Computer classification of linear codes based on lattice point enumeration and integer linear programming

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Abstract. Linear codes play a central role in coding theory and have applications in several branches of mathematics. For error correction purposes the minimum Hamming distance should be as large as possible. Linear codes related to applications in Galois Geometry often require a certain divisibility of the occurring weights. In this paper we present an algorithmic framework for the classification of linear codes over finite fields with restricted sets of weights. The underlying algorithms are based on lattice point enumeration and integer linear programming. We present new enumeration and non-existence results for projective two-weight codes, divisible codes, and additive $\mathbb{F}_4$-codes.

Keywords: linear codes · classification · enumeration · lattice point enumeration · integer linear programming · two-weight codes.

1 Introduction

A linear code of length n is a k-dimensional linear subspace C of the vector space $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is the finite field with q elements. The vectors in C are called codewords. The weight $\mathrm{wt}(c)$ of a codeword $c = (c_1, \ldots, c_n) \in C$ is the number of nonzero positions $\#\{1 \le i \le n : c_i \ne 0\}$. For two codewords $c', c'' \in C$ the Hamming distance is given by $d(c', c'') := \mathrm{wt}(c' - c'')$. With this, the minimum distance $d(C)$ is given by the minimum occurring Hamming distance between two different codewords, i.e., $d(C) := \min\{d(c', c'') : c', c'' \in C, c' \ne c''\}$. An $[n, k, d]_q$-code is a k-dimensional subspace C of $\mathbb{F}_q^n$ with minimum distance at least d. We also speak of an $[n, k]_q$-code if we do not want to specify the minimum distance. Having applications in error correction in mind, a main problem in coding theory is the maximization of d, k, or $-n$ fixing the other two parameters. A famous problem in Galois Geometry is the maximum size of a partial t-spread in $\mathbb{F}_q^n$, where a partial t-spread is a set $\mathcal{T}$ of t-dimensional subspaces whose pairwise intersection is the zero vector. To such a set $\mathcal{T}$ we can associate an $[n, k]_q$-code C whose weights are divisible by $\Delta := q^{t-1}$, where $n = (q^n - 1)/(q - 1) - \#\mathcal{T} \cdot (q^t - 1)/(q - 1)$. So, non-existence results for Δ-divisible codes imply upper bounds on the size of partial spreads. Using the fact that those codes have to be projective we remark that indeed all currently known

upper bounds for partial spreads can be deduced from non-existence results for projective Δ-divisible codes, see e.g [12] for details. Linear codes with few weights have applications in e.g. cryptography, designs, and secret sharing schemes. To sum up, there is a wide interest in the enumeration of linear $[n,k]_q$-codes with certain restrictions on the occurring weights.

Algorithms for the computer classification of linear codes date back at least to 1960 [21], see also [3, 5, 14, 15, 18] for more recent literature. Here we want to focus on the approach using lattice point enumeration algorithms, see e.g. [1, 2, 19], from [5]. We refine some of the algorithmic techniques and apply integer linear programming, see e.g. [20].

The remaining part of this paper is structured as follows. We start introducing the preliminaries in Section 2. Since the geometric representation of linear codes as multisets of points in projective spaces plays a major role we briefly introduce this concept in Subsection 2.1. The main idea of extending linear codes using lattice point enumeration from [5] is briefly outlined in Subsection 2.2. The extension of each linear code is performed in two phases: In `Phase 1` lattice points of a certain Diophantine equation system are enumerated and in `Phase 2` additional checks are executed in order to reduce the number of extension candidates. In Section 3 we state integer linear programming (ILP) formulations for some of those checks from `Phase 2` that are moved to `Phase 0` that is executed prior to `Phase 1`. Since we are interested in the practical performance of our proposed algorithm we discuss computational results in Section 4.

2 Preliminaries

A common representation of an $[n,k]_q$-code is as the row-span of a $k \times n$ matrix over $\mathbb{F}_q$ – called generator matrix. As an example consider the $[56,6]_3$-code C spanned by

$$\begin{pmatrix} 1000002211010011020211110010120102121111122001200201211 \\ 0100001110121010112112001021122211121000021202220022010 \\ 0010002222022102001120010112002020200021112212112220011 12 \\ 0001001011222202210200221001010100222210022211212221200 \\ 0000102012102211211200102110221112100002120222021220 01001 \\ 0000011220200220101212200201102012122222120021002020211222 \end{pmatrix}.$$

It is the code of the famous Hill cap [11]. We say that a linear code is Δ-divisible if the weights of all codewords are divisible by Δ and we speak of a t-weight code is t different non-zero weights occur. Our example is a 9-divisible 2-weight code with weights 36 and 45. If a partial 3-spread of $\mathbb{F}_3^8$ of size 248 exists, then the set of uncovered points need to form a Hill cap, see [12] for the details.

