Bounds on the minimum distance of locally recoverable codes

Sascha Kurz

Mathematisches Institut, Universität Bayreuth, D-95440 Bayreuth, Germany,
sascha.kurz@uni-bayreuth.de
and
Department of Data Science, Friedrich-Alexander-Universitäät Erlangen-Nürnberg,
D-91058 Erlangen, Germany, sascha.kurz@fau.de

Abstract

We consider locally recoverable codes (LRCs) and aim to determine the smallest possible length $n = n_q(k,d,r)$ of a linear $[n,k,d]_q$-code with locality r. For $k \le 7$ we exactly determine all values of $n_2(k,d,2)$ and for $k \le 6$ we exactly determine all values of $n_2(k,d,1)$. For the ternary field we also state a few numerical results. As a general result we prove that $n_q(k,d,r)$ equals the Griesmer bound if the minimum Hamming distance d is sufficiently large and all other parameters are fixed.

Mathematics Subject Classification: 94B27, 94B05
Keywords: linear codes, locally recoverable codes, data storage, bounds for parameters

1 Introduction

A code over a finite alphabet is called *locally recoverable* (LRC) if every symbol in the encoding is a function of a small number of (at most r) other symbols. They have e.g. applications in distributed storage and communications. Here we will consider linear codes over a finite field $\mathbb{F}_q$ as alphabet. An $[n,k,d]_q$-code C is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum Hamming distance (at least) d. We also speak of $[n,k]_q$-codes if we do not want to specify the minimum Hamming distance. A code symbol is said to have r-locality if it can be repaired from at most r other code symbols. An $(n,k,r)_q$-LRC is an $[n,k]_q$-code with r-locality for all code symbols. For given parameters n, k and alphabet size q one would like to have codes with small locality r, allowing e.g. a fast recovery process when the code is used for distributed storage, and a large minimum Hamming distance d, in order to deal with possible errors in the transmission. However, there is a natural tradeoff between minimizing r and maximizing d. To this end we mention the bound

$$d \le n - k - \left\lceil \frac{k}{r} \right\rceil + 2 \tag{1}$$

by Gopalan et al. [GHSY12], which reduces to the classical Singleton bound when $r = k$. Surely we have $1 \le r \le k$ and MDS codes attain the bound for $r = k$. Replicating each symbol in an MDS code twice yields codes attaining the bound for $r = 1$. If the field size is sufficiently large then the upper bound (1) can always be attained with equality [GHSY12]. Further constructions requiring smaller field sizes can e.g. be found in [TB14], i.e. $q \ge n + 1$ assuming that $r + 1$ divides n, see also [LXY18, TBGC15]. However, small finite fields as alphabets are often desirable for practical reasons [GC14]. In [TPD16] the search for constructions meeting the stated bound (1) with equality was stated as an open problem. Constructions for the binary case can e.g. be found in [GC14, HYUS15, HYUS16]. For $q \in \{2,3,4\}$ all cases where Inequality (1) can be attained with equality were characterized in [HXC16], [HXC17], and [XKG22], respectively. For $q \ge n + 1$

codes with $d = n - k - \lceil \frac{k}{r} \rceil + 1$, i.e. one less than the upper bound (1), indeed exist [TB14]. Using the function $k^q_{\text{opt}}(n, d)$ for the maximum possible dimension k of an $[n, k, d]_q$-code the bound

$$k \le \min_{t \in \mathbb{N}} \left\{ rt + k^q_{\text{opt}}(n - t(r+1), d) \right\} \tag{2}$$

for $[n, k, d]_q$-codes with locality r was obtained in [CM15] and e.g. used in [HXS⁺20] to conclude explicit parametric bounds on codes with a given locality.

Another classical bound for linear codes is the Griesmer bound [Gri60] relating the minimal possible length $n_q(k, d)$ of an $[n, k, d]_q$-code to its other parameters:

$$n_q(k, d) \ge g_q(k, d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil. \tag{3}$$

Solomon and Stifler gave a construction showing that this bound can be attained for any given parameters k and q if d is sufficiently large [SS65]. Consequently, the determination of the function $n_q(k, \cdot)$ becomes a finite problem for each pair of parameters k, q. For $k \le 7$, the function $n_2(k, \cdot)$ has been completely determined in [BM73] and [vT81]. After a lot of work of different authors, the determination of $n_2(8, d)$ has been completed in [BJV00]. Here we will show that for any given pair of parameters k, q there exist $(n, k, 2)_q$-LRCs and $(n, k, 1)_q$-LRCs with length $n = n_q(k, d)$ and minimum Hamming distance d assuming that d is sufficiently large. So, also the determination of the largest possible Hamming distance of $(n, k, 2)_q$- and $(n, k, 1)_q$-LRCs becomes a finite problem for any parameters k and q.

In the literature also special subclasses of LRCs, like e.g. maximally recoverable codes [CHL07], have been considered. The required field size of a maximally recoverable codes was e.g. improved in [GWFHH19]. There the authors used matroid theory, see also [GFHWH17, WFHEH16], to show non-existence of some codes with prescribed parameters by characterising linearity over small fields via forbidden uniform minors. In [PKLK12] the authors introduced the notion of (r, δ)-LRCs. Corresponding LP and other bounds can e.g. be found in [GJR23]. In [GFHWH19] the authors introduced a slight variant of the definition of locality, called dimension-locality, and study corresponding bounds.

The remaining part of this paper is structured as follows. In Section 2 we introduce the necessary preliminaries. A geometric reformulation of locality is then discussed in Section 3 and used to bound the minimal possible lengths of $[n, k, d]_q$-codes with locality r. Then we summarize several constructions and non-existence results for LRCs in Section 4. For several small parameter sets we were able to completely determine the minimum possible length of an $[n, k, d]_q$-code with locality $r \in \{1, 2\}$ as a function of the minimum Hamming distance d. We close the paper in Section 5 by stating enumeration results for $[n, k, d]_q$-codes with locality r for some small parameters. The focus on exact values for small parameters is thought as a supplement to the existing literature which mainly considers different general bounds. While we cannot draw asymptotic conclusions from our obtained data, we remark that the number of used nodes n is typically rather small in many real-world applications, see e.g. HDFS-Xorbas used by Facebook [APD⁺13] and Windows Azure storage [HSX⁺12]. Since variants and extensions of LRCs are comprehensive, we refrain from discussing similar results for some of them.

2 Preliminaries

For a given linear code C we denote its minimum Hamming distance by $d(C)$ and the dual code by $C^\perp$. Apart from the field size q, the length n, the dimension k, and the minimum Hamming distance d of a linear $[n, k, d]_q$-code we consider the *locality* as an additional parameter.

