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On the Ohsawa–Takegoshi Extension Theorem

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Abstract
We establish a new extension result for twisted canonical forms defined on a hypersurface with simple normal crossings of a projective manifold. Some of the examples presented in the appendix show that the bounds we obtain for the extension are sharp.

Keyword Extension theorems

1 Introduction

Since it was established in [28], the Ohsawa–Takegoshi extension theorem turned out to be a fundamental tool in complex geometry. As of today, there are uncountable many proofs and refinements of the original result and even more applications to both complex analysis and algebraic geometry. Very roughly, the set-up is as follows: u is a canonical form defined on a sub-variety $Y \subset X$ with values in a Hermitian bundle $F \to X$. We are interested in the following two main questions:

Q_1. Does the section u extend to X?
Q_2. If the answer to the previous question is "yes," can one construct an extension whose L^2 norm is bounded by the L^2 norm of u, up to an universal constant?

If Y is non-singular, then the results in e.g., [23] give—practically optimal—curvature conditions for the bundle F such that the answer to both questions above is

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affirmative. We refer to the articles [2, 3, 5, 8, 11, 13, 14, 17, 24–27, 29–34, 36] for many interesting developments and applications.

The case of a singular sub-variety Y turns out to be significantly more difficult and the most complete results obtained so far only treat the *qualitative* aspect of the extension problem, that is to say the question Q_1, cf. [7].

In this article we are concerned with the question Q_2. We obtain a few *quantitative* results for extension of twisted forms defined on sub-varieties Y, which have simple normal crossings. Our main motivation is the Conjecture in [14]. To begin with we fix some notations/conventions.

Let X be a non-singular, projective manifold and let $Y := \sum_{i=1}^N Y_i$ be a divisor with simple normal crossings. Let (L, h_L) be a Hermitian line bundle on X, endowed with a metric h_L. The following assumptions will be in force throughout this article:

(a) The usual curvature requirements are satisfied

$$\Theta_{h_L}(L) \geq 0, \quad \Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(Y),$$

where $\delta > 0$ is a positive real number and h_Y is a smooth metric on the bundle corresponding to $\mathcal{O}(Y)$. Let s be the canonical section of $\mathcal{O}(Y)$ with the normalization condition $|s|^2_{h_Y} \leq e^{-\delta}$.

(b) We write locally $h_L = e^{-\varphi_L}$ and $h_Y = e^{-\varphi_Y}$. The singularities of the metric h_L of L are of the following type:

$$\varphi_L = \sum_j r_j \log |f_j|^2 + \tau_L,$$

where f_j are local holomorphic functions such that they are not identically zero when restricted to any of the components of Y and $r_j > 0$ is positive number. Moreover, we assume that τ_L is non-singular.

(c) Let $u \in H^0\big(X, (K_X + Y + L) \otimes \mathcal{O}_X/\mathcal{O}_X(-Y)\big)$ be a twisted canonical form defined over Y. There exists a covering (Ω_i) of X with coordinate sets such that the restriction of the section $u|_{\Omega_i}$ of u admits an extension $U_i \in H^0(\Omega_i, K_X + Y + L)$, which belongs to the multiplier ideal of h_L, i.e.,

$$\int_{\Omega_i} |U_i|^2 e^{-\varphi_L - \varphi_Y} < +\infty. \tag{1.0.1}$$

We note that near a non-singular point of Y the existence of U_i follows from the usual L^2 hypothesis of OT theorem provided that u belongs to the multiplier ideal sheaf of $h_L|_Y$. But this may no longer be true in a neighborhood of singular point of Y.

In addition to the natural hypotheses (a), (b), and (c) above we collect next two other requirements we need to impose for some of our statements to hold.

(i) We assume that there exists an open subset V_{sing} of X containing the singularities of Y such that the following hold.

(i.α) There exists a snc divisor $W = \sum W_j$ on X such that the singularities of the restriction of the metric h_L of L to V_{sing} are as follows:

$$\varphi_L = \sum_j \left(1 - \frac{1}{k_j}\right) \log |z_j|^2 + \tau_L,$$

where k_j are positive integers, and z_j are the local equations of W. The local weight τ_L above is assumed to be bounded.

(i.β) The curvature of the restriction of $h_L|_{V_{\text{sing}}}$ is greater than $C_{\text{sing}}\omega_{\mathcal{C}}|_{V_{\text{sing}}}$, where C_{sing} is a positive constant, and $\omega_{\mathcal{C}}$ is a fixed Kähler metric with conic singularities on $\big(X, \sum(1 - 1/k_i)W_i\big)$, i.e., locally we have

$$\omega_{\mathcal{C}} \simeq \sum_{i=1}^{r} \frac{\sqrt{-1}dz_i \wedge d\bar{z}_i}{|z_i|^{2-\frac{2}{k_i}}} + \sum_{i \geq r+1} \sqrt{-1}dz_i \wedge d\bar{z}_i \tag{1.0.1}$$

that is to say, $\omega_{\mathcal{C}}$ is quasi-isometric with the RHS of (1.0.1), where $z_1 \dots z_r = 0$ is the local equation of the divisor $W = \sum_{i=1}^{r} W_i$.

(ii) There exists an open subset V_{sing} of X containing the singularities of Y such that the curvature of the restriction of $h_L|_{V_{\text{sing}}}$ is identically zero.

In this context our first result states as follows:

Theorem 1.1 *We assume that the metric $h_L = e^{-\varphi_L}$ of L and the section u verifies the requirements* (a), (b), (c) *as well as* (i) *above. Then u extends to X, and for each $1 \geq \alpha > 0$ there exists a section $U \in H^0(X, K_X + Y + L)$ such that $U|_Y = u$ and we have the estimates*

$$\frac{1}{C} \int_{X \setminus V_{\text{sing}}} |U|^2 e^{-\varphi_Y - \varphi_L} dV_{\omega_{\mathcal{C}}} \leq \int_{Y \setminus V_{\text{sing}}} \left|\frac{u}{ds}\right|^2 e^{-\varphi_L} + \sum_i \left(\int_{Y_i \cap V_{\text{sing}}} \left|\frac{u}{ds}\right|_{\omega_{\mathcal{C}}}^{\frac{2}{1+\alpha}} e^{-\frac{\varphi_L}{1+\alpha}} dV_{\omega_{\mathcal{C}}} \right)^{1+\alpha} \tag{1.1.1}$$

where $\omega_{\mathcal{C}}$ is the reference metric on X and the constant C depends on α, the geometry of $(V_{\text{sing}}, \omega_{\mathcal{C}})$*, the positivity constant and the upper bound for* $\operatorname{Tr}_{\omega_{\mathcal{C}}} dd^c \tau_L$ *in* (i.α).

Remark 1.2 The precise dependence of the constant C in (1.1.1) of the quantities mentioned in Theorem 1.1 can be easily extracted from the proof we present in Sect. 5.

Remark 1.3 It is very likely that our arguments work under more general circumstances, e.g., one can probably establish the same result in the absence of the hypothesis (b) (via the regularization procedure due to Demailly, cf. [12]). But so far it is unclear to us how to remove the local strict positivity hypothesis in (i.β), or the fact that the singularities of $h_L|_{V_{\text{sing}}}$ are assumed to be of conic type.

A result pointing in the direction of the preceding Remark 1.3 is the following.

Theorem 1.4 *Let X be a projective manifold and let (L, h_L) be a line bundle such that the usual conditions* (a)–(c) *are satisfied, respectively. We assume moreover that $\Theta_{h_L}(L) \leq C_1\omega$ on $V_{\rm sing}$ for some constant C_1. Here ω is a non-singular Kähler metric on X. Then there exists a section $U \in H^0(X, K_X + Y + L)$ such that $U|_Y = u$ with the estimates*

$$\frac{1}{C}\int_{X\setminus V_{\rm sing}} |U|^2 e^{-\varphi_Y-\varphi_L} \leq \int_{Y\setminus V_{\rm sing}} \left|\frac{u}{\mathrm{d}s}\right|^2 e^{-\varphi_L} + \left(\int_{Y\cap V_{\rm sing}} \left|\frac{u}{\mathrm{d}s}\right| e^{-\frac{\varphi_L}{2}} \mathrm{d}V_\omega\right)^2$$

where C depends only on the geometry of $(V_{\rm sing}, \omega)$ and C_1.

Remark 1.5 We stress on the fact that in the statement Theorem 1.4 the strict positivity of the curvature of (L, h_L) on $V_{\rm sing}$ is not part of hypothesis.

In conclusion, Theorems 1.1 and 1.4 are providing an extension of u whose L^2 norm is bounded by the usual quantity outside the singularities of Y, and by an *ad hoc* L^p norm near $Y_{\rm sing}$, for any $p \in [1, 2[$. The example proving Claim 4 in Appendix shows that this type of estimates are sharp.

However, the constant $C_\alpha(V_{\rm sing}, \omega_{\mathcal{C}})$ in Theorem 1.1 involves the geometry of the local pair $(V_{\rm sing}, \omega_{\mathcal{C}})$, or if one prefers, the restriction of h_L to $V_{\rm sing}$. Moreover, we only allow singularities of conic type for $h_L|_{V_{\rm sing}}$. In order to try to "guess" the type of estimates one could hope for in general, we make the following observation. Let $\Omega \subset V_{\rm sing}$ be a coordinate open subset. The restriction of the RHS of (1.1.1) to Ω is given by the following expression:

$$\int_{\Omega\cap Y_i} \frac{1}{\prod_{j\neq i} |f_j|^{\frac{2}{1+\alpha}}} \frac{|f_u|^{\frac{2}{1+\alpha}} \mathrm{d}\lambda}{\prod_i |z_i|^{2(1-1/k_i)}} \tag{1.5.1}$$

where $\prod f_j = 0$ is the local equation of $Y \cap \Omega$ and $\prod z_i = 0$ is the equation of W. Therefore the second term of RHS of (1.1.1) is equivalent to

$$\int_{Y_i\cap\Omega} \left|\frac{u}{\mathrm{d}s_Y}\right|_\omega^{\frac{2}{1+\alpha}} e^{-\varphi_L}\mathrm{d}V_\omega, \tag{1.5.2}$$

where ω is a smooth Kähler metric on X. So from this point of view the following important—and very challenging—problem is natural.

Conjecture 1.6 *We assume that the metric $h_L = e^{-\varphi_L}$ of L and the section u verifies the requirements* (a), (b), (c). *Then for each $1 \geq \alpha > 0$, there exists an section*

$U \in H^0(X, K_X + Y + L)$ such that $U|_Y = u$ with the estimates

$$\frac{1}{C_\alpha(V_{\text{sing}})} \int_{X \setminus V_{\text{sing}}} |U|^2 e^{-\varphi_Y - \varphi_L} \mathrm{d}V_\omega \le \int_{Y \setminus V_{\text{sing}}} \left| \frac{u}{\mathrm{d}s} \right|^2_\omega e^{-\varphi_L} \mathrm{d}V_\omega + \sum_i \left(\int_{Y_i \cap V_{\text{sing}}} \left| \frac{u}{\mathrm{d}s} \right|^{\frac{2}{1+\alpha}}_\omega e^{-\varphi_L} \mathrm{d}V_\omega \right)^{1+\alpha} \tag{1.6.1}$$

where ω is a reference smooth Kähler metric on X and $C_\alpha(V_{\text{sing}})$ only depends on $(V_{\text{sing}}, \omega)$ and the restriction of the metric $h_L|_Y$ to Y.

It is our belief that the most subtle part of the previous conjecture would be to have an accurate estimate for the constant $C_\alpha(V_{\text{sing}})$.

Our next two results are of *experimental* nature and therefore we have decided to formulate them for surfaces only, so that we have $\dim(X) = 2$. The same type of statements hold in arbitrary dimension, as one can easily convince oneself. The method of proof is completely identical to the case we explain here in detail, so for simplicity's sake we stick to the case of surfaces.

We fix next few more notations adapted to the pair (X, Y).

Let $(\Omega_i)_{i \in I}$ be covering of X with open coordinate subsets. By the simple normal crossing hypothesis we can choose coordinates $z_i = (z_i^1, z_i^2)$ such as

$$Y \cap \Omega_i = (z_i^1 \dots z_i^p = 0) \tag{1.6.2}$$

for each $i \in I$ and some p (depending on i). Let $(\theta_i)_{i \in I}$ be a partition of unity subordinate to $(\Omega_i)_{i \in I}$.

Since we assume that X is a complex surface, the components of Y are smooth curves. The singular set of Y (i.e., the mutual intersections of its components) consists of a finite number of points of X, denoted by $p_1, \dots, p_s$.

We assume that Ω_i is refined enough so that the section $u|_{\Omega_i}$ is given by

$$f_i dz_i^1 \wedge dz_i^2 \tag{1.6.3}$$

for some holomorphic function f_i. On overlapping open subsets, different expressions (1.6.3) are gluing only modulo a 2-form divisible by the equation of the divisor Y.

Let p be one of the singular points of Y, assumed to be the center of some Ω_i. We denote by $t_i := z_i^1 \cdot z_i^2$; this is—by our previous conventions—the local equation of *the cross* $Y \cap \Omega_i$. We can interpret the function (= $n-2$-form in general) f_i as a local section of the bundle $L|_{\Omega_i}$, and as such we can consider its derivative

$$\partial_{\varphi_L} f_i \tag{1.6.4}$$

with respect to the Chern connection of L. The result is a 1-form on Ω_i.

Given the hypothesis in our following statement, it is possible to construct an extension of u by applying the result in [23]. However, here we obtain different types of estimates.

Theorem 1.7 *Let X be a smooth projective surface, and let (L, h_L) be a line bundle such that the usual curvature conditions* (a) *and* (c) *are satisfied. Assume moreover that h_L is non-singular and for each $i \in I_{\mathrm{sing}}$ we have $f_i \in (z_i^1, z_i^2)$, in other words our section vanishes on the set singular points of Y.*

Then there exists a section $U \in H^0(X, K_X + Y + L)$ enjoying the following properties:

(1) $U|_Y = u$.
(2) *There exists a constant $C(X, V_{\mathrm{sing}}) > 0$ such that we have*

$$\frac{1}{C(X, V_{\mathrm{sing}})} \int_X |U|^2 e^{-\varphi_Y - \varphi_L} \leq \int_{Y \setminus V_{\mathrm{sing}}} \log^2(\max |s_j|^2) \left|\frac{u}{\mathrm{d}s}\right|^2 e^{-\varphi_L} + \int_{Y \cap V_{\mathrm{sing}}} \log^2(\max |s_j|^2) |\partial_{\varphi_L} u|^2 e^{-\varphi_L}. \tag{1.7.1}$$

We obtain the same type of result provided that the bundle (L, h_L) is flat near Y_{sing}, as follows.

Theorem 1.8 *Let X be a smooth projective surface, and let (L, h_L) be a line bundle such that the curvature and L^2 conditions* (a), (b), (c) *as well as the additional property* (ii) *are satisfied.*

Then there exists a section $U \in H^0(X, K_X + Y + L)$ enjoying the following properties:

(1) $U|_Y = u$.
(2) *We have*

$$\frac{1}{C(X, V_{\mathrm{sing}})} \int_{X \setminus V_{\mathrm{sing}}} |U|^2 e^{-\varphi_Y - \varphi_L} \leq \int_{Y \setminus V_{\mathrm{sing}}} \log^2(\max |s_j|^2) \left|\frac{u}{\mathrm{d}s}\right|^2 e^{-\varphi_L} + \int_{Y \cap V_{\mathrm{sing}}} \log^2(\max |s_j|^2) |\partial_{\varphi_L} u|^2 e^{-\varphi_L}. \tag{1.8.1}$$

Our next statement is confined to the two-dimensional case.

Theorem 1.9 *Let X be a smooth projective surface, and let (L, h_L) be a line bundle such that the usual curvature and L^2 conditions* (a), (b), *and* (c) *are satisfied, respectively. We assume moreover that the following hold.*

(1) *There exists a component Y_1 of Y which intersects $\cup_{j \neq 1} Y_j$ in a unique point p_1, such that $u(p_1) \neq 0$.*
(2) *The restriction $(L, h_L)|_{Y_1}$ is Hermitian flat.*

Then the section u admits an extension U satisfying the same estimates as in Theorem 1.7.

