# Non-linear stability of matter shells surrounding a point mass

Master's thesis

by

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# Contents

# 1 The Vlasov-Poisson system with a point mass

Most galaxies can be seen as a collection of stars with gravitational interaction and a central black hole. As a simplification, we study a model of a point mass surrounded by particles interacting in a gravitational, non-relativistic way.

## 1.1 Motivation of the system

Before we look at the whole galaxy, we consider one particle with unit mass in a conservative force field. Its dynamical behavior is determined by the Newtonian equations of motion:

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -\partial_x U_{\mathrm{eff}}(t, x). \end{aligned}$$

The variable $x \in \mathbb{R}^3$ denotes the location, $v \in \mathbb{R}^3$ the velocity, and $t \in I$ the time. Furthermore, the conservative force field is induced by a potential $U_{\mathrm{eff}} : I \times \mathbb{R}^3 \to \mathbb{R}^3$ with an interval $I \subset \mathbb{R}$. Here, $\partial_x g = (\partial_{x_1} g, ..., \partial_{x_3} g)^t$ denotes the gradient with respect to $x$ of a differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$.

To describe the galaxy as a whole, we introduce the density function $f(t) = f(t, x, v)$ on the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ for $t \in I$. Since we neglect collisions between particles, $f$ has to be constant along particle trajectories. Let $(x, v) : I \to \mathbb{R}^3$ be the path of a particle. This implies that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}(f(t, x(t), v(t))) \\ &= \partial_t f(t, x(t), v(t)) + \partial_x f(t, x(t), v(t)) \cdot \dot{x}(t) + \partial_v f(t, x(t), v(t)) \cdot \dot{v}(t) \\ &= \partial_t f(t, x(t), v(t)) + \partial_x f(t, x(t), v(t)) \cdot v(t) - \partial_v f(t, x(t), v(t)) \cdot \partial_x U_{\mathrm{eff}}(t, x(t)), \end{aligned}$$

which leads to the *Vlasov equation*:

$$0 = \partial_t f + \partial_x f \cdot v - \partial_v f \cdot \partial_x U_{\mathrm{eff}}.$$

Here, $\cdot$ denotes the Euclidean inner product and $\partial_v$ is defined as the gradient with respect to $v$.

It remains determine the potential $U_{\text{eff}}$. Since the particles interact via gravity, their *spatial density*

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\, dv$$

induces the *potential $U$* given by the *Poisson Equation*

$$\Delta U = 4\pi\rho, \qquad \lim_{|x|\to\infty} U(t,x) = 0.$$

Here, $\Delta = \Delta_x$ denotes the Laplace operator on $\mathbb{R}^3$ with respect to $x$. As we stated before, the particles are surrounding a point mass with mass $M_0 \geq 0$. In addition to the force arising from their interaction, the particles are affected by the potential $-\frac{M_0}{|x|}$ that is induced by the point mass, so we define $U_{\text{eff}} := U - \frac{M_0}{|x|}$.

Combining these equations, we obtain the following non-linear system of differential equations, the so-called *Vlasov-Poisson system with a point mass*:

$$\partial_t f + v \cdot \partial_x f - \partial_x \left( U - \frac{M_0}{|x|} \right) \cdot \partial_v f = 0, \tag{1.1}$$

$$\Delta U = 4\pi\rho, \qquad \lim_{|x|\to\infty} U(t,x) = 0, \tag{1.2}$$

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\, dv. \tag{1.3}$$

A point mass with $M_0 > 0$ restricts the phase space to $\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ since we obtain a singularity at $x = 0$. A solution of this system represents the behavior of the collection of particles.

Without the point mass with $M_0 = 0$ the system above is well-known as the Vlasov-Poisson system, which is already well analyzed with regard to local and global existence for special and less special initial conditions. Besides the existence of solutions, the stability of steady states is an interesting and extensively studied topic in the analysis of kinetic systems.

## 1.2 Steady states with a centered point mass

In the thesis, we construct and study the stability of steady states. Steady states are time-independent solutions of the system. To be more specific, we consider solutions $f = f(x,v)$ of the system

$$v \cdot \partial_x f - \partial_x \left( U - \frac{M_0}{|x|} \right) \cdot \partial_v f = 0, \tag{1.4}$$

$$\Delta U = 4\pi\rho, \qquad \lim_{|x|\to\infty} U(x) = 0, \tag{1.5}$$

$$\rho(x) = \int_{\mathbb{R}^3} f(x,v)\, dv. \tag{1.6}$$

Since galaxies have finite mass, we require finite mass and additionally compact support for the steady states we construct. To be more precise, we consider spherically symmetric steady states with compact support and

$$f(x,v) = 0 \text{ for } |x \times v|^2 \leq L_0 \tag{1.7}$$

for some fixed $L_0 > 0$, similar to [11]. This implies that

$$\operatorname{supp} f \subset \{r^* \leq |x| \leq R^*\} \times \{|v| \leq R^*\}$$

for some $0 < r^* < R^*$, so the collection of particles forms a matter shell.

## 1.3 Outline of the thesis

As mentioned before, we consider spherically symmetric time-independent solutions of the Vlasov-Poisson system with a point mass. Hence, we first define in Chapter 2, more precisely in Section 2.1, the concept of spherical symmetry and derive some properties of spherically symmetric functions. We define the concept of a solution for non-differentiable functions in Section 2.2 and discuss why this definition is reasonable and consistent with the classical concept of a solution. In Section 2.3, we deduce certain properties of solutions and obtain some conserved quantities which are fundamental in the following chapters. Furthermore, we define the concept of a steady states.

In Chapter 3, we construct steady states with compact support and finite mass. To do this, we proceed analogously to [9] and transfer the method to the system with a point mass. First, we analyze general anisotropic steady states and derive certain properties. We use these properties in Section 3.2 and specify the class of anisotropic functions by a separation ansatz. Due to the ansatz function, we reduce the system to an ordinary differential equation and investigate in Section 3.3 whether and under which conditions there exists a unique solution of the ordinary differential equation. In Section 3.4, we determine a necessary condition for the compact support of the solution and examine under which conditions this aspect is satisfied. Finally, in Section 3.5, we summarize in Theorem 3.10 the results and consider two examples, the so-called (generalized) polytropic steady states and the (generalized) King-model.

After constructing spherically symmetric steady states, we analyze in Chapter 4 the stability of certain steady states. For this purpose, we proceed analogously to [4] and transfer the methods to anistropic steady states of the Vlasov-Poisson system with a point mass. We first construct in Section 4.1 a useful conserved quantity, namely the energy-Casimir functional which is constant along spherically symmmetric solutions. In the next Section 4.2, we consider perturbations which respect spherical symmetry and specify the class of perturbations. Furthermore, we introduce a map comparable to a metric on the space of perturbations. Afterwards, we discuss in Section 4.3, more precisely in Theorem 4.9, an estimate which shows that the energy-Casimir functional has a local minimum in the steady state to be investigated and show in Theorem 4.10 that the steady stated is stable. Finally, we consider again the examples given in Chapter 3 and show that under additional assumptions these steady states satisfy the required conditions and thus are stable.

Since the proof of Theorem 4.9 is not simple, we dedicate the entire Chapter 5 to prove this result. For this purpose, we proceed analogously to [4]. We assume that Theorem 4.9 were false and prove in Section 5.1 that there exists a function $g$ such that the second order variation of the energy-Casimir functional is negative. In the next Section 5.2, we introduce the Poisson-bracket and the transport operator and show that the second order variation of the energy-Casimir functional is positive for functions induced by the Poisson-bracket. In order to invert the Poisson-bracket or the transport operator, respectively, we proceed analogously to [5] and define the transport operator in a weak sense. Furthermore, we introduce in Section 5.3 the $(\theta, E, L)$-coordinates and express the transport operator in the new coordinates. Thus, we invert the transport operator in Section 5.4 and show that $g$ is induced by the Poisson bracket and a function $h$. In order to create a contradiction, we regularize the function $h$ in Section 5.5 analogously to [4] and show certain convergences. Finally, we summarize the results in Section 5.6 and show that the assumption that Theorem 4.9 were false leads to a contradiction.

Since we allow non-differentiable solution, we introduce in Chapter 6 the concept of strong Lagrangian solutions and discuss that for spherically symmetric continuous initial conditions with cut-off quantity $L_0$, as described before, and compact support there exist a unique strong Lagrangian solution. For this purpose, we proceed analogously to [6] and transfer the argumentation to the Vlasov-Poisson system with a point mass. To show the existence of the characteristics, we use the method in [11].

# 2 Spherical symmetry and the concept of solution

As stated before, we construct and analyze spherically symmetric time-independent solutions of the Vlasov-Poisson system with a point mass. For this reason, we first introduce the terms *spherical symmetry* and *solution*. The last expression seems trivial, but since we allow continuous, not necessarily differentiable functions as solutions, we have to define this term carefully.

## 2.1 Spherical symmetry

Mostly, we consider functions $f : \mathbb{R}^3 \to \mathbb{R}$ or $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. As is generally known, functions on $\mathbb{R}^3$ are called spherically symmetric if they are invariant under rotations (in $x$). Thus, we call functions on $\mathbb{R}^3 \times \mathbb{R}^3$ spherically symmetric if they are invariant under simultaneous rotations in $x$ and $v$. To be more precise, we define *spherical symmetry on* $\mathbb{R}^3 \times \mathbb{R}^3$ as follows:

**Definition 2.1.** *A function* $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *is* spherically symmetric (on $\mathbb{R}^3 \times \mathbb{R}^3$) *if*

$$f(Ax, Av) = f(x, v), \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

*for every* $A \in \mathrm{SO}(3)$. *Here,* $\mathrm{SO}(3)$ *denotes the special orthogonal group of real-valued* $3 \times 3$ *matrices.*

Spherically symmetric functions on $\mathbb{R}^3$ can be identified with a one-dimensional function in $r := |x|$. A similar behavior can be seen for spherically symmetric functions on $\mathbb{R}^3 \times \mathbb{R}^3$ because they are related to a function on a lower dimensional set:

**Lemma 2.2.** *Let* $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *be spherically symmetric. Then there exists a unique function* $\tilde{f} : ]0, \infty[ \times \mathbb{R} \times [0, \infty[ \to \mathbb{R}$ *such that*

$$f(x, v) = \tilde{f}(r, w, L)$$

*with*

$$r := |x|, \quad w := \frac{x \cdot v}{|x|}, \quad L := |x \times v|^2,$$

*for all* $(x, v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$.

*Proof.* Let $f$ be spherically symmetric and $(x, v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$. Then there exists $A \in \mathrm{SO}(3)$ such that

$$Ax = re_3$$

with $e_3 = (0,0,1)^t$ the third unit vector in $\mathbb{R}^3$. Since $A \in \mathrm{SO}(3)$ and $L = r^2|v|^2 - w^2r^2$, the identities $(Av)_3 = e_3 \cdot (Av) = \frac{Ax \cdot Av}{r} = w$ and $|Av|^2 = |v|^2 = \frac{L}{r^2} + w^2$ hold. Therefore, there exists $B \in \mathrm{SO}(3)$ of the structure

$$B = \begin{pmatrix} \tilde{B} & 0 \\ 0 & 1 \end{pmatrix}$$

with $\tilde{B} \in SO(2)$ satisfying

$$BAv = \begin{pmatrix} \frac{\sqrt{L}}{r} \\ 0 \\ w \end{pmatrix}.$$

The structure of $B$ ensures that $BAx = re_3$ still holds. In summary, we obtain

$$f(x, v) = f(Ax, Av) = f(BAx, BAv) = f\Big(re_3, \Big(\frac{\sqrt{L}}{r}, 0, w\Big)^t\Big).$$

The function $\tilde{f} : ]0, \infty[ \times \mathbb{R} \times [0, \infty[ \to \mathbb{R}$ with $\tilde{f}(r, w, L) = f(re_3, (\frac{\sqrt{L}}{r}, 0, w)^t)$ satisfies the assertion. Finally, the uniqueness of $\tilde{f}$ follows directly from the relation between $f$ and $\tilde{f}$. □

**Remark.** *(a) In the following, we identify under slight abuse of notation $\tilde{f}$ with $f$ for spherically symmetric functions on $\mathbb{R}^3 \times \mathbb{R}^3$. The same applies for spherically symmetric functions $f$ on $\mathbb{R}^3$ with $\tilde{f}$ satisfying $f(x) = \tilde{f}(r)$ for $r = |x| \neq 0$.*

*(b) From a physical point of view, $r := |x| \in [0, \infty[$ denotes the* radius, *$w := \frac{x \cdot v}{|x|} \in \mathbb{R}$ the* radial velocity, *and $L := |x \times v|^2 = |x|^2|v|^2 - (x \cdot v)^2 \in [0, \infty[$ the* angular momentum squared *of a particle $(x, v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$.*

*(c) The definition of $\tilde{f}$ in the last step of the proof more closely shows that properties of $f$ like continuity or differentiability transfer to the function $\tilde{f}$. Vice versa, continuity or differentiability of $\tilde{f}$ lead to the same properties of $f$ on the set $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$.*

The last step of the previous proof leads to a map which can be extended to a $C^1$-diffeomorphism for fixed $x \in \mathbb{R}$ with $x \neq 0$. Additionally, this gives us the possibility to transform integrals over spherically symmetric functions into lower-dimensional integrals:

**Lemma 2.3.** *Fix $r > 0$. The map*

$$T : [0, 2\pi[\times\mathbb{R} \times [0, \infty[\to \mathbb{R}^3,\ (\varphi, w, L) \to \begin{pmatrix} \frac{\sqrt{L}}{r}\cos\varphi \\ \frac{\sqrt{L}}{r}\sin\varphi \\ w \end{pmatrix}$$

*is bijective, and the restriction*

$$T : ]0, 2\pi[\times\mathbb{R}\times]0, \infty[\to \mathbb{R}^3 \setminus \{(v_1, v_2, v_3)^t \in \mathbb{R}^3 | v_1 \geq 0,\ v_2 = 0\}$$

*forms a $C^1$-diffeomorphism with $\det(DT) = \frac{1}{2r^2}$. In particular,*

$$w(re_3, T(\varphi, w, L)) = w, \quad L(re_3, T(\varphi, w, L)) = L$$

*for $\varphi \in [0, 2\pi[$, $w \in \mathbb{R}$ and $L \geq 0$. Furthermore, integrals over spherically symmetric functions $f$ transform into integrals in $(r, w, L)$-coordinates via*

$$\int_{\mathbb{R}^3} f(x, v)\ dv = \frac{\pi}{r^2} \int_{\mathbb{R}} \int_0^\infty f(r, w, L)\ dL\ dw, \quad x \in \mathbb{R}^3 \textit{ with } r = |x| > 0,$$
$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x, v)\ dv\ dx = 4\pi^2 \int_0^\infty \int_{\mathbb{R}} \int_0^\infty f(r, w, L)\ dL\ dw\ dr.$$

*Proof.* We first show that $T$ is bijective. Let $v \in \mathbb{R}^3$ be arbitrary. Defining $w := v_3$ and $L := r^2(v_1^2 + v_2^2)$, there exists a unique $\varphi \in [0, 2\pi[$ such that

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\sqrt{L}}{r} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}.$$

Obviously, the parameters $(\varphi, w, L)$ are unique. Furthermore, the function $T$ and its restriction are well-defined. Since $T$ is continuously differentiable, it follows that

$$DT(\varphi, w, L) = \begin{pmatrix} -\frac{\sqrt{L}}{r}\sin\varphi & 0 & \frac{1}{2\sqrt{L}r}\cos\varphi \\ \frac{\sqrt{L}}{r}\cos\varphi & 0 & \frac{1}{2\sqrt{L}r}\sin\varphi \\ 0 & 1 & 0 \end{pmatrix}$$

with $\det(DT) = \frac{1}{2r^2}$. It remains to show how integrals convert into $(r, w, L)$-coordinates: Let $f$ be spherically symmetric. By change of variable, the previous assertions and the proof of Lemma 2.2 imply

$$\begin{aligned}
\int_{\mathbb{R}^3} f(x, v)\ dv &= \int_{\mathbb{R}^3} f(re_3, v)\ dv \\
&= \frac{1}{2r^2} \int_0^{2\pi} \int_{\mathbb{R}} \int_0^\infty f(re_3, T(\varphi, w, L))\ dL\ dw\ d\varphi \\
&= \frac{\pi}{r^2} \int_{\mathbb{R}} \int_0^\infty f(r, w, L)\ dL\ dw
\end{aligned}$$

for $x \in \mathbb{R}^3$ with $r = |x| > 0$ and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x,v)\; dv\; dx = \int_{\mathbb{R}} \frac{\pi}{r^2} \int_{\mathbb{R}} \int_0^\infty f(r,w,L)\; dL\; dw\; d\varphi\; dx$$
$$= 4\pi^2 \int_0^\infty \int_{\mathbb{R}} \int_0^\infty f(r,w,L)\; dL\; dw\; dr. \qquad \square$$

**Remark.** *Unless specified differently, we use in the following argumentation the abbreviation*

$$\int g(z)\; dz = \int_{\mathbb{R}^3} g(z)\; dz$$

*for integrable functions* $g : \mathbb{R}^3 \to \mathbb{R}$.

As discussed before, the Vlasov-Poisson system with a point mass with mass $M_0 > 0$ is only defined on the set $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$, so we consider spherically symmetric functions on $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$. Obviously, the definitions and assertions above remain valid for functions defined on $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$. Even though we consider only solutions $f$ defined on $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$, we can transfer the following results to the case $M_0 = 0$ by extending them appropriately into $x = 0$.

Finally, we use these properties and define spherical symmetry on $\mathbb{R}^3 \times \mathbb{R}^3$ for functions defined almost everywhere (a.e.). As in [12, Lemma & Definition 2.10], we obtain the following equivalences and definition:

**Lemma and Definition 2.4.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)$ be a pointwise defined representative. Then the following assertions are equivalent:*

*(i) For all $A \in \mathrm{SO}(3)$ there exists a null set $N_A \subset \mathbb{R}^3 \times \mathbb{R}^3$ such that $f(x,v) = f(Ax, Av)$ for $(x,v) \in (\mathbb{R}^3 \times \mathbb{R}^3) \setminus N_A$.*

*(ii) There exists a null set $N \subset \mathbb{R}^3 \times \mathbb{R}^3$ such that $f(x,v) = f(Ax, Av)$ for all $A \in \mathrm{SO}(3)$ and $(x,v) \in (\mathbb{R}^3 \times \mathbb{R}^3) \setminus N$.*

*(iii) There exists a function $\tilde{f} : [0,\infty[ \times \mathbb{R} \times [0,\infty[ \to \mathbb{R}$ such that $f(x,v) = \tilde{f}(r,w,L)$ for a.e. $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $(r,w,L)$ as defined in Lemma 2.2.*

*The function $f$ is called* spherically symmetric (almost everywhere) *if these properties are satisfied.*

*Proof.* In [12, Lemma & Definition 2.10], the equivalence of the assertions (i) and (iii) is shown. With similar arguments as in Lemma 2.2, we obtain the equivalence of the assertions (ii) and (iii). $\square$

## 2.2 Concept of a solution

After spherical symmetry, we turn to the concept of a *solution of the Vlasov-Poisson system with a point mass*. We first look at smooth solutions and afterwards generalize the concept of a solution based on these assertions.

At first, we have a closer look on the Poisson equation. We introduce the term *induced density* and *induced potential*:

**Definition 2.5.** *Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be integrable. The* induced density *$\rho_f$ is defined by*

$$\rho_f(x) := \int_{\mathbb{R}^3} f(x,v)\ dv, \quad x \in \mathbb{R}^3.$$

A function $f : \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to \mathbb{R}$ leads to an induced density $\rho_f$ defined on $\mathbb{R}^3 \backslash \{0\}$.

**Definition 2.6.** *Let $\rho \in L^1_{\text{loc}}(\mathbb{R}^3)$ with $\mathbb{R}^3 \ni y \mapsto \frac{\rho(y)}{|x-y|}$ integrable for a.e. $x \in \mathbb{R}^3$. The* induced potential *is defined by*

$$U_\rho(x) := -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\ dy, \quad x \in \mathbb{R}^3.$$

*If $\rho = \rho_f$ for some function $f$, we denote $U_f := U_{\rho_f}$.*

With these definitions, we can solve the Poisson equation for smooth solutions with compact support and obtain the following properties, as discussed in [10, Lemma P1]:

**Lemma 2.7.** *Let $\rho \in C^1_c(\mathbb{R}^3)$. Then $U_\rho$ is the unique solution in $C^2(\mathbb{R}^3)$ of the Poisson equation*

$$\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(x) = 0.$$

*Furthermore, the induced potential has the following properties:*

*(a) The derivative of $U_\rho$ has the form*

$$\nabla U_\rho(x) = \int \frac{x-y}{|x-y|^3}\rho(y)\ dy, \quad x \in \mathbb{R}^3.$$

*(b) The estimate*

$$\|\nabla U_\rho\|_\infty \le c_p \|\rho\|_p^{\frac{p}{3}} \|\rho\|_\infty^{1-\frac{p}{3}}$$

*is satisfied for $1 \le p < 3$ with $c_p$ independent of $\rho$. In particular, $c_1 = 3(2\pi)^{\frac{2}{3}}$.*

*(c) The identities $U_\rho = \mathcal{O}(\frac{1}{|x|})$ and $\nabla U_\rho = \mathcal{O}(\frac{1}{|x|^2})$ hold for $|x| \to \infty$.*

*Proof.* Since $\rho \in C_c^1(\mathbb{R}^3)$, the density $\rho$ satisfies $\rho \in L^1 \cap L^\infty(\mathbb{R}^3)$ and is Hölder continuous. Thus, the induced potential $U_\rho$ solves the Poisson equation and is the unique solution in $C^2(\mathbb{R}^3)$. Furthermore, it is shown in [2, Lemma 4.1] that $\nabla U_\rho$ has the structure as stated in (a). The other properties are proven in [10, Lemma P1], or more detailed in [8, Lemma 2.3]. □

If we consider the Poisson equation in the sense of distributions, the induced potential still remains the solution of the Poisson equation, and under certain conditions we obtain similar properties as in Lemma 2.7:

**Lemma 2.8.** *Let $\rho \in L^1_{\text{loc}}(\mathbb{R}^3)$ with $\mathbb{R}^3 \ni y \mapsto \frac{\rho(y)}{|x-y|}$ integrable for a.e. $x \in \mathbb{R}^3$. Then the induced potential $U_\rho$ is locally integrable and solves*

$$\Delta U_\rho = 4\pi\rho \text{ in } \mathcal{D}'(\mathbb{R}^3).$$

*Furthermore, the following assertions hold:*

*(a) The distributional derivative of $U_\rho$ exists with*

$$\nabla U_\rho(x) = \int \frac{x-y}{|x-y|^3}\rho(y)\, dy \text{ for a.e. } x \in \mathbb{R}^3,$$

*and $\nabla U_\rho \in L^1_{\text{loc}}(\mathbb{R}^3;\mathbb{R}^3)$.*

*(b) If $\rho \in L^1 \cap L^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$, the induced potential $U_\rho$ is continuous on $\mathbb{R}^3$ with $\lim_{|x|\to\infty} U(x) = 0$. If additionally $p > 3$, the induced potential $U_\rho$ is once continuously differentiable, and $U_\rho \in W^{2,q}(\mathbb{R}^3)$ for all $q > 3$ with $\nabla U_\rho \in L^2(\mathbb{R}^3)$.*

*(c) The estimate in Lemma 2.7 (b) remains true for $\rho \in L^1 \cap L^\infty(\mathbb{R}^3)$, i.e., for $1 \le p < 3$, there exists a constant $c_p$ independent of $\rho$ with $c_1 = 3(2\pi)^{\frac{2}{3}}$ such that*

$$\|\nabla U_\rho\|_\infty \le c_p \|\rho\|_p^{\frac{p}{3}} \|\rho\|_\infty^{1-\frac{p}{3}}.$$

*(d) For $\rho \in L^1(\mathbb{R}^3)$ with compact support, the assertion in Lemma 2.7 (c) stays valid, i.e., $U_\rho = \mathcal{O}(\frac{1}{|x|})$ and $\nabla U_\rho = \mathcal{O}(\frac{1}{|x|^2})$ for $|x| \to \infty$.*

*Proof.* As proven in [7, Theorem 6.21], the induced potential $U_\rho \in L^1_{\text{loc}}(\mathbb{R}^3)$ solves the Poisson equation in the sense of distributions, and $\nabla U_\rho$ has the particular form given in (a) with $\nabla U_\rho \in L^1_{\text{loc}}(\mathbb{R}^3;\mathbb{R}^3)$. If $\rho \in L^1 \cap L^p(\mathbb{R}^3)$, we obtain that $U_\rho \in C(\mathbb{R}^3)$ with $\lim_{|x|\to\infty} U_\rho(x) = 0$ for $p > \frac{3}{2}$ and that $U_\rho \in C^1(\mathbb{R}^3)$ for $p > \frac{3}{2}$. The corresponding proof is discussed more detailed in the proof of Lemma 2.15.

Furthermore, if $\rho \in L^{\tilde{q}}(\mathbb{R}^3)$, the Hardy-Littlewood-Sobolev inequality implies that the induced potential and its derivative has the integrability $U_\rho \in L^r(\mathbb{R}^3)$ and $\nabla U_\rho \in L^s(\mathbb{R}^3)$ with $r = (\frac{1}{\tilde{q}} - \frac{2}{3})^{-1}$ and $s = (\frac{1}{\tilde{q}} - \frac{1}{3})^{-1}$ for $1 < \tilde{q} < \frac{3}{2}$ respectively $1 < \tilde{q} < 3$. If we allow $\rho \in L^1 \cap L^p(\mathbb{R}^3)$ with $p > 3$, this implies $U_\rho \in W^{2,q}$ with $q \in ]3, \infty[$, so assertion (b) is proven.

In the proof of the assertions (b) and (c) in Lemma 2.7, it is only used that $\rho$ has a compact support and $\rho \in L^1(\mathbb{R}^3)$ to show the assertion (c). Moreover, $\rho \in L^1 \cap L^\infty(\mathbb{R}^3)$ is sufficient for the proof of assertion (b), so each proof can be transferred literally for each case. $\square$

In the following chapters we consider spherically symmetric continuous densities. Under these conditions, we are able to show that the induced potential solves the Poisson equation in the classical sense:

**Lemma 2.9.** *Let $\rho : \mathbb{R}^3 \backslash \{0\} \to [0, \infty[$ be continuous and spherically symmetric with $\rho \in L^1(\mathbb{R}^3)$ and $\|\rho\|_1 > 0$. Then the induced potential $U = U_\rho$ has the form*

$$U(r) = -\frac{4\pi}{r} \int_0^r s^2 \rho(s)\; ds - 4\pi \int_r^\infty s\rho(s)\; ds$$

*with*

$$U'(r) = \frac{m(r)}{r^2}, \quad m(r) = 4\pi \int_0^r s^2 \rho(s)\; ds,$$

*and*

$$U''(r) = -2\frac{m(r)}{r^3} + 4\pi\rho(r), \quad r > 0.$$

*Furthermore, $U$ solves the Poisson equation with $U \in C^2(\mathbb{R}^3 \backslash \{0\})$.*

The conditions in Lemma 2.9 can be weakened by allowing non-continuous densities, but later we only consider continuous ones.

*Proof.* Standard calculations and the fundamental theorem of calculus yield the claimed representations of $U$, $U'$, and $U''$. Having a closer look on $U''$, we recognize that

$$U''(r) = -2\frac{U'}{r} + 4\pi\rho(r), \quad r > 0.$$

and

$$\Delta U = \frac{1}{r^2}(r^2 U')' = 2\frac{U'}{r} + U'' = 4\pi\rho. \qquad \square$$

After analyzing the Poisson equation, we turn to the Vlasov equation. Since we allow continuous, but not necessarily differentiable functions, we have to find a more general description of the Vlasov equation that the classical one.

Let $f \in C^1(I \times \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ be a smooth solution of the Vlasov-Poisson system with a point mass with an interval $I \subset \mathbb{R}$. Let $(t, x, v) \in I \times \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ be arbitrary, and let $(X, V)(\cdot, t, x, v) : I \to \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ be the solution of the *characteristic system*

$$\dot{x} = v,$$

$$\dot{v} = -\partial_x \left( U(s,x) - \frac{M_0}{|x|} \right)$$

with $(X,V)(t,t,x,v) = (x,v)$. As a smooth solution, the function $f$ satisfies the Vlasov equation, so it follows with $Z = (X,V)$ that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}(f(s,Z(s,t,x,v))) = \partial_t f(s,Z(s,t,x,v)) + \partial_x f(s,Z(s,t,x,v)) \cdot \dot{X}(s,t,x,v) \\
&\qquad\qquad\qquad\qquad + \partial_v f(s,Z(s,t,x,v)) \cdot \dot{V}(s,t,x,v) \\
&= \left( \partial_t f(s,x,v) + \partial_x f(s,x,v) \cdot v - \partial_v f(s,x,v) \cdot \partial_x \left( U(s,x) - \frac{M_0}{|x|} \right) \right) \Big|_{(x,v)=(X,V)(s,t,x,v)} \\
&= 0
\end{aligned}$$

for $s \in I$. This implies that $f$ is constant along *characteristics*, i.e., the solutions of the characteristic system. While the Vlasov equation is only defined for differentiable functions, we can determine whether a function is constant along characteristics without demanding differentiability. Therefore, we can use this observation to generalize the concept of a solution of the Vlasov equation.

Finally, we define the term *solution*:

**Definition 2.10.** *A function $f : I \times \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to [0,\infty[$ with $I$ an interval is a* solution of the Vlasov-Poisson system with a point mass *if the following assertions are satisfied:*

*(i) The induced density $\rho = \rho_f$ and the induced potential $U = U_f$ exist, and $U$ solves the Poisson equation*

$$\Delta U = 4\pi\rho, \qquad \lim_{|x|\to\infty} U(t,x) = 0.$$

*in the classical sense.*

*(ii) For all $(t,x,v) \in I \times \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ with $|x \times v|^2 > 0$, there exists a unique solution $Z(\cdot,t,x,v) : I \to \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ of the* characteristic system

$$\begin{aligned}
\dot{x} &= v, \\
\dot{v} &= -\partial_x \left( U(s,x) - \frac{M_0}{|x|} \right)
\end{aligned}$$

*with $Z(t,t,x,v) = (x,v)$. Then $Z(\cdot,t,x,v)$ is called a* characteristic *and $Z$ the* characteristic flow.

*(iii) The function $f$ is constant along characteristics, i.e., for $(t,x,v) \in I \times \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ with $|x \times v|^2 > 0$,*

$$f(s,Z(s,t,x,v)) = f(t,x,v), \quad s \in I.$$

This definition is only reasonable if it generalizes the classical concept of a solution. Therefore, we have to show that for smooth functions the definition of classical solutions similar to [10, Definition, p.393] and Definition 2.10 are equivalent.

**Remark.** *In [10, Definition, p.393], the boundedness of $\partial_x U$ set in condition (iii) ensures that the characteristic flow exists globally. In our setting, this conditions is not enough to guarantee the existence of the characteristic flow. That is the reason why we require instead of condition (iii) that the characteristic flow $Z(\cdot,t,x,v) : I \to \mathbb{R}^3\backslash\{0\}\times\mathbb{R}^3$ exists for all $t \in I$ and $(x,v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ with $|x \times v|^2 > 0$.*

First, we analyze the characteristic flow. Analogously to the Vlasov-Poisson system without a point mass, we obtain similar properties as in [10, Lemma 1.2]:

**Lemma 2.11.** *Let $I$ be an interval and $U \in C^{0,2}(I \times \mathbb{R}^3\backslash\{0\})$ spherically symmetric. Furthermore, for every $t \in I$ and $(x,v) \in \mathbb{R}^3\backslash\{0\}\times\mathbb{R}^3$ with $|x\times v|^2 > 0$ assume that there exists a unique solution $Z(\cdot,t,x,v) : I \to \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ of the associated characteristic system*

$$\begin{aligned}
\dot{x} &= v,\\
\dot{v} &= -\partial_x \left( U(s,x) - \frac{M_0}{|x|} \right)
\end{aligned}$$

*with $Z(t,t,x,v) = (x,v)$. Then the following assertions hold:*

*(a) The characteristic flow $Z : I\times I\times\{L > 0\} \to \{L > 0\}$ is continuously differentiable with $\{L > 0\} := \{(x,v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 | L(x,v) = |x \times v|^2 > 0\}$.*

*(b) For every $s,t \in I$, the map $Z(s,t,\cdot) : \{L > 0\} \to \{L > 0\}$ is a $C^1$-diffeomorphism with inverse $Z(s,t,\cdot)^{-1} = Z(t,s,\cdot)$. In addition, the map is measure preserving, i.e.,*

$$\det \frac{\partial Z}{\partial z}(s,t,x,v) = 1, \quad s,t \in I,\ z = (x,v) \in \{L > 0\}.$$

*Proof.* The fact that $Z$ is well-defined follows by Lemma 2.13 below where we see that $L$ is conserved along characteristics. The characteristic flow is continuously differentiable, since the right-hand side of the characteristic system is continuous in $t$ and continuously differentiable in $(x,v)$. Furthermore, we require the uniqueness of the solutions of the characteristic system, which implies $Z(r,s,Z(s,t,x,v)) = Z(r,t,x,v)$ for $r,s,t \in I$ and $(x,v) \in \{L > 0\}$. As a result, we obtain $Z(s,t,\cdot)^{-1} = Z(t,s,\cdot)$ for $s,t \in I$, so it follows directly by (a) that $Z(s,t,\cdot)$ is a $C^1$-diffeomorphism on $\{L > 0\}$. To prove the property that $Z$ is measure preserving, we calculate the first derivative of the Jacobian determinant with respect to $s \in I$ for fixed $t \in I$ and $z = (x,v) \in \{L > 0\}$. As discussed in [10],

$$\frac{\mathrm{d}}{\mathrm{ds}} \det \frac{\partial Z}{\partial z}(s,t,z) = (\mathrm{div}_z G)(s, Z(s,t,z)), \quad s \in I,$$

with $G$ the right-hand side of the characteristic system, i.e.,

$$G(s,x,v) := \begin{pmatrix} v \\ -\partial_x \left( U(s,x) - \frac{M_0}{|x|} \right) \end{pmatrix}$$

for $s \in I$ and $(x,v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$. For more detail, we refer to [8, Lemma 2.4]. Since $G$ is free of divergence, this shows that $\det \frac{\partial Z}{\partial z}(\cdot, t, z)$ is constant. In particular, this implies that

$$\det \frac{\partial Z}{\partial z}(s,t,x,v) = \det \frac{\partial Z}{\partial z}(t,t,x,v) = 1, \quad s \in I. \qquad \square$$

Since the characteristic flow $(X,V)(s,t,\cdot)$ is a $C^1$-diffemorphism on $\{L > 0\}$ for fixed $s,t \in I$, the above lemma leads to the desired equivalence of classical solutions and solutions in the sense of 2.10:

**Lemma 2.12.** *Assume that the conditions as in Lemma 2.11 hold. Furthermore, let $f \in C^1(I \times \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$. Then $f$ is constant along characteristics if and only if $f$ solves the Vlasov equation.*

*Proof.* As shown before,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}(f(s, Z(s,t,x,v))) \\ &= \left( \partial_t f(s,x,v) + \partial_x f(s,x,v) \cdot v - \partial_v f(s,x,v) \cdot \partial_x \left( U(s,x) - \frac{M_0}{|x|} \right) \right) \Big|_{(x,v)=(X,V)(s,t,x,v)} \end{aligned}$$

on $\{L > 0\}$. Since $(X,V)(s,t,\cdot)$ is a $C^1$-diffemorphism $\{L > 0\}$, the equivalence holds on $\{L > 0\}$. If the Vlasov equation is satisfied on $\{L > 0\}$, the regularities of $f$ and $U$ yield the assertion on $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$. $\square$

In conclusion, we have shown that classical solutions are also solutions in the sense of Definition 2.10, so our definition of *solutions of the Vlasov-Poisson system with a point mass* is reasonable.

## 2.3 Conserved quantities and steady states

In order to complete the proof of Lemma 2.11, we collect some useful conserved quantities:

**Lemma and Definition 2.13.** *Let $U : I \times \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}$ be a spherically symmetric with $I \subset \mathbb{R}$ an interval such that the characteristic flow $Z$ exists on $\{L > 0\}$. Assume that $U$ is once differentiable with respect to $x$. Then the* angular momentum squared

$$L(x,v) := |x \times v|^2, \quad (x,v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3,$$

*is constant along characteristics.*

*If $U$ is additionally time-independent, the* particle energy

$$E(x,v) := \frac{1}{2}|v|^2 + U(x) - \frac{M_0}{|x|}, \quad (x,v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3,$$

*is also constant along characteristics and spherically symmetric with*

$$E(r,w,L) = \frac{1}{2}w^2 + \frac{L}{2r^2} + U(r) - \frac{M_0}{r}, \quad (r,w,L) \in ]0,\infty[\times\mathbb{R}\times[0,\infty[.$$

*Proof.* Let $t \in I$ and $(x,v) \in \{L > 0\}$. Since $U(t)$ is spherically symmetric, $\partial_x U(t,x) = U'(t,r)\frac{x}{r}$ for $x \in \mathbb{R}^3\backslash\{0\}$. With the abbreviation $(X,V)(s) = (X,V)(s,t,x,v)$, this leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{ds}}(X(s)\times V(s)) &= \dot{X}(s)\times V(s) + X(s)\times\dot{V}(s)\\ &= V(s)\times V(s) - \partial_r\left(U(t,|X(s)|) - \frac{M_0}{|X(s)|}\right)\frac{X(s)}{|X(s)|}\times X(s) = 0.\end{aligned}$$

If $U(t,x) = U(x)$ is time-independent, this yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{ds}}(E(X(s),V(s))) = V(s)\cdot\dot{V}(s) + \partial_x\left(U(x) - \frac{M_0}{|x|}\right)\Big|_{x=X(s)}\cdot\dot{X}(s)\\ &= -V(s)\cdot\partial_x\left(U(x) - \frac{M_0}{|x|}\right)\Big|_{x=X(s)} + \partial_x\left(U(x) - \frac{M_0}{|x|}\right)\Big|_{x=X(s)}\cdot V(s) = 0.\end{aligned}$$

Since $L = r^2|v|^2 - w^2r^2$, we obtain $|v|^2 = \frac{L}{r^2} + w^2$, so the rewriting of the particle energy into $(r,w,L)$-coordinates and the spherical symmetry follow directly. □

Since solutions of the Vlasov-Poisson system with a point mass are constant along characteristics, we obtain similar to [10, Lemma 1.3] the following properties:

**Remark 2.14.** *Let $U \in C^{0,2}(I\times\mathbb{R}^3\backslash\{0\})$ be a spherically symmetric with $I \subset \mathbb{R}$ an interval such that the characteristic flow $Z$ exists on $\{L > 0\}$. Assume that $0 \in I$ and that $f : I\times\mathbb{R}^3\backslash\{0\}\times\mathbb{R}^3 \to [0,\infty[$ is constant along characteristics with $f(0) = \mathring{f}$. Then $f$ has the structure*

$$f(t,x,v) = \mathring{f}((X,V)(0,t,x,v))$$

*for $t \in I$ and $(x,v) \in \mathbb{R}^3\backslash\{0\}\times\mathbb{R}^3$ with $(x,v) \in \{L > 0\}$. Furthermore, the following holds:*

*(a) The support has the property*

$$\operatorname{supp} f(t) = Z(t,0,\operatorname{supp}\mathring{f}), \quad t \in I,$$

*if* $\operatorname{supp}\mathring{f} \subset \{L > 0\}$.

(b) *The $L^p$-norm is invariant under solutions for every $p \in [1, \infty]$, i.e.,*

$$\|f(t)\|_p = \|\mathring{f}\|_p, \quad t \in I.$$

(c) *The induced density can be estimated by*

$$\|\rho(t)\|_p \le \left(\frac{4\pi}{3} P^3(t)\right)^{\frac{1}{q}} \|f(t)\|_p = \left(\frac{4\pi}{3} P^3(t)\right)^{\frac{1}{q}} \|\mathring{f}\|_p,$$

*with $P(t) := \sup\{|v| | (x, v) \in \operatorname{supp} f(t)\}$ and $q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We denote $\frac{1}{\infty} := 0$. In particular, equality holds for $p = 1$. If $\operatorname{supp} \mathring{f} \subset \{L > 0\}$ is compact, then $P(t) < \infty$.*

*Proof.* Since $f$ is constant along characteristics, we obviously obtain the structure $f(t, x, v) = \mathring{f}((X, V)(s, t, x, v))$ for $t \in I$ and $(x, v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ with $L(x, v) > 0$. This implies the formula for the support of $f$ in (a). According to Lemma 2.11, the characteristic flow is a measure preserving $C^1$-differomorphism, so the change of variables yields (b) since $\{L = 0\}$ is a null set. Assertion (c) follows directly from Hölder's inequality. Note that assertion (a) implies

$$P(t) = \sup\{|v| | (x, v) \in \operatorname{supp} f(t)\} = \sup\{|V(t, s, z)| | z \in \operatorname{supp} f(s)\}, \quad t \in I,$$

if $\operatorname{supp} \mathring{f} \subset \{L > 0\}$. □

If the solution of the Vlasov-Poisson system with a point mass is appropriately integrable, the induced potential is additionally differentiable in $t$:

**Lemma 2.15.** *Let $f : I \times \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 \to [0, \infty[$ be a solution of the Vlasov-Poisson system with a point mass with $I$ an open interval. Assume that $\mathring{f} := f(0) \in L^1 \cap L^p(\mathbb{R}^6)$ with compact support $\operatorname{supp} \mathring{f} \subset \{L > 0\}$ for some $p \in ]3, \infty]$. Then the induced potential $U$ is once continuously differentiable in $(t, x) \in I \times \mathbb{R}^3$ with*

$$\partial_t U(t, x) = -\iint \frac{x - y}{|x - y|^3} \cdot u f(t, y, u)\, du\, dy, \quad t \in I,\ x \in \mathbb{R}^3.$$

*Proof.* Let $f$ be a solution as required and $a < 0 < b$ with $J := [a, b] \subset I$. According to Remark 2.14, the induced density $\rho(t)$ is bounded in $L^p(\mathbb{R}^3)$ independently of $t$ on $[a, b]$ because

$$\|\rho(t)\|_p \le C P^{\frac{3}{q}}(t) \|\mathring{f}\|_p \le C (P^*)^{\frac{3}{q}} \|\mathring{f}\|_p, \quad t \in [a, b]$$

with $\frac{1}{p} + \frac{1}{q} = 1$. As in Remark 2.14, $P(t)$ is defined by $P(t) = \sup\{|v| | (x, v) \in \operatorname{supp} f(t)\}$, and we denote $P^* := \sup\{|v| | (x, v) \in \operatorname{supp} f(t),\ t \in [a, b]\}$. Note that the support $\operatorname{supp} \mathring{f} \subset \{L > 0\}$ is compact by assumption, so Remark 2.14 (a) yields

$$P^* = \sup\{|v| | (x, v) \in \operatorname{supp} f(t),\ t \in [a, b]\}$$

$$= \sup\{|V(t,0,z)| | z \in \operatorname{supp} \mathring{f},\ t \in [a,b]\}.$$

Since the characteristic flow $Z$ is a measure preserving $C^1$-diffeomorphism, as shown in Lemma 2.11, we obtain $P^* < \infty$.

