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Covering families of triangles

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Abstract

A cover for a family $\mathcal{F}$ of sets in the plane is a set into which every set in $\mathcal{F}$ can be isometrically moved. We are interested in the convex cover of smallest area for a given family of triangles. Park and Cheong conjectured that any family of triangles of bounded diameter has a smallest convex cover that is itself a triangle. The conjecture is equivalent to the claim that for every convex set $\mathcal{X}$ there is a triangle Z whose area is not larger than the area of $\mathcal{X}$, such that Z covers the family of triangles contained in $\mathcal{X}$. We prove this claim for the case where a diameter of $\mathcal{X}$ lies on its boundary. We also give a complete characterization of the smallest convex cover for the family of triangles contained in a half-disk, and for the family of triangles contained in a square. In both cases, this cover is a triangle.

Keywords Triangles · Smallest area · Universal cover · Convex cover · Crescent · Half-disk · Square

1 Introduction

A *cover* for a family $\mathcal{F}$ of sets in the plane is a set into which every set in $\mathcal{F}$ can be isometrically moved. We call a cover *smallest* if it has smallest possible area. Smallest convex covers have been studied for various families of planar shapes. In 1914, Lebesgue asked for the smallest convex cover for the family of all sets of diameter one. The problem is still open, with the

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best known upper bound on the area being 0.845 [1, 5] and the best known lower bound being 0.832 [3]. Moser's worm problem asks for the smallest convex cover for the family of all curves of length one, with the best known upper bound of 0.271 [9, 11] and the best known lower bound of 0.232 [6]. More variants can be found in [2, 10].

These problems appear to be hard because we do not even have a conjecture on the shape of a smallest convex cover. The best lower bound for Lebesgue's problem, for instance, is based on an approximation to the optimal placement of a circle, a triangle, and a pentagon obtained through an exhaustive computer search [3].

While smallest convex covers have remained elusive for most families, we have a complete answer for some *families of triangles*. Kovalev showed that the smallest convex cover for the family of all triangles of perimeter one is a uniquely determined triangle [4, 7]. Füredi and Wetzel showed that the same holds for the family of all triangles of diameter one [4], and Park and Cheong showed the same for the family of triangles of circumradius one, as well as for any family of two triangles [8]. These known results led Park and Cheong to make the following conjecture:

Conjecture 1 [8] For any bounded family $\mathcal{T}$ of triangles there is a triangle Z that is a smallest convex cover for $\mathcal{T}$.

It is easy to see that this is equivalent to the following conjecture:

Conjecture 2 [8] Let $\mathcal{X}$ be a convex set. Then there is a triangle Z whose area is at most the area of $\mathcal{X}$, such that Z is a convex cover for the family of triangles contained in $\mathcal{X}$.

In this paper, we add to the existing evidence motivating these conjectures. In particular, we prove that Conjecture 1 is true for the family of triangles contained in a given half-disk, and for the family of triangles contained in a given square. The half-disk result is a rather easy warm-up exercise, proven in Sect. 5; see Fig. 1(left).

Theorem 1 *The triangle with sides $\sqrt{2}$, $1 + \sqrt{2}$, and $\sqrt{3}$ is a smallest convex cover for the family of triangles contained in the half-disk of radius one.*

The family of triangles contained in the unit square turns out to be much harder. Intriguingly, there is a "nice" triangle $\mathcal{C}^\star$ with angles $60°$, $75°$, and $45°$ and a longest edge of length $\sqrt{2}$ that covers "most" triangles contained in the unit square. However, some skinny triangles—the worst case being the isosceles triangle with apex angle $\approx 5.6°$—do not fit into $\mathcal{C}^\star$, and the actual smallest convex cover is a triangle $\mathcal{C}$ whose longest edge has length about $\sqrt{2} + 0.005$. We prove that $\mathcal{C}$ indeed covers all triangles contained in the unit square in Sect. 6; see Fig. 1(right).

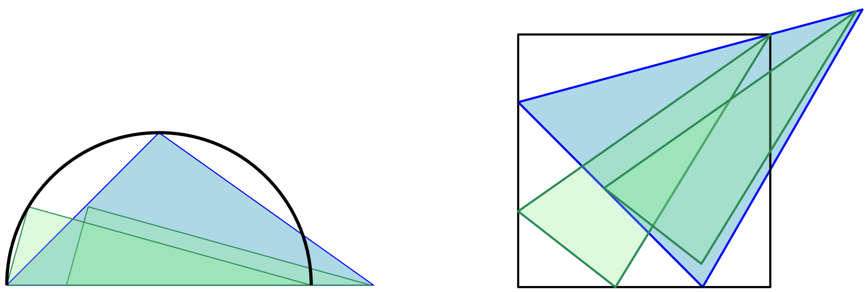

Fig. 1 The smallest convex covers (blue triangles) for triangles in a half-disk or in a square

Theorem 2 *The unique smallest convex cover for the family of triangles contained in the unit square is the triangle $\triangle XYZ$ with $\measuredangle XZY = \frac{\pi}{3}$, $|ZY| = \frac{1}{\cos\frac{\pi}{12}}$, and*

$$|XZ| = \frac{\sin(\frac{\pi}{3} + 2\theta_0)}{\cos\left(\frac{\pi}{4} - \theta_0\right)\sin\left(\frac{\pi}{3}\right)} \approx 1.4195,$$
$$\text{where } \theta_0 = \tan^{-1}\left(\frac{1}{\sqrt[3]{2+\sqrt{3}}}\right) - \frac{\pi}{6} \approx 0.049 \approx 2.81°.$$

In our second main result, we consider Conjecture 2. It is known to hold when $\mathcal{X}$ is a disk [8], a half-disk (Theorem 1), or a square (Theorem 2). In Sect. 3, we prove the following theorem, which extends this to a much larger family of shapes $\mathcal{X}$:

Theorem 3 *Let $\mathcal{X}$ be a crescent, that is, a convex set that contains a diameter on its boundary. Then there is a triangle Z whose area is at most the area of $\mathcal{X}$, such that Z is a convex cover for the family of triangles contained in $\mathcal{X}$.*

Note that we do not claim that the triangle Z is a smallest cover for the family of triangles contained in $\mathcal{X}$. For instance, a half-disk is a crescent, but the triangle Z constructed in the proof of Theorem 3 is larger than the optimal triangle cover of Theorem 1. While proving Conjecture 2 would imply Conjecture 1, the special case of Theorem 3 does therefore not seem to imply any special case of Conjecture 1. In particular, it does not allow us to claim that the family of triangles contained in a given crescent has a triangular smallest convex cover.

The proofs of the three theorems are independent, we start with Theorem 3.

2 Notation

For three points $A, B, C \in \mathbb{R}^2$, we let AB denote the line through A and B, let $\overline{AB}$ denote the line segment connecting A and B, and let $\triangle ABC$ denote the triangle ABC. When AB is not horizontal, then we let $\mathcal{H}_{AB}$ denote the horizontal strip bounded by the horizontal lines through A and through B. For a point $P \in \mathcal{H}_{AB}$, we define $\zeta_{AB}(P)$ as the *horizontal distance* between P and the line AB. Formally, $\zeta_{AB}(P) = |PX|$, where X is the intersection point of AB with the horizontal line through P.

For a point P and a distance $t \geq 0$, we define points $P \ominus t = P - (t, 0)$ and $P \oplus t = P + (t, 0)$. In other words, $P \ominus t$ and $P \oplus t$ lie on the horizontal line through P at distance t to the left and to the right of P.

We say that a triangle T *fits into* a convex planar set $\mathcal{X}$ if there is a triangle $T' \subset \mathcal{X}$ such that T and T' are congruent, that is, T' is the image of T under a combination of translations, rotations, and reflections. We say that T *maximally fits into* $\mathcal{X}$ if T fits into $\mathcal{X}$, but there is no triangle $T' \supsetneq T$ that fits into $\mathcal{X}$.

We define a *crescent* to be a convex shape whose diameter lies on its boundary. Any triangle is itself a crescent. For a convex planar set $\mathcal{X}$, let $|\mathcal{X}|$ denote the area of $\mathcal{X}$.

3 Every crescent has a triangular cover

We start by describing how to construct a triangular cover for the family of all triangles in a given crescent. See Fig. 2 for illustration. Let $\mathcal{X}$ be a crescent with diameter $AB \subset \partial\mathcal{X}$.

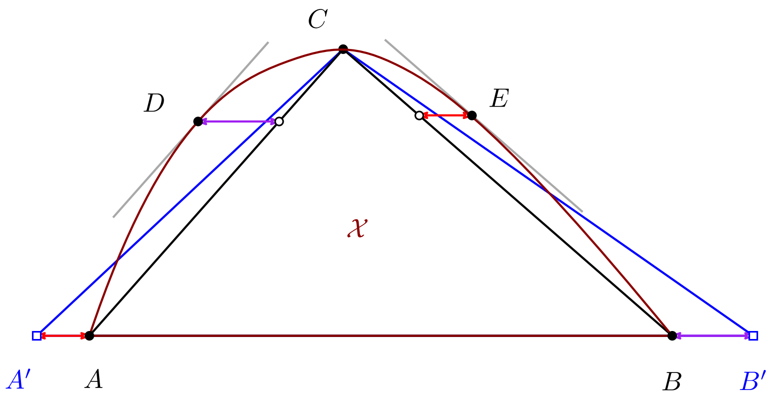

Fig. 2 Construction of a triangular cover from a crescent

We assume that AB is horizontal and A is to the left of B. Let C be a highest point on $\partial\mathcal{X}$, that is, a point maximizing the distance from AB, let D be a point on the curve $AC \subset \partial\mathcal{X}$ maximizing the horizontal distance from AC, and let E be a point on the curve $BC \subset \partial\mathcal{X}$ maximizing the horizontal distance from BC. In other words, $\mathcal{X}$ has a horizontal tangent in C, a tangent parallel to AC in D, and a tangent parallel to BC in E. Let $A' = A \ominus \zeta_{BC}(E)$ and $B' = B \oplus \zeta_{AC}(D)$. We claim that $\triangle A'B'C$ is indeed a cover for the set of triangles in $\mathcal{X}$, and that $|\triangle A'B'C| \leq |\mathcal{X}|$.

Theorem 4 *If a triangle fits into the crescent $\mathcal{X}$, then it fits into the triangle $\triangle A'B'C$.*

Before we prove Theorem 4, we show how it implies the result stated in the introduction.

Proof of Theorem 3 It suffices to observe that for the triangle $Z = \triangle A'B'C$ constructed in Theorem 4 we have $|Z| \leq |\mathcal{X}|$ since

$$Z = \triangle A'AC \cup \triangle ABC \cup \triangle BB'C,$$
$$\mathcal{X} \supset \triangle ADC \cup \triangle ABC \cup \triangle BEC,$$

and $|\triangle A'AC| = |\triangle BEC|$ and $|\triangle BB'C| = |\triangle ADC|$. □

To prove Theorem 4, we first need a few lemmas. The first one characterizes triangles that maximally fit into a crescent.