The *raison d'être* of the previous Theorems 1.7, 1.8, and 1.9 is that the inequality (1.8.1) is meaningful even in the absence of the additional hypothesis these statements contain. Because of the variety of contexts in which an extension of u verifying the estimates of type (2) of Theorem 1.8 can be obtained, it is very tempting to formulate the following.

Conjecture 1.10 *Let (X, Y) be a smooth projective pair, where X is a surface and Y is an snc divisor. Let (L, h_L) be a line bundle such that the properties* (a), (b), *and* (c) *are satisfied. Then there exists a section $U \in H^0(X, K_X + Y + L)$ enjoying the following properties:*

(1) *The section U is an extension of u.*
(2) *We have*

$$\frac{1}{C(X, V_{\text{sing}})} \int_{X \setminus V_{\text{sing}}} |U|^2 e^{-\varphi_Y - \varphi_L} \leq \int_{Y \setminus V_{\text{sing}}} \log^2(\max |s_j|^2) \left| \frac{u}{ds} \right|^2 e^{-\varphi_L} + \lim_{\varepsilon \to 0} \int_{Y \cap V_{\text{sing}}} \log^2(\max |s_j|^2) |\partial_{\varphi_{L,\varepsilon}} u|^2 e^{-\varphi_{L,\varepsilon}}, \tag{1.10.1}$$

where $\varphi_{L,\varepsilon} := \log\left(\varepsilon^2 e^{\phi} + e^{\varphi_L}\right)$ is a non-singular approximation of h_L.

In the sequel of this article we will formulate the precise higher dimensional version of this conjecture, and we will explain its impact on the extension of the pluricanonical forms.

In Appendix A by Bo Berndtsson some examples are given that indicate that the estimates (2) in Conjecture 1.10 are most likely the best one could hope for: without the log factor, this conjecture is simply wrong. Moreover, the example given in order to prove Claim 4 shows that the factor $e^{-\varphi_L}$ in (1.6.1) cannot be replaced by the slightly less singular weight $e^{-(1-\varepsilon)\varphi_L}$, for any $\varepsilon > 0$. Finally, Appendix contains a comparison with a one-dimensional problem (the *fat point*), intended to highlight the origin of the difficulties in a very simple setting.

1.1 Organization of the Paper

In the second section we explain the main ideas involved in the proof of our results. The next section is dedicated to the revision and slight improvement of the usual a-priori inequalities. Our principal contribution to the *Ohsawa–Takegoshi artisan industry* is in Sect. 4, where the necessary tools from geometric analysis are recalled/developed. The proof of the results stated above is presented in Sect. 5.

1.2 In Memoriam Nessim Sibony

This article is dedicated to a mentor and colleague, Nessim Sibony, who has left us in 2021. His untimely death has affected us all deeply, leaving a void in the complex geometry community. He was a highly accomplished mathematician and a colleague who was inspirational to generations. He will be remembered not only as a revered collaborator with his profound understanding of the field of complex geometry but also as a dear friend with his sharp grasp of a wide range of subjects from literature, history, and to life in general.

2 An Overview of the Arguments

Our results are obtained by combining the method in [28] with the method in [2]. In order to highlight the main arguments as well as some of the difficulties, we only discuss here the case of a non-singular metric h_L on L. In general the whole scheme of the proof becomes more technical, since the regularization procedure we have to use for the metric is quite tricky to implement in the presence of the singular hypersurface Y.

We start with a quick review of the usual case.

2.1 The Case of a Non-singular Hypersurface *Y*

Let ξ be a L-valued form of type $(n, 1)$. We denote by $\gamma_\xi := \star\xi$ its Hodge dual (induced by an arbitrary Kähler form on X).

Consider the functional

$$\mathcal{F}(\xi) = \int_X \overline{\partial}\left(\frac{u}{s_Y}\right) \wedge \overline{\gamma_\xi} e^{-\varphi_L} \tag{2.0.2}$$

associated to the current $\overline{\partial}\left(\frac{u}{s_Y}\right)$.

We decompose $\xi = \xi_1 + \xi_2$ according to $\mathrm{Ker}(\overline{\partial})$ and $\mathrm{Ker}(\overline{\partial})^\perp$. It turns out that we have the equality

$$\mathcal{F}(\xi) = \int_Y \frac{u}{\mathrm{d}s_Y} \wedge \overline{\gamma_{\xi_1}} e^{-\varphi_L} \tag{2.0.3}$$

which is not completely obvious, given that the current defining $\mathcal{F}$ is not in L^2.

We have $\frac{u}{\mathrm{d}s_Y} \in L^2(e^{-\varphi_L})$, hence it is enough to find an upper bound for

$$c_{n-1} \int_Y \gamma_{\xi_1} \wedge \overline{\gamma_{\xi_1}} e^{-\varphi_L}. \tag{2.0.4}$$

This is done by the next estimate, which is derived in [4] via the $\partial\overline{\partial}$—Bochner method due to Siu cf. [39]

$$c_{n-1} \int_Y \gamma_{\xi_1} \wedge \overline{\gamma_{\xi_1}} e^{-\varphi_L} \le C \int_X \log^2(|s_Y|^2)|\overline{\partial}^\star \xi_1|^2 e^{-\varphi_L} \mathrm{d}V_\omega. \tag{2.0.5}$$

In conclusion, we have

$$\left|\int_X \bar\partial\left(\frac{u}{s_Y}\right)\wedge\overline{\gamma_\xi}e^{-\varphi_L}\right|^2 \le C\int_X \log^2(|s_Y|^2)|\bar\partial^\star\xi|^2 e^{-\varphi_L}\mathrm{d}V_\omega \tag{2.0.6}$$

and the "estimable" extension will be obtained by using the solution of the equation $\bar\partial\left(\frac{u}{s_Y}\right) = \bar\partial v$. We define $U := s_Y v$ and then we have

$$U|_Y = u, \qquad \int_X \frac{|U|^2}{|s|^2\log^2|s|^2}e^{-\varphi_Y-\varphi_L} \le C\int_Y |\frac{u}{\mathrm{d}s_Y}|^2 e^{-\varphi_L}. \tag{2.0.7}$$

2.1.1 Difficulties in the Case of an snc Hypersurface Y

In our setting we have $Y = \bigcup Y_i$, and the difficulty steams from the fact that the functional

$$\int_Y \frac{u}{\mathrm{d}s_Y}\wedge\overline{\gamma_{\xi_1}}e^{-\varphi_L} = \sum_i\int_{Y_i}\frac{u}{\prod_{j\neq i}s_j\mathrm{d}s_{Y_i}}\wedge\overline{\gamma_{\xi_1}}e^{-\varphi_L} \tag{2.0.8}$$

becomes a sum of expressions involving forms with log poles. We have

$$\frac{u}{\prod_{j\neq i}s_j\mathrm{d}s_{Y_i}} \notin L^2(e^{-\varphi_L}|_{Y_i}) \tag{2.0.9}$$

in general, so the previous arguments are breaking down.

Nevertheless we do have

$$\left|\frac{u}{\mathrm{d}s_Y}\right|_\omega^{\frac{2}{1+\alpha}} \in L^1(Y,\omega|_Y) \tag{2.0.10}$$

near the singularities of Y for any reasonable metric ω. This means that we have to find an estimate of the L^∞ norm of $\gamma_{\xi_1}|_{V_{\mathrm{sing}}\cap Y}$ in terms of the RHS of (2.0.5).

To this end, we use a procedure due to Donaldson–Sun in [15]. This consists in the following simple observation. Assume that the support of ξ is contained in $X\setminus V_{\mathrm{sing}}$. Then we have

$$\bar\partial\xi_1 = 0, \qquad \bar\partial^\star\xi_1|_{V_{\mathrm{sing}}} = 0 \tag{2.0.11}$$

in other words, the restriction of ξ_1 to V_{sing} is harmonic. As we learn from function theory, harmonic functions satisfy the mean value inequality: this is what we implement in our context, and it leads to the proof of Theorem 1.1.

The drawback of this method is that in the end, the constant measuring the L^2 norm of the extension is far from being as universal as in the case $Y_{\mathrm{sing}} = \emptyset$. This is due to the fact that the quantity $\Delta''|\xi_1|^2$ has a term *with wrong sign* involving the trace of the curvature of (L, h_L) with respect to the metric ω on X. This trace is not bounded e.g., if h_L is singular and ω is a fixed, non-singular Kähler metric. It is for this reason that the singularities of h_L and those of $\omega_{\mathcal{C}}$ must be the same in Theorem 1.1.

3 A-Priori Inequalities Revisited

We first recall the following estimate, which is essentially due to [4].

Theorem 3.1 *Let (X, ω) be a Kähler manifold, and Y be simple normal crossing divisor in X. Let L be a line bundle on X with a non-singular metric h_L such that*

$$\Theta_{h_L}(L) \geq 0, \qquad \Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(Y)$$

for some $\delta > 0$ small enough, where h_Y is a smooth metric on $\mathcal{O}_X(Y)$ such that $|s_Y|^2_{h_Y} \leq e^{-\delta}$. Let ξ be a smooth $(n, 1)$ form with compact support and with values in L. We denote by $\gamma_\xi := \star\xi$ the image of ξ by the Hodge operator. Then we have

$$\begin{aligned} c_{n-1} \int_X \frac{\tau^2}{(\tau^2 + |s_Y|^2)^2} \gamma_\xi \wedge \overline{\gamma_\xi} e^{-\varphi_L} \wedge \sqrt{-1} \partial s_Y \wedge \overline{\partial s_Y} \\ \leq C \int_X \log^2(\tau^2 + |s_Y|^2) \left(|\bar{\partial}^\star \xi|^2 + |\bar{\partial} \xi|^2 \right) e^{-\varphi_L} \mathrm{d}V_\omega \end{aligned}$$

where C is a numerical constant and τ is an arbitrary real number.

Before giving the proof of Theorem 3.1 we notice that it implies the following statement:

Theorem 3.2 *Let (X, ω) be a Kähler manifold, and Y be simple normal crossing divisor in X. Let L be a line bundle on X with a non-singular metric h_L such that*

$$\Theta_{h_L}(L) \geq 0, \qquad \Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(Y)$$

for any $\delta > 0$ small enough, where h_Y is a smooth metric on $\mathcal{O}_X(Y)$. Let ξ be a smooth $(n, 1)$ form with compact support and with values in L. We denote by $\gamma_\xi := \star\xi$ the image of ξ by the Hodge operator. Then we have

$$c_{n-1} \int_Y \gamma_\xi \wedge \overline{\gamma_\xi} e^{-\varphi_L} \leq C \int_X \log^2(|s_Y|^2_{h_Y}) \left(|\bar{\partial}^\star \xi|^2 + |\bar{\partial} \xi|^2 \right) e^{-\varphi_L} \mathrm{d}V_\omega, \quad (3.2.1)$$

where s_Y is the canonical section of $\mathcal{O}(Y)$, normalized in a way that works for the proof.

Proof of Theorem 3.1 We note that this improves slightly the estimate of Bo Berndtsson in [4], but the proof is virtually the same. Nevertheless, we will provide a complete treatment for the convenience of the reader.

To start with, we recall the following "$\partial\bar{\partial}$-Bochner formula."

Lemma 3.3 ([38]) *Let ξ be a $(n,1)$-form with values in (L, h_L) and compact support. We denote by $\gamma_\xi = \star\xi$ the Hodge $\star$ of ξ with respect to a Kähler metric ω. Let*

$$T_\xi := c_{n-1}\gamma_\xi \wedge \overline{\gamma}_\xi e^{-\varphi_L} \tag{3.3.1}$$

be the $(n-1, n-1)$-form on X corresponding to ξ, where $c_{n-1} = \sqrt{-1}^{(n-1)^2}$ is the usual constant. Then we have the equality

$$\begin{aligned}\sqrt{-1}\partial\bar\partial T_\xi = &\left(-2\Re\langle\bar\partial\bar\partial_\varphi^\star\xi, \xi\rangle + \|\bar\partial\gamma_\xi\|^2 + \|\bar\partial_\varphi^\star\xi\|^2 - \|\bar\partial\xi\|^2\right)\mathrm{d}V_\omega\\ &+ \Theta_{h_L}(L)\wedge T_\xi.\end{aligned} \tag{3.3.2}$$

We apply this in the following context. Consider the function $w := \log\frac{1}{|s_Y|^2+\tau^2}$. A quick computation gives

$$\sqrt{-1}\partial\bar\partial w = \frac{|s_Y|^2}{|s_Y|^2+\tau^2}\theta_Y - \frac{\tau^2}{(|s_Y|^2+\tau^2)^2}\sqrt{-1}\partial s_Y \wedge \overline{\partial s_Y} \tag{3.3.3}$$

where $\theta_Y := \Theta_{h_Y}(\mathcal{O}(Y))$ is the curvature of the bundle $\mathcal{O}(Y)$ with respect to the metric h_Y.

We multiply the equality (3.3.2) with w and integrate the resulting top form over X. The left-hand side term is equal to the difference of two terms

$$c_{n-1}\int_X \frac{|s_Y|^2}{|s_Y|^2+\tau^2}\theta_Y \wedge \gamma_\xi \wedge \overline{\gamma}_\xi e^{-\varphi_L} \tag{3.3.4}$$

and

$$c_{n-1}\int_X \frac{\tau^2}{(|s_Y|^2+\tau^2)^2}\gamma_\xi \wedge \overline{\gamma}_\xi \wedge \sqrt{-1}\partial s_Y \wedge \overline{\partial s_Y}. \tag{3.3.5}$$

and we see that (3.3.5) is the term we have to estimate.

We drop the positive terms on the right-hand side of (3.3.2) and we therefore get

$$\begin{aligned}c_{n-1}\int_X \frac{\tau^2}{(|s_Y|^2+\tau^2)^2}\gamma_\xi \wedge \overline{\gamma}_\xi \wedge \sqrt{-1}\partial s_Y \wedge \overline{\partial s_Y} \le \int_X w|\bar\partial\xi|^2 e^{-\varphi_L}\mathrm{d}V_\omega\\ + 2\Re\int_X w\langle\partial_\varphi^\star\partial_\varphi\gamma_\xi, \gamma_\xi\rangle e^{-\varphi_L}\mathrm{d}V_\omega\\ - c_{n-1}\int_X\left(w\Theta_{h_L}(L) - \frac{|s_Y|^2}{|s_Y|^2+\tau^2}\theta_Y\right)\gamma_\xi \wedge \overline{\gamma}_\xi e^{-\varphi_L}\end{aligned} \tag{3.3.6}$$

A first observation is that the curvature term (3.3.6) is negative, by the hypothesis of Theorem 3.1. Moreover, by Stokes formula we have

$$\int_X w\langle \partial_\varphi^\star \partial_\varphi \gamma_\xi, \gamma_\xi\rangle e^{-\varphi} dV_\omega = \int_X w|\partial_\varphi \gamma_\xi|^2 e^{-\varphi} dV_\omega + \int_X \langle \partial_\varphi \gamma_\xi, \partial w \wedge \gamma_\xi\rangle e^{-\varphi} dV_\omega \tag{3.3.7}$$

so we see that modulo the second term on the RHS of (3.3.7), we are done.

In order to take care of it we use Cauchy–Schwarz inequality and we obtain

$$\left|\int_X \langle \partial_\varphi \gamma_\xi, \partial w \wedge \gamma_\xi\rangle e^{-\varphi_L} dV_\omega\right| \le \int_X w^2|\partial_\varphi \gamma_\xi|^2 e^{-\varphi_L} dV_\omega + c_{n-1}\int_X \gamma_\xi \wedge \overline{\gamma}_\xi \wedge \frac{\sqrt{-1}\partial w \wedge \bar\partial w}{w^2} e^{-\varphi_L}. \tag{3.3.8}$$

Thus the new term to bound is

$$c_{n-1}\int_X \gamma_\xi \wedge \overline{\gamma}_\xi \wedge \frac{\sqrt{-1}\partial w \wedge \bar\partial w}{w^2} e^{-\varphi_L} \tag{3.3.9}$$

and as observed in [4], the quantity (3.3.9) is less singular that the LHS of (3.3.6), which was our initial problem.