Analogously to the proof [2, Lemma 4.1], we require $w \in C^\infty(\mathbb{R})$ with $0 \le w \le 1$, $0 \le w' \le 1$ and $w(s) = 0$ for $s \le 1$ and $w(s) = 1$ for $s \ge 2$. Let $\varepsilon > 0$ be arbitrary. We define

$$\begin{aligned} U_\varepsilon(t,x) &:= -\iint \frac{f(t,y,u)}{|x-y|} w\left(\frac{|x-y|}{\varepsilon}\right) dv\, dy \\ &= -\iint \frac{\mathring{f}(y,v)}{|x - X(t,0,y,v)|} w\left(\frac{|x-X(t,0,y,v)|}{\varepsilon}\right) dv\, dy. \end{aligned}$$

Since the characteristic flow is continuously differentiable according to Lemma 2.11, $U_\varepsilon \in C^1(I \times \mathbb{R}^3)$ holds with

$$\partial_x U_\varepsilon(t,x) = \iint \left(\frac{x-y}{|x-y|^3} w\left(\frac{|x-y|}{\varepsilon}\right) - \frac{x-y}{\varepsilon|x-y|^2} w'\left(\frac{|x-y|}{\varepsilon}\right)\right) f(t,y,v) dv\, dy$$

and

$$\begin{aligned} \partial_t U_\varepsilon(t,x) = -\iint & \left(\frac{x - X(t,s,y,v)}{|x-X(t,s,y,v)|^3} \cdot \dot{X}(t,s,y,v) w\left(\frac{|x-X(t,0,y,v)|}{\varepsilon}\right)\right. \\ & \left. - \frac{x - X(t,s,y,v)}{\varepsilon|x-X(t,s,y,v)|^2} \cdot \dot{X}(t,s,y,v) w'\left(\frac{|x-X(t,0,y,v)|}{\varepsilon}\right)\right) \mathring{f}(y,v)\, dv\, dy \\ = -\iint & \left(\frac{x-y}{|x-y|^3} \cdot vw\left(\frac{|x-y|}{\varepsilon}\right) - \frac{x-y}{\varepsilon|x-y|^2} \cdot vw'\left(\frac{|x-y|}{\varepsilon}\right)\right) f(t,y,v)\, dv\, dy \end{aligned}$$

for $t \in I$ and $x \in \mathbb{R}^3$. Using that $\operatorname{supp}_{(x,v)\in \operatorname{supp} f(t)} |v| \le P^*$ and $\|\rho(t)\|_p \le C(P^*)^{\frac{3}{q}}$ for $t \in J$, some lines of calculations and estimates yield

$$\begin{aligned} |U_\varepsilon(t,x) - U(t,x)| &\le C\|\rho(t)\|_p \le C(P^*)^{\frac{3}{q}} \varepsilon^{\frac{3-q}{q}} \to 0, \\ |\partial_x U_\varepsilon(t,x) - V(t,x)| &\le C\|\rho(t)\|_p \le C(P^*)^{\frac{3}{q}} \varepsilon^{\frac{3-2q}{q}} \to 0, \\ |\partial_t U_\varepsilon(t,x) - \tilde{V}(t,x)| &\le C\|\rho(t)\|_p \le C(P^*)^{\frac{3}{q}} \varepsilon^{\frac{3-2q}{q}} \to 0 \end{aligned}$$

as $\varepsilon \to 0$ with

$$\begin{aligned} V(t,x) &= \iint \frac{x-y}{|x-y|^3} f(t,y,u)\, du\, dy, \\ \tilde{V}(t,x) &= -\iint \frac{x-y}{|x-y|^3} \cdot u f(t,y,u)\, du\, dy, \quad t \in J,\ x \in \mathbb{R}^3. \end{aligned}$$

Note that $p > 3$ and thus $q < 3$. Since $U$, $\partial_x U$ and $\partial_t U$ are continuous and converges uniformly on $J \times \mathbb{R}^3$, the limiting functions $U$, $V$ and $\tilde{V}$ are continuous as well. In particular, $V = \partial_x U$ and $\tilde{V} = \partial_t U$. Since $a < 0 < b$ with $J = [a,b] \subset I$ are arbitrary, the assertions hold on $I \times \mathbb{R}^3$, so the proof is complete. $\square$

After having introduced the concept of a solution, we now consider time-independent spherically symmetric potentials. Under suitable conditions, we can show that the characteristic flow exists on $\{L > 0\}$:

**Lemma 2.16.** *Let $U \in C^2(\mathbb{R}^3 \backslash \{0\})$ spherically symmetric with $\partial_x U$ bounded on $\mathbb{R}^3 \backslash \{0\}$. Assume that $\lim_{|x| \to \infty} U(x) = 0$. Then for all $t \in \mathbb{R}$ and all $(x, v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ with $|x \times v|^2 > 0$, there exists a unique solution $Z(\cdot, t, x, v) : \mathbb{R} \to \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ of the associated characteristic system*

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -\partial_x \left( U(x) - \frac{M_0}{|x|} \right) \end{aligned}$$

*with $Z(t, t, x, v) = (x, v)$.*

*Proof.* Let $(\mathring{t}, \mathring{x}, \mathring{v}) \in \mathbb{R} \times \{L > 0\}$ arbitrary. The right-hand side of the characteristic system is continuously differentiable and thus locally Lipschitz continuous. By the Picard-Lindelöf theorem, there exists a maximal solution $(x, v) : I \to \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ on an open interval $I = ]t_1, t_2[$ with $-\infty \le t_1 < \mathring{t} < t_2 \le \infty$ such that $(x, v)(\mathring{t}) = (\mathring{x}, \mathring{v})$.

As shown in Lemma 2.13, the angular momentum squared $L$ and the particle energy $E$ are constant along characteristics. Furthermore, $E$ is spherically symmetric with $E(x, v) = E(r, w, L) = \frac{1}{2} w^2 + \frac{L}{2r^2} + U(r) - \frac{M_0}{r}$ for $(x, v) \in \{L > 0\}$.

Assume that there exists a sequence $(t_k) \subset ]t_1, t_2[$ with $r(t_k) \to 0$ and $t_k \to t_{1,2}$ as $k \to \infty$. Then it follows that

$$\begin{aligned} E(r(0), w(0), L(0)) = E(r(t_k), w(t_k), L(t_k)) &= \frac{1}{2} w(t_k)^2 + \frac{L(t_k)}{2r(t_k)^2} + U(r(t_k)) - \frac{M_0}{r(t_k)} \\ &\ge \frac{L(0)}{2r(t_k)^2} \left( 1 - 2\frac{M_0}{L(0)} r(t_k) \right) + U(r(t_k)) \to \infty, \quad k \to \infty, \end{aligned}$$

since $L(0) > 0$ by assumption, which leads to a contraction. Therefore, there exists a radius $R_0 > 0$ such that $r(t) > R_0$ for all $t \in ]t_1, t_2[$.

Inserting $(x, v)(t)$ into the right-hand sight of the characteristic system gives an expression which is bounded on every compact subset $J \subset I$. Hence, the characteristics exist on $I = \mathbb{R}$. □

The purpose of this work is to construct steady states and study their stability. For this purpose, we define time-independent solutions as steady states:

**Definition 2.17.** *A function $f : \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to [0, \infty[$ is a* steady state (of the Vlasov-Poisson system with a point mass) *if $\tilde{f} : \mathbb{R} \times \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to [0, \infty[$ defined by $\tilde{f}(t, x, v) := f(x, v)$ is a solution of the Vlasov-Poisson system with a point mass in the sense of Definition 2.10.*

We are now in the position to start constructing steady states.

# 3 Steady states with compact support and finite mass

In this section, we aim to construct steady states with compact support and finite mass. In Lemma 2.16, we have shown that the characteristic flow exists on $\mathbb{R}$ for $U \in C^2(\mathbb{R}^3 \backslash \{0\})$ spherically symmetric with $\partial_x U$ bounded and $\lim_{r\to\infty} U(r) = 0$. For this reason, we first fix a steady state $f$ such that $f$ solves the Vlasov equation provided that the characteristics exist. Thus, this ansatz leads to a semi-linear Poisson equation for $U$ that we have to solve. Then we verify that $U$ satisfies the conditions of Lemma 2.16, so the characteristic flow actually exists as in Definition 2.10.

In this chapter, we proceed analogously to [9] which shows among other things the existence of anisotropic steady states of the Vlasov-Poisson system without a point mass with compact support and finite mass.

## 3.1 Anisotropic steady states

First, we construct a solution $f$ of the Vlasov equation. Since we are looking for functions which are constant along characteristics, we use two conserved quantities, namely the particle energy $E$ and the angular momentum squared $L$ as described in Definition 2.13: Let $U : \mathbb{R}^3 \to \mathbb{R}$ be spherically symmetric and once differentiable with respect to $x$. We assume that the corresponding characteristic flow exists. Let $\Phi : \mathbb{R} \times [0, \infty[ \to [0, \infty[$ be at first general. We define $f : \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to [0, \infty[$ by

$$f(x, v) := \Phi(E(x, v), L(x, v)), \quad (x, v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3, \tag{3.1}$$

so $f$ obviously solves the Vlasov equation.

**Remark.** *In [4], the ansatz function $\Phi$ only depends on the particle energy $E$. Since $E$ only depends on $|x|$ and $|v|$, there is no preferred direction in $(x, v)$. That is the reason why ansatz functions with $\Phi = \Phi(E)$ are called* isotropic. *In the following, we require an explicit dependence of $L$, so in this case the ansatz function $\Phi = \Phi(E, L)$ is called* anisotropic.

Inserting this ansatz into the Poisson equation leads to the semi-linear Poisson equation

$$\Delta U = 4\pi \int_{\mathbb{R}^3} \Phi\left(\frac{1}{2}|v|^2 + U - \frac{M_0}{|x|}, |x \times v|^2\right) dv, \quad \lim_{|x|\to\infty} U(x) = 0. \tag{3.2}$$

As discussed in [9] and thus in [1], solutions $U \in C^2(\mathbb{R}^3)$ of the semi-linear Poisson equation (3.2) with $M_0 = 0$ are necessarily spherically symmetric. That is the reason why we also require $U$ to be spherically symmetric.

It is necessary that steady states of the form (3.1) have cut-off energies:

**Lemma 3.1.** *Let $U \in C(\mathbb{R}^3)$ be spherically symmetric with $\lim_{r\to\infty} U(r) = 0$. Furthermore, we assume that $f = \Phi(E, L)$ has compact support. Then there exists a cut-off energy $E_0 \in \mathbb{R}$ such that*

$$\Phi(E, L) = 0, \quad E \geq E_0, \tag{3.3}$$

*holds for every $L \geq 0$.*

*Proof.* Since the function $f$ has compact support, there exist a radius $R > 0$ such that $|v|, |x| < R$ for all $(x, v) \in \operatorname{supp} f$, so it follows

$$E(x,v) = \frac{1}{2}|v|^2 + U(x) - \frac{M_0}{|x|} \leq \frac{1}{2}R^2 + \max_{|y|\leq R} U(y) =: E_0$$

for $(x, v) \in \operatorname{supp} f$. Since $U$ is continuous with $\lim_{r\to\infty} U(r) = 0$, the function $U$ is bounded on $\mathbb{R}^3$, and $E$ is continuous. The intermediate value theorem implies

$$\begin{aligned} E((0,0,\mathbb{R}), v) &= \left]-\infty, \frac{1}{2}|v|^2\right[, \quad v \in \mathbb{R}^3, \\ E(x, (0,0,\mathbb{R})) &= \left[U(x) - \frac{M_0}{|x|}, \infty\right[, \quad x \in \mathbb{R}^3\backslash\{0\}, \end{aligned}$$

so the map $\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 \ni (x,v) \mapsto E(x,v) \in \mathbb{R}$ is surjective. Thus,

$$\Phi(E, L) = 0, \quad E \geq E_0,$$

for all $L > 0$. □

Similar to the necessary cut-off energy $E_0$, we introduce the cut-off quantity $L_0 > 0$ which acts in a comparable way, namely let $L_0 > 0$ with

$$\Phi(E, L) = 0, \quad L \leq L_0,$$

for all $E \in \mathbb{R}$. If $E_1 < 0$, this implies that the support of $f = \Phi(E, L)$ has the form

$$\operatorname{supp} f \subset \{(x,v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 | E(x,v) \leq E_0 \wedge L(x,v) \geq L_0\}.$$

Under suitable conditions, these two cut-off quantities create a shell-like solution $f = \Phi(E, L)$ since the set on the right-hand side is compact and in particular bounded away from the singularity:

**Lemma 3.2.** *Let $U \in C(\mathbb{R}^3)$ with $U \leq 0$ and $\lim_{|x|\to\infty} U(x) = 0$. Furthermore, let $E_1 \in \mathbb{R}$ and $L_1 > 0$ be arbitrary. Then the set*

$$\{(x,v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 | E(x,v) \leq E_1 \wedge L(x,v) \geq L_1\}$$

*is compact if and only if $E_1 < 0$. In particular, there exist $0 < r^* < R^*$ such that*

$$\{E \leq E_1\} \cap \{L \geq L_1\} \subset \{r^* \leq |x| \leq R^*\} \times \{|v| \leq R^*\}.$$

*Proof.* At first, we consider $E_1 \geq 0$ and $L_1 > 0$. Let $(v_n) \subset \mathbb{R}^3$ with $v_n := \frac{1}{n} e_1$ for $n \in \mathbb{N}$. We define $x_n := 2M_0 n^2 e_2$ for $n \in \mathbb{N}$ with $n \geq \frac{\sqrt{L_1}}{2M_0}$. Here, $e_1 := (1,0,0)^t$ and $e_2 := (0,1,0)^t$ are the first and the second unit vector in $\mathbb{R}^3$. Then we obtain that

$$\begin{aligned} L(x_n, v_n) &= |x_n \times v_n|^2 = |x_n|^2 |v_n|^2 = 4M_0^2 n^2 \geq L_1, \\ E(x_n, v_n) &= \frac{1}{2}|v_n|^2 + U(x_n) - \frac{M_0}{|x_n|} \leq \frac{1}{2n^2} - \frac{M_0}{2M_0 n^2} = 0 \leq E_1 \end{aligned}$$

for $n \geq \frac{\sqrt{L_1}}{2M_0}$. Obviously, the sequence $(x_n)$ is unbounded, so $\{E \leq E_1\} \cap \{L \geq L_1\}$ is not compact in $\mathbb{R}^3 \times \mathbb{R}^3$ and neither in $\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$.

In the next step, let $E_1 < 0$ and $L_1 > 0$. First, we show that $\{E \leq E_1\} \cap \{L \geq L_1\}$ is bounded in this case, i.e., that there exists a radius $R^* > 0$ such that

$$|x| \leq R^*, \ |v| \leq R^*, \quad (x,v) \in \{E \leq E_1\} \cap \{L \geq L_1\}.$$

Assume that $\{E \leq E_1\} \cap \{L \geq L_1\}$ is unbounded in $x$. Then there exists a sequence $(x_n, v_n) \subset \{E \leq E_1\} \cap \{L \geq L_1\}$ with $|x_n| \to \infty$ as $n \to \infty$. Since $L_1 \leq |x_n|^2|v_n|^2 - (x_n \cdot v_n)^2 \leq |x_n|^2|v_n|^2$, the estimate $|v_n|^2 \geq \frac{L_1}{|x_n|^2}$ and we obtain that

$$E_1 \geq \frac{1}{2}|v_n|^2 + U(x_n) - \frac{M_0}{|x_n|} \geq \frac{1}{2}\frac{L_1}{|x_n|^2} + U(x_n) - \frac{M_0}{|x_n|} \to 0, \quad n \to \infty.$$

Since $E_1 < 0$, this leads to a contradiction. On the other hand, we assume that $\{E \leq E_1\} \cap \{L \geq L_1\}$ is unbounded in $v$. Then there exists a sequence $(x_n, v_n) \subset \{E \leq E_1\} \cap \{L \geq L_1\}$ with $|v_n| \to \infty$ as $n \to \infty$. Analogously, we get $|x_n|^2 \geq \frac{L_1}{|v_n|^2}$ for $n \in \mathbb{N}$ and

$$\begin{aligned} E_1 &\geq \frac{1}{2}|v_n|^2 + U(x_n) - \frac{M_0}{|x_n|} \geq \frac{1}{2}|v_n|^2 + \min_{\mathbb{R}^3} U - \frac{M_0}{|x_n|} \\ &\geq \frac{1}{2}|v_n|^2 + \min_{\mathbb{R}^3} U - |v_n|\frac{M_0}{\sqrt{L_1}} \to \infty, \quad n \to \infty, \end{aligned}$$

so again the assumption yields a contradiction. Note that $U \in C(\mathbb{R}^3)$ with $U \leq 0$ and $\lim_{|x|\to\infty} U(x) = 0$, so $\min_{\mathbb{R}^3} U$ exists. Therefore, the set $\{E \leq E_1\} \cap \{L \geq L_1\}$ is bounded, so there obviously exists the desired radius $R^* > 0$. Furthermore, this results in the estimate $|x| \geq \frac{L_1}{(R^*)^2} =: r^* > 0$ for $(x,v) \in \{E \leq E_1\} \cap \{L \geq L_1\}$. In summary,

$$\{E \leq E_1\} \cap \{L \geq L_1\} \subset \{r^* \leq |x| \leq R^*\} \times \{|v| \leq R^*\}.$$

The maps $(x,v) \mapsto E(x,v)$ and $(x,v) \mapsto L(x,v)$ are continuous on $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$, so the set on the left-hand side is closed. Since the set on the right-hand side is compact in $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$, the closed set $\{E \leq E_1\} \cap \{L \geq L_1\}$ is also compact in $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$. $\square$

**Remark.** *If we have a closer look on the proof of Lemma 3.2, we notice that the proof can be transferred for $U : \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to \mathbb{R}$ with $-\frac{M}{|x|} \leq U(x) \leq 0$ for some $M \geq 0$ and for all $x \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$.*

## 3.2 Separation ansatz and the associated density

Using these observations, we specify the ansatz in equation (3.1) by choosing a separation ansatz, i.e., we separate the dependence of $E$ and $L$ by considering functions $\Phi$ of the form

$$\Phi(E,L) = \xi(E)\zeta(L), \quad (E,L) \in \mathbb{R} \times [0,\infty[,$$

with $\xi : \mathbb{R} \to [0,\infty[$ and $\zeta : [0,\infty[ \to [0,\infty[$. More precisely, we consider in this chapter ansatz functions of the form

$$\Phi(E,L) = \varphi(E_0 - E)(L - L_0)_+^l, \quad (E,L) \in \mathbb{R} \times [0,\infty[, \tag{3.4}$$

with fixed $E_0 \in \mathbb{R}$, $L_0 > 0$ and $l > -\frac{1}{2}$. Here, $(\cdot)_+^l$ denotes

$$(\cdot)_+^l : \mathbb{R} \to [0,\infty[, \quad (x)_+^l := \begin{cases} x^l, & x > 0, \\ 0, & \text{else}, \end{cases}$$

while $l > -\frac{1}{2}$ will guarantee integrability at certain points.

The crucial step is to determine whether the semi-linear Poisson equation (3.2) has a solution. For this reason, we formulate certain conditions on the ansatz function $\varphi$ step by step and proceed analogously to [9]. First of all, we require the following two conditions:

(V1) The function $\varphi : \mathbb{R} \to [0,\infty[$ is measurable.

(V2) There exists $\eta_0 > 0$ such that $\varphi(\eta) = 0$ for $\eta \leq 0$ and $\varphi(\eta) > 0$ for $\eta \in ]0,\eta_0[$.

While the first condition ensures that $\Phi$ of the form (3.4) and thus $f = \Phi(E,L)$ are measurable, the second condition guarantees that for any chosen $E_0$ exactly this $E_0$ is the smallest cut-off energy which fulfills the condition (3.3).

Inserting the ansatz into the definiton of $\rho$ results in the following representation:

**Lemma 3.3.** *Let $U$ be spherically symmetric. Furthermore, assume that $\varphi$ is satisfying (V1) and (V2). Then the density $\rho$ induced by $\varphi$, i.e., induced by the function $f = \varphi(E_0 - E)(L - L_0)_+^l$, has the form*

$$\rho(r) = c_l r^{2l} \int_0^{E_0 - U(r) + \frac{M_0}{r} - \frac{L_0}{2r^2}} \varphi(\eta) \left( E_0 - U(r) + \frac{M_0}{r} - \frac{L_0}{r^2} - \eta \right)^{l+\frac{1}{2}} d\eta$$

*for $r > 0$ with $E_0 - U(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} > 0$, provided the integral exists, and*

$$\rho(r) = 0$$

*otherwise. The constant $c_l$ is defined by*

$$c_l := 2^{l+\frac{3}{2}} \pi \int_0^1 (1-s)^l s^{-\frac{1}{2}} \, ds.$$

*Proof.* Let $r > 0$. We use the spherical symmetry of $E$ as shown in Lemma 2.13 and convert the integration in $v$ into an integral in $(r, w, L)$-coordinates, as discussed in Lemma 2.3. For the sake of clarity, we use the abbreviation $\xi(r) := E_0 - U(r) + \frac{M_0}{r}$, which leads to

$$\begin{aligned}
\rho(r) &= \int \varphi(E_0 - E)(L - L_0)_+^l \, dv \\
&= \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \varphi \left( E_0 - \frac{1}{2} w^2 - \frac{L}{2r^2} - U(r) + \frac{M_0}{r} \right) (L - L_0)_+^l \, dL \, dw \\
&= \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_{L_0}^{\infty} \varphi \left( \xi(r) - \frac{1}{2} w^2 - \frac{L}{2r^2} \right) (L - L_0)^l \, dL \, dw.
\end{aligned}$$

After substituting $\tau = \frac{L}{2r^2} + \frac{1}{2} w^2$ in the $L$-integral for fixed $w$, changing the order of integration by applying Fubini's theorem implies

$$\begin{aligned}
\rho(r) &= 2\pi \int_{-\infty}^{\infty} \int_{\frac{L_0}{2r^2} + \frac{1}{2} w^2}^{\infty} \varphi(\xi(r) - \tau)(2r^2\tau - r^2 w^2 - L_0)^l \, d\tau \, dw \\
&= 2\pi \int_{\frac{L_0}{2r^2}}^{\infty} \int_{-\sqrt{2\tau - \frac{L_0}{r^2}}}^{\sqrt{2\tau - \frac{L_0}{r^2}}} \varphi(\xi(r) - \tau)(2r^2\tau - r^2 w^2 - L_0)^l \, dw \, d\tau.
\end{aligned}$$

If we consider the $w$-integral separately, after some changes of variables and lines of calculations we get

$$\int_{-\sqrt{2\tau - \frac{L_0}{r^2}}}^{\sqrt{2\tau - \frac{L_0}{r^2}}} (2r^2\tau - r^2 w^2 - L_0)^l \, dw = 2^{l+\frac{1}{2}} I_l \left( \tau - \frac{L_0}{2r^2} \right)^{l+\frac{1}{2}} r^{2l}$$

for $\tau \geq \frac{L_0}{2r^2}$, where we denote $I_l := \int_0^1 (1-s)^l s^{-\frac{1}{2}} \, ds$. Note that the integral $I_l$ exists because of the assumption $l > \frac{1}{2}$. This yields

$$\rho(r) = 2^{l+\frac{3}{2}} \pi I_l r^{2l} \int_{\frac{L_0}{2r^2}}^{\infty} \varphi(\xi(r) - \tau) \left( \tau - \frac{L_0}{2r^2} \right)^{l+\frac{1}{2}} d\tau$$

$$= 2^{l+\frac{3}{2}}\pi I_l r^{2l} \int_0^\infty \varphi\left(\xi(r) - \frac{L_0}{2r^2} - \tau\right) \tau^{l+\frac{1}{2}}\, d\tau.$$

Since $\varphi = 0$ on $]-\infty, 0]$, it follows $\rho(r) = 0$ for $\xi(r) - \frac{L_0}{2r^2} \le 0$. Otherwise,

$$\begin{aligned}\rho(r) &= 2^{l+\frac{3}{2}}\pi I_l r^{2l} \int_0^\infty \varphi\left(\xi(r) - \frac{L_0}{2r^2} - \tau\right) \tau^{l+\frac{1}{2}}\, d\tau \\ &= c_l r^{2l} \int_0^{\xi(r)-\frac{L_0}{2r^2}} \varphi(\eta) \left(\xi(r) - \frac{L_0}{2r^2} - \eta\right)^{l+\frac{1}{2}} d\eta. \qquad \square\end{aligned}$$

To ensure the existence of $\rho$, we have to impose a growth condition to $\varphi$, so the integral in Lemma 3.3 exists:

(V3) There exists $\kappa > -1$ such that for all compact sets $K \subset \mathbb{R}$ there holds the estimate

$$\varphi(\eta) \le C\eta^\kappa, \quad \eta \in K,$$

with $C = C(K) > 0$.

As seen in Lemma 3.3, the density $\rho$ induced by $\varphi$ which satisfies the conditions (V1)–(V3) can be represented with

$$\rho(r) = r^{2l} g\left(E_0 - U(r) + \frac{M_0}{r} - \frac{L_0}{2r^2}\right), \quad r \ge 0, \tag{3.5}$$

by using the function $g : \mathbb{R} \to [0, \infty[$ given by

$$g(y) := \begin{cases} c_l \int_0^y \varphi(\eta)(y-\eta)^{l+\frac{1}{2}}\, d\eta, & y > 0, \\ 0, & y \le 0. \end{cases} \tag{3.6}$$

Since $g$ is the same function as in [9, Equation (2.3)], we obtain the same properties:

**Lemma 3.4.** *Let $g$ be as defined in (3.6), and assume that $\varphi$ satisfies the conditions (V1)–(V3). Then $g \in C(\mathbb{R}) \cap C^1(]0, \infty[)$ with*

$$g'(y) = \left(l + \frac{1}{2}\right) c_l \int_0^y \varphi(\eta)(y-\eta)^{l-\frac{1}{2}}\, d\eta, \quad y > 0.$$

*If in addition $l + \kappa + \frac{1}{2} > 0$, the function $g$ is continuously differentiable on $\mathbb{R}$.*

*Proof.* Obviously, $g \in C^\infty(]-\infty, 0[)$. In order to verify $g \in C^1(]0, \infty[)$, we show first the differentiablity of $g$ and then the continuity of $g'$.

Let $y > 0$ and $0 < h < \min\{\frac{y}{4}, 1\}$ be arbitrary. By condition (V3), there exists $\kappa > -1$ and $C^* = C^*(y) > 0$ such that

$$\varphi(\eta) \le C^*\eta^\kappa, \quad \eta \in [0, y+1].$$

To show the differentiability of $g$, we consider the right-hand and left-hand limit of the difference quotient separately. First, we regard the left-hand difference quotient, which leads to two integrals $I_1$ and $I_2$ via

$$\begin{aligned}\frac{g(y)-g(y-h)}{h} &= c_l\Bigg(h^{-1}\int_{y-h}^{y}\varphi(\eta)(y-\eta)^{l+\frac{1}{2}}\,d\eta \\ &\qquad + \int_0^{y-h}\varphi(\eta)\frac{(y-\eta)^{l+\frac{1}{2}}-(y-h-\eta)^{l+\frac{1}{2}}}{h}\,d\eta\Bigg) \\ &=: c_l(I_1+I_2).\end{aligned}$$

In the following, a constant $C$ appears and may change from line to line and depends on $y$, but never on $h$. The integral $I_1$ vanishes as $h\to 0$ since we obtain

$$\begin{aligned}I_1 &= h^{-1}\int_{y-h}^{y}\varphi(\eta)(y-\eta)^{l+\frac{1}{2}}\,d\eta \\ &\le C^*h^{-1}\int_{y-h}^{y}\eta^{\kappa}(y-\eta)^{l+\frac{1}{2}}\,d\eta \\ &\le C^*h^{-1}\max_{s\in[\frac{y}{2},y+1]}s^{\kappa}\int_{y-h}^{y}(y-\eta)^{l+\frac{1}{2}}\,d\eta \\ &\le Ch^{l+\frac{1}{2}}\to 0,\quad h\to 0.\end{aligned}$$

The second integral $I_2$ is more difficult to calculate. We aim to show that $I_2$ converges to the claimed formula of $g'(y)$, so we consider the following expression:

$$\begin{aligned}&\left|\int_0^{y-h}\varphi(\eta)\frac{(y-\eta)^{l+\frac{1}{2}}-(y-h-\eta)^{l+\frac{1}{2}}}{h}\,d\eta-\left(l+\frac{1}{2}\right)\int_0^{y}\varphi(\eta)(y-\eta)^{l-\frac{1}{2}}\,d\eta\right| \\ &\le \int_0^{y-h}\varphi(\eta)\left|\frac{(y-\eta)^{l+\frac{1}{2}}-(y-h-\eta)^{l+\frac{1}{2}}}{h}-\left(l+\frac{1}{2}\right)(y-\eta)^{l-\frac{1}{2}}\right|\,d\eta \\ &\quad+\left(l+\frac{1}{2}\right)\int_{y-h}^{y}\varphi(\eta)(y-\eta)^{l-\frac{1}{2}}\,d\eta \\ &\le C^*\int_0^{\frac{y}{2}}\eta^{\kappa}\left|\frac{(y-\eta)^{l+\frac{1}{2}}-(y-h-\eta)^{l+\frac{1}{2}}}{h}-\left(l+\frac{1}{2}\right)(y-\eta)^{l-\frac{1}{2}}\right|\,d\eta \\ &\quad+C^*\int_{\frac{y}{2}}^{y-h}\eta^{\kappa}\left|\frac{(y-\eta)^{l+\frac{1}{2}}-(y-h-\eta)^{l+\frac{1}{2}}}{h}-\left(l+\frac{1}{2}\right)(y-\eta)^{l-\frac{1}{2}}\right|\,d\eta \\ &\quad+\left(l+\frac{1}{2}\right)\int_{y-h}^{y}\varphi(\eta)(y-\eta)^{l-\frac{1}{2}}\,d\eta \\ &=: C^*J_1+C^*J_2+J_3.\end{aligned}$$

Obviously, $J_3$ vanishes as $h\to 0$. The integrand of $J_1$ vanishes for any $\eta\in[0,\frac{y}{2}]$ as $h\to 0$. Furthermore, the integrand can be estimated for $0<\eta<\frac{y}{2}$ by using the mean value theorem. More precisely, there exists $\xi \in [y-h, y]$ such that

$$
\begin{aligned}
&\eta^\kappa \left| \frac{(y-\eta)^{l+\frac{1}{2}} - (y-h-\eta)^{l+\frac{1}{2}}}{h} - \left(l+\frac{1}{2}\right)(y-\eta)^{l-\frac{1}{2}} \right| \\
&= \left(l+\frac{1}{2}\right)\eta^\kappa |(\xi-\eta)^{l-\frac{1}{2}} - (y-\eta)^{l-\frac{1}{2}}| \\
&\leq \left(l+\frac{1}{2}\right)\eta^\kappa \begin{cases} (y-\eta)^{l-\frac{1}{2}}, & l \geq \frac{1}{2}, \\ (\xi-\eta)^{l-\frac{1}{2}}, & -\frac{1}{2} < l < \frac{1}{2}, \end{cases} \\
&\leq C\eta^\kappa \begin{cases} (y-\eta)^{l-\frac{1}{2}}, & l \geq \frac{1}{2}, \\ (y-h-\eta)^{l-\frac{1}{2}}, & -\frac{1}{2} < l < \frac{1}{2}, \end{cases} \\
&\leq C\eta^\kappa \begin{cases} y^{l-\frac{1}{2}}, & l \geq \frac{1}{2}, \\ (y-\frac{y}{4}-\frac{y}{2})^{l-\frac{1}{2}}, & -\frac{1}{2} < l < \frac{1}{2}, \end{cases} \\
&\leq C\eta^\kappa.
\end{aligned}
$$

Since the right-hand term is integrable, it follows by the dominated convergence theorem that $J_1$ vanishes as $h \to 0$. After changing the variables, we obtain:

$$
\begin{aligned}
J_2 &= \int_{\frac{y}{2}}^{y-h} \eta^\kappa \left| \frac{(y-\eta)^{l+\frac{1}{2}} - (y-h-\eta)^{l+\frac{1}{2}}}{h} - \left(l+\frac{1}{2}\right)(y-\eta)^{l-\frac{1}{2}} \right| d\eta \\
&\leq \max_{s\in[\frac{y}{2},y]} s^\kappa \int_{\frac{y}{2}}^{y-h} \left| \frac{(y-\eta)^{l+\frac{1}{2}} - (y-h-\eta)^{l+\frac{1}{2}}}{h} - \left(l+\frac{1}{2}\right)(y-\eta)^{l-\frac{1}{2}} \right| d\eta \\
&\leq C \int_{\frac{y}{2}+h}^{y} \left| \frac{(y+h-\eta)^{l+\frac{1}{2}} - (y-\eta)^{l+\frac{1}{2}}}{h} - \left(l+\frac{1}{2}\right)(y+h-\eta)^{l-\frac{1}{2}} \right| d\eta \\
&\leq C \int_{\frac{y}{2}}^{y} \left| \frac{(y+h-\eta)^{l+\frac{1}{2}} - (y-\eta)^{l+\frac{1}{2}}}{h} - \left(l+\frac{1}{2}\right)(y+h-\eta)^{l-\frac{1}{2}} \right| d\eta.
\end{aligned}
$$

Again, the integrand vanishes as $h \to 0$ and, similar to the integral $J_1$, the integrand can be bounded for $\eta \in [\frac{y}{2}, y]$ by

$$
\begin{aligned}
&\left| \frac{(y+h-\eta)^{l+\frac{1}{2}} - (y-\eta)^{l+\frac{1}{2}}}{h} - \left(l+\frac{1}{2}\right)(y+h-\eta)^{l-\frac{1}{2}} \right| \\
&\leq \left(l+\frac{1}{2}\right) |(\xi-\eta)^{l-\frac{1}{2}} - (y+h-\eta)^{l-\frac{1}{2}}| \\
&\leq C \begin{cases} (y+h-\eta)^{l-\frac{1}{2}}, & l \geq \frac{1}{2}, \\ (y-\eta)^{l-\frac{1}{2}}, & -\frac{1}{2} < l < \frac{1}{2}, \end{cases} \\
&\leq C \begin{cases} (y+1-\eta)^{l-\frac{1}{2}}, & l \geq \frac{1}{2}, \\ (y-\eta)^{l-\frac{1}{2}}, & -\frac{1}{2} < l < \frac{1}{2}, \end{cases}
\end{aligned}
$$

for some $\xi \in [y, y+h]$. The integrability of the right-hand term and again the dominated convergence theorem lead to $J_2 \to 0$ as $h \to 0$. In summary, we have shown that

$$\frac{g(y) - g(y-h)}{h} \to \left(l + \frac{1}{2}\right) c_l \int_0^y \varphi(\eta)(y-\eta)^{l-\frac{1}{2}} \, d\eta, \quad h \to 0.$$

Considering the right-hand difference quotient, we obtain

$$\begin{aligned}\frac{g(y+h) - g(y)}{h} &= c_l \bigg( h^{-1} \int_y^{y+h} \varphi(\eta)(y+h-\eta)^{l+\frac{1}{2}} \, d\eta \\ &\quad + \int_0^y \varphi(\eta) \frac{(y+h-\eta)^{l+\frac{1}{2}} - (y-\eta)^{l+\frac{1}{2}}}{h} \, d\eta \bigg).\end{aligned}$$

Similar, but simpler arguments than for the integrals $I_1$ and $I_2$ result in the convergence

$$\frac{g(y+h) - g(y)}{h} \to \left(l + \frac{1}{2}\right) c_l \int_0^y \varphi(\eta)(y-\eta)^{l-\frac{1}{2}} \, d\eta, \quad h \to 0.$$

Since $y > 0$ is arbitrary, the function $g$ is differentiable on $]0, \infty[$ with

$$g'(y) = \left(l + \frac{1}{2}\right) c_l \int_0^y \varphi(\eta)(y-\eta)^{l-\frac{1}{2}} \, d\eta, \quad y > 0,$$

and thus $g$ is continuous.

In the next step, we show that $g'$ is continuous on $]0, \infty[$. A similar procedure to before leads to

$$\begin{aligned}g'(y+h) - g'(y) &= c_l \left(l + \frac{1}{2}\right) \bigg( \int_y^{y+h} \varphi(\eta)(y+h-\eta)^{l-\frac{1}{2}} \, d\eta \\ &\quad + \int_0^y \varphi(\eta)((y+h-\eta)^{l-\frac{1}{2}} - (y-\eta)^{l-\frac{1}{2}}) \, d\eta \bigg) \\ &\to 0, \quad h \to 0\end{aligned}$$

and

$$\begin{aligned}g'(y) - g'(y-h) &= c_l \left(l + \frac{1}{2}\right) \bigg( \int_{y-h}^{y} \varphi(\eta)(y-\eta)^{l-\frac{1}{2}} \, d\eta \\ &\quad + \int_0^{y-h} \varphi(\eta)((y-\eta)^{l-\frac{1}{2}} - (y-h-\eta)^{l-\frac{1}{2}}) \, d\eta \bigg) \\ &\to 0, \quad h \to 0,\end{aligned}$$

so $g'$ is also continuous. Note that the appearing integrals have a similar structure to $I_1$ and $I_2$ which gives us the possibility to adapt the previous arguments. Summing up, $g \in C^1(]0, \infty[)$ with $g'(y) = \left(l + \frac{1}{2}\right) c_l \int_0^y \varphi(\eta)(y-\eta)^{l-\frac{1}{2}} \, d\eta$ for $y > 0$.

It remains taking care of the regularity in $y = 0$. Let $0 < h < 1$. Then obviously $g(-h) - g(0) \to 0$ as $h \to 0$ and

$$\begin{aligned} g(h) - g(0) &= c_l \int_0^h \varphi(\eta)(h-\eta)^{l+\frac{1}{2}}\, d\eta \\ &\leq c_l h^{l+\frac{1}{2}} \int_0^1 \varphi(\eta)\, d\eta \to 0, \quad h \to 0, \end{aligned}$$

so $g \in C(\mathbb{R}) \cap C^1(]0,\infty[)$. We now assume that $\kappa + l + \frac{1}{2} > 0$. Then $\frac{g(0)-g(-h)}{h} = 0$ and

$$\begin{aligned} \frac{g(h) - g(0)}{h} &= c_l h^{-1} \int_0^h \varphi(\eta)(h-\eta)^{l+\frac{1}{2}}\, d\eta \\ &\leq C h^{l-\frac{1}{2}} \int_0^h \eta^\kappa\, d\eta = C h^{l+\kappa+\frac{1}{2}} \to 0, \quad h \to 0, \end{aligned}$$

which means that $g$ is differentiable in $y = 0$ with $g'(0) = 0$. The continuity of $g'$ in $y = 0$ follows by

$$\begin{aligned} |g'(h) - g'(0)| &= c_l \left(l + \frac{1}{2}\right) \int_0^h \varphi(\eta)(h-\eta)^{l-\frac{1}{2}}\, d\eta \\ &\leq C \int_0^h \eta^\kappa (h-\eta)^{l-\frac{1}{2}}\, d\eta \\ &= C h^{l+\frac{1}{2}+\kappa} \int_0^1 s^\kappa (1-s)^{l-\frac{1}{2}}\, ds \to 0, \quad h \to 0. \end{aligned}$$

Note that $\kappa > -1$ and $l > -\frac{1}{2}$, so the latter integral exists. This leads to the conclusion that $g \in C^1(\mathbb{R})$ for $\kappa + l + \frac{1}{2} > 0$ which completes the proof. □

Inserting the representation (3.5) into the Poisson equation, the semi-linear Poisson equation (3.2) takes on the form

$$\Delta U = 4\pi r^{2l} g\left(E_0 - U + \frac{M_0}{r} - \frac{L_0}{2r^2}\right), \quad \lim_{r\to\infty} U(r) = 0. \tag{3.7}$$

In the next steps, we require $\kappa + l + \frac{1}{2} \geq 0$ and determine whether the semi-linear Poisson equation (3.7) has a solution and whether the steady state $f = \Phi(E, L)$ with $\Phi$ as in (3.4) has compact support, as described in Lemma 3.2. Assume that these two assumptions are satisfied. Then Lemma 3.3 implies that the induced density $\rho$ is spherically symmetric and continuously differentiable with compact support $\operatorname{supp}\rho \subset \{r^* \leq |x| \leq R^*\}$. In this case, the solution $U$ of the semi-linear Poisson equation is induced by the density $\rho$ and thus $U \in C^2(\mathbb{R}^3)$, as discussed in Lemma 2.7. That is the reason why we search for solutions $U \in C^2(\mathbb{R}^3)$ of the semi-linear Poisson equation.

