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Topical Review

Stability and instability results for equilibria of a (relativistic) self-gravitating collisionless gas—a review

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Abstract

We review stability and instability results for self-gravitating matter distributions, where the matter model is a collisionless gas as described by the Vlasov equation. The focus is on the general relativistic situation, i.e. on steady states of the Einstein–Vlasov system and their stability properties. In order to put things into perspective we include the Vlasov–Poisson (VP) system and the relativistic VP system into the discussion.

Keywords: Einstein–Vlasov system, stability, self-gravitating collisionless mass distribution

Contents

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1. Introduction

Consider a large ensemble of massive particles which interact only through the gravitational field which they create collectively. The density $f \geqslant 0$ on phase space of such a collisionless gas obeys the collisionless Boltzmann or Liouville equation, which in mathematics is usually (and regrettably) called the Vlasov equation. The exact form of this equation depends on the situation at hand—Newtonian, special relativistic, or general relativistic—, but its content is always that f is constant along particle trajectories. This equation is coupled to the field equation for gravity which in the Newtonian case results in the Vlasov–Poisson (VP) system and in the general relativistic one in the Einstein–Vlasov (EV) system; these systems will be formulated in the next section.

The former system has a long history in the astrophysics literature where it is used to model galaxies and globular clusters, and we refer to [12, 28] and the references there. The importance of the latter system is two-fold: on the one hand there are major open, conceptual problems in general relativity such as the cosmic censorship hypotheses for which the choice of a matter model which by itself is well understood is important, and the Vlasov equation is a natural candidate in this context. On the other hand, general relativistic effects become increasingly important in astrophysics, given for example the fact that most galaxies have a massive black hole at their center. Historically, this interest started in the mid 1960s with the discovery of quasars [13, 14, 119].

From a mathematical point of view one gains a better understanding of these systems if one includes the so-called relativistic VP system, a hybrid system which is neither Galilei nor Lorentz invariant. All three systems under consideration share the property that they have a plethora of steady state solutions which represent possible equilibrium configurations of a galaxy or a globular cluster; only steady states with finite mass will be relevant here. A natural question both from the mathematical and the astrophysical point of view is which of these are stable, and how stable or unstable equilibria react, at least qualitatively, to perturbations.

For the VP system several different approaches to the stability question exist in the by now quite extensive mathematical literature on this subject, part of which we will recall later on. Our focus in these notes is on the stability problem for the EV system where much less is known. We aim to bring out the differences between the non-relativistic and the relativistic situation and to discuss why some approaches which were successful in the Newtonian case seem to fail in the general relativistic case and which approaches are at least partially successful also in the relativistic case. In the context of general relativity the stability question differs in an essential and striking way from its Newtonian counterpart: in the latter case one can formulate conditions on the so-called microscopic equation of state which guarantee nonlinear stability of any steady state with that microscopic equation of state, but in the former context such a microscopic equation of state guarantees stability only if the steady state is not too relativistic, while a sufficiently relativistic steady state with the same microscopic equation of state will be unstable. This instability of strongly relativistic steady states has no analogue in the Newtonian context. We should also emphasize that so far the general relativistic case has only been attacked under the assumption of spherical symmetry.

These notes are intended to be such that they can be followed and the main ideas can be appreciated without consulting the original literature. We aim to introduce the necessary concepts and major results self-consistently and explain at least the ideas of most proofs. But while some hopefully instructive proofs are given in detail, we very often have to refer the reader to the original literature for a complete, rigorous analysis. Although we try to do justice to the mathematical literature on the subject, the selection and presentation of the material is without doubt strongly influenced by the author's preferences, prejudices, and limitations; the coverage of the relevant astrophysics literature is certainly quite incomplete.

The paper proceeds as follows. In the next section we state the three systems under discussion—the VP system, the relativistic VP (RVP) system, and the EV system—together with their conserved quantities which play a key role in the stability analysis; we will often use the abbreviations (VP), (RVP), (EV) to refer to these systems. We also point out an important, basic difference between (VP), (RVP), and (EV), according to which they may be dubbed energy 'subcritical', 'critical', or 'supercritical'. In section 3 we review the basic construction of one-parameter families of steady states which in the general relativistic case are parameterized by their central redshift and share the same microscopic equation of state. Section 4 recalls the basic strategies which have lead to stability results for (VP) or (RVP), which we distinguish into global variational methods, local variational methods, and linearization. Section 5 is devoted to a linear stability result for steady states of (EV) with small central redshift, while section 6 discusses a linear, exponential instability result for large central redshift. The spectral properties of the linearized (EV) system are reviewed in section 7, where we in particular discuss a recently derived Birman–Schwinger principle for (EV). In section 8 we review the main numerical observations concerning stability for (EV) and discuss some related conjectures and open problems. The last section provides an example which shows that for infinite dimensional dynamical systems strict global energy minimizers need not be stable.

1.1. Notation

Since these notes are fairly long it may be useful to provide a place where some general notation is collected which is used throughout these notes; some of it will be re-introduced again later.

For vectors like $x, p, v \in \mathbb{R}^3$ we use $|\cdot|$ and $\cdot$ for the Euclidean norm and scalar product,

$$x \cdot v = \sum_{j=1}^{3} x_j v_j, \ |x| = \sqrt{x \cdot x},$$

etc. Gradients with respect to, say, x or p are denoted by ∂_x or ∂_p, and in order not to be too consistent we occasionally write ∇ instead of ∂_x. We also abbreviate

$$\langle v \rangle = \sqrt{1 + |v|^2}, \ w = \frac{x \cdot v}{|x|}, \ L = |x \times v|^2 \text{ for } x, v \in \mathbb{R}^3;$$

this will make more sense when it first comes up. If H is some Hilbert space and $\mathcal{L}$ a linear, bounded or unbounded operator on H we denote by $\mathscr{D}(\mathcal{L})$, $\mathscr{R}(\mathcal{L})$, and $\mathscr{N}(\mathcal{L})$ its domain of definition, its range, and its null-space or kernel, i.e.

$$\mathcal{L}\colon H\supset\mathscr{D}(\mathcal{L})\to H,\ \mathscr{R}(\mathcal{L})=\mathcal{L}(\mathscr{D}(\mathcal{L})),\ \mathscr{N}(\mathcal{L})=\mathcal{L}^{-1}(\{0\})\subset\mathscr{D}(\mathcal{L}).$$

2. The systems under consideration and their conserved quantities

2.1. The VP system

In the Newtonian case the density $f=f(t,x,p)\geqslant 0$ of the particle ensemble on phase space is a function of time t, position $x\in\mathbb{R}^3$, and momentum $p\in\mathbb{R}^3$. It obeys the VP system

$$\partial_t f+p\cdot\partial_x f-\partial_x U\cdot\partial_p f=0, \tag{2.1}$$

$$\Delta U=4\pi\rho,\qquad \lim_{|x|\to\infty}U(t,x)=0, \tag{2.2}$$

$$\rho(t,x)=\int f(t,x,p)\,\mathrm{d}p, \tag{2.3}$$

where $U=U(t,x)$ is the gravitational potential induced by the macroscopic, spatial mass density $\rho=\rho(t,x)$; integrals without explicitly specified domain extend over $\mathbb{R}^3$. The boundary condition in (2.2) corresponds to the fact that we consider an isolated system in an otherwise empty Universe. As usual, we assume that all the particles in the ensemble have the same mass which is normalized to unity so that p is also the velocity of a particle with coordinates (x,p). Up to regularity issues a function f satisfies the Vlasov equation (2.1), iff it is constant along solutions of the equations of motion of a test particle in the potential U, namely

$$\dot{x}=p,\ \dot{p}=-\partial_x U(s,x); \tag{2.4}$$

the latter is the characteristic system of (2.1). If the sign in the Poisson equation is reversed, the system models a plasma, where one will typically add a neutralizing ion background and/or an exterior confining field.

Smooth, compactly supported initial data $f_{|t=0}=\mathring{f}\in C^1_c(\mathbb{R}^6)$ launch classical, smooth solutions to this system, which are known to be global in time [75, 85, 104], see also the review [99]. These solutions conserve the following quantities, which we define as functionals acting on states $f=f(x,p)$:

$$\mathcal{H}(f):=E_{\mathrm{kin}}(f)+E_{\mathrm{pot}}(f):=\frac{1}{2}\iint|p|^2 f(x,p)\,\mathrm{d}p\,\mathrm{d}x-\frac{1}{8\pi}\int|\nabla U_f(x)|^2\mathrm{d}x \tag{2.5}$$

is the total energy of the state f, i.e. the sum of its kinetic and potential energies, where the potential U_f is induced by f via (2.2) and (2.3), and

$$\mathcal{C}(f):=\iint\Phi(f(x,p))\,\mathrm{d}p\,\mathrm{d}x \tag{2.6}$$

is a so-called Casimir functional, which is conserved for any choice of $\Phi\in C^1([0,\infty[)$ with $\Phi(0)=0$. The fact that the energy $\mathcal{H}$ is conserved along solutions of the VP system simply says that the latter is a conservative system, while the conservation of the Casimir functionals corresponds to the fact that the characteristic flow induced by (2.4) preserves Lebesgue measure. In other words, $f(t)$, the state of the system at time t, is related to $\mathring{f}$ via

$$f(t) = \mathring{f} \circ Z(0,t) \tag{2.7}$$

where

$$s \mapsto (X,P)(s,t,x,p) = Z(s,t,z)$$

is the solution to (2.4) with $(X,P)(t,t,x,p) = (x,p)$, which induces a diffeomorphism

$$Z(t,0) = Z(t,0,\cdot): \mathbb{R}^6 \to \mathbb{R}^6$$

with inverse $Z(0,t)$, and

$$\det \partial_z Z(t,0) = 1.$$

Both types of conservation laws are essential for deducing global-in-time existence of solutions and for nonlinear stability issues, see section 2.4.

Before we proceed to relativistic models we mention a different way of writing the Vlasov equation (2.1). To this end we recall the Poisson bracket of two smooth functions $g = g(x,p)$ and $h = h(x,p)$,

$$\{g,h\} := \partial_x g \cdot \partial_p h - \partial_p g \cdot \partial_x h, \tag{2.8}$$

and the energy of a particle with coordinates (t,x,p),

$$E = E(t,x,p) = \frac{1}{2}|p|^2 + U(t,x). \tag{2.9}$$

Then the Vlasov equation (2.1) can be written as

$$\partial_t f + \{f,E\} = 0. \tag{2.10}$$

We recall that $\cdot$ denotes the Euclidean scalar product between vectors in $\mathbb{R}^3$, and the Euclidean norm of such vectors is denoted by $|\cdot|$.

2.2. *The relativistic VP system*

For this system the Vlasov equation takes the form

$$\partial_t f + \frac{p}{\sqrt{1+|p|^2}} \cdot \partial_x f - \partial_x U \cdot \partial_p f = 0, \tag{2.11}$$

where we again assume that all the particles have the same rest mass, normalized to unity, and the speed of light is set to unity as well. The Poisson equation (2.2) together with its boundary condition and the relation (2.3) remain unchanged. The characteristic system now reads

$$\dot{x} = \frac{p}{\sqrt{1+|p|^2}},\ \dot{p} = -\partial_x U(s,x),$$

and the relation (2.7) remains true with the flow map redefined accordingly. The characteristic flow is still measure preserving so that we keep the Casimir functionals (2.6) as conserved quantities, and (RVP) is still conservative with the obvious change that now

$$E_{\mathrm{kin}}(f) := \iint \sqrt{1+|p|^2} f(x,p)\,\mathrm{d}p\,\mathrm{d}x.$$

The Vlasov equation (2.11) can again be put into the form (2.10) with (2.9) replaced by

$$E = E(t,x,p) = \sqrt{1+|p|^2} + U(t,x). \tag{2.12}$$

As mentioned above, this system is neither Galilei nor Lorentz invariant. While it may not be so relevant from the physics point of view it will be useful in illustrating the difficulties which the stability discussion encounters when moving from (VP) to (EV). Initial data as specified for (VP) launch local, classical, smooth solutions of (RVP) which can easily be seen by adapting the proof of [99, theorem 1.1]. But it is known that such solutions can blow up in finite time, see [32]. In section 2.4 we will explain this difference to (VP) and consider the question what this means with respect to stability.

2.3. The EV system

On a smooth spacetime manifold M equipped with a Lorentzian metric $g_{\alpha\beta}$ with signature $(-+++)$ the Einstein equations read

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{2.13}$$

Here $G_{\alpha\beta}$ is the Einstein tensor induced by the metric, and $T_{\alpha\beta}$ is the energy–momentum tensor; Greek indices run from 0 to 3. The world line of a test particle on M obeys the geodesic equation, which can be written either as a first order ODE on the tangent bundle TM of the spacetime manifold, coordinatized by (x^α, p^β) where x^α are general coordinates on M and p^α are the corresponding canonical momenta, or on the cotangent bundle TM^*, coordinatized by (x^α, p_β) where $p_\beta = g_{\beta\gamma}p^\gamma$. If we opt for the latter alternative,

$$\dot{x}^\alpha = g^{\alpha\beta}p_\beta,\ \dot{p}_\alpha = -\frac{1}{2}\partial_{x^\alpha}g^{\beta\gamma}p_\beta p_\gamma;$$

$g^{\alpha\beta}$ denotes the inverse of the metric $g_{\alpha\beta}$, the dot indicates differentiation with respect to proper time along the world line of the particle, and the Einstein summation convention is applied. All the particles are to have the same rest mass which we normalize to unity, and to move forward in time. Their number density f is a non-negative function supported on the mass shell

$$PM^* := \left\{g^{\alpha\beta}p_\alpha p_\beta = -1,\ p^\alpha \text{ future pointing}\right\},$$

a submanifold of the cotangent bundle TM^* which is invariant under the geodesic flow. Letting Latin indices range from 1 to 3 we use coordinates (t, x^a) with zero shift which implies that $g_{0a} = 0$. On the mass shell PM^* the variables p_0 and p^0 then become functions of the variables (t, x^a, p_b):

$$p_0 = -|g_{00}|^{1/2}\sqrt{1+g^{ab}p_a p_b},\ p^0 = |g_{00}|^{-1/2}\sqrt{1+g^{ab}p_a p_b}.$$

Since the number density $f = f(t, x^a, p_b)$ is constant along the geodesics, the Vlasov equation reads

$$\partial_t f + \frac{g^{ab}p_b}{p^0}\partial_{x^a}f - \frac{1}{2p^0}\partial_{x^a}g^{\beta\gamma}p_\beta p_\gamma \partial_{p_a}f = 0. \tag{2.14}$$

The energy–momentum tensor is given as

$$T_{\alpha\beta} = |g|^{-1/2}\int p_\alpha p_\beta f\frac{\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3}{p^0}, \tag{2.15}$$

where $|g|$ denotes the modulus of the determinant of the metric. The system (2.13)–(2.15) is the EV system in general coordinates. As we want to describe isolated systems, we require that the spacetime is asymptotically flat which corresponds to the boundary condition in (2.2).

An obvious steady state of this system is flat Minkowski space with $f = 0$. In [24, 72, 112] nonlinear stability of this trivial steady state was shown for the system above, which is a highly non-trivial result. Under the simplifying assumption of spherical symmetry this result was shown in [94, 100]. Mathematically speaking, these results are small data results which rely on the fact that close to vacuum the characteristic flow of the Vlasov equation disperses the matter in space. When perturbing a non-trivial, i.e. non-vacuum, steady state no such mechanism exists, and the problem becomes completely different. Our discussion is focused exclusively on the stability of non-trivial steady states.

Questions like the stability or instability of non-trivial steady states are at present out of reach of a rigorous mathematical treatment, unless simplifying symmetry assumptions are made. We assume spherical symmetry, use Schwarzschild coordinates (t, r, θ, φ), and write the metric in the form

$$\mathrm{d}s^2 = -\mathrm{e}^{2\mu(t,r)}\mathrm{d}t^2 + \mathrm{e}^{2\lambda(t,r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{2.16}$$

Here $t \in \mathbb{R}$ is a time coordinate, and the polar angles $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ coordinatize the surfaces of constant t and $r > 0$. The latter are the orbits of $\mathrm{SO}(3)$, which acts isometrically on this spacetime, and $4\pi r^2$ is the area of these surfaces. The boundary condition

$$\lim_{r\to\infty} \lambda(t,r) = \lim_{r\to\infty} \mu(t,r) = 0 \tag{2.17}$$

guarantees asymptotic flatness, and in order to guarantee a regular center we impose the boundary condition

$$\lambda(t, 0) = 0. \tag{2.18}$$

Polar coordinates have a tendency to introduce artificial singularities at the center. Hence it is convenient to also use the corresponding Cartesian coordinates

$$x = (x^1, x^2, x^3) = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

and the corresponding canonical covariant momenta $p = (p_1, p_2, p_3)$.

Before we proceed to formulate (EV) in these variables we emphasize the fact that from this point on we will not raise or lower any indices, treat x and p simply as variables in $\mathbb{R}^3$, and use notations like

$$x \cdot p = \sum_{a=1}^{3} x^a p_a, \; |p|^2 = \sum_{a=1}^{3} (p_a)^2$$

for Euclidean scalar products and norms, just as we did for (VP) or (RVP).

In order that the particle distribution function $f = f(t, x, p)$ is compatible with (2.16) it must be spherically symmetric; we call a state $f = f(x, p)$ *spherically symmetric* iff

$$f(x,p) = f(Ax, Ap), \; x, p \in \mathbb{R}^3, \; A \in \mathrm{SO}(3). \tag{2.19}$$

Using the abbreviation

$$\langle p \rangle := -\mathrm{e}^{-\mu} p_0 = \sqrt{1 + |p|^2 + (\mathrm{e}^{2\lambda} - 1)\left(\frac{x \cdot p}{r}\right)^2}, \tag{2.20}$$

(EV) can be put into the following form:

$$\partial_t f + \mathrm{e}^{\mu-2\lambda}\frac{p}{\langle p\rangle}\cdot\partial_x f + \left[\mathrm{e}^{\mu-2\lambda}\lambda'\left(\frac{x\cdot p}{r}\right)^2\frac{1}{\langle p\rangle} - \mathrm{e}^{\mu}\mu'\langle p\rangle + \mathrm{e}^{\mu}\frac{1-\mathrm{e}^{-2\lambda}}{r\langle p\rangle}\left(|p|^2-\left(\frac{x\cdot p}{r}\right)^2\right)\right]\frac{x}{r}\cdot\partial_p f = 0, \quad (2.21)$$

$$\mathrm{e}^{-2\lambda}(2r\lambda'-1)+1 = 8\pi r^2\rho, \quad (2.22)$$

$$\mathrm{e}^{-2\lambda}(2r\mu'+1)-1 = 8\pi r^2\sigma, \quad (2.23)$$

$$\dot\lambda = -4\pi r\mathrm{e}^{\lambda+\mu}\jmath, \quad (2.24)$$

$$\mathrm{e}^{-2\lambda}\left(\mu''+(\mu'-\lambda')\left(\mu'+\frac{1}{r}\right)\right) - \mathrm{e}^{-2\mu}\left(\ddot\lambda+\dot\lambda(\dot\lambda-\dot\mu)\right) = 8\pi\sigma_T, \quad (2.25)$$

where

$$\rho(t,r) = \rho(t,x) = \mathrm{e}^{-\lambda}\int\langle p\rangle f(t,x,p)\,\mathrm{d}p, \quad (2.26)$$

$$\sigma(t,r) = \sigma(t,x) = \mathrm{e}^{-3\lambda}\int\left(\frac{x\cdot p}{r}\right)^2 f(t,x,p)\frac{\mathrm{d}p}{\langle p\rangle}, \quad (2.27)$$

$$\jmath(t,r) = \jmath(t,x) = \mathrm{e}^{-2\lambda}\int\frac{x\cdot p}{r}f(t,x,p)\mathrm{d}p, \quad (2.28)$$

$$\sigma_T(t,r) = \sigma_T(t,x) = \frac{1}{2}\mathrm{e}^{-3\lambda}\int\left|\frac{x\times p}{r}\right|^2 f(t,x,p)\frac{\mathrm{d}p}{\langle p\rangle}. \quad (2.29)$$

Here $\dot{}$ and $'$ denote the derivatives with respect to t and r respectively, ρ is the mass-energy density—its integral is the ADM mass, see (2.30)—, and σ, σ_T are the pressure in the radial or tangential direction, respectively.

The equations (2.21)–(2.29) are a form of the spherically symmetric (EV) which does not look too appealing and has so far not been used in the literature. The fact that the source terms defined in (2.26)–(2.29) depend on the metric, in particular via (2.20), makes it technically unpleasant to handle. But this form of the system has some advantages. The characteristic flow of the Vlasov equation (2.21) is again measure preserving, and hence the Casimir functionals defined exactly as in (2.6) remain conserved quantities. Moreover, the Vlasov equation (2.21) still is of the general form (2.10) with

$$E = E(t,x,p) = \mathrm{e}^{\mu}\langle p\rangle.$$

The total energy, which in this case is usually referred to as the ADM mass, is given as

$$\mathcal{H}(f) := \iint \mathrm{e}^{-\lambda_f}\sqrt{1+|p|^2+(\mathrm{e}^{2\lambda_f}-1)\left(\frac{x\cdot p}{r}\right)^2}f(x,p)\,\mathrm{d}p\,\mathrm{d}x \quad (2.30)$$

where λ_f is the solution to (2.22) subject to the boundary conditions from (2.17) and (2.18) and with ρ satisfying (2.26).