Lemma 1 *Let $\mathcal{X}$ be a crescent with horizontal diameter $\overline{AB}$, A left of B, contained in the upper halfplane bounded by AB. If a triangle $\triangle PQR$ fits maximally into $\mathcal{X}$, then it is of one of the following three forms:*

(i) $P = A$ and $Q = B$, and $R \in \partial\mathcal{X} \setminus \overline{AB}$;
(ii) $P = A$ and R, $Q \in \partial\mathcal{X} \setminus \overline{AB}$, with R to the left of and strictly above Q;
(iii) $P = B$ and R, $Q \in \partial\mathcal{X} \setminus \overline{AB}$, with R to the left of and strictly below Q.

Proof Since $\triangle PQR$ maximally fits into $\mathcal{X}$, we can assume that P, Q, R all lie on the boundary $\partial\mathcal{X}$. If no vertex lies on $\overline{AB}$, we can translate the triangle downwards until it touches $\overline{AB}$, so we can assume that $P \in \overline{AB}$. If $Q \in \overline{AB}$, then $\triangle PQR \subset \triangle ABR$, so the maximality implies that $\triangle PQR = \triangle ABR$ and we are in case (i). It remains to consider the case where

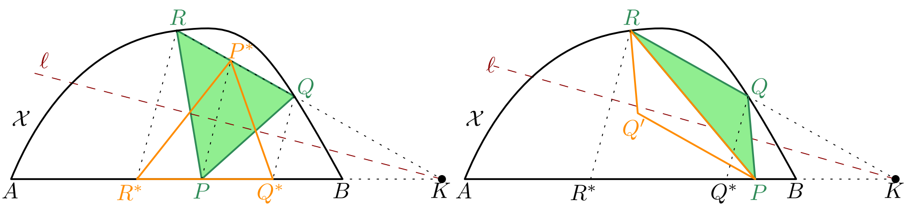

Fig. 3 Proof of Lemma 1

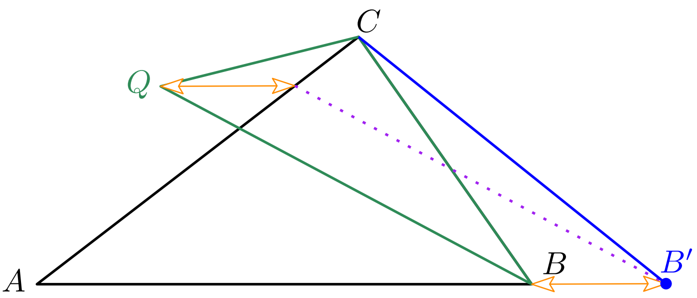

Fig. 4 Lemma 2

$P \in \overline{AB}$, while Q and R lie on the upper chain $\partial\mathcal{X} \setminus \overline{AB}$, so we can assume that R lies to the left of Q.

Let us first assume that R lies above Q. Let K be the intersection point of AB and RQ, and let ℓ be the bisector of the angle $\angle AKR$; see Fig. 3. We reflect the points R and Q about ℓ to obtain points R^* and Q^* on the line AB. Since $|KR^*| = |KR| < |KB| + |BR| \leq |KB| + |BA| = |KA|$, we have $R^* \in \overline{AB}$ but is not equal to A. We also note that Q^* lies strictly between R^* and B and thus $Q^* \in \overline{AB}$.

If P lies between R^* and Q^*, then we can reflect it about ℓ to obtain a point P^* on the segment RQ so that $\triangle P^*Q^*R^*$ is congruent to $\triangle PQR$; see Fig. 3(left). Since $\triangle P^*Q^*R^* \subsetneq \triangle ABP^*$, it does not maximally fit into $\mathcal{X}$.

If P lies to the left of R^* but is not equal to A, then we can slightly rotate $\triangle PQR$ clockwise around R. This moves Q and P into the interior of $\mathcal{X}$, so $\triangle PQR$ does not maximally fit into $\mathcal{X}$.

If P lies to the right of Q^*, then we rotate Q by $180°$ about the midpoint of $\overline{PR}$ to obtain Q', see Fig. 3(right). The quadrilateral $PQRQ'$ is a parallelogram, and $\triangle PRQ'$ is congruent to $\triangle PQR$. Then $Q' \in \triangle PRR^*$ since R above Q implies Q' above P and P right of QQ^* implies Q' right of RR^*. Since $\triangle PRQ' \subsetneq \triangle APR$, $\triangle PQR$ does not maximally fit into $\mathcal{X}$.

It follows that whenever R lies above Q, then $P = A$ and we are in case (ii). By symmetry, whenever R lies below Q, then $P = B$ and we are in case (iii).

Finally, when RQ is horizontal, we let ℓ be the horizontal line equidistant from AB and RQ. Again we mirror R and Q about ℓ to obtain R^* and Q^* on $\overline{AB}$. The arguments above apply literally, and we conclude that $P = A$. By symmetry, however, we can also conclude that $P = B$, a contradiction. It follows that when RQ is horizontal, then $\triangle PQR$ does not maximally fit into $\mathcal{X}$. □

We now state two lemmas, postponing their proofs to Sect. 4.

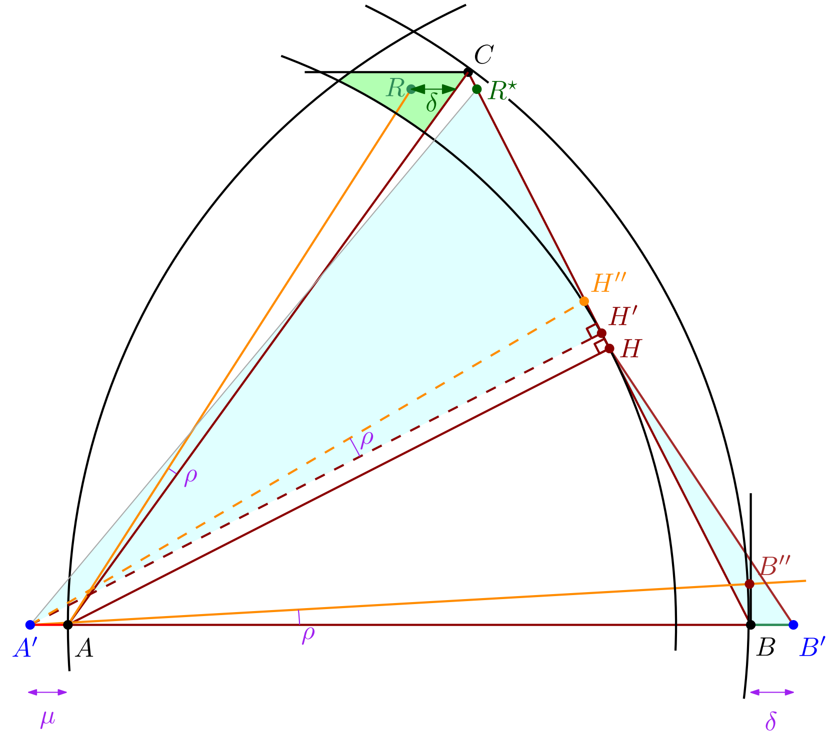

Fig. 5 Lemma 3

Lemma 2 *Let $\triangle ABC$ be a triangle with the longest edge $|AB|$. Let AB be horizontal, A left of B, and let $Q \in \mathcal{H}_{AC}$ lie to the left of AC with $|QB| \le |AB|$. Then $\triangle BCQ$ fits into $\triangle AB'C$, where $B' = B \oplus \zeta_{AC}(Q)$ (see Fig. 4).*

Lemma 3 *Let $\triangle ABC$ be an isosceles triangle with $|AB| = |AC|$. Let AB be horizontal, A left of B, let $\overline{AH}$ be the height of $\triangle ABC$ with respect to $\overline{BC}$, and let $R \in \mathcal{H}_{AC}$ lie to the left of AC with $|AR| \ge |AH|$ and $|BR| \le |AB|$ (that is, R lies in the green area of Fig. 5). Let $A' = A \ominus \mu$ for some $\mu \ge 0$ such that $\frac{|A'B|}{|AB|} \le \frac{|AB|}{|AH|}$, let $B' = B \oplus \zeta_{AC}(R)$, let H' be the orthogonal projection of A' on BC, and let $R^\star$ be the horizontal projection of R on BC. We rotate B and H' around A' by angle $\measuredangle CAR$, obtaining points B'' and H'', respectively. Then B'' lies in $\triangle BB'H$ and $H'' \in \triangle A'H'R^\star$.*

Proof of Theorem 4 It suffices to prove the statement for an arbitrary triangle $\triangle PQR$ that maximally fits into $\mathcal{X}$. By Lemma 1, this implies that $\triangle PQR$ is of one of the three types in the lemma.

Case $\triangle PQR$ of type(i)

If $\triangle PQR$ is of type (i) with $P = A$ and $Q = B$, then, depending on the location of R, we translate it leftwards by $\zeta_{BC}(R)$ or rightwards by $\zeta_{AB}(R)$ to place it in $\triangle A'BC$ or $\triangle AB'C$, which are both included in $\triangle A'B'C$.

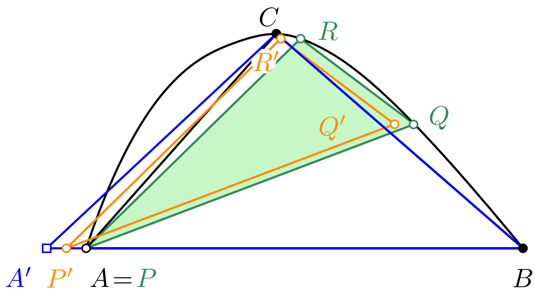

Fig. 6 Proof of Theorem 4: R not left of C

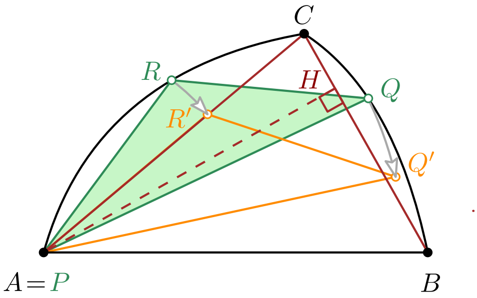

Fig. 7 Proof of Theorem 4: Q below AH

Types (ii) and (iii) are symmetric, so we break the symmetry and assume that $P = A$ and R lies to the left and above Q.

Case *R* not to the left of *C*

If R does not lie to the left of C, we translate $\triangle PQR$ leftwards by $\zeta_{BC}(R)$ to obtain a triangle $\triangle P'Q'R'$ with R' on $\overline{BC}$; see Fig. 6. If $Q' \in \triangle ABC$, we are done since $\triangle P'Q'R' \subset \triangle A'BC$. Otherwise we apply Lemma 2 (after symmetry) to the triangle $\triangle BP'R'$ and the point Q' and obtain that $\triangle PQR$ fits in the triangle $\triangle(P' \ominus \zeta_{BC}(Q'))BR' \subset \triangle A'BC$.

Now we are left with the case where R lies strictly to the left of C. Since Q lies below R, Q lies on the right chain from B up to but not including C. Let H be the foot of the height of $\triangle ABC$ with respect to $\overline{BC}$ and let B^* be the mirror image of B about AH.