The existence of solutions of the semi-linear Poisson equation (3.7) and the solution itself depend on the parameter $E_0$ and the boundary condition $\lim_{|x|\to\infty} U(x) = 0$. Since we require $U \in C^2(\mathbb{R}^3)$, it would be easier to parameterize the solutions by prescribing $U(0)$ instead of $E_0$. In order to introduce the new parameter $U(0)$, we have to give up the fixed parameter $E_0$ and interpret the cut-off energy as a parameter given by the solution of the semi-linear Poisson equation. More precisely: Let $U \in C^2(\mathbb{R}^3)$ be a solution of the semi-linear Poisson equation and define $y(r) := E_0 - U(r)$ for $r \geq 0$. Inserting this expression into the semi-linear Poisson equation yields

$$\frac{1}{r^2}(r^2 y')' = -4\pi r^{2l} g\left(y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2}\right), \quad y(0) = E_0 - U(0).$$

Since we require $U \in C^2(\mathbb{R}^3)$ to be spherically symmetric, this implies $y \in C^2([0,\infty[)$ with $y'(0) = 0$. Using these properties, integrating the above equation leads to the equation

$$\begin{aligned} y' &= -\frac{m(r)}{r^2}, \quad y(0) = \mathring{y}, \\ m(r) &= m(r,y) = 4\pi \int_0^r r^{2l+2} g\left(y(s) + \frac{M_0}{s} - \frac{L_0}{2s^2}\right) ds, \quad r \geq 0, \end{aligned} \tag{3.8}$$

with $\mathring{y} := E_0 - U(0)$. Furthermore, we obtain $\lim_{r\to\infty} y(r) = E_0$.
That means that $U$ generates a solution $y$ of the above equation with $\lim_{r\to\infty} y(r) = E_0$. Vice versa, a solution $y$ of the equation (3.8) generates a solution $U$ of the semi-linear Poisson equation:

**Lemma 3.5.** *Let $y \in C^2([0,\infty[)$ be a solution of the equation 3.8 with $y'(0) = 0$ and $\lim_{r\to\infty} y(r) =: E_0 \in \mathbb{R}$. Furthermore, let $U := E_0 - y$. Then $U \in C^2(\mathbb{R}^3)$ is spherically symmetric with*

$$\begin{aligned} \Delta U = \frac{1}{r^2}(r^2 U')' &= 4\pi r^{2l} g\left(E_0 - U(r) + \frac{M_0}{r} - \frac{L_0}{2r^2}\right), \\ \lim_{|x|\to\infty} U(x) &= 0. \end{aligned}$$

As a result, we consider equation (3.8) and determine whether there exists a solution.

## 3.3 The existence of solutions

In the next step, we investigate the existence of solutions of equation (3.8) and determine whether they are unique and global. Furthermore, we obtain some useful properties:

**Lemma 3.6.** *Let $g \in C^1(\mathbb{R})$ with $g(y) = 0$ for $y \leq 0$ and $g(y) > 0$ for $y > 0$. Furthermore, let $\mathring{y} \in \mathbb{R}$. Then there exists a unique solution $y : [0,\infty[ \to \mathbb{R}$ of the equation (3.8). In addition, $y \in C^2([0,\infty[)$ with $y'(0) = 0$, in particular $y \in C^2(\mathbb{R}^3)$, and the following properties hold:*

(a) *For $\mathring{y} \leq -\frac{M_0^2}{2L_0}$, the solution $y$ is trivial, i.e., $y = \mathring{y}$ on $[0,\infty[$.*

*(b) For $\mathring{y} > -\frac{M_0^2}{2L_0}$, the solution $y$ is constant on $[0, R_0]$ and non-trivial on $[R_0, \infty[$, i.e.,*

$$
\begin{aligned}
y(r) &= \mathring{y}, \quad 0 \le r \le R_0, \\
y(r) &< \mathring{y}, \quad r > R_0,
\end{aligned}
$$

*with*

$$
R_0 := \begin{cases} -\frac{M_0}{2\mathring{y}} + \sqrt{\frac{L_0}{2\mathring{y}} + \frac{M_0^2}{4\mathring{y}^2}}, & \mathring{y} > 0, \\ \frac{L_0}{2M_0}, & \mathring{y} = 0, \\ -\frac{M_0}{2\mathring{y}} - \sqrt{\frac{L_0}{2\mathring{y}} + \frac{M_0^2}{4\mathring{y}^2}}, & -\frac{M_0^2}{2L_0} < \mathring{y} < 0. \end{cases} \tag{3.9}
$$

*Proof.* Let us assume that $y : [0, R[ \to \mathbb{R}$ is a solution of the system (3.8) with $R \in ]0, \infty]$ and analyze its properties in order to reduce the problem to an ordinary differential equation with initial condition at some positive radius. As a solution, $y$ is continuous with $y(0) = \mathring{y}$, so

$$
\lim_{\substack{s \to 0 \\ >}} \left( y(s) + \frac{M_0}{s} - \frac{L_0}{2s^2} \right) = -\infty,
$$

which means that there exists $0 < R^* \le R$ such that

$$
y(s) + \frac{M_0}{s} - \frac{L_0}{2s^2} < 0, \quad 0 < s < R^*.
$$

Thus, $g = 0$ and thus $m = 0$ on $[0, R^*]$ which implies that $y' = 0$ on $[0, R^*]$. Since $y = \mathring{y}$ on $[0, R^*]$, it follows

$$
\mathring{y} + \frac{M_0}{s} - \frac{L_0}{2s^2} < 0, \quad 0 < s < R^*. \tag{3.10}
$$

Furthermore, $\operatorname{supp}_{s>0}(\frac{M_0}{s} - \frac{L_0}{s^2}) = \frac{M_0^2}{2L_0}$, so $y = \mathring{y}$ on $[0, R[$ for $\mathring{y} \le -\frac{M_0^2}{2L_0}$. In particular, $y : [0, \infty[ \to \mathbb{R}$ with $y(r) := \mathring{y}$ is the unique and global solution for $\mathring{y} \le -\frac{M_0^2}{2L_0}$.

For $\mathring{y} > -\frac{M_0^2}{2L_0}$, we denote $R_0 > 0$ as the maximal radius $R^* > 0$ which satisfies the relation (3.10). By calculating the smallest non-negative zero of $\mathring{y} + \frac{M_0}{s} - \frac{L_0}{2s^2}$, the formula (3.9) for the inner radius $R_0$ follows directly. Hence, we have shown that for $\mathring{y} > -\frac{M_0^2}{2L_0}$, every solution $y$ is constant on $[0, R_0[$ with $y = \mathring{y}$, so it suffices considering the following initial value problem:

$$
\begin{aligned}
&y' = -\frac{m(r)}{r^2}, \quad y(R_0) = \mathring{y}, \\
&m(r) = m(r, y) = 4\pi \int_{R_0}^{r} s^{2l+2} g \left( y(s) + \frac{M_0}{s} - \frac{L_0}{2s^2} \right) ds, \quad r \ge R_0.
\end{aligned}
$$

Similar to the previous calculations, this differential equation is equivalent to

$$y'' = -2\frac{y'}{r} - 4\pi r^{2l+2} g\left(y + \frac{M_0}{r} - \frac{L_0}{2r^2}\right),$$
$$y(R_0) = \mathring{y}, \quad y'(R_0) = 0.$$

Because $g$ is continuously differentiable on $\mathbb{R}$ according to the assumptions, the function

$$F : ]0,\infty[ \to \mathbb{R}^2,\ F(r,v) := \left(v_2, -2\frac{v_2}{r} - 4\pi r^{2l+2} g\left(v_1 + \frac{M_0}{r} - \frac{L_0}{2r^2}\right)\right)^t$$

is also continuously differentiable and thus locally Lipschitz continuous. According to the Picard-Lindelöf theorem, there exists a unique and maximal solution $y \in C^2([R_0, \bar{R}[)$ with $\bar{R} \in ]R_0, \infty]$. The initial conditions $y(R_0) = \mathring{y}$ and $y'(R_0) = 0$ lead to $y''(R_0) = 0$, so the solution $y$ can be extended on $[0, \bar{R}[$ by using $y = \mathring{y}$ on $[0, R_0]$. In addition, $y \in C^2([0, \bar{R}[)$ and in particular $y \in C^2(B_{\bar{R}}(0))$ with $B_{\bar{R}}(0) := \{x \in \mathbb{R}^3 | |x| < \bar{R}\}$.

At last, it remains to show $\bar{R} = \infty$. Since $y$ is decreasing and $\operatorname{supp}_{s>0}(\frac{M_0}{s} - \frac{L_0}{s^2}) = \frac{M_0^2}{2L_0}$, the argument of $g$ is bounded by some positive constant $C^* = C^*(M_0, L_0, \mathring{y}) > 0$ because

$$y(s) + \frac{M_0}{s} - \frac{L_0}{s^2} \leq \mathring{y} + \max_{s>0}\left(\frac{M_0}{s} - \frac{L_0}{s^2}\right) < C^*, \quad 0 < s < \bar{R}.$$

Therefore, we obtain

$$\begin{aligned} 0 \leq -y'(r) &= \frac{4\pi}{r^2}\int_0^r s^{2l+2} g\left(y(s) + \frac{M_0}{s} - \frac{L_0}{2s^2}\right)\, ds \\ &\leq \frac{4\pi}{r^2}\|g\|_{L^\infty(]0,C^*])}\int_0^r s^{2l+2}\, ds = Cr^{2l+3-2} = Cr^{2l+1}. \end{aligned}$$

and thus

$$\mathring{y} \geq y(r) = \mathring{y} + \int_0^r y'(s)\, ds \geq \mathring{y} - C\int_0^r s^{2l+1}\, ds \geq \mathring{y} - Cr^{2l+2}$$

for $0 \leq r < \bar{R}$. From the standard theory of ordinary differential equations, it follows directly $\bar{R} = \infty$, so the proof is complete. □

Even if it may not seem so at first glance, it can be easily shown that the inner radius $R_0$ as defined in equation 3.9 is continuous in $\mathring{y}$. In the proof of Lemma 3.6, we examine the expression $y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2}$ which will be useful in the later argumentation. Thus, we summarize its properties:

**Remark 3.7.** *In the situation of Lemma 3.6, it follows by its proof:*

*(a) For $\mathring{y} \leq -\frac{M_0^2}{2L_0}$, the inequality $y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} < 0$ holds for all $r > 0$.*

*(b) For $\mathring{y} > -\frac{M_0^2}{2L_0}$, the inequality $y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} < 0$ holds for all $0 < r < R_0$, and there exists $\delta > 0$ such that $y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} > 0$ for all $R_0 < r < R_0 + \delta$.*

## 3.4 The compact support of solutions

After showing the existence of a unique solution of the equation (3.8) for every $\mathring{y} \in \mathbb{R}$, we determine whether and under which conditions $f$ has a compact support. As a reminder, $f$ has the form

$$f(x,v) = \varphi\left(y(x) - \frac{1}{2}|v|^2 + \frac{M_0}{|x|}\right)(L - L_0)_+^l$$

with $\varphi = 0$ on $]-\infty, 0]$ and $\varphi > 0$ on $]0, \infty[$ according to (V2). Before making statements about the different cases, we have a closer look on the support of $f$. Let $(x,v) \in \operatorname{supp} f$. Then we obtain

$$y(r) - \frac{1}{2}|v|^2 + \frac{M_0}{r} > 0 \quad \wedge \quad L(x,v) > L_0.$$

Using $L_0 \leq L \leq |x|^2|v|^2$, it follows

$$\frac{L_0}{2r^2} \leq \frac{1}{2}|v|^2 \leq y(r) + \frac{M_0}{r}.$$

In order to bound the support of $f$, we analyze the expression $z(r) := y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2}$ for $r > 0$. We investigate under which conditions there exist radii $0 < R_{\min} < R_{\max}$ such that

$$z(r) \begin{cases} < 0, & 0 < r < R_{\min}, \\ > 0, & R_{\min} < r < R_{\min} + \delta, \\ < 0, & r > R_{\max} \end{cases} \tag{3.11}$$

for some $\delta > 0$. Since $y$ is monotonically decreasing, the limit $\lim_{r\to\infty} y(r) =: y_\infty \in [-\infty, \mathring{y}]$ exists. Hence, the existence of $R_{\max}$ is equivalent to $y_\infty < 0$.

Assume that these radii $0 < R_{\min} < R_{\max}$ exist. Then $\frac{L_0}{2r^2} > y(r) + \frac{M_0}{r}$ on $]0, R_{\min}[$ and on $]R_{\max}, \infty[$, so the support of $f$ has the property

$$\operatorname{supp} f \subset (\bar{B}_{R_{\max}} \setminus B_{R_{\min}}) \times \bar{B}_{v_{\max}}$$

with $v_{\max} := \mathring{y} + \frac{M_0}{R_{\min}}$, and $\operatorname{supp} f$ is compact in $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$. Here, $\bar{B}_R := \bar{B}_R(0) = \{\tilde{x} \in \mathbb{R}^3 | |\tilde{x}| \leq R\}$ denotes the ball in $\mathbb{R}^3$ around 0 with radius $R > 0$.

In order to determine whether and under which conditions the required radii exist, we distinguish between different cases for $\mathring{y}$:

**Case 1:** $\mathring{y} \leq -\frac{M_0^2}{2L_0}$
As shown in Lemma 3.6 (a) and Remark 3.7 (a), $y$ is trivial with $y = \mathring{y} < 0$ and, in particular, $z(r) = \mathring{y} + \frac{M_0}{r} - \frac{L_0}{2r^2} \leq 0$ for all $r > 0$. Therefore, $f = 0$ on $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$.

**Case 2:** $-\frac{M_0^2}{2L_0} < \mathring{y} \leq 0$
As discussed in Remark 3.7 (b), the radius $R_0$ in (3.9) satisfies the conditions of the desired radius $R_{\text{min}}$. Since $y$ is monotonically decreasing and $\mathring{y} \leq 0$, we obtain $y(r) < \mathring{y}$ for $r > R_0$ and thus $y_\infty < \mathring{y} \leq 0$. Hence, the required radius $R_{\text{max}}$ exists, and no further conditions need to be imposed on $\varphi$ to ensure the compact support of $f$.

**Case 3:** $\mathring{y} > 0$
Like in Case 2, $R_{\text{min}}$ can be set as the inner radius $R_0$ in (3.9), but now the difficulty is to obtain $y_\infty < 0$. Hence, we proceed similarly as in [9] and set further conditions on the function $g$ to ensure $y_\infty < 0$. Analogously to [9, Lemma 3.1], we obtain the following assertion:

**Lemma 3.8.** *Let $l > -\frac{1}{2}$ and $g \in C(\mathbb{R})$ be monotonically increasing with the following properties:*

*(g1) $g(y) = 0$ for $y \leq 0$ and $g(y) > 0$ for $y > 0$.*

*(g2) There exists $c > 0$, $y^* > 0$ and $n < 3 + l$ such that*

$$g(y) \geq c y^{n+l}, \quad 0 < y < y^*.$$

*Furthermore, let $y \in C^1([0, \infty[)$ and $\mathring{y} > 0$ with*

$$\begin{aligned} y' &= -\frac{m(r)}{r^2}, \quad y(0) = \mathring{y}, \\ m(r) &= m(r, y) = 4\pi \int_0^r s^{2l+2} g\left(y(s) + \frac{M_0}{s} - \frac{L_0}{2s^2}\right) ds, \quad r \geq 0. \end{aligned}$$

*Then,* $\lim_{r\to\infty} y(r) = y_\infty < 0$.

*Proof.* In order to show Lemma 3.8, we proceed analogously to [9]. First, we assume that $y_\infty > 0$. To create a contradiction, we estimate the mass function $m$ appropriately. Then we apply the estimate together with the fundamental theorem of calculus to the differential equation. Since $y$ converges to $y_\infty$, a radius $\bar{R} > R_0$ exists such that

$$z(r) = y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} \geq \frac{y_\infty}{2}, \quad r \geq \bar{R}.$$

In particular, the monotonicity of $g$ and the assumption (g1) imply

$$\begin{aligned} m(r) &= 4\pi \int_0^r s^{2l+2} g\left(y(s) + \frac{M_0}{s} - \frac{L_0}{2s^2}\right) ds \geq 4\pi \int_{\bar{R}}^r s^{2l+2} g(z(s))\, ds \\ &\geq 4\pi g\left(\frac{y_\infty}{2}\right) \int_{\bar{R}}^r s^{2l+2}\, ds \\ &= 4\pi g\left(\frac{y_\infty}{2}\right) \frac{1}{2l+3}(r^{2l+3} - \bar{R}^{2l+3}), \quad r > \bar{R}. \end{aligned}$$

With the fundamental theorem of calculus, we obtain the following estimate:

$$\begin{aligned}
y(r) &\leq \mathring{y} - \int_{\bar{R}}^{r} y'(s)\, ds \\
&\leq \mathring{y} - 4\pi g\left(\frac{y_\infty}{2}\right)\frac{1}{2l+3}\int_{\bar{R}}^{r}(s^{2l+1} - \bar{R}^{2l+3}s^{-2})\, ds \\
&= C_1 - C_2 r^{2l+2} - C_3 r^{-1}, \quad r > \bar{R},
\end{aligned}$$

with constants $C_i = C_i(\mathring{y}, l, g, y_\infty, \bar{R}) > 0$ for $i = 1, ..., 3$. Since the right-hand side converges to $-\infty$ as $r \to \infty$ and $y_\infty > 0$ by assumption, this leads to a contradiction. Thus $y_\infty \leq 0$.

Now, it remains to prove $y_\infty \neq 0$. Therefore, we assume that $y_\infty = 0$. In order to create a contradiction, we use the growth condition (g2) and suitably estimate the integral for $r > 0$ large enough by

$$\int_{y(r)}^{y^*} \frac{1}{g(\eta)} d\eta \leq \frac{1}{c}\int_{y(r)}^{y^*} \frac{1}{\eta^{n+l}} d\eta.$$

Note that $\lim_{r\to\infty} y(r) = y_\infty = 0$, so $0 < y(r) < y^*$ for $r > 0$ large enough. Thus, calculating the right-hand side results for $n + l \neq 1$ in

$$\int_{y(r)}^{y^*} \frac{1}{g(\eta)} d\eta \leq \frac{1}{c}\frac{1}{1-n-l}((y^*)^{1-n-l} - (y(r))^{1-n-l})$$

and for $n + l = 1$ in

$$\int_{y(r)}^{y^*} \frac{1}{g(\eta)} d\eta \leq \frac{1}{c}\ln\left(\frac{y^*}{y(r)}\right).$$

In the next step, we determine a lower bound of the integral on the left-hand side. First, we have a closer look at $\frac{M_0}{r} - \frac{L_0}{2r^2}$. Obviously, the map $]0,\infty[\ni r \mapsto \frac{M_0}{r} - \frac{L_0}{2r^2}$ is strictly monotonically increasing or decreasing on $]0, \frac{L_0}{M_0}[$ or $]\frac{L_0}{M_0}, \infty[$, respectively, with

$$\operatorname{supp}_{r>0}\left(\frac{M_0}{r} - \frac{L_0}{2r^2}\right) = \left(\frac{M_0}{r} - \frac{L_0}{2r^2}\right)\Big|_{r=\frac{L_0}{M_0}} = \frac{M_0^2}{2L_0}.$$

Let $R > \max\{\frac{L_0}{M_0}, R_0\}$ with

$$0 < y(r) < y^*, \quad r > R.$$

Since $R > \frac{L_0}{M_0}$, we obtain $y(r) < z(r)$ and $z'(r) < y'(r)$ for $r > R$. In particular, $z$ is monotonically decreasing on $]R, \infty[$. The change of variable given by $z$ leads to

$$\int_{y(r)}^{z(R)} \frac{1}{g(\eta)}\, d\eta \geq \int_{z(r)}^{z(R)} \frac{1}{g(\eta)}\, d\eta = -\int_{R}^{r} \frac{z'(s)}{g(z(s))}\, ds$$

$$\geq - \int_R^r \frac{y'(s)}{g(z(s))} \, ds, \quad r > R.$$

For further estimates, we calculate an upper bound on $y'$. The monotonicity of $g$ and $z$ on $\mathbb{R}$ and $]R, \infty]$, respectively, yield

$$\begin{aligned} m(r) &= 4\pi \int_0^r s^{2l+2} g(z(s)) \, ds \geq 4\pi \int_R^r s^{2l+2} g(z(s)) \, ds \\ &\geq 4\pi g(z(r)) \int_R^r s^{2l+2} \, ds = \frac{4\pi}{2l+3} g(z(r))(r^{2l+3} - R^{2l+3}) \end{aligned}$$

and therefore

$$y'(r) \leq -\frac{4\pi}{2l+3} g(z(r))(r^{2l+1} - R^{2l+3} r^{-2})$$

for $r > R$. Inserting this estimate implies

$$\begin{aligned} \int_{y(r)}^{z(R)} \frac{1}{g(\eta)} \, d\eta &\geq - \int_R^r \frac{y'(s)}{g(z(s))} \, ds \geq \frac{4\pi}{2l+3} \int_R^r (s^{2l+1} - R^{2l+3} s^{-2}) \, ds \\ &= C_1 r^{2l+2} + C_2 r^{-1} - C_3, \quad r > R, \end{aligned}$$

with $C_i = C_i(l, R) > 0$ for $i = 1, ..., 3$.

In summary, the estimates from above and below result in

$$\begin{aligned} C_1 r^{2l+2} + C_2 r^{-1} - C_3 &\leq \int_{y(r)}^{y^*} \frac{1}{g(\eta)} d\eta + \int_{y^*}^{z(R)} \frac{1}{g(\eta)} d\eta \\ &\leq \frac{1}{c} \frac{1}{1-n-l} ((y^*)^{1-n-l} - (y(r))^{1-n-l}) + C_4 \end{aligned}$$

for $n + l \neq 1$ and in

$$\begin{aligned} C_1 r^{2l+2} + C_2 r^{-1} - C_3 &\leq \int_{y(r)}^{y^*} \frac{1}{g(\eta)} d\eta + \int_{y^*}^{z(R)} \frac{1}{g(\eta)} d\eta \\ &\leq \frac{1}{c} \ln \left( \frac{y^*}{y(r)} \right) + C_4 \end{aligned}$$

for $n+l = 1$ and $r > R$ with $C_4 = C_4(g, y^*, z(R)) > 0$. In order to create a contradiction, we multiply the inequalities with $(y(r))^{2l+2}$ and obtain

$$\begin{aligned} & C_1 r^{2l+2} (y(r))^{2l+2} + C_2 r^{-1} (y(r))^{2l+2} \\ & \leq \frac{1}{c} \frac{1}{1-n-l} ((y^*)^{1-n-l} - (y(r))^{1-n-l}) (y(r))^{2l+2} + C_5 (y(r))^{2l+2} \end{aligned}$$

for $n + l \neq 1$ and

$$C_1 r^{2l+2} (y(r))^{2l+2} + C_2 r^{-1} (y(r))^{2l+2} \leq \frac{1}{c} \ln \left( \frac{y^*}{y(r)} \right) (y(r))^{2l+2} + C_5 (y(r))^{2l+2}$$

for $n+l=1$ and $r>R$ with $C_i>0$ independent of $r$ for $i=1,...,5$. Exploiting $y_\infty=0$, we adapt the inequalities and find an appropriate estimate for the expression $ry(r)$. Since $\mathring{y}>0$, Remark 3.7 (b) implies that there exists $\delta>0$ such that $z(r)>0$ for all $R_0<r<R_0+\delta$. The mass function $m$ is monotonically increasing, and in particular $m>0$ on $]R_0,\infty[$, so it follows

$$m(r)\geq m(R)>0, \quad r>R,$$

which leads to the estimate

$$y(r)=-\int_r^\infty y'(s)\,ds=\int_r^\infty \frac{m(s)}{s^2}\,ds\geq m(r)\int_r^\infty \frac{1}{s^2}\,ds=\frac{m(r)}{r}\geq\frac{m(R)}{r}$$

for $r>R$. In particular, $ry(r)\geq m(R)>0$ for $r>R$. Since $2l+2>0$, we obtain

$$\begin{aligned}C_1+C_2(y(r))^{2l+2}r^{-1}\leq&\frac{1}{c}\frac{1}{1-n-l}((y^*)^{1-n-l}(y(r))^{2l+2}-(y(r))^{l+3-n})\\&+C_5(y(r))^{2l+2}\end{aligned}$$

for $n+l\neq 1$ and

$$C_1+C_2(y(r))^{2l+2}r^{-1}\leq\frac{1}{c}\ln\left(\frac{y^*}{y(r)}\right)(y(r))^{2l+2}+C_5(y(r))^{2l+2}$$

for $n+l=1$ and $r>R$. Together with $l+3-n>0$ and $2l+2>0$, the assumption $y_\infty=\lim_{r\to\infty}y(r)=0$ implies that the left-hand side converges to $C_1$ while the right-hand side vanishes for $r\to\infty$ in both cases. Since $C_1>0$, this is a contradiction, so $y_\infty<0$. □

As shown in Lemma 3.8, we conclude $y_\infty\in[-\infty,0[$. In addition, the limit exists in $\mathbb{R}$:

**Remark 3.9.** *Let $-\frac{M_0^2}{2L_0}\leq\mathring{y}\leq 0$. Then the limit $E_0:=\lim_{r\to\infty}y(r)\in]-\infty,0[$ is finite and negative. In the situation of Lemma 3.8, we also obtain $E_0<0$ for $\mathring{y}>0$. In particular, this is consistent with Lemma 3.2.*

*Proof.* As discussed before, the limit $\lim_{r\to\infty}y(r)\in[-\infty,0[$ exists and is negative. As a result, a radius $R>R_0$ exists such that $z(r)=y(r)+\frac{M_0}{r}-\frac{L_0}{2r^2}<0$ for all $r>R$. Since $g=0$ on $]-\infty,0[$, the mass function $m$ is constant on $]R,\infty[$, i.e., $m(r)=m(R)$ for $r>R$. The monotonicity of $m$ leads to

$$\begin{aligned}y(r)=y(0)-\int_0^r\frac{m(s)}{s^2}\,ds&=\mathring{y}-\int_{R_0}^r\frac{m(s)}{s^2}\,ds\\&\geq\mathring{y}-m(R)\left(\frac{1}{R_0}-\frac{1}{r}\right),\quad r>R.\end{aligned}$$

This estimate implies that $\lim_{r\to\infty}y(r)\geq\mathring{y}-\frac{m(R)}{R_0}>-\infty$. □

## 3.5 The existence of solutions with compact support and examples

We determined in the previous sections which conditions on general functions $g$ are sufficient to ensure the existence of a unique solution $y$ and, especially for $\mathring{y} > 0$, the negativity of the limit $y_\infty$. It remains to transfer these conditions on $g$ to conditions on $\varphi$, so $g$ of the form (3.6) satisfies the required conditions. After that, we discuss some examples for $\varphi$.

We assume that $\varphi$ is satisfying the conditions (V1)–(V3). Then let the function $g$ be as defined in equation (3.6). Obviously, $g = 0$ on $]-\infty, 0]$ and $g > 0$ on $]0, \infty[$. Let $\kappa + l + \frac{1}{2} > 0$. By Lemma 3.4, $g \in C^1(\mathbb{R})$. According to Lemma 3.6, a unique solution $y \in C^2([0, \infty[)$ exists for all $\mathring{y} \in \mathbb{R}$. In the case $\mathring{y} > 0$, we additionally have to determine whether the conditions in Lemma 3.8 are satisfied. Lemma 3.4 implies that $g$ is monotonically increasing, and as discussed before, the condition (g1) is obviously satisfied. In order to fulfill the condition (g2), we impose another growth condition on $\varphi$:

(V4*) There exist $\eta_1 > 0$, $c > 0$ and $-1 < k < l + \frac{3}{2}$ such that

$$\varphi(\eta) \geq c\eta^k, \quad 0 \leq \eta \leq \eta_1.$$

This assumption leads to

$$\begin{aligned} g(y) &= c_l \int_0^y \varphi(\eta)(y-\eta)^{l+\frac{1}{2}} \, d\eta \geq C \int_0^y \eta^k (y-\eta)^{l+\frac{1}{2}} \, d\eta \\ &= C y^{k+l+\frac{3}{2}} \int_0^1 s^k (1-s)^{l+\frac{1}{2}} \, ds = C y^{k+l+\frac{3}{2}} \end{aligned}$$

for $0 < y < \eta_1$. Note that $k > -1$ and $l > -\frac{1}{2}$, so the latter integral exists. Since $0 < k + l + \frac{3}{2} < 2l + 3$, the function $g$ satisfies condition (g2) with $0 < n = k + \frac{3}{2} < 3 + l$.

Finally, we summarize the results of this chapter in the following theorem:

**Theorem 3.10.** *Let $l > -\frac{1}{2}$ and $L_0 > 0$. Assume that $\varphi : \mathbb{R} \to [0, \infty[$ satisfy the conditions (V1)–(V3) with $l + \kappa + \frac{1}{2} > 0$. Furthermore, let $\mathring{y} > -\frac{M_0^2}{2L_0}$. If $\mathring{y} > 0$, let $\varphi$ additionally satisfy (V4*). Then there exists a unique solution $y \in C^2([0, \infty[)$ of*

$$\begin{aligned} y' &= -\frac{m(r)}{r^2}, \quad y(0) = \mathring{y} \\ m(r) &= m(r, y) = 4\pi \int_0^r s^{2l+2} g(y(s) + \frac{M_0}{s} - \frac{L_0}{2s^2}) \, ds, \quad r \geq 0. \end{aligned}$$

*with $y'(0) = 0$ and $E_0 := \lim_{r\to\infty} y(r) < 0$.*

*Let $U := E_0 - y$. Then $U \in C^2(\mathbb{R}^3)$ and the function*

$$f(x,v) = \varphi\left(E_0 - U(r) + \frac{M_0}{r} - \frac{1}{2}|v|^2\right)(|x \times v|^2 - L_0)_+^l$$

*is a spherically symmetric steady state of the Vlasov-Poisson system with a point mass. Furthermore, $f$ has a compact support with $\operatorname{supp} f \subset (\bar{B}_{R_1} \setminus B_{R_0}) \times \bar{B}_{v_{\max}}$. Here, $R_0 := \operatorname{supp}\{\tilde{R} > 0 | y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} < 0,\ 0 < r < \tilde{R}\}$ and $R_1 := \inf\{\tilde{R} > R_0 | y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} < 0,\ r > \tilde{R}\} > R_0$ and $v_{\max} := \mathring{y} + \frac{M_0}{R_0}$. In particular, $R_0$ is given by*

$$R_0 = \begin{cases} -\frac{M_0}{2\mathring{y}} + \sqrt{\frac{L_0}{2\mathring{y}} + \frac{M_0^2}{4y_0^2}}, & \mathring{y} > 0, \\ \frac{L_0}{2M_0}, & \mathring{y} = 0, \\ -\frac{M_0}{2\mathring{y}} - \sqrt{\frac{L_0}{2\mathring{y}} + \frac{M_0^2}{4y_0^2}}, & -\frac{M_0^2}{2L_0} < \mathring{y} < 0. \end{cases}$$

*The compact support in $x$ has a shell-like structure with radii $0 < R_0 < R_1$.*

*Proof.* We already discussed most assertion, so we first show that $R_0$ is the largest radius which satisfies $z(r) = y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2}$ for $0 < r < R$ and that $R_1 > R_0$ exists. Then we turn to prove that $f$ is a time-independent solution in the sense of Definition 2.10 to complete the proof.

By Remark 3.7 (b), $y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} < 0$ for $0 < r < R_0$ and $y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} > 0$ for $R_0 < r < R_0 + \delta$ and for some $\delta > 0$, so $R_0 = \min_{(x,v)\in\operatorname{supp} f} |x|$. Furthermore, $\lim_{r\to\infty} y(r) < 0$, so there exists a radius $\tilde{R} > R_0$ such that $y(r) + \frac{M_0}{r} - \frac{L_0}{2r^2} < 0$ for all $r > \tilde{R}$. In particular, $R_1 = \max_{(x,v)\in\operatorname{supp} f} |x|$ exists.

It remains to show that $f$ is a solution of the Vlasov-Poisson system with a point mass. Since $y \in C^2([0,\infty[)$ with $y'(0) = 0$ and $\lim_{r\to\infty} y(r) =: E_0 \in ]-\infty, 0[$, the potential $U$ given by $U := E_0 - y$ solves the corresponding semi-linear Poisson equation (3.7) as discussed in Lemma 3.5. In particular, $U$ is spherically symmetric with $U \in C^2(\mathbb{R}^3)$. Since we require $\kappa + l + \frac{1}{2} > 0$, the function $g$ is continuously differentiable on $\mathbb{R}$ by Lemma 3.4. Because of the compact support of $f$ and the relation between $g$ and $\rho$ in Lemma 3.3, the induced density $\rho = \rho_f$ is continuously differentiable with compact support. Since $U \in C^2(\mathbb{R}^3)$ solves the Poisson equation, it follows from the uniqueness in Lemma 2.7 that $U = U_f$. Moreover, $\partial_x U$ is bounded according to Lemma 2.7 (b). Therefore, Lemma 2.16 implies that the characteristic flow exists on $\mathbb{R}\times\mathbb{R}\times\{L > 0\}$. As shown in Lemma 2.13, the particle energy $E$ and $L$ are constant along characteristics, so $f = \Phi(E, L)$ obviously solves the Vlasov equation. In summary, the function $f$ is a steady state of the Vlasov-Poisson system with a point mass in the sense of Definition 2.10 respectively Definition 2.17. $\square$

In the previous argumentation, we rarely used $M_0 > 0$, so in just a few steps, we can transfer almost all assertions in this chapter to the case $M_0 = 0$.

One of the two times where we used $M_0 > 0$ was in the proof of Lemma 3.6. In the first step of the proof, we showed that

$$\mathring{y} + \frac{M_0}{s} - \frac{L_0}{s^2} < 0, \quad 0 < s < R^*,$$

for some $R^* > 0$. Since $\sup_{s>0}(-\frac{L_0}{s^2}) = 0$, it follows $\mathring{y} - \frac{L_0}{2r^2} \leq 0$ for all $r > 0$ and $\mathring{y} \leq 0$. For $\mathring{y} > 0$, we obtain $\mathring{y} - \frac{L_0}{2r^2} < 0$ for $0 < r < R_0$ and $\mathring{y} - \frac{L_0}{2r^2} > 0$ for $R_0 < r < R_0 + \delta$ with some $\delta > 0$ and $R_0 := \sqrt{\frac{L_0}{2\mathring{y}}}$. Therefore, we have to distinguish between $\mathring{y} \leq 0$ and $\mathring{y} > 0$.

The second time where we used $M_0 > 0$ was in the proof of Lemma 3.8 in the case $y_\infty = 0$. We used that the map $]0, \infty[ \ni r \to \frac{M_0}{r} - \frac{L_0}{r^2}$ is monotonically decreasing on some interval of the form $]R, \infty[$ to ensure that $z(r) := y(r) + \frac{M_0}{r} - \frac{L_0}{r^2}$ is monotonically decreasing in $r$ on $]R, \infty[$. Since the map $]0, \infty[ \ni r \to -\frac{L_0}{2r^2}$ is monotonically increasing, we appropriately estimate $z'$ to ensure monton at a similar radius $R > 0$:

$$\begin{aligned} z'(r) &= y'(r) + \frac{L_0}{r^3} = -\frac{m(r)}{r^2} + \frac{L_0}{r^3} \\ &\leq -\frac{1}{r^3}(-m(R_0+1)r + L_0), \quad r > R_0 + 1. \end{aligned}$$

Obviously, $z$ is monotonically decreasing on $[R, \infty[$ with $R := \max\{R_0 + 1, \frac{L_0}{m(R_0+1)}\}$. Note that $m(R_0 + 1) > 0$.

Before we analyze the stability of steady states, we introduce two explicit families of steady states, the so-called *(generalized) polytropic steady states* and *(generalized) King models*:

**Example 3.11** (The (generalized) polytropic steady states)**.** *Let $\varphi : \mathbb{R} \to [0, \infty[$ be given by*

$$\varphi(\eta) := (\eta)_+^k$$

*with $l > -\frac{1}{2}$ and $-1 < k < l + \frac{3}{2}$. In addition, let $k + l + \frac{1}{2} > 0$. Then $\varphi$ obviously satisfies the assumptions (V1), (V2), (V3) and (V4*).*

*Steady states induced by $\varphi$ are called* (generalized) polytropic. *In particular, for $\mathring{y} > -\frac{M_0^2}{2L_0}$ (or $\mathring{y} > 0$ in the case $M_0 = 0$) there exists an associated global solution $y$ of equation* (3.8) *with $E_0 := \lim_{r\to\infty} y(r) < 0$ and $U := E_0 - y$. Furthermore,*

$$f(x, v) := (E_0 - E)_+^k (L - L_0)_+^l, \quad (x, v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3,$$

*is a solution of the Vlasov-Poisson system with a point mass.*

**Example 3.12** (The (generalized) King models). *Let $\varphi : \mathbb{R} \to [0, \infty[$ be given by*

$$\varphi(\eta) := (e^\eta - 1)_+^k$$

*with $l > -\frac{1}{2}$ and $-1 < k < l + \frac{3}{2}$. Furthermore, let $k + l + \frac{1}{2} > 0$. This ansatz function $\varphi$ satisfies the assumptions (V1), (V2), (V3) and (V4*) with $\kappa = k$.*

*Steady states induced by this ansatz are called* (generalized) King model*; choosing $k = 1$ corresponds to the usual ansatz for the King models. As seen before, for $\mathring{y} > -\frac{M_0^2}{2L_0}$ (or $\mathring{y} > 0$ for $M_0 = 0$) the associated solution $y$ of the equation* (3.8) *exists globally with $E_0 := \lim_{r\to\infty} y(r) < 0$ and $U := E_0 - y$, so*

$$f(x, v) := (e^{E_0 - E} - 1)_+^k (L - L_0)_+^l, \quad (x, v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3,$$

*is a solution of the Vlasov-Poisson system with a point mass.*

# 4 Stability of steady states

In the previous chapter we constructed steady states with finite mass and compact support. In the next step, we analyze the stability of a certain class of steady states which includes the (generalized) polytropic steady states (cf. Example 3.11) and the (generalized) King models (cf. Example 3.12) under certain conditions. We proceed analogously to [4] and apply the method in [4] to the Vlasov-Poisson system with a point mass and to anisotrophic steady states. As discussed before, we consider anisotropic steady states $f_0$ of the form

$$f_0(x,v) = \Phi(E(x,v), L(x,v)), \quad (x,v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3,$$

with suitable $\Phi : \mathbb{R} \times [0,\infty[ \to [0,\infty[$. Recall that $E$ and $L$ are the particle energy and the angular momentum squared as defined in Definition 2.13. Furthermore, we require the induced potential $U_0 := U_{f_0}$ to be spherically symmetric, so $f_0$ is spherically symmetric according to Lemma 2.13.

In the theory of ordinary differential equations, we can determine the stability of a steady state in a metric space with a Lyapunov function. Lyapunov functions have a local minimum in the equilibrium point to be examined and are decreasing among solutions of the ordinary differential equation. If there exists a Lyapunov function, the equilibrium point is stable. We adapt this procedure and introduce a map similar to a metric and construct a functional which acts like a Lyapunov function in the theory of ordinary differential equation. More precisely, we search for a functional which is constant along solutions of the system and has a local minimum in $f_0$.

## 4.1 The energy-Casimir functional

In order to construct a functional as described before, we consider the *total energy* and the *Casimir functional*, which are both conserved along solutions of the Vlasov-Poisson system with a point mass.

From a physical point of view, the *total energy* of the system is conserved along solutions since we only allow gravitational particle-particle interactions and gravitational interactions between particles and the point mass. Before defining it, we motivate the total energy from a physical point of view.