2.1 Geometric representation of linear codes

Permuting the columns of the stated generator matrix or multiplying arbitrary columns by arbitrary elements in $\mathbb{F}_q^* := \mathbb{F}_q \backslash \{0\}$ yields an isomorphic linear code. We can factor out those symmetries by using the geometric representation of linear codes as multisets of points in a projective space $\mathrm{PG}(k-1, q)$. Here we just give a brief sketch of the most important facts and refer to e.g. [9] for

details. The 1-dimensional subspaces of $\mathbb{F}_q^k$ are the points of $\mathrm{PG}(k-1,q)$ which we denote by $\mathcal{P}_k$. A multiset of points $\mathcal{M}$ in $\mathrm{PG}(k-1,q)$ is a mapping from $\mathcal{P}_k$ to $\mathbb{N}$, i.e., to each point $P \in \mathcal{P}_k$ we assign a point multiplicity $\mathcal{M}(P)$. Starting from a generator matrix of a linear code we obtain a multiset of points by considering the multiset of one-dimensional subspaces spanned by the respective columns. In the other direction, we can take $\mathcal{M}(P)$ arbitrary generators for a point P and place them at arbitrary positions in a generator matrix. Two linear codes are ismorphic iff their associated multisets of points are. We call 2-dimensional subspaces lines. $(k-1)$-dimensional subspaces in $\mathrm{PG}(k-1,q)$ are called hyperplanes and there set is denoted by $\mathcal{H}_k$. For each arbitrary non-zero codeword $c \in C$ also $\alpha \cdot c$ is a codeword for all $\alpha \in \mathbb{F}_q^*$ and we can associate $\mathbb{F}_q^* \cdot c$ with a hyperplane $H \in \mathcal{H}_k$. The weight $\mathrm{wt}(c)$ of a codeword c equals $\#\mathcal{M} - \mathcal{M}(H)$ for the associated hyperplane H, where $\#\mathcal{M} := \sum_{P \in \mathcal{P}_k} \mathcal{M}(P)$ and $\mathcal{M}(H) := \sum_{P \in \mathcal{P}_k \,:\, P \in H} \mathcal{M}(P)$. The residual code of a non-zero codeword c is the restriction $\mathcal{M}|_H$ of $\mathcal{M}$ to the corresponding hyperplane H. We say that a linear $[n,k]_q$-code C is projective iff we have $\mathcal{M}(P) \in \{0,1\}$ for all $P \in \mathcal{P}_k$ for the corresponding multiset of points $\mathcal{M}$.

2.2   Extending linear codes using lattice point enumeration

We say that a generator matrix $G \in \mathbb{F}_q^{k \times n}$ of an $[n,k]_q$-code C is systematic if it is of the form $G = (I_k | R)$, where I_k is the $k \times k$ unit matrix and $R \in \mathbb{F}_q^{k \times (n-k)}$. Our general strategy to enumerate linear codes is to start from a systematic generator matrix G of a code and to extend G to a systematic generator matrix G' of a "larger" code C', c.f. [5, Section III]. Here we assume the form

$$G' = \begin{pmatrix} I_k & 0 \ldots 0 & R \\ 0 & \underbrace{1 \ldots 1}_{r} & \star \end{pmatrix} \tag{1}$$

where $G = (I_k | R)$ and $r \geq 1$, i.e., C' is an $[n+r, k+1]_q$-code. Let $\mathcal{M}$ and $\mathcal{M}'$ be multisets of points corresponding to C and C', respectively. Geometrically, $\mathcal{M}$ arises from $\mathcal{M}'$ by projection through a point $P \in \mathcal{P}_{k+1}$, i.e., for each line L through P we define $\mathcal{M}(L/P) = \mathcal{M}'(L) - \mathcal{M}'(P) = \sum_{Q \in L \,:\, Q \neq P} \mathcal{M}'(Q)$ and use $\mathrm{PG}(k,q)/P \cong \mathrm{PG}(k-1,q)$. Given the assumed shape of G' we have $P = \langle e_{k+1} \rangle$ for the $(k+1)$th unit vector e_{k+1} with $\mathcal{M}'(P) = r$. However, we may choose any point $P \in \mathcal{P}_{k+1}$ with $\mathcal{M}'(P) \geq 1$ to construct $\mathcal{M}$. While the linear code C corresponding to $\mathcal{M}$ may not admit a systematic generator matrix, there always is an isomorphic linear code which does. So, in general there are a lot of extensions ending up in a given code C'. In order to reduce the number of possible paths in [5] the authors speak of "canonical length extension" if

