RESPONSE

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Response to Comment on "On the Evolution Equations of Nonlinearly Permissible, Coherent Hole Structures Propagating Persistently in Collisionless Plasmas" [*Ann. Phys. (Berlin)* 2023, 2300102]

Hans Schamel and Nikhil Chakrabarti*

Our critical analysis of Hutchinson's work, expressed in our recent article in the *Annalen der Physik*, which goes beyond mutual misunderstandings and misrepresentations, is also maintained in light of his Comment. The main reason for the limited yield in structural description is the author's adherence to the BGK method and the associated lack of a nonlinear dispersion relation (NDR). Even with an additional asymptotic constraint, as a supposed replacement for the NDR, the author's theory remains inferior to the pseudo-potential method.

First, to support our statement in ref. [1] that Hutchinson in his review[2] has not adequately appreciated the differences between the BGK method[3] and the pseudo-potential method in Schamel's version,[4,5] we need to delve a little deeper into the problem.

To put the topic of a stationary traveling wave structure $\phi(x - v_0 t)$ in perspective, we first note that it consists of two parts, the determination of the electrostatic structure $\phi(x)$ and the phase velocity v_0, both representing equally important issues. Since Hutchinson apparently did not fully accept (explore) this fact in his work and in his comment[6] on our work,[1] we feel compelled to reconsider and defend the criticism made in our paper.

The following can be said about this.

It is true, as he mentioned in ref. [2], that Bernstein, Greene, and Kruskal[3] also knew about the pseudo-potential method ("differential equation") in principle, but did not pursue it further in favor of the integral method ("integral equation"—BGK method). The main reason was that they were only interested in standing structures ($v_0 = 0$).

For traveling structures $v_0 \neq 0$, it was one of the two current authors (HS) who made the generalization in a work[7,8] that fell into oblivion or at least was not considered by Hutchinson in his review.[2]

Through ref. [7], it became immediately clear that restrictions with regard to the various plasma parameters are necessary in order to achieve physically meaningful distributions, especially in the finite amplitude range. These are not provided by the BGK method on an ad hoc basis as it essentially functions as a black box with an uncertain, unpredictable outcome. The main reason is that in the BGK method, $\phi(x), v_0$ are prescribed rather than derived quantities.

The shortcomings of the BGK method can be in more detail listed as follows:

1) a nonlinear dispersion relation (NDR), which is required to decide on the microscopic quality of the solution, is missing, that is, BGK does not provide a method for determining v_0;
2) one has no prior control over the resulting distribution of the trapped particles f_{et} or f_{it}, respectively, and if the latter is found inappropriate, one has no means of properly correcting it;
3) although $\phi(x)$ can be arbitrary, it requires an analytic expression to run. This restricts the manifold of structures appreciably as no mathematically undisclosed potentials are included, which by far represent the majority of structures,[5]
4) one has no way to find the necessary interrelations among the various free parameters to make the final results physically meaningful.

To make these issues more transparent, we consider the so-called priviledged solitary electron hole solution (SEH) of ref. [9], which is generally acknowledged and obtained by Schamel's pseudo-potential method. This solution is privileged because it is associated with the smoothest possible f_{et} and allows a transition to the infinitesimal amplitude regime.

It assumes the small amplitude limit $0 < \psi << 1$ and reads:

$$\phi(x) = \psi \ \mathrm{sech}^4\left(\frac{\sqrt{B_e}x}{4}\right) \tag{1}$$

H. Schamel
Physikalisches Institut
Universität Bayreuth
D-95440 Bayreuth, Germany
E-mail: hans.schamel@uni-bayreuth.de
N. Chakrabarti
Saha Institute of Nuclear Physics
1/AF Bidhannagar, Calcutta 700064, India

The ORCID identification number(s) for the author(s) of this article can be found under https://doi.org/10.1002/andp.202300441

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where

$$B_e := \frac{16}{15\sqrt{\pi}}(1-\beta-v_0^2)e^{-\frac{v_0^2}{2}}\sqrt{\psi} \tag{2}$$

$$v_0 = 1.307(1-B_e) \tag{3}$$

Equation (3) represents the slow electron acoustic mode, and it holds $0 < B_e << 1$ or $\beta < -0.71$.