In order to obtain a bound for (3.3.9) we consider the function

$$w_1 := \log w.$$

We have

$$\sqrt{-1}\partial\bar\partial w_1 = \frac{\sqrt{-1}\partial\bar\partial w}{w} - \frac{\sqrt{-1}\partial w \wedge \bar\partial w}{w^2} \tag{3.3.10}$$

and we use the same procedure as before, but with w_1 instead of w. The analog of (3.3.4) and (3.3.5) read as

$$c_{n-1}\int_X \frac{\sqrt{-1}\partial\bar\partial w}{w} \wedge \gamma_\xi \wedge \overline{\gamma}_\xi e^{-\varphi_L} - c_{n-1}\int_X \gamma_\xi \wedge \overline{\gamma}_\xi \wedge \frac{\sqrt{-1}\partial w \wedge \bar\partial w}{w^2}, \tag{3.3.11}$$

and this is good, because the second term in (3.3.11) is the one we are now after. We skip some intermediate steps because they are absolutely the same as in the preceding consideration, except that w_1 appears instead of w. After integration by parts, the new "bad term," i.e., the analog of the RHS of (3.3.7) in our current setting is

$$\int_X \langle \partial_\varphi \gamma_\xi, \partial w_1 \wedge \gamma_\xi\rangle e^{-\varphi_L} dV_\omega \tag{3.3.12}$$

for which we use Cauchy–Schwarz inequality and the observation is that $\partial w_1 \wedge \overline{\partial w_1}$ coincides with $\frac{\sqrt{-1}\partial w \wedge \bar\partial w}{w^2}$.

As a result of this second part of the proof we infer that we have

$$\begin{aligned} &c_{n-1}\int_X \gamma_\xi \wedge \overline{\gamma}_\xi \wedge \frac{\sqrt{-1}\partial w \wedge \bar\partial w}{w^2} \\ &\leq C\int_X \log^2(|s_Y|^2+\tau^2)\left(|\bar\partial^\star\xi|^2+|\bar\partial\xi|^2\right)e^{-\varphi_L}\mathrm{d}V_\omega \end{aligned} \tag{3.3.13}$$

Then Theorem 3.1 follows, by combining (3.3.13) with (3.3.8). □

Remark 3.4 Actually we can use the second part of the proof of Theorem 3.2 in order to get the estimates

$$\begin{aligned} &c_{n-1}\int_X \gamma_\xi \wedge \overline{\gamma_\xi}e^{-\varphi_L} \wedge \frac{\partial\sigma \wedge \overline{\partial\sigma}e^{-\varphi_F}}{|\sigma|^2\log^2|\sigma|^2} \\ &\leq C\int_X \log^2(|\sigma|^2)\left(|\xi|^2+|\bar\partial^\star\xi|^2+|\bar\partial\xi|^2\right)e^{-\varphi_L}\mathrm{d}V_\omega \end{aligned} \tag{3.4.1}$$

where σ is a holomorphic section of a line bundle (F, h_F) endowed with a non-singular metric h_F. The constant "C" in (3.4.1) depends on the norm of the curvature of (F, h_F). Thus, we obtain an estimate of the norm of γ_ξ *in the tangential directions of* $\sigma = 0$ with respect to the Poincaré-type measure associated to σ. If the curvature of (L, h_L) is greater than the some (small) multiple of $\Theta_{h_F}(F)$, then we can remove the term $|\xi|^2$ in the formula (3.4.1).

4 Geometric Analysis Methods and Results

In this section, we follow the notations in Theorem 1.1. Moreover, we ask that the metric φ_L satisfies the requirements (a), (b), (c) and $(i.\alpha)$. In particular, we *don't* assume $(i.\beta)$ for φ_L.

Let ξ be a L-valued form of $(n, 1)$ type such that $\mathrm{Supp}(\xi) \subset X \setminus (V_{\mathrm{sing}} \cup |H|)$. We recall that here V_{sing} is an open subset of X containing the singularities of Y, and H is a hyperplane section containing the singularities of the metric h_L.

We consider the orthogonal decomposition

$$\xi = \xi_1 + \xi_2 \tag{4.0.2}$$

where $\xi_1 \in \mathrm{Ker}(\bar\partial)$ and $\xi_2 \in \mathrm{Ker}(\bar\partial)^\perp$ with respect to the fixed Kähler metric ω_C with conic singularities on X and the given metric h_L on L.

• The convention during the current section is that we denote by "C" any constant which depends in an explicit way of the quantities we will indicate.

4.1 Orthogonal Decomposition, I: Approximation

In the following sections we will use an approximation statement, for which the context is as follows:

We can write

$$X \setminus H = \bigcup \Omega_m \tag{4.0.3}$$

where each Ω_m is a Stein domain with smooth boundary. Let ω_m be a complete metric on Ω_m. Corresponding to each positive δ we introduce

$$\omega_{m,\delta} := \omega_{\mathcal{C}} + \delta\omega_m; \tag{4.0.4}$$

it is a complete metric on Ω_m such that $\omega_{m,\delta} > \omega_{\mathcal{C}}$ and $\lim_{\delta\to 0}\omega_{m,\delta} = \omega_{\mathcal{C}}$ for each m.

We remark that the L^2 norm of ξ with respect to $\omega_{m,\delta}$ and $h_L|_{\Omega_m}$ is finite, given the pointwise monotonicity of the norm of $(n, 1)$-forms. Then we can decompose the restriction of ξ to each Ω_m as follows:

$$\xi|_{\Omega_m} = \xi_1^{(m,\delta)} + \xi_2^{(m,\delta)}. \tag{4.0.5}$$

We establish next the following statement.

Lemma 4.1 *We have*

$$\xi_1 = \lim_{m,\delta} \xi_1^{(m,\delta)} \tag{4.1.1}$$

uniformly on compact sets of $X \setminus H$.

The proof is based on the monotonicity of the L^2 norms

$$|\rho|^2_{\omega_{m,\delta}} dV_{\omega_{m,\delta}} < |\rho|^2_{\omega} dV_{\omega} \tag{4.1.2}$$

for each m, δ and for any form ρ of type $(n, 1)$ with values in L. The details are as follows:

Proof Let $K \subset X \setminus (\varphi_L = -\infty)$ be a compact subset. In what follows we are using the notation "ε" to indicate the set of parameters (m, δ), and we assume that $m \gg 0$ so that $K \subset \Omega_m$.

We first notice that for each parameter ε the form $\xi_1^{(\varepsilon)}$ is smooth, and that it verifies the equation

$$\Delta''_\varepsilon(\xi_1^{(\varepsilon)}) = \bar\partial\bar\partial^\star(\xi), \tag{4.1.3}$$

where Δ''_ε is the Laplace operator on $(n, 1)$-forms with values in (L, h_L) and $(\Omega_m, \omega_{m,\delta})$. We also have

$$\int_{\Omega_m} \left|\xi_1^{(\varepsilon)}\right|^2_{\omega_\varepsilon} e^{-\varphi_L} dV_{\omega_\varepsilon} \leq \int_X |\xi|^2_{\omega_{\mathcal{C}}} e^{-\varphi_L} dV_{\omega_{\mathcal{C}}} = \|\xi\|^2 \tag{4.1.4}$$

given the fact that (4.1.1) is orthogonal. It follows that the family

$$\xi_1^{(\varepsilon)}|_K \tag{4.1.5}$$

is uniformly bounded in $\mathcal{C}^\infty$ norm. We can therefore extract a limit $\xi_1^{(0)}$ as $\varepsilon \to 0$, uniform on compact subsets by the usual diagonal process. We remark that we have

$$\bar\partial \xi_1^{(0)} = 0, \qquad \int_X \left|\xi_1^{(0)}\right|^2_{\omega} e^{-\varphi_L} dV_{\omega} < \infty \tag{4.1.6}$$

given that each form $\xi_1^{(\varepsilon)}$ is $\bar\partial$-closed, combined with (4.1.4).

On the other hand, let ρ be a $\bar\partial$-closed form of $(n, 1)$-type with values in L. We assume moreover that ρ is L^2 with respect to $\omega_{\mathcal{C}}$ and h_L. Then we equally have

$$\int_{\Omega_m} |\rho|^2_{\omega_\varepsilon} e^{-\varphi_L} dV_{\omega_\varepsilon} < \int_{\Omega_m} |\rho|^2_{\omega_{\mathcal{C}}} e^{-\varphi_L} dV_{\omega_{\mathcal{C}}} < \infty \tag{4.1.7}$$

for each $\varepsilon = (m, \delta)$, and open subset Ω_m. We infer that

$$\int_{\Omega_m} \langle \xi_2^{(\varepsilon)}, \rho\rangle_{\omega_\varepsilon} e^{-\varphi_L} dV_{\omega_\varepsilon} = 0 \tag{4.1.8}$$

for each value of m and ε.

Let (K_l) be an increasing exhaustion of $X \setminus (\varphi_L = -\infty)$ by relatively compact sets. If $m \gg 0$ (depending on l) then we have

$$\left|\int_{\Omega_m \setminus K_l} \langle \xi_2^{(\varepsilon)}, \rho\rangle_{\omega} e^{-\varphi_L} dV_{\omega}\right|^2 \leq C(\xi) \int_{X \setminus K_l} |\rho|^2_{\omega} e^{-\varphi_L} dV_{\omega} \tag{4.1.9}$$

by Cauchy inequality combined with (4.1.7). It follows that

$$\left|\int_{K_l} \langle \xi_2^{(\varepsilon)}, \rho\rangle_{\omega_\varepsilon} e^{-\varphi_L} dV_{\omega_\varepsilon}\right|^2 \leq C(\xi) \int_{X \setminus K_l} |\rho|^2_{\omega_{\mathcal{C}}} e^{-\varphi_L} dV_{\omega_{\mathcal{C}}}. \tag{4.1.10}$$

By letting $\varepsilon \to 0$ we infer that for each fixed l we have

$$\left|\int_{K_l} \langle \xi_2^{(0)}, \rho\rangle_{\omega_{\mathcal{C}}} e^{-\varphi_L} dV_{\omega_{\mathcal{C}}}\right|^2 \leq C(\xi) \int_{X \setminus K_l} |\rho|^2_{\omega_{\mathcal{C}}} e^{-\varphi_L} dV_{\omega_{\mathcal{C}}}. \tag{4.1.11}$$

Next, the inequality (4.1.6) shows that $\xi_2^{(0)}$ is L^2-integrable with respect to (L, h_L) and $(X, \omega_{\mathcal{C}})$. It follows that we have

$$\int_{X\setminus K_l} \langle \xi_2^{(0)}, \rho\rangle_{\omega_{\mathcal{C}}} e^{-\varphi_L} dV_{\omega_{\mathcal{C}}} \to 0 \tag{4.1.12}$$

as $l \to \infty$ since both ρ and $\xi_2^{(0)}$ are in L^2.

In other words, the form $\xi_2^{(0)}$ is orthogonal to Ker $\bar\partial$ and since we have

$$\xi = \xi_1^{(0)} + \xi_2^{(0)} \tag{4.1.13}$$

our lemma is proved (thanks to the uniqueness of such decomposition). □

4.2 Orthogonal Decomposition, II: Mean Value Inequality

We analyze here the behavior of ξ_1 restricted to the set V_{sing}. During the current subsection we make the following conventions.

(i) We work with respect to the Kähler metric $\omega_{\mathcal{C}}$ exclusively on $V_{\mathrm{sing}} \subset X$ (this will be understood even if we do not mention it explicitly) and with respect to the Hermitian metric h_L defined in the previous section on L.

(ii) We denote by ξ a $(n, 1)$ form with values in L such that we have

$$\mathrm{Supp}(\xi) \subset X \setminus (V_{\mathrm{sing}} \cup |H|).$$

We use the notations in (4.0.2) for its orthogonal decomposition with respect to $(\omega_{\mathcal{C}}, h_L)$.

(iii) We fix $\frac{1}{2}V_{\mathrm{sing}} \Subset V_{\mathrm{sing}}$ an open set of compact support in V_{sing}, and $\frac{1}{2}V_{\mathrm{sing}}$ contains the singular locus of Y.

In this subsection we establish the next result.

Theorem 4.2 *We have the mean value type inequality*

$$\sup_{\frac{1}{2}V_{\mathrm{sing}}} |\xi_1|^2 e^{-\varphi_L} \le C(V_{\mathrm{sing}}) \int_{V_{\mathrm{sing}}} |\xi_1|^2 e^{-\varphi_L} dV_{\omega_{\mathcal{C}}} \tag{4.2.1}$$

where $C(V_{\mathrm{sing}})$ here is a constant which only depends on the allowed quantities, i.e., the geometry of $(V_{\mathrm{sing}}, \omega_{\mathcal{C}})$ as well as τ_L in the assumption (i.α).

The norm of ξ_1 in (4.2.1) is measured with the conic metric $\omega_{\mathcal{C}}$.

The proof of Theorem 4.2 unfolds as follows (cf. [15, 22] for similar computations). In order to simplify the notations, we drop the $e^{-\varphi_L}$ in (4.2.1), and write $|\xi_1|^2$ to express the pointwise norm of ξ_1 with respect to $\omega_{\mathcal{C}}$ and h_L. First we show that there exists a constant C such that

$$\sup_{V_{\mathrm{sing}}\setminus|W|} |\xi_1|^2 \le C < \infty \tag{4.2.2}$$

where we denote by $|W|$ the support of the divisor W. This is the main reason why we have to assume that the singularities of h_L and ω are "the same" in Theorem 1.1.

After this, we establish a differential inequality satisfied by the function $|\xi_1|^2$ *in the complement of the set*

$$(\varphi_L = -\infty) \cap V_{\mathrm{sing}}.$$

This is standard, and it combines nicely with (4.2.2) and Moser iteration process to give (4.2.1).

Proof of (4.2.2) First we establish the crucial boundedness result (4.2.2). Let $z_1, \dots, z_n$ be a set of local coordinates defined on a open subset $\Omega \subset V_{\mathrm{sing}}$. We assume that the $(z_i)_{i=1\dots n}$ are adapted to the pair (X, W), meaning that the local equation of $\Omega \cap W$ is

$$z_1 \dots z_p = 0$$

for some $p \leq n$. By hypothesis, the weight of the metric h_L can be written as

$$\varphi_L = \sum_{i=1}^{p} \left(1 - \frac{1}{k_i}\right) \log |z_i|^2 + \tau, \tag{4.2.3}$$

where k_i are positive integers and τ is a smooth function defined on Ω.

The restriction of ξ_1 to Ω has the following properties:

$$\int_\Omega |\xi_1|^2_{\omega_C} e^{-\varphi_L} \mathrm{d}V_{\omega_C} < \infty, \qquad \bar\partial \xi_1 = 0, \qquad \int_\Omega \langle \xi_1, \bar\partial \phi \rangle_{\omega_C} e^{-\varphi_L} \mathrm{d}V_{\omega_C} = 0 \tag{4.2.4}$$

where ϕ is any smooth $(n, 0)$ form with compact support in Ω which is L^2-integrable, and such that $\bar\partial\phi$ is in L^2 as well.

These properties have a very neat interpretation in terms of ramified covers, as follows. Let

$$\pi : \widehat{\Omega} \to \Omega, \qquad \pi(w_1, \dots, w_n) = (w_1^{k_1}, \dots, w_p^{k_p}, w_{p+1}, \dots, w_n) \tag{4.2.5}$$

be the usual local covering map corresponding to the divisor $\sum_{i=1}^p \left(1 - \frac{1}{k_i}\right) W_i$. We define the $(n, 1)$-form η on $\widehat{\Omega}$ as follows:

$$\eta := \frac{1}{\prod_{j=1}^p w_j^{k_j - 1}} \pi^\star \xi_1, \tag{4.2.6}$$

and a first remark is that we have

$$\int_{\widehat{\Omega}} |\eta|^2_g e^{-\tau\circ\pi} \mathrm{d}V_g < \infty, \tag{4.2.7}$$

where g is the inverse image of the conic metric $g := \pi^\star\omega_{\mathcal{C}}$. The relation (4.2.7) is an immediate consequence of the change of variables formula, combined with the expression of φ_L in (4.2.3). It follows that

$$\bar{\partial}\eta = 0 \tag{4.2.8}$$

on $\widehat{\Omega}$ (this is true pointwise outside the support of W, and it extends across W by using [12, Chap VIII, Lemma 7.3]).