$$\min \{\mathcal{M}'(Q) \,:\, \mathcal{M}'(Q) > 0, Q \in \mathcal{P}_{k+1}\} = r \tag{2}$$

is satisfied, c.f. [5, Corollary 9], i.e., the smallest possible value of r is chosen.

The working horse for the algorithmic approach based on lattice point enumeration is [5, Lemma 7]:

Lemma 1. *Let G be a systematic generator matrix of an $[n,k]_q$ code C whose non-zero weights are contained in $\{i\Delta : a \le i \le b\} \subseteq \mathbb{N}_{\ge 1}$. By $c(P)$ we denote the number of columns of G whose row span equals P for all points $P \in \mathcal{P}_k$ and set $c(\mathbf{0}) = r$ for some integer $r \ge 1$. Let $\mathcal{S}(G)$ be the set of feasible solutions of*

$$\Delta y_H + \sum_{P \in \mathcal{P}_{k+1} : P \le H} x_P = n - a\Delta \ \ \forall H \in \mathcal{H}_{k+1} \tag{3}$$

$$\sum_{q \in \mathbb{F}_q} x_{\langle (u|q) \rangle} = c(\langle u \rangle) \ \ \forall \langle u \rangle \in \mathcal{P}_k \cup \{\mathbf{0}\} \tag{4}$$

$$x_{\langle e_i \rangle} \ge 1 \ \ \forall 1 \le i \le k+1 \tag{5}$$

$$x_P \in \mathbb{N} \ \ \forall P \in \mathcal{P}_{k+1} \tag{6}$$

$$y_H \in \{0, ..., b-a\} \ \ \forall H \in \mathcal{H}_{k+1}, \tag{7}$$

where e_i denotes the ith unit vector in $\mathbb{F}_q^{k+1}$. Then, for every systematic generator matrix G' of an $[n+r, k+1]_q$ code C' whose first k rows coincide with G and whose weights of its non-zero codewords are contained in $\{i\Delta : a \le i \le b\}$, we have a solution $(x,y) \in \mathcal{S}(G)$ such that G' has exactly x_P columns whose row span is equal to P for each $P \in \mathcal{P}_{k+1}$.

Our algorithmic strategy is to enumerate all lattice points satisfying constraints (3)-(7) in `Phase 1` and consider them as extension candidates C' for a given linear $[n,k]_q$-code C, where additional checks may be applied, in `Phase 2`. Of course we have to deal with the problem of eliminating isomorphic copies. On the other hand, there are there are some theoretic insights that allow to directly reject some of the lattice points as candidates in `Phase 2`, see [5] for details.

For our purpose, a few remarks are in order. Equations (3) ensure that C' is Δ-divisible with minimum weight at least $a\Delta$ and maximum weight at most $b\Delta$. Of course we may always choose $\Delta = 1$, but the larger we choose Δ and the tighter we choose a, b the less lattice points will satisfy constraints (3)-(7). Inequalities (4) and (5) model the assumed shape of (1). (Technically, Inequalities (5) are removed in a preprocessing step before calling a software for the enumeration of lattice points.) The variables x_P model $\mathcal{M}'(P)$ and the variables y_H parameterize $\mathcal{M}'(H)$ as detailed in (3). Actually, constraints (6) and (7) are just saying that we are only interested in lattice points, i.e., integral solutions.

Due to the availability of practically fast lattice point enumerations algorithms, the algorithmic strategy to generate many extension candidates in `Phase 1` and filter out suitable candidates afterwards in `Phase 2` turned out to be quite efficient if the number of constraints and variables is not too large, see [5]. It was also observed that considering just a subset of the constraints (3) can reduce computation times in many situations, i.e., generating more candidates in `Phase 1` can pay off if a simpler system allows faster generation of lattice points and the checks in `Phase 2` can be implemented efficiently.