From the BGK point of view, it is realistically impossible to get this $\varphi(\xi) := \phi(\xi)/\psi = \text{sech}^4(\xi)$ solution with $\xi = x/l$ without additional information from the pseudo-potential method. There is an infinite variety of bell-shaped functions, and it is quite unlikely to correctly guess sech(ξ) and its power 4. (By the way the often chosen Gaussian profile $\varphi(\xi) = e^{-\xi^2}$ is less preferable for an appropriate $\varphi(\xi)$ in the sense that it provides no solution valid for $\psi \to 0^+$ and that it has in addition a strongly singular slope behavior of f_{et} at the separatrix [see also ref. [10]]). Even if the scale length l can be identified with B_e through $l := 4/\sqrt{B_e}$, there is no way to guess correctly the relation (3) between v_0 and l, respectivelyB_e, without prior knowledge from the pseudo-potential method or NDR, respectively. From the perspective of the BGK method, approaching this solution would require an endless trial and error process.

The phase velocity v_0 in (3) solves a NDR, which is an unknown relation within BGK. Hutchinson even goes further denying the need of a NDR, which he calls eigensolution, and considers it to be a weakness rather than a strength of the pseudo-potential method. He also ignores that an NDR offers the simplest possibility to determine the constraints for the existence of solutions like $\theta := T_e/T_i > 3.5$ for ion-acoustic structures (see later).

The limitations of the BGK method prompted HS to turn the tables and look for solutions with physically justifiable approaches to the distribution of the trapped particles. By switching to the pseudo-potential method, which HS carried out in ref. [4] and is for convenience abbreviated as the SPP method, the second part of the theory, the determination of v_0, received the attention it deserved, in that a NDR was derived as a central element of the theory.

This equation arises as a necessary consistence condition by demanding that the slope of $\phi(x)$ vanishes at the maximum (minimum) for a solitary electron (ion) hole, an obvious condition. It is neither imposed as an additional constraint (or eigensolution) nor does it need the consideration of the asymptotic form of $\phi(x)$, nor is it necessary to insure that the trapping boundary energy lies at the assumed value, as Hutchinson continues to claim. It simply provides consistency under the assumed parameters. No further assumptions are needed or used. With the SPP method one can establish, by choosing appropriate trapping scenarios, the trapped particle distributions of desired regularity and boundary behavior, control their behavior in terms of trapping scenarios, and obtain the necessary relation among the various plasma parameters to guarantee a consistent physical solution even in cases when $\phi(x)$ can no longer be expressed by known functions.

Thanks to the focus on this theory over the last 50 years, as documented in ref. [5], HS was able to derive new, generally valid statements. These include

(1) the failure of linear Vlasov theories in the context of stationary coherent structures (no Landau behavior of harmonic waves, rather smooth continuous spectra instead of van Kampen's singular spectra),
(2) the fact that there are potentials $\phi(x)$, which cannot be expressed in terms of mathematically known functions, a central assumption in BGK theory, and
(3) the nonlinear proof of the resonance broadening caused by the Γ trapping effect.

In one point we agree with him, namely that the existence condition for solitary ion holes is $\theta < 3.5$ and not $\theta > 3.5$. Our proof in ref. [1] was presented for long-wave length ($k^2 << 1$) ion-acoustic structures (and not for linear ion-acoustic waves). For structures with arbitrary k^2, the constraint reads: $\theta > 3.5(1+k^2)$.

For ion holes, the NDR has to be evaluated differently. For solitary ion holes with $u_0 = 1.307, k_0^2 = 0, q = 0, B_e = 0$ (see Section 4.5 of ref. [1]), the NDR becomes

$$1 - \frac{\theta}{2}Z_r'\left(\frac{u_0}{\sqrt{2}}\right) = \frac{3}{2}\theta^{3/2}B_i > 0 \tag{4}$$

from which follows the inequality

$$-\frac{1}{2}Z_r'\left(\frac{u_0}{\sqrt{2}}\right) > -\frac{1}{\theta} \tag{5}$$

Taking the minimum value of the left side of Equation (5), which is -0.285, we finally get $0.285 < \frac{1}{\theta}$ or $\theta < 3.5$. This corrects $\theta > 3.5$ of refs. [11–13]. We thank Hutchinson for pointing out this error to us. It was first detected by authors of ref. [14] and has been recently confirmed experimentally by the authors of ref.[15]. We agree that sufficiently hot ions are required for ion holes to exist.