We rewrite the above form of (EV) by introducing non-canonical momentum variables via

$$v = p + (\mathrm{e}^{\lambda}-1)\frac{x\cdot p}{r}\frac{x}{r}. \quad (2.31)$$

In these variables (2.20) turns into

$$\langle v\rangle := -\mathrm{e}^{-\mu}p_0 = \sqrt{1+|v|^2}, \quad (2.32)$$

in the definition of spherical symmetry of $f = f(t, x, v)$ we simply replace p by v, and the Vlasov equation (2.21) becomes

$$\partial_t f + e^{\mu-\lambda} \frac{v}{\langle v \rangle} \cdot \partial_x f - \left(\dot{\lambda} \frac{x \cdot v}{r} + e^{\mu-\lambda} \mu' \langle v \rangle \right) \frac{x}{r} \cdot \partial_v f = 0. \tag{2.33}$$

The field equations (2.22)–(2.25) remain unchanged, but the source terms

$$\rho(t,r) = \rho(t,x) = \int \langle v \rangle f(t,x,v)\, dv, \tag{2.34}$$

$$\sigma(t,r) = \sigma(t,x) = \int \left(\frac{x \cdot v}{r} \right)^2 f(t,x,v) \frac{dv}{\langle v \rangle}, \tag{2.35}$$

$$\jmath(t,r) = \jmath(t,x) = \int \frac{x \cdot v}{r} f(t,x,v) dv, \tag{2.36}$$

$$\sigma_T(t,r) = \sigma_T(t,x) = \frac{1}{2} \int \left| \frac{x \times v}{r} \right|^2 f(t,x,v) \frac{dv}{\langle v \rangle} \tag{2.37}$$

are now given completely in terms of f, they do no longer depend on the metric. The price to pay for this simplification is that the characteristic flow of the Vlasov equation (2.33) is not measure preserving, and the Casimir functionals, which are still conserved quantities, take the form

$$\mathcal{C}(f) := \iint e^{\lambda_f} \Phi(f(x,v))\, dv\, dx. \tag{2.38}$$

On the other hand, the ADM mass simplifies to a linear functional that depends only on f,

$$\mathcal{H}(f) := \iint \langle v \rangle f(x,v)\, dv\, dx = \iint \sqrt{1 + |v|^2} f(x,v)\, dv\, dx. \tag{2.39}$$

Taking into account the boundary conditions (2.17) and(2.18) the metric components are given explicitly in terms of ρ and σ, and hence of the state $f = f(x,v)$; we suppress the time variable t for the moment:

$$e^{-2\lambda} = 1 - \frac{2m}{r} \tag{2.40}$$

and

$$\mu' = e^{2\lambda} \left(\frac{m}{r^2} + 4\pi r \sigma \right), \tag{2.41}$$

where

$$m(r) = 4\pi \int_0^r \rho(s) s^2 ds. \tag{2.42}$$

At this point we notice that a spacetime manifold can only be covered by Schwarzschild coordinates if $2m < r$ everywhere; the spacetime must not contain trapped surfaces. Finally, we also mention that the structure (2.10) is lost when using the non-canonical momentum variable v. As in most of the stability-related literature we will use the version of (EV) in non-canonical variables, but since many important aspects of the stability issue are still widely open (even in spherical symmetry), it may be useful to keep the alternative, canonical formulation in mind. It is also possible that other coordinates adapted to spherical symmetry are more suitable for the stability analysis. We will not pursue this issue but mention maximal areal coordinates as one alternative [35].

2.4. A basic difference between (VP), (RVP), and (EV)

Let us suppose that we want to make use of conservation of energy to get insight into global existence issues for the initial value problem or stability issues. Then in the case of (VP) or (RVP) we must deal with the fact that while $E_{\text{kin}} + E_{\text{pot}}$ is conserved, the two terms have opposite signs, and no immediate control of E_{kin} or E_{pot} results.

In what follows we sometimes employ the notation

$$\rho_f(x) := \int f(x,p)\,\mathrm{d}p$$

for the spatial density induced by some measurable phase-space density $f = f(x,p) \geqslant 0$. Similarly, we will write U_ρ or U_f for the potential induced by ρ or ρ_f via (2.2).

Now let $0 \leqslant k \leqslant \infty$ and $n = k + 3/2$. For any $R > 0$,

$$\begin{aligned}\rho_f(x) &= \int_{|p|\leqslant R} f(x,p)\,\mathrm{d}p + \int_{|p|>R} f(x,p)\,\mathrm{d}p \\ &\leqslant \left(\frac{4\pi}{3}R^3\right)^{\frac{1}{k+1}} \|f(x,\cdot)\|_{1+1/k} + \frac{1}{R^2}\int |p|^2 f(x,p)\,\mathrm{d}p,\end{aligned}$$

where $\|\cdot\|_s$ denotes the usual L^s norm, in this case over $\mathbb{R}^3$. We choose

$$R = \left(\int |p|^2 f \mathrm{d}p / \|f(x,\cdot)\|_{1+1/k}\right)^{\frac{1+k}{5+2k}},$$

take the resulting estimate to the power $1+1/n$ and integrate with respect to x to conclude that

$$\|\rho_f\|_{1+1/n} \leqslant C\|f\|_{1+1/k}^{\frac{2+2k}{5+2k}} \left(\iint |p|^2 f(x,p)\,\mathrm{d}p\,\mathrm{d}x\right)^{\frac{3}{5+2k}}. \tag{2.43}$$

On the other hand, the Hardy–Littlewood–Sobolev inequality [71, theorem 4.3] implies that

$$-E_{\text{pot}}(f) \leqslant C\|\rho_f\|_{6/5}^2.$$

We require that

$$1 + \frac{1}{n} \geqslant \frac{6}{5}, \text{ i.e. } n \leqslant 5, \text{ i.e. } k \leqslant \frac{7}{2}.$$

Then we can interpolate the $L^{6/5}$ norm between the L^1 and $L^{1+1/n}$ norms and conclude that

$$-E_{\text{pot}}(f) \leqslant C\|f\|_1^{\frac{7-2k}{6}} \|f\|_{1+1/k}^{\frac{k+1}{3}} E_{\text{kin}}(f)^{1/2}. \tag{2.44}$$

This key estimate has important consequences for (VP). Assume that we have a local-in-time, smooth solution to this system, which conserves the total energy and both $\|f(t)\|_1$ and $\|f(t)\|_\infty$. Then (2.44) with $k = 0$ implies that along this solution both $E_{\text{pot}}(f(t))$ and $E_{\text{kin}}(f(t))$ and hence also $\|\rho(t)\|_{5/3}$ remain bounded, which are key *a-priori* bounds toward global-in-time existence.

Staying with (VP) we now suppose that we want to minimize an energy-Casimir functional

$$\mathcal{H}(f) + \mathcal{C}(f)$$

under the constraint that the mass $\iint f = M$ is prescribed and for a Casimir function Φ which grows sufficiently fast to control $\|f\|_{1+1/k}$ with $k \leqslant 7/2$; a corresponding minimizer will be a candidate for a stable steady state, see section 4. Then the key estimate (2.44) implies that along a corresponding minimizing sequence $E_{\rm kin}(f)$ and hence also $\|\rho_f\|_{1+1/n}$ remain bounded, which is an important step toward a necessary compactness argument along such a minimizing sequence. We see that the success of both global-in-time existence results and stability results via global variational techniques hinges on the estimate (2.44).

Let us check how (2.44) fares in the (RVP) case. Proceeding as before,

$$\|\rho_f\|_{\frac{k+4}{k+3}} \leqslant C\|f\|_{1+1/k}^{\frac{k+1}{k+4}} \left(\iint \langle p\rangle f(x,p)\,\mathrm{d}p\,\mathrm{d}x\right)^{\frac{3}{k+4}},$$

and (2.44) turns into

$$-E_{\rm pot}(f) \leqslant C\|f\|_1^{\frac{2-k}{3}}\,\|f\|_{1+1/k}^{\frac{k+1}{3}} E_{\rm kin}(f),\ 0\leqslant k\leqslant 2. \tag{2.45}$$

This estimate gives no control on $E_{\rm kin}$ along either a local-in-time solution or a minimizing sequence for the variational problem mentioned above. The point here is that for (RVP) the kinetic energy is only a first order moment in p while for (VP) it is a second order moment.

For (EV) the situation is even worse in the following sense. The key point above is that the kinetic energy is a higher-order moment in p than what appears in the definition of ρ so that some L^s norm of ρ with $s>1$ is under control, provided $E_{\rm kin}$ is under control. But as we see from the formula for the energy (2.39) in the (EV) situation, this energy gives us exactly an L^1 bound for ρ and nothing more. This missing 'something more' makes (EV) that much harder to deal with, both with respect to global-in-time existence and with respect to stability.

3. Steady states

Before we address their stability we must recall what typical steady states of (VP), (RVP), or (EV) look like, and how one can establish their existence. To this purpose let us suppose that we are given a time-independent potential $U=U(x)$ or a time-independent metric of the form (2.16). Then the particle energy, defined for (VP) or (RVP) in (2.9) or (2.12) and for the non-canonical form of (EV) as

$$E = E(x,v) = \mathrm{e}^{\mu}\,\langle v\rangle\,, \tag{3.1}$$

is constant along characteristics of the corresponding static Vlasov equation and hence solves that equation. For (VP) or (RVP) we therefore make the ansatz

$$f(x,p) = \phi(E) = \varphi(E_0 - E) \tag{3.2}$$

for the particle distribution function, which for technical reasons we modify to

$$f(x,v) = \phi(E) = \varphi\left(1 - \frac{E}{E_0}\right) \tag{3.3}$$

in the (EV) case. We refer to the relation (3.2) or (3.3) as a *microscopic equation of state*. To keep matters simple we assume that

$$\varphi \in C(\mathbb{R})\cap C^2(]0,\infty[),\quad \varphi = 0 \text{ on }]-\infty,0],\quad \varphi>0 \text{ on }]0,\infty[; \tag{3.4}$$

E_0 is a cut-off energy with $E_0<0$ for (VP) or (RVP) and $0<E_0<1$ for (EV). Such a cut-off energy is necessary to obtain steady states with a localized matter distribution. Notice that the ansatz function ϕ depends on the cut-off energy E_0, which must be specified to in order to specify ϕ, but the ansatz function φ does not depend on E_0, and it has the reversed monotonicity behavior with respect to E.

With this ansatz we satisfy the Vlasov equation, the source terms become functionals of U or μ, respectively, and the static systems reduce the the field equation(s) with this dependence substituted in. In the (EV) case these functions are spherically symmetric by assumption, but also in the (VP) and (RVP) case the ansatz (3.2) leads to steady states which necessarily are spherically symmetric, see [31]. In particular, both U and μ can be viewed as functions of $r = |x|$. Instead of looking for U or μ directly, we define a new unknown

$$y(r) = E_0 - U(r) \text{ or } y(r) = E_0 - U(r) - 1$$

in the (VP) or (RVP) case, respectively, and

$$y(r) = \ln E_0 - \mu(r)$$

in the (EV) case. In the latter case,

$$\rho(r) = g(y(r)), \quad \sigma(r) = h(y(r)) = \sigma_T(r), \tag{3.5}$$

where

$$g(y) := 4\pi\, \mathrm{e}^{4y} \int_0^{1-\mathrm{e}^{-y}} \varphi(\eta)\,(1-\eta)^2 \left((1-\eta)^2 - \mathrm{e}^{-2y}\right)^{1/2} d\eta \tag{3.6}$$

and

$$h(y) := \frac{4\pi}{3} \mathrm{e}^{4y} \int_0^{1-\mathrm{e}^{-y}} \varphi(\eta) \left((1-\eta)^2 - \mathrm{e}^{-2y}\right)^{3/2} \mathrm{d}\eta. \tag{3.7}$$

The functions g and h are continuously differentiable on $\mathbb{R}$, see [101, lemma 2.2], they are strictly decreasing for $y > 0$, and they vanish for $y < 0$. For (VP),

$$\rho(r) = g_N(y(r)), \text{ where } g_N(y) := 4\pi\sqrt{2} \int_0^y \varphi(\eta)\,(y-\eta)^{1/2}\,\mathrm{d}\eta;$$

the subscript N stands for 'Newtonian', and the exact form of the analogous relation for (RVP) is not relevant here. We recall that λ is given in terms of ρ via (2.40), and the static (EV) system is reduced to

$$y'(r) = -\frac{1}{1 - 2m(r)/r} \left(\frac{m(r)}{r^2} + 4\pi\, r\sigma(r) \right),\ y(0) = \kappa \tag{3.8}$$

see (2.41); here m, ρ, and σ are given in terms of y by (3.5) and (2.42), and $\kappa > 0$ is prescribed. The static (VP) or (RVP) systems reduce to

$$y'(r) = -\frac{m(r)}{r^2},\ y(0) = \kappa. \tag{3.9}$$

For any given $\kappa > 0$ a fixed point argument yields a unique, smooth, local solution to (3.8) or (3.9) on some short interval $[0, \delta]$. The solution y is strictly decreasing, can be extended to exist on $[0, \infty[$, and either remains strictly positive, or has a unique zero at some radius $R > 0$ beyond which there is vacuum. The crucial question is for which ansatz functions ϕ respectively φ the latter case holds, because in that case the above procedure yields steady states with compact support and finite mass. Once such a solution y is obtained, $E_0 := \lim_{r\to\infty} y(r)$ and $U(r) := E_0 - y(r)$ defines the cut-off energy and the potential in the (VP) or (RVP) case, while $E_0 := \exp(\lim_{r\to\infty} y(r))$ and $\mu(r) = \ln E_0 - y(r)$ for (EV); in either case the boundary condition at infinity follows. For more details to these arguments we refer to [87] and the references there.

A sufficient condition on φ which guarantees finite mass and compact support of the resulting steady states in all three cases, (VP), (RVP) and (EV), is that

$$\varphi(\eta) \geqslant C\eta^k \text{ for } \eta \in]0, \eta_0[\tag{3.10}$$

for some parameters $C > 0$, $\eta_0 > 0$, and $0 < k < 3/2$, see [87]; in passing we note that conditions on φ or ϕ which are both necessary and sufficient for finite radius and finite mass are not known. To sum up:

Proposition 3.1. *Let φ satisfy (3.4) and (3.10).*

(a) There exists a one-parameter family of steady states $(f_\kappa, U_\kappa)_{\kappa>0}$ of the spherically symmetric (VP) (or (RVP)) system, and $\kappa = U_\kappa(R_\kappa) - U_\kappa(0)$.
(b) There exists a one-parameter family of steady states $(f_\kappa, \lambda_\kappa, \mu_\kappa)_{\kappa>0}$ of the spherically symmetric, asymptotically flat (EV) system, and $\kappa = \mu_\kappa(R_\kappa) - \mu_\kappa(0)$.

The spatial support of such a steady state is an interval $[0, R_\kappa]$ with $0 < R_\kappa < \infty$, $\rho_\kappa, \sigma_\kappa \in C^1([0,\infty[)$, $y_\kappa, U_\kappa, \mu_\kappa, \lambda_\kappa \in C^2([0,\infty[)$, and $\rho'_\kappa(0) = \sigma'_\kappa(0) = y'_\kappa(0) = U'_\kappa(0) = \mu'_\kappa(0) = \lambda'_\kappa(0) = 0$. Moreover, we denote $D = D_\kappa := \{f_\kappa > 0\}$ so that $\mathrm{supp} f_\kappa = \overline{D_\kappa}$, which is compact in $\mathbb{R}^6$.

An essential difference between the (VP) and the (EV) case concerning the stability of steady states is the following: for (VP) one can formulate conditions on the microscopic equations of state—φ in (3.2) should be strictly increasing on $[0,\infty[$—which guarantee that all steady states in the corresponding one-parameter family from proposition 3.1 are nonlinearly stable; this remains true even for the King model $\varphi(\eta) = (\mathrm{e}^\eta - 1)_+$ where the mass-radius diagram, mentioned in item (d) of the remark below, exhibits a spiral structure, see [44]. For (EV) the same type of microscopic equation of state will yield a one-parameter family where the individual steady states change from being stable to being unstable as the central redshift κ increases from small values to larger ones. There is by now ample numerical evidence for this behavior [7, 34, 36], and we will discuss the first steps toward an analytic understanding of this behavior. To do so, we must understand the consequences which very small or very large values of κ have on the structure of the corresponding steady states $(f_\kappa, \lambda_\kappa, \mu_\kappa)_{\kappa>0}$ in the (EV) case.

In what follows we make the dependence of the (EV) steady states on κ explicit in the notation only when we study the limits $\kappa \to 0$ in section 5.2 and $\kappa \to \infty$ in section 6.1 or when the logic of some statement requires this. For other parts of our discussion, in particular for the (VP) case, the value of κ plays no role or is fixed, and we will abuse notation in saying that (f_0, U_0) or (f_0, λ_0, μ_0) is a steady state of (VP) or (EV), which is to be understood in the sense that some $\kappa_0 > 0$ is fixed and $f_0 := f_{\kappa_0}$ etc.

One should also notice that many other quantities depend on κ such as the particle energy

$$E = E(x,v) = \mathrm{e}^{\mu_\kappa} \langle v \rangle,$$

the set $D = \{f_\kappa > 0\}$, the ansatz function ϕ in (3.3) via the cut-off energy E_0, and various operators introduced in sections 5–7. These dependencies on κ will always be suppressed in our notation.

We conclude our steady state discussion with some remarks.

Remark. (a) The steady states obtained in proposition 3.1 are isotropic in the sense that $\sigma = \sigma_T$; we use σ to denote the (radial) pressure also in the Newtonian case. In the (EV) case they satisfy the following identities on $[0,\infty[$, the second of which is known as the Tolman–Oppenheimer–Volkov equation:

$$\lambda_\kappa' + \mu_\kappa' = 4\pi r e^{2\lambda_\kappa}(\rho_\kappa + \sigma_\kappa), \tag{3.11}$$

$$\sigma_\kappa' = -(\rho_\kappa + \sigma_\kappa)\,\mu_\kappa'. \tag{3.12}$$

(b) A remarkable property of these steady states is that their induced macroscopic quantities solve the Euler–Poisson or Einstein–Euler system respectively. Given the fact that the functions g_N or g are one-to-one for $y > 0$ one can write y as a function of ρ, and substituting into the relation for the pressure in (3.5) yields the corresponding macroscopic equation of state $\sigma = \sigma(\rho)$, which is part of the corresponding Einstein–Euler of Euler–Poisson system.

(c) The parameter κ which parameterizes the above steady state families is the difference in the potential between the center and the boundary of the matter distribution. In the (EV) case it is related to the redshift factor z of a photon which is emitted at the center $r = 0$ and received at the boundary R_κ of the steady state; this is not the standard definition of the central redshift where the photon is received at infinity, but it is a more suitable parameter here:

$$z = \frac{e^{\mu_\kappa(R_\kappa)}}{e^{\mu_\kappa(0)}} - 1 = \frac{e^{y_\kappa(0)}}{e^{y_\kappa(R_\kappa)}} - 1 = e^\kappa - 1.$$

Although this is not the standard terminology we refer to κ as the central redshift, and we will see later that it is a measure for how non-relativistic or relativistic a steady state is. In the (VP) case the parameter κ seems to have no effect on the stability properties of the corresponding steady states, but in the (EV) case steady states with sufficiently large κ will be seen to be unstable.

(d) An instructive way to visualize one of these one-parameter families of steady states is to plot, for a certain parameter range, the points (M_κ, R_κ) representing the (ADM) mass and radius of the state with parameter κ. The resulting curve is referred to as a mass-radius curve. For (VP) these curves can be strictly monotonic, for example in the polytropic case $\varphi(\eta) = \eta_+^k$, or they can exhibit a spiral structure, for example for the King model $\varphi(\eta) = (e^\eta - 1)_+$, see [88]. In strong contrast, these curves always have a spiral structure in the (EV) case, see [8, 79]. This is interesting, because according to the so-called turning point principle [110] passing through a turning point on the mass-radius spiral should affect the stability behavior of the steady state. For the Einstein–Euler system a rigorous version of this principle has been proven in [46], see also [47], but the principle does not hold in the (EV) case, see [34, 36]. The principle is known to be false for (VP) where for example all the steady states along the mass-radius spiral for the King model are known to be nonlinearly stable.

(e) Due to spherical symmetry the quantity

$$L := |x \times p|^2,$$

the modulus of angular momentum squared, is conserved along characteristics of both (VP) and (EV); for the latter system, $L = |x \times v|^2$. Hence one may include a dependence on L into the microscopic equation of state (3.2) or (3.3). Resulting steady states are then no longer isotropic, i.e. $\sigma \neq \sigma_T$, and the correspondence to the Euler matter model explained in part (b) above is lost. A common way to include the L-dependence is to generalize (3.3) to

$$f(x,v) = \phi(E,L) = \varphi\left(1 - \frac{E}{E_0}\right)(L - L_0)_+^l. \tag{3.13}$$

Here $l > -1/2$, and the analogous ansatz is used for (VP). If L is bounded away from zero, i.e. $L_0 > 0$, then the resulting steady states have a vacuum region at the center, if $L_0 = 0$ they do not. The static shell solutions with $L_0 > 0$ look somewhat artificial, but they become more interesting if one places a Schwarzschild black hole (or a point mass in the (VP) case) into the vacuum region, which is then surrounded by a static shell of Vlasov matter, see [37, 59, 93, 96].

4. Strategies toward stability in the (VP) and (RVP) case

In this section we recall the main methods which have resulted in stability results for the VP or the relativistic VP system. We do not aim for completeness, but only wish to give some orientation on what approaches one may try for the stability problem in the EV case. Our discussion will be even less complete concerning results from the astrophysics literature. All the available results rely explicitly or implicitly on the condition that the ansatz (3.2) or (3.13) is strictly decreasing in E on its support:

$$\phi'(E) < 0 \text{ for } E < E_0 \text{ or } \partial_E \phi(E,L) < 0 \text{ for } E < E_0, L \geqslant L_0. \tag{4.1}$$

4.1. Global variational methods

Let us consider the following variational problem: minimize the energy-Casimir functional

$$\mathcal{H}_C = \mathcal{H} + \mathcal{C}$$

over the set

$$\mathcal{F}_M := \left\{ f \in L^1(\mathbb{R}^6) \mid f \geqslant 0, \iint f \,\mathrm{d}p\,\mathrm{d}x = M,\ E_{\mathrm{kin}}(f) + \mathcal{C}(f) < \infty \right\}.$$

Here the kinetic, potential, and total energy E_{kin}, E_{pot}, and $\mathcal{H}$ are defined as in (2.5), the Casimir functional $\mathcal{C}$ is defined in (2.6) where for the moment we take

$$\Phi(f) := \frac{k}{1+k} f^{1+1/k} \tag{4.2}$$

with some $k \in]0, 3/2[$, and $M > 0$ is fixed. Since $k < 3/2$ we can choose α such that $0 < \frac{\alpha}{2}, \frac{k}{3}\frac{\alpha}{\alpha-1} < 1$. The key estimate (2.44) and Young's inequality then imply that for any $f \in \mathcal{F}_M$,

$$\begin{aligned} \mathcal{H}_C(f) &\geqslant E_{\mathrm{kin}}(f) + \mathcal{C}(f) - C\mathcal{C}(f)^{k/3} E_{\mathrm{kin}}(f)^{1/2} \\ &\geqslant E_{\mathrm{kin}}(f) - CE_{\mathrm{kin}}(f)^{\frac{\alpha}{2}} + \mathcal{C}(f) - C\mathcal{C}(f)^{\frac{k}{3}\frac{\alpha}{\alpha-1}}, \end{aligned} \tag{4.3}$$

where the constant $C > 0$ depends on M, k, and α. This estimate implies that

$$h_M := \inf_{\mathcal{F}_M} \mathcal{H}_C > -\infty.$$

Now consider a minimizing sequence $(f_j) \subset \mathcal{F}_M$, i.e. $\mathcal{H}_C(f_j) \to h_M$. Then by (4.3), $E_{\mathrm{kin}}(f_j)$ and $\mathcal{C}(f_j)$ remain bounded, in particular, (f_j) is a bounded sequence in $L^{1+1/k}(\mathbb{R}^6)$ which by the Banach–Alaoglu theorem has a weakly convergent subsequence, again denoted by (f_j). Its limit is a natural candidate for a global minimizer of $\mathcal{H}_C$ over $\mathcal{F}_M$. By (2.43) the sequence of induced spatial densities (ρ_j) is bounded and (up to a subsequence) weakly convergent in $L^{1+1/n}(\mathbb{R}^3)$.