Definition 1. *A (linear) code $C \subseteq \mathbb{F}_q^n$ has* locality r *if for every coordinate $i \in \{1, \ldots, n\}$ there exists a set $S_i \subseteq \{1, \ldots, n\}$ with $i \notin S_i$, $|S_i| \le r$, and if $c_j = c'_j$ for all $j \in S_i$ for two codewords $c, c' \in C$ then we have $x_i = y_i$.*

We also speak of a *locally recoverable code* (LRC). The set S_i is called a *recovery set* for coordinate i. Denoting the projection map onto the coordinates from some set $S \subseteq \{1,\ldots,n\}$ by π_S, we can introduce *recovery functions* $f_i\colon \pi_{S_i}(C) \to \mathbb{F}_q$ satisfying $f_i(\pi_{S_i}(c)) = c_i$ for all $c \in C$ and all $1 \le i \le n$.

Lemma 2. *([ABH+18, Lemma 10]) Let C be an $[n,k]_q$-code and S_i a recovery set for coordinate $1 \le i \le n$ with recovery function f_i. Then, f_i is an $\mathbb{F}_q$-linear map.*

So, we can express the locality of a given linear code C via the existence of certain codewords in its dual code $C^\perp$:

Lemma 3. *A linear $[n,k]_q$-code C has locality $r \ge 1$ iff for every coordinate $1 \le i \le n$ there exists a dual codeword $c \in C^\perp$ with weight at most $r+1$ that contains i in its support.*

Definition 4. *An $[n,k]_q$-code C is* non-degenerate*, if there does not exist a coordinate $1 \le i \le n$ such that $c_i = 0$ for all codewords $c \in C$. We call C projective if the coordinates of the codewords are linearly independent; that is, there exists no coordinate $i \neq j \in \{1,\ldots,n\}$ and $\lambda \in \mathbb{F}_q \backslash \{0\}$ such that $c_i = \lambda \cdot c_j$ for every $c \in C$.*

In other words, a linear code C is non-degenerate if every column of a generator matrix G of C is non-zero. If no column of a generator matrix C is a scalar multiple of another column, then the corresponding linear code C is projective.

By $\mathrm{PG}(k-1,q)$ we denote the finite projective geometry of dimension $k-1$ and order q. A well-known and often exploited interpretation of non-degenerated linear codes is the following. The columns of a generator matrix G of a non-degenerated $[n,k]_q$-code C may be interpreted as points of the projective space $\mathrm{PG}(k-1,q)$. In the other direction, a multiset of points $\mathcal{M}$ in $\mathrm{PG}(k-1,q)$ is a mapping from the set of points to the natural integers. Here, for each point P we call $\mathcal{M}(P)$ the multiplicity of P. The cardinality $|\mathcal{M}|$ of our multiset $\mathcal{M}$ is the sum over all point multiplicities and equals the length n of the corresponding (non-degenerate) code C. By $\gamma(\mathcal{M})$ we denote the maximum point multiplicity of $\mathcal{M}$. So, we have $d(C^\perp) = 2$ iff $\gamma(\mathcal{M}) \ge 2$ and $\gamma(\mathcal{M}) = 1$ iff C is projective. Using the geometric language we call $2-$, $3-$, $4-$, and $(k-1)$-dimensional subspaces lines, planes, solids, and hyperplanes, respectively. So, each dual codeword $c \in C^\perp$ of weight three geometrically corresponds to a triple of points spanning a line and each dual codeword of weight 2 geometrically corresponds to a point with multiplicity at least 2. We call a multiset of points $\mathcal{M}$ in $\mathrm{PG}(k-1,q)$ *spanning* if the points with positive multiplicity span the ambient space $\mathrm{PG}(k-1,q)$. The multiset of points corresponding to an $[n,k]_q$-code is always spanning. In the other direction we have that a multiset of points in $\mathrm{PG}(k-1,q)$ might correspond to an $[n,k']_q$-code with $k' < k$. The minimum Hamming distance of C is at least d iff we have $\mathcal{M}(H) \le n-d$ for every hyperplane H, where $\mathcal{M}(H)$ is defined as the sum of all multiplicities of the points contained in H, i.e., $\mathcal{M}(H) = \sum_{P \le H} \mathcal{M}(P)$. Writing $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for the number of k-dimensional subspaces in $\mathrm{PG}(n-1,q)$ we can state that each t-dimensional subspace contains $\begin{bmatrix} t \\ 1 \end{bmatrix}_q$ points and $\mathrm{PG}(n-1,q)$ contains $\begin{bmatrix} n \\ n-1 \end{bmatrix}_q = \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ hyperplanes in total. Since each point is contained in $\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ hyperplanes and each pair of different points is contained in $\begin{bmatrix} k-2 \\ 1 \end{bmatrix}_q$ hyperplanes, assuming $k \ge 3$, we have

$$(n-d) \cdot \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q \ge \sum_{H\,:\,P \le H} \mathcal{M}(H) = n \cdot \begin{bmatrix} k-2 \\ 1 \end{bmatrix}_q + q^{k-2} \cdot \mathcal{M}(P) \tag{4}$$

for every point P. Similarly,

$$(n-d) \cdot q^{k-1} \ge \sum_{H\,:\,P \not\le H} \mathcal{M}(H) = q^{k-2} \cdot (n - \mathcal{M}(P)) \tag{5}$$

for every point P. For two multisets of points $\mathcal{M}, \mathcal{M}' \in \mathrm{PG}(k-1,q)$ we write $\mathcal{M} + \mathcal{M}'$ for the multiset of points in $\mathrm{PG}(k-1,q)$ with multiplicity $\mathcal{M}(P) + \mathcal{M}'(P)$ for every point P. Similarly, for each multiset $\mathcal{M}$ in $\mathrm{PG}(k-1,q)$ and each integer $t \ge 1$ we write $t \cdot \mathcal{M}$ for the multiset of points in $\mathrm{PG}(k-1,q)$ with multiplicity $t \cdot \mathcal{M}(P)$ for every point P.

3 Geometric reformulation of locality and minimal possible lengths of $[\mathbf{n}, \mathbf{k}, \mathbf{d}]_{\mathbf{q}}$-codes with locality $\mathbf{r}$

Directly from Lemma 3 we conclude:

Lemma 5. *Let C be a linear $[n, k]_q$-code and $\mathcal{M}$ be the corresponding multiset of points in $\operatorname{PG}(k-1, q)$ of cardinality n. Then, C has locality $r \ge 1$ iff for every point P with positive multiplicity $\mathcal{M}(P) \ge 1$ either $\mathcal{M}(P) \ge 2$ or there are $t \le r$ points with positive multiplicity and being different from P that span an t-dimensional subspace S with $P \le S$.*

So, locality is also a geometric property of a multiset of points $\mathcal{M}$ and we directly say that $\mathcal{M}$ has locality r is its corresponding code has locality r. We remark that especially the locality of a linear code does not depend on a specific representation in terms of a generator matrix with is e.g. different for private information retrieval (PIR) codes, see [KY21, Proposition 9]. For small values of r we can also spell out the condition of Lemma 5 more directly.