A further selection of controversial points as follows:

a) Continuity of distributions:
With respect to his discussion of (2) in his comment,[6] which involves the NDR (13) of ref. [5] for SEHs, we point out that $\Gamma_e = 0$ is not needed to establish continuity of $f_e(x, v)$ at the separatrix $\epsilon = 0$ as (1) of ref. [5] shows, contradicting his requirement $\Gamma_e = 0$. The reason is that $\Gamma_e\sqrt{-\epsilon}$ vanishes for every finite Γ_e as ϵ tends to zero, giving continuity. However, as indicated in refs. [16, 17], discontinuous distributions in the strict collisionless limit should not be completely neglected.

b) Trapping scenarios:
The concept of trapping scenarios does not appear in his vocabulary. However, this is an important aspect as it sheds light on the effects of the trapping channels opened in the time-dependent formation process. They arise during particle trapping via stochastic processes in the resonance range, destroy any relationship with initial conditions (Cauchy's initial value problem is no longer well-posed), and hence let us forget any linear Vlasov connections. Depending on the variety of trapping scenarios, different forms of $\phi(x)$ and v_0 can be obtained, as detailed in ref. [5]. Incidentally, it is not true that Schamel and co-workers specify a negative β, as claimed in footnote 22 in ref. [6]. Correct is that $\beta < 0$ automatically results as a necessary existence condition if only the β trapping scenario is at work. (As an aside, we add that Gurevich's

$\beta = 0$ distribution[18] consequently fails to describe this SEH solution.)

c) Positively and negatively polarized coupled electron and ion holes of ultra-slow velocity:

This case generally belongs to the class of undisclosed structures, since only in limiting cases can the form of $\phi(x)$ be extracted. It can be handled by four trapping scenarios $(\Gamma_e, \Gamma_i, B_e, B_i)$ and has been considered in Section 7.2 of ref. [5]. The pseudo-potential is given by (52) of ref. [5] and includes for $k_0^2 = 0$ as a special case the extension of the privileged $sech^4$ SEH by the α-trapping scenario, provided that it holds $B_e > \theta^{3/2} B_i$.

If one chooses instead of $k_0^2 = 0$ a k_0^2 satisfying $2k_0^2 + B_e - \theta^{3/2} B_i = 0$, then the corresponding extension of the familiar negatively polarized ion hole, extended by the β-electron trapping effect, is obtained, provided that it holds $\theta^{3/2} B_i > B_e$, which is just the opposite inequality. If k_0^2 is different from these two values, a periodic cnoidal wave is obtained and a more detailed discussion is required, which has not yet been carried out but may be of interest to future generations.

What is of interest in this context is the phase velocity of these structures, as this lies beyond a BGK analysis. Being interested in ultra-slow structures, we set $v_0 = 0 = u_0$ and get with $-\frac{1}{2} Z_r'(0) = 1$ from the NDR (51) of ref. [5], extended by the Γ_e and Γ_i scenario effect, the following NDR for a positively polarized coupled hole solution in a current-carrying plasma with finite drift velocity v_D

$$-\frac{1}{2} Z_r'\left(\frac{v_D}{\sqrt{2}}\right) + \theta + \Gamma_e + \Gamma_i = B_e + \frac{3}{2} \theta^{\frac{3}{2}} B_i \tag{6}$$

or

$$-\frac{1}{2} Z_r'\left(\frac{v_D}{\sqrt{2}}\right) + \theta + \frac{\sqrt{\pi}}{2}\left(e^{-\frac{v_D^2}{2}} \gamma_e + \gamma_i\right) = \frac{16\sqrt{\psi}}{15\sqrt{\pi}}\left[\frac{3}{2}(1-\beta) e^{-\frac{v_D^2}{2}} + \theta^{\frac{3}{2}}(1-\alpha)\right] \tag{7}$$

The necessary constraint becomes $\frac{3}{2}(1-\beta) e^{-v_D^2/2} > \theta^{3/2}(\alpha - 1)$. It is clear that Equation (7) allows a wide class of solutions in the 6D parameter space spanned by the parameters $(v_D, \theta, \alpha, \beta, \gamma_e, \gamma_i)$, albeit only four trapping scenarios have been assumed. This immense class of solutions is beyond the scope of the BGK method.

In the negatively polarized coupled hole case, the right-hand side of Equation (6) has to be replaced by $\frac{3}{2} B_e + \theta^{3/2} B_i$.

It is rather obvious that such a necessary connection between the various free parameters cannot be achieved by specifying a $\phi(x)$, especially if this belongs to the class of mathematically undisclosed functions.