Let α be a smooth $(n, 0)$-form on $\widehat{\Omega}$ with compact support. We claim that we have

$$\int_{\widehat{\Omega}} \langle \eta, \bar{\partial}\alpha\rangle_g e^{-\tau\circ\pi} dV_g = 0. \tag{4.2.9}$$

Indeed this is clear, given the equality

$$\int_{\widehat{\Omega}} \langle \eta, \bar{\partial}\alpha\rangle_g e^{-\tau\circ\pi} dV_g = \int_{\widehat{\Omega}} \langle \pi^\star\xi_i, \bar{\partial}\widehat{\alpha}\rangle_g e^{-\varphi_L\circ\pi} dV_g \tag{4.2.10}$$

where $\widehat{\alpha} := \prod_{j=1}^p w_j^{k_j-1}\alpha$. On the right-hand side of (4.2.10) we can assume that $\widehat{\alpha}$ is the inverse image of a $(n, 0)$ form with compact support on Ω. Indeed, let f be an element of the group G acting on $\widehat{\Omega}$. Then we have

$$\int_{\widehat{\Omega}} \langle \pi^\star\xi_i, \bar{\partial}\widehat{\alpha}\rangle_g e^{-\varphi_L\circ\pi} dV_g = \int_{\widehat{\Omega}} \langle \pi^\star\xi_i, \bar{\partial}(f^\star\widehat{\alpha})\rangle_g e^{-\varphi_L\circ\pi} dV_g \tag{4.2.11}$$

since all the other objects involved in the integral in question are invariant by inverse image. It follows that

$$\int_{\widehat{\Omega}} \langle \pi^\star\xi_i, \bar{\partial}\widehat{\alpha}\rangle_g e^{-\varphi_L\circ\pi} dV_g = \int_{\widehat{\Omega}} \langle \pi^\star\xi_i, \bar{\partial}(\pi^\star\phi)\rangle_g e^{-\varphi_L\circ\pi} dV_g \tag{4.2.12}$$

where $\pi^\star\phi := \frac{1}{|G|}\sum_{f\in G} f^\star\widehat{\alpha}$. Then our claim follows by the third property in (4.2.4).

In conclusion, the form η is harmonic on $\widehat{\Omega}$ with respect to the metric g and the weight $e^{-\tau\circ\pi}$. It is in particular bounded, and this is equivalent to (4.2.2). □

We choose geodesic local coordinates $(z_i)_{i=1,\ldots,n}$ for the Kähler metric $\omega_{\mathcal{C}}$ locally near a point $x_0 \in V_{\mathrm{sing}}\backslash W$. Let e_L be a local holomorphic frame of L, such that the induced weight ϕ of h_L verifies the relations

$$\phi(x_0) = 0, \qquad d\phi(x_0) = 0. \tag{4.2.13}$$

We express ξ_1 locally with respect to these coordinates

$$\xi_1 = \sum \xi_{\overline{\alpha}} dz \wedge dz^{\overline{\alpha}} \otimes e_L \tag{4.2.14}$$

where $dz := dz^1 \wedge \cdots \wedge dz^n$. We then have

$$|\xi_1|^2_{\omega_{\mathcal{C}}} = \sum_{\alpha,\beta} \xi_{\overline{\alpha}} \overline{\xi_{\overline{\beta}}} g^{\overline{\alpha}\beta} \frac{e^{-\phi}}{\det g}. \tag{4.2.15}$$

The formula for the Laplace operator is $\Delta'' = \mathrm{Tr}_{\,\omega_{\mathcal{C}}} \sqrt{-1}\partial\overline{\partial}$ and so we have

$$\Delta''(|\xi_1|^2) = |\nabla \xi_1|^2 + 2\sum_{\alpha,\beta} \Re\left(\xi_{\overline{\alpha},p\overline{q}} g^{\overline{q}p} \overline{\xi_{\overline{\beta}}} g^{\overline{\alpha}\beta}\right) \frac{e^{-\phi}}{\det g} \tag{4.2.16}$$

$$+ \sum_{\alpha,\beta} \xi_{\overline{\alpha}} \overline{\xi_{\overline{\beta}}} g^{\overline{\alpha}\beta}_{,p\overline{q}} g^{\overline{q}p} \frac{e^{-\phi}}{\det g} \tag{4.2.17}$$

$$- \sum_{\alpha,\beta} \xi_{\overline{\alpha}} \overline{\xi_{\overline{\beta}}} (\phi + \log\det g)_{,p\overline{q}} g^{\overline{q}p} g^{\overline{\alpha}\beta} \frac{e^{-\phi}}{\det g} \tag{4.2.18}$$

where we denote by $\left(g_{p\overline{q}}\right)$ the coefficients of the metric $\omega_{\mathcal{C}}$ with respect to the geodesic coordinates (z_i) and by g the corresponding matrix.

In order to obtain an intrinsic expression of the terms containing the second derivative in the RHS of (4.2.16), we recall that we have

$$\overline{\partial}^\star_{\varphi_L} \xi_1 = (-1)^n \left(-\frac{\partial \xi_{\overline{\alpha}}}{\partial z_\beta} g^{\overline{\alpha}\beta} - \frac{\partial g^{\overline{\alpha}\beta}}{\partial z_\beta} \xi_{\overline{\alpha}} + \xi_{\overline{\alpha}} g^{\overline{\alpha}\beta} \frac{\partial \varphi_L}{\partial z_\beta}\right) dz \otimes e_L \tag{4.2.19}$$

hence the next equality holds at x_0

$$\langle \Box \xi_1, \xi_1 \rangle = \left(-\xi_{\overline{\alpha},p\overline{q}} g^{\overline{q}\beta} \overline{\xi_{\overline{\beta}}} g^{\overline{\alpha}p} - \xi_{\overline{\alpha}} \overline{\xi_{\overline{\beta}}} g^{\overline{\alpha}\delta}_{,\delta\overline{\gamma}} g^{\overline{\gamma}\beta} + \xi_{\overline{\alpha}} \overline{\xi_{\overline{\beta}}} \varphi_{L,\delta\overline{\gamma}} g^{\overline{\alpha}\delta} g^{\overline{\gamma}\beta}\right) \frac{e^{-\phi}}{\det g} \tag{4.2.20}$$

where $\Box := [\overline{\partial}, \overline{\partial}^\star]$ is the Laplace operator acting on L-valued forms of $(n, 1)$ type. The formula (4.2.20) is only valid for closed forms, which is the case for ξ_1. Also, we have $\xi_{\overline{\alpha},p\overline{q}} = \xi_{\overline{q},p\overline{\alpha}}$ and therefore (twice the real part of) the first term on the RHS of (4.2.20) coincides with the second one on the RHS of (4.2.16).

Next, since the metric $\omega_{\mathcal{C}}$ is Kähler we have $g^{\overline{\alpha}\delta}_{,\delta\overline{\gamma}} = -g_{\alpha\overline{\gamma},\delta\overline{\delta}}$ hence we obtain

$$\xi_{\overline{\alpha}} \overline{\xi_{\overline{\beta}}} g^{\overline{\alpha}\delta}_{,\delta\overline{\gamma}} g^{\overline{\gamma}\beta} = \mathcal{R}_{\alpha\overline{\beta}} \xi_{\overline{\alpha}} \overline{\xi_{\overline{\beta}}} \tag{4.2.21}$$

where $\mathcal{R}_{\alpha\overline{\beta}}$ are the coefficients of the Ricci tensor of $\omega_{\mathcal{C}}$.

The last term in (4.2.20) is simply $\theta_{\alpha\overline{\beta}} \xi_{\overline{\alpha}} \overline{\xi_{\overline{\beta}}}$ where $\theta_{\alpha\overline{\beta}}$ are the coefficients of $\Theta_{h_L}(L)$.

Again by the Kähler hypothesis the term (4.2.17) is equal to

$$\mathcal{R}_{\alpha\overline{\beta}} \xi_{\overline{\alpha}} \overline{\xi_{\overline{\beta}}} \tag{4.2.22}$$

and therefore we obtain

$$\Delta''(|\xi_1|^2) = |\nabla\xi_1|^2 - 2\Re\langle\Box\xi_1, \xi_1\rangle \tag{4.2.23}$$

$$+ 2\sum_{\alpha,\beta}\left(\theta_{\alpha\overline{\beta}} - \mathcal{R}_{\alpha\overline{\beta}}\right)\xi_{\overline{\alpha}}\overline{\xi_{\overline{\beta}}} + \sum_{\alpha,\beta}\mathcal{R}_{\alpha\overline{\beta}}\xi_{\overline{\alpha}}\overline{\xi_{\overline{\beta}}} \tag{4.2.24}$$

$$- \sum_{\alpha,\beta}\left(\theta_{\alpha\overline{\beta}} - \mathcal{R}_{\alpha\overline{\beta}}\right)g^{\overline{\beta}\alpha}|\xi_1|^2 \tag{4.2.25}$$

by collecting the previous equalities at x_0.

The Ricci curvature of the metric $\omega_{\mathcal{C}}$ is uniformly bounded, so the function

$$f^2 := |\xi_1|^2 \tag{4.2.26}$$

(where the norm is measured with respect to $\omega_{\mathcal{C}}$ and h_L) verifies the following properties:

(1) We have $\sup_{V_{\text{sing}}\setminus W} f < \infty$, and moreover f is smooth on $V_{\text{sing}} \setminus W$.
(2) The following differential inequality holds true

$$\Delta'' f^2 \geq |\nabla f|^2 - Cf^2 \tag{4.2.27}$$

where C is a constant depending on the Ricci curvature of the metric $\omega_{\mathcal{C}}$ and the trace of $dd^c\tau$ with respect to it.

Indeed the inequality at the point (2) follows from (4.2.23), since we have

$$|\nabla\xi_1|^2 = \big|\nabla|\xi_1|\big|^2.$$

Based on (1) and (2) we can conclude in two ways: either show that Schoen–Yau mean value inequality holds for functions f which verify these properties (the proof would be a simple adaptation of the arguments presented in [37]), or use the Moser iteration procedure. In what follows, we use Moser procedure.

We show next that the following statement holds true.

Lemma 4.3 *Let f be the function defined in* (4.2.26). *Then there exists a constant C_1 depending on C and $(V_{\text{sing}}, \omega_{\mathcal{C}})$ only such that we have*

$$\sup_{\frac{1}{2}V_{\text{sing}}\setminus W} f^2 \leq C_1 \int_{V_{\text{sing}}} f^2 \mathrm{d}V_{\omega_{\mathcal{C}}}. \tag{4.3.1}$$

Remark that the main point here is that the constant C_1 is *independent of the* sup *norm in (1)*. After establishing this statement we are basically done, i.e., this implies Theorem 4.2 announced at the beginning of the current section.

Proof of Lemma 4.3 Let ρ be a function which is equal to 1 on $1/2V_{\text{sing}}$ and whose support is in V_{sing}. Then we have

$$|\Delta\rho| \leq C, \qquad |\mathrm{d}\rho| \leq C \tag{4.3.2}$$

where the norm of the differential in (4.3.2) is measured with respect to the reference metric ω on X.

Following [3], there exists a family of functions $(\Xi_\varepsilon)_{\varepsilon>0}$ associated to the analytic subset $W = (h_L = \infty) \cap V_{\text{sing}}$ such that $\mathrm{Supp}(\Xi_\varepsilon) \subset V_{\text{sing}} \backslash W$ and for each compact subset $K \subset X \setminus W$ we have $\Xi_\varepsilon|_K = 1$ if $\varepsilon < \varepsilon(K)$ is small enough. Moreover we have

$$\int_X |d(\Xi_\varepsilon)|^2 \mathrm{d}V_\omega \to 0, \qquad \int_X |\Delta(\Xi_\varepsilon)| \mathrm{d}V_\omega \to 0 \tag{4.3.3}$$

as $\varepsilon \to 0$. We recall very briefly the construction: let ρ_ε be a function which is equal to one on the interval $[0, \varepsilon^{-1}]$ and which equals zero on $[1+\varepsilon^{-1}, \infty[$. Then we define

$$\Xi_\varepsilon := \rho_\varepsilon \left(\log \left(\log \frac{1}{|s_W|^2} \right) \right) \tag{4.3.4}$$

where s_W is the sections whose zero set is W. Then with respect to the conic metric $\omega_{\mathcal{C}}$ we have

$$|d(\Xi_\varepsilon)|^2_{\omega_{\mathcal{C}}} \leq \frac{\rho'_\varepsilon}{\log^2 |s_W|^2} \sum \frac{1}{|z_j|^{2/k_j}}, \tag{4.3.5}$$

up to a constant, from which (4.3.3) follows (we get a similar inequality for the Laplacian of Ξ_ε).

The existence of $(\Xi_\varepsilon)_{\varepsilon>0}$ combined with the second property in (4.3.2) allows us to deal with the fact that f is not necessarily smooth.

The proof which follows is rather standard, but we will nevertheless provide a complete treatment for convenience. We refer to [16] for a general discussion about Moser iteration method, and more specifically to [22] where this is implemented in a context which is very similar to ours here.

We multiply the differential inequality (4.2.27) with $\Xi_\varepsilon \rho^2$ and integrate the result over X; we infer that we have

$$\int_X \Xi_\varepsilon \rho^2 \Delta(f^2) \mathrm{d}V_{\omega_{\mathcal{C}}} \geq \int_X \Xi_\varepsilon \rho^2 |d(f)|^2 \mathrm{d}V_{\omega_{\mathcal{C}}}. \tag{4.3.6}$$

On the LHS we integrate by parts. The terms containing derivatives of Ξ_ε are

$$\int_X |\Delta(\Xi_\varepsilon)| \rho^2 f^2 \mathrm{d}V_{\omega_{\mathcal{C}}}, \int_X \langle \mathrm{d}\Xi_\varepsilon, d(\rho^2) \rangle f^2 \mathrm{d}V_{\omega_{\mathcal{C}}} \tag{4.3.7}$$

and they tend to zero *precisely* because of the uniform boundedness property (1) of f, together with (4.3.3). These terms are vanishing as $\varepsilon \to 0$, and the inequality (4.3.6) becomes

$$\int_X f^2 \Delta(\rho^2) \mathrm{d}V_{\omega_{\mathcal{C}}} \geq \int_X \rho^2 |d(f)|^2 \mathrm{d}V_{\omega_{\mathcal{C}}}. \tag{4.3.8}$$

On the other hand we write

$$\int_X \rho^2 |df|^2 \mathrm{d}V_{\omega_{\mathcal{C}}} \geq \frac{1}{2} \int_X \left|d(\rho f)\right|^2 \mathrm{d}V_{\omega_{\mathcal{C}}} - \int_X f^2 |d\rho|^2 \mathrm{d}V_{\varepsilon}, \tag{4.3.9}$$

which combined with (4.3.8) gives

$$\int_X \left|\nabla(\rho f)\right|^2 \mathrm{d}V_{\omega_{\mathcal{C}}} \leq C \int_{V_{\mathrm{sing}}} f^2 \mathrm{d}V_{\omega_{\mathcal{C}}}, \tag{4.3.10}$$

where the constant C in (4.3.10) only depends on the norm of the first and second derivatives of ρ.

The following version of Sobolev inequality is a direct consequence of [21], page 153.

Theorem 4.4 *There exists a constant $C > 0$ such that the following holds*

$$\frac{1}{C}\left(\int_X |f|^{\frac{2n}{n-1}} \mathrm{d}V_{\omega_{\mathcal{C}}}\right)^{\frac{n-1}{n}} \leq \int_X |f|^2 \mathrm{d}V_{\omega_{\mathcal{C}}} + \int_X |\nabla f|^2 \mathrm{d}V_{\omega_{\mathcal{C}}} \tag{4.4.1}$$

for any function f on X.