The key difficulty now is to upgrade these weak convergences in such a way that one can pass to the limit in the potential energy; the kinetic energy is not a problem since it is linear in f. More generally speaking, some sort of compactness argument must be applied to the minimizing sequence. The following lemma, which is proven for example in [99, lemma 2.5], captures the compactness property of the solution operator to the Poisson equation; recall that U_ρ or U_f denotes the Newtonian gravitational potential induced by a spatial density ρ or a phase-space density f.

Lemma 4.1. *Let $0 < n < 5$. Let $(\rho_j) \subset L^{1+1/n}(\mathbb{R}^3)$ be such that*

$$\begin{aligned} &0 \leqslant \rho_j \rightharpoonup \rho_0 \text{ weakly in } L^{1+1/n}(\mathbb{R}^3), \text{ and} \\ &\forall \epsilon > 0\, \exists R > 0 : \limsup_{j\to\infty} \int_{|x|\geqslant R} \rho_j(x)\,\mathrm{d}x < \epsilon. \end{aligned} \tag{4.4}$$

Then $\nabla U_{\rho_j} \to \nabla U_{\rho_0}$ strongly in $L^2(\mathbb{R}^3)$.

Under our assumption on k it holds that $n = k + 3/2 < 3$, so the key issue is to verify (4.4), i.e. the minimizing sequence must in essence remain concentrated. To do this, one may employ the concentration-compactness principle introduced by Lions [74] combined with an analysis of how $E_{\mathrm{pot}}(f)$ behaves under scalings and splittings, or one may rely on the latter arguments exclusively, and all this is greatly simplified if one restricts the discussion to spherical symmetry; we refer to [99] and the references there for details. At this point one should note that while the steady states under consideration are spherically symmetric anyway, an *a-priori* restriction to spherical symmetry in the variational problem limits a resulting stability result to spherically symmetric perturbations and is thus undesirable.

In the concentration argument it turns out that in order to achieve (4.4) the ball in which the mass remains concentrated must be allowed to shift with the sequence; notice that all the functionals used above are invariant under translations in x. The resulting existence result for the above variational problem reads as follows.

Theorem 4.2. *Let $(f_j) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}_C$. Then there exists a function $f_0 \in \mathcal{F}_M$, a subsequence, again denoted by (f_j) and a sequence $(a_j) \subset \mathbb{R}^3$ of shift vectors such that*

$$\begin{aligned} f_j(\cdot + a_j, \cdot) &\rightharpoonup f_0 \text{ weakly in } L^{1+1/k}(\mathbb{R}^6),\ j \to \infty, \\ \nabla U_{f_j}(\cdot + a_j) &\to \nabla U_{f_0} \text{ strongly in } L^2(\mathbb{R}^3),\ j \to \infty, \end{aligned}$$

and the state f_0 minimizes the energy-Casimir functional $\mathcal{H}_C$ over $\mathcal{F}_M$.

One should realize that the compactness along minimizing sequences captured in the theorem above is indispensable for concluding that the state f_0 is a nonlinearly stable steady state of (VP); its minimizer property is not sufficient for stability. To appreciate this point, we now discuss how stability is obtained; a pedagogical example which further illustrates this issue, which is typical for infinite dimensional dynamical systems, will be given in section 9.

First we remark that by standard arguments which can for example be found in [99, theorem 5.1] the minimizer obtained in theorem 4.2 is of the form

$$f_0(x,p) = (E_0 - E)_+^k \tag{4.5}$$

with E defined as in (2.9) with the induced potential $U_0 = U_{f_0}$; E_0 arises as a Lagrange multiplier. So f_0 is a polytropic steady state of (VP).

For $f \in \mathcal{F}_M$,

$$\mathcal{H}_C(f) - \mathcal{H}_C(f_0) = \mathrm{d}(f,f_0) - \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2 \mathrm{d}x, \tag{4.6}$$

where

$$\begin{aligned}\mathrm{d}(f,f_0) &:= \iint [\Phi(f) - \Phi(f_0) + E(f - f_0)]\, \mathrm{d}p\, \mathrm{d}x \\ &= \iint [\Phi(f) - \Phi(f_0) + (E - E_0)(f - f_0)]\, \mathrm{d}p\, \mathrm{d}x \\ &\geqslant \iint [\Phi'(f_0) + (E - E_0)]\, (f - f_0)\, \mathrm{d}p\, \mathrm{d}x \geqslant 0\end{aligned}$$

with $\mathrm{d}(f,f_0) = 0$ iff $f = f_0$. Let us define

$$\mathrm{dist}(f,f_0) := \mathrm{d}(f,f_0) + \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2 \mathrm{d}x. \tag{4.7}$$

Notice the switch in the sign between (4.6) and (4.7); $\mathrm{dist}(f,f_0)$ is a perfectly fine measure for the distance of a perturbation f from f_0. We obtain the following nonlinear stability result.

Theorem 4.3. *Let f_0 be a minimizer as obtained in theorem 4.2. Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every classical solution $t \mapsto f(t)$ of the VP system with $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$ the initial estimate*

$$\mathrm{dist}(f(0),f_0) < \delta$$

implies that for every $t \geqslant 0$ there is a shift vector $a \in \mathbb{R}^3$ such that

$$\mathrm{dist}(f(t,\cdot + a,\cdot),f_0) < \epsilon.$$

Proof. Assume the assertion is false. Then there exist $\epsilon > 0$, $t_j > 0$, $f_j(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$ such that for $j \in \mathbb{N}$,

$$\mathrm{dist}(f_j(0),f_0) < \frac{1}{j},$$

but for any shift vector $a \in \mathbb{R}^3$,

$$\mathrm{dist}(f_j(t_j,\cdot + a,\cdot),f_0) \geqslant \epsilon.$$

Since $\mathcal{H}_C$ is conserved, (4.6) and the assumption on the initial data imply that $\mathcal{H}_C(f_j(t_j)) = \mathcal{H}_C(f_j(0)) \to \mathcal{H}_C(f_0)$, i.e. $(f_j(t_j)) \subset \mathcal{F}_M$ is a minimizing sequence. Hence by theorem 4.2, $\int |\nabla U_{f_j(t_j)} - \nabla U_0|^2 \to 0$ up to subsequences and shifts in x, provided that there is no other minimizer to which this sequence can converge. By (4.6), $d(f_j(t_j),f_0) \to 0$ as well, which is the desired contradiction.

For the polytropic case (4.5) there exists for each value of the total mass M exactly one corresponding steady state—up to shifts in x—which provides the uniqueness of the minimizer f_0 used above. □

The spatial shifts in the above arguments are necessary due to the Galilei invariance of the problem, and a stability assertion of the form above is sometimes referred to as orbital stability [81].

At the end of this subsection we will briefly comment on various variations and extensions of the basic theme discussed so far, but one variation deserves some attention. As seen from lemma 4.1 the basic compactness mechanism along minimizing sequences operates on spatial densities ρ. Following [97] we define for $r \geqslant 0$,

$$\mathcal{G}_r := \left\{ g \in L^1(\mathbb{R}^3) \mid g \geqslant 0,\ \int \left(\frac{1}{2}|p|^2 g(p) + \Phi(g(p)) \right) \mathrm{d}p < \infty,\ \int g(p)\,\mathrm{d}p = r \right\}$$

and

$$\Psi(r) := \inf_{g \in \mathcal{G}_r} \int \left(\frac{1}{2}|p|^2 g(p) + \Phi(g(p)) \right) \mathrm{d}p.$$

We consider the problem of minimizing the reduced functional

$$\mathcal{H}_r(\rho) := \int \Psi\left(\rho(x)\right) \mathrm{d}x + E_{\mathrm{pot}}(\rho) \tag{4.8}$$

over the set

$$\mathcal{R}_M := \left\{ \rho \in L^1(\mathbb{R}^3) \mid \rho \geqslant 0,\ \int \Psi\left(\rho(x)\right) \mathrm{d}x < \infty,\ \int \rho(x)\,\mathrm{d}x = M \right\};$$

the potential energy $E_{\mathrm{pot}}(\rho)$ is defined in the obvious way and is finite for states in this constraint set. For the polytropic choice (4.2),

$$\Psi(r) = c_n r^{1+1/n},\ r \geqslant 0,$$

which should be compared with the estimates introduced in section 2.4; here $n = k + 3/2$ as before, and $c_n > 0$ is some constant. There is a close relation between the reduced variational problem and the original one. For every function $f \in \mathcal{F}_M$,

$$\mathcal{H}_C(f) \geqslant \mathcal{H}_r(\rho_f),$$

and if $f = f_0$ is a minimizer of $\mathcal{H}_C$ over $\mathcal{F}_M$ then equality holds, i.e. the reduced functional 'supports' the original one from below. Moreover, if $\rho_0 \in \mathcal{R}_M$ is a minimizer of $\mathcal{H}_r$ with induced potential U_0 then it can be lifted to a minimizer f_0 of $\mathcal{H}_C$ in $\mathcal{F}_M$ as follows: the Euler–Lagrange equation for the reduced functional says that

$$\rho_0 = \begin{cases} (\Psi')^{-1}(E_0 - U_0) & , \quad U_0 < E_0, \\ 0 & , \quad U_0 \geqslant E_0, \end{cases}$$

where E_0 is the corresponding Lagrange multiplier. With the particle energy E defined as before the function

$$f_0 := \begin{cases} (\Phi')^{-1}(E_0 - E) & , \quad E < E_0, \\ 0 & , \quad E \geqslant E_0, \end{cases}$$

is a minimizer of $\mathcal{H}_C$ in $\mathcal{F}_M$; for the details see [99, theorem 2.1].

To attack the variational problem through the reduced functional has several advantages. The minimizer of the reduced functional can be shown to be a nonlinearly stable steady state of the Euler–Poisson system with macroscopic equation of state $\sigma = \sigma(\rho) = c_n \rho^{1+1/n}$. The relation between the latter system and (VP) which was noted for isotropic steady states carries over to their stability properties, see [98, 99]. More important for the present context, compactness properties are easier to study for the reduced functional, because the latter lives on a space of functions of x and, in case of spherical symmetry, of the $1d$ variable $r = |x|$. In addition, a result of Burchard and Guo [16, theorem 1] shows that if one minimizes the reduced functional only over spherically symmetric densities $\rho = \rho(|x|)$, the resulting minimizer is actually a minimizer over the full set $\mathcal{R}_M$, and the stability result theorem 4.3 is recovered.

A somewhat different reduced functional which acts on the mass functions $m_f(r) = 4\pi \int_0^r \rho_f(s)\, s^2 \mathrm{d}s$ induced by spherically symmetric phase-space densities f was used in [116]. This was historically the first rigorous stability result for (VP), but for (VP) the approach was not explored any further. The approach may become useful for (EV), see [5, 6, 117].

As mentioned before, there are many variations to the basic theme discussed above, and we mention some:

Remark. (a) The form of the Casimir functional can be much more general than the prototypical form (4.2). Strict convexity of Φ and growth conditions for small and for large arguments compatible with (4.2) are sufficient. Strict convexity of Φ corresponds to the main stability condition (4.1).

(b) For such more general Casimir functionals the uniqueness of the minimizer, which played a role in the proof of theorem 4.3, will in general be lost, but this is not essential for the stability argument, see [106].

(c) Instead of minimizing the energy-Casimir functional $\mathcal{H} + \mathcal{C}$ under the mass constraint $\iint f = M$, one can also minimize the energy $\mathcal{H}$ under the mass-Casimir constraint $\iint f + \mathcal{C}(f) = M$. This has the advantage that one can cover the polytropes (4.5) for $0 < k < 7/2$, see [42, 99], and, with some extra effort, also the limiting case $k = 7/2$, the so-called Plummer sphere; for $k > 7/2$ finite mass and physical relevance are lost.

(d) The reduction mechanism does no longer work for the situation described in (c), but this is as it should be, since for $k > 3/2$, i.e. $n > 3$, stability of the corresponding Euler–Poisson steady states is lost, see [60]. That the (VP) steady states remain stable also for $k > 3/2$ shows that the parallels between the Euler and the Vlasov matter models have their (obscure) limitations.

(e) By making the Casimir functional depend on the angular momentum variable L, in which case it should no longer be called 'Casimir' functional, steady states depending on L can be covered, see [38, 40, 41]. Besides such spherically symmetric, non-isotropic states one can apply the method also to states with axial symmetry, with a point mass at the center, or to flat steady states with or without a dark matter halo, see [25–27, 43, 95, 108].

(f) One can also minimize the energy $\mathcal{H}$ under two separate constraints, a mass constraint and a Casimir constraint, see [103]. Along these lines the arguably strongest result on global minimizers for (VP) was obtained by Lemou *et al* [68].

4.2. Local minimizers; the structure of $D^2\mathcal{H}_C$

The global minimizer approach reviewed in the previous subsection has been quite successful, but it also has limitations. Suppose we want to investigate the stability of the King model, an important steady state of (VP) which appears in the astrophysics literature, obtained via

$$\varphi(\eta) = (\mathrm{e}^{\eta} - 1)_+.$$

Then the function Φ in the corresponding Casimir functional becomes

$$\Phi(f) = (1 + f) \ln(1 + f) - f,$$

which grows too slowly to control any L^s norm of f with $s > 1$, and hence the key estimate (2.44) cannot be brought into play. If instead we consider (RVP), then the corresponding estimate (2.45) provides no control in the context of the global variational problem to begin with, so the method from the previous subsection fails. Notice further that the global minimizer method provides the existence of a steady state which then turns out to be stable. The method is not really one for addressing the stability of some given steady state, obtained by some other method.

In the present subsection we discuss an approach which aims to show that a given steady state f_0 is a local minimizer of an energy-Casimir functional by examining the structure of the latter near f_0. The method was first applied to the King model in the (VP) context, see [44]. Following [48] we review this approach in the context of (RVP), which is a little closer to (EV) where for analogous reasons the global approach seems to fail as well.

We consider some fixed, isotropic steady state (f_0, ρ_0, U_0) of (RVP) given by an ansatz like (3.2), and an energy-Casimir functional defined as before. By a (formal) expansion,

$$\begin{aligned}\mathcal{H}_C(f) = \mathcal{H}_C(f_0) + \iint (E + \Phi'(f_0))\,(f - f_0)\,\mathrm{d}v\,\mathrm{d}x \\ - \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2 \mathrm{d}x + \frac{1}{2}\iint \Phi''(f_0)(f-f_0)^2 \mathrm{d}p\,\mathrm{d}x + \ldots,\end{aligned}$$

and we now define $\Phi\colon\, [0,\infty[\to \mathbb{R}$ such that f_0 becomes a critical point of $\mathcal{H}_C$, namely

$$\Phi(f) := -\int_0^f \phi^{-1}(z)\,\mathrm{d}z,\ f\in[0,\infty[,$$

so that $\Phi \in C^2([0,\infty[)$ with $\Phi'(f_0) = -E$ on $\mathrm{supp} f_0$. To simplify the discussion we restrict ourselves to the polytropic form (4.5) with $1 \leqslant k < 7/2$ where the above becomes rigorous; the key assumption is again that on its support the ansatz strictly decreases in the particle energy E, see (4.1). The question whether f_0 is a strict local minimizer of $\mathcal{H}_C$ obviously depends on the behavior of the quadratic term in the expansion above, i.e. on

$$D^2\mathcal{H}_C(f_0)(g,g) := \frac{1}{2}\iint_{\{f_0>0\}} \frac{1}{|\phi'(E)|} g^2\,\mathrm{d}p\,\mathrm{d}x - \frac{1}{8\pi}\int |\nabla U_g|^2 \mathrm{d}x;$$

we write the argument g twice to emphasize that this is a term which is quadratic in g, and we notice that $\phi' < 0$ where $f_0 > 0$. It was a remarkable insight in the astrophysics community and for the (VP) case that on so-called linearly dynamically accessible states $g = \{f_0, h\} = \phi'(E)\{E,h\}$ the quadratic term $D^2\mathcal{H}_C(f_0)(g,g)$ is positive definite, see [62, 111], and the analogous result holds for (RVP); the Poisson bracket $\{\cdot,\cdot\}$ was introduced in (2.8).

Lemma 4.4. *Let $h \in C_c^\infty(\mathbb{R}^6)$ be spherically symmetric with $\mathrm{supp}\, h \subset \{f_0 > 0\}$ and such that $h(x,-p) = -h(x,p)$. Then the following inequality holds:*

$$D^2\mathcal{H}_C(f_0)(\{E,h\},\{E,h\}) \geqslant \frac{1}{2}\iint \frac{1}{|\phi'(E)|}\left[|x\cdot p|^2\left|\left\{E, \frac{h}{x\cdot p}\right\}\right|^2 + \frac{U_0'}{r(1+|p|^2)^{3/2}}h^2\right]\mathrm{d}p\,\mathrm{d}x.$$

This lemma is proven in [48, lemma 3.4]. It provides positive definiteness of $D^2\mathcal{H}_C(f_0)$ on dynamically accessible states in a quantified manner. A crucial step in any stability analysis is to specify the set of admissible perturbations. In astrophysical reality, perturbations arise by some exterior force acting on the steady state ensemble. It redistributes the particles in phase space by a measure preserving flow, leading to perturbations of the form $f = f_0 \circ T$ with $T\colon\, \mathbb{R}^6 \to \mathbb{R}^6$ a measure preserving diffeomorphism. Such perturbations are called *dynamically accessible from* f_0. For the case at hand we restrict ourselves to spherically symmetric such perturbations and require that the diffeomorphism $T\colon\, \mathbb{R}^6 \to \mathbb{R}^6$ *respects spherical symmetry*, i.e. for all $x, p \in \mathbb{R}^3$ and all rotations $A \in \mathrm{SO}(3)$,

$$T(Ax, Ap) = (Ax', Ap') \text{ and } |x' \times p'| = |x \times p|, \text{ where } (x', p') = T(x,p).$$

From a physics point of view this restriction is undesirable. The set of admissible perturbations is defined as

$$\mathcal{D}_{f_0} := \Big\{ f = f_0 \circ T \mid \ T\colon \mathbb{R}^6 \to \mathbb{R}^6 \text{ is a measure preserving } C^1\text{-diffeomorphism} \\ \text{which respects spherical symmetry} \Big\}.$$

This set is invariant under classical solutions of (RVP). At least formally, states of the bracket form $g = \{f_0, h\}$ are tangent vectors to the manifold $\mathcal{D}_{f_0}$ at the point f_0, and the set of these states is invariant under the linearized dynamics; this terminology is borrowed from Hamiltonian dynamics, see [83].

We are going to measure the distance of a state $f \in \mathcal{D}_{f_0}$ from the steady state f_0 by the same quantity which we used in the previous subsection, namely

$$\mathrm{dist}(f,f_0) = \iint [\Phi(f) - \Phi(f_0) + E(f - f_0)]\,\mathrm{d}p\,\mathrm{d}x + \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2\,\mathrm{d}x,$$

see (4.7). Then

$$\mathrm{dist}(f,f_0) = \mathcal{H}_C(f) - \mathcal{H}_C(f_0) + \frac{1}{4\pi}\int |\nabla U_f - \nabla U_0|^2\,\mathrm{d}x. \tag{4.9}$$

It can be shown by Taylor expansion that there exists a constant $C > 0$ which depends only on the steady state f_0 such that

$$\|f - f_0\|_2^2 + \|\nabla U_f - \nabla U_0\|_2^2 \leqslant C\ \mathrm{dist}(f,f_0),\ f \in \mathcal{D}_{f_0},$$

see [48, lemma 3.1]. The key result is the following theorem which says—in a precise, quantified manner—that the steady state is a local minimizer of the energy-Casimir functional in the set $\mathcal{D}_{f_0}$.

Theorem 4.5. *There exist constants* $\delta_0 > 0$ *and* $C_0 > 0$ *such that for all* $f \in \mathcal{D}_{f_0}$ *with* $\mathrm{dist}(f,f_0) \leqslant \delta_0$ *the following estimate holds:*

$$\mathcal{H}_C(f) - \mathcal{H}_C(f_0) \geqslant C_0 \|\nabla U_f - \nabla U_0\|_2^2.$$

The proof goes by contradiction: if the theorem were false, one could eventually construct a linearly dynamically accessible state that would contradict the positive definiteness of $D^2\mathcal{H}_C(f_0)$ obtained in lemma 4.4; for the quite technical and non-trivial details we refer to [48]. Stability of f_0 is an immediate corollary.

Theorem 4.7. *There exist constants* $\delta > 0$ *and* $C > 0$ *such that every solution* $t \mapsto f(t)$ *of (RVP) which starts close to* f_0 *in the sense that* $f(0) \in \mathcal{D}_{f_0}$ *with* $\mathrm{dist}(f(0),f_0) < \delta$*, exists globally in time and satisfies the estimate*

$$\mathrm{dist}(f(t),f_0) \leqslant C\,\mathrm{dist}(f(0),f_0),\ t \geqslant 0.$$

Proof. With δ_0 and C_0 from theorem 4.5, define $\delta := \delta_0(1 + 1/(4\pi C_0))^{-1}$, and consider a solution $[0,T[\ni t \mapsto f(t)$ of (RVP) with $f(0) \in \mathcal{D}_{f_0}$ on some maximal interval of existence; a suitable local existence result can be found in [64]. Assume that

$$\mathrm{dist}(f(0),f_0) < \delta < \delta_0.$$

By continuity,

$$\mathrm{dist}(f(t),f_0) < \delta_0,\ t \in [0,t^*[,$$

where $0 < t^* \leqslant T$ is chosen maximal. Since $f(t) \in \mathcal{D}_{f_0}$ for all $t \in [0,T[$, theorem 4.5, the relation (4.9), and the fact that $\mathcal{H}_C$ is conserved yield the following chain of estimates for $t \in [0,t^*[$:

$$\begin{aligned}\mathrm{dist}(f(t),f_0) &= \mathcal{H}_C(f(t)) - \mathcal{H}_C(f_0) + \frac{1}{4\pi}\|\nabla U_{f(t)} - \nabla U_0\|_2^2\\ &\leqslant \mathcal{H}_C(f(t)) - \mathcal{H}_C(f_0) + \frac{1}{4\pi C_0}\left(\mathcal{H}_C(f(t)) - \mathcal{H}_C(f_0)\right)\\ &= \left(1+\frac{1}{4\pi C_0}\right)\left(\mathcal{H}_C(f(0)) - \mathcal{H}_C(f_0)\right) \leqslant \left(1+\frac{1}{4\pi C_0}\right)\mathrm{dist}(f(0),f_0) < \delta_0.\end{aligned}$$

This implies that $t^* = T$. Thus $E_{\mathrm{kin}}(f(t))$ is bounded on $[0,T[$ which for spherically symmetric solutions is sufficient to conclude that $T = \infty$, see [48, proposition 4.1] and [64].