We consider $N \in \mathbb{N}$ particles with unit mass and a central point mass $M_0$ and $t \geq 0$. The particle-particle interaction is given by the potential $\tilde{U}(x,y) := -\frac{1}{|x-y|}$ for $x, y \in \mathbb{R}^3$ with $x \neq y$ and the interaction between particles and the point mass by the potential $U_{M_0}(x) = -\frac{M_0}{|x|}$ for $x \in \mathbb{R}^3$ with $x \neq 0$. Then the many-body system has the total energy

$$\begin{aligned} H(t) &= \sum_{i=1}^{N} \frac{1}{2}|v_i|^2 - \sum_{i=1}^{N} \frac{M_0}{|x_i|} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} \tilde{U}(|x_i - x_j|) \\ &= \sum_{i=1}^{N} \Big( \frac{1}{2}|v_i|^2 - \frac{M_0}{|x_i|} + \frac{1}{2} \sum_{\substack{j=1 \\ j\neq i}}^{N} \tilde{U}(|x_i - x_j|) \Big) \end{aligned}$$

where $(x_i, v_i) = (x_i, v_i)(t)$ denotes the location and the velocity, respectively, of the $i$th particle. The total energy consists of the total kinetic energy given by the sum over all kinetic energies and of the potential energy given by the sum over all interactions of the particles and the point mass and by the sum over all particle-particle interactions. Converting the discrete into a continuous setting, we introduce the phase density $f(t) = f(t, x, v)$ which denotes the amount of particles with location $x$ and velocity $v$ at the time $t$. This turns the total energy into

$$\begin{aligned} H(t) &= \Big( \frac{1}{2}|v|^2 - \frac{M_0}{|x|} + \frac{1}{2} \iint \tilde{U}(|x - x'|) f(t, x', v') \, dv' \, dx' \Big) f(t, x, v) \, dv \, dx \\ &= \iint \Big( \frac{1}{2}|v|^2 - \frac{M_0}{|x|} + \frac{1}{2} U(t, x) \Big) f(t, x, v) \, dx \, dv \end{aligned}$$

with $U(t,x) := \iint \tilde{U}(|x - x'|) f(t, x', v') \, dv' \, dx' = U_{f(t)(x)}$ for $x \neq 0$. This leads to the *total energy* of the Vlasov-Poisson system with a point mass:

**Lemma and Definition 4.1.** *Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty[$ with compact support* $\operatorname{supp} f(t) \subset \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ *for $t \geq 0$. Assume that $f \in L^1 \cap L^p(\mathbb{R}^6)$ for some $p \in ]3, \infty]$. Then the* total energy *is defined by*

$$\begin{aligned} \mathcal{H}(f) &:= \iint \Big( \frac{1}{2}|v|^2 - \frac{M_0}{|x|} \Big) f(x, v) \, dv \, dx + \frac{1}{2} \int U_f(x) \rho_f(x) \, dx \\ &= \frac{1}{2} \iint |v|^2 f(x, v) \, dv \, dx - \frac{1}{8\pi} \int \Big( |\nabla U_f(x)|^2 + 8\pi \frac{M_0}{|x|} \Big) \rho_f(x) \, dx. \end{aligned}$$

*In particular, all integrals are finite.*

*Proof.* Since $f \in L^1 \cap L^p(\mathbb{R}^6)$ has compact support, the induced density $\rho_f \in L^1 \cap L^p(\mathbb{R}^3)$ has compact support. This implies that $U_f \in C^1(\mathbb{R}^3)$ by Lemma 2.8 and that the integrals in the definition of $\mathcal{H}(f)$ exist. Furthermore, $U_f \in L^p(\mathbb{R}^3)$ and $\nabla U_f \in L^2(\mathbb{R}^3)$. Using Friedrich's mollification, there exists a sequence $(\chi_j) \subset C_c^\infty(\mathbb{R}^3)$ such that $\chi_j \to U_f$ in $L^p(\mathbb{R}^3)$ and $\nabla \chi_j \to \nabla U_f$ in $L^2(\mathbb{R}^3)$. Since $U_f$ solves the Poisson equation in a distributional sense, it follows that

$$\int U_f(x) \rho_f(x) \, dx = \lim_{j \to \infty} \int \chi_j \rho_f \, dx = \lim_{j \to \infty} \frac{1}{4\pi} \int \Delta \chi_j U_f \, dx$$

$$= \lim_{j\to\infty} \frac{1}{4\pi} \lim_{R\to\infty} \int_{r\le|R|} \Delta\chi_j U_f \, dx$$
$$= \lim_{j\to\infty} \frac{1}{4\pi} \lim_{R\to\infty} \left( \int_{r=R} U_f \nabla\chi_j \cdot \frac{x}{|x|} \, dS(x) - \int_{r\le R} \nabla\chi_j \cdot \nabla U_f \, dx \right)$$
$$= - \lim_{j\to\infty} \frac{1}{4\pi} \int \nabla\chi_j \cdot \nabla U_f \, dx = -\frac{1}{4\pi} \int |\nabla U_f(x)|^2 \, dx.$$

Because of $p > 3$, the induced density $\rho$ is in $L^1 \cap L^p(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$ with $q = (1-\frac{1}{p})^{-1} < \frac{3}{2}$ which shows the validity of the first limit. In summary, this leads to

$$\mathcal{H}(f) = \iint \left(\frac{1}{2}|v|^2 - \frac{M_0}{|x|}\right) f(x,v) \, dv \, dx + \frac{1}{2} \int U_f(t,x)\rho_f(x) \, dx$$
$$= \iint \left(\frac{1}{2}|v|^2 - \frac{M_0}{|x|}\right) f(x,v) \, dv \, dx - \frac{1}{8\pi} \int |\nabla U_f(x)|^2 \, dx. \qquad \square$$

The total energy is a candidate for the Lyapunov functional since it is conserved along spherically symmetric solutions of the Vlasov-Poisson system with a point mass in the sense of Definition 2.10:

**Lemma 4.2.** *Let $f : I \times \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 \to [0,\infty[$ be a spherically symmetric solution of the Vlasov-Poisson system with a point mass with $I$ an open interval and $0 \in I$. Suppose that $\mathring{f} := f(0) \in L^1 \cap L^p(\mathbb{R}^6)$ for some $p \in ]3,\infty]$ and that $\operatorname{supp} \mathring{f} \subset \{L \ge L_0\}$ is compact with some $L_0 > 0$. Assume that $\rho = \rho_f \in C(I \times \mathbb{R}^3)$. Then the total energy $\mathcal{H}(f)$ is constant in time, i.e.,*

$$\mathcal{H}(f(t)) = \mathcal{H}(\mathring{f}), \quad t \in I.$$

*Proof.* Let $f$ be a solution as required. Since the angular momentum squared $L$ is constant along characteristics according to Lemma 2.13, it follows that

$$f(t,x,v) = 0, \quad t \in I, \ (x,v) \in \{L \le L_0\}.$$

Let $a < 0 < b$ be with $[a,b] \subset I$. Furthermore, let $t \in ]a,b[$ be arbitrary. Since $\rho \in C(I \times \mathbb{R}^3)$ and $\rho$ is spherically symmetric, we obtain $U^{0,2}(I \times \mathbb{R}^3)$ according to Lemma 2.9. Thus the assumptions in Lemma 2.11 are satisfied, so the characteristic flow $X(t,s,\cdot)$ is a measure preserving $C^1$-diffeomorphism on $\{L > 0\}$. Because $\{L = 0\}$ is a null set, it follows by the change of variables:

$$\mathcal{H}(f(t)) = \iint \left(\frac{1}{2}|v|^2 - \frac{M_0}{|x|} + \frac{1}{2}U_{f(t)}(x)\right) f(t,x,v) \, dv \, dx$$
$$= \iint_{\{L>0\}} \left(\frac{1}{2}|V(t)|^2 - \frac{M_0}{|X(t)|} + \frac{1}{2}U_{f(t)}(X(t))\right) f(t,(X,V)(t)) \, dv \, dx$$
$$= \iint_{\{L>0\}} \left(\frac{1}{2}|V(t)|^2 - \frac{M_0}{|X(t)|} + \frac{1}{2}U_{f(t)}(X(t))\right) \mathring{f}(x,v) \, dv \, dx.$$

For the reason of clarity, we write $(X,V)(t)$ instead of $(X,v)(t,0,x,v)$ for $(x,v)\in\{L>0\}$. According to Lemma 2.15, the induced potential $U$ is continuously differentiable on $I\times\mathbb{R}^3$, so the integrand is differentiable with respect to $t$ and it follows

$$\begin{aligned}\partial_t\mathcal{H}(f(t)) &= \iint\left(V(t)\cdot\dot V(t)+\partial_x\left(U_{f(t)}-\frac{M_0}{|x|}\right)\Big|_{x=X(t)}\cdot\dot X(t)\right)\mathring{f}(x,v)\;dv\;dx\\ &\quad-\frac{1}{2}\iint\partial_x(U_{f(t)})\Big|_{x=X(t)}\cdot\dot X(t)\mathring{f}(x,v)\;dv\;dx\\ &\quad+\frac{1}{2}\iint\partial_t(U_{f(t)})(X(t))\mathring{f}(x,v)\;dv\;dx.\end{aligned}$$

Note that $\mathring{f}$ has compact support and the integrand is continuous on $[a,b]\times\operatorname{supp}\mathring{f}$. Since $L$ is constant along characteristics and $(X,V)$ is continuous, there exists a constant $C>0$ such that $|V(t,0,x,v)|\le C$ for every $t\in[a,b]$ and $(x,v)\in\operatorname{supp}\mathring{f}$. In particular,

$$\begin{aligned}L_0&\le|x\times v|^2=|X(t,0,x,v)\times V(t,0,x,v)|^2\le|X(t,0,x,v)|^2|V(t,0,x,v)|^2\\&\le C|X(t,0,x,v)|^2,\quad (x,v)\in\operatorname{supp}\mathring{f},\ t\in[a,b],\end{aligned}$$

so the change of integration and differentiation is valid. Obviously, the first integral vanishes since the characteristic flow $(X,V)$ solves the characteristic system. According to Lemma 2.8 and 2.15, the induced potential $U=U_{f(t)}(x)$ is once continuously differentiable on $I\times\mathbb{R}^3$ with

$$\begin{aligned}\partial_xU_{f(t)}(x)&=\iint\frac{x-y}{|x-y|^3}f(t,y,u)\;du\;dy\\&=\iint\frac{x-X(t,s,y,u)}{|x-X(t,0,y,u)|^3}\mathring{f}(y,u)\;du\;dy\end{aligned}$$

and

$$\begin{aligned}\partial_tU_{f(t)}(x)&=-\iint\frac{x-y}{|x-y|^3}\cdot u\;f(t,y,u)\;du\;dy\\&=-\iint\frac{x-X(t,0,y,u)}{|x-X(t,0,y,u)|^3}\cdot V(t,0,y,u)\mathring{f}(y,u)\;du\;dy\end{aligned}$$

for $x\in\mathbb{R}^3$ and $t\in I$. In particular, this leads to

$$\begin{aligned}&\iint\partial_tU_{f(t)}(X(t,0,x,v))\mathring{f}(x,v)\;dv\;dx\\&=-\iiiint\frac{X(t,0,x,v)-X(t,0,y,u)}{|X(t,0,x,v)-X(t,0,y,u)|^3}\cdot\dot X(t,0,y,u)\mathring{f}(y,u)\;du\;dy\mathring{f}(x,v)\;dv\;dx\\&=\iiiint\frac{X(t,0,y,u)-X(t,0,x,v)}{|X(t,0,y,u)-X(t,0,x,v)|^3}f(s,x,v)\;dv\;dx\cdot\dot X(t,0,y,u)\mathring{f}(y,u)\;du\;dy\\&=\iint\partial_xU_{f(t)}(X(t,0,y,u))\cdot\dot X(t,0,y,u)\mathring{f}(y,u)\;du\;dy,\end{aligned}$$

so it follows that

$$\partial_t \mathcal{H}(f(t)) = 0, \quad t \in I.$$

Note that we choose $a < 0 < b$ arbitrary with $[a, b] \subset I$. □

Since the total energy is constant along spherically symmetric solutions of the Vlasov-Poisson system with a point mass, we examine whether the energy functional has a minimum in $f_0$. Let $f_0$ and $f$ be as required in Lemma 4.2 with the induced potentials $U_0$ and $U_f$. Then we can expand the energy around the state $f_0$:

$$\mathcal{H}(f) = \mathcal{H}(f_0) + \iint \Big(\frac{1}{2}|v|^2 - \frac{M_0}{|x|}\Big)(f - f_0)\ dv\ dx - \frac{1}{8\pi}\int (|\nabla U_f|^2 - |\nabla U_0|^2)\ dx$$

Similar approximations as in the proof of Lemma 4.1 and Green's identity lead to

$$\begin{aligned}\int (|\nabla U_f|^2 - |\nabla U_0|^2)\ dx &= \int |\nabla U_f - \nabla U_0|^2\ dx + 2\int \nabla U_0 \cdot (\nabla U_f - \nabla U_0)\ dx \\ &= \int |\nabla U_f - \nabla U_0|^2\ dx - 8\pi \iint U_0(f - f_0)\ dv\ dx.\end{aligned}$$

In summary, this yields

$$\mathcal{H}(f) = \mathcal{H}(f_0) + \iint \Big(\frac{1}{2}|v|^2 + U_0(x) - \frac{M_0}{|x|}\Big)(f - f_0)\ dv\ dx - \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2\ dx. \tag{4.1}$$

Notice that the factor in the linear term is the particle energy, which is introduced in Definition 2.13. In general, the linear term does not vanish, so the total energy can not act like a Lyapunov function. For this reason, we try to extend the functional and look at another conserved quantity, the so-called *Casimir functional*:

**Lemma and Definition 4.3.** *Let $\tilde{L} > 0$ and $\Psi : [0, \infty[ \times [0, \infty[ \to \mathbb{R}$ with $\Psi(f, L) = 0$ for $f = 0$ or $L \leq \tilde{L}$. Furthermore, let $\Psi$ be continuous on $[0, \infty[ \times ]\tilde{L}, \infty[$. The* Casimir functional *is defined by*

$$\mathcal{C}(f) := \iint \Psi(f(x, v), L(x, v))\ dv\ dx$$

*for $f : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty[$ provided that the integral exists. Then the Casimir functional is constant along the spherically symmetric solutions of the Vlasov-Poisson system with a point mass as given in Lemma 4.2.*

*Proof.* Since $L$ is conserved along characteristics by Lemma 2.13 and the characteristic flow is measure preserving by Lemma 2.11, the Casimir functional $\mathcal{C}(f(t))$ is obviously constant in $t$. □

We combine the energy and the Casimir functional to the so-called *energy-Casimir functional*:

$$\mathcal{H}_\mathcal{C} := \mathcal{H} + \mathcal{C}.$$

Since the function $\Psi$ is initially still arbitrary, we choose $\Psi$ such that the linear term of the energy-Casimir functional vanishes. We assume that $\Psi = \Psi(f, L) \in C^{2,0}([0,\infty[\times[0,\infty[)$ and that $\mathcal{H}_\mathcal{C}(f)$ and $\mathcal{H}_\mathcal{C}(f_0)$ exists. Taylor expansion of $\Psi(f, L)$ with respect to $f$ around $f_0$ leads to

$$\begin{aligned}\mathcal{H}_\mathcal{C}(f) = \mathcal{H}_\mathcal{C}(f_0) + \iint (E + \Psi'(f_0, L))(f - f_0)\, dv\, dx - \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2\, dx \\ + \frac{1}{2}\iint \Psi''(f_0, L)(f - f_0)^2\, dv\, dx + \iint \mathcal{O}((f - f_0)^3)\, dv\, dx.\end{aligned} \tag{4.2}$$

In the following, $\Psi'$ denotes the $f$-derivative of $\Psi$, i.e., $\Psi' = \frac{\mathrm{d}}{\mathrm{d}f}\Psi = \partial_f \Psi$. In order that the linear term of the energy-Casimir functional $\mathcal{H}_\mathcal{C}$ vanishes, the choice of $\Psi$ depends on the steady state $f_0$ which we are investigating. For this reason, we set some assumptions on $f_0$:

Let $\Phi \in C(\mathbb{R} \times [0,\infty[)$. In addition, we assume that there exist $E_0 \in \mathbb{R}$ and $L_0 > 0$ such that

$$\Phi(E, L) = 0, \quad E \geq E_0 \text{ or } L \leq L_0$$

and $\lim_{E\to+\infty} \Phi(E, L) = -\infty$ for $L \geq L_0$. Furthermore, let $\Phi \in C^{2,0}(]-\infty, E_0[\times]L_0, \infty[)$ with

$$\Phi'(E, L) < 0, \quad E < E_0,$$

for all $L > L_0$. Here, $\Phi' = \frac{\mathrm{d}}{\mathrm{d}E}\Phi = \partial_E \Phi$.

Let $f_0 = \Phi(E, L) \in C_c(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$ be a non-trivial spherically symmetric solution of the Vlasov-Poisson system with a point mass. Then $U_0 = U_{f_0} \in C^{0,2}(\mathbb{R}^3)$ by Lemma 2.9. Furthermore, we require that the support of $f_0$, which has the form

$$\begin{aligned}\operatorname{supp} f_0 = \Big\{(x, v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 \Big| E(x, v) = \frac{1}{2}|v|^2 + U_0(x) - \frac{M_0}{|x|} \leq E_0 \\ \wedge\, L(x, v) = |x \times v|^2 \geq L_0\Big\},\end{aligned}$$

is compact. In particular, the cut-off energy $E_0$ is necessarily negative, as seen in Lemma 3.2.

We aim to choose $\Psi$ such that

$$\Psi'(f_0, L) = \Psi'(\Phi(E, L), L) = -E, \quad (x, v) \in \operatorname{supp} f_0.$$

Since $\Phi :]-\infty, E_0]\times]0,\infty[\to [0,\infty[$ is continuous and monotonically decreasing in $E$, the inverse with respect to $E$ exists for all $L > L_0$ and is continuous:

**Lemma 4.4.** *Let $h : I \times A \to J$ be continuous with $I$ and $J$ intervals and $A \subset \mathbb{R}^n$ with $n \in \mathbb{N}$. Additionally, let $h = h(x,z)$ be strictly increasing or decreasing in $x$ with $h(I,z) = J$ for all $z \in A$. Then there exists the map $h_x^{-1} : J \times A \to I$ with $h(h_x^{-1}(y,z),z) = y$ and $h_x^{-1}(h(x,z),z) = x$ for all $x \in I$, $y \in J$ and $z \in A$. Furthermore, the map $h_x^{-1}$ is continuous.*

*Proof.* Since the map $h(\cdot,z) : I \to J$ is monotonic for every $z \in A$, the inverse $h_x^{-1} : J \times A \to I$ exists. Without loss of generality, let $h$ be strictly monotonically decreasing in $x$. Assume that $h_x^{-1} : J \times A \to I$ is not continuous in $(y,z) \in J \times A$. Then there exists a convergent sequence $((y_k,z_k))_{k\in\mathbb{N}} \subset J \times A$ and $\varepsilon > 0$ with $\lim_{k\to\infty} y_k = y$ and $\lim_{k\to\infty} z_k = z$ such that

$$|h_x^{-1}(y_k,z_k) - h_x^{-1}(y,z)| \geq \varepsilon, \quad k \in \mathbb{N}.$$

Let $x := h_x^{-1}(y,z)$ and $x_k := h_x^{-1}(y_k,z_k)$ for $k \in \mathbb{N}$. By the Bolzano-Weierstraß theorem, there exists a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} x_{k_j} \in \bar{I} \cup \{\pm\infty\}$. In particular, $\tilde{x} := \lim_{j\to\infty} x_{k_j} \in I$, as we show at the end of the proof. Since $h$ is continuous and $x_{k_j} \to \tilde{x}$ as $j \to \infty$, we obtain

$$h(x,z) = y = \lim_{j\to\infty} y_{k_j} = \lim_{j\to\infty} h(x_{k_j}, z_{k_j}) = h(\tilde{x}, z).$$

Since $h(\cdot,z)$ is bijective, this leads to $x = \tilde{x}$, so the sequence $(x_{k_j})_{j\in\mathbb{N}}$ converges to $x$. By construction,

$$|x_{k_j} - x| \geq \varepsilon, \quad j \in \mathbb{N},$$

so this leads to a contraction. Thus, $h_x^{-1}$ is continuous in every $(y,z) \in J \times A$.

At least, it remains to show that $\tilde{x} = \lim_{j\to\infty} x_{k_j} \in I$. If $\inf I \notin I$, we assume that $\lim_{j\to\infty} x_{k_j} = \inf I$. Then $[x-\varepsilon, x] \subset I$ holds after shrinking $\varepsilon > 0$ if necessary. Furthermore, we obtain $x_{k_j} < x - \varepsilon$ for $j$ large enough. The monotonicity and the continuity of $h$ lead to the following contradiction:

$$\begin{aligned} y = \lim_{j\to\infty} y_{k_j} &= \lim_{j\to\infty} h(x_{k_j}, z_{k_j}) \geq \lim_{j\to\infty} h(x-\varepsilon, z_{k_j}) \\ &= h(x-\varepsilon, z) > h(x,z) = y. \end{aligned}$$

Analogously, it can be shown that $\lim_{j\to\infty} x_{k_j} < \sup I$ if $\sup I \notin I$. □

We denote the inverse of $\Phi$ with respect to $E$ by $\Phi_E^{-1} : [0,\infty[\times]L_0,\infty[\to]-\infty,E_0]$. In the following arguments, we need some additional technical assumptions on $\Phi$ to ensure a certain boundedness. The compact support $\operatorname{supp} f_0 \subset \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ implies that the minimal energy $E_{\min} := \min_{\operatorname{supp} f_0} E < E_0$ and the maximal angular momentum squared $L_{\max} := \max_{\operatorname{supp} f_0} L > L_0$ exist since $E$ and $L$ are continuous maps on $\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$. Let $\Phi$ respectively $\Phi_E^{-1}$ satisfy the following conditions:

(A1) For all $k \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$|\Phi'(\Phi_E^{-1}(y, L), L)| < C$$

for all $0 < y \leq k$ and $L_0 < L \leq L_{\max}$.

(A2) For all $k \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$|\Phi'(\Phi_E^{-1}(y, L), L)| > C$$

for all $\frac{1}{k} \leq y \leq k$ and $L_0 + \frac{1}{k} \leq L \leq L_{\max}$.

(A3) For all $k \in \mathbb{N}$, there exists constants $C_{1,2} > 0$ such that

$$C_1 \leq |\Phi'(E, L)| < C_2$$

for all $E_{\min} \leq E \leq E_0 - \frac{1}{k}$ and $L_0 + \frac{1}{k} \leq L \leq L_{\max}$.

(A4) The map $\Phi(E, \cdot)$ is monotonically increasing for all $E \in \mathbb{R}$.

We will later provide examples of steady states satisfying these conditions.

To ensure that the linear term of the energy-Casimir functional vanishes, we choose $\Psi$ by

$$\Psi(f, L) := -\int_0^f \Phi_E^{-1}(z, L)\, dz, \quad f \geq 0,\ L > L_0,$$

and $\Psi(f, L) := 0$ for $L \leq L_0$ or $f = 0$ similar to [4, Equation (2.2)]. Later in Lemma 5.1, we examine this choice of $\Psi$ in more detail and verify that $\Psi$ satisfies the conditions in Lemma 4.3.

## 4.2 The class of perturbations

In order to determine whether the energy-Casimir functional $\mathcal{H}_\mathcal{C}$ has a local minimum in $f_0$, we introduce analogously to [4] the space of perturbations and a map comparable to a metric. We restrict the class of perturbations to perturbations which are *dynamically accessible from* $f_0$:

**Definition 4.5.** *A function of the form $f = f_0 \circ T$ with $T : \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ a measure preserving $C^1$-diffeomorphism is called* dynamically accessible from $f_0$.

Dynamically accessible functions from $f_0$ rearrange particles by a $C^1$-diffeomorphism $T$, so particles are given a new position and velocity. Since no particle disappears, physical quantities like the total mass are unchanged. For this reason, we require $T$ to be measure preserving. To keep the spherical symmetry of $f_0$, we additionally require that the measure preserving $C^1$-diffeomorphism *respects spherical symmetry*:

**Definition 4.6.** *A function* $T : \mathbb{R}^3\backslash\{0\}\times\mathbb{R}^3 \to \mathbb{R}^3\backslash\{0\}\times\mathbb{R}^3$ respects spherical symmetry *if for all* $(x, v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ *and* $A \in \mathrm{SO}(3)$ *holds*

$$T(Ax, Av) = (Ax', Av'),$$
$$|x' \times v'| = |x \times v|,$$

*with* $(x', v') = T(x, v)$.

Note that $L$ is invariant under transformations which respect spherical symmetry.

Finally, we define the class of perturbations $\mathcal{D}_{f_0}$ as the functions which are dynamically accessible from $f_0$ and respect spherical symmetry, more precisely, we define, analogously to [4],

$$\begin{aligned}\mathcal{D}_{f_0} := \{f = f_0 \circ T | T : \{L > 0\} \to \{L > 0\} &\text{ is a}\\ &\text{measure preserving } C^1\text{-diffeomorphism}\\ &\text{which respects spherical symmetry}\}.\end{aligned} \tag{4.3}$$

In the following, we extend $f$ by 0 on $\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$; since $L$ is invariant under transformations which respect spherical symmetry and $f_0 = 0$ on $\{L \leq L_0\}$, this extension is reasonable.

An example for a measure preserving $C^1$-diffeomorphism which respects spherical symmetry is the characteristic flow:

**Lemma 4.7.** *Let $I$ be an interval and $U \in C^{0,2}(I \times \mathbb{R}^3\backslash\{0\})$ be spherically symmetric. Furthermore, assume that the characteristic flow $Z$ of the associated characteristic system exists. Then the map $Z(s, t, \cdot) : \{L > 0\} \to \{L > 0\}$ is a measure preserving $C^1$-diffeomorphism which respects spherical symmetry for $s, t \in I$.*

*Proof.* As discussed in Lemma 2.11, $Z(\tau, t, \cdot)$ is a measure preserving $C^1$-diffeomorphism $\tau, t \in I$, so it is sufficient to show that $Z(\tau, t, \cdot)$ respects spherical symmetry for $\tau, t \in I$. Let $\tau, t \in I$ and $A \in \mathrm{SO}(3)$ be arbitrary and $(x, v) \in \{L > 0\}$. For the sake of clarity, we write $(X, V)(s) := X(s, t, x, v)$ for $s \in I$. Since $U$ is spherically symmetric, we obtain

$$\begin{aligned}(\dot{AX})(s) &= A\dot{X}(s) = AV(s),\\ (\dot{AV})(s) &= A\dot{V}(s) = A\left(-\partial_r U(|X(s)|) - \frac{M_0}{|x|}\right)\frac{X(s)}{|X(s)|}\\ &= \left(-\partial_r U(|AX(s)|) - \frac{M_0}{|AX(s)|}\right)\frac{AX(s)}{|AX(s)|},\end{aligned}$$

for $s \in I$ and $(AX, AV)(t, t, x, v) = (Ax, Av)$. Therefore, $(AX, AV)(\cdot, t, x, v)$ solves the characteristic system with initial condition $(t, Ax, Av)$. The solution of the characteristic system is unique according to the assumption, so

$$(AX, AV)(\tau, t, x, v) = (X, V)(\tau, t, Ax, Av).$$

The angular momentum squared $L$ is constant along characteristics according to Lemma 2.13 which implies

$$|X(\tau) \times V(\tau)|^2 = L((X,V)(\tau)) = L(x,v) = |x \times v|^2.$$

Because $(x,v) \in \{L > 0\}$ and $A \in \mathrm{SO}(3)$ are arbitrary, the map $Z(\tau,t,\cdot)$ respects spherical symmetry. □

In particular, the class of perturbations $\mathcal{D}_{f_0}$ is invariant under spherically symmetric solutions of the Vlasov-Poisson system with a point mass:

**Lemma 4.8.** *The set $\mathcal{D}_{f_0}$ is invariant under spherically symmetric solutions of the Vlasov-Poisson system with a point mass, i.e., let $I \ni t \to f(t)$ be a spherically symmetric solution of the Vlasov-Poisson system with a point mass with $I$ an interval and $f(0) =: \mathring{f} \in \mathcal{D}_{f_0}$. Then $f(t) \in \mathcal{D}_{f_0}$ for all $t \in I$.*

*Proof.* Since $\mathring{f} \in \mathcal{D}_{f_0}$, there exists a measure preserving $C^1$-diffeomorphism $T$ which respects spherical symmetry such that $\mathring{f} = f_0 \circ T$ on $\{L > 0\}$. By Definition 2.10, the characteristic flow exists. The spherical symmetry of the solution $f$ leads to the spherical symmetry of $U$ and $U \in C^1(I \times \mathbb{R}^3)$ by Lemma 2.15. Note that $\mathring{f} \in C_c(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ since by assumption $f_0$ is continuous on $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ with compact support. By Lemma 4.7, $Z(0,t,\cdot)$ is a measure preserving $C^1$-diffeomorphism which respects spherical symmetry for $t \in I$. Since $f$ is constant along characteristics, we obtain

$$f(t,z) = f(0, Z(0,t,z)) = \mathring{f}(Z(0,t,z))$$

for $t \in I$ and $z \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$. Therefore, $f(t)$ can be written as the composition $f(t) = \mathring{f} \circ Z(0,t,\cdot) = f_0 \circ T \circ Z(0,t,\cdot)$ for $t \in I$. Obviously, the composition of two measure preserving $C^1$-diffeomorphisms which respect spherical symmetry retains these properties, so $f(t) \in \mathcal{D}_{f_0}$ holds for all $t \in I$. □

To determine whether the energy-Casimir functional has a local minimum in $f_0$, we need an analogue to a metric on the space of perturbations which calculates the distance between $f_0$ and $f \in \mathcal{D}_{f_0}$. Therefore, we define

$$d(f, f_0) := \iint (\Psi(f,L) - \Psi(f_0,L) + E(f - f_0)) \, dv \, dx + \frac{1}{8\pi} \int |\nabla U_f - \nabla U_0|^2 \, dx$$

for $f \in \mathcal{D}_{f_0}$ and $\Psi$ as defined above. Note that $f_0 \in C_c(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ and $f_0$ is spherically symmetric, so $f \in \mathcal{D}_{f_0}$ is spherically symmetric and continuous with compact support $\operatorname{supp} f \subset \{L \geq L_0\}$. This implies that $U_f \in C^1(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and furthermore $\mathcal{H}_\mathcal{C}(f_0)$ and $\mathcal{H}_\mathcal{C}(f)$ exist. In summary, $d = d(\cdot, f_0)$ is well-defined on $\mathcal{D}_{f_0}$.

Since the particle energy $E = E(x,v) = \frac{1}{2}|v|^2 + U_0(x) - \frac{M_0}{|x|}$ depends on $U_0 = U_{f_0}$, the whole definition of $d = d(\cdot, f_0)$ depends on the steady state $f_0$ to be investigated, so $d$

is not a metric in the classical sense. In Lemma 5.1, we will see that there exists some $C = C(f_0) > 0$ such that

$$\|f - f_0\|_2^2 + \|\nabla U_f - \nabla U_0\|^2 \leq Cd(f, f_0), \quad f \in \mathcal{D}_{f_0}.$$

This estimate guarantees that $d$ is positive definite. With the expansion of energy functional around $f_0$ in equation (4.1), we see that the distance $d$ is connected with the energy-Casimir functional by

$$d(f, f_0) = \mathcal{H}_\mathcal{C}(f) - \mathcal{H}_\mathcal{C}(f_0) + \frac{1}{4\pi} \int |\nabla U_f - \nabla U_0|^2 \, dx, \quad f \in \mathcal{D}_{f_0}. \tag{4.4}$$

## 4.3 Stability of steady states and examples

With this at hand, we turn to the main result and show that the energy-Casimir functional $\mathcal{H}_\mathcal{C}$ has a local minimum in $f_0$ on the set $\mathcal{D}_{f_0}$. We obtain analogously to [4, Theorem 2.1]:

**Theorem 4.9.** *There exist $C_0 = C_0(f_0) > 0$ and $\delta_0 = \delta_0(f_0) > 0$ such that for all $f \in \mathcal{D}_{f_0}$ with $d(f, f_0) < \delta_0$ holds*

$$\mathcal{H}_\mathcal{C}(f) - \mathcal{H}_\mathcal{C}(f_0) \geq C_0 \|\nabla U_f - \nabla U_0\|_2^2.$$

We dedicate the entire following chapter to the, not easy, proof of this result. But before that, we assume that $\mathcal{H}_\mathcal{C}$ has a local minimum in $f_0$ in the sense of Theorem 4.9 and show the final result that provides the stability of steady states. Analogously to Lyapunov functions for ordinary differential equations, the energy-Casimir functional is constant along spherically symmetric solutions of the Vlasov-Poisson system with a point mass and has a local minimum in $f_0$, so we obtain, similar to ordinary differential equations, the stability of $f_0$. Analogously to [4, Theorem 2.2]:

**Theorem 4.10.** *There exist $C = C(f_0) > 0$ and $\delta = \delta(f_0) > 0$ such that for all $\mathring{f} \in \mathcal{D}_{f_0}$ with*

$$d(\mathring{f}, f_0) < \delta,$$

*the spherically symmetric continuous solution $I \ni t \to f(t)$ with $f(0) = \mathring{f}$ and $f(t) \in C_c(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ satisfies*

$$d(f(t), f_0) \leq Cd(\mathring{f}, f_0), \quad t \in I.$$

*Here, $I$ denotes an interval with $0 \in I$.*

*Proof.* Let $\mathring{f} \in \mathcal{D}_{f_0}$ be at first arbitrary and $I \ni t \to f(t)$ the corresponding solution as required in Theorem 4.10. As shown in Lemma 4.8, $f(t) \in \mathcal{D}_{f_0}$ for all $t \in I$, so the map $I \ni t \to d(f(t), f_0)$ is well-defined. Furthermore, it can be shown that the map is continuous:

Let $s, \tau \in I$ and $J \subset I$ be a compact interval with $s, \tau \in J$. According to Lemma 4.3, the energy-Casimir functional is invariant under spherically symmetric solutions of the Vlasov-Poisson system with a point mass, so by (4.4)

$$|d(f(s), f_0) - d(f(\tau), f_0)| = \frac{1}{4\pi} \left| \|\nabla U_{f(s)} - \nabla U_0\|_2^2 - \|\nabla U_{f(\tau)} - \nabla U_0\|_2^2 \right|.$$

Since $f(t) \in C_c(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ and $f(t)$ is spherically symmetric, the induced density $\rho$ is continuous and $\rho(t)$ is spherically symmetric for $t \in I$. According to Lemma 2.9, the induced potential $U(\tau) = U_{f(\tau)}$ is twice continuously differentiable with respect to $x$ and $U \in C^{0,2}(I \times \mathbb{R}^3)$. Hence, the characteristic flow is continuous by Lemma 2.11, and the solution $f$ has the support $\operatorname{supp} f(t) = Z(t, 0, \operatorname{supp} \mathring{f})$ for $t \in I$ as discussed in Remark 2.14. Therefore, the support of $f(t)$ is bounded uniformly in $t \in J$. Together with Remark 2.14 (c), this implies

$$\begin{aligned}
&\left| \|\nabla U_{f(s)} - \nabla U_0\|_2 - \|\nabla U_{f(t)} - \nabla U_0\|_2 \right| \leq \|\nabla U_{f(s)} - \nabla U_{f(t)}\|_2 \\
&\leq C\|\rho(s) - \rho(t)\|_{\frac{6}{5}} \leq C\|f(s) - f(t)\|_{\frac{6}{5}}.
\end{aligned}$$

Since $f$ is continuous, the term $f(\tau) - f(s)$ vanishes pointwise as $\tau \to s$ and is bounded uniformly on $J \times \mathbb{R}^3$, so $f(\tau) - f(s)$ vanishes in $L^{\frac{6}{5}}(\mathbb{R}^6)$ as $\tau \to s$ by the dominated convergence theorem. In conclusion, this convergence leads to

$$|d(f(s), f_0) - d(f(\tau), f_0)| = \frac{1}{4\pi} \left| \|\nabla U_{f(s)} - \nabla U_0\|_2^2 - \|\nabla U_{f(\tau)} - \nabla U_0\|_2^2 \right| \to 0$$

as $\tau \to s$, so the map $I \ni t \to d(f(t), f_0)$ is continuous.

Let $\delta_0 = \delta_0(f_0)$ and $C_0 = C_0(f_0)$ be as in Theorem 4.9. We define $\delta := \delta_0(1 + \frac{1}{4\pi C_0})^{-1} > 0$ and $C := 1 + \frac{1}{4\pi C_0} > 0$. Now let $\mathring{f} \in \mathcal{D}_{f_0}$ with $d(\mathring{f}, f_0) < \delta$. Since $I \ni t \to d(f(t), f_0)$ is continuous and $\delta < \delta_0$, there exists $t^* \in \bar{I} \cup \{\infty\}$ such that

$$d(f(t), f_0) < \delta_0, \quad t \in [0, t^*[;$$

let $t^*$ be the maximal value with this property. The choice of $\delta$ and $C$ and the estimate in Theorem 4.9 yield together with equation (4.4) to

$$\begin{aligned}
d(f(t), f_0) &= \mathcal{H}_\mathcal{C}(f(t)) - \mathcal{H}_\mathcal{C}(f_0) + \frac{1}{4\pi}\|\nabla U_{f(t)} - \nabla U_0\|_2^2 \\
&\leq \left(1 + \frac{1}{4\pi C_0}\right) (\mathcal{H}_\mathcal{C}(f(t)) - \mathcal{H}_\mathcal{C}(f_0)) \\
&= \left(1 + \frac{1}{4\pi C_0}\right) (\mathcal{H}_\mathcal{C}(\mathring{f}) - \mathcal{H}_\mathcal{C}(f_0)) \\
&\leq \left(1 + \frac{1}{4\pi C_0}\right) \left(\mathcal{H}_\mathcal{C}(\mathring{f}) - \mathcal{H}_\mathcal{C}(f_0) + \frac{1}{4\pi}\|\nabla U_{\mathring{f}} - \nabla U_0\|_2^2\right)
\end{aligned}$$

$$= \left(1 + \frac{1}{4\pi C_0}\right) d(\mathring{f}, f_0) < \left(1 + \frac{1}{4\pi C_0}\right) \delta = \delta_0$$

for all $t \in [0, t^*[$. This implies $t^* = \sup I$, so the proof is complete. □

As the attentive reader will have noticed, the above theorem assumes that for all $\mathring{f} \in \mathcal{D}_{f_0}$ there exists a spherically symmetric continuous solution $I \ni t \to f(t)$ with $f(0) = \mathring{f}$ and $f(t) \in C_c(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ for $t \in I$ and an interval $I$ with $0 \in I$. In the last chapter, more precisely, in Theorem 6.2, we show that for all $\mathring{f} \in C_c(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ with $\mathring{f}(x, v) = 0$ for $(x, v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ with $|x \times v|^2 \leq L_0$ with some $L_0 > 0$, there exists a suitable spherically symmetric solution, the so-called Lagrangian solution, which has the required properties and exists on $\mathbb{R}$.

Theorem 4.10 regarding stability is only useful if there exist anisotropic steady states $f_0$ which satisfy the assumptions. In the previous chapter, we constructed a class of anisotropic steady states with compact support and finite mass. Two examples we considered in the previous chapter are the (generalized) polytropic steady states and the (generalized) King model. If we restrict the parameters appropriately, we obtain their stability by Theorem 4.10.

**Example 4.11** (Stability of the (generalized) polytropic steady states)**.** *Let $k \geq 1$ and $l > 0$. Furthermore, let $L_0 > 0$ and $E_0 < 0$. Analogously to Example 3.11, we define*

$$\Phi(E, L) := (E_0 - E)_+^k (L - L_0)_+^l.$$

*Then, $\Phi \in C(\mathbb{R} \times [0, \infty[) \cap C^{2,0}(]-\infty, E_0[\times]L_0, \infty[)$ with*

$$\Phi'(E, L) = -k(E_0 - E)^{k-1}(L - L_0)^l, \quad E < E_0,\ L > L_0$$

*and $\Phi_E^{-1} : [0, \infty[\times]L_0, \infty[\to [E_0, \infty[$ with*

$$\Phi_E^{-1}(y, L) = E_0 - (L - L_0)^{-\frac{l}{k}} y^{\frac{1}{k}}.$$

*In summary, this leads to*

$$\Phi'(\Phi_E^{-1}(y, L), L) = -k y^{1-\frac{1}{k}} (L - L_0)^{\frac{l}{k}}, \quad y \geq 0,\ L > L_0.$$

*Conditions (A1)–(A4) are obviously satisfied with arbitrary $E_{\min} < E_0$ and $L_{\max} > L_0$. Hence, Theorem 4.10 shows the stability of (generalized) polytropic steady states for $l > 0$, $k \geq 1$, $L_0 > 0$ and $E_0 < 0$ if $f_0 = \Phi(E, L)$ solves the Vlasov-Poisson system with a point mass. Note that* $\operatorname{supp} f = \{E \leq E_0\} \cap \{L \geq L_0\}$ *is compact according to Lemma 3.2.*

*If we further require $k < l + \frac{3}{2}$, then $f_0 = \Phi(E, L)$ solves the Vlasov-Poisson system with a point mass with compact support and finite mass for suitable $E_0 < 0$ as discussed in Example 3.11 which leads to the stability of the (generalized) polytropic steady states.*

**Example 4.12** (Stability of the (generalized) King models)**.** *Let $k \geq 1$ and $l > 0$. We define, analogously to Example 3.12,*

$$\Phi(E, L) := (e^{E_0 - E} - 1)_+^k (L - L_0)_+^l$$

*for $E_0 < 0$ and $L_0 > 0$ arbitrary. Obviously, $\Phi \in C(\mathbb{R} \times [0, \infty[) \cap C^{2,0}(]-\infty, E_0[\times]L_0, \infty[)$ with*

$$\Phi'(E, L) = -k e^{E_0 - E} (e^{E_0 - E} - 1)^{k-1} (L - L_0)^l, \quad E < E_0, \ L > L_0,$$

*and $\Phi_E^{-1} : [0, \infty[\times]L_0, \infty[ \to [E_0, \infty[$ with*

$$\Phi_E^{-1}(y, L) = E_0 - \ln\left(y^{\frac{1}{k}} (L - L_0)^{-\frac{l}{k}} + 1\right).$$

*This implies*

$$\Phi'(\Phi_E^{-1}(y, L), L) = -k \left(y + y^{1 - \frac{1}{k}} (L - L_0)^{\frac{l}{k}}\right), \quad y \geq 0, \ L > L_0.$$

*With similar arguments as in Example 4.11, conditions (A1)–(A4) are obviously satisfied for arbitrary $E_{\min} < E_0$ and $L_{\max} > L_0$. Again, Theorem 4.10 yields the stability of (generalized) King models with $l > 0$, $k \geq 1$, $L_0 > 0$ and $E_0 < 0$ if $f_0 = \Phi(E, L)$ solves the Vlasov-Poisson system with a point mass.*

*For $k < l + \frac{3}{2}$, Example 3.12 shows that $f_0 = \Phi(E, L)$ solves the Vlasov-Poisson system with a point mass with compact support and finite mass for suitable $E_0 < 0$ and thus is stable.*

# 5 Proof of Theorem 4.9

For the proof of Theorem 4.9, we apply the methods in [4] to our setting. While many parts work analogously, some steps can be done more directly with the $(\theta, E, L)$-coordinates as described in [5]. That is the reason why we consider the arguments in [5] and transfer certain helpful results to our setting.