In the implementation described in [5], the check of condition (2) as well as checks based on possible gaps in the assumed weight spectrum $\{i\Delta : a \le i \le b\}$ are moved to `Phase 2`. The idea of this paper is to demonstrate that it sometimes

can pay off to move such checks to integer linear programming computations in a `Phase 0` prior to `Phase 1`.

3   Enhancing the algorithmic approach via integer linear programming computations

Of course one can check the feasibility or infeasibility of an ILP using lattice point enumeration, after possibly transforming inequalities into equalities. In the other direction, many ILP solvers can also enumerate all lattice points of a polytope. However, in many situations ILP solvers can find a single feasible solution or show infeasibility faster than the full enumeration by a lattice point enumeration algorithm. However, the situation changes if one wants to enumerate a larger set of solutions exhaustively.

So, a first idea is to check feasibility of constraints (3)-(7) using an ILP solver in a `Phase 0` prior to `Phase 1`. This pays off in those situations where some extension problems don't have a solution. We remark that several calls of ILP solvers for different random target functions can also be used in a heuristic approach if it is not necessary to classify all possible codes with certain parameters but just to find some examples.

The second idea is to move some of the checks of `Phase 2` into an initial feasibility test based on ILP computations. Starting with Inequality (2) we observe that it can be rewritten as

$$x_P = 0 \quad \lor \quad x_P \geq r \tag{8}$$

for every point $P \in \mathcal{P}_{k+1}$. Having an upper bound $x_P \leq \Lambda_P$ at hand, which we have in most applications, we can linearize Inequality (8), as

$$x_P \leq \Lambda_P \cdot u_P \quad \land \quad x_P \geq r \cdot u_P \tag{9}$$

using an additional binary variable u_P.

Instead of solving a larger ILP some cases can also be eliminated in a simple preprocessing step. Suppose that an instance of Inequality (4) reads

$$\sum_{i=1}^{q} x_{P_i} = c$$

and we have $x_{P_i} \leq \Lambda$ as well as $x_{P_i} = 0 \ \lor \ x_{P_i} \geq r$ for all $1 \leq i \leq r$. If

$$\left\lfloor \frac{c}{r} \right\rfloor < \left\lceil \frac{c}{\Lambda} \right\rceil, \tag{10}$$

then no solution exists since at most $\left\lfloor \frac{c}{r} \right\rfloor$ variables x_{P_i} have to be non-zero and at least $\left\lceil \frac{c}{\Lambda} \right\rceil$ variables x_{P_i} have to be non-zero. We have observed the applicability of this criterion in practice for parameters $(q, r, \Lambda, c) = (2, 3, 4, 5)$.

With respect to gaps in the weight spectrum we assume that the possible non-zero weights are contained in

$$\{a_1\Delta_1, \ldots, b_1\Delta_1, a_2\Delta_2, \ldots, b_2\Delta_2, \ldots, a_l\Delta_l, \ldots, b_l\Delta_l\}, \tag{11}$$

where $l \geq 2$, $a_i < b_i$ for $1 \leq i \leq l$, and $b_{i-1}\Delta_{i-1} < a_i\Delta_i$ for $2 \leq i \leq l$. With this, we can replace Inequalities (3) and Inequalities (7) by

$$\sum_{i=1}^{l} \Delta_i y_H^i + \sum_{P \in \mathcal{P}_{k+1}: P \leq H} x_P = n - \sum_{i=1}^{l} a_i \Delta_i z_H^i \Delta \ \ \forall H \in \mathcal{H}_{k+1}, \tag{12}$$

$$y_H^i \leq (b_i - a_i) z_H^i \ \ \forall H \in \mathcal{H}_{k+1}, \forall 1 \leq i \leq l, \tag{13}$$

$$\sum_{i=1}^{l} z_H^i = 1 \ \ \forall H \in \mathcal{H}_{k+1}, \tag{14}$$

$$z_H^i \in \{0,1\} \ \ \forall H \in \mathcal{H}_{k+1}, \forall 1 \leq i \leq l, \tag{15}$$

$$y_H^i \in \mathbb{N} \ \ \forall H \in \mathcal{H}_{k+1}, \forall 1 \leq i \leq l. \tag{16}$$

The ILP for `Phase 0` consists of inequalities (4)-(6) and (12)-(16) if we have a possible weight spectrum as in (11) with $l \geq 2$ or, alternatively, of inequalities (3)-(7). If $r \geq 2$, see (1), then we additionally add the constraints (9) and additional variables $u_P \in \{0,1\}$ for all $P \in \mathcal{P}_{k+1}$.