□

A nice feature of this theorem, also in view of the (EV) case, is that the stability estimate provided by the theorem implies global existence of spherically symmetric solutions which start close enough to f_0, while spherically symmetric solutions of (RVP) with $\mathcal{H}(f(t)) < 0$ are known to blow up in finite time, see [32]. The only previously known global solutions of (RVP) were small data solutions (and steady states). We should also point out that the need for the spatial shifts which were necessary in theorem 4.3 is eliminated by the restriction to spherical symmetry.

We close the discussion of the local minimizer approach by some comments on possible variations and extensions.

Remark. (a) We restricted ourselves to the polytropic case (4.5) in order to avoid formulating the general conditions on the steady state which are needed for the above arguments, see [48].

(b) The method works for (VP) just as well and was introduced in [44] to deal with the King model for which the global method fails.

(c) The local minimizer method can be combined with a suitable reduction of the energy functional. In [69] this was done for (VP) by exploiting the monotonicity of $\mathcal{H}$ under generalized symmetric rearrangements. This analysis was inspired by results from the astrophysics literature [2, 78, 86, 115] and was restricted to spherical symmetry. The latter restriction was removed in [70] which provides arguably the strongest result on (VP) in the spirit of the present subsection.

4.3. Linearization

Why is it that linearization has up to this point not shown up in this review, when this approach is probably the first that one encounters in the relevant mathematics courses and when it figures most prominently in the relevant astrophysics literature [9, 12, 19, 28, 62]? For a possible answer we have to look at the linearization of (VP) about some given steady state f_0, which we take as isotropic; $f_0 = \phi(E)$.

If we substitute $f = f_0 + \delta f$ into (VP), use the fact that f_0 is a stationary solution, and drop the term which is quadratic in δf with the justification that δf is very small, the result is the equation

$$\partial_t \delta f + \mathcal{T}\delta f - \nabla U_{\delta f(t)} \cdot p\,\phi'(E) = 0, \tag{4.10}$$

where

$$\mathcal{T} := p \cdot \partial_x - \nabla U_0 \cdot \partial_p = \{\cdot, E\}$$

is the transport operator associated to the steady state f_0, i.e. the operator which generates the characteristic flow in the steady state potential $U_0 = U_{f_0}$, see [102]. Following Antonov [9] we split $\delta f = \delta f_+ + \delta f_-$ into its even and odd parts with respect to p,

$$\delta f_\pm(t,x,p) = \frac{1}{2}\left(\delta f(t,x,p) \pm \delta f(t,x,-p)\right).$$

Since $U_{\delta f(t)} = U_{\delta f_+(t)}$,

$$\partial_t \delta f_- + \mathcal{T}\delta f_+ = \nabla U_{\delta f_+} \cdot p\,\phi'(E),$$
$$\partial_t \delta f_+ + \mathcal{T}\delta f_- = 0.$$

We differentiate the first equation with respect to t and substitute the second one in order to eliminate δf_+. If we write g instead of δf_- the linearized (VP) system takes the form

$$\partial_t^2 g + \mathcal{L} g = 0, \tag{4.11}$$

where the *Antonov operator* $\mathcal{L}$ is defined as

$$\mathcal{L} g := -\mathcal{T}^2 g - \mathcal{R} g = -\mathcal{T}^2 g + \nabla U_{\mathrm{div}\, j_g} \cdot p\,\phi'(E)$$

with $j_g := \int p g \mathrm{d}p$; notice that $\partial_t U_{\delta f_+} = U_{\partial_t \delta f_+} = -U_{\mathcal{T}\delta f_-}$ and $\rho_{\mathcal{T}\delta f_-} = \mathrm{div}\, j_{\delta f_-}$. The operator $\mathcal{R}$ is the *gravitational response operator*. We will not go into the functional analysis details of properly defining these operators on suitable Hilbert spaces. This has been done in [52], in particular, $\mathcal{L}$ can be realized as a self-adjoint operator on some Hilbert space; since the latter is a weighted L^2 space on $\{f_0 > 0\}$ with weight $|\phi'(E)|^{-1}$ the key assumption on the steady state is again that (4.1) holds.

One can now check that a state $f = f(x,p)$ is an eigenfunction of the operator in (4.10) with eigenvalue λ iff $g = f_-$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $\mu = \lambda^2$. Since the spectrum of $\mathcal{L}$ is real, the eigenvalues of (4.10) come in pairs of the form $\pm\lambda$ and $\pm \mathrm{i}\lambda$ with $\lambda \in \mathbb{R}$. But this means that the best possible situation as to stability is that the spectrum of (4.10) sits on the imaginary axis, which is the situation when even in finite dimensions no conclusion to nonlinear stability is possible; the example in section 9 will show that in such a situation stability (in the Lyapunov sense) cannot even be concluded for the linearized system.

Given the fact that the variational methods by-pass all spectral considerations and yield nonlinear stability directly, linearization seemed, for the author, of little value for the questions at hand. But this conclusion turned out to be too rash for two reasons. Firstly, variational methods so far do not seem to succeed for (EV), while linearization has lead to some interesting, non-trivial results. Secondly, once a steady state is known to be nonlinearly stable the question arises how exactly it responds to perturbations: does it start to oscillate in a time-periodic way or are such oscillations damped? Very recently, progress on this question was made via linearization [51, 52, 65], and hence we now take a closer look at (4.11).

To do so we restrict ourselves to spherically symmetric functions; $f = f(x,p)$ is spherically symmetric in the sense of (2.19) iff be abuse of notation

$$f(x,p) = f(r,w,L), \text{ where } r = |x|,\ w = \frac{x \cdot p}{r},\ L = |x \times p|^2.$$

In order to understand the transport operator $\mathcal{T}$ we must understand the characteristic flow in the stationary potential $U_0 = U_0(r)$.

Lemma 4.7. *(a) Under the assumption of spherical symmetry the characteristic system (2.4) for the stationary potential U_0 takes the form*

$$\dot{r} = w,\ \dot{w} = -\Psi_L'(r),\ \dot{L} = 0, \tag{4.12}$$

where the effective potential Ψ_L is defined as

$$\Psi_L\colon\ ]0,\infty[\ \to \mathbb{R},\ \Psi_L(r) := U_0(r) + \frac{L}{2r^2}.$$

The particle energy E is conserved and takes the form

$$E = E(r,w,L) = \frac{1}{2}w^2 + \Psi_L(r).$$

(b) For any $L>0$ there exists a unique $r_L>0$ such that $\min\Psi_L = \Psi_L(r_L) < 0$, and for any $E \in]\Psi_L(r_L),0[$ there exist two unique radii $r_\pm(E,L)$ satisfying

$$0 < r_-(E,L) < r_L < r_+(E,L) < \infty \text{ and } \Psi_L(r_\pm(E,L)) = E.$$

(c) Let $t \mapsto (r(t),w(t),L)$ be a solution of (4.12) with $\Psi_L(r_L) < E = E(r(t),w(t),L) < 0$. Then r(t) oscillates between $r_-(E,L)$ and $r_+(E,L)$, and the period of this motion, i.e. the time needed for r(t) to travel from $r_-(E,L)$ to $r_+(E,L)$ and back, is given by the period function of the steady state,

$$T(E,L) := 2\int_{r_-(E,L)}^{r_+(E,L)} \frac{\mathrm{d}r}{\sqrt{2E-2\Psi_L(r)}}.$$

These assertions a fairly easy to see. The key property of the effective potential is that it has a single well structure. Since

$$\Psi_L'(r) = \frac{1}{r^3}\left(rm(r) - L\right)$$

and $r \mapsto rm(r)$ is strictly increasing from 0 to ∞, Ψ_L' has a unique zero $r_L > 0$, and Ψ_L is strictly decreasing on $]0,r_L]$ with $\lim_{r\to 0}\Psi_L(r) = \infty$, and strictly increasing on $[r_L,\infty[$ with $\lim_{r\to\infty}\Psi_L(r) = 0$; the mass function $m(r)$ is defined as in (2.42).

The structure of the stationary characteristic flow can be used to introduce action-angle variables on the set

$$D := \{(r,w,L) \in \mathbb{R}^3 \mid f_0(r,w,L) > 0\};$$

this is a slight abuse of the notation introduced in proposition 3.1. For $(r,w,L) \in D$ let $(R,W)(\cdot,r,w,L)$ be the solution to (4.12) with $(R,W)(0,r,w,L) = (r,w)$; $(R,W)(\cdot,r,w,L)$ is periodic with period $T(E,L)$, where $E = E(r,w,L)$. We supplement the action variables (E,L) with the angle variable $\theta \in [0,1]$ defined by

$$(r,w,L) = ((R,W)(\theta T(E,L), r_-(E,L),0,L),L);$$

the mapping $[0,\frac{1}{2}] \ni \theta \mapsto R(\theta T(E,L), r_-(E,L),0,L) \in [r_-(E,L), r_+(E,L)]$ is bijective with inverse

$$\theta(r,E,L) := \frac{1}{T(E,L)}\int_{r_-(E,L)}^{r} \frac{\mathrm{d}s}{\sqrt{2E-2\Psi_L(s)}}.$$

Functions defined on D can now be written as functions of the action-angle variables (E,L,θ). By the chain rule,

$$\left(\mathcal{T}^2 g\right)(E,L,\theta) = \frac{1}{T^2(E,L)}(\partial_\theta^2 g)(E,L,\theta) \tag{4.13}$$

for suitable functions g defined on D.

One can now analyze the spectra of $-\mathcal{T}^2$ and $\mathcal{L}$. Using (4.13) and the fact that $\mathcal{R}$ is relatively $\mathcal{T}^2$-compact, one can show that the essential spectra of $\mathcal{L}$ and $-\mathcal{T}^2$ coincide, and

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma_{\mathrm{ess}}(-\mathcal{T}^2) = \sigma(-\mathcal{T}^2) = \overline{\left\{ \frac{4\pi^2 k^2}{T^2(E,L)} \;\middle|\; k \in \mathbb{N}_0,\ (E,L) \in \mathring{D}^{EL} \right\}}, \tag{4.14}$$

where $D^{EL} = (E,L)(D)$, see [52, theorems 5.7 and 5.9]. For suitable steady states the period function $T(E,L)$ is bounded from above and bounded away from zero on the support of the steady state. Thus (4.14) shows that the essential spectrum has a gap between 0 and the value $\frac{4\pi^2}{\sup^2(T)}$, the *principal gap* G.

The spectrum of $-\mathcal{T}^2$ is purely essential, but the spectrum of $\mathcal{L}$ may contain isolated eigenvalues, in particular, eigenvalues in the principal gap G. To obtain such eigenvalues a version of the Birman–Schwinger principle has been developed, inspired by a paper by Mathur [80]. It is easily checked that $\lambda \in G$ is an eigenvalue of $\mathcal{L}$ iff 1 is an eigenvalue of the operator

$$Q_\lambda = \mathcal{R}\left(-\mathcal{T}^2 - \lambda\right)^{-1}.$$

The operator Q_λ is not easy to analyze directly, but due to spherical symmetry,

$$(\mathcal{R}g)(r,w,L) = -\frac{4\pi^2}{r^2}\, w\,\phi'(E) \int_{-\infty}^{\infty}\int_0^{\infty} \tilde{w}\, g(r,\tilde{w},\tilde{L})\, \mathrm{d}\tilde{L}\, \mathrm{d}\tilde{w}.$$

Hence $\mathcal{R}$ and Q_λ map onto functions of the form $|\phi'(E)|\, w\, F(r)$ which allows the definition of an operator

$$\mathcal{M}_\lambda \colon \mathcal{F} \to \mathcal{F}$$

on a Hilbert space of functions of the radial variable r such that any eigenvalue of $\mathcal{M}_\lambda$ gives an eigenvalue of Q_λ. When considered on the appropriate function space $\mathcal{F}$ this *Mathur operator* is a symmetric Hilbert–Schmidt operator with an integral kernel representation. The largest element in its spectrum, which is an eigenvalue, is given by

$$M_\lambda = \sup\left\{ \langle h, \mathcal{M}_\lambda h\rangle_{\mathcal{F}} \mid h \in \mathcal{F},\ \|h\|_{\mathcal{F}} = 1 \right\}.$$

It follows that the operator $\mathcal{L}$ has an eigenvalue in the principal gap G iff there exists $\lambda \in G$ such that $M_\lambda \geqslant 1$, see [52, theorem 8.11]. This criterion can be verified for certain examples of steady states by rigorous proof, and for more general examples with numerical support, see [52, section 8.2].

A positive eigenvalue of $\mathcal{L}$ gives rise to a time-periodic, oscillating solution of the linearized (VP) system (4.10), and this explains—at least on the linear level—numerical observations made in [89]; the fact that the latter oscillating solutions pulse in the sense that their support expands and contracts can be understood by linearization in mass-Lagrange variables, which leads to the same spectral problem, see [52, section 3.2].

But in [89] it was also observed that some steady states upon perturbation start to oscillate in a damped way. In [51] such damping phenomena are for the first time rigorously analyzed in the gravitational situation. A family of steady states of (VP) with a point mass at the center is constructed, which are parameterized by their polytropic index $k > 1/2$, so that the phase space density of the steady state is C^1 at the vacuum boundary if and only if $k > 1$; see remark (e) at the end of section 3. The following dichotomy result is established: if $k > 1$, linear perturbations damp, and if $1/2 < k \leqslant 1$ they do not. The undamped oscillations for $1/2 < k \leqslant 1$ are obtained by Birman–Schwinger type arguments as above. The damping for $k > 1$ occurs on the level of macroscopic quantities and is (up to now) non-quantitative: no damping rate is established, but (for example)

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\nabla U_{\mathcal{T}f(t,\cdot)}\|_{L^2}^2 \mathrm{d}t = 0,$$

where $[0,\infty[\ni t \to f(t)$ is any solution to (4.11) with initial data $f(0)$ in the domain of $\mathcal{L}$.

This type of damping is obtained by an application of the RAGE theorem [17]. The main fact which has to be established in order to apply this theorem is that the operator $\mathcal{L}$ has no eigenvalues, and the key difficulty is to exclude eigenvalues embedded in the essential spectrum, see [51, theorem 4.5].

The damping result can also be viewed as a result on macroscopic, asymptotic stability for the corresponding steady states on the linearized level. The importance of relaxation processes in astrophysics can be seen from the discussion in [12] and the references there; we explicitly mention the pioneering work of Lynden–Bell [76, 77].

In the plasma physics situation an analogous damping phenomenon around spatially homogeneous steady states was discovered by Landau [66] on the linearized level, and on the non-linear level in the celebrated work of Mouhot and Villani [82], see also [11, 33]. It should be noticed that in this case the characteristic flow of the unperturbed steady state is simple free streaming, so the corresponding result for the gravitational case has to deal with substantial and qualitatively new difficulties due to the non-trivial characteristic steady state flow.

5. Stability for (EV)—steady states with small central redshift

5.1. The set-up

We consider the spherically symmetric (EV) system as formulated in section 2.3 and choose the formulation which employs the non-canonical momentum variable v, see (2.31). Functions or states $f = f(x,v) \geqslant 0$ are always spherically symmetric, i.e.

$$f(x,v) = f(r,w,L) \text{ with } r = |x|,\ w = \frac{x\cdot v}{r},\ L = |x\times v|^2, \tag{5.1}$$

and induce the mass-energy density

$$\rho_f(r) = \rho_f(x) = \int \langle v\rangle f(x,v)\,\mathrm{d}v$$

and the metric component $\lambda = \lambda_f$ via

$$\mathrm{e}^{-2\lambda_f(r)} = 1 - \frac{2m_f(r)}{r} = 1 - \frac{8\pi}{r}\int_0^r \rho_f(s)\,s^2\mathrm{d}s; \tag{5.2}$$

only states f with $2m_f(r) < r$ are admissible.

Let us fix some steady state (f_0, λ_0, μ_0) of (EV) of the form (3.3) with (3.4); for the moment the central redshift κ of this steady state is not relevant and suppressed. Let us also fix a function $\Phi \in C^1([0,\infty[)$ with $\Phi(0) = 0$. In section 2.3 we introduced the energy and Casimir functionals, see (2.38), (2.39), and we define the energy-Casimir functional

$$\mathcal{H}_C(f) := \mathcal{H}(f) + \mathcal{C}(f) = \iint \langle v\rangle f(x,v)\,\mathrm{d}v\,\mathrm{d}x + \iint e^{\lambda_f}\Phi(f(x,v))\,\mathrm{d}v\,\mathrm{d}x.$$

We formally expand $\mathcal{H}_C$ about f_0:

$$\mathcal{H}_C(f_0 + \delta f) = \mathcal{H}_C(f_0) + D\mathcal{H}_C(f_0)(\delta f) + D^2\mathcal{H}_C(f_0)(\delta f, \delta f) + \mathrm{O}((\delta f)^3).$$

To proceed we again make the standard stability assumption that on the support of the steady state ϕ is strictly decreasing, see (4.1). If the function Φ is such that

$$\Phi'(f_0) = \Phi'(\phi(E)) = -E, \text{ i.e. } \Phi' = -\phi^{-1},$$

then a non-trivial, formal computation [49], see also [61], shows that

$$D\mathcal{H}_C(f_0)(\delta f) = 0$$

and

$$D^2\mathcal{H}_C(f_0)(\delta f, \delta f) = \frac{1}{2}\iint \frac{\mathrm{e}^{\lambda_0}}{|\phi'(E)|}(\delta f)^2 \,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2}\int_0^\infty \mathrm{e}^{\mu_0-\lambda_0}\,(2r\mu_0' + 1)\,(\delta\lambda)^2\,\mathrm{d}r. \tag{5.3}$$

Here $\delta\lambda$ is to be expressed in terms of δf through the variation of (5.2), see (5.7) below. Only perturbations δf which are supported in the support of the steady state f_0 are considered—δf must be small compared to f_0—which is important for the first integral in (5.3). We see that the steady state is a critical point of the energy-Casimir functional $\mathcal{H}_C$, but like for (VP) the quadratic term (5.3) is the sum of two terms with opposite signs, which is the central difficulty in the stability analysis; one should notice that since $\mu_0' \geqslant 0$,

$$2r\mu_0' + 1 \geqslant 1,\ r \geqslant 0. \tag{5.4}$$

For (VP), one way to by-pass this difficulty was the global minimizer approach explained in section 4.1, but so far this strategy has not been successful for (EV) for reasons which we indicated in section 2.4. But we also saw in section 4.2 how for (RVP) $D^2\mathcal{H}_c(f_0)$ is positive definite on linearly dynamically accessible states, and how this fact can lead to a stability result as well. We follow this route in the present (EV) case. To do so we first need to discuss the concept of dynamically accessible states for (EV).

An admissible state f is *nonlinearly dynamically accessible from* f_0 iff for all $\chi \in C^1(\mathbb{R})$ with $\chi(0) = 0$,

$$\mathcal{C}_\chi(f) = \mathcal{C}_\chi(f_0), \tag{5.5}$$

where $\mathcal{C}_\chi$ is defined like $\mathcal{C}$, but with the general function χ instead of Φ, the latter being specific for the steady state under consideration. Property (5.5) is preserved by the flow of the Einstein–Vlasov system. Taking the first variation in (5.5), a definition for δf to be linearly dynamically accessible could be that

$$D\mathcal{C}_\chi(f_0)(\delta f) = \iint \mathrm{e}^{\lambda_0}\,(\chi'(f_0)\delta f + \chi(f_0)\delta\lambda)\,\mathrm{d}v\,\mathrm{d}x = 0 \tag{5.6}$$

for all $\chi \in C^1(\mathbb{R})$ with $\chi(0) = 0$, where

$$\delta\lambda = \mathrm{e}^{2\lambda_0}\frac{4\pi}{r}\int_0^r s^2\rho_{\delta f}(s)\,\mathrm{d}s. \tag{5.7}$$

This needs to be turned into a more explicit and workable definition. A suitable integration by parts turns (5.6) into

$$D\mathcal{C}_\chi(f_0)(\delta f) = \iint \mathrm{e}^{\lambda_0}\chi'(f_0)\left[\delta f - \mathrm{e}^{\mu_0}\delta\lambda\,\phi'(E)\frac{w^2}{\langle v\rangle}\right]\mathrm{d}v\,\mathrm{d}x = 0, \tag{5.8}$$

see [49, lemma 3.1]. Hence a variation δf satisfies (5.6), if

$$e^{\lambda_0}\delta f - e^{\mu_0+\lambda_0}\delta\lambda\,\phi'(E)\frac{w^2}{\langle v\rangle} = \{h, f_0\} \tag{5.9}$$

for some spherically symmetric generating function $h \in C^2(\mathbb{R}^6)$; note that for any such h,

$$\iint \chi'(f_0)\,\{h, f_0\}\,\mathrm{d}v\,\mathrm{d}x = 0.$$

We make the definition more explicit; recall that $D = \{f_0 > 0\}$.

Definition 5.1. A state δf is *linearly dynamically accessible from* f_0 if there exists some spherically symmetric generating function $h \in C^1(\overline{D})$ such that

$$\delta f = f_h := e^{-\lambda_0}\{h, f_0\} + 4\pi r e^{2\mu_0+\lambda_0}\phi''(E)\frac{w^2}{\langle v\rangle}\int \phi'(E(x,\tilde{v}))\,h(x,\tilde{v})\,\tilde{w}\,\mathrm{d}\tilde{v}. \tag{5.10}$$

Notice that possible values of the generating function h outside D would not influence δf which vanishes outside D. The justification for this definition is the following result, see [49, proposition 3.2]; we will see later that this form of δf is preserved under the linearized (EV) dynamics, and we will give a slightly more general, functional-analysis type definition of this concept.

Proposition 5.2. *If δf is linearly dynamically accessible from f_0 and $\delta\lambda$ is defined by (5.7), then*

$$\delta\lambda = \lambda_h := 4\pi r e^{\mu_0+\lambda_0}\int \phi'(E)\,h(x,v)\,w\,\mathrm{d}v, \tag{5.11}$$

δf satisfies both (5.6) and (5.9), and

$$\delta f = f_h = \phi'(E)\left(e^{-\lambda_0}\{h, E\} + e^{\mu_0}\lambda_h\frac{w^2}{\langle v\rangle}\right). \tag{5.12}$$

The key feature of linearly dynamically accessible states is that if we substitute such a state into $D^2\mathcal{H}_C(f_0)$, then, for sufficiently non-relativistic steady states, this quadratic form becomes positive definite, just as for (RVP), see lemma 4.4. To see this we have to understand the behavior of the steady states obtained in proposition 3.1 for small redshift κ.