In Theorem 4.9, we claim, among other things, that $\mathcal{H}_\mathcal{C}$ has a local minimum in $f_0$. In one-dimensional analysis, we can detect a local minimum if the first derivative vanishes and the second derivative is positive in the minimum point. We adapt this behavior to our setting and have a closer look on the expansion (4.2) of the energy-Casimir functional $\mathcal{H}_\mathcal{C}$ around $f_0$. By the choice of $\Psi$, recall

$$\Psi(f, L) = -\int_0^f \Phi_E^{-1}(z, L)\, dz, \quad f \geq 0,\ L > L_0,$$

and $\Psi(f, L) = 0$ for $L \leq L_0$, the linear term vanishes. Considering the quadratic term, we define the *the second order variation of* $\mathcal{H}_\mathcal{C}$ with $\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)$ by

$$\begin{aligned} \mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[g] &:= \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0, L) g^2\, dv\, dx - \frac{1}{8\pi}\int |\nabla U_g|^2\, dx \\ &= \frac{1}{2}\iint_{\{f_0>0\}} \frac{1}{|\Phi'(E, L)|} g^2\, dv\, dx - \frac{1}{8\pi}\int |\nabla U_g|^2\, dx \end{aligned} \tag{5.1}$$

for $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ if the single expressions exist. To prove Theorem 4.9, we assume that the assertion in Theorem 4.9 were false, and as a result, we construct a function $g$ with $\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[g] \leq 0$. Furthermore, we will show that $\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[\{-E, \cdot\}] > 0$ for suitable functions. As we will later introduce, $\{\cdot, \cdot\}$ denotes the Poisson bracket. By showing that $g = \{-E, h\}$ and with regularization, we derive a contradiction.

In order to calculate terms like $\Psi''(f_0, L)$ which appears in the definition of $\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[g]$, we examine $\Psi$ in more detail and establish useful identities and estimates as in [4, Lemma 3.1]:

**Lemma 5.1.** *(a) The function $\Psi$ is continuous on $[0, \infty[\times]L_0, \infty[$ and $\Psi \in C^{1,0}([0, \infty[\times]L_0, \infty[) \cap C^{3,0}(]0, \infty[\times]L_0, \infty[)$ with*

$$\begin{aligned} \Psi'(f, L) &= -\Phi_E^{-1}(f, L), \\ \Psi''(f, L) &= -\frac{1}{\Phi'(\Phi_E^{-1}(f, L), L)}, \end{aligned}$$

$$\Psi'''(f,L) = \frac{\Phi''(\Phi_E^{-1}(f,L),L)}{(\Phi'(\Phi_E^{-1}(f,L),L))^3}$$

*for $f > 0$ and $L > L_0$.*

(b) *The estimate $\Psi(f,L) \geq -E_0 f$ holds for $f \geq 0$ and $L > L_0$. Furthermore, $f_0 = \Phi(E,L)$ leads to*

$$\Psi'(f_0,L) = -E \text{ on } \operatorname{supp} f_0,$$
$$\Psi''(f_0,L) = -\frac{1}{\Phi'(E,L)} \text{ on } \{f_0 > 0\}.$$

(c) *There exists a constant $C = C(f_0) > 0$ such that*

$$\iint [\Psi(f,L) - \Psi(f_0,L) + E(f-f_0)]\, dv\, dx \geq C \iint |f-f_0|^2\, dv\, dx$$

*for all $f \in \mathcal{D}_{f_0}$. This yields*

$$\|f-f_0\|_2^2 + \|\nabla U_f - \nabla U_0\|_2^2 \leq \tilde{C} d(f,f_0)$$

*for all $f \in \mathcal{D}_{f_0}$ with $\tilde{C} = \max\{\frac{1}{C}, 8\pi\}$.*

*Proof.* We require $\Phi \in C(\mathbb{R} \times [0,\infty[) \cap C^{2,0}(]-\infty,E_0[\times]L_0,\infty[)$ with $\Phi' < 0$ and $\Phi(]-\infty,E_0],L) = [0,\infty[$ for $L > L_0$, as stated in the previous chapter. Hence, Lemma 4.4 implies that $\Phi_E^{-1}$ is continuous on $[0,\infty[\times]L_0,\infty[$. By the fundamental theorem of calculus and the inverse function theorem, the formulas of $\Psi$, $\Psi'$ and $\Psi''$ follow directly and lead to the regularity asserted in (a).

Since $\Phi_E^{-1} : [0,\infty[\times]L_0,\infty[\to]-\infty,E_0]$ and therefore $\Phi_E^{-1} \leq E_0$, the fundamental theorem of calculus implies that $\Psi(f,L) \geq -E_0 f$ for $f \geq 0$ and $L > L_0$. Because of the definition of $\Phi_E^{-1}$, the other assertions in (b) follow directly. According to the assumption imposed in the previous chapter, $\{f_0 > 0\} = \{E < E_0\} \cap \{L > L_0\}$ and $\Phi'(E,L) < 0$ on $\{f_0 > 0\}$, so the expressions in (b) are well-defined.

In order to show the assertions in (c), we estimate the expression $E(f-f_0)$ accordingly. Let $f \in \mathcal{D}_{f_0}$ be arbitrary. Based on the assertions in (b), we obtain

$$E(f-f_0) = -\Psi'(f_0,L)(f-f_0) \text{ on } \operatorname{supp} f_0$$

and

$$E(f-f_0) \geq E_0 f = \Phi_E^{-1}(0,L) f = -\Psi'(f_0,L)(f-f_0) \text{ on } \{E \geq E_0\} \cap \{L > L_0\}.$$

In summary, this leads to

$$\Psi(f,L) - \Psi(f_0,L) + E(f-f_0) \geq \Psi(f,L) - \Psi(f_0,L) - \Psi'(f_0,L)(f-f_0)$$

$$= \lim_{\varepsilon\to 0}(\Psi(f+\varepsilon, L) - \Psi(f_0+\varepsilon, L) - \Psi'(f_0+\varepsilon, L)(f-f_0))$$
$$= \lim_{\varepsilon\to 0}\frac{1}{2}\Psi''(\xi_{\varepsilon,L}, L)(f-f_0)^2$$

with some $\xi_{\varepsilon,L}(x,v)$ between $f(x,v)+\varepsilon$ and $f_0(x,v)+\varepsilon$ for $(x,v) \in \{L > L_0\}$ and $\varepsilon > 0$. Hence, we obtain $0 < \xi_{\varepsilon,L} \leq \|f_0\|_\infty + 1$ for $\varepsilon$ small enough. Note that $\|f\|_\infty = \|f_0\|_\infty$ since $f \in \mathcal{D}_{f_0}$. According to the assumption (A1) and the formula of $\Psi''$, the infimum

$$C := \frac{1}{2}\inf\{\Psi''(f,L) | 0 < f < \|f\|_\infty + 1,\ L_0 < L \leq L_{\max}\} > 0$$

exists with $L_{\max} = \max_{\operatorname{supp} f_0} L$, and it follows

$$\Psi(f,L) - \Psi(f_0,L) + E(f-f_0) \geq \lim_{\varepsilon\to 0}\frac{1}{2}\Psi''(\xi_{\varepsilon,L}, L)(f-f_0)^2 \geq C|f-f_0|^2$$

on $\{L_0 < L \leq L_{\max}\}$. Integrating this estimate over $\{L_0 < L \leq L_{\max}\}$ yields the desired expression but only with integrals over the subset $\{L_0 < L \leq L_{\max}\}$. Since $f \in \mathcal{D}_{f_0}$, there exists a measure preserving $C^1$-diffeomorphism $T : \{L > 0\} \to \{L > 0\}$ which respects spherical symmetry such that $f = f_0 \circ T$. The angular momentum squared $L$ is invariant under transformations which respect spherical symmetry, so $f = 0$ on $\{L \leq L_0\}$ and $\{L > L_{\max}\}$. Therefore, the proof is complete. □

**Remark.** *(a) The function $\Psi \in C([0,\infty[\times]L_0,\infty[)$, so $\Psi$ satisfies the condition in Lemma 4.3.*

*(b) For the expansion* (4.2) *of the energy-Casimir functional around $f_0$, we require the function $\tilde{\Psi}$ which induces the Casimir functional to be $C^{2,0}([0,\infty[\times[0,\infty[)$. Since $f = 0 = f_0$ on $\{L \leq L_0\}$ for $f \in \mathcal{D}_{f_0}$, it is sufficient to demand $C^{2,0}([0,\infty[\times]L_0,\infty[)$. The function $\Psi$ as defined above has at least the regularity $C^{2,0}(]0,\infty[\times]L_0,\infty[)$.*

*(c) Lemma 5.1 (a) shows that the identity* (5.1) *is valid.*

## 5.1 The construction of $g$

As mentioned before, under the assumption that Theorem 4.9 were false, we are able to construct, analogously to [4, Lemma 3.2], a function $g$ such that the second derivative of the energy-Casimir functional is negative:

**Lemma 5.2.** *Assume that Theorem 4.9 were false. Then there exists $g \in L^2(\mathbb{R}^6)$ with the following properties:*

*(g1) The function $g$ is spherically symmetric.*

*(g2) The support of $g$ is in $\operatorname{supp} f_0$, i.e., $\operatorname{supp} g \subset \operatorname{supp} f_0$.*

*(g3) The function $g$ is even in $v$, i.e., $g(x,-v) = g(x,v)$ a.e. $(x,v) \in \mathbb{R}^6$.*

*(g4)* *The derivative* $\nabla U_g \in L^2(\mathbb{R}^3)$ *exists with*

$$\frac{1}{8\pi}\|\nabla U_g\|_2^2 = 1.$$

*(g5)* *The second order variation of the energy-Casimir functional exists and is negative, i.e.,*

$$\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[g] = \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0, L) g^2 \; dv \; dx - 1 \leq 0.$$

*(g6)* *Let* $G = G(f, L) \in C^{2,0}([0, \infty[\times[0, \infty[)$ *with* $G(0, L) = 0 = \partial_f G(0, L)$ *for* $L \geq 0$ *and* $\partial_f^2 G$ *bounded. Then*

$$\iint \partial_f G(f_0, L) g \; dv \; dx = 0.$$

**Remark.** *In [4], the function* $g$ *is interpreted as a vector tangent to* $\mathcal{D}_{f_0}$ *in the point* $f_0$*. For more detail, we refer to [4] and in particular to [4, Remark 3.3]*

*Proof of Lemma 5.2.* Assume that Theorem 4.9 were false. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_{f_0}$ with

$$d(f_n, f) < \frac{1}{n}, \quad n \in \mathbb{N},$$

and

$$\mathcal{H}_\mathcal{C}(f_n) - \mathcal{H}_\mathcal{C}(f_0) < \frac{1}{8\pi n}\|\nabla U_{f_n} - \nabla U_0\|_2^2.$$

Note that $\|\nabla U_{f_n} - \nabla U_0\|_2^2 > 0$ for $n \in \mathbb{N}$. Otherwise, $\mathcal{H}_\mathcal{C}(f_n) - \mathcal{H}_\mathcal{C}(f_0) < 0$ and thus $d(f_n, f_0) = \mathcal{H}_\mathcal{C}(f_n) - \mathcal{H}_\mathcal{C}(f_0) < 0$ would hold which were a contradiction to Lemma 5.1 (c).

**Construction of the sequence** $(g_n)_{n\in\mathbb{N}}$

First of all, we have to manipulate $(f_n)$ suitably to construct a sequence which converges in a proper sense. Then we can use its limiting function to examine whether the above conditions are satisfied.

In (g4), we require $\frac{1}{8\pi}\|\nabla U_g\|_2^2 = 1$. Hence, we define $g_n$ in such a way that this condition is satisfied for all $n \in \mathbb{N}$. Therefore, we define

$$\sigma_n := \frac{1}{\sqrt{8\pi}}\|\nabla U_{f_n} - \nabla U_0\|_2,$$
$$g_n := \frac{1}{\sigma_n}(f_n - f_0), \quad n \in \mathbb{N}.$$

This implies

$$\begin{aligned}
&f_n = f_0 + \sigma_n g_n, \\
&\frac{1}{8\pi}\|\nabla U_{g_n}\|_2^2 = 1, \quad n \in \mathbb{N}.
\end{aligned}$$

To show the convergence of $(\sigma_n)$, we recall the proof of Lemma 5.1 (c) in which we have shown that

$$\Psi(f_n, L) - \Psi(f_0, L) + E(f_n - f_0) \geq C|f_n - f_0|^2 \geq 0 \text{ on } \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3.$$

Note that $f_n = f = 0$ on $\{L \leq L_0\}$ and $\{L > L_{\max}\}$ with $L_{\max} := \max_{\operatorname{supp} f_0} L$. Hence, it follows that

$$\begin{aligned}
\sigma_n^2 &\leq \iint (\Psi(f_n, L) - \Psi(f_0, L) + E\sigma_n g_n)\ dv\ dx + \frac{1}{8\pi}\|\nabla U_{f_n} - \nabla U_0\|_2^2 \\
&= d(f_n, f_0) < \frac{1}{n}
\end{aligned}$$

for $n \in \mathbb{N}$.

**The existence of the weak limit $g$**
We now show that the sequence $(g_n)$ converges weakly. To apply the Banach-Alaoglu theorem and thus ensure convergence, we show that the sequence $(g_n)$ is bounded in $L^2(\mathbb{R}^6)$. Lemma 5.1 (c) and the definition of $d$ yield together with Equation (4.4) the following estimate:

$$\begin{aligned}
C \iint |g_n|^2\ dv\ dx &\leq \frac{1}{\sigma_n^2} \iint (\Psi(f_n, L) - \Psi(f_0, L) + E\sigma_n g_n)\ dv\ dx \\
&= \frac{1}{\sigma_n^2}\left(d(f_n, f_0) - \frac{1}{8\pi}\|\nabla U_{f_n} - \nabla U_0\|_2^2\right) \\
&= \frac{1}{\sigma_n^2}\left(\mathcal{H}_\mathcal{C}(f_n) - \mathcal{H}_\mathcal{C}(f_0) + \frac{1}{8\pi}\|\nabla U_{f_n} - \nabla U_0\|_2^2\right) \\
&< \frac{1}{\sigma_n^2}\left(\frac{1}{8\pi n}\|\nabla U_{f_n} - \nabla U_0\|_2^2 + \frac{1}{8\pi}\|\nabla U_{f_n} - \nabla U_0\|_2^2\right) \\
&= 1 + \frac{1}{n} \leq 2 \qquad (5.2)
\end{aligned}$$

for $n \in \mathbb{N}$. The constant $C$ denotes the one in Lemma 5.1 (c), and thus it is independent of $n \in \mathbb{N}$, so $(g_n)$ is bounded in $L^2(\mathbb{R}^6)$. The Banach-Alaoglu theorem yields the existence of a subsequence of $(g_n)$ such that it converges weakly to a limiting function $g$ in $L^2(\mathbb{R}^6)$. Under abuse of notation, we denote the subsequence with $(g_n)$ for the sake of clarity.

In the next steps, we analyze whether the required conditions are satisfied.

**Spherical symmetry of $g$**
Let $A \in \mathrm{SO}(3)$. Since $(g_n)$ converges weakly to $g$ in $L^2(\mathbb{R}^6)$, the sequence $g_n(A\cdot, A\cdot)$ converges to $g(A\cdot, A\cdot)$ as well. By assumption, $f_0$ and thus $(f_n) \subset \mathcal{D}_{f_0}$ is spherically symmetric, so $g_n = \frac{1}{\sigma_n}(f_n - f_0)$ preserves the spherical symmetry. This leads to

$$g_n = g_n(A\cdot, A\cdot) \rightharpoonup g(A\cdot, A\cdot)$$

as $n \to \infty$. As the unique weak limit of $(g_n)$, the limiting function $g$ has to satisfies $g = g(A\cdot, A\cdot)$ a.e.. In summary, this means that for all $A \in \mathrm{SO}(3)$ there exists a measurable set $N \subset \mathbb{R}^6$ with $\lambda(N) = 0$ such that $g = g(A\cdot, A\cdot)$ on $N^c$. With Lemma 2.4, this yields spherical symmetry of $g$, so (g1) is satisfied.

**The support of $g$**
We aim to show that $\operatorname{supp} g \subset \operatorname{supp} f_0 = \{E \leq E_0\} \cap \{L \geq L_0\}$. For this purpose, let $E_0 < E_1 < 0$ and $0 < L_1 < L_0$ be arbitrary. As we have discussed in the proof of Lemma 5.1 (c), $f = 0$ on $\{L \leq L_0\}$ for all $f \in \mathcal{D}_{f_0}$. By definition, $g_n$ also vanishes on the set $\{L \leq L_0\}$ which implies that

$$\begin{aligned}\iint_{\{E>E_1\}\cup\{L<L_1\}} g_n \, dv \, dx &= \iint_{\{E>E_1\}} g_n \, dv \, dx \\ &\leq \iint_{\{E>E_1\}} \frac{E-E_0}{E_1-E_0} g_n \, dv \, dx \\ &\leq \frac{1}{E_1-E_0} \iint_{\{E>E_0\}} (E-E_0) g_n \, dv \, dx\end{aligned}$$

for $n \in \mathbb{N}$. The steady state $f_0$ vanishes on $\{E > E_0\}$, so $f_n = \sigma_n g_n$ on $\{E > E_0\}$. Lemma 5.1 (b) yields the following estimate on $\{E > E_0\} \cap \{L > L_0\}$:

$$\begin{aligned}\Psi(f_n, L) - \Psi(f_0, L) + E\sigma_n g_n = \Psi(f_n, L) + E\sigma_n g_n &\geq -E_0 f_n + E\sigma_n g_n \\ &= (E - E_0)\sigma_n g_n, \quad n \in \mathbb{N}.\end{aligned}$$

The inequality (5.2) leads to the estimate

$$\begin{aligned}&\iint_{\{E>E_0\}} (E-E_0) g_n \, dv \, dx = \iint_{\{E>E_0\}\cap\{L>L_0\}} (E-E_0) g_n \, dv \, dx \\ &\leq \frac{1}{\sigma_n} \iint_{\{E>E_0\}\cap\{L>L_0\}} (\Psi(f_n, L) - \Psi(f_0, L) + E\sigma_n g_n) \, dv \, dx \\ &\leq \frac{1}{\sigma_n} \iint (\Psi(f_n, L) - \Psi(f_0, L) + E\sigma_n g_n) \, dv \, dx < 2\sigma_n\end{aligned}$$

for $n \in \mathbb{N}$. Since $(\sigma_n)$ tends to zero, we obtain the convergence

$$\iint_{\{E>E_1\}\cup\{L<L_1\}} g_n \, dv \, dx \leq \frac{1}{E_1-E_0} \iint_{\{E>E_0\}} (E-E_0) g_n \, dv \, dx$$

$$\leq \frac{2}{E_1 - E_0}\sigma_n \to 0, \quad n \to \infty.$$

Note that $g_n = \frac{1}{\sigma_n} f_n \geq 0$ on $\{E \geq E_0\} \cup \{L \leq L_0\}$, so this can be interpreted as

$$1_{\{E>E_1\}\cup\{L<L_1\}} g_n \to 0 \text{ in } L^1(\mathbb{R}^6).$$

The weak convergence of $g_n$ yields

$$1_{\{E>E_1\}\cup\{L<L_1\}} g_n \rightharpoonup 1_{\{E>E_1\}\cup\{L<L_1\}} g \text{ in } L^2(\mathbb{R}^6).$$

This implies that

$$1_{\{E>E_1\}\cup\{L<L_1\}} g = 0.$$

Therefore, $\operatorname{supp} g \subset \{E \leq E_1\} \cap \{L \geq L_1\}$. Since $E_0 < E_1 < 0$ and $0 < L_1 < L_0$ are arbitrary, we obtain the assertion (g2), namely $\operatorname{supp} g \subset \{E \leq E_0\} \cap \{L \geq L_0\} = \operatorname{supp} f_0$.

In addition to the boundedness in $L^2(\mathbb{R}^6)$, we use the $L^1$-convergence of $1_{\{E\leq E_1\}\cap\{L\geq L_1\}} g_n$ for arbitrary $E_1 < 0$ and $L_1 > 0$ to show the boundedness of $(g_n)$ in $L^1(\mathbb{R}^6)$:

**Boundedness of $(g_n)$ in $L^1(\mathbb{R}^6)$**
Let $E_0 < E_1 < 0$ and $0 < L_1 < L_0$. Since $f_0 \in C_c(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ is spherically symmetric, the set $\{E \leq E_1\} \cap \{L \geq L_1\}$ is compact by Lemma 3.2. With the convergence shown before and the boundedness of $(g_n)$ in $L^2(\mathbb{R}^6)$, this leads to

$$\begin{aligned}\iint |g_n| \, dv \, dx &= \iint_{\{E\leq E_1\}\cap\{L\geq L_1\}} g_n \, dv \, dx + \iint_{\{E>E_1\}\cup\{L<L_1\}} g_n \, dv \, dx \\ &\leq C\|g_n\|_2 + C \leq C, \quad n \in \mathbb{N}.\end{aligned}$$

By using the boundedness of the sequence $(g_n)$ in $L^1(\mathbb{R}^3)$, we next examine the condition (g4):

**The condition (g4), i.e., $\frac{1}{8\pi}\|\nabla U_g\|_2^2 = 1$**
In order to show the existence of $\nabla U_g$ and the required identity, we first examine the sequence $(\rho_{g_n})$ of the induced densities for weak convergence and then show that the sequence remains concentrated.

At first, we have a closer look on the kinetic energy induced by $g_n$. Since $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$ is spherically symmetric with finite mass, Lemma 2.9 implies that

$$0 \geq U_0(r) = -\int_r^\infty \frac{m(s)}{s^2} \, ds \geq -\|f_0\|_1 \int_r^\infty \frac{1}{s^2} \, ds = -\frac{M}{r}, \quad r > 0,$$

with mass $M := \|f_0\|_1$. As a result, we estimate the kinetic energy on the set $\{E > E_0\}$ by

$$\iint_{\{E>E_0\}} \frac{1}{2}|v|^2 |g_n| \, dv \, dx =$$

$$
\begin{aligned}
&= \iint_{\{E>E_0\}} (E-E_0)|g_n|\;dv\;dx + \iint_{\{E>E_0\}} \left(E_0 - U_0(|x|) + \frac{M_0}{|x|}\right)|g_n|\;dv\;dx \\
&\leq \iint_{\{E>E_0\}} (E-E_0)|g_n|\;dv\;dx + |E_0| \iint |g_n|\;dv\;dx + \iint_{\{E>E_0\}} \frac{M_{\text{tot}}}{|x|}|g_n|\;dv\;dx
\end{aligned}
$$

for $n \in \mathbb{N}$ with the total mass $M_{\text{tot}} := M_0 + M$. While the first two integrals are bounded by previous calculations, the third term is the crucial one. As already discussed, $g_n = 0$ on $\{L < L_0\}$ and $|x| \geq \frac{\sqrt{L_0}}{|v|}$ for all $(x,v) \in \{L \geq L_0\}$. Let $a > 0$ be at first arbitrary. Then this implies

$$
\begin{aligned}
\iint_{\{E>E_0\}} \frac{M_{\text{tot}}}{|x|}|g_n|\;dv\;dx &= \iint_{\{L\geq L_0\}\cap\{E>E_0\}} \frac{M_{\text{tot}}}{|x|}|g_n|\;dv\;dx \\
&\leq \frac{M_{\text{tot}}}{\sqrt{L_0}} \iint_{\{E>E_0\}\cap\{L\geq L_0\}} |v||g_n|\;dv\;dx \\
&\leq \frac{M_{\text{tot}}}{\sqrt{L_0}} \iint_{\{E>E_0\}} \left(a + \frac{1}{a}|v|^2\right)|g_n|\;dv\;dx
\end{aligned}
$$

and thus

$$
\begin{aligned}
\iint_{\{E>E_0\}} \frac{1}{2}|v|^2|g_n|\;dv\;dx \leq{}& \iint_{\{E>E_0\}} (E-E_0)|g_n|\;dv\;dx + |E_0| \iint |g_n|\;dv\;dx \\
&+ \frac{M_{\text{tot}}}{\sqrt{L_0}} \iint_{\{E>E_0\}} \left(a + \frac{1}{a}|v|^2\right)|g_n|\;dv\;dx, \quad n \in \mathbb{N}.
\end{aligned}
$$

If we choose $a := 4\frac{M_{\text{tot}}}{\sqrt{L_0}}$ such that $\frac{1}{2} - \frac{M_{\text{tot}}}{a\sqrt{L_0}} = \frac{1}{4}$, this leads to

$$
\begin{aligned}
&\iint_{\{E>E_0\}} \frac{1}{2}|v|^2|g_n|\;dv\;dx = 2\left(\frac{1}{2} - \frac{M_{\text{tot}}}{a\sqrt{L_0}}\right) \iint_{\{E>E_0\}} |v|^2|g_n|\;dv\;dx \\
&\leq 2\left(\iint_{\{E>E_0\}} (E-E_0)|g_n|\;dv\;dx + |E_0| \iint |g_n|\;dv\;dx + \frac{aM_{\text{tot}}}{\sqrt{L_0}} \iint |g_n|\;dv\;dx\right)
\end{aligned}
$$

for $n \in \mathbb{N}$. As shown before, the first term tends to zero for $n \to \infty$. Since $(g_n)$ is bounded in $L^1(\mathbb{R}^6)$, the second and the third term are bounded independently of $n \in \mathbb{N}$ as well. Now, it remains to examine the kinetic energy on $\{E \leq E_0\}$. The support $\operatorname{supp} f_0 = \{E \leq E_0\} \cap \{L \geq L_0\}$ is compact, so

$$
\begin{aligned}
&\iint_{\{E\leq E_0\}} \frac{1}{2}|v|^2|g_n|\;dv\;dx = \iint_{\{E\leq E_0\}\cap\{L\geq L_0\}} \frac{1}{2}|v|^2|g_n|\;dv\;dx \\
&\leq C \iint_{\{E\leq E_0\}\cap\{L\geq L_0\}} |g_n|\;dv\;dx \leq C \iint |g_n|\;dv\;dx, \quad n \in \mathbb{N}.
\end{aligned}
$$

In summary, the kinetic energy is bounded independently of $n \in \mathbb{N}$.

As a result, $(\rho_{g_n})$ is bounded in certain $L^p$ spaces: Let $x \in \mathbb{R}^3\backslash\{0\}$ with $\rho_{g_n}(x) \neq 0$. It follows by Hölder's inequality that

$$\begin{aligned}
|\rho_{g_n}(x)| = \int |g_n(x,v)|\ dv &\leq \int_{|v|\leq R} |g_n(x,v)|\ dv + \frac{2}{R^2}\int_{|v|\geq R} \frac{1}{2}|v|^2|g_n(x,v)|\ dv \\
&\leq C\left(R^{\frac{3}{2}}\left(\int |g_n(x,v)|^2\ dv\right)^{\frac{1}{2}} + R^{-2}\int \frac{1}{2}|v|^2|g_n(x,v)|\ dv\right) \\
&= C\left(\int |g_n(x,v)|^2\ dv\right)^{\frac{2}{7}}\left(\int \frac{1}{2}|v|^2|g_n(x,v)|\ dv\right)^{\frac{3}{7}}
\end{aligned}$$

with $R = R(x) := (\int |g_n(x,v)|^2\ dv)^{-\frac{1}{7}}(\int \frac{1}{2}|v|^2|g_n(x,v)|\ dv)^{\frac{2}{7}}$ and $C > 0$ independently of $n \in \mathbb{N}$ and $x \in \mathbb{R}^3\backslash\{0\}$. Again, Hölder's equality yields

$$\begin{aligned}
\int |\rho_{g_n}|^{\frac{7}{5}}\ dx &\leq C\int\left(\int |g_n(x,v)|^2\ dv\right)^{\frac{2}{5}}\left(\int \frac{1}{2}|v|^2|g_n(x,v)|\ dv\right)^{\frac{3}{5}}\ dx \\
&\leq C\left(\iint |g_n(x,v)|^2\ dv\ dx\right)^{\frac{2}{5}}\left(\iint \frac{1}{2}|v|^2|g_n(x,v)|\ dv\ dx\right)^{\frac{3}{5}}
\end{aligned}$$

for $n \in \mathbb{N}$. Since the kinetic energy is bounded independently of $n \in \mathbb{N}$ and the sequence $(g_n)$ is bounded in $L^1 \cap L^2(\mathbb{R}^6)$, the sequence $(\rho_{g_n})$ is bounded in $L^1 \cap L^{\frac{7}{5}}(\mathbb{R}^3)$ and thus in particular in $L^{\frac{6}{5}}(\mathbb{R}^3)$. The Banach-Alaoglu theorem provides the existence of a subsequence of $(\rho_{g_n})$ which we denote again with $(\rho_{g_n})$ such that

$$\rho_{g_n} \rightharpoonup \rho^* \text{ in } L^{\frac{6}{5}}(\mathbb{R}^3)$$

with $\rho^* \in L^{\frac{6}{5}}(\mathbb{R}^3)$. In order to examine $\rho^*$, let $\chi \in C_c^\infty(\mathbb{R}^3)$ and $E_0 < E_1 < 0$ and $0 < L_1 < L_0$. Then

$$\begin{aligned}
&\int \rho^*\chi\ dx = \lim_{n\to\infty}\int \rho_{g_n}(x)\chi(x)\ dx = \lim_{n\to\infty}\iint g_n(x,v)\chi(x)\ dv\ dx \\
&= \lim_{n\to\infty}\left(\iint g_n(x,v)1_{\{E\leq E_1\}\cap\{L\geq L_1\}}\chi(x)\ dv\ dx + \iint_{\{E>E_1\}} g_n(x,v)\chi(x)\ dv\ dx\right) \\
&= \iint g(x,v)1_{\{E\leq E_1\}\cap\{L\geq L_1\}}\chi(x)\ dv\ dx = \iint g(x,v)\chi(x)\ dv\ dx.
\end{aligned}$$

Note that $g_n \rightharpoonup g$ in $L^2(\mathbb{R}^6)$ and $\operatorname{supp} g \subset \{E \leq E_0\} \cap \{L \geq L_0\}$ is compact. Furthermore, we have shown that $\iint_{\{E>E_1\}} |g_n(x,v)|\ dv\ dx \to 0$ as $n \to \infty$. Since $\chi \in C_c^\infty(\mathbb{R}^3)$ is arbitrary, it follows with the fundamental lemma of the calculus of variations that $\rho^* = \rho_g$, so

$$\rho_{g_n} \rightharpoonup \rho_g \text{ in } L^{\frac{6}{5}}(\mathbb{R}^3).$$

It remains to show that $(\rho_{g_n})$ is concentrated. For this purpose, let $E_0 < E_1 < 0$. Then a radius $R_1 > 0$ exists such that $U_0(R_1) - \frac{M_0}{R_1} = E_1$ and thus

$$E(x,v) = \frac{1}{2}|v|^2 + U_0(r) - \frac{M_0}{r} \geq U_0(R_1) - \frac{M_0}{R_1} = E_1,$$

for $(x,v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ with $|x| \geq R_1$. In summary, we obtain

$$\int_{\{|x|\geq R_1\}} |\rho_{g_n}| \; dx \leq \iint_{\{E\geq E_1\}} |g_n| \; dv \; dx \to 0, \quad n \to \infty.$$

This means that the sequence $(\rho_{g_n})$ remains concentrated.

By [10, Lemma 2.5], it follows

$$\nabla U_{g_n} \to \nabla U_g \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^3).$$

The conditions as stated in [10, Lemma 2.5] are only almost satisfied since $\rho_{g_n}$ can be negative. Anyway, the proof remains true in every step if $\rho_{g_n}$ is negative. In particular, it is shown that $\nabla U_g$ exists. Since we construct $g_n$ in such a way that $\frac{1}{8\pi}\|\nabla U_{g_n}\|_2^2 = 1$, the limiting function $g$ obviously retains the required property $\frac{1}{8\pi}\|\nabla U_g\|_2^2 = 1$, and we conclude (g4).

**The second order variation of the energy-Casimir functional** $\mathrm{D}^2\mathcal{H}_\mathcal{C}$ ((g5))
In the next step, we show that the second order variation of the energy-Casimir functional $\mathrm{D}^2\mathcal{H}_\mathcal{C}[g]$ is negative. For this purpose we introduce increasing sets $(K_j)$ on which the sequence $(\sigma_n g_n)$ convergences uniformly and increasing sets $(S_m)$ which fix the distance from the boundary of $\operatorname{supp} f_0$.

Since $(g_n)$ is bounded in $L^2(\mathbb{R}^6)$ and $\sigma_n \to 0$ as $n \to \infty$, we observe

$$\|\sigma_n g_n\|_2 \leq \sigma_n \|g_n\|_2 \leq C\sigma_n \to 0, \quad n \to \infty,$$

so there exists a subsequence which we denote again with $(\sigma_n g_n)$ such that $\sigma_n g_n$ convergences to 0 pointwise a.e.. Since $\operatorname{supp} f_0$ is compact, Egorov's theorem yields an increasing sequence $(K_j)$, i.e., $K_j \subset K_{j+1} \subset ... \subset \operatorname{supp} f_0$ for all $j \in \mathbb{N}$, such that

$$\operatorname{vol}(\operatorname{supp} f_0 \setminus K_j) < \frac{1}{j}$$

and

$$\lim_{n\to\infty} \sigma_n g_n = 0 \text{ uniformly on } K_j$$

for all $j \in \mathbb{N}$. Furthermore, we define

$$S_m := \left\{ (x,v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 \middle| E(x,v) \leq E_0 - \frac{1}{m} \wedge L(x,v) \geq L_0 + \frac{1}{m} \right\}$$

for $m \in \mathbb{N}$ to fix the distance to the boundary of $\operatorname{supp} f_0$. Note that $(S_m)$ is increasing with $S_m \subset S_{m+1} \subset ... \subset \operatorname{supp} f_0$ for all $m \in \mathbb{N}$.

Since $S_m$ has fixed distance to the boundary of $\operatorname{supp} f_0$, the steady state $f_0$ is bounded from below on $S_m$, so we can show $\delta_m := \inf_{S_m} f_0 > 0$ for $m \in \mathbb{N}$: Let $(x, v) \in S_m$, so $E := E(x, v) \leq E_0 - \frac{1}{m}$ and $L := L(x, v) \geq L_0 + \frac{1}{m}$. This leads to

$$f_0(x, v) = \Phi(E, L) \geq \Phi\left(E_0 + \frac{1}{m}, L_0 - \frac{1}{m}\right).$$

Here, we applied that $\Phi$ is monotonically decreasing in $E$ because of $\Phi' < 0$ on $S_m$ and monotonically increasing in $L$ by assumption (A4). Hence, it follows $\delta_m > \Phi(E_0 + \frac{1}{m}, L_0 - \frac{1}{m}) > 0$.

Fix $m \in \mathbb{N}$ and $j \in \mathbb{N}$. As we have shown before, the $\frac{1}{8\pi}\|\nabla U_g\|_2^2 = 1$ holds, so it remains to show that $\frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0, L)g^2 \; dv \; dx$ exists and is smaller or equal than 1. Before we can apply the Taylor expansion, we have to clarify that the conditions are satisfied. Since $\Psi \in C^{3,0}(]0, \infty[\times]L_0, \infty[)$, the Taylor expansion yields

$$\begin{aligned}
&\Psi(f_n, L) - \Psi(f_0, L) - \Psi'(f_0, L)(f_n - f_0) - \frac{1}{2}\Psi''(f_0, L)(f_n - f_0)^2 \\
&= \lim_{\varepsilon\to 0}(\Psi(f_n + \varepsilon, L) - \Psi(f_0 + \varepsilon, L) - \Psi'(f_0 + \varepsilon, L)(f_n - f_0) \\
&\qquad - \frac{1}{2}\Psi''(f_0 + \varepsilon, L)(f_n - f_0)^2) \\
&= \lim_{\varepsilon\to 0}\frac{1}{6}\Psi'''(f_0 + \varepsilon + \xi(f_n - f_0), L)(f_n - f_0)^3 \\
&= \lim_{\varepsilon\to 0}\frac{1}{6}\Psi'''(f_0 + \varepsilon + \xi\sigma_n g_n, L)(f_n - f_0)^3, \quad n \in \mathbb{N}.
\end{aligned}$$

with some $0 \leq \xi = \xi_n(x, v) \leq 1$ for $(x, v) \in S_m \cap K_j$. Because of the construction of $K_j$, the sequence $(\sigma_n g_n)$ converges uniformly on $K_j$, so there exists $n_0 \in \mathbb{N}$ such that $-\frac{\delta_m}{2} < \sigma_n g_n < \frac{1}{2}$ a.e. on $K_j$ for $n \geq n_0$. This implies for $n \geq n_0$ and $0 < \varepsilon < \frac{1}{2}$ and $0 \leq \xi \leq 1$ that

$$\frac{\delta_m}{2} \leq \delta_m - \xi\frac{\delta_m}{2} \leq f_0 + \xi\sigma_n g_n \leq f_0 + \varepsilon + \xi\sigma_n g_n \leq \|f_0\|_\infty + 1$$

a.e. on $S_m \cap K_j$. Since $S_m \cap K_j \subset \operatorname{supp} f_0 \subset \{L > L_{\max}\}$, we obtain

$$\begin{aligned}
&|\Psi'''(f_0 + \varepsilon + \xi\sigma_n g_n, L)| \\
&\leq \sup\left\{|\Psi'''(z, L)| \middle| \frac{\delta_m}{2} \leq z \leq \|f_0\|_\infty + 1,\ L_0 + \frac{1}{m} \leq L \leq L_{\max}\right\} =: C_m < \infty
\end{aligned}$$

on $S_m \cap K_j$ for $n \geq n_0$. The existence of the supremum follows from Lemma 5.1 (a) and the assumption (A2). Inserting these assertions, we obtain for $n \geq n_0$ that

$$\frac{1}{2}\iint_{S_m\cap K_j} \Psi''(f_0, L)|g_n|^2 \; dv \; dx$$

$$
\begin{aligned}
&= \frac{1}{2\sigma_n^2} \iint_{S_m \cap K_j} \lim_{\varepsilon \to 0} \Psi''(f_0+\varepsilon, L)|f_n - f_0|^2 \, dv \, dx \\
&= \frac{1}{\sigma_n^2} \iint_{S_m \cap K_j} \lim_{\varepsilon \to 0} (\Psi(f_n+\varepsilon, L) - \Psi(f_0+\varepsilon, L) - \Psi'(f_0+\varepsilon, L)(f_n - f_0) \\
&\qquad\qquad\qquad - \frac{1}{6}\Psi'''(f_0+\varepsilon+\xi\sigma_n g_n, L)(f_n-f_0)^3) \, dv \, dx \\
&\leq \frac{1}{\sigma_n^2} \iint_{S_m \cap K_j} (\Psi(f_n, L) - \Psi(f_0, L) - \Psi'(f_0, L)(f_n - f_0) \\
&\qquad\qquad + \frac{1}{6} C_m |f_n - f_0|^3) \, dv \, dx \\
&= \frac{1}{\sigma_n^2} \iint_{S_m \cap K_j} (\Psi(f_n, L) - \Psi(f_0, L) + E(f_n - f_0)) \, dv \, dx \\
&\quad + \frac{1}{6} C_m \iint_{S_m \cap K_j} \sigma_n |g_n|^3 \, dv \, dx \\
&\leq 1 + \frac{1}{n} + \frac{1}{6} C_m \sup_{K_j} |\sigma_n g_n| \iint |g_n|^2 \, dv \, dx,
\end{aligned}
$$

where we have again used equation (5.2). To make statements about $g$, we examine both sides for convergence. Since $(\sigma_n g_n)$ converges uniformly to 0 on $K_j$ and $(g_n)$ is bounded in $L^2(\mathbb{R}^6)$, the right-hand side converges to 1. On the other hand, according to the assumptions, $\Psi''(f_0, L) = \frac{1}{|\Phi'(E,L)|} > 0$ is bounded on $S_m \cap K_j$ because of $S_m \cap K_j \subset \{E_{\min} \leq E \leq E_0 - \frac{1}{m}\} \cap \{L_0 + \frac{1}{m} \leq L \leq L_{\max}\}$ with $E_{\min} := \min_{\operatorname{supp} f_0} E$. Furthermore, the boundedness of $S_m \cap K_j$ implies $1_{S_m \cap K_j}(\Psi''(f_0, L))^{\frac{1}{2}} \in L^1 \cap L^\infty(\mathbb{R}^6)$. The fact that $g_n \rightharpoonup g$ converges weakly in $L^2(\mathbb{R}^6)$ leads to

$$
1_{S_m \cap K_j}(\Psi''(f_0, L))^{\frac{1}{2}} g_n \rightharpoonup 1_{S_m \cap K_j}(\Psi''(f_0, L))^{\frac{1}{2}} g \text{ in } L^2(\mathbb{R}^6).
$$

The lower semi-continuity of $\|\cdot\|_2$ implies that

$$
\begin{aligned}
&\frac{1}{2} \iint_{S_m \cap K_j} \Psi''(f_0, L)|g|^2 \, dv \, dx \leq \liminf_{n \to \infty} \frac{1}{2} \iint_{S_m \cap K_j} \Psi''(f_0, L)|g_n|^2 \, dv \, dx \\
&\leq \lim_{n \to \infty} \left( 1 + \frac{1}{n} + \frac{1}{6} C_m \sup_{K_j} |\sigma_n g_n| \iint |g_n|^2 \, dv \, dx \right) = 1.
\end{aligned}
$$

Since $(K_j)$ and $(S_m)$ are increasing sets and the integrand is non-negative, the monotone convergence theorem applied first for $j \to \infty$ and then for $m \to \infty$ yields

$$
\frac{1}{2} \iint_{\{f_0 > 0\}} \Psi''(f_0, L)|g|^2 \, dv \, dx \leq 1.
$$

Together with (g4), this shows that the second order variation of the energy-Casimir functional $\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[g] = \frac{1}{2} \iint_{\{f_0>0\}} \Psi''(f_0, L) g^2 \, dv \, dx - 1 \leq 0$ is negative, so the condition (g5) is satisfied.