Lemma 6. *Let C be a linear $[n, k]_q$-code and $\mathcal{M}$ be the corresponding multiset of points in $\operatorname{PG}(k-1, q)$ of cardinality n. Then, C has locality 1 iff every point P with positive multiplicity $\mathcal{M}(P) \ge 1$ has multiplicity at least 2.*

Or in other words, a non-empty multiset of points has locality 1 iff no point has multiplicity exactly 1.

Lemma 7. *Let C be a linear $[n, k]_q$-code and $\mathcal{M}$ be the corresponding multiset of points in $\operatorname{PG}(k-1, q)$ of cardinality n. Then, C has locality 2 iff for every point P with positive multiplicity $\mathcal{M}(P) \ge 1$ we have $\mathcal{M}(P) \ge 2$ or there exist points Q, R with $|\{P, Q, R\}| = 3$, $\dim(\langle P, Q, R\rangle) = 2$, and $\mathcal{M}(Q), \mathcal{M}(R) \ge 1$.*

A multiset of points $\mathcal{M}$ over the binary field $\mathbb{F}_2$ has locality 2 iff every point with multiplicity 1 is contained in a full line, i.e., all three points of the line have positive multiplicity.

Lemma 8. *Let C be an $[n, k]_q$-code with dual minimum distance $d^\perp = 3$. If the number of dual codewords of weight 3 is less than $(q-1) \cdot n/3$, then the locality of C is larger than 2.*

Proof. Due to Lemma 3 the locality of C is at least 2. So, due to Lemma 7, any point with positive multiplicity w.r.t. the corresponding multiset of points $\mathcal{M}$ has to be contained in a line that contains at least three points with positive multiplicity. Our n points with positive multiplicity cannot be covered by fewer than $n/3$ such lines. However, since each such line corresponds to $(q-1)$ dual codewords of weight 3, we obtain a contradiction. □

Similar to the notion of $n_q(k, d)$ let $n_q(k, d, r)$ denote the minimal possible length of an $[n, k, d]_q$-code with locality r, i.e., the minimum possible cardinality of a spanning multiset of points in $\operatorname{PG}(k-1, q)$ with locality r. So, clearly we have

$$n_q(k, d, r) \ge n_q(k, d) \ge g_q(k, d) = \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i}. \right\rceil \tag{6}$$

and $n_q(k, d, r) \le n_q(k, d, r')$ for all $r \ge r'$. In Theorem 10 we will show $n_q(k, d, r) = g_q(k, d)$ for sufficiently large d and fixed parameters q, k, and r.

Example 9. *Let $\mathcal{M}$ be the (multi-)set of points in $\mathcal{P}(k-1, q)$ where each point has multiplicity exactly 1. Then, $\mathcal{M}$ has cardinality $\frac{q^k-1}{q-1} = \begin{bmatrix} k \\ 1 \end{bmatrix}_q$ and each hyperplane H has multiplicity $\mathcal{M}(H) = \frac{q^{k-1}-1}{q-1} = \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$, so that $|\mathcal{M}| - \mathcal{M}(H) = q^{k-1}$. The corresponding linear code is called k-dimensional q-ary simplex code and has parameters $[n, k, d]_q = \left[\begin{bmatrix} k \\ 1 \end{bmatrix}_q, k, q^{k-1}\right]_q$. If $k \ge 2$ then $\mathcal{M}$ has locality $r = 2$ since each point has multiplicity 1 and each point is contained in at least one line (whose points also have multiplicity 1 each). For each* *integer $t \geq 1$ the multiset of points $t \cdot \mathcal{M}$ has cardinality $t \cdot \begin{bmatrix} k \\ 1 \end{bmatrix}_q$ and locality 1 iff $t \geq 2$. The corresponding linear code, called t-fold simplex code, has parameters $[n,k,d]_q = \left[t \cdot \begin{bmatrix} k \\ 1 \end{bmatrix}_q, k, t \cdot q^{k-1}\right]_q$. Note that we have $n = g_q(k,d)$ for all $t \geq 1$.*

Theorem 10. *Let q be an arbitrary prime power, $k \in \mathbb{N}_{\geq 2}$, and $r \in \mathbb{N}_{\geq 1}$. If d is sufficiently large, then we have $n_q(k,d,r) = g_q(k,d)$.*

Proof. It suffices to prove the statement for locality $r = 1$. Due to [SS65] there exists a constant d' (depending on k and q) such that for all $d \geq d'$ we have $n_q(k,d) = g_k(k,d)$.[1] For $d \geq d'$ consider an $[n,k,d]_q$-code with $n = g_q(k,d)$ and let $\mathcal{M}$ be the corresponding multiset of points in $\mathrm{PG}(k-1,q)$. Setting $t := \lfloor d/q^{k-1} \rfloor$, we have $d \geq t \cdot q^{k-1}$ and $n \leq (t+1) \cdot \begin{bmatrix} k \\ 1 \end{bmatrix}_q$. Using this and Inequality (5) we conclude

$$\mathcal{M}(P) \geq n(1-q) + dq \geq t - q^k$$

for every point P. So, if d is sufficiently large, then we have $t \geq q^k + 2$ and $\mathcal{M}$ has locality 1 due to Lemma 6. □

So for every set of parameters q, k, and r the determination of $n_q(k,d,r)$, as a function of d, is a finite problem. In principle we can determine the exact value of $n_q(k,d,r)$ for each given set of parameters as the optimum target value of an integer linear program (ILP). We will spell out the details for $r \in \{1,2\}$ and leave the general problem as an exercise for the interested reader.

Proposition 11. *Let q, k, and d be arbitrary but fixed parameters. Then, $n_q(k,d,1)$ is given as the optimum target value of the following ILP:*

$$\begin{aligned} \min n & \quad \textit{subject to} \\ \sum_{P \in \mathcal{P}} x_P &= n \\ x_{\langle e_i \rangle} &\geq 1 \quad \forall 1 \leq i \leq k \\ \sum_{P \in \mathcal{P}\colon P \leq H} x_P &\leq n - d \quad \forall H \in \mathcal{H} \\ x_P &\geq 2y_P \quad \forall P \in \mathcal{P} \\ x_P &\leq \Lambda y_P \quad \forall P \in \mathcal{P} \\ x_P &\in \mathbb{N} \quad \forall P \in \mathcal{P} \\ y_P &\in \{0,1\} \quad \forall P \in \mathcal{P}, \end{aligned}$$

where e_i denotes the ith unit vector, $\mathcal{P}$ denotes the set of points in $\mathrm{PG}(k-1,q)$, $\mathcal{H}$ denotes the set of hyperplanes in $\mathrm{PG}(k-1,q)$, and Λ is a sufficiently large constant.