Case *Q* below *AH*

If $\measuredangle BAQ \leq \measuredangle BAH$, then we rotate $\triangle PQR$ clockwise around A. During the rotation, both $\zeta_{AC}(R)$ and $\zeta_{BC}(Q)$ are decreasing. We continue the rotation to obtain $\triangle PQ'R'$ until either $\zeta_{AC}(R') = 0$ or $Q' \in \overline{AB}$; see Fig. 7. If $\zeta_{AC}(R') > 0$, then $Q' \in \overline{AB}$. We then translate the triangle rightwards by $\zeta_{AC}(R') < |BB'|$ to place it in $\triangle AB'C$. Otherwise, $R' \in \overline{AC}$. Since $\triangle AR'Q' \subset \triangle ACQ'$ and $\zeta_{BC}(Q') < |AA'|$, we apply Lemma 2 to $\triangle ABC$ and Q' to conclude that $\triangle AR'Q'$ fits in $\triangle A'BC$.

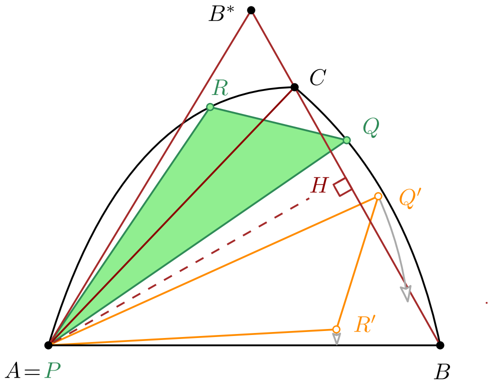

Fig. 8 Proof of Theorem 4: $R \in \triangle AHB^*$

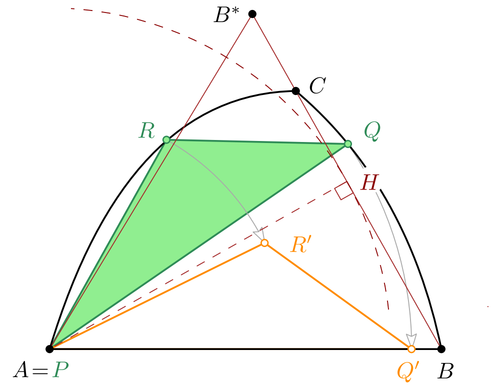

Fig. 9 Proof of Theorem 4: $|AR| \le |AH|$

Case $R \in \triangle AHB^*$.

We now have that $\measuredangle BAQ > \measuredangle BAH$. If $R \in \triangle AHB^*$, then we mirror $\triangle PQR$ about AH to obtain a new triangle $\triangle PR'Q'$ with $R' \in \triangle ABH \subset \triangle ABC$ and $\zeta_{BC}(Q') = \zeta_{BC}(Q)$; see Fig. 8. We can then rotate the triangle clockwise. The rotation decreases $\zeta_{BC}(Q')$. We stop when either $Q' \in \overline{BC}$ or $R' \in \overline{AB}$ and denote by $\triangle PQ''R''$ the triangle in the new position. In the first case, we have $\triangle PR''Q'' \subset \triangle ABC$, in the second case we can translate leftwards by $\zeta_{BC}(Q'') < \zeta_{BC}(Q) \le |AA'|$ to place the triangle in $\triangle A'BC$.

Case $|AR| \le |AH|$

Consider now the case where $|AR| \le |AH|$. We can rotate $\triangle PQR$ clockwise around A to obtain a new triangle $\triangle PQ'R'$ with $Q' \in AB$; see Fig. 9. Since R' lies in the interior of $\mathcal{X}$, the triangle $\triangle PQ'R'$ does not maximally fit into $\mathcal{X}$.

Final case

We are now left with the final case where $R \in \mathcal{H}_{AC} \subset \mathcal{H}_{AB^*}$ lying to the left of AB^*, $|AR| \ge |AH|$, $|BR| \le |AB|$, $\measuredangle BAQ > \measuredangle BAH$, and $Q \in \mathcal{H}_{AC} \subset \mathcal{H}_{AB^*}$ lying to the right of BC; see Fig. 10. We let $\rho = \measuredangle B^*AR$. We first mirror $\triangle PQR$ about AH to obtain $\triangle PR'Q'$,

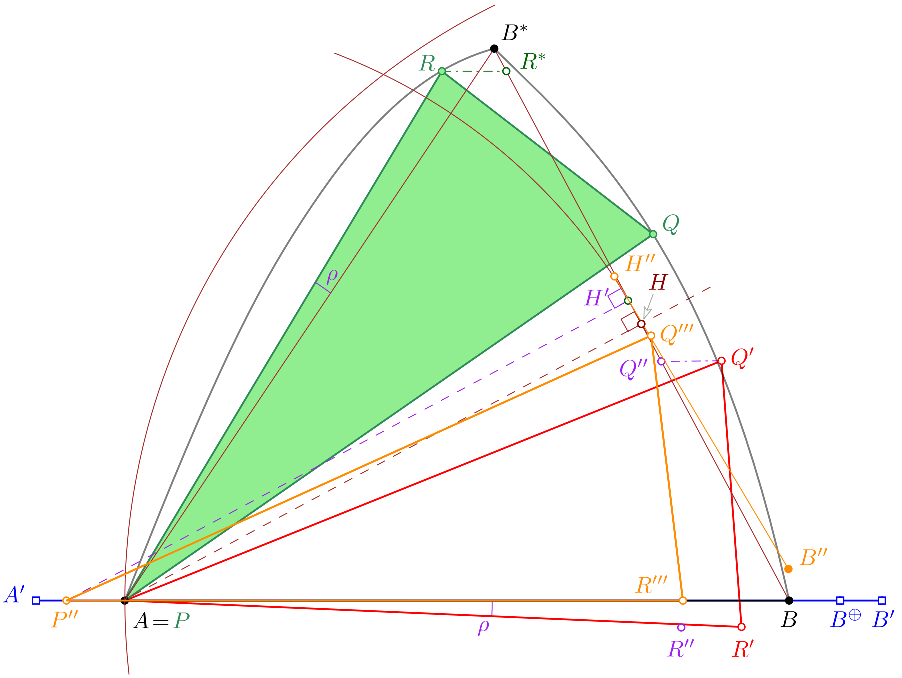

Fig. 10 Proof of Theorem 4: the final case

with R' below AB, $\measuredangle R'AB = \rho$, and $\zeta_{BC}(Q') = \zeta_{BC}(Q)$. We then translate $\triangle PR'Q'$ leftwards by $\zeta_{BC}(Q')$ to obtain $\triangle P''Q''R''$ and finally rotate counter-clockwise about P'' by angle ρ to obtain $\triangle P''Q'''R'''$. To see that $\triangle P''R'''Q'''$ fits into $\triangle A'B'C$, it remains to show that $Q''' \in \triangle A'B'C$, since $P'' \in \overline{A'A}$ and $R''' \in \overline{AB}$.

Let H' be the foot of the perpendicular from P'' to BC. Since the line $P''H'$ is the image of a leftward translation of the line PH by $\zeta_{BC}(Q')$, and Q' lies below the line AH, we have $Q'' \in \overline{H'B}$. We rotate H' and B about P'' by angle ρ to obtain H'' and B'', respectively, so that $Q''' \in \overline{H''B''}$.

We now apply Lemma 3 to $\triangle ABB^*$ and R, with $\mu = \zeta_{BC}(Q) = \zeta_{BB^*}(Q)$. Note that C in the lemma is our B^*, A' in the lemma is our P''. The B' in the lemma will be denoted here $B^\oplus = B \oplus \zeta_{AB^*}(R) \in \overline{BB'}$ since $\zeta_{AB^*}(R) \le \zeta_{AC}(R) \le \zeta_{AC}(D)$. To check the lemma's condition on μ, let Q_0 be the point on the line AH at distance $|AB|$ from A and let $A_0 = A \ominus \zeta_{BB^*}(Q_0)$; see Fig. 11. Since $\mu \le \zeta_{BB^*}(Q_0)$ under the constraint $|AQ| \le |AB|$, we have that $\frac{\mu + |AB|}{|AB|} \le \frac{|A_0B|}{|AB|}$. Let $Y = Q_0 \ominus \zeta_{BB^*}(Q_0)$ and let $Z = Q_0 \oplus |AB|$. The quadrilateral $ABZQ_0$ is a rhombus, and the triangle $\triangle BZY$ is right-angled at B, and therefore similar to $\triangle ABH$. It follows that $\frac{|A_0B|}{|AB|} = \frac{|YZ|}{|AB|} = \frac{|AB|}{|AH|}$, and the condition in Lemma 3 is satisfied. The lemma implies that $B'' \in \triangle BB^\oplus H$, and that $H'' \in \triangle H'P''R^*$, where R^* is the horizontal projection of R on the line BB^*. Since $R \in \mathcal{H}_{AC}$, the point C must lie on the segment $\overline{R^*B^*}$, and thus $H'' \in \triangle H'P''R^* \subset \triangle H'P''C \subset \triangle A'BC$.

Since $Q''' \in \overline{H''B''}$, $H'' \in \triangle A'BC \subset \triangle A'B'C$ and $B'' \in \triangle BB^\oplus H \subset \triangle A'B'C$, convexity of $\triangle A'B'C$ implies that $Q''' \in \triangle A'B'C$, completing the proof. □

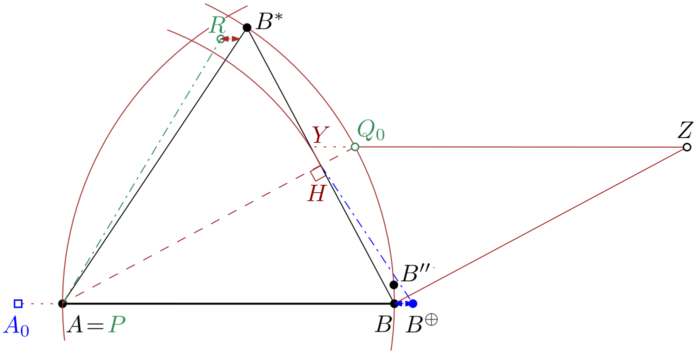

Fig. 11 Proof of Theorem 4: verifying the condition of Lemma 3

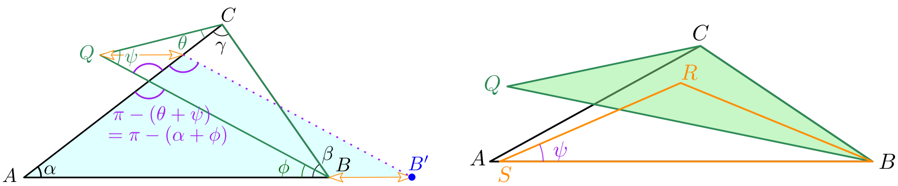

Fig. 12 Lemma 2. Angle notation (left); Case 1 (right)

4 Proofs with trigonometry and calculus

In this section we provide the postponed proofs of Lemma 2 and Lemma 3.

Proof of Lemma 2 We denote angles as in Fig. 12(left). Note that $\alpha + \phi = \psi + \theta$. We claim that $\triangle BCQ$ fits into $\triangle AB'C$. We distinguish two cases.

Case 1: $\theta \geq \phi$

In this case, we rotate $\triangle BCQ$ by angle ϕ around B, resulting in $\triangle BRS$ with S on the line AB; see Fig. 12(right). Since $|BS| = |BQ| \leq |AB|$, S lies on $\overline{AB}$. On the other hand $\psi = \alpha + (\phi - \theta) \leq \alpha$, so R lies below AC, and therefore in $\triangle ABC$.