We therefore infer that we have

$$\left(\int_{1/2 V_{\mathrm{sing}}} |f|^{\frac{2n}{n-1}} \mathrm{d}V_{\omega_{\mathcal{C}}}\right)^{\frac{n-1}{n}} \leq C \int_{V_{\mathrm{sing}}} |f|^2 \mathrm{d}V_{\omega_{\mathcal{C}}}. \tag{4.4.2}$$

In order to obtain estimates for higher norms, we use (4.4.1) for $f := \Xi_\varepsilon \rho f^{\frac{p}{2}}$ and we obtain

$$\begin{aligned} \frac{1}{C}\left(\int_X (\Xi_\varepsilon \rho)^{\frac{2n}{n-1}} f^{\frac{pn}{n-1}} \mathrm{d}V_{\omega_{\mathcal{C}}}\right)^{\frac{n-1}{n}} &\leq \int_X (\Xi_\varepsilon \rho)^2 f^p \mathrm{d}V_{\omega_{\mathcal{C}}} \\ &\quad + \int_X \left|\nabla\left(\Xi_\varepsilon \rho f^{\frac{p}{2}}\right)\right|^2 \mathrm{d}V_{\omega_{\mathcal{C}}}. \end{aligned} \tag{4.4.3}$$

We show now that the second term of the right-hand side of (4.4.3) verifies the inequality

$$\int_X \left|\nabla\left(\rho f^{\frac{p}{2}}\right)\right|^2 \mathrm{d}V_{\omega_{\mathcal{C}}} \leq Cp \int_X (\rho^2 + |\nabla \rho|^2) f^p \mathrm{d}V_{\omega_{\mathcal{C}}}. \tag{4.4.4}$$

This is done using integration by parts: we have

$$\nabla\left(\rho\,\Xi_\varepsilon f^{\frac{p}{2}}\right) = f^{\frac{p}{2}}\nabla(\rho\,\Xi_\varepsilon) + \frac{p}{2}\Xi_\varepsilon\rho f^{\frac{p-2}{2}}\nabla f,$$

so we have to obtain a bound for the term

$$\int_X (\rho\,\Xi_\varepsilon)^2 f^{p-2}|\nabla f|^2 \mathrm{d}V_{\omega_C} = \frac{1}{2}\int_X (\rho\,\Xi_\varepsilon)^2 f^{p-3}\langle\nabla f^2, \nabla f|\rangle \mathrm{d}V_{\omega_C}. \tag{4.4.5}$$

We have

$$\begin{aligned}
(p-2)&\int_X (\rho\,\Xi_\varepsilon)^2 f^{p-3}\langle\nabla f^2, \nabla f\rangle \mathrm{d}V_{\omega_C}\\
&= -\int_X (\rho\,\Xi_\varepsilon)^2 f^{p-2}\Delta f^2 \mathrm{d}V_{\omega_C} - 2\int_X f^{p-2}\langle(\rho\,\Xi_\varepsilon)\nabla f, f\nabla(\rho\,\Xi_\varepsilon)\rangle \mathrm{d}V_{\omega_C}\\
&\leq -\int_X (\rho\,\Xi_\varepsilon)^2 f^{p-2}|\nabla(f)|^2 \mathrm{d}V_{\omega_C} - 2\int_X f^{p-2}\langle(\rho\,\Xi_\varepsilon)\nabla f, f\nabla(\rho\,\Xi_\varepsilon)\rangle \mathrm{d}V_{\omega_C}\\
&\quad + C\int_X (\rho\,\Xi_\varepsilon)^2 f^p \mathrm{d}V_{\omega_C}\\
&\leq C\int_X ((\rho\,\Xi_\varepsilon)^2 + |\nabla(\rho\,\Xi_\varepsilon)|^2) f^p \mathrm{d}V_{\omega_C}
\end{aligned} \tag{4.4.6}$$

and as before, the terms involving $\nabla(\Xi_\varepsilon)$ tend to zero as $\varepsilon \to 0$. We therefore get the inequality (4.4.4). Remark that we are using the inequality (4.2.27) in order to obtain (4.4.6).

We define $V_i := (1/2 + 1/2^i)V_{\mathrm{sing}}$ and let ρ_i be a cutoff function such that $\rho_i = 1$ on V_{i+1} and such that $\mathrm{Supp}(\rho_i) \subset V_i$. Then we have $|\nabla\rho_i| \leq C2^i$, and by using (4.4.4) combined with the usual iteration process, Lemma 4.3 follows. □

Theorem 4.2 is proved. □

Remark 4.5 Actually a careful examination of the proof shows that one can obtain a constant "C" in Lemma 4.3 as follows:

$$C = \frac{C(X,\omega)}{\mathrm{Vol}\,(V_{\mathrm{sing}})}. \tag{4.5.1}$$

If necessary, this can be obtained by adapting the arguments of Schoen–Yau in [37, p. 75].

5 Proof of the Main Results

5.1 Proof of Theorem 1.1

We consider the "usual" family of cutoff functions

$$\rho_\varepsilon : X \to \mathbb{R}, \qquad \rho_\varepsilon(z) = \rho\left(\frac{|s_Y|^2}{\varepsilon^2}\right) \tag{5.0.2}$$

where ρ is a function defined on the set of positive real numbers such that $\rho = 1$ on $[0, 1]$ and $\rho = 0$ on $[2, \infty[$.

We will show here that the following a-priori inequality holds

$$\left|\int_X \bar\partial\,(\rho_\varepsilon U_0) \wedge \overline{\gamma_\xi} e^{-\phi_L}\right|^2 \leq C_\varepsilon(U_0) \int_X \prod \log^2(|s_{Y_j}|^2 + \varepsilon^2)|\bar\partial^\star \xi|^2 e^{-\phi_L} dV_\omega \tag{5.0.3}$$

where ξ is a $(n, 1)$-form with values in $Y + L$ whose support is contained in the complement of $V_{\text{sing}} \cup H$, and U_0 is an arbitrary holomorphic extension of u, cf. [7]. Also, we denote by ϕ_L the metric

$$\phi_L := \varphi_L + \log|f_Y|^2 \tag{5.0.4}$$

on the bundle $L + Y$. We will see that the constant $C_\varepsilon(U_0)$ in (5.0.3) is explicit, and it converges to the RHS of (1.6.1) as $\varepsilon \to 0$. Note that all the integrals above are at least well defined, given the condition we impose on the support of ξ.

The proof of (5.0.3) will be presented along the following line of arguments.

• Consider a $(n, 1)$-form ξ as above together with the orthogonal decomposition $\frac{\xi}{s_Y} = \xi_1 + \xi_2$ we have already discussed in detail in the previous section. Then we show that we have

$$\begin{aligned} c_n \int_X \frac{\varepsilon^2}{(\varepsilon^2 + |s_Y|^2)^2} \gamma_{\xi_1} \wedge \partial s_Y \wedge \overline{\gamma_{\xi_1} \wedge \partial s_Y} e^{-\varphi_L} \\ \leq \int_X \prod \log^2(|s_{Y_j}|^2 + \varepsilon^2)|\bar\partial^\star \xi|^2 e^{-\phi_L} dV_{\omega_L}. \end{aligned} \tag{5.0.5}$$

up to a numerical constant. This will be done by an approximation argument, using Lemma 4.1 as well as Theorem 3.1.

• The norm of the functional on the LHS of (5.0.3) is evaluated in two steps on the set V_{sing} we use Theorem 4.2, combined with a few simple calculations. In the complement $X \setminus V_{\text{sing}}$ the arguments are rather standard: we will use (5.0.5).

The remaining part of the current section is organized as follows. We first show that (5.0.3) implies the existence of an "estimable extension" of u. Then we prove that the estimate (5.0.3) holds true.

5.1.1 Functional Analysis

Our method relies on the next statement.

Theorem 5.1 *Let $u \in H^0(Y, (K_X + Y + L)|_Y)$ be a holomorphic section. We assume that there is a constant $C_\varepsilon(U)$ such that for any L-valued smooth test form ξ of type $(n, 1)$ with compact support in $X \setminus (V_{\text{sing}} \cup |W|)$ the a-priori inequality (5.0.3) holds. Then u admits an extension $U \in H^0(X, K_X + Y + L)$ such that*

$$\int_{X \setminus V_{\text{sing}}} \frac{|U|^2}{|s_Y|^2 \prod \log^2(|s_{Y_j}|^2)} e^{-\varphi_L - \varphi_Y} \mathrm{d}V_\omega \leq \lim_{\varepsilon \to 0} C_\varepsilon(U). \tag{5.1.1}$$

Proof This is done as in the classical case, by considering the vector subspace

$$E := \left\{ \bar\partial^\star \xi : \xi \in C^2_c\big(X \setminus (V_{\text{sing}} \cup H)\big) \right\} \tag{5.1.2}$$

of the $L^2_{n,0}(X, Y + L)$ forms, endowed with the scalar product induced by

$$\|\rho\|^2 := \int_X |\rho|^2 e^{-\phi_L} \prod \log^2(|s_{Y_j}|^2 + \varepsilon^2) \mathrm{d}V_\omega. \tag{5.1.3}$$

The functional

$$\bar\partial^\star \xi \to \int_X \bar\partial\left(\rho_\varepsilon U_0\right) \wedge \overline{\gamma_\xi} e^{-\phi_L} \tag{5.1.4}$$

is well defined and bounded on E by (5.0.3), hence it extends by Hahn–Banach. The representation theorem of Riesz implies that there exists some form

$$v \in L^2_{n,0}(X, Y + L)$$

such that we have

$$\int_X \left\langle \bar\partial\left(\rho_\varepsilon U_0\right), \xi \right\rangle e^{-\phi_L} = \int_X \left\langle v, \bar\partial^\star \xi \right\rangle e^{-\phi_L} \prod \log^2(|s_{Y_j}|^2 + \varepsilon^2) \mathrm{d}V_{\omega_L} \tag{5.1.5}$$

for all test forms $\xi \in E$ and such that

$$\int_X |v|^2 e^{-\phi_L} \prod \log^2(|s_{Y_j}|^2 + \varepsilon^2) \mathrm{d}V_{\omega_L} \leq C_\varepsilon(U_0). \tag{5.1.6}$$

Equation (5.1.5) shows that we have

$$\bar\partial\left(\rho_\varepsilon U_0\right) = \bar\partial\left(\prod \log^2(|s_{Y_j}|^2 + \varepsilon^2) v\right) \tag{5.1.7}$$

on $X \setminus V_{\rm sing}$. On the other hand, the form

$$\rho_\varepsilon U_0 - \left(\prod \log^2(|s_{Y_j}|^2 + \varepsilon^2)\right) v \tag{5.1.8}$$

is in $L^2(X \setminus V_{\rm sing})$: this is clear for the first term, as for the second one it is a consequence of (5.1.6).

We infer that the form

$$U_\varepsilon := \rho_\varepsilon U_0 - \left(\prod \log^2(|s_{Y_j}|^2 + \varepsilon)\right) v \tag{5.1.9}$$

extends holomorphically on $X \setminus V_{\rm sing}$. This implies that $v|_{X\setminus V_{\rm sing}}$ is non-singular, in particular v equal zero when restricted to $Y \setminus V_{\rm sing}$—given the estimates in (5.1.6).

Therefore we infer the equality

$$U_\varepsilon|_{Y\setminus V_{\rm sing}} = u. \tag{5.1.10}$$

We remark that U_ε extends to X by theorem of Hartog's. This is clear if X is a surface cf. e.g., [19, Theorem 2.3.2]. The general case follows as a consequence of this, by a simple argument of slicing which we will not detail here.

Finally, the estimate for the L^2 norm of U is deduced from (5.1.6): we have

$$\int_{X\setminus V_{\rm sing}} \frac{|U_\varepsilon|^2}{|s_Y|^2 \prod \log^2(|s_{Y_j}|^2 + \varepsilon)} e^{-\varphi_L - \varphi_Y} \leq C_\varepsilon(U_0) \tag{5.1.11}$$

modulo a quantity which tends to zero as $\varepsilon \to 0$. The conclusion follows. □

5.1.2 End of the Proof

We prove now the inequality (5.0.3). As we have already mentioned, one of the main part of the proof is based on the a-priori estimate (5.0.5) which we derive here from Theorem 3.2 combined with the results established in the first part of Sect. 3.

We start with the following technical result, which plays a key role in the arguments to come. In order to simplify the notations, we write ξ instead of the quotient $\dfrac{1}{s_Y}\xi$.

Proposition 5.2 *Consider the orthogonal decomposition $\xi = \xi_1 + \xi_2$. Then the following hold: for each positive ε we have*

$$\sum_i c_{n-1} \int_X \frac{\varepsilon^2}{(\varepsilon^2 + |s_Y|^2)^2} \gamma_{\xi_1} \wedge \overline{\gamma_{\xi_1}} e^{-\varphi_L} \wedge \sqrt{-1}\partial s_i \wedge \overline{\partial s_i}$$
$$\leq C \int_X \prod \log^2(\varepsilon^{2/N} + |s_j|^2) \left(|\bar\partial^\star \xi|^2\right) e^{-\varphi_L} dV_{\omega_L} \tag{5.2.1}$$

where N is the number of components of Y.

Remark that we have the equality $|\bar\partial^\star \mu|^2 e^{-\varphi_L} = |\bar\partial^\star \xi|^2 e^{-\phi_L}$ if $\mu = \frac{1}{s_Y}\xi$, so the estimate (5.2.1) is precisely what we have to prove.

Proof We recall the context in Sect. 3: we have considered an exhaustion

$$X \setminus H = \bigcup \Omega_m \tag{5.2.2}$$

where each Ω_m was a Stein domain with smooth boundary, endowed with the family of complete metrics $\omega_{m,\delta}$ cf. (4.0.4). The restriction of ξ to each Ω_m decomposes as follows:

$$\xi|_{\Omega_m} = \xi_1^{(m,\delta)} + \xi_2^{(m,\delta)} \tag{5.2.3}$$

according to $(\Omega_m, \omega_{m,\delta})$ and (L, h_L).

We apply the inequality in Theorem 3.1 for $\xi_1^{(m,\delta)}$ and we get

$$\begin{aligned}\sum_i c_{n-1} \int_{\Omega_m} &\frac{\varepsilon^2}{(\varepsilon^2+|s_Y|^2)^2} \gamma_{\xi_1^{(m,\delta)}} \wedge \overline{\gamma_{\xi_1^{(\varepsilon)}}} e^{-\varphi_L} \wedge \sqrt{-1}\partial s_Y \wedge \overline{\partial s_Y} \\ &\leq C \int_{\Omega_m} \prod \log^2(\varepsilon^{2/N} + |s_j|^2) \left|\bar\partial^\star(\xi_1^{(m,\delta)})\right|^2 e^{-\varphi_L} dV_{\omega_{m,\delta}}. \end{aligned} \tag{5.2.4}$$

Indeed we can use Theorem 3.1 in this context even if the form does not have compact support because the metric $\omega_{m,\delta}$ is complete. This has another consequence: we have the equality $\bar\partial^\star(\xi_1^{(m,\delta)}) = \bar\partial^\star(\xi)$. For the inequality (5.2.4) we have used the inequality

$$\log^2(\varepsilon^2 + |s_Y|^2) \leq C \prod \log^2(\varepsilon^{2/N} + |s_j|^2),$$

where N is the number of components of Y.