These solutions and their necessary constraints apply to single-hump background distributions for electrons and ions. From this, we can immediately conclude that double-hump ion distributions are not necessary for their existence, so we do not even need to discuss their existence, their negative energy states associated with them, and their self-acceleration.

To the best of our knowledge, double-hump ion distributions have not yet been used to search for ion hole solutions. How can he then conclude that ion holes prefer to locate between the humps?

d) Stability and self-acceleration:

The stability of solitary holes is, strictly speaking, a mathematically unsolved problem[5] (including the transverse instability,[19–22] introduced by HS in ref. [20]). No reliable statement can hence be made about the transient behavior of holes such as self-acceleration. The background to self-acceleration is a hole's tendency to reach a lower, typically negative energy state (see Section 9 of ref. [5] and ref. [23]. The equilibrium hole theory can describe the initial and final state, but not the transition between them. By applying the NDR,[23] it could be shown that the final transition from the slower to the faster equilibrium state involves a jump from the negative to the positive slope branch of the $-\frac{1}{2} Z_r'(v_0/\sqrt{2})$ function, that is, from the slow electron acoustic to the Langmuir branch.

This acceleration mechanism is supported by the numerical simulations of ref. [23], which showed that the finally settled hole velocity is in a wide range independent of the initial hole velocity.

e) Cnoidal waves:

Regarding his discomfort with finite wavelength cnoidal waves, we can say that they only represent problems of a physical nature when viewed as idealized solutions. (Mathematically it is the curvature of $\phi(x)$ at the potential minimum, i.e., the normalization at that point $(-\phi''(0) = n_e(0) - n_i(0))$ that determines the wavelength of the structure.) However, as localized wavelets with a finite number of humps of varying amplitudes decreasing from the center, they have legitimacy. To ensure that each hump propagates at the same speed v_0, its trapping scenario must be adjusted accordingly. For further details, for example, their occurrence or group speed, we refer to ref. [24].

Similar physical caveats could be made for SEHs of negative polarity, for example that the trapping regions extend to infinity. However, embedded in a favorable environment of intermittent plasma, they are quite justified, as their observation indicates.[1]

To test the strength of his theory without reference to the concept of a cnoidal wave, he might try to independently derive our solution for a SEH of negative polarity.[1] We are sorry to say he will fail.

f) Asymptotic constraint versus NDR:

Since he is obviously aware that something is missing in the BGK theory to obtain a complete solution, he tries to add an additional restriction through asymptotic analysis to overcome this disadvantage. His attempt refers to the special case of a regular, continuous, monotonic $f_{et}(-\epsilon)$, which is represented in the SPP theory by the specific B_e trapping scenario. By applying an asymptotic $e^{-|x|/\lambda}$ ansatz, assuming correct execution, he obtains an equation that is identical to the corresponding NDR of the SPP theory except for k_0^2 and Γ_e. He hence reproduces the SPP result for this special case but does not take periodic structures into account and fails to achieve a broader v_0 due to the missing Γ_e trapping scenario. Moreover, all other trapping scenarios are ignored by him. He, for example, misses the often used Gaussian $e^{-(x/\lambda)^2}$ solitary structure, which is connected with the $\sqrt{-\epsilon}\ln(-\epsilon)$ trapping scenario, or the second-order Gaussian $e^{-\sinh^2(x/\lambda)}$ solitary mode, which is related to the $\sqrt{-\epsilon}\ln^2(-\epsilon)$ trapping scenario.

Our central statement remains hence valid: The BGK method is incomplete in capturing the diversity of electrostatic structures, which can only be achieved by the SPP method by providing $\mathcal{V}(\phi)$ and NDR as necessary tools. For further information, in particular on the numerical proof of a two-parametric (γ_e, β) SEH in which the electron density is centrally depressed and in which f_{et} possess shoulders near the separatrices, we refer to refs.[5, 10, 25–27].

In summary, our criticism of Hutchinson's work is primarily due to the fact that he used the wrong method with the BGK method and thus set the wrong course. However, he is not alone with this, because anyone who believes that the linear Vlasov description is a valid approximation for describing coherent structures is also on a wrong course. Although the BGK theory includes the trapping aspect at full strength, the horse is "bridled from behind." With its macroscopic specifications, one cannot delve deeply enough into the world of microscopic physics. Only by using the pseudo-potential method, and thus the concept of trapping scenarios, can a deeper insight into structure formation including its chaotic parts be achieved.

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Conflict of Interest

The authors declare no conflict of interest

Keywords

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