Case 2: $\theta < \phi$

In this case, we rotate $\triangle BCQ$ by angle θ around B, resulting in $\triangle BRS$ with RS parallel to AC; see Fig. 13(left). We let $\triangle BS'R'$ be the image of $\triangle BRS$ mirrored about the angular bisector of $\angle ABR$, which means that R' lies on $\overline{AB}$; see Fig. 13(right).

We claim that $\min\{\zeta_{AC}(S), \zeta_{AC}(S')\} \leq \zeta_{AC}(Q) = |BB'|$, which implies that at least one of $\triangle BRS$ or $\triangle BS'R'$ can be translated rightward to fit into $\triangle AB'C$. By the law of sines, applied to triangles $\triangle A(Q \oplus \zeta_{AC}(Q))(B \oplus \zeta_{AC}(Q))$, $\triangle A(S \oplus \zeta_{AC}(S))(B \oplus \zeta_{AC}(S))$, and $\triangle A(S' \oplus \zeta_{AC}(S'))(B \oplus \zeta_{AC}(S'))$ (see blue shaded triangles in Figs. 12 and 13), we have

$$\frac{|BQ|}{\sin\alpha} = \frac{\zeta_{AC}(Q) + |AB|}{\sin(\alpha + \phi)} = \frac{\zeta_{AC}(S) + |AB|}{\sin\psi} = \frac{\zeta_{AC}(S') + |AB|}{\sin(\alpha + \beta - \phi)},$$

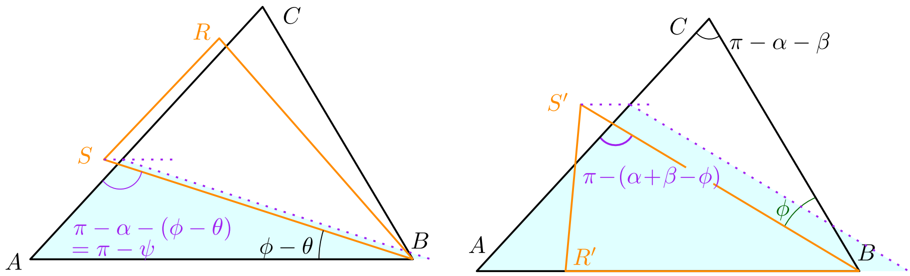

Fig. 13 Lemma 2. Case 2

so we need to prove $\min\{\sin(\psi), \sin(\alpha+\beta-\phi)\} \le \sin(\alpha+\phi)$. Suppose this is not the case. Then we have $\sin(\alpha+\phi) < \sin\psi$. Since $\alpha+\phi = \psi+\theta > \psi$ and $x \mapsto \sin x$ is monotonously increasing on $[0, \frac{\pi}{2}]$, we must have $\alpha+\phi > \frac{\pi}{2}$ and $\pi-(\alpha+\phi) < \frac{\pi}{2}$. On the other hand, we also have $\sin(\alpha+\beta-\phi) > \sin(\alpha+\phi) = \sin(\pi-(\alpha+\phi))$, which implies $\alpha+\beta-\phi > \pi-(\alpha+\phi)$ and therefore $(\alpha+\phi)+(\alpha+\beta-\phi) > \pi$. However, since γ is the largest angle in $\triangle ABC$, $(\alpha+\phi)+(\alpha+\beta-\phi) = 2\alpha+\beta \le \alpha+\beta+\gamma = \pi$, a contradiction. □

Proof of Lemma 3 We scale the problem such that $|AB| = |AC| = 1$ and place A at the origin, so that $B = (1, 0)$. Let $\beta = \measuredangle ABC$, $\alpha = \measuredangle BAC = \pi - 2\beta$, and $\rho = \measuredangle CAR$. We have $|AH| = \sin\beta$, and $|A'B| \le \frac{1}{\sin\beta}$; see Fig. 14.

We first observe that we can replace R by the point at distance $\sin\beta$ from A on $\overline{AR}$. This keeps ρ unchanged, decreases $|BR|$, decreases $\zeta_{AC}(R)$, and decreases $|H'R^\star|$ so that $\triangle A'H'R^\star$ becomes strictly smaller. So in the following, $|AR| = |AH| = \sin\beta$.

Let next $\delta = \zeta_{AC}(R)$, and let $X \in \overline{AC}$ be the point $R \oplus \delta$. Applying the law of sines to $\triangle AXR$, we have

$$\frac{\delta}{\sin\rho} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin\beta}{\sin 2\beta} = \frac{\sin\beta}{2\sin\beta\cos\beta} = \frac{1}{2\cos\beta} \quad \text{so} \quad \delta = \frac{\sin\rho}{2\cos\beta}. \tag{1}$$

We now analyse the interval of angles β for which the conditions of the lemma can be satisfied. Consider the point $R_0 = (-\sin\beta\cos 2\beta, \sin\beta\sin 2\beta)$ on $\overline{AC}$ with $|AR_0| = \sin\beta$, and let

$$\phi(\beta) := |R_0B|^2 = (1+\sin\beta\cos 2\beta)^2 + (\sin\beta\sin 2\beta)^2.$$

Notice[1] that $\frac{d}{d\beta}\phi(\beta) = 2\cos\beta(6\cos^2\beta + \sin\beta - 5)$, which is negative on $[\frac{\pi}{4}, \frac{\pi}{2}]$ since $6\cos^2\beta + \sin\beta - 5 \le 6\cos^2\frac{\pi}{4} - 4 = -1$. Thus, as β increases from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, $\phi(\beta)$ decreases monotonously from $\phi(\frac{\pi}{4}) = \frac{3}{2}$ to $\phi(\frac{\pi}{2}) = 0$, so there is a $\beta_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$ with $\phi(\beta) = 1$. For $\beta < \beta_0 \approx 1.003 \approx 57.47°$, no point R left of AC at distance smaller than one from B can exist; see Fig. 15. In the following we therefore have $\beta \ge \beta_0$.

The point R lies on an arc of circle around A with right endpoint R_0. There are two other critical points on this circle: let R_1 be the point with $|AR_1| = \sin\beta$ and $|BR_1| = 1$, and let R_2 be the point on the horizontal line through C with $|AR_2| = \sin\beta$ and a positive x-coordinate. A point R satisfying the conditions of the lemma cannot lie to the left of R_1 because $|BR| \le 1$, and cannot lie to the left of R_2 since $R \in \mathcal{H}_{AC}$.

[1] The computations in Maple can be found in the appendix.

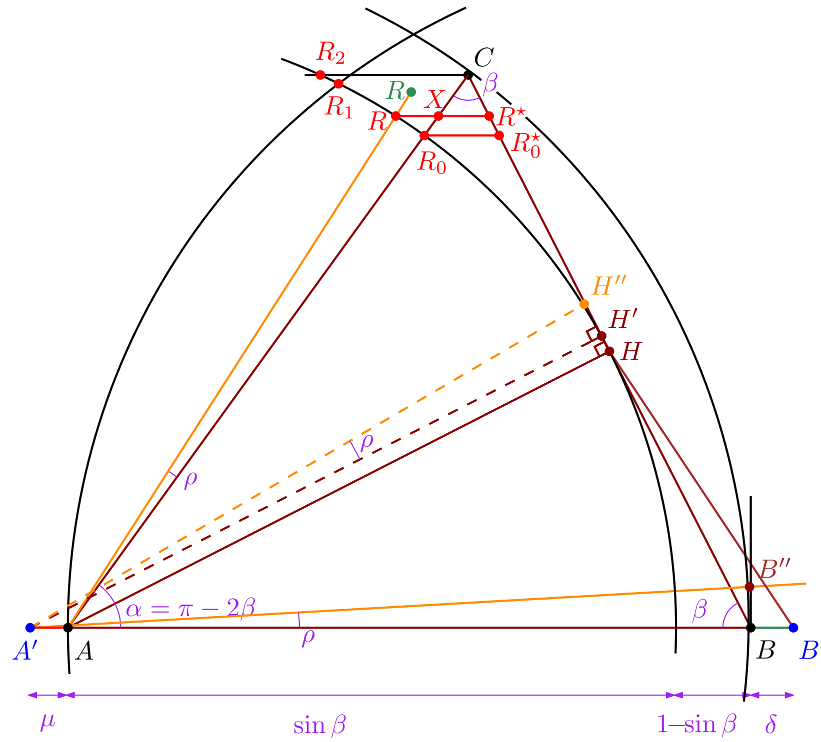

Fig. 14 Lemma 3: Notation

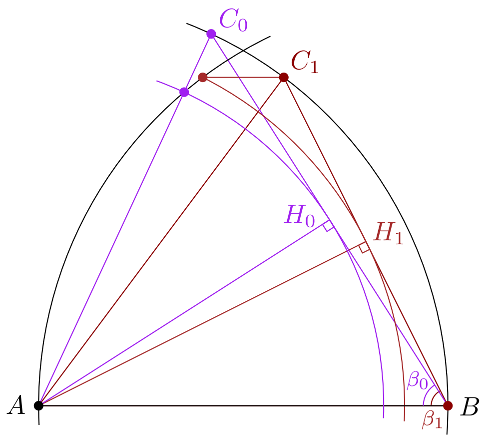

Fig. 15 Lemma 3: definitions of β_0 and β_1

The triangle $\triangle ABR_1$ is isosceles with two sides of length one and a short side $\overline{AR_1}$ of length $\sin\beta$, so $\measuredangle ABR_1 = 2\sin^{-1}(\frac{1}{2}\sin\beta)$. The law of sines applied to triangle $\triangle(A \ominus \zeta_{AC}(R_1))R_1B$ now shows that

$$\frac{\zeta_{AC}(R_1) + 1}{\sin(\pi - (\pi - 2\beta) - 2\sin^{-1}(\frac{1}{2}\sin\beta))} = \frac{1}{\sin 2\beta}.$$

We set

$$h(\beta) := \zeta_{AC}(R_1) = \frac{\sin(2\beta - 2\sin^{-1}(\frac{1}{2}\sin\beta))}{\sin 2\beta} - 1. \tag{2}$$

Since $C = (-\cos 2\beta, \sin 2\beta)$, the x-coordinate of R_2 is $\sqrt{\sin^2\beta - \sin^2 2\beta}$. For $\beta = \beta_1 := \cos^{-1}(\frac{1}{\sqrt{5}}) \approx 1.107 \approx 63.4°$, we have $C = (\frac{3}{5}, \frac{4}{5})$, $R_2 = (\frac{2}{5}, \frac{4}{5})$, implying $|BR_2| = 1$, that is, $R_1 = R_2$; see Fig. 15. We set

$$g(\beta) := \zeta_{AC}(R_2) = |R_2C| = -\cos 2\beta - \sqrt{\sin^2\beta - \sin^2 2\beta}. \tag{3}$$

To summarize:

- For $\beta_0 \le \beta \le \beta_1$, R lies on the arc between R_0 and R_1. The angle ρ is maximized when $R = R_1$. For $\beta = \beta_0$, we have $R_1 = R_0$ (so there is only a single choice for R), for $\beta = \beta_1$ we have $R_1 = R_2 = (\frac{2}{5}, \frac{4}{5})$. Since R cannot lie to the left of R_1, we have $\delta \le h(\beta)$.
- For $\beta_1 \le \beta < \frac{\pi}{2}$, R lies on the arc between R_0 and R_2, with ρ maximized when $R = R_2$. Since R cannot lie to the left of R_2, we have $\delta \le g(\beta)$.