Let $K \subset X$ be any open set with compact closure in $X \setminus H$; for any $m \geq m_0(K)$ we have $\overline{K} \subset \Omega_m$ so the inequality (5.2.4) implies

$$\begin{aligned}\sum_i c_{n-1} \int_K &\frac{\varepsilon^2}{(\varepsilon^2+|s_Y|^2)^2} \gamma_{\xi_1^{(m,\delta)}} \wedge \overline{\gamma_{\xi_1^{(m,\delta)}}} e^{-\varphi_L} \wedge \sqrt{-1}\partial s_Y \wedge \overline{\partial s_Y} \\ &\leq C \int_{\Omega_m} \prod \log^2(\varepsilon^{2/N} + |s_j|^2) \left|\bar\partial^\star \xi\right|^2_{\omega_{m,\delta}} e^{-\varphi_L} dV_{\omega_{m,\delta}}. \end{aligned} \tag{5.2.5}$$

Now the support of ξ is a compact contained in $X \setminus H$, so if m is large enough the boundary of Ω_m is disjoint from $\mathrm{Supp}(\xi)$. A limit process (i.e., $\delta \to 0, m \to \infty$), together with Lemma 4.1 implies that we have

$$
\begin{aligned}
&\sum_i c_{n-1} \int_K \frac{\varepsilon^2}{(\varepsilon^2 + |s_Y|^2)^2} \gamma_{\xi_1} \wedge \overline{\gamma_{\xi_1}} e^{-\varphi_L} \wedge \sqrt{-1} \partial s_Y \wedge \overline{\partial s_Y} \\
&\quad \leq C \int_X \prod \log^2(\varepsilon^{2/N} + |s_j|^2) \left|\bar\partial^\star \xi\right|^2_{\omega_C} e^{-\varphi_L} dV_{\omega_C}.
\end{aligned} \tag{5.2.6}
$$

The compact subset K in (5.2.6) is arbitrary, so Proposition 5.2 is proved. □

We are now ready to finish the proof of Theorem 1.1. Consider the integral

$$
\int_X \left\langle \bar\partial \left(\rho_\varepsilon U_0\right), \xi \right\rangle e^{-\phi_L} dV_{\omega_C} \tag{5.2.7}
$$

which up to a sign equals

$$
\int_X \rho' \left(\frac{|s_Y|^2}{\varepsilon^2}\right) \left\langle U_0 \wedge \frac{\overline{\partial s_Y}}{\varepsilon^2}, \frac{\xi}{s_Y} \right\rangle e^{-\varphi_L} dV_{\omega_C}. \tag{5.2.8}
$$

We decompose as usual $\frac{\xi}{s_Y} = \xi_1 + \xi_2$ and then (5.2.8) becomes

$$
\int_X \rho' \left(\frac{|s_Y|^2}{\varepsilon^2}\right) \left\langle U_0 \wedge \frac{\overline{\partial s_Y}}{\varepsilon^2}, \xi_1 \right\rangle e^{-\varphi_L} dV_{\omega_C} = \int_X \rho' \left(\frac{|s_Y|^2}{\varepsilon^2}\right) U_0 \wedge \overline{\partial s_Y \wedge \gamma_{\xi_1}} \frac{e^{-\varphi_L}}{\varepsilon^2} \tag{5.2.9}
$$

We split its evaluation into two parts. The first one is

$$
\int_{X \setminus V_{\text{sing}}} \rho' \left(\frac{|s_Y|^2}{\varepsilon^2}\right) U_0 \wedge \overline{\partial s_Y \wedge \gamma_{\xi_1}} \frac{e^{-\varphi_L}}{\varepsilon^2} \tag{5.2.10}
$$

and by Cauchy–Schwarz inequality the square of its absolute value is smaller than

$$
\int_{K_\varepsilon} |U_0|^2 \frac{e^{-\varphi_L}}{\varepsilon^2} dV_{\omega_C} \cdot \int_{K_\varepsilon} |\partial s_Y \wedge \gamma_{\xi_1}|^2_{\omega_C} \frac{e^{-\varphi_L}}{\varepsilon^2} dV_{\omega_C}, \tag{5.2.11}
$$

where K_ε is the support of the function $\rho'\left(\frac{|s_Y|^2}{\varepsilon^2}\right)$. We remark that we have

$$
\frac{1}{\varepsilon^2} \simeq \frac{\varepsilon^2}{(\varepsilon^2 + |s_Y|^2)^2} \tag{5.2.12}
$$

on the set K_ε. Therefore, the second factor of the product (5.2.11) is smaller than

$$
C \int_X \prod \log^2(\varepsilon^2 + |s_j|^2) \left(|\bar\partial^\star \xi|^2\right) e^{-\varphi_L} dV_{\omega_C} \tag{5.2.13}
$$

by Proposition 5.2.

The rest of the integral (5.2.9) is analyzed as follows. For simplicity we assume that $V_{\mathrm{sing}} = \Omega$ is a coordinate subset and the expression we have to evaluate is

$$\left| \frac{1}{\varepsilon^2} \int_\Omega \rho' \left(\frac{|s_Y|^2}{\varepsilon^2} \right) U_0 \wedge \overline{\partial s_Y \wedge \gamma_{\xi_1}} e^{-\varphi_L} \mathrm{d}V_{\omega_C} \right|. \tag{5.2.14}$$

This is bounded by the quantity

$$\sup_\Omega (|\xi_1|^\alpha) \left| \frac{1}{\varepsilon^2} \int_\Omega \rho' \left(\frac{|s_Y|^2}{\varepsilon^2} \right) |\partial s_Y|^\alpha |U_0|_{\omega_C} |\partial s_Y \wedge \gamma_{\xi_1}|^{1-\alpha} e^{-(1-\alpha/2)\varphi_L} \mathrm{d}V_{\omega_C} \right| \tag{5.2.15}$$

and Hölder inequality shows that (5.2.15) is smaller than the product of

$$\sup_\Omega (|\xi_1|^\alpha) \left(\int_{\Omega \cap K_\varepsilon} |\partial s_Y \wedge \gamma_{\xi_1}|^2 \frac{e^{-\varphi_L}}{\varepsilon^2} \mathrm{d}V_{\omega_C} \right)^{\frac{1-\alpha}{2}} \tag{5.2.16}$$

with

$$\left(\int_\Omega \rho' \left(\frac{|s_Y|^2}{\varepsilon^2} \right) |\partial s_Y|^{\frac{2\alpha}{1+\alpha}} |U_0|_{\omega_C}^{\frac{2}{1+\alpha}} \frac{e^{-\frac{\varphi_L}{1+\alpha}}}{\varepsilon^2} \mathrm{d}V_{\omega_C} \right)^{\frac{1+\alpha}{2}} \tag{5.2.17}$$

The limit of the quantity (5.2.17) as $\varepsilon \to 0$ is equal to

$$\left(\int_{\Omega \cap Y} \left| \frac{u}{\partial s_Y} \right|_{\omega_C}^{\frac{2}{1+\alpha}} e^{-\frac{\varphi_L}{1+\alpha}} \mathrm{d}V_{\omega_C} \right)^{\frac{1+\alpha}{2}}. \tag{5.2.18}$$

As for the (5.2.16), we use Theorem 4.2 together with our previous considerations and it follows that it is smaller than

$$C \left(\int_\Omega |\xi_1|^2 \mathrm{d}V_{\omega_C} \right)^{\alpha/2} \left(\int_X \prod \log^2(\varepsilon^{2/N} + |s_j|^2) \left(|\bar\partial^\star \xi|^2 \right) e^{-\varphi_L} \mathrm{d}V_{\omega_C} \right)^{\frac{1-\alpha}{2}}. \tag{5.2.19}$$

It is at this point that we are using the positivity assumption (i): we have

$$\int_{V_{\mathrm{sing}}} |\xi_1|^2 e^{-\varphi_L} \mathrm{d}V_{\omega_C} \leq \frac{1}{C_{\mathrm{sing}}} \int_{V_{\mathrm{sing}}} \langle [\Theta_{h_L}(L), \Lambda_{\omega_C}] \xi_1, \xi_1 \rangle e^{-\varphi_L} \mathrm{d}V_{\omega_C}, \tag{5.2.20}$$

where C_{sing} is the (positive) lower bound for the positivity of $(L, h_L)|_{V_{\text{sing}}}$. By Bochner formula we get

$$\int_{V_{\text{sing}}} |\xi_1|^2 e^{-\varphi_L} \mathrm{d}V_{\omega_{\mathcal{C}}} \leq \frac{1}{C_{\text{sing}}} \int_X \left|\bar{\partial}^\star \xi_1\right|^2 e^{-\varphi_L} \mathrm{d}V_{\omega_{\mathcal{C}}}. \tag{5.2.21}$$

Thus we obtain the expected estimate for the functional (5.2.9) and Theorem 1.1 is proved. We remark that the contribution of the singularities of Y to the estimate in this result is

$$C\left(1+\frac{1}{C_{\text{sing}}^{\alpha}}\right)\left(\int_{\Omega\cap Y}\left|\frac{u}{\partial s_Y}\right|_{\omega_{\mathcal{C}}}^{\frac{2}{1+\alpha}} e^{-\frac{\varphi_L}{1+\alpha}} \mathrm{d}V_{\omega_{\mathcal{C}}}\right)^{1+\alpha}, \tag{5.2.22}$$

where C is a constant depending on $(X, V_{\text{sing}}, \omega_{\mathcal{C}})$ and $\alpha \in [0, 1]$ is an arbitrary positive real which is smaller than 1.

Remark 5.3 The quantity (5.2.22) is part of the term estimating

$$\int_{X\setminus V_{\text{sing}}} \frac{|U|^2}{|s_Y|^2 \prod \log^2(|s_j|^2)} e^{-\varphi_L-\varphi_Y} \mathrm{d}V_{\omega_{\mathcal{C}}}. \tag{5.3.1}$$

A slight modification of the proof shows that we can get a similar estimate for the integral

$$\int_{X\setminus V_{\text{sing}}} \frac{|U|^2}{|s_Y|^2 \log^{2+\tau}(1/|s_Y|^2)} e^{-\varphi_L-\varphi_Y} \mathrm{d}V_{\omega_{\mathcal{C}}} \tag{5.3.2}$$

for any strictly positive real τ.

Remark 5.4 Actually one can replace the curvature condition (i) with the following: *there exists a constant* $C_{\text{sing}} > 0$ *such that we have*

$$\Theta_{h_L}(L) \geq \frac{C_{\text{sing}}}{\log \frac{1}{|s_Y|^2}} \omega_{\mathcal{C}} \tag{5.4.1}$$

pointwise on V_{sing}. The estimate for the extension we obtain in the end is the same, but we are using a twisted Bochner formula instead of (5.2.21).

5.2 Proof of Theorem 1.4

We follow the notations in Theorem 1.8. Since $\bar{\partial}\xi_1 = 0$, the Hodge relation $\partial^\star_{\varphi_L} = [\bar{\partial}, \Lambda_\omega]$ implies that

$$\bar{\partial}(\gamma_{\xi_1}) = \partial^\star_{\varphi_L}\xi_1 \qquad \text{on } X.$$

Let $\cup\Omega_i$ be a small Stein cover of V_{sing}. The standard L^2-estimate as well as the Bochner inequality (5.5.7) imply that we can find G_i satisfying

$$\bar{\partial} G_i = \partial^\star_{\varphi_L}\xi_1 \text{ on } \Omega_i \qquad \text{and} \qquad \int_{\Omega_i} |G_i|^2_{\omega,h_L} dV_\omega \leq C\int_X |\bar{\partial}^\star\xi_1|^2_{\omega,h_L} \mathrm{d}V_\omega, \tag{5.4.2}$$

where C is a uniform constant independent of h_L.

We would like to control the C^0-norm of G_i. Note first that, as $i\Theta_{h_L}(L)$ is bounded on V_{sing}, we can found a bounded function φ'_L on Ω_i such that

$$dd^c\varphi_L = dd^c\varphi'_L.$$

Therefore $\varphi_L - \varphi'_L$ is the real part of some holomorphic function f_i on Ω_i. Replacing ξ_1 by $\xi_1 \cdot e^{-\frac{f_L}{2}}$, we can suppose in the beginning that the C^2-norm of φ_L is bounded by the constant C_1.

Thanks to (5.4.2), $F_i := \gamma_{\xi_1} - G_i$ is a holomorphic $(n-1,0)$-form on Ω_i. Recall that ξ_1 is Δ''-harmonic on V_{sing}. Then

$$\Delta''(\omega\wedge G_i) = \Delta''(\omega\wedge F_i) \qquad \text{on } \Omega_i. \tag{5.4.3}$$

Using (5.4.2), the H^{-2}-norm of $\Delta''(\omega\wedge F_i)$ on $\frac{3}{4}\Omega_i$ is bounded by $C\int_X |\bar{\partial}^\star\xi_1|^2_{\omega,h_L}\mathrm{d}V_\omega$. Since F_i is holomorphic and the C^2-norm of φ_L is bounded, $\Delta''(\omega\wedge F_i)$ is continuous. Therefore the C^0-norm of $\Delta''(\omega\wedge F_i)$ on $\frac{3}{4}\Omega_i$ is bounded by $C\int_X |\bar{\partial}^\star\xi_1|^2_{\omega,h_L}\mathrm{d}V_\omega$. As a consequence, the elliptic regularity implies that the

$$\begin{aligned}\sup_{\frac{1}{2}\Omega_i} |G_i|^2_{\omega,h_L} &\leq C\left(\int_{\Omega_i} |G_i|^2_{\omega,h_L} dV_\omega + \sup_{\frac{3}{4}\Omega_i} |\Delta''(\omega\wedge F_i)|^2_{\omega,h_L}\right)\\ &\leq C'\int_X |\bar{\partial}^\star\xi_1|^2_{\omega,h_L}\mathrm{d}V_\omega,\end{aligned} \tag{5.4.4}$$

where C' is a uniform constant.

Now we can prove the theorem. Theorem 3.2 and (5.4.4) imply that

$$\int_{Y\cap\frac{1}{2}\Omega_i} F_i\wedge\overline{F_i}e^{-\varphi_L} \leq C'\int_X \log^2|s_Y|^2|\bar{\partial}^\star\xi_1|^2_{\omega,h_L}\mathrm{d}V_\omega.$$

Then the mean value inequality for holomorphic functions implies that

$$\sup_{Y\cap\frac{1}{3}\Omega_i} |F_i|_Y|^2_\omega e^{-\varphi_L} \leq C'\int_X \log^2|s_Y|^2|\bar{\partial}^\star\xi_1|^2_{\omega,h_L}\mathrm{d}V_\omega.$$

Together with (5.4.4), we have thus

$$\sup_{Y\cap\frac{1}{3}\Omega_i} |\gamma_{\xi_1}|_Y|^2_\omega e^{-\varphi_L} \leq C' \int_X \log^2 |s_Y|^2 |\bar\partial^\star \xi_1|^2_{\omega,h_L} dV_\omega. \tag{5.4.5}$$

As a consequence, we have

$$\left| \int_{Y\cap V_{\text{sing}}} \frac{u}{ds_Y} \wedge \overline{\gamma_{\xi_1}} e^{-\varphi_L} \right| \leq C' \int_{Y\cap V_{\text{sing}}} |\frac{u}{ds_Y}|_\omega e^{-\frac{\varphi_L}{2}} dV_\omega \\ \times \left(\int_X \log^2 |s_Y|^2 |\bar\partial^\star \xi_1|^2_{\omega,h_L} dV_\omega \right)^{\frac{1}{2}}.$$

Like the arguments as in the above theorems, this implies the existence of an holomorphic extension U with the estimates

$$\frac{1}{C'} \int_{X\setminus V_{\text{sing}}} |U|^2 e^{-\varphi_Y - \varphi_L} \leq \int_{Y\setminus V_{\text{sing}}} \left|\frac{u}{ds}\right|^2 e^{-\varphi_L} + \left(\int_{Y\cap V_{\text{sing}}} \left|\frac{u}{ds}\right| e^{-\frac{\varphi_L}{2}} dV_\omega \right)^2, \tag{5.4.6}$$

where C' depends only on the geometry of $(V_{\text{sing}}, \omega)$ and C_1.

5.3 Proof of Theorem 1.7

We are using the notations from the previous section, so ω is a fixed reference Kähler metric on X.

By hypothesis, the metric h_L is non-singular and in this case the equality

$$\int_Y \frac{u}{ds_Y} \wedge \overline{\gamma_\xi} e^{-\varphi_L} = \int_Y \frac{u}{ds_Y} \wedge \overline{\gamma_{\xi_1}} e^{-\varphi_L} \tag{5.4.7}$$

is immediate.

We remark that the restriction $\left.\frac{u}{ds_Y}\right|_{Y_j}$ is holomorphic, for each component Y_j of Y. This is where the vanishing of u on the singularities of Y is used. We decompose the restriction of γ_{ξ_1} to Y_j as follows:

$$\gamma_{\xi_1}|_{Y_j} = \alpha_j + \beta_j \tag{5.4.8}$$

where α_j is holomorphic and β_j is orthogonal to the space of L-valued holomorphic top forms on Y_j. Then we have

$$\int_{Y_j} \frac{u}{ds_Y} \wedge \overline{\gamma_{\xi_1}} e^{-\varphi_L} = \int_{Y_j} \frac{u}{ds_Y} \wedge \overline{\alpha_j} e^{-\varphi_L} \tag{5.4.9}$$

Let x_0 be a singular point of Y. We then have coordinates (z_1, z_2) defined on a open subset $x_0 \in V$, centered at x_0 and such that $(z_k = 0) = Y_k \cap V$ for each $k = 1, 2$. We equally fix a trivialization of $L|_V$ and let φ_L be the corresponding weight of the metric h_L. We write

$$u|_V = f_u dz_1 \wedge dz_2 \otimes e_L \tag{5.4.10}$$

and let θ be a function which is equal to 1 near x_0 and such that $\mathrm{Supp}(\theta) \subset V$.