5.2. Steady states for κ small—the non-relativistic limit

We fix an ansatz function φ satisfying (3.4), define

$$\varphi_N(\eta) := C\eta^k \text{ for } \eta > 0$$

with $0 < k < 3/2$ and $C > 0$, and require that

$$\varphi(\eta) = \varphi_N(\eta) + \mathrm{O}(\eta^{k+\delta}) \text{ for } \eta \to 0+, \tag{5.13}$$

with some $\delta > 0$; notice that this condition implies (3.10). For $\kappa > 0$ small we wish to relate y_κ and the induced steady state $(f_\kappa, \lambda_\kappa, \mu_\kappa)$ obtained in proposition 3.1(b) to the solution y_N of the Newtonian problem (3.9), with $y_N(0) = 1$ and φ_N as Newtonian microscopic equation of state, and the induced steady state (f_N, U_N) of (VP). We define

$$a := \frac{k + 1/2}{2}.$$

Proposition 5.3. *There exist constants* $\kappa_0 > 0$*,* $S_0 > 0$*, and* $C > 0$ *such that for all* $\kappa \in]0, \kappa_0]$,

$$\mathrm{supp}\, \rho_\kappa \subset [0, \kappa^{-a} S_0],$$

and for all $r \geqslant 0$,

$$\left|\kappa^{-1} y_\kappa(r) - y_N(\kappa^a r)\right| \leqslant C\kappa^\delta,$$

$$\left|\mathrm{e}^{2\lambda_\kappa(r)} - 1\right| \leqslant C\kappa,$$

$$\left|\kappa^{-1}\mu_\kappa(r) - U_N(\kappa^a r)\right| + \left|\kappa^{-1-2a}\rho_\kappa(r) - \rho_N(\kappa^a r)\right| \leqslant C\kappa^\delta.$$

This result was shown in [50]. For the proof one introduces a rescaled function $\bar{y}_\kappa$ and a rescaled radial variable s by

$$y_\kappa(r) = \kappa \bar{y}_\kappa(\kappa^a r) = \kappa \bar{y}_\kappa(s), \quad s = \kappa^a r.$$

One can then derive an equation for the function $\bar{y}_\kappa$ which corresponds to the equation (3.8) for y_κ. In this rescaled version of (3.8) the microscopic equation of state becomes

$$\varphi_\kappa(\eta) := \kappa^{-k} \varphi(\kappa\eta),$$

which by (5.13) converges to φ_N for $\kappa \to 0$. In addition, the 'relativistic corrections' in the rescaled version of (3.8) like the pressure term σ and the term $2m/r$ in the denominator pick up multiplicative factors of κ, while $\bar{y}_\kappa(0) = 1 = y_N(0)$. A lengthy Gronwall-type argument implies that there exist constants $\kappa_0 > 0$ and $C > 0$ such that for all $0 < \kappa \leqslant \kappa_0$ and $s \geqslant 0$,

$$|\bar{y}_\kappa(s) - y_N(s)| \leqslant C\kappa^\delta.$$

The assertions in proposition 5.3 then follow.

In section 4.3 we saw that action-angle variables are an essential tool for understanding the linearized dynamics in the (VP) case. Introducing these variables relied on the single-well structure of the effective potential Ψ_L discussed in lemma 4.7. For (EV), the steady state characteristics obey the equations

$$\dot{r} = \mathrm{e}^{-\lambda_\kappa(r)}\, \partial_w E_\kappa(r, w, L),\ \dot{w} = -\mathrm{e}^{-\lambda_\kappa(r)}\, \partial_r E_\kappa(r, w, L)$$

with

$$E_\kappa(r, w, L) = \mathrm{e}^{\mu_\kappa(r)} \sqrt{1 + w^2 + \frac{L}{r^2}}.$$

Let us define the analogue of the Newtonian effective potential as

$$\Psi_{\kappa,L}(r) := \mathrm{e}^{\mu_\kappa(r)} \sqrt{1 + \frac{L}{r^2}}$$

and assume that

$$\frac{2m_\kappa(r)}{r} \leqslant \frac{1}{3},\ r > 0. \tag{5.14}$$

Then one can show that $\Psi_{\kappa,L}$ has a single-well structure analogous to lemma 4.7(b), and the conclusions of lemma 4.7(c) and its action-angle consequences remain valid, see [37, section 3]. By proposition 5.3, the condition (5.14) holds for κ small.

Remark. The question whether the steady state characteristic flow has the single-well structure is intimately related to the question whether for spherically symmetric steady states the phase space density can always be written in the form $f = \phi(E, L)$. For (VP) this result, which is sometimes called Jeans' theorem, is a direct consequence of the single-well structure of the effective potential. For (EV), Jeans' theorem is known to be false, see [105]. Numerical evidence strongly suggests that for isotropic steady states of (EV), $2m(r)/r < 1/2$, which is a considerably sharper bound than the general Buchdahl inequality [3, 4, 15], but it is unclear whether this is sufficient to yield the single-well structure.

The information provided by proposition 5.3 can be used to show that on linearly dynamically accessible states the quadratic form $D^2\mathcal{H}_C(f_\kappa)$ is positive definite for κ sufficiently small.

5.3. An energy-Casimir coercivity estimate

As in the previous section, let the microscopic equation of state φ satisfy (3.4) and (5.13), and let (f_0, λ_0, μ_0) be a steady state as obtained in proposition 3.1(b). As an abbreviation, let

$$\begin{aligned}\mathcal{A}(\delta f, \delta f) &:= D^2\mathcal{H}_C(f_0)(\delta f, \delta f)\\ &= \frac{1}{2}\iint \frac{\mathrm{e}^{\lambda_0}}{|\phi'(E)|}(\delta f)^2\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2}\int_0^\infty \mathrm{e}^{\mu_0-\lambda_0}\left(2r\mu_0'+1\right)(\delta\lambda)^2\,\mathrm{d}r.\end{aligned}$$

The following result is the desired energy-Casimir coercivity estimate.

Theorem 5.4. *There exist constants $C^* > 0$ and $\kappa^* > 0$ such that for any $0 < \kappa \leqslant \kappa^*$ and any spherically symmetric function $h \in C^1(\overline{D})$ which is odd in v,*

$$\mathcal{A}(\delta f, \delta f) \geqslant C^* \iint |\phi'(E)| \left((rw)^2 \left|\left\{E, \frac{h}{rw}\right\}\right|^2 + \kappa^{1+2a}|h|^2\right)\mathrm{d}v\,\mathrm{d}x,$$

where h generates the dynamically accessible perturbation δf according to (5.10).

For a dynamically accessible perturbation $\delta f = f_h$ defined by (5.10) and (5.11),

$$\mathcal{A}(\delta f, \delta f) = \mathcal{A}(h,h) = \frac{1}{2}\mathcal{A}_1(h) + \frac{1}{2}\mathcal{A}_2(h),$$

where

$$\begin{aligned}\mathcal{A}_1(h) &:= \iint \mathrm{e}^{-\lambda_0}|\phi'(E)||\{E,h\}|^2\mathrm{d}v\,\mathrm{d}x - \int_0^\infty \mathrm{e}^{\mu_0-\lambda_0}(2r\mu_0{}'+1)(\lambda_h)^2\mathrm{d}r,\\ \mathcal{A}_2(h) &:= -2\iint |\phi'(E)|\{E,h\}\lambda_h \mathrm{e}^{\mu_0}\frac{w^2}{\langle v\rangle}\,\mathrm{d}v\,\mathrm{d}x + \iint |\phi'(E)|\mathrm{e}^{2\mu_0+\lambda_0}\frac{w^4}{\langle v\rangle^2}(\lambda_h)^2\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

It turns out that $\mathcal{A}_1$ yields the desired lower bound while $\mathcal{A}_2$ is of higher order in κ and can be controlled by the positive contribution from $\mathcal{A}_1$. For more details we refer to [50, theorem 5.1], but we emphasize that for the proof the complete structure of the stationary (EV) system must be exploited, in particular, the static version of (2.25) and (3.11) come into play.

The assumption in theorem 5.4 that h is odd in v can be removed. We split a general, spherically symmetric function $h \in C^2(\mathbb{R}^6)$ into its even and odd-in-v parts, $h = h_+ + h_-$. Then

$$\lambda_h = 4\pi\, r\mathrm{e}^{\mu_0+\lambda_0}\int \phi'(E)\,h_-(x,v)\,w\,\mathrm{d}v = \lambda_{h_-},$$

and

$$\delta f_+ = (f_h)_- = \mathrm{e}^{-\lambda_0}\{h_+, f_0\},\ \delta f_- = (f_h)_+ = \mathrm{e}^{-\lambda_0}\{h_-, f_0\} + \gamma \mathrm{e}^{\mu_0}\phi'(E)\frac{w^2}{\langle v\rangle}\lambda_{h_-}.$$

Hence

$$\begin{aligned}\mathcal{A}(\delta f,\delta f) &= \mathcal{A}(\delta f_+,\delta f_+) + \iint e^{\lambda_0}\frac{\delta f_+\,\delta f_-}{|\phi'(E)|}\,dv\,dx + \frac{1}{2}\iint e^{\lambda_0}\frac{|\delta f_-|^2}{|\phi'(E)|}\,dv\,dx\\ &= \mathcal{A}(\delta f_+,\delta f_+) + \frac{1}{2}\iint e^{-\lambda_0}|\phi'(E)|\,|\{E,h_+\}|^2\,dv\,dx\\ &\geqslant C^*\iint|\phi'(E)|\left((rw)^2\left|\left\{E,\frac{h_-}{rw}\right\}\right|^2+\kappa^{1+2a}|h_-|^2\right)dv\,dx\\ &\quad+\frac{1}{2}\iint e^{-\lambda_0}|\phi'(E)|\,|\{E,h_+\}|^2\,dv\,dx. \end{aligned}\tag{5.15}$$

If one tries to proceed from this positive definiteness result on the second variation of $\mathcal{H}_C$ towards nonlinear stability, analogously to section 4.2 for (RVP), serious difficulties in deriving an analogue of theorem 4.5 arise, again from the inherent lack of compactness for (EV). We believe that some analogue of theorem 4.5 remains correct for (EV), but so far the result of the present section has only been used to derive linear stability.

5.4. Linear stability

In order to deal with this issue we need to linearize (EV) about some given steady state (f_0,λ_0,μ_0); for the moment, no assumption is made on the size of κ, because we will use the linearized system also for large κ. We substitute

$$f(t)=f_0+\delta f(t),\ \lambda(t)=\lambda_0+\delta\lambda(t),\ \mu(t)=\mu_0+\delta\mu(t)$$

into the system, use the fact that (f_0,λ_0,μ_0) is a solution, and drop all terms beyond the linear ones in $(\delta f,\delta\lambda,\delta\mu)$. In addition the boundary conditions $\delta\lambda(t,0)=\delta\lambda(t,\infty)=\delta\mu(t,\infty)=0$ are imposed. We observe that

$$\delta\lambda=\lambda_{\delta f}:=e^{2\lambda_0}\frac{4\pi}{r}\int_0^r s^2\rho_{\delta f}(s)\,ds \tag{5.16}$$

is the corresponding solution to the linearized version of the field equation (2.22), see (5.7). The linearized versions of the field equations (2.23) and (2.24) yield

$$\delta\mu'=\mu'_{\delta f}:=4\pi\,re^{2\lambda_0}\sigma_{\delta f}+\left(2\mu_0'+\frac{1}{r}\right)\lambda_{\delta f}, \tag{5.17}$$

$$\dot{\delta\lambda}=-4\pi\,re^{\mu_0+\lambda_0}\jmath_{\delta f}, \tag{5.18}$$

where as before,

$$\rho_{\delta f}=\int\langle v\rangle\,\delta f\,dv,\ \sigma_{\delta f}=\int\frac{w^2}{\langle v\rangle}\delta f\,dv,\ \jmath_{\delta f}=\int w\,\delta f\,dv.$$

If we substitute into the linearization of the Vlasov equation (2.33),

$$\begin{aligned}&\partial_t\delta f+e^{-\lambda_0}\{\delta f,E\}+4\pi\,re^{2\mu_0+\lambda_0}\phi'(E)\left(\frac{w^2}{\langle v\rangle}\jmath_{\delta f}-w\sigma_{\delta f}\right)\\ &\quad-e^{2\mu_0-\lambda_0}\left(2\mu_0'+\frac{1}{r}\right)\lambda_{\delta f}\phi'(E)\,w=0;\end{aligned}\tag{5.19}$$

it can be shown that (5.18) follows from the other equations.

In order to proceed we need to observe that linear dynamic accessibility propagates under the dynamics of the linearized (EV) system. We recall that some $h\in C^1(\overline{D})$ generates a linearly dynamically accessible perturbation $\delta f = f_h$ according to (5.10), and (5.16) turns into $\delta\lambda = \lambda_h$, defined by (5.11). One can check that if $t \mapsto h(t)$ solves the transport equation

$$\partial_t h + \mathrm{e}^{-\lambda_0}\{h, E\} + \mathrm{e}^{\mu_0}\lambda_{h(t)}\frac{w^2}{\langle v\rangle} - \mathrm{e}^{\mu_0}\langle v\rangle\,\mu_{h(t)} = 0 \tag{5.20}$$

with spherically symmetric initial data $h(0) = \mathring{h}$, then $\delta f(t) = f_{h(t)}$ defined according to (5.10) is the solution of the above linearized (EV) system to the linearly dynamically accessible data $\mathring{\delta f} = f_{\mathring{h}}$. In particular, $\delta f(t)$ is the linearly dynamically accessible state generated by $h(t)$. A simple iteration argument shows that for any $\mathring{h} \in C^1(\overline{D})$ there exists a unique solution $h \in C^1([0,\infty[; C(\overline{D})) \cap C([0,\infty[; C^1(\overline{D}))$ of (5.20) with $h(0) = \mathring{h}$. The induced linearly dynamically accessible solution δf needs to be only continuous (unless one demands more regularity of $\mathring{h}$ and φ) and solves the linearized (EV) system integrated along the steady state characteristics, see [10] for the analogous concept for the linearized (VP) system. We do not discuss these issues further, since in the context of the more functional analytic approach in section 7 we solve the linearized (EV) system by a suitable C_0 group.

The important fact here is that such linearly dynamically accessible solutions preserve the energy

$$\mathcal{A}(f_h, f_h) = \mathcal{A}(h,h) = \frac{1}{2}\iint \frac{\mathrm{e}^{\lambda_0}}{|\phi'(E)|}(f_h)^2\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2}\int_0^\infty \mathrm{e}^{\mu_0-\lambda_0}\,(2r\mu_0'+1)\,(\lambda_h)^2\,\mathrm{d}r. \tag{5.21}$$

Combining this fact with theorem 5.4 or with the more general estimate (5.15) proves the following stability result.

Theorem 5.5. *Let C^* and κ^* be as in theorem 5.4, and let $0 < \kappa \leqslant \kappa^*$. Then the steady state $(f_\kappa, \lambda_\kappa, \mu_\kappa)$ is linearly stable in the following sense. For any spherically symmetric function $\mathring{h} \in C^1(\overline{D})$ the solution of the linearized (EV) system with dynamically accessible data $\mathring{\delta f}$ generated by $\mathring{h}$ according to (5.10) satisfies for all times $t \geqslant 0$ the estimate*

$$C^* \iint |\phi'(E)| \left((rw)^2 \left| \left\{ E, \frac{h_-(t)}{rw} \right\} \right|^2 + \kappa^{1+2a}|h_-(t)|^2 \right) \mathrm{d}v\,\mathrm{d}x$$
$$+ \frac{1}{2}\iint \mathrm{e}^{-\lambda_\kappa}|\phi'(E)|\,|\{E, h_+(t)\}|^2\,\mathrm{d}v\,\mathrm{d}x \leqslant \mathcal{A}(\mathring{\delta f}).$$

The restriction to perturbations $\mathring{\delta f}$ of the form (5.10) may seem a bit special. Condition (5.8) suggests that the natural set of perturbations for the linear problem are functions $\mathring{\delta f} \in C^1(\mathbb{R}^6)$ supported on the support of f_0 with the property that

$$\mathrm{e}^{\lambda_0}\mathring{\delta f} - \mathrm{e}^{\mu_0+\lambda_0}\mathring{\delta\lambda}\phi'(E)\frac{w^2}{\langle v\rangle} \text{ is } L^2\text{-orthogonal to any } \psi(f_0) \in L^2(\mathbb{R}^6),\ \psi \in C(\mathbb{R}).$$

For any such perturbation there exists a generating function $h \in C^2(\mathbb{R}^6)$ so that

$$\{h, f_0\} = \mathrm{e}^{\lambda_0}\mathring{\delta f} - \mathrm{e}^{\mu_0+\lambda_0}\mathring{\delta\lambda}\phi'(E)\frac{w^2}{\langle v\rangle},$$

which by proposition 5.2 says that $\mathring{\delta f}$ is linearly dynamically accessible. The proof is analogous to the proof of the parallel fact for (VP) given in [44, section 3.2] and relies on the fact that for κ small the stationary characteristic flow (or rather its effective potential) has a single-well structure, see the end of section 5.2.

6. Instability for (EV)—steady states with large central redshift

We continue to use the set-up which we discussed in section 5.1. We saw in section 5.3 that the second variation $D^2\mathcal{H}_C$ is positive definite on linearly dynamically accessible states, provided the central redshift κ of the steady state $(f_\kappa, \lambda_\kappa, \mu_\kappa)$ in question is small, and we saw in section 5.4 that this fact implies linearized stability of the corresponding steady state. The key to this was a good understanding of the behavior of the steady state in the limit $\kappa \to 0$, the Newtonian limit. It turns out that for κ sufficiently large, such steady states become unstable. The major step towards this result is that there is a direction in which $D^2\mathcal{H}_C$ becomes negative, provided κ is sufficiently large, and the key to this is a good understanding of the behavior of the steady state in the limit $\kappa \to \infty$. This is more challenging and more interesting, because no $\kappa \to \infty$ limiting system seems to suggest itself for the role that (VP) plays as the $\kappa \to 0$ limiting system. But such a system exists.

6.1. Steady states for κ large—the ultrarelativistic limit

We again consider steady states of (EV) as obtained in proposition 3.1. As indicated in part (b) of the remark following that proposition a microscopic equation of state φ gives rise to a macroscopic equation of state which relates pressure and mass-energy density, more precisely,

$$\sigma_\kappa = P(\rho_\kappa), \text{ where } P := h \circ g^{-1} \tag{6.1}$$

with g and h defined by (3.6) and (3.7). When κ is very large also $y_\kappa(r)$ and $\rho_\kappa(r)$ become very large at least for r close to 0. For y very large,

$$g(y) = 4\pi\,\mathrm{e}^{4y} \int_0^{1-\mathrm{e}^{-y}} \varphi(\eta)\,(1-\eta)^2\,\left((1-\eta)^2 - \mathrm{e}^{-2y}\right)^{1/2} \mathrm{d}\eta \approx \mathrm{e}^{4y} =: g^*(y), \tag{6.2}$$

and

$$h(y) = \frac{4\pi}{3}\mathrm{e}^{4y} \int_0^{1-\mathrm{e}^{-y}} \varphi(\eta)\,\left((1-\eta)^2 - \mathrm{e}^{-2y}\right)^{3/2} \mathrm{d}\eta \approx \frac{1}{3}\mathrm{e}^{4y} =: h^*(y), \tag{6.3}$$

where for the sake of notational simplicity we normalize

$$4\pi \int_0^1 \varphi(\eta)\,(1-\eta)^3\,\mathrm{d}\eta = 1.$$

Hence for κ very large and close to the center the equation of state (6.1) asymptotically turns into

$$\sigma_\kappa = P^*(\rho_\kappa) = \frac{1}{3}\rho_\kappa \tag{6.4}$$

which is known in astrophysics and cosmology as the equation of state for radiation. It can be shown that

$$\left|P(\rho) - \frac{1}{3}\rho\right| \leqslant C\rho^{1/2},\ \rho \geqslant 0$$

for some constant $C > 0$, see [47], which is the precise version of the limiting behavior of the equation of state.

Of course now the question arises how the limiting equation of state (6.4) fits into the Vlasov context, since that equation of state cannot come from an isotropic steady state particle distribution of the form (3.3):

$$\sigma(r) = \int f(x,v)\left(\frac{x\cdot v}{r}\right)^2 \frac{\mathrm{d}v}{\langle v\rangle} = \frac{1}{3}\int f(x,v)\,|v|^2 \frac{\mathrm{d}v}{\langle v\rangle} < \frac{1}{3}\int f(x,v)\,(1+|v|^2)\frac{\mathrm{d}v}{\langle v\rangle} = \frac{1}{3}\rho(r);$$

massive particles do not behave like radiation. To obtain a Vlasov-type system which captures the limiting behavior as $\kappa\to\infty$ we must pass to a collisionless ensemble of massless particles. Mathematically, this means that throughout the (EV) system the term $\langle v\rangle$ must be replaced by $|v|$. In particular, (3.3) turns into the ansatz

$$f(x,v) = \phi(\mathrm{e}^{\mu(r)}|v|) = \varphi\left(1-\frac{\mathrm{e}^{\mu(r)}|v|}{E_0}\right).$$

This ansatz satisfies the massless version of (2.33), we get exactly the radiative equation of state (6.4), and

$$\rho(r) = \int \phi(\mathrm{e}^{\mu(r)}|v|)\,|v|\,\mathrm{d}v = 4\pi\int_0^\infty \phi(\eta)\,\eta^3\mathrm{d}\eta\,\mathrm{e}^{-4\mu(r)}$$

which is as expected from (6.2). Hence if y is a solution of (3.8) where g and h are replaced by g^* and h^*, and μ,λ,ρ,σ are induced by y, then these quantities satisfy the stationary Einstein equations together with the radiative equation of state and the above f is a consistent, stationary solution of the massless EV system.

Proceeding as in [93, theorem 3.4] one can obtain the following result.

Lemma 6.1. *For every $\kappa>0$ there exists a unique solution $y^* = y^*_\kappa \in C^1([0,\infty[)$ to the problem*

$$y'(r) = -\frac{1}{1-2m^*(r)/r}\left(\frac{m^*(r)}{r^2}+4\pi r\sigma^*(r)\right),\ y(0)=\kappa>0, \tag{6.5}$$

where $\rho^* = g^*(y)$, $\sigma^* = h^*(y)$ with (6.2), (6.3) and

$$m^*(r) = m^*(r,y) = 4\pi\int_0^r s^2\rho^*(s)\,\mathrm{d}s.$$

For κ very large and close to the center the behavior of the massive steady state is indeed captured by the massless one, more precisely:

Lemma 6.2. *There exists a constant $C>0$ such that for all $\kappa>0$ and $r\geqslant 0$,*

$$|y_\kappa(r)-y^*_\kappa(r)| \leqslant C\mathrm{e}^{2\kappa}\left(r^2+\mathrm{e}^{4\kappa}r^4\right)\exp\left(C\left(\mathrm{e}^{4\kappa}r^2+\mathrm{e}^{8\kappa}r^4\right)\right).$$

In [47, lemma 3.10] this result is proven for the pressures σ instead of the functions y, because that allows one to treat the EV and the Einstein–Euler cases simultaneously. The result above is actually more easy to obtain. Essentially, the proof consists of a lengthy Gronwall-type estimate, based on the equations satisfied by y_κ and y^*_κ, but rewritten in the rescaled radial variable $\tau = \mathrm{e}^{2\kappa}r$, and using the facts that due to the Buchdahl inequality [3, 4],

$$\frac{2m(r)}{r},\ \frac{2m^*(r)}{r} < \frac{8}{9},$$

and that the asymptotics in (6.2) and (6.3) take the quantitative form

$$|g(y)-g^*(y)|+|h(y)-h^*(y)| \leqslant C\mathrm{e}^{2y},\ y\in\mathbb{R},$$

This is indeed a good approximation for large y, since all the terms on the left are then of order e^{4y}.