***B′′* position**

Consider now the point B''. Since $|A'B| \le \frac{1}{\sin\beta}$, it has y-coordinate at most $\frac{\sin\rho}{\sin\beta}$. We will prove that HB' intersects the vertical line $x = 1$ through B at y-coordinate at least $\frac{\sin\rho}{\sin\beta}$, implying that B'' lies below HB', and therefore is in $\triangle BB'H$.

Since $H = (\sin^2\beta, \sin\beta\cos\beta)$ and $B' = (1+\delta, 0)$, the line $x = 1$ intersects HB' at y-coordinate

$$\delta \cdot \frac{\sin\beta\cos\beta}{1+\delta-\sin^2\beta} = \frac{\sin\rho}{2\cos\beta} \cdot \frac{\sin\beta\cos\beta}{\cos^2\beta+\delta} = \frac{\sin\rho\sin\beta}{2(\cos^2\beta+\delta)}.$$

This is at least $\frac{\sin\rho}{\sin\beta}$ if and only if

$$\frac{1}{\cos^2\beta+\delta} \ge \frac{2}{\sin^2\beta},$$

which is equivalent to

$$\delta \le \frac{1}{2}\sin^2\beta - \cos^2\beta = \frac{3}{2}\sin^2\beta - 1.$$

Setting $f(\beta) = \frac{3}{2}\sin^2\beta - 1$, it remains to show that $\delta = \zeta_{AC}(R) \le f(\beta)$ under the conditions of the lemma.

We first consider the case $\beta \ge \beta_1$, where $\delta \le g(\beta)$. Since $g(\beta)$ is a decreasing function, while $f(\beta)$ is an increasing function, this implies that $\delta \le g(\beta) \le g(\beta_1) = \frac{1}{5} = f(\beta_1) \le f(\beta)$.

We next consider $\beta_0 \le \beta \le \beta_1$. For $\beta = \beta_1$, $R_2 = R_1$, so $h(\beta_1) = g(\beta_1) = \frac{1}{5} = f(\beta_1)$. We consider the function $\beta \mapsto f(\beta) - h(\beta)$. Plotting its derivative on the interval $[\beta_0, \beta_1]$ shows that it is smaller than -0.2, so $f(\beta) - h(\beta)$ is decreasing on the interval. This implies that $\delta \le h(\beta) \le f(\beta)$ for $\beta \in [\beta_0, \beta_1]$, completing the proof of $B'' \in \triangle BB'H$.

H'' position

We now turn to the point H''. It is obtained by rotating H' counter-clockwise around A' by angle ρ. Since $A'H'$ is orthogonal to BC, H'' lies below the line BC. Since $\rho < \pi$, H'' lies above the line $A'H'$. To show that $H'' \in \triangle A'H'R^\star$, it remains to prove that H'' lies below the line $A'R^\star$. This is equivalent to proving $\rho \leq \measuredangle H'A'R^\star$.

Let $R_0^\star$ be the horizontal projection of R_0 on the line BC; see Fig. 14. Since the y-coordinate of R_0 is $\sin\beta\sin 2\beta$, we have $R_0^\star = (1 - \cos\beta\sin 2\beta, \sin\beta\sin 2\beta)$. We have

$$\begin{aligned}\measuredangle H'A'R^\star &= \measuredangle BA'R^\star - \measuredangle BA'H' = \measuredangle BA'R^\star - \measuredangle BAH \\ &= \measuredangle BA'R^\star - (\tfrac{\pi}{2} - \beta) \geq \measuredangle BA'R_0^\star - (\tfrac{\pi}{2} - \beta)\end{aligned}$$

Since $|A'B| \leq \frac{1}{\sin\beta}$, we can therefore bound from below $\measuredangle H'A'R^\star$ by $r(\beta)$, where

$$r(\beta) := \tan^{-1}\left(\frac{\sin\beta\sin 2\beta}{\big(1 - \cos\beta\sin 2\beta\big) + \big(\frac{1}{\sin\beta} - 1\big)}\right) - \frac{\pi}{2} + \beta.$$

Plotting $r(\beta)$ shows that it is larger than 0.25 on the interval $[\beta_0, \frac{2\pi}{5}]$.

We consider the case $\beta_0 \leq \beta \leq \beta_1$. This implies that ρ is maximized when $R = R_1$. Combining (1) and (2), this gives us $\sin\rho \leq 2h(\beta)\cos\beta$. Plotting $\sin^{-1}(2h(\beta)\cos\beta)$ on the interval $[\beta_0, \beta_1]$ shows that $\rho < 0.2 < 0.25 < r(\beta)$.

Finally, we turn to the case $\beta_1 \leq \beta < \frac{\pi}{2}$. Here, ρ is maximized when $R = R_2$. Combining (1) and (3), this gives us $\sin\rho \leq 2g(\beta)\cos\beta$. Plotting the function $\beta \mapsto \sin^{-1}(2g(\beta)\cos\beta)$ on the interval $[\beta_1, \frac{\pi}{2}]$ shows that $\rho < 0.2$ on that interval. For $\beta \leq \frac{2\pi}{5}$, this implies $\rho < 0.25 < r(\beta)$. For $\beta \geq \frac{2\pi}{5}$, we consider the function $t(\beta) = r(\beta) - \sin^{-1}(2g(\beta)\cos\beta)$. We plot the derivative of $t(\beta)$ on the interval $[\frac{2\pi}{5}, \frac{\pi}{2}]$ to show that it is smaller than -0.05, so $t(\beta)$ is a decreasing function on that interval. Since $t(\frac{\pi}{2}) = 0$, this implies that $t(\beta) \geq 0$, and therefore $\rho \leq r(\beta)$ for $\frac{2\pi}{5} \leq \beta < \frac{\pi}{2}$. To summarize, we have shown $\rho \leq r(\beta) \leq \measuredangle H'A'R^\star$, so $H'' \in \triangle A'H'R^\star$ for all values of β. □

5 Triangles contained in a half-disk

As a warm-up exercise to the square case, we determine the smallest convex cover for the family of triangles contained in the half-disk that is the intersection of the unit disk with the halfplane $y \geq 0$. The half-disk is a crescent, but the triangular cover constructed in Theorem 4 is in this case not the smallest one.

Proof of Theorem 1 Let T be a triangle that maximally fits into the half-disk, and so T falls into one of the three cases of Lemma 1. Cases (ii) and (iii) cannot occur, since such a triangle can rotate around its bottom vertex. Thus we are in case (i): T is a right-angled triangle whose hypotenuse is the diameter of the half-disk.

By symmetry, we in fact only have to consider the triangles T_x whose vertices are $(-1, 0)$, $(1, 0)$, $(-x, \sqrt{1 - x^2})$, for $x \in [0, 1]$; see Fig. 16(left). When translating T_x horizontally so that its upper vertex is on the line segment $(-1, 0)(0, 1)$, the right vertex of the translation of T_x is at coordinate $(x + \sqrt{1 - x^2}, 0)$. The x-coordinate of this point is maximized for $x = \frac{1}{\sqrt{2}}$, so the triangle Z with vertices $(-1, 0)$, $(\sqrt{2}, 0)$, and $(0, 1)$ is a cover for all T_x; see the blue triangle in Fig. 16(right). To complete the proof of Theorem 1, we need to argue

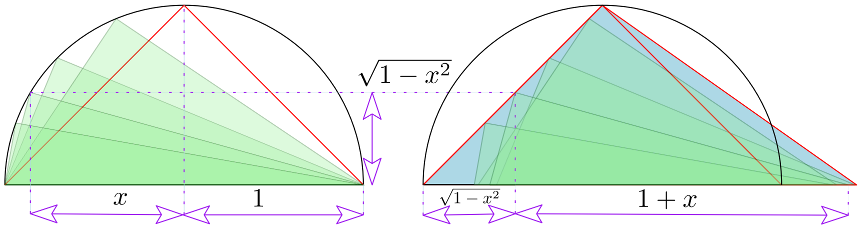

Fig. 16 Right triangles of diameter two

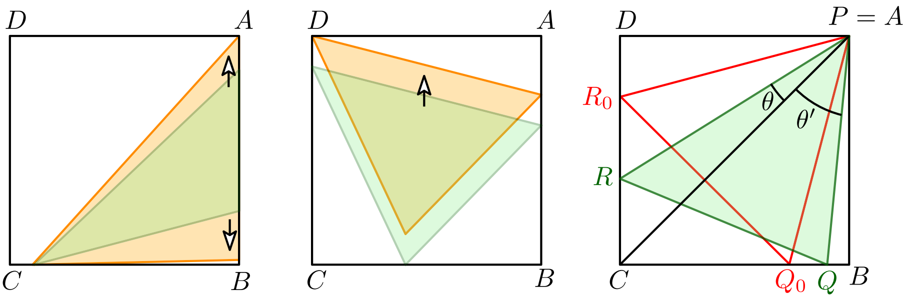

Fig. 17 Triangles that fit in a square

that Z is a smallest cover for the family T_x. This is true since it is already a smallest cover for the two triangles T_0 and $T_{\frac{1}{\sqrt{2}}}$, as can be seen using Corollary 10 of Park and Cheong [8]. □

6 Triangles contained in the unit square

In this section, we prove Theorem 2. We start again by characterizing triangles that maximally fit into the square.

Lemma 4 *Let $\mathcal{X} = ABCD$ be a square. If a triangle $T = \triangle PQR$ fits maximally into $\mathcal{X}$, then without loss of generality, we can assume that $P = A$, Q lies on BC, and R lies on CD.*

Proof Since T maximally fits into $\mathcal{X}$, we can assume that P, Q, R all lie on the boundary $\partial\mathcal{X}$. Suppose two vertices of T lie on the same side of $\mathcal{X}$, say, P, Q lie on AB. Then $T \subset \triangle ABR \subset \mathcal{X}$ as in Fig. 17(left). Since T maximally fits into $\mathcal{X}$, this implies $P = A$, $Q = B$. Suppose next that no vertex of T coincides with a vertex of $\mathcal{X}$. Then P, Q, R lie on three different sides of $\mathcal{X}$, so we can assume that no vertex lies on AD. We can then translate T upwards so that it no longer touches BC, which implies that T does not maximally fit into $\mathcal{X}$; see Fig. 17(middle). It follows that we can assume that $P = A$ and that Q, R lie on the sides BC and CD. □

By Lemma 4, it suffices to study the triangles with $P = A$, $Q \in \overline{BC}$, and $R \in \overline{CD}$. We parameterize these triangles $\triangle PQR$ by the two angles θ and θ' made by the edges $\overline{PQ}$ and $\overline{PR}$ with the diagonal $\overline{AC}$ of the square. We denote this triangle $T_{\theta,\theta'}$; see Fig. 17(right). These parameters range in $[0, \frac{\pi}{4}]$ and the case $\theta = \theta' = \frac{\pi}{6}$ corresponds to the largest

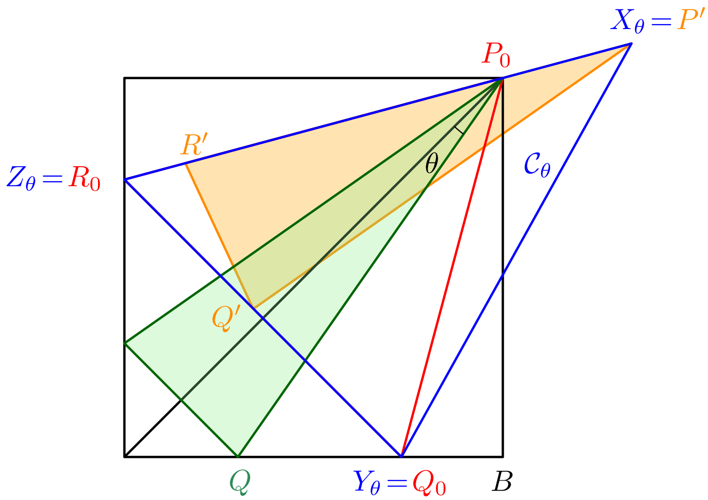

Fig. 18 Construction of the smallest convex cover for the equilateral triangle T_0 and the isosceles triangle $T_{\theta,\theta}$

equilateral triangle that can fit in the square. We denote this equilateral triangle as $T_0 = \triangle P_0 Q_0 R_0 = T_{\frac{\pi}{6},\frac{\pi}{6}}$; see the red triangle in Fig. 17(right).