Thus we have

$$\frac{u}{\mathrm{d}s_Y}\Big|_{V\cap Y_1} = f_u \frac{dz_2}{z_2} \otimes e_L \tag{5.4.11}$$

together with a similar equality on $V \cap Y_2$. We can write

$$\begin{aligned} \partial_{\varphi_L}\left(\theta f_u \log |z_2|^2 \otimes e_L\right) &= \theta f_u \frac{dz_2}{z_2} \otimes e_L \\ &\quad + \theta \log |z_2|^2 \partial_{\varphi_L}(f_u \otimes e_L) + f_u \log |z_2|^2 \partial\theta \otimes e_L \end{aligned} \tag{5.4.12}$$

and then we observe that the left-hand side term of (5.4.12) is ∂_{φ_L}—exact on Y_1. Therefore we have

$$\int_{Y_1} \partial_{\varphi_L}\left(\theta f_u \log |z_2|^2 \otimes e_L\right) \wedge \overline{\alpha_1} e^{-\varphi_L} = 0 \tag{5.4.13}$$

since α_1 is holomorphic. We infer that we have

$$\begin{aligned} -\int_{Y_1} \theta \frac{u}{\mathrm{d}s_Y} \wedge \overline{\alpha_1} e^{-\varphi_L} &= \int_{Y_1} \theta \log |z_2|^2 \partial_{\varphi_L}(f_u \otimes e_L) \wedge \overline{\alpha_1} e^{-\varphi_L} \\ &\quad + \int_{Y_1} f_u \log |z_2|^2 \partial\theta \otimes e_L \wedge \overline{\alpha_1} e^{-\varphi_L} \end{aligned}$$

and all that we still have to do is to apply the Cauchy–Schwarz inequality to each of the two terms of the RHS of the inequality above.

A last remark is that we have

$$\int_{Y_1} |\alpha_1|^2 e^{-\varphi_L} \leq \int_{Y_1} |\gamma_{\xi_1}|^2 e^{-\varphi_L} \tag{5.4.14}$$

by the definition of α_1 and β_1. We use the a-priori inequality and we conclude as in Theorem 1.1. □

Remark 5.5 In the absence of hypothesis $u|_{Y_{\mathrm{sing}}} \not\equiv 0$ the evaluation of the term (5.4.7) near the singularities of Y is problematic. In the decomposition (5.4.8), we write

$\beta_j = \bar\partial^\star(\tau_j)$, and then the question is to estimate the quotient

$$f_j := \frac{\tau_j}{\omega}$$

at the points of $Y_{\rm sing}$. This does not seem to be possible, since we only have the norm $W^{1,2}$ of f_j at our disposal. Indeed, the quantity $\bar\partial\beta_j$ is equal to the restriction of the form $\bar\partial\gamma_{\xi_1}$ to Y_j. In Question 5.7 we provide a few more precisions about this matter.

5.4 Proof of Theorem 1.8

This is another set-up in which the considerations above work, as follows. We recall that the metric of h_L of L satisfies the hypotheses (a) and (b) at the beginning *and* moreover (L, h_L) is flat near the singularities of Y, i.e.,

$$\Theta_{h_L}(L)\big|_{V_{\rm sing}} = 0. \tag{5.5.1}$$

Then we get an estimable extension as follows. Let ω be a fixed Kähler metric on X. As in the proof of the preceding result Theorem 1.7, we will use the method of Berndtsson [2], so the quantity to be bounded is

$$\int_{Y\cap V_{\rm sing}} \frac{u}{ds_Y} \wedge \overline{\gamma_{\xi_1}} e^{-\varphi_L}. \tag{5.5.2}$$

Integration by parts shows that it is enough to obtain a mean value inequality for the function

$$\sup_{1/2V_{\rm sing}} |\bar\partial\gamma_{\xi_1}|^2 = \sup_{1/2V_{\rm sing}} |\partial^\star_{\varphi_L}\xi_1|^2. \tag{5.5.3}$$

This is done according to the same principle as before. In the first place the differential inequality satisfied by $|\partial^\star_{\varphi_L}\xi_1|^2$ is as follows:

$$\Delta''\left(|\partial^\star_{\varphi_L}\xi_1|^2\right) \geq |\nabla\left(\partial^\star_{\varphi_L}\xi_1\right)|^2 - C|\partial^\star_{\varphi_L}\xi_1|^2 \tag{5.5.4}$$

for some constant $C > 0$ which only depends on (the curvature of) $\omega|_{V_{\rm sing}}$. We will not detail the calculation here because this is very similar with the one in the proof of Theorem 1.1. However, we highlight next the main differences:

(1) It is not necessary to introduce any regularization of the metric, since by hypothesis (5.5.1) the restriction $h_L|_{V_{\rm sing}}$ is non-singular.
(2) Without any additional information about (L, h_L), the term $\left\langle\partial^\star_{\varphi_L}\xi_1, \Box\partial^\star_{\varphi_L}\xi_1\right\rangle$ is problematic. Actually (5.5.1) is needed precisely in order to deal with it: the curvature of $(L, h_L)|_{V_{\rm sing}}$ equals zero, then we have $\Box\partial^\star_{\varphi_L}\xi_1 = 0$ pointwise on $V_{\rm sing}$.

In general we have the term

$$\left\langle [\bar\partial, \Lambda_{\Theta_{h_L}(L)}]\xi_1, \partial^\star_{\varphi_L}\xi_1 \right\rangle$$

which appears in the computation and seems impossible to manage.

(3) In the evaluation of the Laplacian of the norm of a harmonic tensor we have two terms: the gradient of the tensor, and several curvature terms corresponding to the metric on the ambient manifold and to the twisting, respectively. Here we do not have any contribution from L, and the term involving the curvature of ω is taken care by the constant $-C$ in (5.5.4).

Anyway, the inequality (5.5.4) can be re-written as

$$\Delta''\left(|\partial^\star_{\varphi_L}\xi_1|^2\right) \geq \left|\nabla\left|\partial^\star_{\varphi_L}\xi_1\right|\right|^2 - C|\partial^\star_{\varphi_L}\xi_1|^2 \tag{5.5.5}$$

and this combined with Moser iteration procedure shows that we have

$$\sup_{1/2V_{\mathrm{sing}}} |\partial^\star_{\varphi_L}\xi_1|^\alpha_{\omega,h_L} \leq C\int_{V_{\mathrm{sing}}} |\partial^\star_{\varphi_L}\xi_1|^\alpha_{\omega,h_L}\mathrm{d}V_\omega. \tag{5.5.6}$$

Finally, the term that one (almost) never uses in Bochner formula shows that we have

$$\int_X |\partial^\star_{\varphi_L}\xi_1|^2_{\omega,h_L}\mathrm{d}V_\omega \leq \int_X |\bar\partial^\star\xi_1|^2_{\omega,h_L}\mathrm{d}V_\omega \tag{5.5.7}$$

and we thus obtain the inequality

$$\sup_{1/2V_{\mathrm{sing}}} |\partial^\star_{\varphi_L}\xi_1|^\alpha_{\omega,h_L} \leq C\int_{V_{\mathrm{sing}}} |\bar\partial^\star_{\varphi_L}\xi_1|^\alpha_{\omega,h_L}\mathrm{d}V_\omega. \tag{5.5.8}$$

Then we conclude as in Theorem 1.7. □

Remark 5.6 Actually in the proof of Theorem 1.8 only needs to evaluate the L^2 norm

$$\int_Y |\bar\partial\gamma_{\xi_1}|^2\mathrm{d}V_\omega \tag{5.6.1}$$

of $\bar\partial\gamma_{\xi_1}|_Y$. One might try to use a similar method as the one in Sect. 2, but there are serious difficulties to overcome.

5.5 Proof of Theorem 1.10

By hypothesis we know that Y has one component Y_1 which only intersects $\cup_{i\neq 1}Y_i$ in a unique point p_0 such that $u(p_0)\neq 0$. We also assume that $L|_{Y_1}$ is flat, in the sense that there exists a section τ such that $\tau(p_0) \neq 0$ and $\partial_{\varphi_L}\tau = 0$. Then we argue as follows:

Let ω be a fixed, reference metric on X. On each component Y_j of Y we solve the equation

$$\gamma_{\xi_1}|_{Y_j} = \alpha_j + \bar{\partial}^\star \beta_j, \tag{5.6.2}$$

where α_j is holomorphic $(1, 0)$ form and β_j is of type $(1, 1)$ on Y_j. We note that by elliptic regularity the form β_j is smooth.

We have $\beta_j = f_j \omega|_{Y_j}$ and then the equality

$$\int_{Y_j} \Big\langle \frac{u}{\sigma_j \mathrm{d}s_{Y_j}}, \bar{\partial}^\star \beta_j \Big\rangle_\omega e^{-\varphi_L} \mathrm{d}V_\omega = \sum_{x \in Y_{\mathrm{sing}} \cap Y_j} f_u(x) \overline{f_j(x)} e^{-\varphi_L(x)} \tag{5.6.3}$$

follows by the residues formula. Here we denote by $\sigma_j := \prod_{i \neq j} s_{Y_i}$.

In case $j = 1$, the sum above only has one term, by hypothesis. Since we have $\bar{\partial}^\star(\tau\omega) = 0$, we can modify the solution f_1 so that the global sum of residues is zero.

Question 5.7 Let p be one of the intersection points of two curves $Y_1 \cap Y_2$ in X. The analog of the a-priori inequality in Sect. 2 gives

$$|f_j(p)|^2 e^{-\varphi_L(p)} \leq C \int_X \frac{\log^2 \|s\|^2}{\|s\|^2} |\partial_{\varphi_L} f_j|^2 e^{-\varphi_L} \mathrm{d}V_\omega \tag{5.7.1}$$

provided that the bundle (L, h_L) has the right curvature hypothesis, let us assume this holds for the moment. In (5.7.1) we denote by C a constant which we can compute explicitly. This a-priori inequality is obtained by considering the $(2, 2)$-form with values in L

$$f_j \omega^2 \tag{5.7.2}$$

whose $\star$ coincides with the section f_j, and use the procedure Theorem 2.1 for the function $w := \frac{1}{|s_1|^2 + |s_2|^2}$. The curvature requirements this induces will most likely be

$$\frac{1}{\delta} \Theta_{h_L}(L) \wedge \omega \geq \frac{|s_1|^2 \Theta(Y_1) + |s_2|^2 \Theta(Y_2)}{|s_1|^2 + |s_2|^2} \wedge \omega. \tag{5.7.3}$$

We have

$$\bar{\partial} f_j \omega^2 = 0, \qquad \bar{\partial}^\star f_j \omega^2 = -\star \left(\partial_{\varphi_L} f_j\right) \tag{5.7.4}$$

which explains (5.7.1).

Anyway, by equality (5.6.2) we control the norm

$$\int_Y |\partial_{\varphi_L} f_j|^2 e^{-\varphi_L} \mathrm{d}V_\omega. \tag{5.7.5}$$

Then the question is: can we find a smooth section $\widetilde{f_j}$ of L such that it equals f_j on Y and such that

$$\int_X \frac{\log^2 \|s\|^2}{\|s\|^2} |\partial_{\varphi_L} \widetilde{f_j}|^2 e^{-\varphi_L} \mathrm{d}V_\omega \le C \int_Y |\partial_{\varphi_L} f_j|^2 e^{-\varphi_L} \mathrm{d}V_\omega, \tag{5.7.6}$$

where C in (5.7.6) is universal?

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Appendix A: Further Results and Examples (by Bo Berndtsson)

In this appendix we will study two very simple model examples of L^2-extension from a non-reduced or singular variety. It is the second example (in Sect. 3) that is most relevant to the subject of the main paper. The main point is to show that it is not possible to obtain an estimate that is substantially better than Theorems 1.1 and 1.7. The role of the first example is to show that similar difficulties appear already in an even simpler situation, that can be analyzed in a more complete way.

In the first example we consider the space

$$A^2_\phi(\Delta) = \{h \in H(\Delta); \int_\Delta |h|^2 e^{-\phi} \mathrm{d}\lambda := \|h\|^2 < \infty\}$$

of holomorphic functions in the disk Δ that are square integrable against a weight $e^{-\phi}$. Given numbers a_k, $k = 0, 1, ...N-1$, we will compute the minimal norm of a function $h \in A^2$ that satisfies $h^{(k)}(0) = a_k$. The formula we give is exact but not very explicit; it contains the Bergman kernel for the space and various metrics derived from it. Therefore we will also discuss to what extent it is possible to estimate it in more concrete terms, and show that the most optimistic estimates fail.

In the second example we consider the unit polydisk U in $\mathcal{C}^2$ and the singular variety $V = \{z \in U; z_1 z_2 = 0\}$ in U. We again denote by $A^2_\phi(U)$ the Bergman space of holomorphic functions in U that are square integrable against the weight $e^{-\phi}$. The extension problem is now to find a function $h \in A^2$ that restricts to a given function on V, i.e., satisfies $h = f_1$ when $z_1 = 0$ and $h = f_2$ when $z_2 = 0$, where f_1 and f_2 are holomorphic functions of one variable satisfying $f_1(0) = f_2(0)$. In this case we get only an *estimate* for the minimal extension, and we conclude by an example (basically due to Ohsawa, [35]) which (perhaps) can serve as a motivation for the statement in Theorem 1.4.

Extension from a (Fat) Point in the Unit Disk

Let $A^2 = A^2_\phi(\Delta)$ be defined as above and let for $k = 0, 1, 2...$

$$E_k = \{h \in A^2; h^{(j)}(0) = 0, j < k\}.$$

It follows from elementary Hilbert space theory that there is a unique function h of minimal norm in A^2 satisfying $h^{(k)}(0) = a_k$ for $k = 0, ...N-1$. Write

$$h = h_0 + r_1,$$

where $r_1 \in E_1$ and $h_0 \perp E_1$. Then we write

$$r_1 = h_1 + r_2,$$

with $r_2 \in E_2$ and $h_1 \perp E_2$. Continuing this way we get

$$h = h_0 + h_1 + ...h_{N-1} + r_N,$$

with $h_k \in E_k \ominus E_{k+1}$ and r_N in E_N. That h has minimal norm means that h is orthogonal to E_N, so $r_N = 0$. By orthogonality we have

$$\|h\|^2 = \sum_0^{N-1} \|h_k\|^2,$$

so the problem amounts to estimating the norms of h_k.

The spaces $E_k \ominus E_{k+1}$ are one dimensional. Let e_k be an element of unit length. We start with a simple lemma from [18] whose proof follows almost directly from the definitions.