Proof. For a feasible solution of the stated ILP we can define a multiset of points $\mathcal{M}$ via $\mathcal{M}(P) = x_P \in \mathbb{N}$ for all points $P \in \mathcal{P}$. The cardinality of $\mathcal{M}$ is given by $\sum_{P \in \mathcal{P}} \mathcal{M}(P) = \sum_{P \in \mathcal{P}} x_P = n$. Since $\mathcal{M}(H) = \sum_{P \in \mathcal{P}\colon P \leq H} x_P \leq n - d$ the corresponding linear code C has Hamming distance at least d. Since $\mathcal{M}(\langle e_i \rangle) = x_{\langle e_i \rangle} \geq 1$ the multiset $\mathcal{M}$ is spanning, i.e., the linear code C has dimension k. For any point $P \in \mathcal{P}$ the constraints $x_P \geq 2y_P$ and $x_P \leq \Lambda y_P$ are equivalent to $x_P = 0$ for $y_P = 0$ and to $2 \leq x_P \leq \Lambda$ for $y_P = 1$, i.e., we have $\mathcal{M}(P) \neq 1$. Thus, $\mathcal{M}$ and C have locality 1.

For the other direction consider an $[n,k,q]_q$-code C' and its corresponding multiset of points $\mathcal{M}'$. Since $\mathcal{M}'$ is spanning there exists an isomorphic multisets of points $\mathcal{M}$ with $\mathcal{M}(\langle e_i \rangle) \geq 1$ for all $1 \leq i \leq k$. Let C denote the $[n,k,d]_q$-code corresponding to $\mathcal{M}$. Setting $x_P = \mathcal{M}(P) \in \mathbb{N}$ for all points $P \in \mathcal{P}$ all constraints that do not involve a y-variable are satisfied. If C' has locality 1, so does C and $\mathcal{M}$. From Lemma 6 we conclude $x_P = \mathcal{M}(P) \neq 1$ for all $P \in \mathcal{P}$. So, if $\mathcal{M}(P) = 0$ we can set $y_P = 1$. If $2 \leq \mathcal{M}(P) \leq \Lambda$

[1] For $k \geq 3$, e.g. $d \geq (k-2)q^{k-1} - (k-1)q^{k-2} + 1$ is sufficient [Mar97].

for a sufficiently large constant Λ we can set $y_P = 1$. With this all constraints are satisfied, i.e., we have constructed a feasible solution of the above ILP. We remark that choosing $\Lambda = \lceil d/q^{k-1} \rceil \cdot {k \brack 1}_q \ge n$ always works considering a t-fold simplex code with $t = \lceil d/q^{k-1} \rceil$. □

We remark that we may also start with a rather small value for Λ. If we find a feasible solution with target value n, then we can deduce $n_q(k, d, r) \le n$. Using this n we can utilize Inequality (4) to deduce an upper bound for Λ.

Using indicator variables $u_P \in \{0, 1\}$ for $\mathcal{M}(P) \ge 1$ and $z_L \in \{0, 1\}$ for lines L containing at least three points with positive multiplicity we can adjust the previous ILP to model multisets of points with locality 2.

Proposition 12. *Let q, k, and d be arbitrary but fixed parameters. Then, $n_q(k, d, 2)$ is given as the optimum target value of the following ILP:*

$$\begin{array}{rcl}
\min n & & \textit{subject to} \\
\sum_{P \in \mathcal{P}} x_P & = & n \\
x_{\langle e_i \rangle} & \ge & 1 \quad \forall 1 \le i \le k \\
\sum_{P \in \mathcal{P}\,:\, P \le H} x_P & \le & n - d \quad \forall H \in \mathcal{H} \\
x_P & \ge & u_P \quad \forall P \in \mathcal{P} \\
x_P & \le & \Lambda u_P \quad \forall P \in \mathcal{P} \\
x_P & \ge & 2y_P \quad \forall P \in \mathcal{P} \\
\sum_{P \in \mathcal{P}\,:\, P \le L} u_P & \ge & 3z_L \quad \forall L \in \mathcal{L} \\
y_P + \sum_{L \in \mathcal{L}\,:\, P \le L} z_L & \ge & u_P \quad \forall P \in \mathcal{P} \\
x_P & \in & \mathbb{N} \quad \forall P \in \mathcal{P} \\
y_P & \in & \{0, 1\} \quad \forall P \in \mathcal{P} \\
u_P & \in & \{0, 1\} \quad \forall P \in \mathcal{P} \\
z_L & \in & \{0, 1\} \quad \forall L \in \mathcal{L},
\end{array}$$

where e_i denotes the ith unit vector, $\mathcal{P}$ denotes the set of points in $\mathrm{PG}(k-1, q)$, $\mathcal{L}$ denotes the set of lines in $\mathrm{PG}(k-1, q)$, $\mathcal{H}$ denotes the set of hyperplanes in $\mathrm{PG}(k-1, q)$, and Λ is a sufficiently large constant.

Lemma 13. *For each $n \ge n_q(k, d, r)$ there exists an $[n, k, d]_q$-code with locality r.*

Proof. Assume $n_q(k, d, r) < \infty$. Let C' be an $[n', k, d]_q$-code with $n' = n_q(k, d, r)$ and locality r. Consider the corresponding multiset of points $\mathcal{M}'$ in $\mathrm{PG}(k-1, q)$ and let P by an arbitrary point with positive multiplicity. Define the multiset of points $\mathcal{M}$ in $\mathrm{PG}(k-1, q)$ by setting $\mathcal{M}(Q) = \mathcal{M}'(Q)$ for all points $Q \ne P$ and $\mathcal{M}(P) = \mathcal{M}'(P) + n - n' \ge \mathcal{M}'(P)$. by construction we have $|\mathcal{M}| = n$ and $\mathcal{M}$ has locality r. The linear code C corresponding to $\mathcal{M}$ also has minimum Hamming distance at least d. □

The previous results are in principle sufficient to determine $n_q(k, d, r)$ for small parameters q, k, and r, so that we may just give tables of the obtained computational results. However, we prefer to give general constructions and non-existence results in the next section first.

4 Constructions and non-existence results

In this section we want to study some general constructions to upper bound $n_q(k, d, r)$. To this end we denote the ith unit vector by e_i, whenever the dimension of the ambient space is clear from the context, and for each subspace S its characteristic function is denoted by χ_S, i.e., $\chi_S(P) = 1$ if $P \le S$ and $\chi_S(P) = 0$ otherwise. In a few cases we can also give theoretical non-existence proofs for certain parameters to obtain lower bounds for $n_q(k, d, r)$. First, let us define $n'_q(k, r)$ as the minimum length n of an $[n, k]_q$-code with locality r. Clearly, we have $n_q(k, d, r) \ge n_q(k, r)$. As observed earlier, the maximum possible locality is given by $r = k$ and we have $n_q(k, d, 1) \ge n_q(k, d, 2) \ge \cdots \ge n_q(k, d, k)$.