6.1 The isosceles case: construction of the cover

We first consider the isosceles triangle $T_{\theta,\theta}$ with $\theta \le \frac{\pi}{6}$. A convex cover $\mathcal{C}_\theta = \triangle X_\theta Y_\theta Z_\theta$ for $T_{\theta,\theta}$ and T_0 is obtained when $P'R'$ is aligned with P_0R_0, and Q' is on Q_0R_0; see Fig. 18. We have $|P'Q'| = |P_0Q| = \frac{1}{\cos(\frac{\pi}{4}-\theta)}$. Hence we compute the distance $\ell(\theta)$ between P' and R_0 by the law of sines in $\triangle X_\theta Q' Z_\theta$:

$$\ell(\theta) = \frac{|P'Q'| \cdot \sin \measuredangle Z_\theta Q' X_\theta}{\sin \measuredangle X_\theta Z_\theta Q'} = \frac{\sin(\frac{\pi}{3}+2\theta)}{\cos(\frac{\pi}{4}-\theta)\sin(\frac{\pi}{3})}.$$

When $\theta = 0$, $T_{0,0}$ degenerates to the diagonal of the square and $\ell(0) = \sqrt{2}$. As θ increases from zero, $\ell(\theta)$ increases to a maximum[2] at

$$\theta_0 = \tan^{-1}\left(\frac{1}{\sqrt[3]{2+\sqrt{3}}}\right) - \frac{\pi}{6} \approx 0.049 \approx 2.81°,$$

then decreases to $\ell(\frac{\pi}{6}) = 1/\cos\frac{\pi}{12}$. We have $\ell(\theta_0) \approx 1.4195$.

It follows that the triangle $\mathcal{C} = \triangle XYZ$, where $X = X_{\theta_0}$, $Y = Y_{\theta_0}$, $Z = Z_{\theta_0}$, is a cover for the family of all isosceles triangles $T_{\theta,\theta}$ for $0 < \theta \le \frac{\pi}{6}$. We note that $\measuredangle XZY = \frac{\pi}{3}$, $|XZ| = \ell(\theta_0)$, and $|ZY| = \ell(\frac{\pi}{6})$.

It is intriguing that $\mathcal{C}$ is just slightly larger than the much "nicer" triangle $\triangle X_0Y_0Z_0$ obtained for $\theta = 0$. We will denote this triangle as $\mathcal{C}^\star = \triangle X^\star Y^\star Z^\star$. The angles of $\mathcal{C}^\star$ are $\frac{\pi}{4} = 45°$, $\frac{5\pi}{12} = 75°$, and $\frac{\pi}{3} = 60°$. The longest side is $|X^\star Z^\star| = \sqrt{2}$, and, by construction, we have $\mathcal{C}^\star \subset \mathcal{C}$.

[2] The computations in Maple can be found in the appendix.

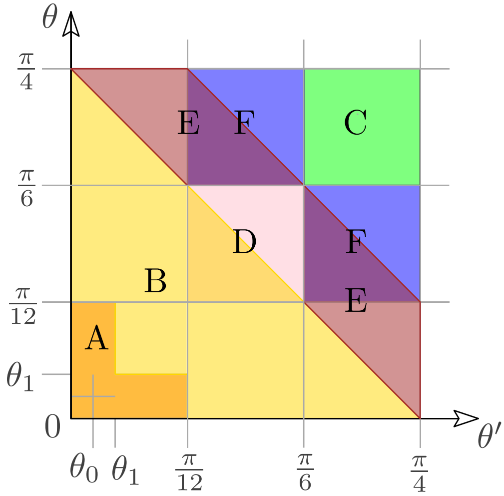

Fig. 19 Six cases cover all possible triangles. All cases but Case A fit in $\mathcal{C}^\star$

We have $\ell(\theta) \geq \sqrt{2}$ when $\theta \in [0, \theta_1]$ and $\ell(\theta) \leq \sqrt{2}$ when $\theta \in [\theta_1, \frac{\pi}{6}]$ with

$$\theta_1 = \tan^{-1}\left(\frac{4\sin^2\frac{\pi}{12}+1}{8\sin^2\frac{\pi}{12}-6+\sqrt{16\sin^4\frac{\pi}{12}-72\sin^2\frac{\pi}{12}+57}}\right)-\frac{\pi}{6} \approx 0.0996 \approx 5.7^\circ,$$

so the triangle $T_{\theta,\theta}$ actually fits into $\mathcal{C}^\star$ for $\theta_1 \leq \theta \leq \frac{\pi}{6}$.

In the following six sections, we discuss why each triangle $T_{\theta,\theta'}$ indeed fits into $\mathcal{C}$. Fig. 19 shows how the six cases cover the entire domain of (θ, θ'). It turns out that only case A requires the cover $\mathcal{C}$, in all other cases $T_{\theta,\theta'}$ fits into the nicer triangle $\mathcal{C}^\star$—so in a sense $\mathcal{C}^\star$ is a cover for "most" triangles contained in the unit square.

It follows from the complete characterization of the smallest convex cover for two given triangles by Park and Cheong [8] that $\mathcal{C}$ is a smallest convex cover for T_0 and T_{θ_0,θ_0}. This makes $\mathcal{C}$ a smallest convex cover for the family of all triangles contained in the unit square.

Moreover, $\mathcal{C}$ is indeed the *unique* smallest cover for this family. To show this, we can directly adapt the proof of Lemma 13 by Park and Cheong [8] to argue that a smallest cover for T_0 and T_{θ_0,θ_0} that is different from $\mathcal{C}$ is a quadrilateral, and that this quadrilateral does not cover either $T_{\theta_0+\varepsilon,\theta_0+\varepsilon}$ or $T_{\theta_0-\varepsilon,\theta_0-\varepsilon}$ for small enough ε.

We now turn to the six cases. Without loss of generality we will always assume that $\theta \leq \theta'$.

6.2 Case A

We start with the triangles where $\theta \leq \theta_1 \approx 5.7^\circ$ and $\theta' \leq \frac{\pi}{12}$. This is the only case where we need to use the cover $\mathcal{C}$—that should not come as a surprise, since T_{θ_0,θ_0} falls into this case.

Let $\triangle PQR = T_{\theta,\theta'}$ be a triangle with $\theta \leq \theta_1$. Let $Q' \in \overline{BC}$ be such that $\triangle PQ'R = T_{\theta,\theta}$. We have seen in Sect. 6.1 that $\mathcal{C}_\theta \subset \mathcal{C}$ covers $\triangle PQ'R$ as in Fig. 20. The point Q lies on the segment $\overline{BQ'}$, so $\triangle PQR \subset \mathcal{C}_\theta$ as long as $\theta + \theta' \leq \measuredangle Y_\theta X_\theta Z_\theta$. Since the angle $\measuredangle Y_\theta X_\theta Z_\theta$ is minimized by $\measuredangle Y_{\theta_0} X_{\theta_0} Z_{\theta_0} > 44.8^\circ$, this holds by $\theta + \theta' \leq \theta_1 + \frac{\pi}{12} < 21^\circ$.

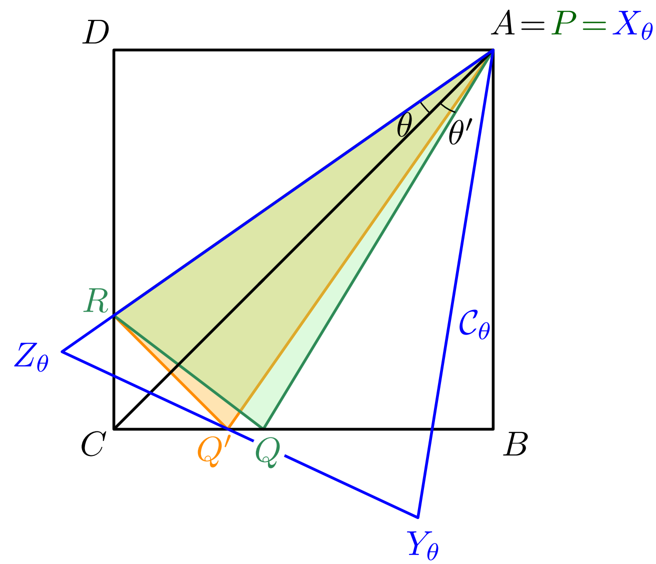

Fig. 20 Case A: covering $\triangle PQR$ with $\theta \leq \theta_1$ and $\theta \leq \theta' \leq \pi/12$

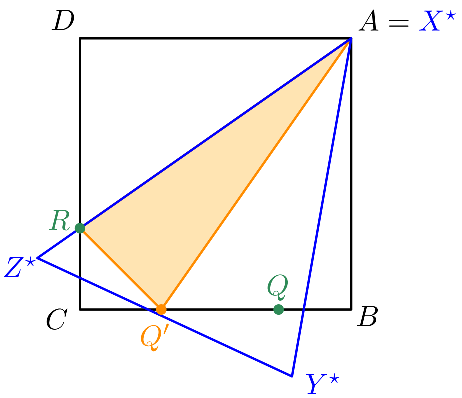

Fig. 21 Case B: covering $\triangle PQR$ when $\theta + \theta' \leq \frac{\pi}{4}$ and $\theta_1 \leq \theta \leq \theta'$

6.3 Case B

Case B covers those triangles where $\theta + \theta' \leq \frac{\pi}{4}$, except for those triangles we treated in case A. It is nearly identical to case A, but now we can use our "nice" cover $\mathcal{C}^\star$. We place $\mathcal{C}^\star$ with $X^\star = A$ and such that R is on $\overline{X^\star Z^\star}$.

