6,[1,0,0,0,0,0,0,-1,0,0,0,0,0,0,1,0,0,0,0,0,0,-1,0,0,0,0,0,0,1,0,0,0,0,0,0,-1]);
```

```
// The group of semilinearities SR=<D1,...,D4> as a subgroup of GL(6,F).

N:=sub<GL(6,F)|RealMat(C1),RealMat(C2),RealMat(C3),C4>;
a:=DiagonalMatrix([-ze,-ze,-ze]);
A:=sub<N | RealMat(a)>;

S, pi:=N/A;
I1:=DiagonalMatrix([1]);
CM_N := CohomologyModule(N,[0],[I1,I1,I1,-I1]);
CM_S := CohomologyModule(S,[0],[I1,I1,I1,-I1]);

// These are the cohomology groups H^1(S,C^*) and H^1(N,C^*). They have different order,
// violating the third condition of Proposition 3.24

CohomologyGroup(CM_S,2);
CohomologyGroup(CM_N,2);
```

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Eigene Publikationen

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