Proposition 14. *For each integer $k \ge 1$ we have $n'_q(k, 1) = 2k$ and $n'_q(k, 2) = \left\lceil \frac{3k}{2} \right\rceil$.*

Proof. Let $\mathcal{M}$ be a spanning multiset of points in $\mathrm{PG}(k-1, q)$ with locality 1. Due to Lemma 6 every point P with positive multiplicity $\mathcal{M}(P) \ge 1$ satisfies $\mathcal{M}(P) \ge 2$. Since $\mathcal{M}$ is spanning, it contains at least k points with positive multiplicity, so that $|\mathcal{M}| \ge 2k$. An attaining example is e.g. given by $\mathcal{M} = \sum_{i=1}^{k} 2 \cdot \chi_{\langle e_i \rangle}$.

For the other case let $\mathcal{M}$ be a spanning multiset of points in $\mathrm{PG}(k-1, q)$ with locality 2. Due to Lemma 7 every point P with positive multiplicity $\mathcal{M}(P) \ge 1$ either has multiplicity at least 2 or is contained in a line L that contains at least three points of positive multiplicity. So, let $\{L_1, \dots, L_m\}$ be the set of those lines. For each index $1 \le i \le m$ let S_i be the subspace spanned by the points of the lines $L_1, \dots, L_i$. To also capture the case $m = 0$ let us denote the empty space by S_0. With this we have $\dim(S_0) = 0$, $\mathcal{M}(S_0) = 0$, $\dim(S_1) = 2$, and $\mathcal{M}(S_1) \ge 3$. For $i \ge 2$ we show $\mathcal{M}(S_i) \le \lceil 3\dim(S_i)/2 \rceil$ and $\dim(S_{i-1}) \le \dim(S_i) \le \dim(S_{i-1}) + 2$ by induction. So, if L_i is completely contained in S_{i-1}, then we have $\dim(S_i) = \dim(S_{i-1})$ and $\mathcal{M}(S_i) = \mathcal{M}(S_{i-1})$. If the intersection of S_{i-1} and L_i is non-empty but L_i is not completely contained in S_{i-1}, then $Q := S_{i-1} \cap L_i$ is a point. So, we have $\dim(S_i) = \dim(S_{i-1}) + 1$ and $\mathcal{M}(S_i) \ge \mathcal{M}(S_{i-1}) + 2$. If the intersection of L_i and S_{i-1} is empty, then we have $\dim(S_i) = \dim(S_{i-1}) + 2$ and $\mathcal{M}(S_i) \ge \mathcal{M}(S_{i-1}) + 3$. Now let us consider the set of points P with positive multiplicity that are not contained in S_m. By construction, none of these points is contained in a line L that contains at least three points with positive multiplicity. So, we have $\mathcal{M}(P) \ge 2$ for each such point. Since $\mathcal{M}$ is spanning we have at least $k - \dim(S_m)$ of those points and conclude

$$|\mathcal{M}| \ge \left\lceil \frac{3\dim(S_m)}{2} \right\rceil + 2\left(k - \dim(S_m)\right) \ge \left\lceil \frac{3k}{2} \right\rceil .$$

For even k an attaining example is given by the set of points

$$\{\langle e_{2i-1} \rangle, \langle e_{2i} \rangle, \langle e_{2i-1} + e_{2i} \rangle : 1 \le i \le k/2\}$$

and for odd k and example is given the sum of $2 \cdot \chi_{\langle e_k \rangle}$ and the characteristic function of the set of points

$$\{\langle e_{2i-1} \rangle, \langle e_{2i} \rangle, \langle e_{2i-1} + e_{2i} \rangle : 1 \le i \le (k-1)/2\} .$$

□

Corollary 15. *For each integer $k \ge 1$ we have $n_q(k, 1, 2) = n_q(k, 2, 2) = \left\lceil \frac{3k}{2} \right\rceil$.*

Proof. Due to Proposition 14 it suffices to state attaining examples and indeed we will just verify that the examples from the proof of Proposition 14 have minimum Hamming distance $d = 2$.

If k is even let $\mathcal{M} = \sum_{i=1}^{k} \chi_{\langle e_i \rangle} + \sum_{i=1}^{k/2} \chi_{\langle e_{2i-1} + e_{2i} \rangle}$, so that $|\mathcal{M}| = \frac{3k}{2}$, $\mathcal{M}$ is spanning, and has locality $r = 2$. For each $1 \le i \le k/2$ let L_i the the line spanned by the points $\langle e_{2i-1} \rangle$, $\langle e_{2i} \rangle$, and $\langle e_{2i-1} + e_{2i} \rangle$. For each hyperplane H at most $k/2 - 1$ lines L_i can be fully contained in H and the others intersect in at most a point, so that $\mathcal{M}(H) \le \frac{3k}{2} - 2$. Thus, the linear code C corresponding to $\mathcal{M}$ has minimum Hamming distance $d = 2$.

If k is odd let $\mathcal{M} = \sum_{i=1}^{k-1} \chi_{\langle e_i \rangle} + \sum_{i=1}^{(k-1)/2} \chi_{\langle e_{2i-1} + e_{2i} \rangle} + 2 \cdot \chi_{\langle e_k \rangle}$, so that $|\mathcal{M}| = \left\lceil \frac{3k}{2} \right\rceil = \frac{3k+1}{2}$, $\mathcal{M}$ is spanning, and has locality $r = 2$. For each $1 \le i \le (k-1)/2$ let L_i the the line spanned by the points $\langle e_{2i-1} \rangle$, $\langle e_{2i} \rangle$, and $\langle e_{2i-1} + e_{2i} \rangle$. By P be denote the point $\langle e_k \rangle$. For each hyperplane H that contains P at most $(k-1)/2 - 1$ lines L_i can be fully contained in H and the others intersect in at most a point, so that $\mathcal{M}(H) \le \frac{3k+1}{2} - 2$. For each hyperplane H that does not contain P we also have $\mathcal{M}(H) \le \frac{3k+1}{2} - 2$, so that the linear code C corresponding to $\mathcal{M}$ has minimum Hamming distance $d = 2$. □

Proposition 16. *For each integer $k \geq 2$ we have $n'_q(k,k) = k+1$.*

Proof. Let $\mathcal{M}$ be a spanning multiset of points in $\mathrm{PG}(k-1,q)$ with locality k. Since $\mathcal{M}$ is spanning we have $|\mathcal{M}| \geq k$. Up to symmetry the unique spanning multiset of points in $\mathrm{PG}(k-1,q)$ of cardinality k is given by the set of points $\{\langle e_i\rangle : 1 \leq i \leq k\}$ which does have a finite locality. Thus, we have $n'_q(k,k) \geq k+1$. An attaining example is given by the set of points

$$\{\langle e_i\rangle : 1 \leq i \leq k\} \cup \left\{\left\langle \sum_{i=1}^{k} e_i \right\rangle\right\}.$$

Here we can easily check that each subset of k points spans the entire ambient space. □

We remark that the point set of our construction is called *projective base* or *frame* in the literature. Due to Proposition 14 we also have $n'_q(k,k) = k+1$ for $k=1$ and we may also consider the double-point $2 \cdot \chi_{\langle e_1\rangle}$ as a degenerated cases of a projective base.