For $\theta \geq \theta_1$, we again let $Q' \in \overline{BC}$ be such that $\triangle PQ'R = T_{\theta,\theta}$. We argued in Sect. 6.1 that $\mathcal{C}^\star$ covers $\triangle PQ'R$ as in Fig. 21. Since $\measuredangle RAQ \leq \frac{\pi}{4} = \measuredangle Z^\star X^\star Y^\star$, $Q \in \mathcal{C}^\star$ and so $\triangle PQR \subset \mathcal{C}^\star$.

It remains to consider the situation where $\theta < \theta_1$ and $\theta' \geq \frac{\pi}{12}$. Let Q'' be the point on BC with $\measuredangle BAQ'' = \frac{\pi}{6}$. Since $|AQ''| = \frac{2}{3}\sqrt{3}$ is less than the height of $X^\star$ in $\mathcal{C}^\star$, the point Q'' lies in $\mathcal{C}^\star$. From $\theta' \geq \frac{\pi}{12}$ follows that Q lies on $\overline{BQ''}$, and since $\theta + \theta' \leq \frac{\pi}{4}$, it therefore lies in $\mathcal{C}^\star$ as in Fig. 22.

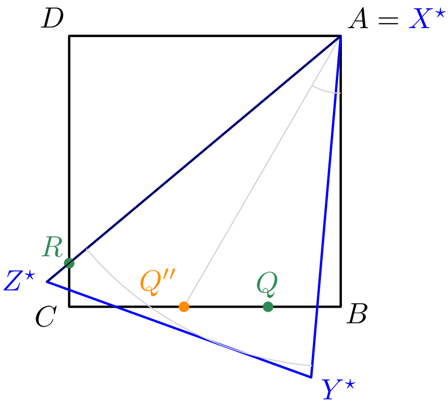

Fig. 22 Case B: covering $\triangle PQR$ when $\theta + \theta' \leq \frac{\pi}{4}$ and $\theta < \theta_1 < \frac{\pi}{12} \leq \theta'$

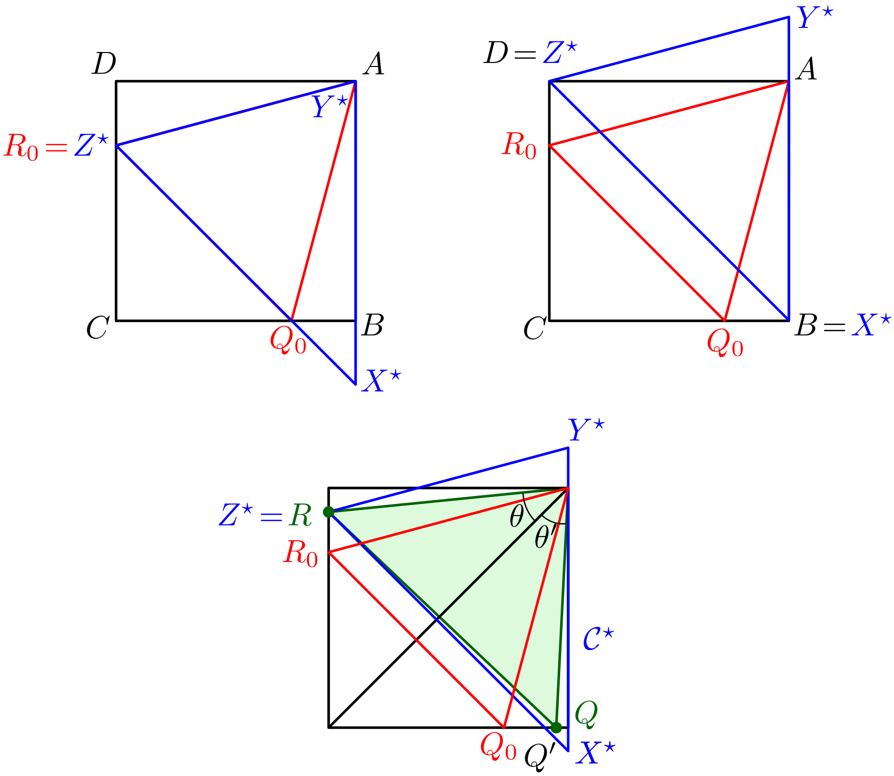

Fig. 23 Case C: covering $\triangle PQR$ when $\frac{\pi}{6} \leq \theta \leq \theta'$

6.4 Case C

We now consider the triangles where $\theta, \theta' \geq \frac{\pi}{6}$. In other words, $Q \in \overline{BQ_0}$, $R \in \overline{R_0 D}$.

We first observe that $\mathcal{C}^\star$ can be placed such that $\overline{X^\star Y^\star}$ is vertical and lies on the line AB, while $Z^\star$ lies on the line CD (recall that $\measuredangle Y^\star X^\star Z^\star = \frac{\pi}{4}$ while $|X^\star Z^\star| = \sqrt{2}$). When

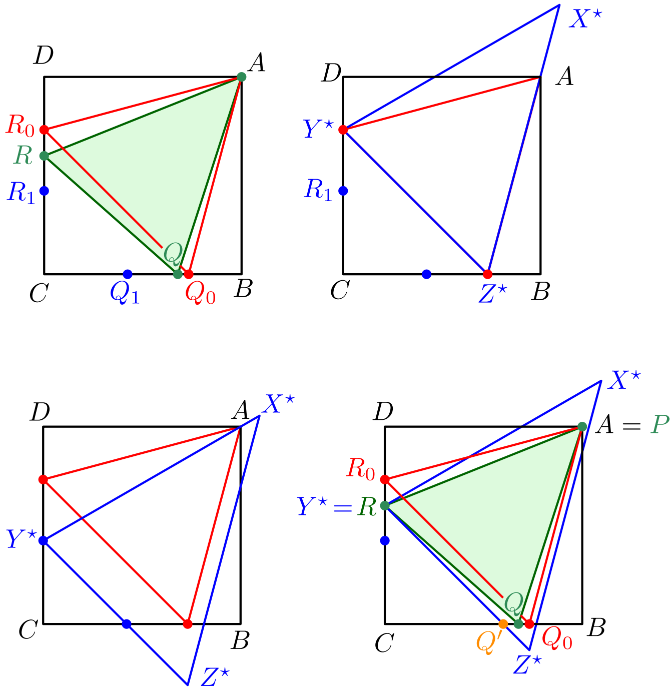

Fig. 24 Case D: covering $\triangle PQR$ when $\frac{\pi}{12} < \theta \leq \theta' \leq \frac{\pi}{6}$

$Z^\star \in \overline{R_0 D}$, then the side $\overline{X^\star Y^\star}$ covers the entire square edge $\overline{AB}$. Figure 23 shows the two extreme cases where $Z^\star = R_0$ (top left) and where $Z^\star = D$ (top right).

Consider now our triangle $\triangle PQR$. We place $\mathcal{C}^\star$ such that $Z^\star = R$; see Fig. 23(bottom). Since the line $Z^\star X^\star$ has slope -1, it intersects $\overline{BC}$ in a point Q' such that $\triangle PQ'R = T_{\theta,\theta}$. Since $\theta' \geq \theta$, we have $Q \in \overline{BQ'} \subset \mathcal{C}^\star$ and thus $\triangle PQR \subset \mathcal{C}^\star$.

6.5 Case D

We now look at the situation where we have $\frac{\pi}{12} = 15^\circ < \theta \leq \theta' \leq \frac{\pi}{6}$. In other words, we have $Q \in \overline{Q_0 Q_1}$ and $R \in \overline{R_1 R_0}$ where $\measuredangle Q_1 AC = \measuredangle R_1 AC = \frac{\pi}{12}$ as in Fig. 24(top left).

We observe that $\mathcal{C}^\star$ can be placed to cover $T_0 = \triangle PQ_0 R_0$ as in Fig. 24(top right). Starting in this position, we can translate $\mathcal{C}^\star$ downwards until $Y^\star = R_1$. Since $X^\star Y^\star$ is parallel to AR_1, A lies in $\mathcal{C}^\star$ during the entire translation; see Fig. 24(bottom left).

Among these positions for $\mathcal{C}^\star$, we choose the one where $Y^\star = R$; see Fig. 24(bottom right). Since the line $Y^\star Z^\star$ has slope -1, it intersects $\overline{BC}$ in a point Q' such that $\triangle PQ'R = T_{\theta,\theta}$. Since $\theta' \geq \theta$, we have $Q \in \overline{Q_0 Q'} \subset \mathcal{C}^\star$ and thus $\triangle PQR \subset \mathcal{C}^\star$.

6.6 Case E

We consider the situation where $\theta \leq \frac{\pi}{6} \leq \theta'$, with the constraints $\frac{\pi}{4} - \theta \leq \theta' \leq \frac{\pi}{3} - \theta$. In other words, R lies on $\overline{CR_0}$, while Q lies on $\overline{BQ_0}$ in Fig. 25, with $\frac{\pi}{4} \leq \measuredangle RAQ \leq \frac{\pi}{3}$.

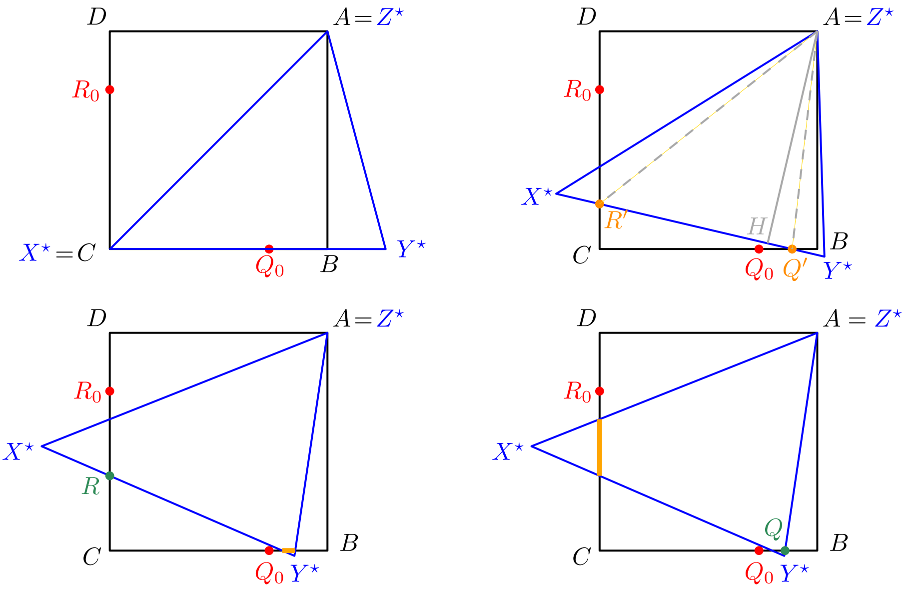

Fig. 25 Case E: covering $\triangle PQR$ when $\theta \leq \frac{\pi}{6} \leq \theta'$ and $\frac{\pi}{4} - \theta \leq \theta' \leq \frac{\pi}{3} - \theta$

We place $\mathcal{C}^\star$ with $Z^\star = A$ and $X^\star = C$. Rotating $\mathcal{C}^\star$ clockwise around A, the line $X^\star Y^\star$ intersects BC and CD in two points Q' and R', respectively; see Fig. 25(top right).