Lemma A.1

$$|e_k^{(k)}(0)|^2 = \sup_{f \in E_k} \frac{|f^{(k)}(0)|^2}{\|f\|^2}.$$

For $k = 0$, $|e_0(0)|^2 = B_0(0)$, the (diagonal) Bergman kernel at the origin. For $k \geq 1$ $|e_k^{(k)}(0)|^2 =: B_k(0)$ can be viewed as a 'higher order Bergman kernel' and we refer to

[18] for interesting applications of this idea. By a classical formula of Bergman, [1], we have

$$B_1(0) = |e_1'(0)|^2 = B_0(0)\frac{\partial^2 \log B_0(z)}{\partial z \partial \bar{z}} =: B_0\omega_B, \tag{A.1.1}$$

so the first-order Bergman kernel is strongly related to the Bergman *metric*. (We are abusing notation by identifying the metric with its density; more properly we should write

$$\omega_B = \frac{\partial^2 \log B_0(z)}{\partial z \partial \bar{z}} i dz \wedge d\bar{z}.)$$

Since $h^{(k)}(0) = a_k$ and $h_j^{(k)}(0) = 0$ for $j > k$, we get

$$h_0(0) =: b_0 = a_0, \quad h_1'(0) =: b_1 = a_1 - h_0'(0),$$
$$h_2''(0) =: b_2 = a_2 - h_1''(0) - h_0''(0), \ldots$$

Then, since h_k is a multiple of e_k and $h_k^{(k)}(0) = b_k$, we have

$$h_k = \frac{b_k}{e_k^{(k)}(0)} e_k.$$

Recalling that e_k has norm 1 and that $|e_k^{(k)}(0)|^2 = B_k(0)$ we find that the norm of the minimal extension is given by

$$\|h\|^2 = \sum_0^{N-1} |b_k|^2 / B_k(0). \tag{A.1.2}$$

When $N = 1$ this is just the standard formula

$$\|h\|^2 = |a_0|^2 / B_0(0),$$

which shows that estimates from above of the norm of the minimal extension are equivalent to estimates from below of the (usual) Bergman kernel. The next case is $N = 2$. Then we use (A.1.1) and find (since $h_0 = (a_0/e_0(0))e_0$)

$$\|h\|^2 = (|a_0|^2 + |a_1 - a_0 e_0'(0)/e_0(0)|^2_{\omega_B}) / B_0(0).$$

Here we think of $a_1 - a_0 e_0'(0)/e_0(0)$ as a 1-form and the second term in the right-hand side is its norm for the Bergman metric. Since the off-diagonal Bergman kernel $B_0(z, w)$ is holomorphic in z and antiholomorphic in w, and $e_0(z) = B_0(z, 0)$, we have

$$e_0'(0)/e_0(0) = (\partial/\partial z)|_0 \log B_0(z, z).$$

Hence we can also write

$$\|h\|^2 = (|a_0|^2 + |a_1 - a_0\partial \log B_0(0)|^2_{\omega_B})/B_0(0).$$

By the standard Ohsawa–Takegoshi theorem, $B_0(0) \geq C^{-1}e^{\phi(0)}$, with a universal constant C, so we get the slightly more explicit estimate

$$\|h\|^2 \leq C(|a_0|^2 + |a_1 - a_0\partial \log B_0(0)|^2_{\omega_B})e^{-\phi(0)}. \tag{A.1.3}$$

Because of the following lemma we can replace the norm with respect to the Bergman metric by the Euclidean norm.

Lemma A.2 *For any (1-form) a, we have at the origin of Δ*

$$|a|^2_{\omega_B} \leq |a|^2,$$

where $|a|^2$ denotes the norm with respect to the Euclidean metric.

Proof By (A.1.1) and Lemma A.1, at the origin

$$\omega_B = B_1(0)/B_0(0) = \sup_{f\in E_1} \frac{|f'(0)|^2}{\|f\|^2 B_0(0)}.$$

Now choose $f = ze_0$. Since $|z| < 1$ and e_0 has norm 1, f has norm less that 1. Moreover, $f'(0) = e_0(0)$. Hence $\omega_B \geq 1$, which proves the lemma. □

Thinking of $\log B_0(z)$ as an approximation of ϕ, (A.1.3) suggests that one might also have the inequality

$$\|h\|^2 \leq C(|a_0|^2 + |a_1 - a_0\partial\phi(0)|^2 e^{-\phi(0)},$$

but we shall see later that this does not hold.

We next discuss briefly estimates for larger values of N. We first note that there is a version of Lemma A.2 for all k, which is proved in much the same way.

Lemma A.3

$$B_k(0) \geq (k!)^2 B_0(0).$$

Proof Recall that

$$B_k(0) = |e_k(0)|^2 = \sup_{f\in E_k} \frac{|f^{(k)}(0)|^2}{\|f\|^2}.$$

Take $f = z^k e_0$. Then f has norm less than 1 and $f^{(k)}(0) = k!e_0(0)$. Thus

$$B_k(0) \geq (k!)^2 |e_0(0)|^2 = (k!)^2 B_0(0).$$

□

From the lemma and (A.1.2) we get the estimate for the norm of the L^2-minimal extension

$$\|h\|^2 \leq \left(\sum_{k=0}^{N-1} |b_k|^2 \right) / B_0(0) \leq C \left(\sum |b_k|^2 \right) e^{-\phi(0)},$$

where

$$b_k = a_k - \sum_{j=0}^{k-1} h_j^{(k)}(0) = a_k - \sum_{j=o}^{k-1} \frac{b_j}{e_j(0)} e_j^{(k)}(0).$$

As we have seen, this is difficult to estimate even when $N = 2$ and it is clear that the complexity grows with higher values of k and N.

Examples

We focus on the estimates for $N = 2$, i.e., extension of a first-order jet. The most naive conjecture for an explicit estimate would be

$$\|h\|^2 \leq C(|a_0|^2 + |a_1|^2) e^{-\phi(0)}. \tag{A.3.1}$$

Claim 1: *There is no constant C independent of ϕ such that for all subharmonic ϕ, (A.3.1) holds.*

For this, take $\phi(z) = -2m\Re(z)$ and put $g = e^{mz}h$. Take $a_0 = 1$, $a_1 = 0$. If (A.3.1) held we would get

$$\|h\|^2 \leq Ce^{-\phi(0)} = C.$$

Hence

$$\int_\Delta |g|^2 \mathrm{d}\lambda = \|h\|^2 \leq C,$$

and $g'(0) = m$. This contradicts Cauchy's estimates for the derivative.

The estimate

$$\|h\|^2 \leq C(|a_0|^2 + |a_1 - a_0 \partial\phi(0)|^2) e^{-\phi(0)}. \tag{A.3.2}$$

might seem more plausible since we estimate $\|h\|^2$ by the connection in A^2_ϕ, $h' - h\partial\phi$ instead of just $h'(0)$.

Claim 2: *There is no constant C independent of ϕ such that for all subharmonic ϕ the minimal extension satisfies* (A.3.2).

Assume that $\phi(0) = 0$ and that ϕ is smooth. Then there is a small annulus $U = \{r_1 < |z| < r_2\}$ around the origin, where $\phi > -1$. Choose $\varepsilon > 0$ so small that $\phi + \varepsilon \log |z| > -1$ in the annulus too. Define a function ψ in the disk by $\psi = \max(\phi + \varepsilon \log |z|, -1)$, in the disk where $|z| < r_2$, and $\psi = \phi + \varepsilon \log |z|$ where $|z| > r_1$. Since the two definitions agree on overlaps, ψ is a well-defined subharmonic function, and $\psi = -1$ in a small neighborhood of the origin.

If (A.3.2) holds for ψ, then (A.3.1) also holds for ψ since $\partial\psi$ vanishes near the origin. Letting $\varepsilon \to 0$ (or just taking ε sufficiently small) we get an extension that satisfies (A.3.1) for ϕ. By the first claim, this is impossible.

A Singular Variety in the Bidisk

In this section we study L^2-extension from the variety

$$V = \{z \in U; z_1 z_2 = 0\}$$

in the unit bidisk U, and we use the notation from the introduction. Following the scheme in the previous section we let

$$E_1 = \{h \in A^2_\phi(U); h(0) = 0\},$$

and

$$E_2 = \{h \in A^2_\phi(U); h|_V = 0\}.$$

Let f be a holomorphic function on V, and let h be the holomorphic extension of f to U of minimal norm. Again we write

$$h = h_0 + r_1,$$

where $h_0 \perp E_1$ and $r_1 \in E_1$. Continuing as before we write

$$r_1 = h_1 + r_2$$

where $h_1 \in E_1 \ominus E_2$ and $r_2 \in E_2$. Then $h = h_0 + h_1 + r_2 = h_0 + h_1$ since, by minimality, h is orthogonal to E_2. Moreover, $h_0 \perp h_1$, so

$$\|h\|^2 = \|h_0\|^2 + \|h_1\|^2.$$

The holomorphic function f on V is given by a pair (f_1, f_2), where f_1 is holomorphic on $\{z_1 = 0\}$ and f_2 is holomorphic on $\{z_2 = 0\}$, and $f_1(0) = f_2(0) =: a_0$.

Since h_0 is orthogonal to E_1 and $h_0(0) = a_0$ we have that

$$h_0 = \frac{a_0}{e_0(0)} e_0,$$

whereas in the previous section e_0 is a function of unit norm orthogonal to E_1. Then $|e_0(0)|^2 = B_0(0)$, the (diagonal) Bergman kernel at the origin, and we get

$$\|h_0\|^2 = |a_0|^2/B_0(0),$$

just as before.

We next turn to h_1, which is the L^2-minimal extension of $\tilde{f} := f - h_0$. We can not give an exact formula for the norm of h_1 but it is easy to give an estimate. Since $\tilde{f}$ vanishes at the origin we have $\tilde{f} = (f_1 - h_0, f_2 - h_0) =: (z_2 g_1, z_1 g_2)$, where $g_1(z_2)$ and $g_2(z_1)$ are holomorphic functions of one variable. Let G_1 and G_2 be the minimal extensions of g_1 and g_2, from $V_1 = \{z_1 = 0\}$ and $V_2 = \{z_2 = 0\}$, respectively. By the Ohsawa–Takegoshi theorem

$$\|G_i\|^2 \leq C \int_{V_i} |g_i|^2 e^{-\phi} \mathrm{d}\lambda.$$

Let $H = z_2 G_1 + z_1 G_2$. This is an extension of $\tilde{f}$ and

$$\|H/|z|\|^2 \leq \|G_1\|^2 + \|G_2\|^2 \leq C \int_V |f - h_0|^2/|z|^2 e^{-\phi} \mathrm{d}\lambda.$$

Hence

$$\|H\|^2 \leq \|H/|z|\|^2 \leq C \int_V |f - h_0|^2/|z|^2 e^{-\phi} \mathrm{d}\lambda.$$

All in all we get the estimate for the minimal extension h of f,

$$\|h\|^2 \leq C(|a_0|^2/B_0(0) + \int_V |f - h_0|^2/|z|^2 e^{-\phi} \mathrm{d}\lambda). \tag{A.3.3}$$

When comparing this to the classical Ohsawa–Takegoshi theorem, let us first assume that $a_0 = f_1(0) = f_2(0) = 0$, so we get

$$\|h\|^2 \leq C \int_V |f|^2/|z|^2 e^{-\phi} \mathrm{d}\lambda.$$

Translating this to the notation in the main text of this article, $|f|^2/|z|^2 = |u/\mathrm{d}s|^2$, so this means that

$$\|U\|^2 \leq C \int_V |\frac{u}{\mathrm{d}s}|^2 e^{-\phi} \mathrm{d}\lambda,$$

which is exactly what the Ohsawa–Takegoshi theorem predicts. The problem with this estimate is that if the right-hand side is finite, it conversely forces a_0 to be zero, so the ‘classical’ theorem deals only with this case. The purpose of the theorems in the introduction is to weaken the norm in the right-hand side so that this can become finite even if $f = u$ does not vanish on the singular set.

Here one might hope that the quotient $\frac{|f-h_0|^2}{|z|^2}$ in the right-hand side could be replaced by the squared norm of a derivative acting on f. Asymptotically as $z \to 0$ on e.g., V_2,

$$\frac{f - h_0}{z_1} \to f'(0) - h_0'(0) = f'(0) - a_0 \frac{e_0'(0)}{e_0(0)} = \frac{\partial f(0)}{\partial z_1} - a_0 \frac{\partial \log B_0(0)}{\partial z_1}$$

(the last equality follows as in the discussion leading to (A.1.3)). Again, thinking of the logarithm of the Bergman kernel, $\log B_0$ as an approximation of ϕ, one is led to look for estimates in terms of $\partial^\phi f$, like in Theorem 1.7. Theorem 1.7, however, also contains a factor $\log^2(\max |z_j|^2)$, and we next give an example showing that something of this kind is necessary.

More Examples

We first give a counterexample (cf. [35]) to the most naive conjecture that the same estimate as for smooth varieties holds.

Claim 3: *There is no universal constant, independent of the plurisubharmonic weight ϕ, such that the minimal extension satisfies*

$$\|h\|^2 \leq C \int_V |f|^2 e^{-\phi} \mathrm{d}\lambda \tag{A.3.4}$$

for all functions f that vanish at the origin.

To see this, take $f = z_1$ when $z_2 = 0$ and $f = 0$ when $z_1 = 0$. Take $\phi = \log |z_1 - z_2|^2$. Any extension H must have the form $H = z_1 G$. If H has finite norm, then $G = 0$ when $z_1 = z_2$. Hence H vanishes to second degree at the origin, which is not possible. This example also motivates the choice of weight $e^{-\phi_L}$ in the last term of the right-hand side in formula (1.6.1). Indeed, replacing that by the more natural $e^{-\phi_L/(1+\alpha)}$, the right-hand side would be finite for our choice of f, and as we have seen extension is not possible.

This same example also shows that it does not help to add the L^2-norm of the twisted derivative of f in the right-hand side. We use the notation

$$\partial^\phi f = e^\phi \partial e^{-\phi} f = \partial f - f \partial \phi.$$

Claim 4: *There is no universal constant, independent of the plurisubharmonic weight ϕ, such that the minimal extension satisfies*

$$\|h\|^2 \leq C \left(\int_V |f|^2 e^{-\phi} \mathrm{d}\lambda + \int_V |\partial^\phi f|^2 e^{-\phi} \mathrm{d}\lambda \right) \tag{A.3.5}$$

for all functions f that vanish at the origin.

Indeed, with the same choice of ϕ and f as above, we have on V_2 outside the origin

$$\partial^{\phi} f = dz_1 - z_1(1/z_1)dz_1 = 0.$$

Since the weight ϕ has a singularity at the origin we look at regular approximations. Let ϕ_ε be the convolution of ϕ with $(\pi\varepsilon^2)^{-1}\chi_\varepsilon$, where χ_ε is the characteristic function of the disk with radius ε. Explicitly,

$$\phi_\varepsilon(\zeta) = \frac{|\zeta|^2 - \varepsilon^2}{\varepsilon^2} + \log \varepsilon^2$$

when $|\zeta| < \varepsilon$ and $\phi_\varepsilon(\zeta) = \log|\zeta|^2$ when $|\zeta| \geq \varepsilon$. This gives a sequence of subharmonic functions decreasing to ϕ on V_2. We have

$$\partial\phi_\varepsilon = \frac{\bar{\zeta}}{\varepsilon^2}\mathrm{d}\zeta$$

when $|\zeta| < \varepsilon$. Hence

$$\partial^{\phi_\varepsilon} f = \frac{\varepsilon^2 - |z_1|^2}{\varepsilon^2}\chi_\varepsilon dz_1$$

on V_2. Since $e^{-\phi_\varepsilon}$ is of size roughly ε^{-2} when $|\zeta| < \varepsilon$, the right-hand side in (A.3.5) stays bounded as $\varepsilon \to 0$. Hence, if (A.3.5) held, we would again get an extension of finite norm in $L^2(e^{-\phi})$, which we have seen is impossible. This motivates the logarithmic factor in the estimate (1.7.1) of Theorem 1.7.

It is easy to construct a compact analog of the example used to prove the claims 3 and 4 above, as we briefly indicate next.

Consider two transverse lines L_1 and L_2 in $X := \mathbb{P}^2$, and the adjoint bundle $K_X + L_1 + L_2 + \mathcal{O}(2)$. We define the metric h on $\mathcal{O}(2)$ by its weights

$$\varphi_\varepsilon := \log(\varepsilon^2 e^{\phi_{\mathrm{FS}}} + |f_1 - f_2|^2) + \phi_{\mathrm{FS}}, \tag{A.3.6}$$

where f_i is the local expression of the section σ_i which defines the line L_i.

We define the section u which equals σ_2 on L_1 and zero on L_2. By Theorem 1.1 we can construct an extension U_ε of u for whose L^2 norm is bounded by

$$\int_{(\mathbb{C},0)} \left|\frac{u}{z}\right|^{\frac{2}{1+\alpha}} e^{-\varphi_\varepsilon}$$

which equals $\int_{(\mathbb{C},0)} \frac{1}{\varepsilon^2+|z|^2}\mathrm{d}\lambda$. In particular, we see that the bound (1.6.1) cannot be improved by replacing the weight $e^{-\varphi_L}$ with $e^{-(1-\delta_0)\varphi_L}$, for any positive δ_0.

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