For small dimensions $k \leq 2$ the determination of $n_q(k,d,r)$ can be resolved completely analytically:

Proposition 17. *We have $n_q(1,d,r) = \max\{2,d\}$, $n_q(2,d,1) = 2\left\lceil \frac{d}{2}\right\rceil + 2$ if $d < 2q$, $n_q(2,d,1) = d + \left\lceil \frac{d}{q}\right\rceil$ if $d \geq 2q$, and $n_q(2,d,r) = d + \left\lceil \frac{d}{q}\right\rceil$ for $r \geq 2$.*

Proof. In $\mathrm{PG}(1-1,q)$ the unique multiset of points with cardinality n is given by $\mathcal{M} = n \cdot \chi_{\langle e_1\rangle}$ which does not have a finite locality if $n=1$ and has locality 1 if $n \geq 2$. Since there are no hyperplanes in $\mathrm{PG}(1-1,q)$ we need to observe that the non-zero weights of the codewords of the corresponding linear code C all are equal to n. Thus, we conclude $n_q(1,d,r) = \max\{2,d\}$.

Let $\mathcal{M}$ be a spanning multiset of points of cardinality n in $\mathrm{PG}(2-1,q)$ and C its corresponding linear code $[n,k]_q$-code. The minimum Hamming distance of C is at least d iff we have $\mathcal{M}(P) \leq n-d$ for every point P. Let us uniquely write $d = aq+b$ with $a \in \mathbb{N}$ and $b \in \{0,1,\dots,q-1\}$. With this we have

$$n \geq n_q(k,d,r) \geq n_q(k,d) \geq g_q(k,d) = \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = d + \left\lceil \frac{d}{q} \right\rceil = a(q+1) + b + \left\lceil \tfrac{b}{q} \right\rceil. \qquad (7)$$

Let L denote the ambient space, which is a line in our situation. If $b=0$ and $a \geq 1$, then $a \cdot \chi_L$ attains this bound and has locality 2 (or 1 if $a \geq 2$). If $b \geq 1$ and $a \geq 0$, then the multiset of points given by the sum of $a \cdot \chi_L$ and the characteristic function of arbitrary $b+1 \leq q$ different points on L attains this bound and has locality 2 if $a \geq 1$ or $b \geq 2$; we even have locality 1 if $a \geq 2$. So, for locality $r=2$ it remains to consider the case $d=1$ where we can consider the multiset of points $2 \cdot \chi_P$ for an arbitrary point P. Thus, for $r \geq 2$ we have $n_q(2,d,r) = d + \left\lceil \frac{d}{q}\right\rceil$ and $n_q(2,d,1) = d + \left\lceil \frac{d}{q}\right\rceil$ if $d \geq 2q$.

For locality $r=1$, dimension $k=2$, and $1 \leq d < 2q$ each point P in a multiset $\mathcal{M}$ with these parameters satisfying $\mathcal{M}(P) \geq 1$ indeed has to satisfy $\mathcal{M}(P) \geq 2$. If we have $l \leq 2$ points with positive multiplicity, then we have $n \geq 2l + \mathcal{M}(P) - 2 \geq 2l$ and $d \leq n - \mathcal{M}(P) \leq 2l-2$ for every point P with positive multiplicity. Thus, for even $2 \leq d < 2q$ we have $n_q(2,d,1) = d+2$ and for odd $1 \leq d < 2q$ we have $n_q(2,d,1) = d+3$. □

For $k \geq 3$ we remark that even the determination of $n_q(k,d)$ is a long-standing open problem for $q > 9$, so that we do not expect a closed-form solution for $n_q(k,d,r)$ when $k \geq 3$.

A well-known construction for distance-optimal linear codes is due to Solomon and Stiffler [SS65].

Lemma 18. *Let $k \geq 2$,*

$$n = \sigma[k]_q - \sum_{i=0}^{k-2} \varepsilon_i \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q,$$

and

$$n - d = \sigma[k-1]_q - \sum_{i=1}^{k-2} \varepsilon_i \begin{bmatrix} i \\ 1 \end{bmatrix}_q,$$

where $\sigma \in \mathbb{N}$ and $\varepsilon_i \in \mathbb{N}$ for all $0 \le i \le k-2$. If there exist subspaces $S_1, \ldots, S_l$ in $\mathrm{PG}(k-1,q)$ such that

$$\#\left\{1 \le j \le l \,:\, \dim(S_j) = i\right\} = \varepsilon_{i+1} \tag{8}$$

for $1 \le i \le k-1$ and

$$\#\left\{1 \le j \le l \,:\, P \in S_j\right\} \le \sigma \tag{9}$$

for each point P in $\mathrm{PG}(k-1,q)$, then an $[n,k,d]_q$-code exists.

In terms of a multiset of points the underlying construction is given by $\mathcal{M} = \sigma \cdot \chi_V - \sum_{j=1}^{l} \chi_{S_j}$, where V denotes the ambient space $\mathrm{PG}(k-1,q)$. In the literature mostly the case $n = g_k(k,d)$ is considered, while the construction works of course in general. There are also many criteria available in the literature when those subspaces S_j exist given the other numerical parameters. Here we will just speak of a *Solomon-Stifler construction of type* $[\sigma; \varepsilon_{k-2}, \ldots, \varepsilon_1, \varepsilon_0]$ and will mostly leave the existence proof for subspaces satisfying the conditions of Lemma 18 to the reader.

Example 19. *Let $\mathcal{M}$ be the multiset of points in $\mathrm{PG}(k-1,q)$, where $k \ge 2$, obtained from the Solomon-Stifler construction of type $[1; 1, 0, \ldots, 0]$ and C its corresponding $[n,k,d]_q$-code. Then, we have $n = {k \brack 1}_q - {k-1 \brack 1}_q = q^{k-1}$, $d = (q-1)q^{k-2}$, and $n = g_q(k,d)$. Note that the maximum point multiplicity of $\mathcal{M}$ is 1 and that very line L that contains two points of positive multiplicity has multiplicity $\mathcal{M}(L) = q$ since L intersects the subspace S_i in exactly a point. Thus, we have $n_q(k,(q-1)q^{k-2},2) = q^{k-1}$ for all $q \ge 3$.*

The just considered set of points is also known under the term *affine subspace*. For the binary case $q = 2$, where our construction does not give a multiset of points with locality 2, we can use the well-known result that each code with the above parameters can be obtained from the Solomon-Stifler construction of type $[1; 1, 0, \ldots, 0]$, see e.g. [KY21, Lemma 12], to conclude:

Lemma 20. *For each $k \ge 2$ we have $n_2(k, 2^{k-2}, 2) \ge 2^{k-1} + 1$.*

Lemma 21. *Let C' be a projective $[n', k', d']_2$-code with $k' \ge 2$. Then, we have $n_2(k'+1, \min\{2d', n'+1\}, 2) \le 2n' + 1$.*

Proof. Let $\mathcal{M}'$ be a spanning multiset of points in $\mathrm{PG}(k'-1,2)$ corresponding to C', so that $\mathcal{M}'$ has maximum point multiplicity 1. For each hyperplane H' of $\mathrm{PG}(k'-1,2)$ we have $\mathcal{M}'(H') \le n' - d'$. Let S be a k'-dimensional subspace in $\mathrm{PG}(k-1,2)$, where $k = k'+1$, and $\mathcal{P}$ be the set of points in $\mathrm{PG}(k-1,2)$ that arise from an embedding of $\mathcal{M}'$ in S. Choose an arbitrary point P outside of S and let $\mathcal{L}$ be the set of lines spanned by P and each element of $\mathcal{P}$, so that $|\mathcal{L}| = n'$. With this, we define the multiset of points $\mathcal{M}$ in $\mathrm{PG}(k,2)$ as the characteristic function of the points contained in at least one line in $\mathcal{L}$, so that $|\mathcal{M}| = 2n'+1$. For each hyperplane H of $\mathrm{PG}(k-1,2)$ that contains P we have $\mathcal{M}(H) \le 1 + 2(n'-d')$ since at most $n'-d'$ lines of $\mathcal{L}$ can be fully contained in H. (The other lines intersect the hyperplane H just in point P.) For every other hyperplane H in $\mathrm{PG}(k-1,2)$, i.e. P is not contained in H, we have $\mathcal{M}(H) = n'$ since any line in $\mathcal{L}$ intersects H in precisely a point. Thus, we have $\mathcal{M}(H) \le \max\{1 + 2(n'-d'), n'\}$. So, for the linear code C corresponding to $\mathcal{M}$ is an $[n,k,d]_2$-code with $n = 2n'+1$, $k = k'+1$ and $d = \min\{2d', n'+1\}$. By construction, every point in $\mathrm{PG}(k-1,2)$ with positive multiplicity w.r.t. $\mathcal{M}$ lies on a line consisting of three points with positive multiplicity each. Thus, $\mathcal{M}$ as well as C have locality 2 and we can deduce the stated upper bound. □

Proposition 22. *For each $k \ge 2$ we have $m_2(k, 2^{k-2}, 2) = 2^{k-1} + 1$.*

Proof. Due to Lemma 20 it suffices give a construction showing $m_2(k, 2^{k-2}, 2) \le 2^{k-1} + 1$. For $k = 2$ such an example is given by the characteristic function of a line. For $k \ge 3$ we apply Lemma 21 with the first order Reed-Muller code with parameters $(n', k', d') = \left(2^{k-2}, k-1, 2^{k-3}\right)$ (which corresponds to the characteristic function of an affine space in geometrical terms). □

Next we want to give an easy sufficient criterion when a code obtained from the Solomon-Stifler construction has locality 2:

Lemma 23. *Let $\mathcal{M}$ be a spanning multiset of points in* $\operatorname{PG}(k-1,q)$ *obtained from the Solomon-Stifler construction with type* $[\sigma;\varepsilon_{k-2},\dots,\varepsilon_1,\varepsilon_0]$. *If* $\sum_{i=0}^{k-1}\varepsilon_i\cdot\left[{i+1 \atop 1}\right]_q<\sigma\cdot\left[{k-1 \atop 1}\right]_q$, *then* $\mathcal{M}$ *has locality* $r=2$.

Proof. Let $\mathcal{M}=\sigma\cdot\chi_V-\sum\limits_{j=1}^{l}\chi_{S_j}$ using the notation from Lemma 18, where V denotes the ambient space $\operatorname{PG}(k-1,q)$. In $\operatorname{PG}(k-1,q)$ each point P is on $\left[{k-1 \atop 1}\right]_q$ many lines and so χ_V as well as $\sigma\cdot\chi_V$ contain $\left[{k-1 \atop 1}\right]_q$ lines through P in its support. Since $\mathcal{M}$ arises from $\sigma\cdot\chi_V$ by decreasing point multiplicities by $\sum_{i=0}^{k-1}\varepsilon_i\cdot\left[{i+1 \atop 1}\right]_q<\sigma\cdot\left[{k-1 \atop 1}\right]_q$ and removing a point from the support costs multiplicity σ, at least one full line through P remains if $\mathcal{M}(P)\ge 1$. □

Theorem 24. *For each* $t\in\mathbb{N}$ *we have* $n_2(3,3+4t,2)=6+7t$, $n_2(3,4+4t,2)=7+7t$, $n_2(3,5+4t,2)=10+7t$, *and* $n_2(3,6+4t,2)=11+7t$. *Moreover, we have* $n_2(3,1,2)=n_2(3,2,2)=5$.

Proof. In $\operatorname{PG}(3-1,2)$ consider Solomon-Stifler constructions with types $[t+1;0,1]$, $[t+1;0,0]$, $[t+2;1,1]$, and $[t+2;1,0]$, respectively. The lengths n and minimum distances d as well as the dimension $k=3$ are as stated. Using Lemma 23 we can easily check that all those examples have locality $r=2$ (and in some cases even locality $r=1$). Proposition 22 yields $n_2(3,2,2)=5$, so that it remains to show $n_2(3,1,2)\ge 5$, which is implied by Proposition 14. □

Corollary 25. *We have* $n_2(3,1,2)=g_2(3,1)+2$, $n_2(3,2,2)=g_2(3,2)+1$, *and* $n_2(3,d,2)=g_2(3,d)$ *for all* $d\ge 3$.

We remark that we have $n_2(3,d)=g_2(3,d)$ for all $d\ge 1$.

Lemma 26. *For* $k\ge 4$ *we have* $n_2(k,3,2)\le 2k$.

Proof. For $1\le i\le k-1$ let $L_i=\langle e_i,e_{i+1}\rangle$ and $L_k=\langle e_k,e_1\rangle$. With this, let $\mathcal{M}$ be the characteristic function of all points that are contained in one of the lines L_i, so that $|\mathcal{M}|=2k$, $\mathcal{M}$ has locality $r=2$, and $\mathcal{M}$ is spanning. For each hyperplane H there exists and index $1\le i\le k$ such that $P:=\langle e_i\rangle$ is not contained in H. The two lines L_j that contain P intersect H in precisely a point so that $\mathcal{M}(H)\le 2k-3$. Thus, the linear code C corresponding to $\mathcal{M}$ has minimum Hamming distance d. □

Theorem 27. *For each* $t\in\mathbb{N}$ *we have* $n_2(4,5+8t,2)=11+15t$, $n_2(4,6+8t,2)=12+15t$, $n_2(4,7+8t,2)=14+15t$, $n_2(4,8+8t,2)=15+7t$, $n_2(4,9+8t,2)=19+15t$, $n_2(4,10+8t,2)=20+15t$, $n_2(4,11+8t,2)=22+15t$, *and* $n_2(4,12+8t,2)=23+15t$. *Moreover, we have* $n_2(4,1,2)=n_2(4,2,2)=6$, $n_2(4,3,2)=8$, *and* $n_2(4,4,2)=9$.