We claim that $\measuredangle R'AQ' = \frac{\pi}{4}$. To see this, consider the point $H \in \overline{X^\star Y^\star}$ such that $\overline{Z^\star H}$ is a height of $\mathcal{C}^\star$. Since the height $|AH| = 1$, we have $\triangle ADR' \equiv \triangle AHR'$ and $\triangle ABQ' \equiv \triangle AHQ'$.

We continue rotating $\mathcal{C}^\star$ until either R lies on $\overline{X^\star Y^\star}$ or Q lies on $\overline{Y^\star Z^\star}$.

In the first case, $R = R'$; see Fig. 25(bottom left). Then $\measuredangle RAQ \geq \frac{\pi}{4} = \measuredangle R'AQ'$ implies that Q lies to the right of Q' in $\mathcal{C}^\star$. Since the line $Y^\star Z^\star$ has not yet passed the point Q, Q lies on the highlighted segment in $\mathcal{C}^\star$.

The second case is illustrated in Fig. 25(bottom right). The line $X^\star Y^\star$ has not yet reached R, so R lies above that line. Since $\measuredangle QAR \leq \frac{\pi}{3} = \measuredangle Y^\star Z^\star X^\star$, R lies below the line $X^\star Z^\star$, and therefore on the highlighted segment in $\mathcal{C}^\star$.

6.7 Case F

In the final case we consider the angles $\frac{\pi}{12} = 15^\circ \leq \theta \leq \frac{\pi}{6}$ and $\frac{\pi}{6} \leq \theta' \leq \frac{\pi}{4}$. In other words, Q lies on $\overline{BQ_0}$, while $R \in \overline{R_1 R_0}$; see Fig. 26(top left).

We again start by covering $T_0 = \triangle PQ_0R_0$ with $\mathcal{C}^\star$, but this time we need to cover it in two different ways; see Fig. 26(top right). The first copy $\mathcal{C}_1^\star$ has $Y_1^\star = Q_0$ and $Z_1^\star = A$ and the second copy $\mathcal{C}_2^\star$ has $Z_2^\star = Q_0$ and $Y_2^\star = A$. Note that $\overline{X_1^\star Z_1^\star}$ and $\overline{X_2^\star Z_2^\star}$ intersect exactly at R_0.

Consider now the point $Q \in \overline{BQ_0}$. We rotate $\mathcal{C}_2^\star$ counter-clockwise around A until $Q \in \overline{X_2^\star Z_2^\star}$ and translate $\mathcal{C}_1^\star$ to the right until $Y_1^\star = Q$. This places A outside of $\mathcal{C}_1^\star$, so we then rotate $\mathcal{C}_1^\star$ counter-clockwise around Q until $A \in \overline{X_1^\star Z_1^\star}$. Fig. 26 depicts the situation for different positions of Q.

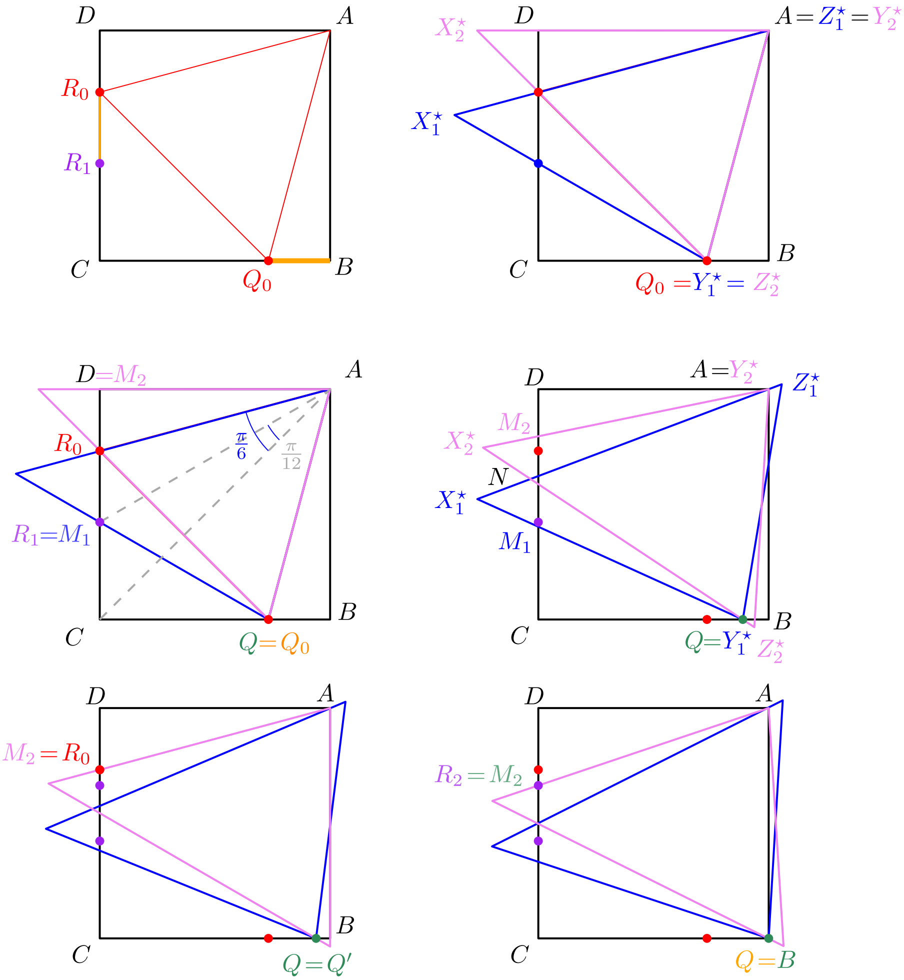

Fig. 26 Case F: top left: locations of Q and R; **top right**: double-covering of T_0; **middle left**: $Q = Q_0$; middle right: Q moving right, M_2 moving down, bottom left: when M_2 reaches R_0, let Q' be the position of Q, **bottom right**: $Q = B$

Let M_1 be the intersection of $\overline{X_1^\star Y_1^\star}$ and $\overline{CD}$ and let M_2 be the intersection of $\overline{X_2^\star Y_2^\star}$ and $\overline{CD}$; see Fig. 26(middle right). When Q moves from Q_0 to B, the line AM_2 rotates around A and thus M_2 moves downwards monotonously. We let Q' be the position of Q when $B \in \overline{Y_2^\star Z_2^\star}$ and $M_2 = R_0$; see Fig. 26(bottom left).

Let N be the intersection of $\overline{X_1^\star Z_1^\star}$ and $\overline{X_2^\star Z_2^\star}$. We will show below that for $Q \in \overline{BQ_0}$, the point N always lies on or to the left of CD. This will imply that the segment $\overline{M_1 M_2}$ lies entirely in $\mathcal{C}_1^\star \cup \mathcal{C}_2^\star$, so as long as $R \in \overline{M_1 M_2}$, we have $\triangle PQR \subset \mathcal{C}_1^\star$ or $\triangle PQR \subset \mathcal{C}_2^\star$.

Assume now that R lies above M_2, that is on $\overline{M_2 R_0}$. This implies that $R \in \overline{R_2 R_0}$, where R_2 is the position of M_2 when $Q = B$ as in Fig. 26(bottom right). Such a triangle $\triangle PQR$ is covered by $\mathcal{C}_2^\star$ in its position when $Q = Q'$, as illustrated in Fig. 26(bottom left).

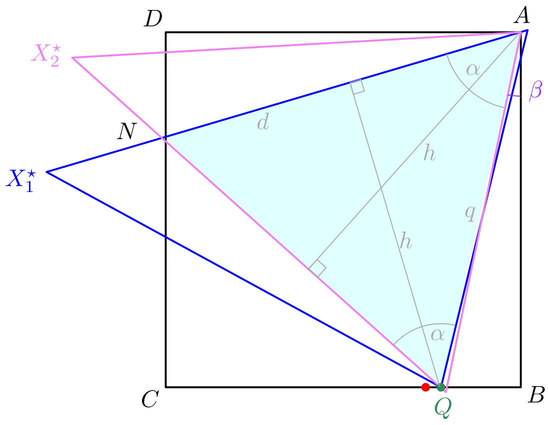

Fig. 27 Case F: computing the x-coordinate of N

Otherwise R lies below M_1, that is on $\overline{R_1 M_1}$. (This is indeed possible: while Q moves from Q_0 to B, M_1 initially moves slightly upwards above R_1 before starting a monotone movement downwards.) In this case we rotate $\mathcal{C}_1^\star$ further counter-clockwise until $R \in X_1^\star Y_1^\star$. Since $\measuredangle RQA \leq \measuredangle R_1 QA \leq \measuredangle R_1 Q_0 A = 75° = \measuredangle X^\star Y^\star Z^\star$, we then have $P = A \in \mathcal{C}_1^\star$ and $\triangle PQR \subset \mathcal{C}_1^\star$.

It remains to prove the claim that the point N lies on or to the left of the line CD. We will compute the x-coordinate of N as a function of $q := |AQ|$. As Q ranges from B to Q_0, q ranges from 1 to $1/\cos\frac{\pi}{12} \approx 1.035$. Let h be the height of $Y^\star$ in $\triangle X^\star Y^\star Z^\star$. We have

$$h = |Z^\star Y^\star| \sin\tfrac{\pi}{3} = |AQ_0| \frac{1}{2}\sqrt{3} = \frac{1}{\cos\frac{\pi}{12}} \frac{1}{2}\sqrt{3} = \sqrt{6 - 3\sqrt{3}}.$$

We next observe that $Z_1^\star X_1^\star$ is the line at distance h from Q through A, while $Z_2^\star X_2^\star$ is the line at distance h from A through Q. This implies that $\triangle AQN$ is isosceles, with two equal heights of length h; see Fig. 27. Let $\alpha := \measuredangle QAN = \measuredangle AQN$ and $d := |AN| = |QN|$. We have $\sin\alpha = \frac{h}{q}$ and $\cos\alpha = \frac{q}{2d}$. Let $\beta := \measuredangle BAQ$. Then, $\cos\beta = \frac{1}{q}$.

Now we compute the horizontal distance $f(q)$ between A and N:

$$\begin{aligned} f(q) &= d\cos\left(\frac{\pi}{2} - \alpha - \beta\right) = d\sin(\alpha+\beta) = d\sin\alpha\cos\beta + d\cos\alpha\sin\beta \\ &= \frac{q}{2\cos\alpha}\frac{h}{q}\frac{1}{q} + \frac{q}{2}\sin\beta = \frac{h}{2q\sqrt{1-\frac{h^2}{q^2}}} + \frac{q}{2}\sqrt{1-\frac{1}{q^2}} \\ &= \frac{1}{2}\left(\frac{h}{\sqrt{q^2-h^2}} + \sqrt{q^2-1}\right). \end{aligned}$$

Plotting the function $f(q)$ shows that $f(q) > 1.01$ on the interval $1 \leq q \leq 1.02$. Plotting the derivative $f'(q)$ shows that $f'(q) < -0.9$ on the interval $1.01 \leq q \leq 1.05$, so $f(q)$ is decreasing on this interval. We also know that $f(|AQ_0|) = 1$ since then $N = R_0$. This implies that $f(q) \geq 1$ for any $Q \in \overline{BQ_0}$. It follows that N lies on or to the left of $\overline{CD}$, completing this case and the entire proof.

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