Since g^* and h^* are strictly positive, a steady state of the massless system is never compactly supported. The massless system has a scaling invariance which is important for what follows.

Lemma 6.3. *Let y_0^* denote the solution of (6.5) with initial data $y_0^*(0) = 0$. Then for all $\kappa > 0$,*

$$y_\kappa^*(r) = \kappa + y_0^*(e^{2\kappa} r), \ r \geqslant 0.$$

We see that in order to understand the behavior of $y_\kappa^*(r)$ for positive, small r and very large κ we need to understand the behavior of the special solution $y_0^*(s)$ for $s \to \infty$. The key point here is that the massless steady state equation (6.5) can be turned into a planar, autonomous dynamical system. We let $w_1(\tau) = r^2\rho(r),\ w_2(\tau) = m(r)/r$ with $\tau = \ln r$. Then the Tolman–Oppenheimer–Volkov equation (3.12) and the relation between ρ and m imply that

$$\frac{dw_1}{d\tau} = \frac{2w_1}{1-2w_2}\left(1 - 4w_2 - \frac{8\pi}{3}w_1\right), \tag{6.6}$$

$$\frac{dw_2}{d\tau} = 4\pi w_1 - w_2. \tag{6.7}$$

The system has two steady states,

$$(0,0) \text{ and } Z := \left(\frac{3}{56\pi}, \frac{3}{14}\right).$$

Using Poincaré–Bendixson theory it can be shown that there is a unique trajectory which corresponds to one branch T of the unstable manifold of $(0,0)$ and converges to Z with a rate determined by the real parts of the eigenvalues of the linearization at Z, which equal $-\frac{3}{2}$. For the solution induced by y_0^* it holds that $w(\tau) \to (0,0)$ for $\tau \to -\infty$, and its trajectory coincides with T. The result is that for any $0 < \gamma < 3/2$ and all τ sufficiently large,

$$|w(\tau) - Z| \leqslant Ce^{-\gamma\tau}.$$

When we rewrite this in terms of the original variables and combine it with the previous three lemmata, we obtain the following result:

Proposition 6.4. *There exist parameters $0 < \alpha_1 < \alpha_2 < \frac{1}{4}$, $\kappa_0 > 0$ sufficiently large, and constants $\delta > 0$ and $C > 0$ such that on the critical layer*

$$[r_\kappa^1, r_\kappa^2] := [\kappa^{\alpha_1} e^{-2\kappa}, \kappa^{\alpha_2} e^{-2\kappa}]$$

and for every $\kappa \geqslant \kappa_0$ the following estimates hold:

$$\left|r^2\rho_\kappa(r) - \frac{3}{56\pi}\right|, \left|r^2\sigma_\kappa(r) - \frac{1}{56\pi}\right|, \left|\frac{m_\kappa(r)}{r} - \frac{3}{14}\right|, |2r\mu_\kappa' - 1|, \left|e^{2\lambda_\kappa} - \frac{7}{4}\right|, |r\lambda_\kappa'| \leqslant C\kappa^{-\delta}.$$

For more details on the proof of this result we refer to [47, propositions 3.13 and 3.14], but we wish to discuss the limiting object which corresponds to the stationary state Z of the dynamical system (6.6) and (6.7). Indeed, Z corresponds to the macroscopic data

$$\rho(r) = \frac{3}{56\pi}r^{-2},\ \sigma(r) = \frac{1}{56\pi}r^{-2},\ m(r) = \frac{3}{14}r,\ \frac{2m(r)}{r} = \frac{3}{7},\ e^{2\lambda} = \frac{7}{4},\ \mu'(r) = \frac{1}{2r},$$

which represent a particular steady state of the massless (EV) system; there is a free constant when defining μ which we take such that $e^{2\mu(r)} = \frac{7}{4}r$. We refer to this solution as the BKZ solution, because these macroscopic quantities are the same as for a certain massive solution found by Bisnovatyi-Kogan and Zel'dovich [13]. It does not represent a regular, isolated system: it violates both the condition (2.18) for a regular center and for asymptotic flatness (2.17), and it has infinite mass. Its Ricci scalar vanishes, while its Kretschmann scalar

$$K(r) := R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}(r) = \frac{72}{49}r^{-4}$$

blows up at the center; the BKZ solution has a spacetime singularity at $r = 0$. The curves

$$r(t) = (c + t/2)^2,\ t > -2c$$

with $c > 0$ represent radially outgoing null geodesics which start at the singularity and escape to $r = \infty$, i.e. the singularity is visible for observers away from the singularity. Hence it violates the strong cosmic censorship hypothesis; the concept of weak cosmic censorship is not applicable to this solution, since it is not asymptotically flat. According to the cosmic censorship hypothesis such 'naked' singularities should be 'non-generic' and/or 'unstable'. The analysis which we review in the present section shows that regular steady states, which in the critical layer are close to the BKZ solution for large central redshift κ, seem to inherit this instability and are indeed unstable themselves.

We also point out that the BKZ solution can for obvious reasons not capture the behavior of the massive (EV) steady state at the center or for large radii. But the information provided in proposition 6.4 on the critical layer $[r_\kappa^1, r_\kappa^2]$ turns out to be what is needed for the next step.

6.2. A negative energy direction for κ large

When κ is sufficiently large there exists a linearly dynamically accessible direction in which the second variation of $\mathcal{H}_C$, i.e. the bilinear form $\mathcal{A}$, becomes negative.

Theorem 6.5. *There exists $\kappa_0 > 0$ such that for all $\kappa > \kappa_0$ there exists a spherically symmetric, odd-in-v function $h \in C^2(\mathbb{R}^6)$ such that*

$$\mathcal{A}(h,h) = \mathcal{A}(f_h,f_h) < 0,$$

where f_h is given by (5.12).

The negative energy direction h is of the form

$$h(x,v) = g(r)w, \tag{6.8}$$

with a suitable function $g \in C^2([0,\infty[)$. Clearly, h is spherically symmetric and odd in v, where we recall (5.1). A suitable integration by parts implies that

$$\int \phi'(E)\, w^2\, \mathrm{d}v = -\mathrm{e}^{-\mu_\kappa}(\rho_\kappa + \sigma_\kappa). \tag{6.9}$$

Combining this with (6.8) and (3.11) the expression (5.11) for λ_h can be simplified:

$$\lambda_h = 4\pi\, r\mathrm{e}^{\mu_\kappa+\lambda_\kappa} g \int \phi'(E)\, w^2\, \mathrm{d}v = -\mathrm{e}^{-\lambda_\kappa} g\,(\lambda_\kappa' + \mu_\kappa')\,.$$

Moreover,

$$\{h,f_\kappa\} = \phi'(E)\,\mathrm{e}^{\mu_\kappa}\left(g'(r)\frac{w^2}{\langle v\rangle} - \mu_\kappa' g(r)\langle v\rangle + g(r)\frac{|v|^2 - w^2}{r\langle v\rangle}\right),$$

and thus

$$f_h = \mathrm{e}^{\mu_\kappa-\lambda_\kappa}\phi'(E)\left((g' - g(\mu_\kappa' + \lambda_\kappa'))\frac{w^2}{\langle v\rangle} - \mu_\kappa' g\langle v\rangle + g\frac{|v|^2 - w^2}{r\langle v\rangle}\right).$$

On the critical layer $[r^1_\kappa, r^2_\kappa]$ the steady state $(f_\kappa, \lambda_\kappa, \mu_\kappa)$ is well approximated by the BKZ solution, provided κ is sufficiently large, see proposition 6.4. We localize the perturbation h given by (6.8) to this interval by setting

$$g = \mathrm{e}^{\frac{1}{2}\mu_\kappa + \lambda_\kappa}\chi,$$

where $0 \leqslant \chi \leqslant 1$ is a smooth cut-off function supported in the interval $[r^1_\kappa, r^2_\kappa]$ and equal to 1 on $[2r^1_\kappa, r^2_\kappa/2]$; the latter interval is non-trivial for κ sufficiently large. In addition, $|\chi'(r)|$ is to satisfy certain bounds which are not relevant here. The perturbation f_h now takes the form

$$f_h = \mathrm{e}^{\frac{3}{2}\mu_\kappa}\phi'(E)\left(-\mu'_\kappa\chi\left[\frac{w^2}{2\langle v\rangle} + \langle v\rangle - \frac{1}{\mu'_\kappa r}\frac{|v|^2 - w^2}{\langle v\rangle}\right] + \chi'\frac{w^2}{\langle v\rangle}\right).$$

Substitution of this expression into (5.21) yields the following identity:

$$\begin{aligned}\mathcal{A}(h,h) = &\int_{r^1_\kappa}^{r^2_\kappa} \mathrm{e}^{2\mu_\kappa - \lambda_\kappa}\chi^2\Bigg[4\pi r^2 \mathrm{e}^{\mu_\kappa + 2\lambda_\kappa}(\mu'_\kappa)^2\int|\phi'(E)|\left(\frac{w^2}{2\langle v\rangle} + \langle v\rangle - \frac{1}{\mu'_\kappa r}\frac{|v|^2 - w^2}{\langle v\rangle}\right)^2 \mathrm{d}v \\ &- (2r\mu'_\kappa + 1)(\mu'_\kappa + \lambda'_\kappa)^2\Bigg]\mathrm{d}r + 4\pi\int_{\mathrm{supp}\chi'} r^2\mathrm{e}^{3\mu_\kappa + \lambda_\kappa}(\chi')^2\int|\phi'(E)|\frac{w^4}{\langle v\rangle^2}\mathrm{d}v\,\mathrm{d}r \\ &+ 8\pi\int_{\mathrm{supp}\chi'} r^2\mathrm{e}^{3\mu_\kappa + \lambda_\kappa}\mu'_\kappa\chi\chi'\int|\phi'(E)|\left(\frac{w^4}{2\langle v\rangle^2} + w^2 - \frac{1}{\mu'_\kappa r}\frac{w^2|v|^2 - w^4}{\langle v\rangle^2}\right)\mathrm{d}v\,\mathrm{d}r.\end{aligned}$$

The key point now is that if in the first integral the steady state quantities are replaced by their corresponding limiting quantities according to proposition 6.4, a strictly negative term arises, together with error terms, which, being like the second and third integral of lower order in κ, do not destroy the negative sign of $\mathcal{A}(h,h)$, provided κ is sufficiently large; for the details we have to refer to [47, theorem 4.3][1].

6.3. Linear exponential instability

An adaptation of an argument by Laval *et al* [67] shows that the existence of a negative energy direction as provided by theorem 6.5 implies a linear exponential instability result. At first glance this may seem surprising, since the energy $\mathcal{A}$ could be negative definite in which case its conservation should imply stability. In order to appreciate the role of the negative energy direction, h in (5.20) must be split into even and odd parts with respect to v, which turns the latter equation into the system

$$\partial_t h_- + \mathcal{T}h_+ = 0, \qquad (6.10)$$

$$\partial_t h_+ + \mathcal{T}h_- = \mathcal{C}h_-. \qquad (6.11)$$

Here

$$\mathcal{T}h := \mathrm{e}^{-\lambda_\kappa}\{h, E\}, \qquad \mathcal{C}h := -\mathrm{e}^{\mu_\kappa}\lambda_h\frac{w^2}{\langle v\rangle} + \mathrm{e}^{\mu_\kappa}\mu_h\langle v\rangle.$$

Let $L^2_W = L^2_W(D)$ denote the weighted L^2 space on the set $D = \{f_0 > 0\}$ with the weight $W := \mathrm{e}^{\lambda_\kappa}|\phi'(E)|$, and let $\langle\cdot,\cdot\rangle_{L^2_W}$ denote the corresponding scalar product. As we noted before,

[1] The proof published in [47] contains an error which has been corrected in arXiv:1810.00809.

solutions to (5.20) conserve the energy $\mathcal{A}(h,h)$. But substituting $h = h_+ + h_-$ and using the fact that $\lambda_h = \lambda_{h_-}$ it follows that

$$\mathcal{A}(h,h) = \mathcal{A}(h_-,h_-) + \langle \mathcal{T}h_+, \mathcal{T}h_+ \rangle_{L^2_W}.$$

Hence for the system (6.10) and (6.11) conservation of energy takes the form

$$\langle \mathcal{T}h_+, \mathcal{T}h_+ \rangle_{L^2_W} + \mathcal{A}(h_-,h_-) = const,$$

and $\mathcal{A}$ now plays the role of potential energy. A negative direction for the latter together with the positive definiteness of the kinetic energy gives a saddle point structure for the total energy, and instability is expected.

Using the fact that solutions of the system (6.10) and (6.11) also satisfy the virial identity

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle h_-, h_- \rangle_{L^2_W} = -\mathcal{A}(h_-,h_-) + \langle \mathcal{T}h_+, \mathcal{T}h_+ \rangle_{L^2_W}$$

one can now follow the idea in [67] to derive the following linear, exponential instability result; for details we refer to [47, theorem 4.8].

Theorem 6.6. *There exist initial data* $\mathring{h}_+, \mathring{h}_- \in C^1(\overline{D})$ *and constants* $c_1, c_2 > 0$ *such that for the corresponding solution to the system (6.10) and (6.11),*

$$\|h_-(t)\|_{L^2_W}, \|\mathcal{T}h_+(t)\|_{L^2_W} \geqslant c_1 \mathrm{e}^{c_2 t}.$$

A much stronger result, namely the existence of an exponentially growing mode, is discussed in the next section.

7. Spectral properties of the linearized (EV) system

7.1. The functional-analytic structure of the linearized (EV) system

Important aspects of the linearized (EV) system such as the existence of exponentially growing modes for κ sufficiently large can only be properly understood, if the linearized system is put into the proper functional-analytic framework. The latter is set up on the real Hilbert space

$$H := \{f\colon D \to \mathbb{R}\text{measurable and spherically symmetric} \mid \|f\|_H < \infty\},$$

where the norm $\|f\|_H$ is defined in terms of the scalar product

$$\langle f, g \rangle_H := \iint_D \frac{\mathrm{e}^{\lambda_0}}{|\phi'(E)|} f g \,\mathrm{d}v\,\mathrm{d}x, \quad g, h \in H;$$

for the moment we consider some fixed steady state (f_0, λ_0, μ_0) and ignore the dependence on the central redshift κ; we recall that $D = \{f_0 > 0\}$. We need to define the transport operator $\mathcal{T}f = \mathrm{e}^{-\lambda_0}\{f, E\}$ where, say, $f \in C^1(D)$, as an operator on H.

We say that for a function $f \in H$ the *transport term* $\mathcal{T}f$ *exists weakly* if there exists $h \in H$ such that for every spherically symmetric test function $\xi \in C^1_c(D)$,

$$\langle f, \mathcal{T}\xi \rangle_H = -\langle h, \xi \rangle_H.$$

If such a function h exists, it is unique, and we set $\mathcal{T}f = h$*in a weak sense*. The domain of $\mathcal{T}$ is defined as

$$\mathscr{D}(\mathcal{T}) := \{f \in H \mid \mathcal{T}f \in H\text{exists weakly}\},$$

and the resulting operator $\mathcal{T}\colon \mathscr{D}(\mathcal{T}) \to H$ is the *transport operator*. In view of (5.19) we also define $\mathcal{B}\colon \mathscr{D}(\mathcal{T}) \to H$ by

$$\mathcal{B}f := -\mathcal{T}f - 4\pi r|\phi'|\mathrm{e}^{2\mu_0+\lambda_0}\left(w\sigma_f - \frac{w^2}{\langle v\rangle}\jmath_f\right), \tag{7.1}$$

and the *residual operator* $\mathcal{R}\colon H \to H$ by

$$\mathcal{R}f := 4\pi|\phi'|\mathrm{e}^{3\mu_0}(2r\mu_0'+1)w\jmath_f.$$

These operators have the following properties[2]:

Lemma 7.1. *(a) The transport operator $\mathcal{T}\colon \mathscr{D}(\mathcal{T}) \to H$ is densely defined and skew-adjoint, i.e. $\mathcal{T}^* = -\mathcal{T}$, and $\mathcal{T}^2\colon \mathscr{D}(\mathcal{T}^2) \to H$ with*

$$\mathscr{D}(\mathcal{T}^2) := \{f \in H \,|\, f \in \mathscr{D}(\mathcal{T}), \mathcal{T}f \in \mathscr{D}(\mathcal{T})\}$$

is self-adjoint.

(b) The operator $\mathcal{B}\colon \mathscr{D}(\mathcal{T}) \to H$ is densely defined and skew-adjoint, and $\mathcal{B}^2\colon \mathscr{D}(\mathcal{T}^2) \to H$ is self-adjoint.

(c) The operator $\mathcal{R}\colon H \to H$ is bounded, symmetric, and non-negative, i.e. $\langle \mathcal{R}f, f\rangle_H \geqslant 0$ for $f \in H$.

That the transport operator is symmetric with respect to the scalar product on the Hilbert space H is easy to see; for the details of the above results we refer to [37] or [47]. We use these operators to put the linearized (EV) system, i.e. (5.19), into the form

$$\partial_t f = \mathcal{B}f - \mathrm{e}^{2\mu_0-\lambda_0}\left(2\mu_0' + \frac{1}{r}\right)\lambda_f|\phi'(E)|w; \tag{7.2}$$

note that we simply write f instead of δf here and in what follows. As before and following Antonov we split $f = f_+ + f_-$ into its even and odd parts with respect to v. Since $\mathcal{B}$ reverses v-parity,

$$\partial_t f_+ = \mathcal{B}f_-,$$
$$\partial_t f_- = \mathcal{B}f_+ - \mathrm{e}^{2\mu_0-\lambda_0}\left(2\mu_0' + \frac{1}{r}\right)\lambda_f|\phi'(E)|w.$$

Differentiating the second equation with respect to t and substituting the first one implies that

$$\begin{aligned}\partial_t^2 f_- &= \mathcal{B}f_+ - \mathrm{e}^{2\mu_0-\lambda_0}\left(2\mu_0' + \frac{1}{r}\right)\partial_t\lambda_f|\phi'(E)|w\\ &= \mathcal{B}^2 f_- + 4\pi\mathrm{e}^{3\mu_0}(2r\mu_0'+1)\jmath_{f_-}|\phi'(E)|w\\ &= \mathcal{B}^2 f_- + \mathcal{R}f_-,\end{aligned}$$

where we used (5.18), the fact that $\jmath_f = \jmath_{f_-}$ and the definition of the residual operator $\mathcal{R}$. Since this second-order formulation of the linearized system lives on the odd-in-v parts of the perturbations, we define

$$H^{\mathrm{odd}} := \{f \in H \,|\, f \text{ is odd in } v\},$$

[2] In the literature the sign in front of $\mathcal{T}$ is not always chosen as consistently as we try to.

which is a Hilbert space with the same scalar product as before. The properties of the operators stated in lemma 7.1 remain true on H^{odd}, and we define the *Antonov operator*

$$\mathcal{L}\colon \mathscr{D}(\mathcal{L}) \to H^{\text{odd}},\ \mathscr{D}(\mathcal{L}) := \mathscr{D}(\mathcal{T}^2) \cap H^{\text{odd}},\ \mathcal{L} := -\mathcal{B}^2 - \mathcal{R}.$$

This is again a self-adjoint operator, and the linearized (EV) system is put into the form

$$\partial_t^2 f_- + \mathcal{L} f_- = 0, \tag{7.3}$$

which has the same structure as the corresponding equation (4.11) for (VP).

The above second-order formulation has been used in the astrophysics literature, see [55, 56, 58] (without precise spaces, domains etc). Based on (7.3) we call a steady state of the EV system *linearly stable* if the spectrum of $\mathcal{L}$ is strictly positive, i.e.

$$\gamma := \inf \sigma(\mathcal{L}) > 0;$$

notice that the spectrum of $\mathcal{L}$ is real since $\mathcal{L}$ is self-adjoint. By [54, proposition 5.12] this spectral condition implies the Antonov-type inequality

$$\langle f, \mathcal{L} f \rangle_H \geqslant \gamma \|f\|_H^2,\ f \in D(\mathcal{L}).$$

Since

$$\|\partial_t f_-\|_H^2 + \langle f_-, \mathcal{L} f_- \rangle_H = \mathcal{A}(\partial_t f_-, \partial_t f_-)$$

is conserved along solutions of the linearized equation (7.3), this implies linear stability in the corresponding norm.

Assume on the other hand that $\alpha < 0$ is an eigenvalue of $\mathcal{L}$ with eigenfunction $f \in H^{\text{odd}}$. Then $g := \mathrm{e}^{\sqrt{-\alpha}\, t} f$ solves (7.3), and we get an exponentially growing solution of the linearized (EV) system. Hence an eigenfunction $f \in H^{\text{odd}}$ to a negative eigenvalue $\alpha < 0$ of $\mathcal{L}$ is called an *exponentially growing mode*. Using theorem 6.5 one can show that for κ sufficiently large, such exponentially growing modes exist.

To see this one needs some further tools which are also used to obtain a first-order formulation of the linearized (EV) system with good functional-analytic properties. This first-order formulation has to our knowledge not appeared in the physics literature and was introduced in [47]. A key ingredient is a modified potential induced by a state $f \in H$:

$$\bar{\mu}(r) = \bar{\mu}_f(r) := -\mathrm{e}^{-\mu_0 - \lambda_0} \int_r^\infty \frac{1}{s}\, \mathrm{e}^{\mu_0(s) + \lambda_0(s)} (2s\mu_0'(s) + 1)\, \lambda_f(s)\, \mathrm{d}s \tag{7.4}$$

is the *modified potential induced by* $f \in H$, where λ_f is defined by (5.16). It has the following properties, where $\dot{H}_r^1$ denotes the subspace of spherically symmetric functions in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$, see [30].

Lemma 7.2. *(a) For* $f \in H$, $\bar{\mu} = \bar{\mu}_f \in C([0,\infty[) \cap C^1(]0,\infty[) \cap \dot{H}_r^1$, *and* $|\bar{\mu}(r)| \leqslant C\|f\|_H$, $r \geqslant 0$, *with some* $C > 0$ *independent of* f.