**The condition $\iint \partial_f G(f_0, L)g \; dv \; dx = 0$ ((g6))**
Let $G = G(f, L) \in C^{2,0}([0, \infty[\times[0, \infty[)$ with $G(0, L) = 0 = \partial_f G(0, L)$ for $L \geq 0$ and $\partial_f^2 G$ bounded. The Taylor expansion with respect to $f$ yields

$$G(f_n, L) - G(f_0, L) = \partial_f G(f_0, L)(f_n - f_0) + \frac{1}{2}\partial_f^2 G(f_0 + \tau(f_n - f_0), L)(f_n - f_0)^2$$

with some $\tau = \tau_n(x, v) \in [0, 1]$ for $(x, v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ and $n \in \mathbb{N}$. Integrating this identity leads to

$$\begin{aligned}&\iint (G(f_n, L) - G(f_0, L)) \; dv \; dx \\ &= \sigma_n \iint \partial_f G(f_0, L)g_n \; dv \; dx + \frac{\sigma_n^2}{2} \iint \partial_f^2 G(f_0 + \tau\sigma_n g_n, L)g_n^2 \; dv \; dx\end{aligned}$$

for $n \in \mathbb{N}$. Since $f_n \in \mathcal{D}_{f_0}$, there exists a measure preserving $C^1$-diffeomorphism $T_n : \{L > 0\} \to \{L > 0\}$ which respects spherical symmetry such that $f_n = f_0 \circ T_n$. Note that $\{L = 0\}$ is a null set. Thus, a change of variables provides

$$\iint G(f_n, L) \; dv \; dx = \iint G(f_0, L) \; dv \; dx, \quad n \in \mathbb{N}.$$

Note that $L$ is invariant under transformations which respect spherical symmetry. This results in

$$\iint \partial_f G(f_0, L)g_n \; dv \; dx = -\frac{\sigma_n}{2} \iint \partial_f^2 G(f_0 + \tau\sigma_n g_n, L)g_n^2 \; dv \; dx, \quad n \in \mathbb{N}.$$

By assumption, $\partial_f^2 G$ is bounded, and we have proven that $(g_n)$ is bounded in $L^2(\mathbb{R}^6)$. Therefore, the integral on the right-hand side is bounded, so the convergence $\sigma_n \to 0$ as $n \to \infty$ provides that the right-hand side vanishes as $n \to \infty$.

Since $\partial_f G \in C([0, \infty[\times[0, \infty[)$ with $\partial_f G(0, L) = 0$ for $L \geq 0$, the function $\partial_f G(f_0, L)$ is bounded with compact support in $\operatorname{supp} f_0$. In particular, $\partial_f G(f_0, L) \in L^1 \cap L^\infty(\mathbb{R}^6)$. Since $g_n \rightharpoonup g$ converges weakly in $L^2(\mathbb{R}^6)$ as $n \to \infty$, this yields

$$\iint \partial_f G(f_0, L)g_n \; dv \; dx \to \iint \partial_f G(f_0, L)g \; dv \; dx, \quad n \to \infty.$$

In summary,

$$\iint \partial_f G(f_0, L)g \; dv \; dx = 0,$$

so the assertion (g6) is satisfied.

**The construction of an even $g$**
Finally, we have to examine whether g is even in $v$. With the above construction, we

are not able to determine this property, but we can manipulate $g$ such that the resulting function is even in $v$ and retains the conditions we have already proven. We expand $g$ into its odd and even part, i.e.,

$$g = g_{\text{odd}} + g_{\text{even}}$$

with $g_{\text{odd}}(x,v) := \frac{1}{2}(g(x,v) - g(x,-v))$ and $g_{\text{even}} := \frac{1}{2}(g(x,v) + g(x,-v))$ for a.e. $(x,v) \in \mathbb{R}^6$. The function $g_{\text{even}}$ is even in $v$ and has the properties which we require: Clearly, $g_{\text{even}} \in L^2(\mathbb{R}^6)$ is spherically symmetric with support in $\operatorname{supp} f_0$. Furthermore, $\rho_g = \rho_{g_{\text{even}}}$ since $\rho_{g_{\text{odd}}} = \iint g_{\text{odd}}(\cdot, v)\, dv = 0$. Therefore, $U_g = U_{g_{\text{even}}}$, so the condition $\frac{1}{8\pi}\|\nabla U_{g_{\text{even}}}\|_2^2 = 1$ remains true. It is easy to see that $E = E(x,v)$ and $L = L(x,v)$ are even in $v$, so $f_0 = \Phi(E,L)$ and hence $\Psi''(f_0, L)$ and $\partial_f G(f_0, L)$ are also even in $v$ if $G$ has the properties required in (g6). This results in

$$\begin{aligned}
1 &\geq \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0,L)|g|^2\, dv\, dx \\
&= \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0,L)(|g_{\text{even}}|^2 + |g_{\text{odd}}|^2)\, dv\, dx + \iint_{\{f_0>0\}} \Psi''(f_0,L) g_{\text{even}} g_{\text{odd}}\, dv\, dx \\
&= \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0,L)|g_{\text{even}}|^2\, dv\, dx + \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0,L)|g_{\text{odd}}|^2\, dv\, dx \\
&\geq \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0,L)|g_{\text{even}}|^2\, dv\, dx
\end{aligned}$$

and

$$\begin{aligned}
0 &= \iint \partial_f G(f_0,L) g_{\text{even}}\, dv\, dx + \iint \partial_f G(f_0,L) g_{\text{odd}}\, dv\, dx \\
&= \iint \partial_f G(f_0,L) g_{\text{even}}\, dv\, dx.
\end{aligned}$$

Note that the set $\{f_0 > 0\} = \{E < E_0 \wedge L > L_0\}$ is even in $v$. In summary, $g_{\text{even}}$ has the required properties, so the proof of Lemma 5.2 is finally complete. □

## 5.2 The Poisson bracket and the transport operator

Under the assumption that Theorem 4.9 were wrong, we are able to construct a function $g$ as shown in Lemma 5.2 with $\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[g] \leq 0$. On the other hand, we can show that $\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)$ is positive on a certain class of functions, namely the ones induced by the particle energy and the *Poisson bracket*, respectively, the *transport operator*. We first define these terms rigorously:

**Definition 5.3.** *The* Poisson bracket $\{\cdot,\cdot\}$ *is defined by*

$$\{f,g\} := \partial_x f \cdot \partial_v g - \partial_v f \cdot \partial_x g$$

*for $f, g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ differentiable. We use the same definition if $f$ and $g$ are only defined on $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$.*

*The transport operator $\mathcal{T}$ is defined by*

$$\mathcal{T} f := \{-E, f\}$$

*for $f : \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to \mathbb{R}$ differentiable. Here, $E = E(x, v)$ denotes the particle energy with the potential $U_0$ induced by the steady state $f_0$.*

Note that $U_0$ and thus $E$ are continuously differentiable on $\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$, so the transport operator $\mathcal{T}$ is well-defined. The transport operator is related to the Vlasov equation and the characteristics:

**Remark 5.4.** *For $f : I \times \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to \mathbb{R}$ differentiable with $I$ an interval, the Vlasov equation can be expressed with the transport operator by*

$$\begin{aligned} \partial_t f + \mathcal{T} f &= \partial_t f + \{-E, f\} \\ &= \partial_t f + v \cdot \partial_x f - \partial_x \left( U_0(x) - \frac{M_0}{|x|} \right) \cdot \partial_v f = 0. \end{aligned}$$

*More generally, the following holds for $h : \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 \to \mathbb{R}$ differentiable:*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}(h((X, V)(s, t, x, v))) \\ &= \partial_x h(Z(s, t, x, v)) \cdot \dot{X}(s, t, x, v) + \partial_v h(Z(s, t, x, v)) \cdot \dot{V}(s, t, x, v) \\ &= \left( \partial_x h(x, v) \cdot v - \partial_v h(x, v) \cdot \partial_x \left( U_0(x) - \frac{M_0}{|x|} \right) \right) \Big|_{(x,v)=(X,V)(s,t,x,v)} \\ &= -\{E, h\}((X, V)(s, t, x, v)) \end{aligned}$$

*for $s, t \in \mathbb{R}$ and $(x, v) \in \{L > 0\}$.*

Moreover, the Poisson bracket has some helpful properties which we will use in later argumentation:

**Remark 5.5.** *(a) The Poisson bracket is anti-symmetric, i.e.,*

$$\{f, g\} = -\{g, f\}$$

*for $f, g : \mathbb{R}^6 \to \mathbb{R}$ differentiable.*

*(b) By the product rule, the identity*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

*holds for $f, g, h : \mathbb{R}^6 \to \mathbb{R}$ differentiable.*

(c) *Let $f, g : \mathbb{R}^6 \to \mathbb{R}$ be differentiable and spherically symmetric. Then $\{f, g\}$ is spherically symmetric as well.*
*Let $A \in \mathrm{SO}(3)$ and $f_A := f(A\cdot, A\cdot)$ and $g_A := g(A\cdot, A\cdot)$. By assumption, $f$ and $g$ are spherically symmetric, so $f = f_A$ and $g = g_A$. Therefore, it follows*

$$\begin{aligned}
&\{f, g\}(Ax, Av) = \partial_x f(Ax, Av) \cdot \partial_v g(Ax, Av) - \partial_v f(Ax, Av) \cdot \partial_x g(Ax, Av) \\
&= (A \cdot \partial_x f_A(x, v)) \cdot (A \cdot \partial_v g_A(x, v)) - (A \cdot \partial_v f_A(x, v)) \cdot (A \cdot \partial_x g_A(x, v)) \\
&= \partial_x f_A(x, v) \cdot \partial_v g_A(x, v) - \partial_v f_A(x, v) \cdot \partial_x g_A(x, v) \\
&= \{f_A, g_A\}(x, v) = \{f, g\}(x, v)
\end{aligned}$$

*for all $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$.*

We apply the Poisson bracket and consider the class of states $f = \{-E, h\} = \mathcal{T}h$ for certain functions $h$. Analogously to [4, Lemma 3.4] and [3, Lemma 1.1], we obtain that $\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[f]$ is positive definite:

**Lemma 5.6.** *Let $h \in C_c^\infty(\mathbb{R}^3\backslash\{0\}\times\mathbb{R}^3)$ be spherically symmetric with $\operatorname{supp} h \subset \{f_0 > 0\}$ and odd in $v$, i.e., $h(x, -v) = -h(x, v)$ for all $(x, v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$. Then*

$$\begin{aligned}
\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[\{-E, h\}] &\geq -\frac{1}{2} \iint \frac{1}{\Phi'(E, L)} \left( |x \cdot v|^2 \left\{ -E, \frac{h}{x \cdot v} \right\}^2 + \left( \frac{U_0'}{r} + \frac{M_0}{r^3} \right) h^2 \right) \, dv \, dx \\
&= \frac{1}{2} \iint \frac{1}{|\Phi'(E, L)|} \left( |x \cdot v|^2 \left\{ E, \frac{h}{x \cdot v} \right\}^2 + \frac{m(r) + M_0}{r^3} h^2 \right) \, dv \, dx
\end{aligned}$$

*with $m(r) := 4\pi \int_0^r s^2 \rho(s) \, ds$ for $r > 0$. The integrals extend over $\{f_0 > 0\}$.*

*Proof.* In order to prove Lemma 5.6, we proceed analogously to [4]. First, we estimate the second term of $\mathrm{D}^2\mathcal{H}_\mathcal{C}(f_0)[\{-E, h\}]$. We take a closer look at the potential and thus the density induced by $-\{E, h\}$. The definition of the Poisson bracket and an integration by parts lead to

$$\begin{aligned}
-\int \{E, h\} \, dv &= \int \partial_v E \cdot \partial_x h \, dv - \int \partial_x E \cdot \partial_v h \, dv \\
&= \sum_{i=1}^3 \left( \partial_{x_i} \int v_i h \, dv + \int \partial_{v_i} \partial_{x_i} \left( U_0(x) - \frac{M_0}{|x|} \right) h \, dv \right) \\
&= \nabla_x \cdot \int vh \, dv = \nabla_x \cdot K
\end{aligned}$$

with $K(x) := \int vh(x, v) \, dv$ for $x \in \mathbb{R}^3\backslash\{0\}$. Note that $h \in C_c^\infty(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$, so integrating by parts and the change of integration and differentiation are permitted. Since $h$ is spherically symmetric, the vector field $K$ has a specific structure which simplifies the calculation of the divergence: Let $x \in \mathbb{R}^3\backslash\{0\}$ and $A \in \mathrm{SO}(3)$ with $Ax = re_3$ and $e_3 := (0, 0, 1)^t$. Then it follows by Lemma 2.3 that

$$K(x) = \int vh(x, v) \, dv = \int vh(Ax, Av) \, dv = \int vh(re_3, Av) \, dv$$

$$
\begin{aligned}
&= A^t \int vh(re_3, v)\ dv \\
&= A^t \int_0^{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \begin{pmatrix} \frac{\sqrt{L}}{r} \cos\varphi \\ \frac{\sqrt{L}}{r} \sin\varphi \\ w \end{pmatrix} h(r, w, L) \frac{1}{2r^2}\ dw\ dL\ d\varphi \\
&= A^t \int_0^{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} we_3 h(r, w, L) \frac{1}{2r^2}\ dw\ dL\ d\varphi \\
&= A^t \int wh(x, v)\ dv\ e_3 = K(r) \frac{x}{r}
\end{aligned}
$$

with $K(r) := \int wh(x, v)\ dv$. Using the symmetry, we obtain that

$$
\begin{aligned}
\nabla_x \cdot K(x) &= \sum_{i=1}^{3} \partial_{x_i} \left( K(r) \frac{x_i}{r} \right) = \sum_{i=1}^{3} \left( \partial_r K(r) \frac{x_i^2}{r^2} + K(r) \frac{1}{r} - K(r) \frac{x_i^2}{r^3} \right) \\
&= \partial_r K(r) + \frac{2}{r} K(r) = \frac{1}{r^2} \partial_r (r^2 K(r)), \quad x \in \mathbb{R}^3 \backslash \{0\}.
\end{aligned}
$$

Since $h$ and thus $-\{E, h\}$ are spherically symmetric, $U_h := U_{-\{E,h\}}$ with

$$
U_h(x) := \iint \frac{\{E, h\}(y, v)}{|x - y|}\ dv\ dy, \quad x \in \mathbb{R}^3,
$$

is spherically symmetric as well with

$$
\begin{aligned}
U_h'(r) &= -\frac{1}{r^2} \int_{B_r(0)} \int \{E, h\}(y, v)\ dv\ dy = -\frac{1}{r^2} \int_{B_r(0)} \nabla_y \cdot K(y)\ dy \\
&= \frac{4\pi}{r^2} \int_0^r \partial_s (s^2 K(s))\ ds = 4\pi K(r) = 4\pi \int wh(x, v)\ dv, \quad r > 0.
\end{aligned}
$$

Note that $h \in C_c^\infty(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ and hence $r^2 K(r) \to 0$ as $r \to 0$. Together with the Cauchy-Schwarz inequality this yields the following estimate:

$$
\begin{aligned}
\frac{1}{8\pi} \int |\nabla_x U_h|^2\ dx &= \frac{1}{8\pi} \int |U_h'|^2\ dx = 2\pi \int \left| \int w \sqrt{-\Phi'(E, L)} \frac{h}{\sqrt{-\Phi'(E, L)}}\ dv \right|^2\ dx \\
&\leq 2\pi \iint (-w^2 \Phi'(E, L))\ dv \int \left( -\frac{h^2}{\Phi'(E, L)} \right)\ dv\ dx.
\end{aligned}
$$

Since $\Phi'(E, L) < 0$ on $\operatorname{supp} h \subset \{f_0 > 0\}$, the integrals exist, and the argument of the square root and thus the integrands are non-negative. The derivative in $E$ which appears in the first integral transforms into one in $w$. More precisely,

$$
\begin{aligned}
w^2 \Phi'(E, L) &= w^2 \frac{\mathrm{d}}{\mathrm{d}E} \Phi(E, L) = w \frac{\mathrm{d}}{\mathrm{d}w} \left( \Phi \left( \frac{1}{2} w^2 + \frac{L}{2r^2} - \frac{M_0}{r}, L \right) \right) \\
&= w \frac{\mathrm{d}}{\mathrm{d}w} (\Phi(E, L))
\end{aligned}
$$

for $(x,v) \in \{f_0 > 0\}$. We fix $r > 0$ and $L > L_0$. Then $E(r,w,L) < E_0$ if and only if $w \in ]-\kappa(r,L), \kappa(r,L)[$ with $\kappa(r,L) := \sqrt{2(E_0 + \frac{M_0}{r} - U_0(r)) - \frac{L}{r^2}} > 0$. Since $E(r, \pm\kappa(r,L), L) = E_0$, the identity $\Phi(E(r,\pm\kappa(r,L),L),L) = 0$ holds for $L \geq L_0$ and $r > 0$, provided that $\kappa(r,L) > 0$ exists. Integration by parts yields

$$\begin{aligned}
-\int_{\{f_0(x,\cdot)>0\}} w^2\Phi'(E,L)\,dv &= -\frac{\pi}{r^2}\int_{L_0}^{\infty}\int_{-\kappa(r,L)}^{\kappa(r,L)} w\frac{\mathrm{d}}{\mathrm{d}w}(\Phi(E,L))\,dw\,dL \\
&= \frac{\pi}{r^2}\int_{L_0}^{\infty}\int_{-\kappa(r,L)}^{\kappa(r,L)} \Phi(E,L)\,dw\,dL \\
&= \int \Phi(E,L)\,dv = \rho_0(x), \quad x \in \mathbb{R}^3\backslash\{0\}.
\end{aligned}$$

If $\{f_0(x,\cdot) > 0\} = \emptyset$, the above identity is also valid since $\rho_0(x) = 0$. Hence, we obtain that

$$\begin{aligned}
\frac{1}{8\pi}\int |\nabla_x U_h|^2\,dx &\leq 2\pi \iint (-w^2\Phi'(E,L))\,dv \int \left(-\frac{h^2}{\Phi'(E,L)}\right)\,dv\,dx \\
&\leq 2\pi \iint \rho_0(x)\frac{h^2}{|\Phi'(E,L)|}\,dv\,dx.
\end{aligned}$$

In summary, these arguments lead to the following estimate:

$$\begin{aligned}
&\mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[\{-E,h\}] = \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}|\{-E,h\}|^2\,dv\,dx - \frac{1}{8\pi}\int |\nabla U_h|^2\,dx \\
&\geq \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}|\{E,h\}|^2\,dv\,dx - 2\pi\iint \rho_0(x)\frac{h^2}{|\Phi'(E,L)|}\,dv\,dx \\
&= \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}(|\{E,h\}|^2 - 4\pi\rho_0(x)h^2)\,dv\,dx.
\end{aligned}$$

So far, we cannot say anything about the sign of $\mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[\{E,h\}]$, so we rewrite $|\{E,h\}|^2$ by introducing the function $\mu$ given by

$$\mu(r,w,L) := \frac{1}{rw}h(r,w,L), \quad r > 0,\ w \in \mathbb{R},\ L \geq 0.$$

Since $h \in C_c^\infty(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$, the auxiliary function $\mu \in C^\infty(]0,\infty[\times\mathbb{R} \setminus \{0\} \times [0,\infty[)$ as discussed before. Hence, it remains to show that $\mu$ is continuously differentiable in $(r,0,L)$ with $r > 0$ and $L \geq 0$. The function $h$ is odd in $v$ and thus in $w$, so $h$ and its second derivative with respect to $w$ vanish in $(r,0,L)$. Taking this into account, the Taylor expansion with respect to $w$ yields

$$\begin{aligned}
\mu(\bar{r},\bar{w},\bar{L}) &= \frac{1}{\bar{r}\bar{w}}h(\bar{r},\bar{w},\bar{L}) \\
&= \frac{1}{\bar{r}\bar{w}}\Big(h(\bar{r},0,\bar{L}) + \partial_w h(\bar{r},0,\bar{L})\bar{w} + \frac{1}{2}\partial_w^2 h(\bar{r},\xi_0,\bar{L})\bar{w}^2\Big)
\end{aligned}$$

$$= \frac{1}{\bar{r}}\partial_w h(\bar{r},0,\bar{L}) + \mathcal{O}(|\bar{w}|) \to \frac{1}{\bar{r}}\partial_w h(r,0,L), \quad (\bar{r},\bar{w},\bar{L}) \to (r,0,L),$$

and

$$\begin{aligned}
\partial_w \mu(\bar{r},\bar{w},\bar{L}) &= \frac{1}{\bar{r}\bar{w}}\partial_w h(\bar{r},\bar{w},\bar{L}) - \frac{1}{\bar{r}\bar{w}^2}h(\bar{r},\bar{w},\bar{L}) \\
&= \frac{1}{\bar{r}\bar{w}}\Big(\partial_w h(\bar{r},0,\bar{L}) + \partial_w^2 h(\bar{r},0,\bar{L})\bar{w} + \frac{1}{2}\partial_w^3 h(\bar{r},\xi_1,\bar{L})\bar{w}^2\Big) \\
&\quad - \frac{1}{\bar{r}\bar{w}^2}\Big(h(\bar{r},0,\bar{L}) + \partial_w h(\bar{r},0,\bar{L})\bar{w} + \frac{1}{2}\partial_w^2 h(\bar{r},0,\bar{L})\bar{w}^2 + \frac{1}{6}\partial_w^3 h(\bar{r},\xi_2,\bar{L})\bar{w}^3\Big) \\
&= \frac{1}{\bar{r}\bar{w}^2}\Big( - h(\bar{r},0,\bar{L}) + \frac{1}{2}\partial_w^2 h(\bar{r},0,\bar{L})\bar{w}^2 + \mathcal{O}(|\bar{w}|^3)\Big) \\
&= \mathcal{O}(|\bar{w}|) \to 0, \quad (\bar{r},\bar{w},\bar{L}) \to (r,0,L).
\end{aligned}$$

Note that $\xi_i = \xi_i(\bar{r},\bar{w},\bar{L}) \in [\bar{w},w]$ or $\xi_i(\bar{r},\bar{w},\bar{L}) \in [w,\bar{w}]$, respectively, so $\xi^i \to 0$ as $(\bar{r},\bar{w},\bar{L}) \to (r,0,L)$ for $i = 0,...,2$. With similar argumentation, we can show

$$\begin{aligned}
&\partial_r \mu(\bar{r},\bar{w},\bar{L}) \to \frac{1}{\bar{r}}\partial_w\partial_r h(r,0,L) - \frac{1}{\bar{r}^2}\partial_w h(r,0,L), \quad (\bar{r},\bar{w},\bar{L}) \to (r,0,L), \\
&\partial_L \mu(\bar{r},\bar{w},\bar{L}) \to \frac{1}{\bar{r}}\partial_L\partial_r h(r,0,L), \quad (\bar{r},\bar{w},\bar{L}) \to (r,0,L),
\end{aligned}$$

The mean value theorem yields $\mu \in C^1(]0,\infty[\times\mathbb{R}\times[0,\infty[)$. The definition of $\mu$ and the product rule in Remark 5.5 (b) lead to the identity

$$\begin{aligned}
|\{E,h\}|^2 = |\{E,rw\mu\}|^2 &= (rw\{E,\mu\} + \mu\{E,rw\})^2 \\
&= (rw)^2(\{E,\mu\})^2 + 2\mu rw\{E,rw\}\{E,\mu\} + \mu^2(\{E,rw\})^2 \\
&= (rw)^2(\{E,\mu\})^2 + rw\{E,rw\}\{E,\mu^2\} + \mu^2\{E,rw\}\{E,rw\} \\
&= (rw)^2(\{E,\mu\})^2 + \{E,\mu^2 rw\{E,rw\}\} - \mu^2 rw\{E,\{E,rw\}\}.
\end{aligned}$$

Since $rw = |x \cdot v|$, first term corresponds to the first term in the claimed inequality. To examine the second term, we define $q(x,v) := \mu^2 rw\{E,rw\}$ for $(x,v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$. Let $s > 0$ be arbitrary. Since $f_0$ is a spherically symmetric steady state with $U_0 \in C^{0,2}(I \times \mathbb{R}^3)$, the characteristic flow $Z(t,0,\cdot)$ exists and is a measure preserving $C^1$-diffeomorphism on $\{L > 0\}$ by Lemma 2.11. The particle energy $E$ and the angular momentum squared $L$ are constant along characteristics, so we obtain with Remark 5.4 that

$$\begin{aligned}
&\iint_{\{f_0>0\}} \frac{1}{\Phi'(E,L)}\{E,\mu^2 rw\{E,rw\}\}\, dv\, dx = \iint_{\{f_0>0\}} \frac{1}{\Phi'(E,L)}\{E,q\}\, dv\, dx \\
&= \iint_{\{f_0>0\}} \frac{1}{\Phi'(E(Z),L(Z))}\{E,q\}(Z(s,0,\tilde{x},\tilde{v}))\, d\tilde{v}\, d\tilde{x} \\
&= -\iint_{\{f_0>0\}} \frac{1}{\Phi'(E(Z),L(Z))}\frac{\mathrm{d}}{\mathrm{d}s}(q(Z(s,0,\tilde{x},\tilde{v})))\, d\tilde{v}\, d\tilde{x} \\
&= -\iint_{\{f_0>0\}} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Phi'(E(Z),L(Z))}q(Z(s,0,\tilde{x},\tilde{v}))\right)\, d\tilde{v}\, d\tilde{x}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mathrm{d}}{\mathrm{d}s} \iint_{\{f_0>0\}} \frac{1}{\Phi'(E(Z),L(Z))} q(Z(s,0,\tilde{x},\tilde{v}))\, d\tilde{v}\, d\tilde{x} \\
&= -\frac{\mathrm{d}}{\mathrm{d}s} \iint_{\{f_0>0\}} \frac{1}{\Phi'(E,L)} q(x,v)\, dv\, dx = 0.
\end{aligned}
$$

Here, for the sake of clarity, $E(Z)$ and $L(Z)$ denotes $E(Z(s,0,\tilde{x},\tilde{v}))$ and $L(Z(s,0,\tilde{x},\tilde{v}))$ for $(\tilde{x},\tilde{v}) \in \{f_0 > 0\}$. Note that the support of $q$ is bounded with $\operatorname{supp} q \subset \operatorname{supp} h$ since $\operatorname{supp} h \subset \{f_0 > 0\}$ is compact. Hence, $\frac{1}{\Phi'(E,L)} q((X,V)(s,0,\cdot))$ is bounded on $[0,T] \times \operatorname{supp} h$ for all $T > 0$, so the change of integration and differentiation is permitted. Therefore, the second term vanishes, so it remains to analyze the third term. By calculating

$$
\begin{aligned}
\{E, rw\} &= \left(U_0'(r) + \frac{M_0}{r^2}\right) \frac{x}{r} \cdot x - v \cdot v \\
&= \left(U_0'(r) + \frac{M_0}{r^2}\right) r - |v|^2,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&- \mu^2 rw\{E,\{E,rw\}\} = -\mu^2 rw(\partial_x E \cdot \partial_v\{E,rw\} - \partial_v E \cdot \partial_x\{E,rw\}) \\
&= -\mu^2 rw \left( \left(U_0'(r) + \frac{M_0}{r^2}\right) \frac{x}{r} \cdot (-2v) - v \cdot \left(\partial_r(rU_0'(r)) - \frac{M_0}{r^2}\right) \frac{x}{r} \right) \\
&= \mu^2 rw^2 \left( 2\left(U_0'(r) + \frac{M_0}{r^2}\right) + \left(\partial_r(rU_0'(r)) - \frac{M_0}{r^2}\right)\right) \\
&= \mu^2 r^2 w^2 \left( 2\frac{U_0'(r)}{r} + 2\frac{M_0}{r^3} + \frac{1}{r}\partial_r(rU_0'(r)) - \frac{M_0}{r^3} \right) \\
&= \mu^2 r^2 w^2 \left( 2\frac{U_0'(r)}{r} + \frac{1}{r}\partial_r(rU_0'(r)) + \frac{M_0}{r^3} \right).
\end{aligned}
$$

Since $U_0$ solves the Poisson equation $\Delta U_0 = 4\pi\rho_0$ and $U_0$ and $\rho_0$ are spherically symmetric, we obtain that

$$
\begin{aligned}
&2\frac{U_0'(r)}{r} + \frac{1}{r}\partial_r(rU_0'(r)) = 2\frac{U_0'(r)}{r} + \frac{1}{r}(U_0'(r) + rU_0''(r)) \\
&= \frac{U_0'(r)}{r} + \frac{1}{r^2}(2rU_0'(r) + r^2 U_0'') = \frac{U_0'(r)}{r} + \frac{1}{r^2}\partial_r(r^2 U_0'(r)) \\
&= \frac{U_0'(r)}{r} + \Delta U_0(r) = \frac{U_0'(r)}{r} + 4\pi\rho_0.
\end{aligned}
$$

This results in

$$
\begin{aligned}
&- \mu^2 rw\{E,\{E,rw\}\} = \mu^2 r^2 w^2 \left( 2\frac{U_0'(r)}{r} + \frac{1}{r}\partial_r(rU_0'(r)) + \frac{M_0}{r^3} \right) \\
&= \mu^2 r^2 w^2 \left( \frac{U_0'(r)}{r} + 4\pi\rho_0 + \frac{M_0}{r^3} \right) = h^2 \left( \frac{U_0'(r)}{r} + 4\pi\rho_0 + \frac{M_0}{r^3} \right).
\end{aligned}
$$

If we collect all assertions, we finally get the desired statement:

$$\begin{aligned}
\mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[\{-E,h\}] &= \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}|\{-E,h\}|^2 \, dv\, dx - \frac{1}{8\pi}\int |\nabla U_h|^2 \, dx \\
&\geq \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}(|\{E,h\}|^2 - 4\pi\rho_0(x)h^2)\, dv\, dx \\
&= \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}((rw)^2\{E,\mu\}^2 + \{E,\mu^2 rw\{E,rw\}\} \\
&\qquad - \mu^2 rw\{E,\{E,rw\}\} - 4\pi\rho_0(x)h^2)\, dv\, dx \\
&= \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}\left((x\cdot v)^2\left\{E,\frac{h}{x\cdot v}\right\}^2 + h^2\left(\frac{U_0'(r)}{r}+\frac{M_0}{r^3}\right)\right) dv\, dx \\
&= \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}\left((x\cdot v)^2\left\{E,\frac{h}{x\cdot v}\right\}^2 + h^2\frac{m(r)+M_0}{r^3}\right) dv\, dx.
\end{aligned}$$

Note that $U_0'(r) = \frac{m(r)}{r^2}$ with $m(r) = 4\pi\int_0^r s^2\rho_0(s)\, ds$ for $r > 0$ as discussed in Lemma 2.9, so the proof is complete. $\square$

Lemma 5.6 says that the second order variation of the energy-Casimir functional $\mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)$ is positive definite for states of the form $\mathcal{T}h = \{-E,h\}$ induced by smooth $h$. In order to derive a contradiction, we first invert the transport operator and show that $g$ has the structure $g = \mathcal{T}h$ for some $h \in L^2(\mathbb{R}^6)$. To do this, we proceed analogously to [5]. In the most cases, the arguments in [5] can be transferred literally, so we will only have only a look at important steps and at arguments which are not obvious at the first glance.

Since $g \in L^2(\mathbb{R}^6)$ with $\operatorname{supp} g \subset \operatorname{supp} f_0$ is not even differentiable in the classical sense, we define $\mathcal{T}h = g$ in a weak sense. The functions $g$ and $E$ are spherically symmetric. Since the Poisson bracket is spherically symmetric, as discussed in Remark 5.5, we only consider spherically symmetric functions on $\Omega_0 := \{f_0 > 0\}$ and define the transport operator in a weak sense as in [5, Definition 4.1]:

**Definition 5.7.** *Let $h,\mu \in L^1_{loc}(\Omega_0)$ be spherically symmetric with*

$$\iint_{\Omega_0} h\mathcal{T}\xi \, dv\, dv = -\iint_{\Omega_0} \mu\xi \, dv\, dx$$

*for all $\xi \in C^1_{c,r}(\Omega_0) := \{\zeta \in C^1_c(\Omega_0) | \zeta$ spherically symmetric$\}$. Then $\mathcal{T}h := \mu$ exists weakly. Furthermore, the domain of the operator $\mathcal{T}$ is defined by*

$$D(\mathcal{T}) := \{h \in L^2(\Omega_0) | \mathcal{T}h \text{ exists weakly and } \mathcal{T}h \in L^2(\Omega_0)\}.$$

In the following, we denote function spaces restricted to spherically symmetric functions with an index $r$.

## 5.3 The $(\theta, E, L)$-coordinates and the transport operator

Similar to the argumentation in Remark 5.4, we define new coordinates to get a more convenient representation of the transport operator.

We define the *effective potential* $\psi_L : ]0, \infty[ \to \mathbb{R}$ as

$$\psi_L(r) := U_0(r) - \frac{M_0}{r} + \frac{L}{2r^2}$$

for $L > L_0$. Analogously to [5, Lemma 2.1], we obtain the following assertions and quantities:

**Lemma 5.8.** *(a) For all $L > 0$ there exists a unique radius $r_L$ with*

$$\min_{]0,\infty[} \psi_L(r) = \psi_L(r_L) < 0.$$

*(b) For all $L > 0$ and $\psi_L(r_L) < E < 0$, there exist two unique radii $0 < r_-(E, L) < r_L < r_+(E, L) < \infty$ such that*

$$\psi_L(r_\pm(E, L)) = E.$$

*Furthermore, the map*

$$\{(E, L) \in ]-\infty, 0[ \times ]0, \infty[ \,|\, \psi_L(r_L) < E\} \ni (E, L) \mapsto r_\pm(E, L)$$

*is continuously differentiable.*

*(c) The radius $r_+(E, L)$ is bounded from above by*

$$r_+(E, L) < -\frac{M + M_0}{E} = -\frac{M_{\text{tot}}}{E}$$

*with $M := \|f_0\|_1$ and $M_{\text{tot}} := M + M_0$ for $L > 0$ and $E \in ]\psi_L(r_L), 0[$ and*

$$E - \psi_L(r) \geq L \frac{(r_+(E, L) - r)(r - r_-(E, L))}{2r^2 r_-(E, L) r_+(E, L)}$$

*for all $r_-(E, L) < r < r_+(E, L)$.*

Via the transformation $R := |X|$, $W := \frac{X \cdot V}{|X|}$ and $L := |X \times V|^2 = \text{const}$, the characteristics $(X, V)$ solve the system

$$\dot{r} = w, \quad \dot{w} = -\psi_L'(r), \quad \dot{L} = 0. \tag{5.3}$$

Similar to Lemma 2.13, the particle energy is constant along solution of the characteristic system in $(r, w, L)$-coordinates. Let $\mathbb{R} \ni t \to (r(t), w(t), L)$ be a global solution of the system (5.3). Then

$$\psi_L(r_L) \leq \psi_L(r(t)) \leq \frac{1}{2}w(t)^2 + \psi_L(r(t)) = E$$

with $E = E(r(t), w(t), L) = E(r(0), w(0), L)$ for $t \in \mathbb{R}$. We assume, $\psi_L(r_L) < E < 0$. This leads to

$$\dot{r}(t) = w(t) = \pm\sqrt{2E - 2\psi_L(r(t))}, \quad t \in \mathbb{R}.$$

Since $\psi_L$ is monotonically decreasing or increasing on $]0, r_L[$ or $]r_L, \infty[$, respectively, and $\psi_L(r_L) \leq \psi_L(r_\pm(E, L))$, we obtain

$$r_-(E, L) \leq r(t) \leq r_+(E, L), \quad t \in \mathbb{R}.$$

Thus, the solution oscillates between $r_-(E, L)$ and $r_+(E, L)$ with some period $T(E, L)$. As in [5, Definition 2.2], we can derive the period function explicitly:

**Definition 5.9.** *For $L > L_0$ and $\psi_L(r_L) < E < 0$ the* period function $T(E, L)$ *of the steady state $f_0$ is defined by*

$$T(E, L) := 2\int_{r_-(E,L)}^{r_+(E,L)} \frac{1}{\sqrt{2E - 2\psi_L(r)}}\, dr.$$

Analogously to [5], the period function is bounded:

**Lemma 5.10.** *The period function satisfies the following estimates:*

$$\left(4\pi\|\rho_0\|_\infty + 3\frac{L}{r_L^4}\right)^{-\frac{1}{2}} \leq T(E, L) \leq 2\pi\frac{(M + M_0)^2}{\sqrt{L}E^2}$$

*for all $(E, L) \in \mathring{\Omega}_0^{E,L}$ with*

$$\Omega_0^{E,L} := \{(E, L) \in \mathbb{R} \times [0, \infty[ | E = E(x, v) \wedge L = L(x, v) \text{ for some } (x,v) \in \Omega_0\}.$$

*In particular, there exist $0 < C_1 < C_2$ with $C_{1,2} = C_{1,2}(f_0)$ such that*

$$C_1 \leq T(E, L) \leq C_2, \quad (E, L) \in \mathring{\Omega}_0^{E,L}.$$

*Proof.* With Lemma 5.8, the first estimates can be proven analogously to [5, Equation (2.12)] and [5, Lemma B.4]. This is why we omit their proof. These estimates implies the existence of $C_{1,2}$: Since $\operatorname{supp} f_0 = \{E \leq E_0\} \cap \{L \geq L_0\}$ is compact with $E_0 < 0$ and $L_0 > 0$, the quantities $L_{\max} := \max_{\operatorname{supp} f_0} L$ and $E_{\min} := \min_{\operatorname{supp} f_0} E$ exist and

$$\Omega_0^{E,L} \subset [E_{\min}, E_0] \times [L_0, L_{\max}].$$

Furthermore, by Lemma 3.2, there exist $0 < r^* < R^*$ with

$$\operatorname{supp} f_0 \subset \{r^* \leq |x| \leq R^*\} \times \{|v| \leq R^*\}.$$

Therefore, $r_L \geq r^*$ for $(E, L) \in \Omega_0^{E,L}$. In summary, we obtain

$$\left(4\pi\|\rho_0\|_\infty + 3\frac{L_{\max}}{(r^*)^4}\right)^{-\frac{1}{2}} \leq T(E, L) \leq 2\pi\frac{(M + M_0)^2}{\sqrt{L_0}E_0^2}$$

for all $(E, L) \in \mathring{\Omega}_0^{E,L}$. □

Analogously to [5], we use the spherical symmetry and the characteristic flow to introduce the $(\theta, E, L)$-coordinates, the so-called *action-angle coordinates*: Let $(r, w, L) \in ]0, \infty[\times\mathbb{R}\times]L_0, \infty[$ with $E(r, w, L) < E_0$, and let $\mathbb{R} \ni t \to (R, W)(t, r, w, L)$ be the unique solution of the system:

$$\dot{R} = W, \quad \dot{W} = -\psi_L'(R)$$

with $(R, W)(0, r, w, L) = (r, w, L)$. Note that the right-hand side is continuously differentiable which implies the uniqueness. Because of the relation between $(R, W)$ and $(X, V)$, the solution $(R, W)(\cdot, r, w, L)$ also exists globally for all $(r, w, L) \in ]0, \infty[\times\mathbb{R}\times]L_0, \infty[$. Furthermore, we discussed before that

$$r_-(E, L) \leq R(t, r, w, L) \leq r_+(E, L), \quad t \in \mathbb{R}.$$

Therefore, for all $(r, w, L) \in ]0, \infty[\times\mathbb{R}\times]L_0, \infty[$ with $\psi_L(r_L) < E(r, w, L) < E_0$, there exist $\theta \in [0, 1[$ and $(E, L) \in \mathring{\Omega}_0^{E,L}$ such that

$$(r, w, L) = ((R, W)(\theta T(E, L), r_-(E, L), 0, L), L).$$

Since $R(\cdot, r, w, L)$ oscillates between $r_-(E, L)$ and $r_+(E, L)$ with period $T(E, L)$, the map

$$\left[0, \frac{1}{2}\right] \ni \theta \mapsto R(\theta T(E, L), r_-(E, L), 0, L) \in [r_-(E, L), r_+(E, L)]$$

is bijective for all $(E, L) \in \mathring{\Omega}_0^{E,L}$. Inserting $r = R(\theta T(E, L), r_-(E, L), 0, L)$ with $\theta \in [0, \frac{1}{2}]$ into the system (5.3) leads to

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \dot{R}(\theta T(E, L), r_-(E, L), 0, L)T(E, L) \\ &= W(\theta T(E, L), r_-(E, L), 0, L)T(E, L) = \sqrt{2E - 2\psi_L(r)}T(E, L),\end{aligned}$$

so the inverse of the previous function is given by

$$\theta(r, E, L) := \frac{1}{T(E, L)} \int_{r_-(E,L)}^{r} \frac{1}{\sqrt{2E - 2\psi_L(s)}}\, ds$$

and exists on $[r_-(E,L), r_+(E,L)]$ for all $(E,L) \in \mathring{\Omega}_0^{E,L}$. Obviously, $\theta(r_-(E,L),0,L) = 0$.