4   Computational results

In this section we want to present a few computational results that have been obtained with our refined algorithmic approach of computer classification of linear codes based on lattice point enumeration and integer linear programming. We start with non-existence results for projective two-weight codes, see e.g. [6, 7] for surveys. To this end, we slightly modify the notion of an $[n,k,d]_q$-code by replacing d with a set of occurring non-zero weights and writing $\leq n$ if the length is at most n. For each projective two-weight code over $\mathbb{F}_q$ with characteristic p there exist integers u and t such that the two occurring non-zero weights can be written as $u \cdot p^t$ and $(u+1) \cdot p^t$, i.e. the code is p^t-divisible, see [8]. Under some mild extra conditions this is also true for non-projective two-weight codes [17].

Proposition 1. *No projective $[66, 5, \{48, 56\}]_4$-code exists.*

Proof. By exhaustive enumeration we have determined all three non-isomorphic $[\leq 65, 3, \{48, 56\}]_4$-codes. They have lengths 63, 64, 65 and orders 362880, 1728, 36 of their automorphism groups, respectively. None of them can be extended to an $[65, 4, \{48, 56\}]_4$-code, so that no projective $[66, 5, \{48, 56\}]_4$-code exists.□

Proposition 2. *No projective $[35, 4, \{28, 32\}]_8$-code exists.*

Proof. We have computationally determined the unique $[\leq 34, 3, \{28, 32\}]_8$-code. It has length 34 and an automorphism group of order 43008. We have computationally checked that no extension exists. The corresponding ILP computation took 5.6 hours of computation time and checked $1\,633\,887$ B&B-nodes.   □

We remark that we have also enumerated all $[\leq 122, 3, \{108, 117\}]_9$-codes (with maximum point multiplicity 9). All of these 1147 non-isomorphic codes have length $n = 122$. It is an interesting open question whether one of these can be extended to a projective $[123, 4, \{108, 117\}]_9$-code, which is currently unknown. Applying the so-called subfield construction, see e.g. [7], would also yield a projective $[492, 8, \{324, 351\}]_3$-code – again currently unknown.

On the way to enumerate all projective $[77, 5, \{56, 64\}]_4$-codes we need to consider the extension of $[74, 3, \{56, 64\}]_4$- to $[76, 4, \{56, 64\}]_4$-codes. Without the application of `Phase 0` exactly 1 087 803 linear codes are constructed from lattice points and eliminated by Inequality (2) afterwards. We remark that there are exactly 5 nonisomorphic $[76, 4, \{56, 64\}]_4$-codes.

Next we consider projective Δ-divisible codes. The characterization of all possible length n such that there exists a projective 8-divisible $[n, k]_2$-code for some k was completed in [13]. For projective 5-divisible codes over $\mathbb{F}_5$ the only undecided length is $n = 40$, see e.g. [16] for a survey.

Proposition 3. *No projective 5-divisible $[40, 4]_5$-code exists.*

Proof. By exhaustive enumeration we have determined all 371 non-isomorphic 5-divisible $[39, 3]_5$-codes with maximum point multiplicity 4. None of them can be extended to a projective 5-divisible $[40, 4]_5$-code. Note that extending a $[39, 3]_5$-code with maximum point multiplicity to a projective $[40, 4]_5$-code would imply that the resulting code contains a two-dimensional simplex code, in geometrical terms a line, in its support. However, no projective 5-divisible $[34, k]_5$-code exists [16, Lemma 7.12]. $\qquad\square$

As a last example we consider additive codes over $\mathbb{F}_4$. In general, each k-dimensional additive code of length n with minimum Hamming distance d over $\mathbb{F}_{q^2}$ geometrically corresponds to a multiset of lines in $\mathrm{PG}(2k-1, q)$ such that each hyperplanes contains at most $n - d$ lines, see e.g. [4]. For $\mathbb{F}_4$ the case $k = 3.5$ was partially studied in [10]. Our aim is to construct examples for $(n, d) = (51, 38)$ which are special $[153, 7, 76]_2$-codes.