(b) It holds that

$$\bar{\mu}' = -(\mu_0' + \lambda_0')\, \bar{\mu} + \frac{2r\mu_0' + 1}{r} \lambda_f, \tag{7.5}$$

$$\frac{\mathrm{e}^{-\mu_0 - \lambda_0} r}{2r\mu_0' + 1} (\mathrm{e}^{\mu_0 + \lambda_0} \bar{\mu})' = \lambda_f,\ r \geqslant 0, \tag{7.6}$$

and in the weak sense,

$$\frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{e}^{-\mu_0-3\lambda_0}r^2}{2r\mu_0'+1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathrm{e}^{\mu_0+\lambda_0}\bar{\mu}\right)\right)=\rho_f \; a.\ e. \tag{7.7}$$

(c) The operator $\mathcal{K}\colon H\to H$, $\mathcal{K}f:=\phi'(E)E\bar{\mu}_f$ *is bounded, self-adjoint, and compact.*

One should at this point recall (5.4). The field equation (2.23) and the boundedness of σ_0,μ_0,λ_0 imply that the quantity $\mathrm{e}^{\mu_0+\lambda_0}(2r\mu_0'+1)$ is bounded. The estimate for $\bar{\mu}$ then follows by the Cauchy–Schwarz inequality. For the remaining assertions one should observe that by (3.11), $\mu_0'+\lambda_0'=0$ outside D, and that $r^2\rho\in L^1([0,\infty[)$. That $\mathcal{K}$ is bounded follows from part (a), integration-by-parts and (7.7) imply its self-adjointness, and compactness follows using the the Arzela-Ascoli theorem, where it is important that the steady state has compact radial support. For the details we refer to [47, lemmata 4.17 and 4.18].

The compactness of the map $\mathcal{K}$ is important for the operator $\bar{\mathcal{L}}\colon H\to H$ defined by

$$\bar{\mathcal{L}}f:=f-\phi'(E)E\bar{\mu}_f. \tag{7.8}$$

By lemma 7.2(c):

Lemma 7.3. *The operator* $\bar{\mathcal{L}}$ *is bounded and symmetric on H.*

The linearized (EV) system can now be put into the following first order Hamiltonian form which means that the general theory developed in [73] can be applied.

Proposition 7.4. *The linearized (EV) system takes the form*

$$\partial_t f=\mathcal{B}\bar{\mathcal{L}}f, \tag{7.9}$$

$\mathscr{D}(\mathcal{B}\bar{\mathcal{L}})=\mathscr{D}(\mathcal{T})$, *the operator* $\bar{\mathcal{L}}$ *induces the quadratic form*

$$\langle\bar{\mathcal{L}}f,f\rangle_H=\iint\frac{\mathrm{e}^{\lambda_0}}{|\phi'(E)|}f^2\,\mathrm{d}v\,\mathrm{d}x-\int_0^\infty \mathrm{e}^{\mu_0-\lambda_0}\left(2r\mu_0'+1\right)\lambda_f^2\,\mathrm{d}r=\mathcal{A}(f,f) \tag{7.10}$$

on H, and the flow of (7.9) preserves $\mathcal{A}(f,f)$. *The relation of the first-order formulation (7.9) to the second-order one in (7.3) is captured in the relation*

$$\mathcal{L}=-\mathcal{B}\bar{\mathcal{L}}\mathcal{B}. \tag{7.11}$$

We refer to [47, lemma 4.20] for a rigorous proof and highlight only some instructive aspects. For $f\in H$, at least formally,

$$\mathcal{T}(\phi'(E)E\bar{\mu}_f)=\mathrm{e}^{2\mu_0-\lambda_0}\phi'(E)\,w\,\bar{\mu}_f'.$$

Together with (6.9) and (7.5) this implies that

$$\mathcal{B}\left(\phi'E\bar{\mu}_f\right)=-\phi'(E)\mathrm{e}^{2\mu_0-\lambda_0}w\frac{2r\mu_0'+1}{r}\lambda_f.$$

If we combine this with the form (7.2) of the linearized (EV) system we obtain

$$\partial_t f=\mathcal{B}\left(f-\phi'(E)E\bar{\mu}_f\right)=\mathcal{B}\bar{\mathcal{L}}f.$$

If we differentiate $\langle\bar{\mathcal{L}}f,f\rangle_H$ with respect to t and use the symmetry of $\bar{\mathcal{L}}$, (7.9), and the skew-adjointness of $\mathcal{B}$ the conservation law follows. By the definition of $\bar{\mathcal{L}}$,

$$
\begin{aligned}
\langle \bar{\mathcal{L}} f, f\rangle_H &= \langle f - \phi'(E) E \bar{\mu}_f, f\rangle_H \\
&= \iint \frac{\mathrm{e}^{\lambda_0}}{|\phi'(E)|} f^2 \,\mathrm{d}v\,\mathrm{d}x + \int \mathrm{e}^{\mu_0+\lambda_0} \bar{\mu}_f \int f \langle v\rangle \,\mathrm{d}v\,\mathrm{d}x \\
&= \iint \frac{\mathrm{e}^{\lambda_0}}{|\phi'(E)|} f^2 \,\mathrm{d}v\,\mathrm{d}x + 4\pi \int_0^\infty r^2 \bar{\mu}_f \mathrm{e}^{\mu_0+\lambda_0} \rho_f \,\mathrm{d}r,
\end{aligned}
$$

and since $4\pi r^2 \rho_f = \left(\mathrm{e}^{-2\lambda_0} r \lambda_f\right)'$ the assertion (7.10) follows by using (7.6).

If we split some element $f \in \mathscr{D}(\mathcal{B}\bar{\mathcal{L}}) = \mathscr{D}(\mathcal{T})$ into its even and odd parts with respect to v, it follows that $\lambda_f = \lambda_{f_+}$, hence also $\bar{\mu}_f = \bar{\mu}_{f_+}$, and $\bar{\mathcal{L}} f = \bar{\mathcal{L}} f_+ + f_-$. Since $\bar{\mathcal{L}}$ preserves v parity and $\mathcal{B}$ reverses it, the first order formulation (7.9) splits into

$$
\partial_t f_+ = \mathcal{B} f_-, \ \partial_t f_- = \mathcal{B}\bar{\mathcal{L}} f_+
$$

which directly implies

$$
\partial_t^2 f_- = \mathcal{B}\bar{\mathcal{L}}\mathcal{B} f_-
$$

as desired; of course the relation (7.11) can be checked directly.

The spectral properties of the operators $\mathcal{L}$ or $\mathcal{B}\bar{\mathcal{L}}$ are difficult to analyze, and a key idea to do so is to find simpler, macroscopic Schrödinger-type operators by which for example $\mathcal{L}$ is bounded from above and below. These reduced operators act on functions of only the radial variable r, which makes them easier to analyze.

The construction which we explain below was developed in [47] and relies on the modified potential $\bar{\mu}_f$ as a key ingredient. An earlier, but not really satisfactory attempt to construct such a reduced operator was made in [55].

The *modified Laplacian* $\bar{\Delta}$ is given by

$$
\bar{\Delta}\psi := \frac{\mathrm{e}^{\mu_0+\lambda_0}}{4\pi r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\mathrm{e}^{-\mu_0-3\lambda_0} r^2}{2r\mu_0'+1} \frac{\mathrm{d}}{\mathrm{d}r} \left(\mathrm{e}^{\mu_0+\lambda_0}\psi\right)\right).
$$

On a flat background, i.e. for $\lambda_0 = \mu_0 = 0$ the operator $4\pi\bar{\Delta}$ is the Laplacian applied to spherically symmetric functions. The *reduced operator* S is given by

$$
S\psi := -\bar{\Delta}\psi - \mathrm{e}^{\lambda_0} \int |\phi'(E)| E^2 \,\mathrm{d}v\,\psi, \tag{7.12}
$$

and the *non-local reduced operator* $\tilde{S}$ is

$$
\tilde{S}\psi := -\bar{\Delta}\psi - \mathrm{e}^{\lambda_0} \int (\mathrm{id} - \Pi)\left(|\phi'(E)| E\psi\right) E \,\mathrm{d}v, \tag{7.13}
$$

where Π denotes the projection onto $\mathscr{R}(\mathcal{B})^\perp$, the orthogonal complement in H of the range of the operator $\mathcal{B}$, and id is the identity.

For what follows, $\bar{\mu}_f$, which only belongs to $\dot{H}^1_r$, must lie in the domain of S and $\tilde{S}$. Hence one must be careful to define these operators between the proper spaces, which is done using duality; $(\dot{H}^1_r)'$ denotes the dual space of $\dot{H}^1_r$ and $\langle\cdot,\cdot\rangle$ denotes the corresponding dual pairing.

Lemma 7.5. *The operator* $S\colon \dot{H}^1_r \to (\dot{H}^1_r)'$ *defined by*

$$
\langle S\psi, \chi\rangle := \int_0^\infty \frac{\mathrm{e}^{-\mu_0-3\lambda_0}}{2r\mu_0'+1} \frac{\mathrm{d}}{\mathrm{d}r}\left(\mathrm{e}^{\mu_0+\lambda_0}\psi\right) \frac{\mathrm{d}}{\mathrm{d}r}\left(\mathrm{e}^{\mu_0+\lambda_0}\chi\right) r^2 \,\mathrm{d}r - \iint_D \mathrm{e}^{\lambda_0} |\psi'(E)| E^2 \psi\chi \,\mathrm{d}v\,\mathrm{d}x
$$

is self-dual, and it is given by (7.12) on sufficiently regular functions. The operator $\tilde{S}\colon \dot{H}^1_r \to (\dot{H}^1_r)'$ *is defined analogously and has the analogous properties.*

These operators bound the quadratic form $\mathcal{A}$ which according to (7.10) is induced by $\tilde{\mathcal{L}}$ from above and below in the following precise sense:

Proposition 7.6. *(a) For every* $\mu \in \dot{H}^1_r$ *and* $f = f_\mu := \phi'_\kappa(E_\kappa)E_\kappa\mu$,

$$\langle S\mu, \mu\rangle \geqslant \mathcal{A}(f_\mu, f_\mu).$$

For every $f \in H$ *and* $\bar{\mu}_f$ *as defined in (7.4),*

$$\mathcal{A}(f,f) \geqslant \langle S\bar{\mu}_f, \bar{\mu}_f\rangle.$$

(b) For every $\mu \in \dot{H}^1_r$ *and* $\tilde{f} = \tilde{f}_\mu := (\mathrm{id} - \Pi)(\phi'(E)E\mu) \in \mathscr{R}(\mathcal{B})$,

$$\langle \tilde{S}\mu, \mu\rangle \geqslant \mathcal{A}(\tilde{f}_\mu, \tilde{f}_\mu).$$

For every $f \in H$,

$$\mathcal{A}(f,f) \geqslant \langle \tilde{S}\bar{\mu}_f, \bar{\mu}_f\rangle.$$

The proof relies on the observations that for $\mu \in \dot{H}^1_r$,

$$\langle -\bar{\Delta}\mu, \mu\rangle = \int_0^\infty \frac{\mathrm{e}^{-\mu_0 - 3\lambda_0}}{2r\mu_0' + 1} \left(\left(\mathrm{e}^{\mu_0+\lambda_0}\mu\right)'\right)^2 r^2\,\mathrm{d}r,$$

in particular, for $f \in H$, using (7.4) and (7.6),

$$\begin{aligned}\langle -\bar{\Delta}\bar{\mu}_f, \bar{\mu}_f\rangle &= \int_0^\infty \mathrm{e}^{\mu_0-\lambda_0}\left(2r\mu_0'+1\right)\left(\frac{\mathrm{e}^{-\mu_0-\lambda_0}}{2r\mu_0'+1} r\left(\mathrm{e}^{\mu_0+\lambda_0}\bar{\mu}_f\right)'\right)^2 \mathrm{d}r \\ &= \int_0^\infty \mathrm{e}^{\mu_0-\lambda_0}\left(2r\mu_0'+1\right)\lambda_f^2\,\mathrm{d}r.\end{aligned}$$

On the other hand by (7.7),

$$\langle -\bar{\Delta}\bar{\mu}_f, \bar{\mu}_f\rangle = -4\pi \int_0^\infty \mathrm{e}^{\mu_0+\lambda_0}\bar{\mu}_f\rho_f r^2\,\mathrm{d}r = -\iint \mathrm{e}^{\mu_0+\lambda_0}\bar{\mu}\,\langle v\rangle f\,\mathrm{d}v\,\mathrm{d}x.$$

The definitions of $\mathcal{A}$ and the operators S and $\tilde{S}$ lead to the desired results; for details see [47, theorem 4.24].

7.2. (In)stability for the linearized (EV) system

The results of the previous section imply the existence of an exponentially growing mode when κ is large enough, more precisely:

Theorem 7.7. *For* κ *sufficiently large, there exists at least one negative eigenvalue of* $\mathcal{L}$ *and therefore an exponentially growing mode for the linearization (7.3) of (EV) around* $(f_\kappa, \lambda_\kappa, \mu_\kappa)$*; such a steady state is (linearly) unstable. For general* $\kappa > 0$*, the negative part of the spectrum of* $\mathcal{L}$ *is either empty or consists of at most finitely many eigenvalues with finite multiplicities.*

Before we sketch the proof we need to introduce some more notation. Let $L\colon H \supset \mathscr{D}(L) \to H$ be a linear, self-adjoint operator on some Hilbert space H. Its *negative Morse index* $n^-(L)$ is the maximal dimension of subspaces of H on which $\langle L\cdot,\cdot\rangle_H < 0$. The analogous terminology applies to a self-dual operator $L\colon H \to H'$.

Proof. The operator $\mathcal{L}$ is self-adjoint. For $f \in \mathscr{D}(\mathcal{L}_\kappa)$,

$$\langle \mathcal{L}f, f\rangle_H = \langle \bar{\mathcal{L}}\mathcal{B}f, \mathcal{B}f\rangle_H,$$

see (7.11). By proposition 7.6,

$$n^-(\mathcal{L}) \leqslant n^-\left(\bar{\mathcal{L}}\right) \leqslant n^-(S) < \infty;$$

the general argument behind the first two estimates is reviewed in [47, lemma A.1]. To show that $n^-(S) < \infty$ is easier than showing this for $\mathcal{L}$ directly, since S has a much simpler structure and acts on functions of only the radial variable; this is the key point in introducing the reduced operators. For $\psi \in \dot{H}^1_r$, $\langle S\psi, \psi\rangle \geqslant C\langle S'\psi, \psi\rangle$, where the self-dual operator $S' \colon \dot{H}^1_r \to (\dot{H}^1_r)'$ is formally given as $S' = -\Delta - V$ with a non-negative, continuous, compactly supported potential V. This follows from suitable bounds on λ_κ and μ_κ. Now the mapping $(-\Delta)^{1/2} \colon \dot{H}^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, $\psi \mapsto (2\pi|\xi|\hat{\psi})\check{}$ is an isomorphism which respects spherical symmetry. Passing to $\chi = (-\Delta)^{1/2}\psi$ the relation $4\pi\langle S'\psi, \psi\rangle = \langle(\mathrm{id} - \mathcal{K})\chi, \chi\rangle_{L^2}$ follows. Here $\mathcal{K} = (-\Delta)^{-1/2}V(-\Delta)^{-1/2} \colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is compact, since V is bounded and supported on the compact set $\bar{B}_R(0)$ with $[0,R]$ the radial support of the steady state, and the map $\dot{H}^1(\mathbb{R}^3) \ni f \mapsto \mathbf{1}_{\bar{B}_R(0)} f \in L^2(\mathbb{R}^3)$ is compact; notice that $h = V(-\Delta)^{-1/2}\chi \in L^1 \cap L^2(\mathbb{R}^3)$ so that $\hat{h} \in L^\infty \cap L^2(\mathbb{R}^3)$, and hence $\frac{1}{2\pi|\xi|}\hat{h}$ and its inverse Fourier transform are in $L^2(\mathbb{R}^3)$. The spectral properties of compact operators imply that $n^-(\mathrm{id} - \mathcal{K}) < \infty$, and invoking [47, lemma A.1] again it follows that $n^-(S) < \infty$.

The assertion on the spectrum of $\mathcal{L}$ now follows from the spectral representation of this operator; the argument is discussed in detail in [47, proposition A.2].

Let us now assume that κ is large enough to apply theorem 6.5. That theorem provides a negative energy direction, i.e. there exists a spherically symmetric function $h \in C^2(\mathbb{R}^6)$ which is odd in v such that $\mathcal{A}(h,h) = \mathcal{A}(f_h, f_h) < 0$. Here f_h is the linearly dynamically accessible perturbation generated by h according to (5.12). If we compare this relation to the definition (7.1) of the operator $\mathcal{B}$ it follows that $f_h = -\mathcal{B}(\phi' h)$. Hence by (7.10),

$$\begin{aligned} 0 > \mathcal{A}(h,h) &= \mathcal{A}(\mathcal{B}(\phi' h), \mathcal{B}(\phi' h)) = \langle \bar{\mathcal{L}}\mathcal{B}(\phi' h), \mathcal{B}(\phi' h)\rangle_H \\ &= -\langle \mathcal{B}\bar{\mathcal{L}}\mathcal{B}(\phi' h), \phi' h\rangle_H = \langle \mathcal{L}(\phi' h), \phi' h\rangle_H; \end{aligned}$$

for the last two equalities notice that by lemma 7.1(b) the operator $\mathcal{B}$ is skew adjoint and the relation (7.11) holds. Hence by definition $n^-(\mathcal{L}) \geqslant 1$, and invoking the spectral representation of this operator again shows that $\mathcal{L}$ has a negative eigenvalue $\alpha < 0$ of finite multiplicity, see [47, proposition A.2]. Since the operator $\mathcal{L}$ is non-negative when restricted to the subspace all even-in-v functions in H, eigenfunctions associated to α must be odd-in-v, and the existence of an exponentially growing mode is established. □

Remark. (a) In the proof above the following observation concerning the concept of linearly dynamic accessibility was important: the fact a state $f = f_h$ is linearly dynamically accessible and generated by h according to (5.12) is equivalent to saying that $f_h = -\mathcal{B}(\phi' h)$. This motivates the following generalization of this concept: a function $f \in H$ is a *linearly dynamically accessible perturbation* if $f \in \overline{\mathscr{R}(\mathcal{B})}$.

(b) The condition (5.8) for linear dynamic accessibility requires that

$$\left\langle \chi'(f_0)\,|\phi'(E)|, f + e^{\mu_0}\lambda_f |\phi'(E)| \frac{w^2}{\langle v\rangle} \right\rangle_H = 0 \text{ for all } \chi \in C^1(\mathbb{R}) \text{ with } \chi(0) = 0,$$

and it can be shown that $f \in \overline{\mathscr{R}(\mathcal{B})}$ satisfies this orthogonality condition which further justifies the generalization of the definition of linear dynamic accessibility.

(c) If the steady state allows the introduction of action-angle variables which according to the discussion in section 5.2 is true in particular when κ is not too large so that the condition (5.14) holds, then $H = \mathscr{R}(\mathcal{B}) \oplus \mathscr{N}(\mathcal{B})$, see [37, proposition 5.9], and $\mathscr{R}(\mathcal{B})$ is closed. It is not clear if this is true in general.

(d) An important feature of linear dynamic accessibility was that it is preserved under the linearized flow, see (5.20). The generalized concept shares this property which can be seen as follows. The exponential formula for C_0 semigroups [84, theorem 8.3] shows that

$$\mathrm{e}^{t\mathcal{B}\bar{\mathcal{L}}} f = \lim_{n\to\infty} \left(\mathrm{id} - \frac{t}{n}\mathcal{B}\bar{\mathcal{L}}\right)^{-n} f.$$

If $f \in \mathscr{R}(\mathcal{B})$ then induction shows that each of the n-dependent functions on the right is again an element of $\mathscr{R}(\mathcal{B})$, and so is the limit if we are in the situation where that space is closed. If needed this argument extends to the $\overline{\mathscr{R}(\mathcal{B})}$-case.

The same arguments as in the proof above show that

$$n^-(\mathcal{L}) \leqslant n^-(\bar{\mathcal{L}}|_{\overline{\mathscr{R}(\mathcal{B})}}) \leqslant n^-(\tilde{S}) < \infty.$$

Based on the first-order formulation (7.9) these estimates and the machinery developed above can be used to derive a detailed picture of the linearized flow for general κ. The important point here is that (7.9) is a linear Hamiltonian PDE in the sense of [73]. We list some of the key features here and refer to [47, theorem 4.28] for details.

The operator $\mathcal{B}\bar{\mathcal{L}}$ generates a C_0 group $(\mathrm{e}^{t\mathcal{B}\bar{\mathcal{L}}})_{t\in\mathbb{R}}$ of bounded linear operators on H. The Hilbert space H can be decomposed into stable, unstable, and center space, i.e.

$$H = E^s \oplus E^u \oplus E^c,$$

where E^u and E^s is the linear subspace spanned by the eigenvectors corresponding to positive or negative eigenvalues of $\mathcal{B}\bar{\mathcal{L}}$ respectively. Moreover,

$$\dim E^u = \dim E^s = n^-(\tilde{S}) < \infty.$$

The subspaces E^c, E^u, E^s are invariant under $\mathrm{e}^{t\mathcal{B}\bar{\mathcal{L}}}$. If $\tilde{S} > 0$, then the steady state is linearly stable in the sense that there exists a constant $C > 0$ such that for all perturbations $f \in H$ and all times $t \in \mathbb{R}$,

$$\left\| \mathrm{e}^{t\mathcal{B}\bar{\mathcal{L}}} f \right\|_H \leqslant M \|f\|_H. \tag{7.14}$$

In the Newtonian limit $\kappa \to 0$ the operator $\tilde{S}$ converges to its Newtonian counterpart, which was proven to be positive e.g. in [39]. By (7.14) one can therefore obtain linear stability against general initial data in H, which improves theorem 5.5.

7.3. A Birman–Schwinger principle for (EV)

An important step in the previous two sections was to relate the generator(s) of the linearized (EV) dynamics to some operator(s) defined on functions which depend only on the radial variable r. An alternative way to do this is the Birman–Schwinger principle, which for (VP) we discussed in section 4.3. In [37] a Birman–Schwinger type principle was developed for (EV), and we now discuss the main features of this approach.