Via the above transformation between $(r,w,L)$- and $(\theta,E,L)$-coordinates, spherically symmetric functions can be expressed in the new coordinates: Let $h \in L^2(\Omega_0)$ be spherically symmetric. By Lemma 2.2, there exists a function $\tilde{h}$ with $h(x,v) = \tilde{h}(r,w,L)$ which we denote with $h$. Under slight abuse of notation, we define

$$h(\theta,E,L) := h((R,W)(\theta,r_-(E,L),0,L),L)$$

for a.e. $(\theta,E,L) \in [0,1] \times \mathring{\Omega}_0^{E,L}$. A change of variables yields that

$$\begin{aligned}
\iint_{\Omega_0} h(x,v)\, dx\, dv &= 4\pi^2 \int_0^\infty \int_{\mathbb{R}} \int_0^\infty h(r,w,L)\, dL\, dw\, dr \\
&= 8\pi^2 \int_0^\infty \int_0^\infty \int_{\psi_L(r)}^{E_0} h(\theta,E,L) \frac{1}{\sqrt{2E-2\psi_L(r)}}\, dE\, dL\, dr \\
&= 8\pi^2 \int_{r_-(E,L)}^{r_+(E,L)} \iint_{\Omega_0^{E,L}} h(\theta,E,L) \frac{1}{\sqrt{2E-2\psi_L(r)}}\, dE\, dL\, dr \\
&= 8\pi^2 \int_0^{\frac{1}{2}} \iint_{\Omega_0^{E,L}} h(\theta,E,L) T(E,L)\, dL\, dE\, d\theta \\
&= 4\pi^2 \int_0^1 \iint_{\Omega_0^{E,L}} h(\theta,E,L) T(E,L)\, dL\, dE\, d\theta,
\end{aligned}$$

so integrals in $(x,v)$-coordinates convert into ones in $(\theta,E,L)$-coordinates. Furthermore, we can investigate how parity in $v$ behaves in varies coordinates:

**Lemma 5.11.** *Let $h \in L^2(\Omega_0)$ be spherically symmetric. Then the following assertions are equivalent:*

*(i) $h = h(x,v)$ is even (or odd) in $v$.*

*(ii) $h = h(r,w,L)$ is even (or odd) in $w$.*

*(iii) $h(\cdot,E,L) \in L^{2,even}(]0,1[)$ (or $h(\cdot,E,L) \in L^{2,odd}(]0,1[)$) for a.e. $(E,L) \in \Omega_0^{E,L}$ with*

$$L^{2,even}(]0,1[) := \{y \in L^2(]0,1[) | y(\theta) = y(1-\theta) \text{ for a.e. } \theta \in ]0,1[\}$$

*and*

$$L^{2,odd}(]0,1[) := \{y \in L^2(]0,1[) | y(\theta) = -y(1-\theta) \text{ for a.e. } \theta \in ]0,1[\}.$$

*Proof.* Since $w = \frac{x \cdot v}{|x|}$ is odd and $L = |x \times v|^2$ is even in $v$, the first equivalence is valid. Furthermore,

$$(R,W)(s,r,w,L) = (R,-W)(T(E,L)-s,r,w,L), \quad s \in [0,T(E,L)],$$

for all $(r, w, L) \in \Omega_0^r$ with

$$\Omega_0^r := \{(r, w, L) \in ]0, \infty[\times \mathbb{R} \times [0, \infty[ | r = r(x, v),\ w = w(x, v),\ L = L(x, v) \\ \text{for some } (x, v) \in \Omega_0\}.$$

This yields

$$(R, W)(\theta, r_-(E, L), 0, L) = (R, -W)(1 - \theta, r_-(E, L), 0, L), \quad \theta \in [0, 1],$$

for all $(E, L) \in \mathring{\Omega}_0^{E,L}$, so also the second equivalence is valid. □

As mentioned before, we use these coordinates to derive a useful representation of $\mathcal{T}$. With Remark 5.4, we obtain for $h \in C_r^1(\Omega_0)$ that

$$\begin{aligned}(\mathcal{T}h)(\theta, E, L) &= \{-E, h\}((X, V)(\theta T(E, L), x(E, L), v(E, L))) \\ &= \frac{1}{T(E, L)} \frac{\mathrm{d}}{\mathrm{d}\theta}(h((X, V)(\theta T(E, L), x(E, L), v(E, L)))) \\ &= \frac{1}{T(E, L)} \partial_\theta h(\theta, E, L), \quad (\theta, E, L) \in [0, 1[\times \mathring{\Omega}_0^{E,L},\end{aligned}$$

with $x(E, L) := r_-(E, L)e_1$ and $v(E, L) := \frac{\sqrt{L}}{r_-(E,L)} e_2$ for all $(E, L) \in \mathring{\Omega}_0^{E,L}$. Similar to [5, Lemma 5.1], we summarize the assertion in the following lemma:

**Lemma 5.12.** *The transport operator $\mathcal{T}$ has the form*

$$(\mathcal{T}h)(\theta, E, L) = \frac{1}{T(E, L)}(\partial_\theta h)(\theta, E, L), \quad \theta \in [0, 1],\ (E, L) \in \Omega_0^{E,L}$$

*for all $h \in C_r^1(\Omega_0)$.*

Analogously to [5], we can extend Lemma 5.12 from $C_r^1(\Omega_0)$ to $D(\mathcal{T})$:

**Lemma 5.13.** *The domain of the transport operator $\mathcal{T}$ is given by*

$$D(\mathcal{T}) = \{g \in L^2(\Omega_0) | g(\cdot, E, L) \in H_\theta^1 \text{ for a.e. } (E, L) \in \Omega_0^{E,L} \\ \text{and } \iint_{\Omega_0^{E,L}} \frac{1}{T(E, L)} \int_0^1 |\partial_\theta g(\theta, E, L)|^2 \, d\theta \, dE \, dL < \infty\}$$

*with*

$$H_\theta^1 := \{y \in H^1(]0, 1[) | y(0) = y(1)\}.$$

*For $h \in D(\mathcal{T})$, the transport operator $\mathcal{T}$ has the form*

$$(\mathcal{T}h)(\theta, E, L) = \frac{1}{T(E, L)}(\partial_\theta h)(\theta, E, L)$$

*for a.e. $(\theta, E, L) \in [0, 1] \times \Omega_0^{E,L}$.*

Note that $H^1(]0,1[) \hookrightarrow C([0,1])$ compactly embedded, so $H^1_\theta$ is well-defined. Besides the additional potential induced by the point mass $M_0$, the other difference between our setting and the one [5] is that we consider $L^2_r(\Omega_0)$ instead of the weighted $L^2$-space $L^2_{r,\frac{1}{|\Phi'|}}(\Omega_0)$. Even so, the proof in [5, Lemma 5.2] can be transferred almost literally if we ignore the factor $\frac{1}{|\Phi'(E,L)|}$. This is why we omit the proof here.

The representations of $\mathcal{T}$ and $D(\mathcal{T})$ in the $(\theta, E, L)$-coordinates allow us to examine the kernel of the operator $\mathcal{T} : D(\mathcal{T}) \to D(\mathcal{T})$. We obtain similar to [5, Proposition 4.2] the following representation of the kernel:

**Lemma 5.14.** *The kernel of $\mathcal{T}$ is characterized by*

$$\begin{aligned}\ker(\mathcal{T}) &= \{h \in L^2_r(\Omega_0) | h(x,v) = f(E(x,v), L(x,v)) \textit{ for a.e. } (x,v) \in \Omega_0 \\ &\qquad \textit{and for some } f : \mathbb{R} \times [0,\infty[ \to \mathbb{R}\} \\ &= \{h \in L^2_r(\Omega_0) | h(\theta, E, L) = f(E,L) \textit{ for a.e. } (\theta, E, L) \in [0,1] \times \Omega_0^{E,L} \\ &\qquad \textit{and for some } f : \mathbb{R} \times [0,\infty[ \to \mathbb{R}\}.\end{aligned}$$

*Proof.* Let $h \in D(\mathcal{T})$ with $\mathcal{T}h = 0$. Lemma 5.13 implies that $h(\cdot, E, L) \in H^1_\theta$ with

$$0 = (\mathcal{T}h)(\theta, E, L) = \frac{1}{T(E,L)} (\partial_\theta h)(\theta, E, L)$$

for a.e. $(\theta, E, L) \in [0,1] \times \Omega_0^{E,L}$. Since $H^1_\theta \subset C([0,1])$, the function $h(\cdot, E, L)$ is continuous in $\theta$ with weak derivative $\partial_\theta h(\cdot, E, L) = 0$ for a.e. $(E, L) \in \Omega_0^{E,L}$. The weak version of the fundamental theorem of calculus leads to

$$h(\theta, E, L) = h(1, E, L), \quad 0 \le \theta \le 1,$$

for a.e. $(E, L) \in \Omega_0^{E,L}$. Therefore, we define $f : \mathbb{R} \times [0,\infty[ \to \mathbb{R}$ with $f(E,L) := h(1,E,L)$ for $(E,L) \in \Omega_0^{E,L}$ and $f(E,L) := 0$ otherwise. Here, $h$ is a pointwise defined representative. This definition yields

$$\begin{aligned}&\iint |h(x,v) - f(E(x,v), L(x,v))| \, dv \, dx \\ &= 4\pi^2 \int_0^1 \iint_{\Omega_0^{E,L}} |h(\theta, E, L) - h(1, E, L)| T(E,L) \, dL \, dE \, d\theta = 0,\end{aligned}$$

so $h(x,v) = f(E(x,v), L(x,v))$ holds for a.e. $(x,v) \in \Omega_0$.

On the other hand, if $h \in L^2(\Omega_0)$ with $h(\theta, E, L) = f(E,L)$ for a.e. $(\theta, E, L) \in [0,1] \times \Omega_0^{E,L}$ and for some $f : \mathbb{R} \times [0,\infty[ \to \mathbb{R}$, Lemma 5.13 implies that $h \in D(\mathcal{T})$ and in particular $h \in \ker \mathcal{T}$. $\square$

## 5.4 The inverse of the transport operator

As stated before, we aim to invert the transport operator to obtain $g = \mathcal{T}h$ for $g$ given by Lemma 5.2 and some $h \in L^2(\mathbb{R}^6)$. The previous assertions and argumentation culminate in the following lemma that ensures the existence of the inverse for certain functions as described in [5, Lemma 5.5]:

**Lemma 5.15.** *Let $g \in L^2_r(\Omega_0)$ with $g \perp \ker\mathcal{T}$. Then there exists $h \in D(\mathcal{T})$ such that*

$$\mathcal{T}h = g.$$

*In particular,*

$$(\ker\mathcal{T})^\perp = \mathrm{im}(\mathcal{T}).$$

Similar to Lemma 5.13, we can transfer the proof in [5] almost literally if we neglect the factor $\frac{1}{|\Phi'(E,L)|}$. Therefore, we omit the proof here and refer to [5, Lemma 5.5].

The proof of Lemma 5.15, as described in [5, Lemma 5.5], is constructive and shows that for $g \in L^2_r(\mathbb{R}^6)$ with $g \perp \ker\mathcal{T}$ the function $h$ given by

$$h(\theta, E, L) := T(E, L) \int_0^\theta g(s, E, L)\ ds$$

for a.e. $(\theta, E, L) \in [0,1] \times \Omega_0^{E,L}$ satisfies $\mathcal{T}h = g$. Additionally, the proof of [5, Lemma 5.5] shows the identity

$$0 = \int_0^1 g(s, E, L)\ ds$$

for a.e. $(E, L) \in \Omega_0^{E,L}$. If $g$ is additionally even in $v$, we obtain $0 = \int_0^1 g(s, E, L)\ ds = 2\int_0^{\frac{1}{2}} g(s, E, L)\ ds$ which leads to

$$\begin{aligned} h(1-\theta, E, L) &= T(E, L)\left(\int_0^{\frac{1}{2}} g(s, E, L)\ ds + \int_{\frac{1}{2}}^{1-\theta} g(s, E, L)\ ds\right) \\ &= -T(E, L) \int_{\frac{1}{2}}^{\theta} g(1-s, E, L)\ ds \\ &= -T(E, L)\left(\int_0^{\frac{1}{2}} g(s, E, L)\ ds + \int_{\frac{1}{2}}^{\theta} g(s, E, L)\ ds\right) \\ &= -h(\theta, E, L) \end{aligned}$$

for a.e. $\theta \in [0,1]$ and a.e. $(E, L) \in \Omega_0^{E,L}$. In summary, if $g \in L^2_r(\Omega_0)$ is even in $v$ with $g \perp \ker\mathcal{T}$, the associated function $h$ which satisfies $\mathcal{T}h = g$ is odd in $v$.

We aim to an $h \in L^2(\Omega_0)$ such that $g = \mathcal{T}h$ with $g$ given by Lemma 5.2. Obviously, $g \in L^2_r(\Omega_0)$ since $g \in L^2(\mathbb{R}^6)$ is spherically symmetric with $\operatorname{supp} g \subset \operatorname{supp} f_0$. To apply Lemma 5.15, it remains to show that $g \perp \ker \mathcal{T}$. The kernel $\ker \mathcal{T}$ is given by Lemma 5.14.

Let $p \in \ker\mathcal{T}$. By Lemma 5.14, there exists $\tilde{p} : \mathbb{R} \times [0, \infty[ \to \mathbb{R}$ with $p(x, v) = \tilde{p}(E(x, v), L(x, v))$ for a.e. $(x, v) \in \Omega_0$. Since $T$ is bounded on $\mathring{\Omega}_0^{E,L}$ as discussed in Lemma 5.10, the function $\tilde{p}$ satisfies $\tilde{p} \in L^2(\Omega_0^{E,L})$ because

$$\begin{aligned}\infty > \iint_{\Omega_0} |p(x, v)|^2 \, dv \, dx &= 4\pi^2 \iint_{\Omega_0^{E,L}} |\tilde{p}(E, L)|^2 T(E, L) \, dL \, dE \\ &\geq 4\pi^2 \inf_{\mathring{\Omega}_0^{E,L}} T \iint_{\Omega_0^{E,L}} |\tilde{p}(E, L)|^2 \, dL \, dE.\end{aligned}$$

Hence, there exists a sequence $(\tilde{p}_k) \subset C_c^\infty(\mathring{\Omega}_0^{E,L})$ with $\tilde{p}_k \to \tilde{p}$ in $L^2(\Omega_0^{E,L})$ as $k \to \infty$. In order to show $\iint pg \, dv \, dx = 0$, we exploit property (g6) and define

$$G_k(f, L) := \int_0^f \tilde{p}_k(\Phi_E^{-1}(y, L), L) \, dy, \quad (f, L) \in [0, \infty[^2,$$

for $k \in \mathbb{N}$. Since $\tilde{p}_k \in C_c^\infty(\mathring{\Omega}_0^{E,L})$ with $\operatorname{supp} \tilde{p}_k \subset \mathring{\Omega}_0^{E,L} \subset \{E < E_0 \wedge L > L_0\}$, we extend for $k \in \mathbb{N}$ the function $\tilde{p}_k(\Phi_E^{-1}(y, L), L)$ continuously by zero to $[0, \infty[^2$. Note that $\min_{\operatorname{supp} \tilde{p}_k} L > L_0$, so $G_k$ is well-defined and twice differentiable with

$$\begin{aligned}\partial_f G_k(f, L) &= \tilde{p}_k(\Phi_E^{-1}(f, L), L), \\ \partial_f^2 G_k(f, L) &= \frac{1}{\Phi'(\Phi_E(f, L), L)} \tilde{p}_k'(\Phi_E^{-1}(f, L), L)\end{aligned}$$

for $(f, L) \in [0, \infty[$. Since $\tilde{p}_k \in C_c^\infty(\mathring{\Omega}_0^{E,L})$ and $\Phi_E^{-1} \in C^{2,0}(]0, \infty[\times]L_0, \infty[)$, we obtain $G_k \in C^{2,0}([0, \infty[\times[0, \infty[)$ for $k \in \mathbb{N}$ and $G_k(0, L) = 0$ and $\partial_f G_k(0, L) = 0$ for $L \geq 0$ and $k \in \mathbb{N}$. Furthermore, $(|\Phi'(E, L)|)^{-1}$ and $\tilde{p}_k'$ are bounded on the compact support $\operatorname{supp} \tilde{p}_k \subset \mathring{\Omega}_0^{E,L}$, so the second derivative $\partial_f^2 G_k$ is bounded for $k \in \mathbb{N}$. In summary, $G_k$ fulfills the conditions imposed in (g6) in Lemma 5.2, which implies

$$0 = \iint \partial_f G_k(f_0, L) g \, dv \, dx = \iint \tilde{p}_k(E(x, v), L(x, v)) g(x, v) \, dv \, dx, \quad k \in \mathbb{N}.$$

Since $g \in L^2(\Omega_0)$, we analyze the sequence $(p_k)$ given by $p_k(x, v) := \tilde{p}_k(E(x, v), L(x, v))$ for $(x, v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ and $k \in \mathbb{N}$ for convergence. Again, the boundedness of the period function $T$ yields that

$$\begin{aligned}\iint_{\Omega_0} |p_k(x, v) - p(x, v)|^2 \, dv \, dx &= 4\pi^2 \iint_{\Omega_0^{E,L}} |\tilde{p}_k(E, L) - \tilde{p}(E, L)|^2 T(E, L) \, dE \, dL \\ &\leq 4\pi^2 \sup_{\mathring{\Omega}_0^{E,L}} T \iint_{\Omega_0^{E,L}} |\tilde{p}_k(E, L) - \tilde{p}(E, L)|^2 \, dE \, dL,\end{aligned}$$

so the convergence of $(\tilde{p}_k)$ implies that $(p_k)$ tends to $p$ in $L^2(\Omega_0)$. In conclusion, we obtain that

$$\begin{aligned}
0 = \lim_{k\to\infty} \iint \partial_f G_k(f_0, L) g \; dv \; dx &= \lim_{k\to\infty} \iint \tilde{p}_k(E(x,v), L(x,v)) g(x,v) \; dv \; dx \\
&= \iint \tilde{p}(E(x,v), L(x,v)) g(x,v) \; dv \; dx \\
&= \iint p(x,v) g(x,v) \; dv \; dx,
\end{aligned}$$

so $g \perp \ker \mathcal{T}$. Hence, the conditions of Lemma 5.15 are fulfilled, and thus there exists $h \in L^2_r(\Omega_0)$ such that

$$\mathcal{T} h = g \text{ weakly}$$

in the sense of Definition 5.7. Furthermore, $g$ is even in $v$ because of the condition (g3) in Lemma 5.2, so $h$ is odd in $v$ as we discussed earlier.

## 5.5 The regularization of the inverse

Although we have shown that $g = \mathcal{T}h = \{-E, h\}$ in a weak sense, we are not allowed to apply Lemma 5.6 and use that $\mathrm{D}^2\mathcal{H}_{\mathcal{C}}[f_0](\{-E, \cdot\})$ is positive definite to derive a contradiction since $h \in L^2_r(\Omega_0)$ is not smooth enough. Thus, we approximate $h$ appropriately by smooth functions.

We first create a cut-off version of $h$ by reducing the support of $h$. More precisely, we reuse the increasing sets $(S_m)$ from the proof of Lemma 5.2 and consider $1_{S_m} h$. As a reminder, the increasing sequence $(S_m)$ with $S_m \subset S_{m+1} \subset \ldots \subset \operatorname{supp} f_0$ has a positive distance to the boundary of $\operatorname{supp} f_0$ and is defined by

$$S_m := \left\{ (x,v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3 | E(x,v) \leq E_0 - \frac{1}{m} \wedge L(x,v) \geq L_0 + \frac{1}{m} \right\}, \quad m \in \mathbb{N}.$$

The distance gives us some space to approximate $1_{S_m} h$ by smooth functions with support in $\operatorname{supp} f_0$. In the following argumentation, we will switch between different coordinates, so we define

$$\begin{aligned}
S_m^r := \{(r, w, L) \in ]0,\infty[ \times \mathbb{R} \times [0,\infty[ | r = |x| \wedge w = \frac{x \cdot v}{|x|} \\
\wedge L = |x \times v|^2 \text{ for some } (x,v) \in S_m\}
\end{aligned}$$

and

$$S_m^{E,L} := \{(E, L) \in \mathbb{R} \times [0,\infty[ | E = E(x,v) \wedge L = L(x,v) \text{ for some } (x,v) \in S_m\}$$

for $m \in \mathbb{N}$. The cut-off version $1_{S_m} h$ is useful since $\mathcal{T}(1_{S_m} h) = 1_{S_m} g$ in the weak sense of Definition 5.7:

**Lemma 5.16.** *For all* $m \in \mathbb{N}$,

$$\mathcal{T}(1_{S_m} h) = 1_{S_m} g \text{ weakly,}$$

*i.e.,* $1_{S_m^r} h, 1_{S_m^r} g \in L^2(\Omega_0^r)$ *with*

$$\iiint_{\Omega_0^r} 1_{S_m^r} h\{-E, \xi\} \, dr \, dw \, dL = -\iiint_{\Omega_0^r} 1_{S_m^r} g \xi \, dr \, dw \, dL$$

*for all* $\xi \in C^1_{c,r}(\Omega_0^r)$.

*Proof.* Since $S_m$ is spherically symmetric, obviously $1_{S_m} h, 1_{S_m} g \in L^2(\Omega_0)$ are spherically symmetric with $1_{S_m^r} h, 1_{S_m^r} g \in L^2(\Omega_0^r)$. Let $\xi \in C^1_{c,r}(\Omega_0)$. The representation of $\mathcal{T}$ in Lemma 5.13 and an integration by parts yield

$$\begin{aligned}
&\iint \{-E, \xi\} 1_{S_m} h \, dv \, dx = \iint_{S_m} \{-E, \xi\} h \, dv \, dx \\
&= 4\pi^2 \iint_{S_m^{E,L}} \int_0^1 (\mathcal{T}\xi)(\theta, E, L) h(\theta, E, L) T(E, L) \, d\theta \, dL \, dE \\
&= 4\pi^2 \iint_{S_m^{E,L}} \int_0^1 (\partial_\theta \xi)(\theta, E, L) h(\theta, E, L) \, d\theta \, dL \, dE \\
&= -4\pi^2 \iint_{S_m^{E,L}} \int_0^1 \xi(\theta, E, L)(\partial_\theta h)(\theta, E, L) \, d\theta \, dL \, dE \\
&= -4\pi^2 \iint_{S_m^{E,L}} \int_0^1 \xi(\theta, E, L)(\mathcal{T} h)(\theta, E, L) T(E, L) \, d\theta \, dL \, dE \\
&= -4\pi^2 \iint_{S_m^{E,L}} \int_0^1 \xi(\theta, E, L) g(\theta, E, L) T(E, L) \, d\theta \, dL \, dE \\
&= -\iint_{S_m} \xi g \, dv \, dx = -\iint \xi 1_{S_m} g \, dv \, dx.
\end{aligned}$$

Note that $(R, W)(\cdot, r_-(E, L), 0, L)$ is periodic with frequency $T(E, L)$. Since $\xi(\cdot, E, L)$ and also $h(\cdot, E, L) \in H^1_\theta(]0, 1[)$ are continuous, we obtain $h(0, E, L) = h(1, E, L)$ and $\xi(0, E, L) = \xi(1, E, L)$ for a.e. $(E, L) \in \mathring{\Omega}_0^{E,L}$, so the boundary terms of the integrating by parts vanish. $\square$

In the next step, we fix $m \in \mathbb{N}$. We aim to construct spherically symmetric approximations $h_n \in C_c^\infty(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$ with $\operatorname{supp} h_n \subset \Omega_0$ which retain the oddness in $v$ for $n \in \mathbb{N}$. Furthermore, we desire the convergences $h_n \to 1_{S_m} h$ in $L^1 \cap L^2(\mathbb{R}^6)$ and $\{-E, h_n\} \to 1_{S_m} g$ in $L^2(\mathbb{R}^6)$. Since we require the sequence to be spherically symmetric, it is sufficient to construct a smooth $h_n = h_n(r, w, L)$ with $\operatorname{supp} h_n \subset \Omega_0^r$ which is odd in $w$ and with convergences $h_n \to 1_{S_m} h$ in $L^1 \cap L^2([0, \infty[ \times \mathbb{R} \times [0, \infty[)$ and $\{-E, h_n\} \to 1_{S_m} g$ in $L^2([0, \infty[ \times \mathbb{R} \times [0, \infty[)$. Note that it follows by Lemma 2.2 and 2.3 that regularity and convergence in $L^p$ in $(r, w, L)$-coordinates transfer to the same properties in $(x, v)$-coordinates.

Even though the operator $\mathcal{T}$ has a simple structure in the $(\theta, E, L)$-coordinates, we prefer the $(r, w, L)$-coordinates in this part of the argumentation. In the $(\theta, E, L)$-coordinates, it would be easier to construct a suitable sequence $h_n = h_n(\theta, E, L)$ by smoothing $1_{S_m} h = (1_{[0,1]\times S_m^{E,L}} h)(\theta, E, L)$. Then both convergences $h_n \to 1_{S_m} h$ in $L^1 \cap L^2(\mathbb{R}^6)$ and $\{-E, h_n\} \to 1_{S_m} g$ in $L^2(\mathbb{R}^6)$ could be shown directly. But other than the $(r, w, L)$-coordinates, it would be more challenging or maybe impossible to show that $h_n = h_n(x, v)$ is regular enough to apply Lemma 5.6. That is the reason why we mollify $1_{S_m} h = (1_{S_m^r} h)(r, w, L)$ by using the $(r, w, L)$-coordinates.

As shown in Lemma 3.2, the set $S_m$ is compact, and there exist radii $0 < r^* < R^*$ with

$$S_m = \left\{ E \leq E_0 - \frac{1}{m} \right\} \cap \left\{ L \geq L_0 + \frac{1}{m} \right\} \subset \{ r^* \leq |x| \leq R^* \} \times \{ |v| \leq R^* \}.$$

Hence there exist $0 < \bar{R}_0 < \bar{R}_1$, $\bar{W}_0 > 0$ and $0 < L_0 < \bar{L}_0 < \bar{L}_1$ such that

$$S_m^r \subset [\bar{R}_0, \bar{R}_1] \times [-\bar{W}_0, \bar{W}_0] \times [\bar{L}_0, \bar{L}_1] =: Q.$$

With the Friedrichs mollification, we smooth the function $1_{S_m} h = (1_{S_m^r} h)(r, w, L)$ in $(r, w, L)$-coordinates. For this purpose, let $\zeta \in C_c^\infty(\mathbb{R}^3)$ with $\operatorname{supp} \zeta \subset B_1(0)$, $\zeta \geq 0$ and $\int \zeta = 1$. Furthermore, we assume that $\zeta$ is even in all variables, i.e., $\zeta(z_1, z_2, z_3) = \zeta(|z_1|, |z_2|, |z_3|)$ for all $z = (z_1, z_2, z_3) \in \mathbb{R}^3$. We define $\zeta_n := n^3 \zeta(n\cdot)$ and

$$h_n(r, w, L) := \iiint (1_{S_m^r} h)(\bar{r}, \bar{w}, \bar{L}) \zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L}) \, d\bar{L} \, d\bar{w} \, d\bar{r}$$

for $(r, w, L) \in \mathbb{R}^3$ and $n \in \mathbb{N}$.

Since $\zeta \in C_c^\infty(\mathbb{R}^3)$ with $\operatorname{supp} \zeta \subset B_1(0)$, the approximate $h_n = h_n(r, w, L) \in C_c^\infty(\mathbb{R}^3)$ has the support $\operatorname{supp} h_n \subset B_{\frac{1}{n}}(S_m^r)$ with $B_{\frac{1}{n}}(S_m^r) := \{(r, w, L) \in \mathbb{R}^3 | |(r, w, L) - (\bar{r}, \bar{w}, \bar{L})| < \frac{1}{n} \text{ for some } (\bar{r}, \bar{w}, \bar{L}) \in S_m\}$ for $n \in \mathbb{N}$. In particular, we obtain $h_n \in C_c^\infty(]0, \infty[ \times \mathbb{R} \times ]0, \infty[)$ with $\operatorname{supp} h_n \subset Q \cap \Omega_0^r$ for $n \in \mathbb{N}$ large enough. By assumption, the mollifier $\zeta$ is even in every variable, while the function $h$ and thus $1_{S_m^r} h$ are odd in $w$. Hence, $h_n$ retains the parity of $h$ and is odd in $w$. The function $\zeta$ represents a Friedrichs mollifier, so it follows that

$$h_n \to 1_{S_m^r} h \text{ in } L^1 \cap L^2([0, \infty[ \times \mathbb{R} \times [0, \infty[)$$

as $n \to \infty$. Note that $S_m^r$ is compact, so $1_{S_m} h \in L^1 \cap L^2(\mathbb{R}^6)$. In particular, $h_n \to 1_{S_m} h$ in $L^1 \cap L^2(\mathbb{R}^6)$ by Lemma 2.3. As discussed before, we obtain the regularity $h_n \in C_c^\infty(\mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3)$, but it takes more effort to show

$$\{-E, h_n\} \to 1_{S_m^r} g \text{ in } L^2([0, \infty[ \times \mathbb{R} \times [0, \infty[)$$

and thus in $L^2(\mathbb{R}^6)$ as $n \to \infty$.

Some lines of straight-forward calculations show how the transport operator $\mathcal{T} = \{-E, \cdot\}$ ca be expressed in the $(r, w, L)$-coordinates. Fix $(r, w, L) \in ]0, \infty[ \times \mathbb{R} \times [0, \infty[$. By inserting the definition of $h_n$, we obtain similar to [4] that

$$
\begin{aligned}
&\mathcal{T} h_n = \{-E, h_n\} = w\partial_r h_n - \psi_L'(r)\partial_w h_n \\
&= \int (1_{S_m^r} h)(\bar{r}, \bar{w}, \bar{L})(w\partial_r - \psi_L'(r)\partial_w)\zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L})\ d\bar{L}\ d\bar{w}\ d\bar{r} \\
&= -\int (1_{S_m^r} h)(\bar{r}, \bar{w}, \bar{L})(w\partial_{\bar{r}} - \psi_L'(r)\partial_{\bar{w}})\zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L})\ d\bar{L}\ d\bar{w}\ d\bar{r} \\
&= -\int (1_{S_m^r} h)(\bar{r}, \bar{w}, \bar{L})(\bar{w}\partial_{\bar{r}} - \psi_{\bar{L}}'(\bar{r})\partial_{\bar{w}})\zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L})\ d\bar{L}\ d\bar{w}\ d\bar{r} \\
&\quad + \int (1_{S_m^r} h)(\bar{r}, \bar{w}, \bar{L})((\bar{w} - w)\partial_{\bar{r}} - (\psi_{\bar{L}}'(\bar{r}) - \psi_L'(r))\partial_{\bar{w}})\zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L})\ d\bar{L}\ d\bar{w}\ d\bar{r} \\
&=: J_{1,n} + J_{2,n}
\end{aligned}
$$

for $n \in \mathbb{N}$ large enough. For the sake of clarity, we write $\int \tilde{f}\ dL\ dw\ dr := \int_0^\infty \int_{\mathbb{R}} \int_0^\infty f\ dL\ dw\ dr$ for integrable functions $\tilde{f}$. Note that the change of integration and differentiation is permitted by general rules of convolutions and the Friedrichs mollification since $J \in C_c^\infty(\mathbb{R}^3)$.

We consider the two integrals separately. Lemma 5.16 converts the first integral $J_{1,n}$ into a Friedrichs mollification of $1_{S_m^r} g$ which ensures convergence because

$$
\begin{aligned}
&-\int (1_{S_m^r} h)(\bar{r}, \bar{w}, \bar{L})(\bar{w}\partial_{\bar{r}} - \psi_{\bar{L}}(\bar{r})\partial_{\bar{w}})\zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L})\ d\bar{L}\ d\bar{w}\ d\bar{r} \\
&= -\int (1_{S_m^r} h)\{-E, \zeta_n(r - \cdot, w - \cdot, L - \cdot)\}\ d\bar{L}\ d\bar{w}\ d\bar{r} \\
&= -\int (1_{S_m^r} h)\mathcal{T}\zeta_n(r - \cdot, w - \cdot, L - \cdot) d\bar{L}\ d\bar{w}\ d\bar{r} \\
&= \int (1_{S_m^r} g)\zeta_n(r - \cdot, w - \cdot, L - \cdot) d\bar{L}\ d\bar{w}\ d\bar{r} \to 1_{S_m^r} g
\end{aligned}
$$

as $n \to \infty$ in $L^2([0, \infty[ \times \mathbb{R} \times [0, \infty[)$.

In the next step, we consider $J_{2,n}$ and show $J_{2,n} \to 0$ as $n \to \infty$. For this purpose, we introduce, analogously to [4], new coordinates for fixed $n \in \mathbb{N}$. Let $\tilde{r} = n(r - \bar{r})$, $\tilde{w} = n(w - \bar{w})$ and $\tilde{L} = n(L - \bar{L})$. Since $\bar{r} = r - \frac{\tilde{r}}{n}$, $\bar{w} = w - \frac{\tilde{w}}{n}$ and $\bar{L} = L - \frac{\tilde{L}}{n}$, the derivative $\partial_{\bar{r}}\zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L})$ transfers into a derivative with respect to $\tilde{r}$ via

$$
\begin{aligned}
\partial_{\bar{r}}(\zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L})) &= n^3\partial_{\bar{r}}(\zeta(\tilde{r}, \tilde{w}, \tilde{L})) = n^3\partial_{\tilde{r}}\zeta(\tilde{r}, \tilde{w}, \tilde{L})\frac{\mathrm{d}\tilde{r}}{\mathrm{d}\bar{r}} \\
&= -n^4\partial_{\tilde{r}}\zeta(\tilde{r}, \tilde{w}, \tilde{L}).
\end{aligned}
$$

The derivative $\partial_{\bar{w}}\zeta_n(r-\bar{r}, w-\bar{w}, L-\bar{L})$ converts into a derivative with respect to $\tilde{w}$ in the same way. In summary, we obtain

$$\begin{aligned}
&\int (1_{S^r_m}h)(\bar{r},\bar{w},\bar{L})((\bar{w}-w)\partial_{\bar{r}} - (\psi'_{\bar{L}}(\bar{r}) - \psi'_L(r))\partial_{\bar{w}})\zeta_n(r-\bar{r}, w-\bar{w}, L-\bar{L})\,d\bar{L}\,d\bar{w}\,d\bar{r} \\
&= -n^4 \int (1_{S^r_m}h)(\bar{r},\bar{w},\bar{L}) \left(\frac{\tilde{w}}{n}\partial_{\tilde{r}} - (\psi'_{\bar{L}}(\bar{r}) - \psi'_L(r))\partial_{\tilde{w}}\right)\zeta(\tilde{r},\tilde{w},\tilde{L})\,d\bar{L}\,d\bar{w}\,d\bar{r} \\
&= -n \int_{B_1(0)} (1_{S^r_m}h)(\bar{r},\bar{w},\bar{L}) \left(\frac{\tilde{w}}{n}\partial_{\tilde{r}} - (\psi'_{\bar{L}}(\bar{r}) - \psi'_L(r))\partial_{\tilde{w}}\right)\zeta(\tilde{r},\tilde{w},\tilde{L})\,d\tilde{L}\,d\tilde{w}\,d\tilde{r}.
\end{aligned}$$

In the next step, we analyze the first term $(1_{S^r_m}h)(z-\frac{\tilde{z}}{n})$ with $z := (r,w,L)$ and $\tilde{z} := (\tilde{r},\tilde{w},\tilde{L})$ for convergence:

**Lemma 5.17.** *The convergence*

$$(1_{S^r_m}h)\Big(\cdot - \frac{\tilde{z}}{n}\Big) \to 1_{S^r_m}h \text{ in } L^2(\mathbb{R}^3)$$

*is uniformly in $\tilde{z} \in B_1(0)$.*

*Proof.* Let $\varepsilon > 0$. Since $1_{S^r_m}h \in L^2(\mathbb{R}^3)$, there exists $\xi \in C_c^\infty(\mathbb{R}^3)$ with

$$\|1_{S^r_m}h - \xi\|_2 < \varepsilon.$$

By a change of variables, we obtain that

$$\begin{aligned}
&\|(1_{S^r_m}h)\Big(\cdot - \frac{\tilde{z}}{n}\Big) - 1_{S^r_m}h\|_2 \\
&\le \|(1_{S^r_m}h)\Big(\cdot - \frac{\tilde{z}}{n}\Big) - \xi\Big(\cdot - \frac{\tilde{z}}{n}\Big)\|_2 + \|\xi\Big(\cdot - \frac{\tilde{z}}{n}\Big) - \xi\|_2 + \|1_{S^r_m}h - \xi\|_2 \\
&= 2\|1_{S^r_m}h - \xi\|_2 + \|\xi\Big(\cdot - \frac{\tilde{z}}{n}\Big) - \xi\|_2.
\end{aligned}$$

The mean value theorem applied to $\xi \in C_c^\infty(\mathbb{R}^3)$ leads to

$$\begin{aligned}
\int \Big|\xi\Big(z - \frac{\tilde{z}}{n}\Big) - \xi(z)\Big|^2\,dz &\le \int_{B_1(\operatorname{supp}\xi)} \|\nabla\xi\|^2_\infty \Big|z - \frac{\tilde{z}}{n} - z\Big|^2\,dz \\
&\le \operatorname{vol}(B_1(\operatorname{supp}\xi))\|\nabla\xi\|^2_\infty \frac{1}{n^2}, \quad n \in \mathbb{N}.
\end{aligned}$$

Note that $\tilde{z} \in B_1(0)$. So, there exists $n_0 \in \mathbb{N}$ with

$$\|(1_{S^r_m}h)\Big(\cdot - \frac{\tilde{z}}{n}\Big) - 1_{S^r_m}h\|_2 = 2\|1_{S^r_m}h - \xi\|_2 + \|\xi\Big(\cdot - \frac{\tilde{z}}{n}\Big) - \xi\|_2 < 3\varepsilon$$

for all $\tilde{z} \in B_1(0)$ and $n \ge n_0$. $\square$

Now, we turn to the second term of $J_{2,n}$ and show that it converges uniformly:

**Lemma 5.18.** *Let $\tilde{Q} := [\tilde{R}_0, \tilde{R}_1] \times [-\tilde{W}_0, \tilde{W}_0] \times [\tilde{L}_0, \tilde{L}_1]$ be a cuboid with $0 < \tilde{R}_0 < \tilde{R}_1$, $\tilde{W}_0 > 0$ and $0 < \tilde{L}_0 < \tilde{L}_1$ arbitrary. Then the convergence*

$$\begin{aligned} n(\psi'_{\bar{L}}(\bar{r}) - \psi'_L(r)) \to &- U_0''(r)\tilde{r} + 2\frac{M_0}{r^3}\tilde{r} - 3\frac{L}{r^4}\tilde{r} + \frac{\tilde{L}}{r^3} \\ &= -\psi''_L(r)(-\tilde{r}) + \frac{\tilde{L}}{r^3} \end{aligned}$$

*is uniformly in $(\tilde{r}, \tilde{w}, \tilde{L}) \in B_1(0)$ and $(r, w, L) \in \tilde{Q}$ as $n \to \infty$. Here, $\bar{r} = r - \frac{\tilde{r}}{n}$, $\bar{w} = w - \frac{\tilde{w}}{n}$ and $\bar{L} = L - \frac{\tilde{L}}{n}$ for $n \in \mathbb{N}$ and $(\tilde{r}, \tilde{w}, \tilde{L})$.*

*Proof.* Let $(\tilde{r}, \tilde{w}, \tilde{L}) \in B_1(0)$, $(r, w, L) \in \tilde{Q}$ and $(\bar{r}, \bar{w}, \bar{L})$ be as described. The mean value theorem leads to the identity

$$\begin{aligned} &n(\psi'_{\bar{L}}(\bar{r}) - \psi'_L(r)) = n\Big(U_0'(\bar{r}) + \frac{M_0}{\bar{r}^2} - \frac{\bar{L}}{\bar{r}^3} - U_0'(r) - \frac{M_0}{r^2} + \frac{L}{r^3}\Big) \\ &= n\Big(U_0'\Big(r - \frac{\tilde{r}}{n}\Big) - U_0'(r) + M_0\Big(\Big(r - \frac{\tilde{r}}{n}\Big)^{-2} - r^{-2}\Big) - \frac{\tilde{L}}{n(r - \frac{\tilde{r}}{n})^3} - L\Big(\Big(r - \frac{\tilde{r}}{n}\Big)^{-3} - r^{-3}\Big)\Big) \\ &= \frac{U_0'(r - \frac{\tilde{r}}{n}) - U_0'(r)}{\frac{-\tilde{r}}{n}}(-\tilde{r}) + M_0\frac{(r - \frac{\tilde{r}}{n})^{-2} - r^{-2}}{\frac{-\tilde{r}}{n}}(-\tilde{r}) + \frac{\tilde{L}}{(r - \frac{\tilde{r}}{n})^3} - L\frac{(r - \frac{\tilde{r}}{n})^{-3} - r^{-3}}{\frac{-\tilde{r}}{n}}(-\tilde{r}) \\ &= -U_0''(\xi_1)\tilde{r} + 2M_0\xi_2^{-3}\tilde{r} + \frac{\tilde{L}}{(r - \frac{\tilde{r}}{n})^3} - 3L\xi_3^{-4}\tilde{r} \end{aligned}$$

with $\xi_i \in [r - \frac{|\tilde{r}|}{n}, r + \frac{|\tilde{r}|}{n}] \subset [\frac{\tilde{R}_0}{2}, \tilde{R}_1 + 1]$ for $n \geq \frac{2}{\tilde{R}_0}$ and $i = 1, ..., 3$. Hence, this yields that

$$\begin{aligned} &\Big|n(\psi'_{\bar{L}}(\bar{r}) - \psi'_L(r)) + U_0''(r)\tilde{r} - 2\frac{M_0}{r^3}\tilde{r} + 3\frac{L}{r^4}\tilde{r} - \frac{\tilde{L}}{r^3}\Big| \\ &= \Big| - U_0''(\xi_1)\tilde{r} + 2M_0\xi_2^{-3}\tilde{r} + \frac{\tilde{L}}{(r - \frac{\tilde{r}}{n})^3} - 3L\xi_3^{-4}\tilde{r} + U_0''(r)\tilde{r} - 2\frac{M_0}{r^3}\tilde{r} + 3\frac{L}{r^4}\tilde{r} - \frac{\tilde{L}}{r^3}\Big| \\ &\leq |U_0''(\xi_1) - U_0''(r)||\tilde{r}| + 2M_0|\xi_2^{-3} - r^{-3}||\tilde{r}| + |\tilde{L}|\Big|\Big(r - \frac{\tilde{r}}{n}\Big)^{-3} - r^{-3}\Big| + 3L|\xi_3^{-4} - r^{-4}||\tilde{r}| \\ &\leq C\Big(|U_0''(\xi_1) - U_0''(r)| + |\xi_2^{-3} - r^{-3}| + \Big|\Big(r - \frac{\tilde{r}}{n}\Big)^{-3} - r^{-3}\Big| + |\xi_3^{-4} - r^{-4}|\Big) \end{aligned}$$

with $C = C(\tilde{Q}, M_0) > 0$. Since $|\xi_i - r| \leq \frac{|\tilde{r}|}{n} \leq \frac{1}{n}$, we obtain $\xi_i \to r$ and $r - \frac{\tilde{r}}{n} \to r$ converges uniformly. The maps $]0, \infty[\ni s \to U_0''(s)$, $]0, \infty[\ni s \to s^{-3}$ and $]0, \infty[\ni s \to s^{-4}$ are continuous and hence uniformly continuous on $[\frac{\tilde{R}_0}{2}, \tilde{R}_1 + 1]$, so it follows that

$$\begin{aligned} &\Big|n(\psi'_{\bar{L}}(\bar{r}) - \psi'_L(r)) + U_0''(r)\tilde{r} - 2\frac{M_0}{r^3}\tilde{r} + 3\frac{L}{r^4}\tilde{r} - \frac{\tilde{L}}{r^3}\Big| \\ &\leq C\Big(|U_0''(\xi_1) - U_0''(r)| + |\xi_2^{-3} - r^{-3}| + \Big|\Big(r - \frac{\tilde{r}}{n}\Big)^{-3} - r^{-3}\Big| + |\xi_3^{-4} - r^{-4}|\Big) \to 0. \end{aligned}$$