Lemma 2. *Let $\mathcal{M}$ be the multiset of points in $\mathrm{PG}(6, 2)$ formed by 51 lines such that every hyperplane contains at most 13 lines and C denote the corresponding binary code. Then, C is $[153, 7, 76]_2$-code with $\mathrm{wt}(c) \leq 102$ for every $c \in C$.*

Proof. Replacing the 51 lines by their 3 points yields a multiset $\mathcal{M}$ in $\mathrm{PG}(6, 2)$ with cardinality 153. Since each hyperplane H contains between 0 and 13 lines, we have $51 \leq \mathcal{M}(H) \leq 77$, so that $76 \leq \mathrm{wt}(c) \leq 102$ for all $c \in C \backslash \{\mathbf{0}\}$. $\qquad\square$

Lemma 3. *No projective 2-divisible $[65, 6, 32]_2$-code with maximum point multiplicity exists.*

Proof. By exhaustive enumeration. $\qquad\square$

We remark that there exists a unique $[63, 6, 32]_2$-code C which is projective and 32-divisible. If $\mathcal{M}$ denotes the corresponding multiset of points, then adding two arbitrary points yields $[65, 6, 32]_2$-codes.

Lemma 4. *Let C be a $[153, 7, 76]_2$-code with $\mathrm{wt}(c) \le 102$ for every $c \in C$. Then, the occurring non-zero weights are contained in $\{76, 80, 92, 96, 100\}$.*

Proof. Note that C is a Griesmer code and that 4 divides the minimum distance 76, so that we can conclude that C is 4-divisible [22]. The residual code of a codeword of weight 84 would be a $[69, 6, 34]_2$-code which does not exist. Now let C' be the residual code of a codeword of weight 88, so that C' is $[65, 6, 32]_2$-code. Since C is 4-divisible and has maximum point multiplicity 2, C' has to be 2-divisible with maximum point multiplicity 2. However, we have just shown that such a code does not exist. □

Proposition 4. *There are exactly two non-isomorphic $[153, 7, 76]_2$-codes C with $\mathrm{wt}(c) \le 102$ for every $c \in C$.*

Proof. We have exhaustively enumerated all $[\le 150, 5]_2$-codes with weight restrictions as in Lemma 4. There are 5 such codes with length 149 and 27 such codes with length 150. Extending them to $[151, 6]_2$-codes with weight restrictions as in Lemma 4 leaves just two possibilities (with maximum point multiplicity 4). Noting that C has maximum point multiplicity 2, it has to be an extension of such a $[151, 6]_2$-code and we have computationally verified the stated results. Those two codes have weight distributions $76^{107}80^{15}92^5$ and $76^{108}80^{14}92^496^1$. The automorphism groups have orders 16128 and 32256, respectively. □

A generator matrix of the $[153, 7, \{76, 80, 92\}]_2$-code with an automorphism group of order 16128 is given by the concatenation of

$$
\begin{pmatrix}
111111111111111111111111111111111111111111111111111111111111111111111111111111 \\
000000000000000000000000000000000000000001111111111111111111111111111111111111 \\
000000000000000000001111111111111111111110000000000000000000001111111111111111 \\
000000000001111111100000000000001111111000000011111110000000000000000000011111 \\
000001111110000111100000011111100001110000011111100001111000001111100000011111100001 \\
001110001110011001100011100011001001100011100011100110011000111000111001100110 \\
010010010110101010001101101100101010101101100101101101101010100101100100101000
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
111000000000000000000000000000000000000000000000000000000000000000000010000000 \\
111111111111111111111111111111111111111110000000000000000000000000000001000000 \\
111000000000000000000001111111111111111111111111111111111111110000000000100000 \\
111000000001111111111110000000011111111111110000000111111111111111111110000010000 \\
111000111100000011111110000001110000001111110001111000011111100011110001111000011000 \\
011011001100011100011100110011000110001101100110001100011011001110000100 \\
101101000100100101100101011101011011001011001010101100100100110101111100000011
\end{pmatrix}.
$$

The corresponding multisets of points can be partitioned into 51 lines. We remark that there is also an $[153, 7, 76]_2$-code with weight distribution $76^{108}80^{15}92^3108^1$ which cannot be partitioned in such a way due to the codeword of weight 108. Without excluding the weights 84 and 88 the intermediate codes would be too numerous to perform an exhaustive enumeration in reasonable time.

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