The general aim is to derive a criterion for the existence of negative eigenvalues of $\mathcal{L} = -\mathcal{B}^2 - \mathcal{R}$. The tool we consider here is the *Birman–Schwinger operator*

$$Q := -\sqrt{\mathcal{R}}\mathcal{B}^{-2}\sqrt{\mathcal{R}}\colon H^{\mathrm{odd}} \to H^{\mathrm{odd}} \tag{7.15}$$

associated to $\mathcal{L}$; notice that we work on the space H^{odd} of odd-in-v functions since the Antonov operator $\mathcal{L}$ governs the evolution of that part of the perturbation, see (7.3). We recall lemma 7.1 for the basic properties of the operators $\mathcal{B}^2$ and $\mathcal{R}$. In order to define the operator Q one first has to show that the non-negative operator $\mathcal{R}$ has a square root $\sqrt{\mathcal{R}}\colon H^{\mathrm{odd}} \to H^{\mathrm{odd}}$. Indeed, this operator can be defined explicitly:

$$\sqrt{\mathcal{R}}f := 4\pi\sqrt{r}|\phi'(E)|\mathrm{e}^{2\mu_0+\lambda_0}\sqrt{\frac{2r\mu_0'+1}{\mu_0'+\lambda_0'}}\,w\jmath_f \tag{7.16}$$

defines a bounded and symmetric operator on H^{odd} with the property that $\sqrt{\mathcal{R}}\sqrt{\mathcal{R}} = \mathcal{R}$; we recall that $\jmath_f = \int wf\mathrm{d}v$ and that by (3.11) the denominator is positive in the interior of the radial support $[0,R_0]$ of the steady state under consideration, see [37, lemma 5.15]. Secondly, one can show that the operator $\mathcal{B}^2\colon \mathscr{D}(\mathcal{T}^2)\cap\mathscr{N}(\mathcal{B}^2)^\perp \to \mathscr{R}(\mathcal{B}^2)$ is bijective; the inverse of the latter operator cannot be written down explicitly, which makes its analysis tricky, but it exists, see [37, proposition 5.14]. The key properties of the Birman–Schwinger operator Q are captured in the following result.

Proposition 7.8. *(a) The Birman–Schwinger operator Q is linear, bounded, symmetric, non-negative, and compact.*

(b) The number of negative eigenvalues of $\mathcal{L}$ counting multiplicities equals the number of eigenvalues > 1 of Q.

Quite some machinery goes into proving this result, and we try to explain the main points. First one introduces a family of auxiliary operators

$$\mathcal{L}_\gamma := -\mathcal{B}^2 - \frac{1}{\gamma}\mathcal{R}\colon \mathscr{D}(\mathcal{T}^2)\cap H^{\mathrm{odd}} \to H^{\mathrm{odd}},\ \gamma > 0.$$

Since $\mathcal{B}^2|_{H^{\mathrm{odd}}}$ is self-adjoint and $\mathcal{R}$ is bounded and symmetric, $\mathcal{L}_\gamma$ is self-adjoint by the Kato-Rellich theorem [92, theorem X.12]. Now Q is constructed such that 0 is an eigenvalue of $\mathcal{L}_\gamma$ if and only if γ is an eigenvalue of Q, and the multiplicities of these eigenvalues are equal: if $f\in\mathscr{D}(\mathcal{T}^2)\cap H^{\mathrm{odd}}$ solves $\mathcal{L}_\gamma f = 0$, i.e. $-\gamma\mathcal{B}^2 f = \mathcal{R}f$, then applying $-\sqrt{\mathcal{R}}\mathcal{B}^{-2}$ to the latter Equation and writing $\mathcal{R} = \sqrt{\mathcal{R}}\sqrt{\mathcal{R}}$ yields

$$\gamma g = \gamma\sqrt{\mathcal{R}}f = Q\left(\sqrt{\mathcal{R}}f\right) = Qg,$$

with $g := \sqrt{\mathcal{R}}f \in H^{\mathrm{odd}}$. The converse direction is similar.

Next, the operator $\mathcal{R}|_{H^{\mathrm{odd}}}$ can be shown to be relatively $(\mathcal{B}^2|_{H^{\mathrm{odd}}})$-compact so that by Weyl's theorem [54, theorem 14.6],

$$\sigma_{\mathrm{ess}}(\mathcal{L}_\gamma) = \sigma_{\mathrm{ess}}(-\mathcal{B}^2|_{H^{\mathrm{odd}}}) = \sigma_{\mathrm{ess}}(\mathcal{L}).$$

In addition, one can show that $\sigma(-\mathcal{B}^2|_{H^{\mathrm{odd}}})$ is positive and bounded away from 0, and hence $\inf(\sigma_{\mathrm{ess}}(\mathcal{L})) > 0$. It remains to understand the behavior of the isolated eigenvalues of $\mathcal{L}_\gamma$ when varying γ. This can be done by their variational characterization, and one can show that the number of negative eigenvalues of $\mathcal{L}$ equals the number of γ's for which 0 is an eigenvalue of $\mathcal{L}_\gamma$. This establishes the relation in proposition 7.8(b); for details we refer to [37, sections 6.1 and 6.2].

By proposition 7.8 the original question of negative eigenvalues of $\mathcal{L}$ is translated into an eigenvalue problem for the Birman–Schwinger operator which has quite favorable qualities. But following Mathur's idea encountered for (VP) in section 4.3 one can exploit the structure which manifests itself in (7.15) and (7.16) to pass to an even simpler operator. By (7.15) an eigenfunction of Q which corresponds to a non-zero eigenvalue lies in the range of $\sqrt{\mathcal{R}}$, and by (7.16),

$$\mathscr{R}(\sqrt{\mathcal{R}}) \subset \left\{f = f(x,v) = |\phi'(E)|\, w\, \alpha_0(r)\, F(r) \,\text{a.e.} \mid F \in L^2([0,R_0])\right\},$$

where

$$\alpha_0(r) := \frac{\mathrm{e}^{\frac{1}{2}(\lambda_0+\mu_0)(r)}}{\sqrt{r(\lambda_0'+\mu_0')(r)}}, \quad r \in]0,R_0[.$$

In addition, if $f(x,v) = |\phi'(E)|\, w\, \alpha_0(r)\, F(r)$ and $g(x,v) = |\phi'(E)|\, w\, \alpha_0(r)\, G(r)$, then

$$\langle f,g\rangle_H = \langle F,G\rangle_{L^2([0,R_0])}; \tag{7.17}$$

a key ingredient here is the identity

$$\iint w^2 |\phi'(E)|\, \mathrm{d}v = \frac{\mathrm{e}^{-2\lambda_0(r)-\mu_0(r)}}{4\pi r}(\lambda_0'+\mu_0')(r),\ r>0,$$

which follows from (6.9) and (3.11). Based on these observations, the *reduced operator* or *Mathur operator*

$$\mathcal{M}\colon L^2([0,R_0]) \to L^2([0,R_0]),\ F \mapsto G$$

is defined as follows. First map $F \in L^2([0,R_0])$ to $f \in H^{\mathrm{odd}}$ defined by

$$f(x,v) := |\phi'(E)|\, w\, \alpha_0(r)\, F(r) \quad \text{for a.e. } (x,v) \in D.$$

Next map this f to $Qf \in \mathscr{R}(\sqrt{\mathcal{R}})$. Then there exists a unique $G \in L^2([0,R_0])$ such that

$$Qf(x,v) = |\phi'(E)|\, w\, \alpha_0(r)\, G(r) \quad \text{for a.e. } (x,v) \in D,$$

which completes the construction of the map $\mathcal{M}$.

The relation of $\mathcal{M}$ with Q immediately implies that $\gamma \neq 0$ is an eigenvalue of Q if and only if it is an eigenvalue of $\mathcal{M}$, and the multiplicities are equal; concerning the latter notice that by (7.17) orthogonality of eigenfunctions is preserved. By the same relation it is easy to verify that $\mathcal{M}$ inherits the functional analytic properties of Q: $\mathcal{M}$ is a bounded, linear, symmetric, non-negative, compact operator, see proposition 7.8.

A draw-back of the construction seems to be that the Birman–Schwinger operator Q and hence also the Mathur operator $\mathcal{M}$ contain the inverse operator of $\mathcal{B}$. As noted before, this inverse cannot be given explicitly, which seems to make it unclear how to apply the machinery above to specific examples. However, the right inverse $\tilde{\mathcal{B}}^{-1}$ of $\mathcal{B}$ can actually be given explicitly, see [37, definition 5.7]. The operator $\mathcal{B}\colon \mathscr{D}(\mathcal{T}) \cap \mathscr{N}(\mathcal{B})^\perp \to \mathscr{R}(\mathcal{B})$ is bijective with bounded inverse given by

$$\mathcal{B}^{-1} = (\mathrm{id} - \Pi)\widetilde{\mathcal{B}}^{-1},$$

where $\Pi: H \to \mathscr{N}(\mathcal{B})$ is the orthogonal projection onto the kernel $\mathscr{N}(\mathcal{B})$ of $\mathcal{B}$, see for example [54, section 5.4]. This information turns out to be sufficient to derive an integral representation of $\mathcal{M}$ which is quite workable in applications.

Proposition 7.9. *For $G \in L^2([0,R_0])$,*

$$(\mathcal{M}G)(r) = \int_0^{R_0} K(r,s) G(s)\, \mathrm{d}s, \quad r \in [0,R_0],$$

where the kernel $K \in L^2([0,R_0]^2)$ is defined as

$$K(r,s) = \mathrm{e}^{\frac{1}{2}(\mu_0(r)+3\lambda_0(r))}\mathrm{e}^{\frac{1}{2}(\mu_0(s)+3\lambda_0(s))}\,\frac{\sqrt{2r\mu_0'(r)+1}\sqrt{2s\mu_0'(s)+1}}{rs}\,I(r,s),$$

with

$$I(r,s) = \left\langle (\mathrm{id}-\Pi)\left(|\phi'|Ee^{-\lambda_0-\mu_0}1_{[0,r]}\right), |\phi'|Ee^{-\lambda_0-\mu_0}1_{[0,s]}\right\rangle_H,\ 0 \leqslant r,s \leqslant R_0.$$

The kernel is symmetric, i.e. $K(r,s) = K(s,r)$, and $\mathcal{M}$ is a Hilbert–Schmidt operator, see [91, theorem VI.22 et seq.].

If one now combines the relations between the spectra of the Antonov operator $\mathcal{L}$, the Birman–Schwinger operator Q, and the Mathur operator $\mathcal{M}$ with general results on Hilbert-Schmidt operators the following linear (in)stability information on (EV) results.

Theorem 7.10. *(a) The steady state is linearly stable if, and only if,*

$$\sup_{G\in L^2([0,R_0]),\,\|G\|_2=1}\int_0^{R_0}\int_0^{R_0} K(r,s)G(r)G(s)\,\mathrm{d}s\mathrm{d}r < 1.$$

If equality holds, there exists a zero-frequency mode but no exponentially growing mode.

(b) The number of exponentially growing modes of the steady state is finite and strictly bounded by $\|K\|^2_{L^2([0,R_0]^2)}$.

(c) The steady state is linearly stable if $\|K\|_{L^2([0,R_0]^2)} < 1$.

For a detailed proof we refer to [37]. Here we want to discuss an application of these techniques yielding a result which was not obtained by the methods in the previous sections, namely, we want to consider the stability of a shell of Vlasov matter surrounding a Schwarzschild black hole. To this end we generalize the steady state ansatz to (3.13), we place a Schwarzschild singularity of fixed mass $M > 0$ at the center, multiply the ansatz for the particle distribution f with a parameter $\delta > 0$ and keep the condition (4.1). One can show that there exist corresponding steady states of (EV) where the Vlasov shell has finite mass, finite extension, and is of course situated outside the Schwarzschild radius of the black hole; for the details of the construction of these steady states we refer to [37, section 2.2]. If one keeps the ansatz function with its cut-off energy and cut-off angular momentum and the mass M of the Schwarzschild singularity fixed one can show that for $\delta > 0$ sufficiently small, the corresponding effective potential for the particle motion still has a single-well structure in the sense of lemma 4.7 and section 5.2. This allows the introduction of action-angle variables for the stationary characteristic flow, which was the tool behind many of the constructions in the present section, so that these constructions continue to function also for the case with a central black hole, provided the mass of the black hole dominates the mass in the surrounding Vlasov shell. If one applies these constructions, one obtains the following result.

Theorem 7.11. *There exist families of steady states $(f^\delta, \lambda^\delta, \mu^\delta)_{\delta>0}$ of (EV) with a Schwarzschild singularity of mass $M > 0$ at the center surrounded by a shell of Vlasov matter with particle distribution f^δ, where the parameter $\delta > 0$ controls the size of the Vlasov shell. These steady states are linearly stable for $\delta > 0$ sufficiently small. For $\delta \to 0$ the metric converges to the vacuum Schwarzschild metric of mass M, uniformly on $]2M,\infty[$, and the density f^δ converges to zero pointwise.*

One should note that the characteristic flow for the particles in the shell of Vlasov matter is very different from the flow induced by null geodesics, which governs the propagation of massless particles and perturbations of the metric. The result in theorem 7.11 is very different from the result in [18].

8. Numerical observations, conjectures, and open problems

Maybe the most important single fact about the stability problem for the EV system is that along a one-parameter family $(f_\kappa, \lambda_\kappa, \mu_\kappa)_{\kappa>0}$ of steady states with some microscopic equation of state φ with $\varphi' > 0$, see proposition 3.1, the steady states change from being stable to being unstable when the central redshift κ changes from being small to being large. This is a genuinely relativistic feature which has no parallel for the non-relativistic VP system.

We discussed some first steps toward understanding this κ-dependence of the stability behavior on the linearized level in sections 5–7, and there is ample numerical evidence that this behavior is very general for (EV) and is true on the nonlinear level, see [7, 34, 36]. Hence any successful, comprehensive stability analysis for (EV) will have to take this phenomenon properly into account.

One key step toward understanding this behavior would obviously be to find a criterion for exactly when the change from stability to instability occurs. For the Einstein–Euler system the turning-point principle clearly specifies the points along the so-called mass-radius curve of a one-parameter steady state family, where stability changes to instability or the other way, see [53, 79, 107]; in [46] Hadžić and Lin give a rigorous proof of the turning-point principle for the Einstein–Euler system. But with the Vlasov matter model instead of a compressible, ideal fluid numerical evidence shows the analogous turning-point principle to be false [34]. This issue has also been discussed in the astrophysics literature [1, 21, 56–58, 90, 109, 118, 119], where the behavior of the so-called binding energy has been suggested as an alternative stability indicator. The (fractional) binding energy of a steady state $(f_\kappa, \lambda_\kappa, \mu_\kappa)$ is defined as

$$E_{b,\kappa} = \frac{N_\kappa - M_\kappa}{N_\kappa},$$

where

$$M_\kappa = \iint f_\kappa \langle v \rangle \, \mathrm{d}v \, \mathrm{d}x, \quad N_\kappa = \iint \mathrm{e}^{\lambda_\kappa} f_\kappa \, \mathrm{d}v \, \mathrm{d}x$$

are its ADM-mass and particle number. One can distinguish two forms of the binding energy hypothesis. The *weak binding energy hypothesis* claims that steady states are stable at least up to the first local maximum of the binding energy curve parameterized by the redshift. The *strong binding energy hypothesis* claims that steady states are stable precisely up to the first local maximum of the binding energy curve and become unstable beyond this maximum. In [36] numerical evidence against the strong binding energy hypothesis is given and it is shown that along the binding energy curve several stability changes can occur. The question from which quantity one can predict the stability behavior of the corresponding steady state is open even on the level of numerical simulations, and a good candidate could indicate how to make progress on the rigorous analysis of the stability issue.

A further question which has been investigated numerically in [7, 34] is how a stable or an unstable steady state reacts to perturbation. Upon perturbation, a stable steady state typically starts to oscillate with an undamped or damped amplitude, very similarly to what we discussed for the (VP) case. The reaction of an unstable steady state to perturbations is much more interesting. Depending on the 'direction' of the perturbation it collapses to a black hole or it seems to follow some sort of heteroclinic orbit to a different, stable steady state about which (the bulk of) it starts to oscillate. Steady states with a very large central redshift may upon perturbation also disperse towards flat Minkowski space instead of following a heteroclinic orbit as described above.

Obviously, there are in this context plenty of challenging questions awaiting rigorous mathematical analysis. We emphasize that the stability question for (EV) has also received a lot of attention in the astrophysics literature; in addition to the citations above we mention [20, 22, 23, 63, 113].

One aim certainly must be to prove nonlinear stability of steady states with small redshift, i.e. for steady states where linear stability holds according to theorem 5.5 and the results in section 7. As we explained in section 2.4 it seems doubtful whether the global variational approach based on the energy-Casimir functional $\mathcal{H}_C$ can succeed in the (EV) case, although it is very successful for the (VP) case as we saw in section 4.1. Probably a better chance for generalizing it to the (EV) case exists for the local minimizer approach discussed in section 4.2 for the (RVP) case. The situation may improve if one considers a suitable reduced functional derived from $\mathcal{H}_C$, such as we discussed for (VP) in section 4.1, see (4.8). In [117] such a functional was derived for (EV), but the approach there suffers from two defects. Firstly, some of the arguments in [117] are wrong; the main assertions in [117] have not been proven, see [5]. Secondly, even if correct the results in [117] would not imply any stability assertion since the existence of minimizers to the reduced functional relies on certain barrier conditions which are not known to be respected by the time-dependent solutions. This second, conceptual problem persists even though in [6] the authors were able to rigorously prove some of the assertions in [117]. In any case, nonlinear stability for (EV) is open, and we believe that new types of (conserved) functionals, probably involving derivatives of the metric coefficients, or/and new types of barrier conditions which are respected by time-dependent solutions are needed.

A second aim should be to prove nonlinear instability in situations where the existence of an exponentially growing mode is known by theorem 7.7. It seems inconceivable that an exponentially growing mode exists and the steady state is nonlinearly stable anyway, but saying this is no proof. We point out that in the (VP) plasma physics case the step from an exponentially growing mode to nonlinear instability has been made in [45], see also [29]. We believe that it is a non-trivial and worthwhile project to prove the analogous result for (EV), even though the outcome will probably not be surprising; notice that such unstable states should upon proper perturbation collapse to a black hole, and initial data which lead to the formation of black holes are very important in themselves. An interesting aspect here is that for the gravitational (VP) case the existence of exponentially growing modes has so far not been rigorously proven for potentially unstable steady states—those with sufficiently non-monotone microscopic equation of state φ—but we refer to [114] for a numerical construction, see also [39].

To conclude this section we point out that the stability problem reviewed above also poses some open problems which refer to the structure of the steady states themselves, but which have some bearing on the dynamic stability problem. It would be interesting to know for which steady states the effective potential has a single-well structure and allows for the introduction of action-angle variables for the stationary characteristic flow. Numerical evidence seems to suggest that this is true for isotropic steady states, but not necessarily for general ones. A proof exists only for isotropic steady states which satisfy the condition $\sup \frac{2m(r)}{r} \leqslant \frac{1}{3}$, while numerical evidence suggests that this Buchdahl quotient is bounded by $\frac{1}{2}$, but whether this improved Buchdahl bound indeed holds for all isotropic (EV) steady states and whether it implies a single-well structure is open.

9. Strict, global energy minimizers need not be stable

Consider the Hilbert space

$$H := \left\{ (z_k)_{k\in\mathbb{N}} \mid z_k = (x_k, p_k) \in \mathbb{R}^2,\ k \in \mathbb{N},\ \text{and}\ \sum_{k=1}^{\infty} (x_k^2 + p_k^2) < \infty \right\}$$

equipped with the norm

$$\|z\| := \left(\sum_{k=1}^{\infty} (x_k^2 + p_k^2) \right)^{1/2},\ z = (z_k)_{k\in\mathbb{N}} = ((x_k, p_k))_{k\in\mathbb{N}}.$$

On this space we define a linear dynamical system via

$$\dot{x}_k = p_k,\ \dot{p}_k = -\frac{1}{k^2} x_k,\ \text{i.e.}\ \ddot{x}_k = -\frac{1}{k^2} x_k,\ k \in \mathbb{N},$$

which can be written as

$$\dot{z} = \mathcal{L} z \tag{9.1}$$

with the linear, bounded operator

$$\mathcal{L}\colon H \to H,\ \mathcal{L} z := \left(\left(p_k, -\frac{1}{k^2} x_k \right) \right)_{k\in\mathbb{N}}.$$

The operator $\mathcal{L}$ generates a uniform C_0 group $(e^{t\mathcal{L}})_{t\in\mathbb{R}}$ of bounded operators on H. The energy functional

$$\mathcal{H}\colon H \to \mathbb{R},\ \mathcal{H}(z) := \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{k^2} x_k^2 + p_k^2 \right)$$

is Fréchet differentiable with

$$\langle D\mathcal{H}(z), \delta z \rangle = \sum_{k=1}^{\infty} \left(\frac{1}{k^2} x_k\, \delta x_k + p_k\, \delta p_k \right),$$

and $\mathcal{H}$ is a conserved quantity: $\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}(\mathrm{e}^{t\mathcal{L}} z) = 0$ for any $z \in H$. Since the system is linear, 0 is a stationary solution, and it is the unique, strict minimizer of the energy $\mathcal{H}$. However, 0 is dynamically unstable in the sense of Lyapunov. To see this, we choose $\epsilon = 1$ and let $\delta > 0$ be arbitrary. Fix some $n \in \mathbb{N}$ such that $\frac{1}{2} n\delta > 1$ and let $\mathring{z} := \left(\delta_{nk}(0, \frac{\delta}{2}) \right)_{k\in\mathbb{N}} \in H$. The solution with these initial data is given by

$$z_k(t) := \delta_{nk} \frac{\delta}{2} \left(-k \cos\left(\frac{t}{k} + \frac{\pi}{2} \right), \sin\left(\frac{t}{k} + \frac{\pi}{2} \right) \right),\ k \in \mathbb{N},$$

and $\|\mathring{z}\| = \frac{\delta}{2} < \delta$, while $\|z(\frac{3\pi}{2} n)\| = |z_n(\frac{3\pi}{2} n)| = \frac{1}{2} n\delta > 1$, which shows that the steady state is unstable.

The nice thing about this example, which in some form or other is certainly known and is obvious enough, is that the instability is not triggered by some nonlinear correction to the linear(ized) dynamics, but solely by the infinitely many directions in which a solution can escape.

It is also obvious that there is no compactness along minimizing sequences of $\mathcal{H}$, such as we exploited in section 4.1. Let $z^n := ((\delta_{nk}, 0))_{k\in\mathbb{N}}$. Then $(z^n)_{n\in\mathbb{N}}$ is a minimizing sequence of $\mathcal{H}$, $\mathcal{H}(z^n) = \frac{1}{2n^2} \to 0$, but it converges no better than weakly.

Finally, the operator has the spectrum $\sigma(\mathcal{L}) = \{\pm\frac{i}{k} \mid k \in \mathbb{N}\}$ consisting only of isolated eigenvalues of multiplicity 1. When viewed as a second order system, (9.1) takes the form

$$\ddot{x} = \widetilde{\mathcal{L}}x,$$

where the bounded, self-adjoint operator $\widetilde{\mathcal{L}}\colon l^2 \to l^2$ is defined by

$$\widetilde{\mathcal{L}}x := \left(-\frac{1}{k^2}x_k\right)_{k\in\mathbb{N}}$$

and has spectrum $\sigma(\widetilde{\mathcal{L}}) = \{-\frac{1}{k^2} \mid k \in \mathbb{N}\}$. One should compare this with the situation for (VP) or (EV) where we also had a first and a second order version of the linearized system with a self-adjoint operator governing the latter.

Data availability statement

No new data were created or analyzed in this study.

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