In particular, the above convergence is uniformly in $(r, w, L) \in \tilde{Q}$ and $(\tilde{r}, \tilde{w}, \tilde{L}) \in B_1(0)$ as $n \to \infty$. $\square$

In summary, Lemma 5.17 and 5.18 lead to the convergence of $J_{2,n}$:

**Lemma 5.19.** *The convergences*

$$J_{2,n} = \int (1_{S_m^r} h)(\bar z)((\bar w - w)\partial_{\bar r} - (\psi'_{\bar L}(\bar r) - \psi'_L(r))\partial_{\bar w})\zeta_n(z - \bar z)\ d\bar L\ d\bar w\ d\bar r$$
$$\to -(1_{S_m^r} h)(r,w,L)\int_{B_1(0)} \Big( -\tilde w \partial_{\tilde r} + \Big(\psi''_L(r)(-\tilde r) - \frac{\tilde L}{r^3}\Big)\partial_{\tilde w}\Big)\zeta(\tilde r, \tilde w, \tilde L)\ d\tilde L\ d\tilde w\ d\tilde r = 0$$

*holds in $L^2(\mathbb{R}^3)$ with $z := (r,w,L)$ for $(r,w,L) \in \mathbb{R}^3$.*

*Proof.* As discussed before, we obtain that

$$\begin{aligned} J_{2,n} &= \int (1_{S_m^r} h)(\bar z)((\bar w - w)\partial_{\bar r} - (\psi'_{\bar L}(\bar r) - \psi'_L(r))\partial_{\bar w})\zeta_n(z - \bar z)\ d\bar L\ d\bar w\ d\bar r \\ &= -n\int_{B_1(0)} (1_{S_m^r} h)(\bar r, \bar w, \bar L)\Big(\frac{\tilde w}{n}\partial_{\tilde r} - (\psi'_{\bar L}(\bar r) - \psi'_L(r))\partial_{\tilde w}\Big)\zeta(\tilde r, \tilde w, \tilde L)\ d\tilde L\ d\tilde w\ d\tilde r \\ &= -\int_{B_1(0)} (1_{S_m^r} h)\Big(z - \frac{\tilde z}{n}\Big)(\tilde w \partial_{\tilde r} - n(\psi'_{\bar L}(\bar r) - \psi'_L(r))\partial_{\tilde w})\zeta(\tilde r, \tilde w, \tilde L)\ d\tilde L\ d\tilde w\ d\tilde r \\ &= -\int_{B_1(0)} (1_{S_m^r} h)\Big(z - \frac{\tilde z}{n}\Big) K_n(z, \tilde z)\ d\tilde L\ d\tilde w\ d\tilde r \end{aligned}$$

with $K_n(z,\tilde z) := \tilde w \partial_{\tilde r}\zeta(\tilde z) - n(\psi'_{\bar L}(\bar r) - \psi'_L(r))\partial_{\tilde w}\zeta(\tilde r, \tilde w, \tilde L)$ and $\bar z = z - \frac{\tilde z}{n}$ for $z, \tilde z \in \mathbb{R}^3$ and $n \in \mathbb{N}$ large enough. We define $K(z,\tilde z) := \tilde w \partial_{\tilde r}\zeta(\tilde z) - (\psi''_L(r)(-\tilde r) + \frac{\tilde L}{r^3})\partial_{\tilde w}\zeta(\tilde r, \tilde w, \tilde L)$ for $z, \tilde z \in \mathbb{R}^3$. Since $S_m^r \subset [\bar R_0, \bar R_1] \times [-\bar W_0, \bar W_0] \times [\bar L_0, \bar L_1]$, the support of $1_{S_m^r} h$ satisfies $\mathrm{supp}(1_{S_m^r} h)(\cdot - \frac{\tilde z}{n}) \subset \tilde Q$ with

$$\tilde Q := \Big[\frac{\bar R_0}{2}, \bar R_1 + 1\Big] \times \big[-\bar W_0 - 1, \bar W_0 + 1\big] \times \Big[\frac{\bar L_0}{2}, 2\bar L_1 + 1\Big]$$

for $n \in \mathbb{N}$ large enough and $\tilde z \in B_1(0)$. Lemma 5.18 shows that

$$K_n(z,\tilde z) \to K(z,\tilde z) \text{ uniformly in } z \in \tilde Q,\ \tilde z \in B_1(0),$$

and Lemma 5.17 says

$$(1_{S_m^r} h)\Big(\cdot - \frac{\tilde z}{n}\Big) \to 1_{S_m^r} h \text{ in } L^2(\mathbb{R}^3) \text{ uniformly in } \tilde z \in B_1(0).$$

Note that $\mathrm{supp}\,\zeta \subset B_1(0)$, so $\mathrm{supp}\,K(z,\cdot), \mathrm{supp}\,K_n(z,\cdot) \subset B_1(0)$ for $z \in \tilde Q$ and $n \in \mathbb{N}$. Using Hölder's inequality, this leads to the following convergence:

$$\begin{aligned} &\int \Big| \int_{B_1(0)} (1_{S_m^r} h)\Big(z - \frac{\tilde z}{n}\Big) K_n(z,\tilde z)\ d\tilde z - \int (1_{S_m^r} h)(z) K(z,\tilde z)\ d\tilde z\Big|^2\ dz \\ &= \int \Big| \int_{B_1(0)} \Big((1_{S_m^r} h)\Big(z - \frac{\tilde z}{n}\Big) - (1_{S_m^r} h)(z)\Big) K_n(z,\tilde z)\ d\tilde z \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{B_1(0)} (1_{S_m^r} h)(z)(K_n(z,\tilde z) - K(z,\tilde z))\, d\tilde z\Big|^2\, dz \\
&\le 4\Big( \int_{\tilde Q} \Big| \int_{B_1(0)} \Big( (1_{S_m^r} h)(z - \frac{\tilde z}{n}) - (1_{S_m^r} h)(z) \Big) K_n(z,\tilde z)\, d\tilde z \Big|^2\, dz \\
&\quad + \int_{\tilde Q} \Big| \int_{B_1(0)} (1_{S_m^r} h)(z)(K_n(z,\tilde z) - K(z,\tilde z))\, d\tilde z\Big|^2\, dz \Big) \\
&\le 4\Big( \int_{\tilde Q} \int_{B_1(0)} \Big| (1_{S_m^r} h)\Big(z - \frac{\tilde z}{n}\Big) - (1_{S_m^r} h)(z) \Big|^2\, d\tilde z \int_{B_1(0)} |K_n(z,\tilde z)|^2\, d\tilde z\, dz \\
&\quad + \int_{\tilde Q} \int_{B_1(0)} |(1_{S_m^r} h)(z)|^2\, d\tilde z \int_{B_1(0)} |K_n(z,\tilde z) - K(z,\tilde z)|^2\, d\tilde z\, dz \Big) \\
&\le 4\Big( \mathrm{supp}_{z\in\tilde Q, \tilde z \in B_1(0), n\in\mathbb N} |K_n(z,\tilde z)|^2 \int_{B_1(0)} \|(1_{S_m^r} h)\Big(\cdot - \frac{\tilde z}{n}\Big) - (1_{S_m^r} h)\|_2^2\, d\tilde z \\
&\quad + \mathrm{supp}_{z\in\tilde Q, \tilde z\in B_1(0)} |K_n(z,\tilde z) - K(z,\tilde z)|^2 \|(1_{S_m^r} h)\|_2^2 \Big) \to 0, \quad n\to\infty.
\end{aligned}
$$

Note that $K_n$ is bounded independently of $n \in \mathbb N$ since $K_n(z,\tilde z)$ converges to $K(z,\tilde z)$ uniformly in $\tilde z \in \operatorname{supp}\zeta$ and $z \in \operatorname{supp}\tilde Q$. Furthermore, the sets $\operatorname{supp}\zeta \subset B_1(0)$ and $\tilde Q$ are compact. Moreover, $\|(1_{S_m^r} h)(\cdot - \frac{\tilde z}{n}) - (1_{S_m^r} h)\|_2$ converges to zero uniformly in $\tilde z \in B_1(0)$, so the claimed convergence is proven.

It remains to evaluate the limiting integral. Since $\zeta \in C_c^\infty(\mathbb R^3)$ with $\operatorname{supp}\zeta \subset B_1(0)$, integrating by parts yields

$$
\begin{aligned}
&\int \Big( -\tilde w \partial_{\tilde r} + \Big(\psi_L''(r)(-\tilde r) - \frac{\tilde L}{r^3}\Big)\partial_{\tilde w}\Big)\zeta(\tilde z)\, d\tilde z \\
&= -\int \tilde w \partial_{\tilde r}\zeta(\tilde r, \tilde w, \tilde L)\, d\tilde r\, d\tilde w\, d\tilde L + \int \Big(\psi_L''(r)(-\tilde r) - \frac{\tilde L}{r^3}\Big)\partial_{\tilde w}\zeta(\tilde r,\tilde w,\tilde L)\, d\tilde w\, d\tilde r\, d\tilde L \\
&= \int \partial_{\tilde r}(\tilde w)\zeta(\tilde r,\tilde w,\tilde L)\, d\tilde r\, d\tilde w\, d\tilde L - \int \partial_{\tilde w}\Big(\psi_L''(r)(-\tilde r) - \frac{\tilde L}{r^3}\Big)\zeta(\tilde r,\tilde w,\tilde L)\, d\tilde w\, d\tilde r\, d\tilde L = 0,
\end{aligned}
$$

so the proof is complete. □

Hence, we have shown that $\lim_{n\to\infty}\{-E, h_n\} = 1_{S_m^r} g$ in $L^2([0,\infty[\times\mathbb R\times[0,\infty[)$ and thus in $L^2(\mathbb R^6)$. In summary, for fixed $m \in \mathbb N$ we have constructed a sequence $(h_n)$ of spherically symmetric functions which are odd in $v$. Furthermore, $h_n \in C_c^\infty(\mathbb R^3\backslash\{0\}\times\mathbb R^3)$ and $\operatorname{supp} h_n \subset S_m$ for $n$ large enough. Note that $S_m \subset Q \cap \{E < E_0\} \cap \{L > L_0\} \subset \{f_0 > 0\}$. In particular, $h_n$ satisfies the conditions in Lemma 5.6 for $n \in \mathbb N$ large enough.

## 5.6 The contradiction

In order to obtain the convergence of $\mathrm D^2\mathcal H_{\mathcal C}(f_0)[\{-E, h_n\}]$ and then exploit that the expression is positive as established in Lemma 5.6, it remains to examine $\nabla U_{\{-E,h_n\}}$ for convergence:

**Lemma 5.20.** *Let $K$ be a compact set in $\mathbb{R}^6$ and $f_n, f \in L^2(\mathbb{R}^6)$ with $\operatorname{supp} f_n, \operatorname{supp} f \subset K$ for $n \in \mathbb{N}$. Assume that $f_n \to f$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$. Then the gradients of the induced potentials exist with $\nabla U_{f_n}, \nabla U_f \in L^2(\mathbb{R}^3)$ for $n \in \mathbb{N}$ and*

$$\nabla U_{f_n} \to \nabla U_f \ \text{in}\ L^2(\mathbb{R}^6).$$

*Proof.* Since $K \subset \mathbb{R}^6$ is compact and $f_n, f \in L^1 \cap L^2(\mathbb{R}^3)$ with $\operatorname{supp} f, f_n \subset K$, we obtain $f, f_n \in L^1 \cap L^2(\mathbb{R}^6)$. Moreover, $\rho_f, \rho_{f_n} \in L^1 \cap L^2(\mathbb{R}^3)$, and there exists a compact set $K' \subset \mathbb{R}^3$ such that $\operatorname{supp} \rho_{f_n}, \operatorname{supp} \rho \subset K'$ for $n \in \mathbb{N}$. In particular, $\rho_f, \rho_{f_n} \in L^{\frac{6}{5}}(\mathbb{R}^3)$, so Lemma 2.8 yields that $\nabla U_f, \nabla U_{f_n} \in L^2(\mathbb{R}^3)$ exist for $n \in \mathbb{N}$, and the Hardy-Littlewood-Sobolev lemma (cf. [7, Theorem 4.3]) yields that

$$\begin{aligned}
&\|\nabla U_{f_n} - \nabla U_f\|_2^2 \le C\|\rho_{f_n} - \rho_f\|_{\frac{6}{5}}^2 = C\Big(\int \Big| \int f_n(x,v) - f(x,v)\, dv\Big|^{\frac{6}{5}}\, dx\Big)^{\frac{10}{6}} \\
&\le C\Big(\iint |f_n(x,v) - f(x,v)|^{\frac{6}{5}}\, dv\, dx\Big)^{\frac{10}{6}} \\
&\le C\Big(\iint |f_n(x,v) - f(x,v)|^2\, dv\, dx\Big) = C\|f_n - f\|_2^2 \to 0, \quad n \to \infty. \qquad \square
\end{aligned}$$

As discussed in the previous section, $\{-E, h_n\} \to 1_{S_m} g$ converges in $L^2(\mathbb{R}^6)$ with $\operatorname{supp} h_n \subset Q$ and $\operatorname{supp}(1_{S_m} g) \subset S_m \subset Q$. According to Lemma 5.20, the gradients of the induced potentials $\nabla U_{\{-E,h_n\}}$, $\nabla U_{1_{S_m} g} \in L^2(\mathbb{R}^3)$ exist for $n \in \mathbb{N}$ with

$$\nabla U_{\{-E,h_n\}} \to U_{1_{S_m} g} \text{ in } L^2(\mathbb{R}^6).$$

In summary, this culminates in the convergence

$$\begin{aligned}
&\mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[\{-E,h_n\}] = \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0,L)|\{-E,h_n\}|^2\, dv\, dx - \frac{1}{8\pi}\int |\nabla U_{\{-E,h_n\}}|^2\, dx \\
&\to \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0,L)|1_{S_m} g|^2\, dv\, dx - \frac{1}{8\pi}\int |\nabla U_{1_{S_m} g}|^2\, dx = \mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[1_{S_m} g]
\end{aligned}$$

as $n \to \infty$. Note that $\operatorname{supp}\{-E, h_n\} \subset S_m$ with $S_m \subset \{f_0 > 0\}$ for $n \in \mathbb{N}$, so $\Psi''(f_0, L) = |\Phi'(E,L)|^{-1}$ is bounded on $S_m$ and the above convergence is legitimate.

Since $g \in L^2(\mathbb{R}^6)$ with $\operatorname{supp} g \subset \operatorname{supp} f_0$, as proven in Lemma 5.2 (g2), and $\operatorname{supp} f_0$ is compact by assumption, we obtain $g \in L^1 \cap L^2(\mathbb{R}^6)$. Furthermore, the set $(S_m)$ is increasing with $\bigcup_{m\in\mathbb{N}} S_m = \{f_0 > 0\}$, so the monotone convergence theorem implies $\lim_{m\to\infty} 1_{S_m} g = g$ in $L^1 \cap L^2(\mathbb{R}^6)$. By assumption, $\operatorname{supp} f_0$ is compact, and thus again Lemma 5.20 yields that

$$\begin{aligned}
&\mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[1_{S_m} g] = \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0,L)|1_{S_m} g|^2\, dv\, dx - \frac{1}{8\pi}\int |\nabla U_{1_{S_m} g}|^2\, dx \\
&\to \frac{1}{2}\iint_{\{f_0>0\}} \Psi''(f_0,L)|g|^2\, dv\, dx - \frac{1}{8\pi}\int |\nabla U_g|^2\, dx = \mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[g]
\end{aligned}$$

as $m \to \infty$.

Since $\frac{1}{8\pi}\|\nabla U_g\|_2^2 = 1$ by Lemma 5.2 (g4), there exists $m_0 \in \mathbb{N}$ such that $1_{S_{m_0}} g \neq 0$. In particular, we obtain $1_{S_{m_0}} h \neq 0$ because otherwise Lemma 5.16 would imply $1_{S_{m_0}} g = 0$.

Since $\{f_0 > 0\} \neq \emptyset$ is an open set, the mass function $m(r)$ and thus $U_0'(r) = \frac{m(r)}{r^2}$ are positive for $(x, v) \in \{f_0 > 0\}$. Finally, we apply Lemma 5.6 and obtain that

$$\begin{aligned}
\mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[\{-E, h_n\}] &\geq \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}\left(|x\cdot v|^2 |\left\{-E, \frac{h_n}{x\cdot v}\right\}|^2 + \frac{m(r)+M_0}{r^3} h_n^2\right) dv\, dx \\
&\geq \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}\frac{m(r)}{r^3} h_n^2\, dv\, dx \\
&\to \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}\frac{m(r)}{r^3} 1_{S_m} h^2\, dv\, dx
\end{aligned}$$

as $n \to \infty$. Note $|\Phi'(E,L)|^{-1}$ and $\frac{m(r)}{r^3}$ are bounded on $S_m \subset Q$, so the last convergence follows by the convergence $h_n \to h$ in $L^2(\mathbb{R}^6)$. By construction, $S_{m_0} \subset S_m$ for all $m \geq m_0$. We apply all assertions about convergence which we have discussed before, and thus we conclude

$$\begin{aligned}
&\mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[1_{S_m} g] = \lim_{n\to\infty} \mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[\{-E, h_n\}] \\
&\geq \lim_{n\to\infty}\frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}\frac{m(r)}{r^3} h_n^2\, dv\, dx = \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}\frac{m(r)}{r^3} 1_{S_m} h^2\, dv\, dx \\
&\geq \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}\frac{m(r)}{r^3} 1_{S_{m_0}} h^2\, dv\, dx > 0, \quad m \geq m_0.
\end{aligned}$$

Finally, Lemma 5.2 (g5) and the limit $m \to \infty$ yield the desired contradiction:

$$\begin{aligned}
0 \geq \mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[g] &= \lim_{m\to\infty} \mathrm{D}^2\mathcal{H}_{\mathcal{C}}(f_0)[1_{S_m} g] \\
&\geq \frac{1}{2}\iint \frac{1}{|\Phi'(E,L)|}\frac{m(r)}{r^3} 1_{S_{m_0}} h^2\, dv\, dx > 0.
\end{aligned}$$

By this, the proof of Theorem 4.9 is complete.

Note that we did not use the assumption $M_0 > 0$ in this whole chapter, so the assertions remain valid for $M_0 = 0$.

# 6 The existence of strong Lagrangian solutions

In Theorem 4.10, we claim that for all $\mathring{f} \in \mathcal{D}_{f_0}$, as defined in (4.3), there exists a (global) solution $f$ of the Vlasov-Poisson system with a point mass with initial condition $f(0) = \mathring{f}$. In the previous chapters, we consider anisotropic spherically symmetric steady states of the form $f_0 = \Phi(E, L)$ with

$$\Phi(E, L) = 0, \quad 0 \leq L \leq L_0, \ E \in \mathbb{R}$$

for some $L_0 > 0$. Let $\mathring{f} \in \mathcal{D}_{f_0}$. Then there exists a measure preserving $C^1$-diffeomorphism $T$ which respects spherical symmetry, as defined in Definition 4.6, such that $\mathring{f} = f_0 \circ T$. Since $f_0$ is spherically symmetric, so is $\mathring{f}$. The angular momentum squared $L$ is invariant under diffemorphisms which respect spherical symmetry, so

$$\mathring{f}(x, v) = 0, \quad 0 \leq L(x, v) \leq L_0.$$

Among other things, we require $f_0 \in C_c(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$ in the previous chapters which yields $\mathring{f} \in C_c(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$.

To show the existence of solutions of the Vlasov-Poisson system with a point mass, we consider the following two papers: In [11], the global existence of solutions of the Vlasov-Poisson system with a point mass is shown for smooth spherically symmetric initial conditions with cut-off $L_0$, whereas in [6] the global existence of so-called strong Lagrangian solutions of the Vlasov-Poisson system with spherically symmetric initial conditions is proven. Combining the methods from these two papers appropriately, we can show that there exists a unique global strong Lagrangian solution of the Vlasov-Poisson system with a point mass for suitable initial data. First, we define the term *strong Lagrangian solution* analogously to [6, Definition 2.1]:

**Definition 6.1.** *A solution $f : I \times \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 \to [0, \infty[$ of the Vlasov-Poisson system with a point mass with $I$ an interval is called a* strong Lagrangian solution *if $\partial_x U$ is Lipschitz continuous in $x$ locally uniformly in $t$, i.e., for all compact intervals $J \subset I$ there exists $C^* > 0$ such that*

$$|\partial_x U(t, x) - \partial_x U(t, x')| \leq C^* |x - x'|$$

*for all $x, x' \in \mathbb{R}^3\backslash\{0\}$ and $t \in J$.*

Combing [11, Theorem 2.1] and [6, Theorem 2.2], we obtain the global existence of strong Lagrangian solutions and furthermore their uniqueness:

**Theorem 6.2.** *Let* $\mathring{f} \in C_c(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$ *be spherically symmetric and non-negative. In addition, assume that there exists* $L_0 > 0$ *such that*

$$\mathring{f}(x, v) = 0, \quad 0 \leq L(x, v) \leq L_0.$$

*Then there exists a unique continuous spherically symmetric strong Lagrangian solution* $f : [0, \infty[\times\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3 \to [0, \infty[$ *with* $f(0) = \mathring{f}$*. Furthermore,* $f(t) \in C_c(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$ *for* $t \geq 0$ *and*

$$f(t, x, v) = 0, \quad 0 \leq L(x, v) \leq L_0.$$

*Proof.* Analogously to [6], we define the following iteration: The 0th iterate of the potential is given by

$$U_0(t, x) := 0, \quad x \in \mathbb{R}^3,\ t \geq 0.$$

Additionally, we define

$$P_{-1}(t) := 0, \quad t \geq 0.$$

Assume that the $n$th iterate $U_n \in C^{0,2}([0, \infty[\times\mathbb{R}^3)$ of the potential and $P_{n-1}(t)$ is already defined for $t \geq 0$ and for some $n \in \mathbb{N}_0$. Furthermore, we assume that $\partial_x U_n(t)$ is spherically symmetric for $t \geq 0$ and $\partial_x U_n$ is globally Lipschitz continuous in $x$ locally uniformly in $t$. In addition, we require that

$$\|\partial_x U_n(t)\|_\infty \leq C_{\mathring{f}} P^2_{n-1}(t), \quad t \geq 0,$$

with $C_{\mathring{f}} := 4 \cdot 3^{\frac{1}{3}} \pi^{\frac{4}{3}} \|\mathring{f}\|_1^{\frac{1}{3}} \|\mathring{f}\|_\infty^{\frac{2}{3}}$ and that $P_{n-1}$ and thus $\partial_x U_n$ is bounded on $[0, T']$ for every $T' > 0$.

Hence, $\partial_x U_{n,\text{eff}} := U_n - \frac{M_0}{|x|}$ is locally Lipschitz continuous in $x$, so there exists a unique solution $Z_n(\cdot, t, z)$ for all $t \geq 0$ and $z \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ of the characteristic system

$$\dot{X}_n = V_n, \quad \dot{V}_n = -\partial_x U_{n,\text{eff}}$$

with $Z(t, t, z) = z$. Later, we will show that the characteristics exist globally for $z = (x, v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ with $L(x, v) > 0$ and $t \geq 0$. Then we define the $n$th iterate $f_n$ is given by

$$f_n(t, z) := \begin{cases} \mathring{f}(Z_n(0, t, z)), & z = (x, v) \text{ with } |x \times v|^2 > 0, \\ 0, & \text{else}, \end{cases}$$

and the induced density by

$$\rho_n(t, x) := \rho_{f_n(t)}(x) = \int f_n(t, x, v)\, dv, \quad x \in \mathbb{R}^3,\ t \geq 0.$$

We define the $(n+1)$st iterate $U_{n+1}$ of the potential by

$$U_{n+1}(t,x) := U_{\rho_n(t)}(x) = -\int \frac{\rho_n(t,y)}{|x-y|}\, dy, \quad x \in \mathbb{R}^3,\ t \geq 0,$$

and

$$P_n(t) := \operatorname{supp}\{|V_n(s,0,z)|\,|z \in \operatorname{supp} \mathring{f},\ 0 \leq s \leq t\}, \quad t \geq 0.$$

Before we turn to the convergence of the iteration, we verify the assumptions we made in the iteration by induction. Obviously, the iterate $U_0$ has the required properties. Assume that $U_n$ for $n \in \mathbb{N}_0$ has the required properties. In the following steps, we show that the iteration is well-defined and that $U_{n+1}$ satisfies the assumptions in the iteration as well.

**Step 1:** The existence of the characteristic flow $Z_n$ on $\{L > 0\}$ and the boundedness of $P_n$.
Since $U_n \in C^{0,2}([0,\infty[\times\mathbb{R}^3\backslash\{0\})$, there exists a unique solution $Z_n(\cdot,t,z)$ of the characteristic system for every $t \geq 0$ and $z \in \{L > 0\}$. Similar to [11], we introduce following quantity:

$$\begin{aligned}
\bar{P}_n(t) &:= \operatorname{supp}\{|v|\,|(x,v) \in \operatorname{supp} \mathring{f}_k(s),\ 0 \leq s \leq t,\ 1 \leq k \leq n\} \\
&\quad = \operatorname{supp}\{|V_k(s,0,z)|\,|z \in \operatorname{supp} \mathring{f},\ 0 \leq s \leq t,\ 1 \leq k \leq n\}, \\
\bar{R}^n_{\min}(t) &:= \inf\{|x|\,|(x,v) \in \operatorname{supp} f_n(s),\ 0 \leq s \leq t\} \\
&\quad = \inf\{|X_n(s,0,z)|\,|z \in \operatorname{supp} \mathring{f},\ 0 \leq s \leq t\}.
\end{aligned}$$

By assumption,

$$\|\partial_x U_n(t)\|_\infty \leq C_{\mathring{f}} P^2_{n-1}(t) \leq C_{\mathring{f}} \bar{P}^2_n(t), \quad t \geq 0,$$

so we can literally transfer the proof of [11] and obtain

$$\bar{P}_n(t) \leq \mathring{P} + 4\sqrt{6C_*}\,\bar{P}_n^{\frac{1}{2}}(t), \quad t \geq 0,$$

and

$$\bar{R}^n_{\min}(t) \geq \frac{\sqrt{L_0}}{\bar{P}_n(t)}, \quad t \geq 0,$$

with $C_* = C_*(\|\mathring{f}\|_1, \|\mathring{f}\|_\infty, M_0, L_0) > 0$. Therefore, there exist $C_{0,1} = C_{0,1}(\mathring{f}, M_0, L_0) > 0$ with

$$P_n(t) \leq C_0, \quad R^n_{\min}(t) \geq C_1, \quad t > 0.$$

Note that $C_{0,1}$ is independent of $n \in \mathbb{N}$. These estimates yield that the characteristics $Z_n(\cdot,0,z)$ with $z \in \operatorname{supp} f_0$ exist globally. It remains to show the global existence of the characteristics $Z_n(\cdot, 0, z)$ for general $z \in \{L > 0\}$. Hence, we fix $R > 0$ and replace $\operatorname{supp} \mathring{f}$ in the definition of $\bar{P}_n$ and $\bar{R}^n_{\min}$ with the compact set

$$K_R := \left\{ L \geq \frac{1}{R^2} \right\} \cap (\bar{B}_R(0) \times \bar{B}_R(0)).$$

The same procedure as described in [11] yields constants $\tilde{C}_{0,1} = \tilde{C}_{0,1}(f_0, M_0, R)$ with $P_n(t) \leq \tilde{C}_0$ and $R^n_{\min} \geq \tilde{C}_1$, so $Z_n(\cdot, 0, z)$ exists globally for $z \in K_R$. Since $R > 0$ is arbitrary, we obtain the global existence of the characteristics $Z(\cdot, 0, z)$ for $z \in \{L > 0\}$. If we repeat the whole procedure by replacing $V_k(s, 0, z)$ and $X_n(s, 0, z)$ with $V_k(s, t, z)$ and $X_n(s, t, z)$ for $t > 0$, it follows that characteristic flow $Z : [0, \infty[\times[0, \infty[\times\{L > 0\} \to \{L > 0\}$ exists.

**Step 2:** The spherical symmetry of $U_{n+1}$ and $U_{n+1} \in C^{0,2}([0, \infty[\times\mathbb{R}^3)$.
By assumption, $U_n \in C^{0,2}([0, \infty[\times\mathbb{R}^3)$ is spherically symmetric, so the characteristic flow $Z_n$ is continuous on $[0, \infty[\times[0, \infty[\times\{L > 0\}$ by Lemma 2.11 (a). Therefore, $f_n$ is continuous on $[0, \infty[\times\{L > 0\}$. Recall that $\mathring{f} \in C_c(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$ and $\mathring{f}(x, v) = 0$ for $(x, v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ with $0 \leq L(x, v) < L_0$. Since $L$ is constant along characteristics as shown in Lemma 2.13, we obtain $f_n(t, x, v) = 0$ for $(x, v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ with $0 \leq L(x, v) \leq L_0$, so $f_n$ is continuous. We discussed in Lemma 4.7 that the characteristic flow $Z_n$ respects spherical symmetry, so $f_n$ is spherically symmetric. Remark 2.14 (a) and the fact that $\operatorname{supp} \mathring{f} \subset \{L \geq L_0\}$ is compact yield that $\operatorname{supp} f_n(t) = Z_n(t, 0, \operatorname{supp} \mathring{f}) \subset \{L \geq L_0\}$, so $f_n(t) \in C_c(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$ with $\operatorname{supp} f_n \subset \{L \leq L_0\}$ for $t \geq 0$. Thus, $\rho_n$ is spherically symmetric and continuous with $\rho_n(t) \in L^1 \cap L^\infty(\mathbb{R}^3)$, so $U_n \in C^{0,2}([0, \infty[\times\mathbb{R}^3\backslash\{0\})$ is spherically symmetric by Lemma 2.9.

**Step 3:** The estimate $\|\partial_x U_{n+1}(t)\|_\infty \leq C_{\mathring{f}} P_n^2(t)$ for $t \geq 0$.
As discussed in Step 2, $\rho_n(t) \in L^1 \cap L^\infty(\mathbb{R}^3)$, so Lemma 2.8 (c) and Remark 2.14 (c) yield that

$$\begin{aligned}
\|\nabla U_{n+1}(t)\|_\infty &\leq 3(2\pi)^{\frac{2}{3}} \|\rho_n(t)\|_1^{\frac{1}{3}} \|\rho_n\|_\infty^{\frac{2}{3}} \leq 3(2\pi)^{\frac{2}{3}} \left( \frac{4\pi}{3} P^3(t) \right)^{\frac{2}{3}} \|\mathring{f}\|_1^{\frac{1}{3}} \|\mathring{f}\|_\infty^{\frac{2}{3}} \\
&= 4 \cdot 3^{\frac{1}{3}} \pi^{\frac{4}{3}} \|\mathring{f}\|_1^{\frac{1}{3}} \|\mathring{f}\|_\infty^{\frac{2}{3}} P_n^2(t) = C_{\mathring{f}} P_n^2(t), \quad t \geq 0.
\end{aligned}$$

**Step 4:** The gradient $\partial_x U_{n+1}$ is globally Lipschitz continuous in $x$ locally uniformly in $t$.
As discussed in Step 2, $U_{n+1} \in C^{0,2}([0, \infty[\times\mathbb{R}^3\backslash\{0\})$ and

$$\partial_x U_{n+1}(t, x) = G_{n+1}(t, r) \frac{x}{r} \text{ with } G_{n+1}(t, r) := \frac{4\pi}{r^2} \int_0^r s^2 \rho_n(t, s)\, ds$$

for $x \in \mathbb{R}^3\backslash\{0\}$ and $t \geq 0$, as discussed in [6, Equation (3.4)]. Note that $G_{n+1}(t, r) = U'_{n+1}(t, r)$ for $r > 0$ and $t \geq 0$. We obtain the same estimate as in [6, Equation (3.5)] for

$0 < u < r$:

$$|G_{n+1}(t,r) - G_{n+1}(t,u)| \leq \frac{4\pi}{r^2} \int_r^u s^2 \rho_n(t,s)\ ds + 4\pi \Big| \frac{1}{r^2} - \frac{1}{u^2} \Big| \int_0^u s^2 \rho_n(t,s)\ ds$$
$$\leq \frac{20\pi}{3} \|\rho_n(t)\|_\infty |r-u|.$$

Since $P_n(t)$ is bounded independently of $t$ according to Step 2, $\|\rho_n(t)\|_\infty$ is also bounded independently of $t$ by Remark 2.14 (c). Just like in [6], it follows directly that $\partial_x U_n$ is Lipschitz in $x$ locally uniformly in $t$ by the above estimate. In particular, we obtain that $\partial_x U_n$ is globally Lipschitz continuous in $x$ uniformly in $t$.

In summary, the above iteration is well-defined. Now we show that the iterates converge. As shown in step 2, the quantities $\bar{P}_n(t)$ are bounded by $C_0$ independently of $n \in \mathbb{N}$ and $t \geq 0$, so

$$P_n(t) \leq \bar{P}_n(t) \leq C_0, \quad t \geq 0,\ n \in \mathbb{N}.$$

Therefore, we neglect the function $Q$ which is introduced in [6, Equation (3.6)] to bound $P_n$ independently of $n \in \mathbb{N}$ and use $C_0$ instead. The constant $C_0$ yields boundedness on $[0,\infty[$ instead of on some interval $[0,\delta[$ which simplifies the proof here slightly.

**Step 5:** The convergence of $f_n$ and the limiting function $f$.
Let $\delta > 0$ be arbitrary. We consider the compact subset $[0,\delta] \subset [0,\infty[$. As discussed above, the induced potential $U_n \in C^{0,2}([0,\infty[\times\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$ is spherically symmetric, so the characteristic flow $Z_n(t,0,\cdot)$ is a $C^1$-diffeomorphism on $\{L > 0\}$ according to Lemma 2.11. Therefore, the change of variables used in the proof in [6] is allowed, and we obtain analogously to [6, Equation (3.12)] that

$$\|G_{n+1}(t) - G_n(t)\|_\infty \leq C^* \int_0^t \|G_n(s) - G_{n-1}(s)\|_\infty\ ds$$

and, in particular,

$$\|\partial_x U_{n+1}(t) - \partial_x U_n(t)\|_\infty \leq C^* \int_0^t \|\partial_x U_n(s) - \partial_x U_{n-1}(s)\|_\infty\ ds$$

for $t \in [0,\delta]$ and $n \in \mathbb{N}$. The constant $C^* > 0$ is independent of $n \in \mathbb{N}$ and $t \in [0,\delta]$, but may depend on $\delta$. As a result, the sequence $(\partial_x U_n)$ is a Cauchy sequence in $(C([0,\delta] \times \mathbb{R}^3), \|\cdot\|_\infty)$, so there exists a continuous map $F : [0,\delta] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that $\partial_x U_n \to F$ converges uniformly on $[0,\delta] \times \mathbb{R}^3$. Furthermore, $F$ is spherically symmetric and, analogously to [6], Lipschitz continuous in $x$ locally uniformly in $t$. As proven in [6, Equation (3.7)],

$$|Z_{n+1}(t,0,z) - Z_n(t,0,z)| \leq C \int_0^t \|G_{n+1}(s) - G_n(s)\|_\infty\ ds$$

for $n \in \mathbb{N}$, $z = (x, v) \in \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ with $L(x, v) > 0$ and $t \geq 0$. The uniform convergence $\partial_x U_n \to F$ on $[0, \delta] \times \mathbb{R}^3$ yields that $(Z_n)$ and $(\dot{Z}_n)$ converges to some $\tilde{Z}$ and $\dot{\tilde{Z}}$, respectively, uniformly on $[0, \delta] \times \{L > 0\}$. Since $F$ is continuous and globally Lipschitz continuous in $x$ locally uniformly in $t$, there exists a maximal solution $Z(\cdot, t, z) : I \to \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ of the system

$$\dot{X} = V, \quad \dot{V} = -F(t, X) + \frac{M_0}{|X|^2} \frac{X}{|X|}$$

with $Z(t, t, z) = z$ and $I = I(t, z)$ an interval for $z \in \{L > 0\}$ and $t \in [0, \delta]$. As the uniform limit of $(Z_n)$, the limit $\tilde{Z}$ also solves the above system with $\tilde{Z}(t, t, z) = z$ for $z \in \{L > 0\}$ and $0 \leq t \leq \delta$. The uniqueness yields $Z = \tilde{Z}$, so $Z(\cdot, t, z)$ exists on $[0, \delta]$ for $z \in \{L > 0\}$ and $t \in [0, \delta]$. To this end, Step 1 implies $|X(s, 0, z)| \geq C_1$ for $s \in [0, \delta]$ and $z \in \operatorname{supp} \mathring{f}$. Furthermore, the characteristic flow $Z$ is continuous on $[0, \delta] \times [0, \delta] \times \{L > 0\}$ as shown in [6, Lemma 3.1].

Because $\mathring{f} \in C_c(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$ is continuous, we obtain for $t \in [0, \delta]$

$$f(t, z) := \lim_{n\to\infty} f_n(t, z) = \lim_{n\to\infty} \mathring{f}(Z_n(t, 0, z)) = \mathring{f}(Z(0, t, z))$$

for $z \in \{L > 0\}$ and $f(t, z) := \lim_{n\to\infty} f_n(t, z) = 0$ otherwise. Since the characteristic flow is continuous and $\operatorname{supp} \mathring{f} \subset \{L \geq L_0\}$, the limiting function $f$ is continuous with compact support $\operatorname{supp} f(t) \subset \{L \geq L_0\}$ for $0 \leq t \leq \delta$. Analogously to [6], we can show that

$$G(t, r) := \lim_{n\to\infty} G_n(t, r) = \frac{4\pi}{r^2} \int_0^r s^2 \rho(t, s)\, ds$$

with $\rho = \rho_f$ for $t \in [0, \delta]$ and $r > 0$. Similar arguments as in Step 2 lead to

$$F(t, x) = G(t, r)\frac{x}{r} = \iint \frac{x - y}{|x - y|^3} f(t, y, v)\, dv\, dy$$

for $t \in [0, \delta]$ and $x \in \mathbb{R}^3\backslash\{0\}$. Finally, we define $U := U_f$ by

$$U(t, x) := -\iint \frac{f(t, y, v)}{|x - y|}\, dv\, dy, \quad 0 \leq t \leq \delta,\ x \in \mathbb{R}^3.$$

Since $f$ is continuous with $f(t) \in C_c(\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3)$ for $0 \leq t \leq \delta$, Lemma 2.8 (a) implies follows that $\partial_x U = F$. Furthermore, $\rho$ is continuous with $\rho(t) \in C_c(\mathbb{R}^3\backslash\{0\})$ and $\rho(t)$ spherically symmetric for $t \in [0, \delta]$. As discussed in Lemma 2.9, $U_f \in C^{0,2}([0, \delta] \times \mathbb{R}^3\backslash\{0\})$ solves the corresponding Poisson equation, so $f$ is a strong Lagrangian solution on $[0, \delta] \times \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$. Since $\delta > 0$ is arbitrary, the constructed strong Lagrangian solution exists globally, i.e., on $[0, \infty[\times \mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$.

**Step 6:** The uniqueness
Let $\tilde{f}$ be a strong Lagrangian solution of the Vlasov-Poisson system with a point mass

satisfying $f(0) = \mathring{f}$. Then there exists $\tilde{\delta} > 0$ such that the solutions $f$ and $\tilde{f}$ exist on the set $[0, \tilde{\delta}]$. Then we obtain with the same arguments as in Step 4, respectively, as in [6, Equation (3.12)] the following estimate:

$$\|\partial_x \tilde{U}(t) - \partial_x U(t)\|_\infty \leq C \int_0^t \|\partial_x \tilde{U}(s) - \partial_x U(s)\|_\infty \, ds$$

for $t \in [0, \tilde{\delta}]$ and $C > 0$ independently of $t$. Therefore, Grönwall's inequality leads to $\partial_x U = \partial_x \tilde{U}$ and thus $\tilde{Z} = Z$. This yields

$$\tilde{f}(t,z) = \tilde{f}(t, Z(t,t,z)) = \tilde{f}(0, Z(0,t,z)) = \mathring{f}(Z(0,t,z)) = f(t,z)$$

for $t \in [0, \tilde{\delta}]$ and $z \in \{L > 0\}$. Since $\tilde{f}$ and $f$ are continuous with support in $\{L \geq L_0\}$, we obtain $f(t,z) = 0 = \tilde{f}(t,z)$ for $t \geq 0$ and $z = (x,v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$ with $L(x,v) = 0$, so the solution is unique. □

Similar to [11, Theorem 2.1], we have shown in the previous proof in Step 1 that the support of $f(t)$ is bounded in $x$ away from $x = 0$ by a fixed radius which is independent of $t \geq 0$.

**Remark.** *Let $\mathring{f}$ be as described in Theorem 6.2 and $f$ be the corresponding strong Lagrangian solution. Then there exists $R_{\min} = R_{\min}(M_0, L_0, \mathring{f}) > 0$ such that*

$$f(t,x,v) = 0, \quad |x| < R_{\min},$$

*for $t \geq 0$ and $(x,v) \in \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}^3$.*

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