Proof. In $\operatorname{PG}(4-1,2)$ consider Solomon-Stifler constructions with types $[t+1;0,1,1]$, $[t+1;0,1,0]$, $[t+1;0,0,1]$, $[t+1;0,0,0]$, $[t+2;1,1,1]$, $[t+2;1,1,0]$, $[t+2;1,0,0]$, and $[t+2;1,0,0]$, respectively. The lengths n and minimum distances d as well as the dimension $k=4$ are as stated. Using Lemma 23 we can easily check that all those examples have locality $r=2$ (and in some cases even locality $r=1$).

Proposition 22 yields $n_2(4,4,2)=9$. Proposition 14 yields $n_2(4,3,2)\ge n_2(4,2,2)\ge n_2(4,1,2)\ge\left\lceil\frac{3\cdot 4}{2}\right\rceil=6$. Corollary 15 yields $n_2(4,1,2)=n_2(4,2,2)=6$ and Lemma 26 yields $n_2(4,3,2)\le 8$.

Due to Lemma 13 it suffices to assume that $\mathcal{M}$ is a spanning multiset of points in $\operatorname{PG}(4-1,2)$ with cardinality 7, locality 2, $\mathcal{M}(H)\le 4$ for every hyperplane H and conclude a contradiction. First, assume that P_1 is a point with multiplicity at least 2. If L is a line in the support of $\mathcal{M}$, then we have $P_1\le L$ since otherwise the hyperplane spanned by P_1 and L would have multiplicity at least 5. However, for any point P_2 with positive multiplicity that is not contained in L the hyperplane spanned by P_2 and L has multiplicity at least 5 – contradiction. So, if one point P_1 has multiplicity at least two, then all points with positive multiplicity have multiplicity at least two. However, a hyperplane spanned by three such points, that are not contained in a line, has multiplicity at least 6 – contradiction. Thus, the maximum point multiplicity of $\mathcal{M}$ is 1 and for each point P with positive multiplicity there exists a line L_P in the support of $\mathcal{M}$. No two different such lines can intersect in a point since otherwise the hyperplane spanned by these two lines would have multiplicity at least 5. However, since 7 is not divisible by 3 the points with positive multiplicity cannot be partitioned into pairwise disjoint lines. □

Corollary 28. *We have* $n_2(4,1,2) = g_2(4,1) + 2$*,* $n_2(4,2,2) = g_2(4,2) + 1$*,* $n_2(4,2,3) = g_2(4,3) + 1$*,* $n_2(4,2,4) = g_2(4,4) + 1$*, and* $n_2(4,d,2) = g_2(4,d)$ *for all* $d \geq 5$*.*

We remark that we have $n_2(4,d) = g_2(4,d)$ for all $d \geq 1$.

Theorem 29. *For each* $t \in \mathbb{N}$ *we have* $n_2(5, 9+16t, 2) = 20 + 31t$*,* $n_2(5, 10+16t, 2) = 21+31t$*,* $n_2(5, 11+16t, 2) = 23 + 31t$*,* $n_2(5, 12+16t, 2) = 24 + 31t$*,* $n_2(5, 13+16t, 2) = 27 + 31t$*,* $n_2(5, 14+16t, 2) = 28 + 31t$*,* $n_2(5, 15+16t, 2) = 30 + 31t$*,* $n_2(5, 16+16t, 2) = 31 + 31t$*,* $n_2(5, 17+16t, 2) = 36 + 31t$*,* $n_2(5, 18+16t, 2) = 37 + 31t$*,* $n_2(5, 19+16t, 2) = 39 + 31t$*,* $n_2(5, 20+16t, 2) = 40 + 31t$*,* $n_2(5, 21+16t, 2) = 43 + 31t$*,* $n_2(5, 22+16t, 2) = 44 + 31t$*,* $n_2(5, 23+16t, 2) = 46 + 31t$*, and* $n_2(5, 24+16t, 2) = 47 + 31t$*. Moreover, we have* $n_2(5,1,2) = n_2(5,2,2) = 8$*,* $n_2(5,3,2) = 10$*,* $n_2(5,4,2) = 11$*,* $n_2(5,5,2) = 13$*,* $n_2(5,6,2) = 14$*,* $n_2(5,7,2) = 16$*, and* $n_2(5,8,2) = 17$*.*

Proof. In $\text{PG}(5-1,2)$ consider Solomon-Stifler constructions with types $[t+1;0,1,1,1]$, $[t+1;0,1,1,0]$, $[t+1;0,1,0,1]$, $[t+1;0,1,0,0]$, $[t+1;0,0,1,1]$, $[t+1;0,0,1,0]$, $[t+1;0,0,0,1]$, $[t+1;0,0,0,0]$, $[t+2;1,1,1,1]$, $[t+2;1,1,1,0]$, $[t+2;1,1,0,1]$, $[t+2;1,1,0,0]$, $[t+2;1,0,1,1]$, $[t+2;1,0,1,0]$, $[t+2;1,0,0,1]$, and $[t+2;1,0,0,0]$, respectively. The lengths n and minimum distances d as well as the dimension $k = 5$ are as stated. Using Lemma 23 we can easily check that all those examples have locality $r = 2$ (and in some cases even locality $r = 1$).

Proposition 22 yields $n_2(5,8,2) = 17$. Proposition 14 yields $n_2(5,3,2) \geq n_2(5,2,2) \geq n_2(5,1,2) \geq \left\lceil \frac{3 \cdot 5}{2} \right\rceil = 8$. Corollary 15 yields $n_2(5,1,2) = n_2(5,2,2) = 8$. Lemma 26 implies $n_2(5,3,2) \leq 10$. Applying Lemma 21 to an $[5,4,2]_2$-code gives $n_2(5,4,2) \leq 11$. The generator matrices

$$\begin{pmatrix} 1111111010000 \\ 0001111101000 \\ 0110011100100 \\ 1011100000010 \\ 1000111000001 \end{pmatrix}, \quad \begin{pmatrix} 11111110010000 \\ 00011111101000 \\ 01100110100100 \\ 10101011100010 \\ 11010011000001 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0000000001111111 \\ 0001111110000011 \\ 0010001110001101 \\ 0100010110010110 \\ 1000111000101001 \end{pmatrix}$$

give $[13,5,5]_2$-, $[14,5,6]_2$-, and $[16,5,7]_2$-codes with locality 2, so that we have $n_2(5,5,2) \leq 13$, $n_2(5,6,2) \leq 14$, and $n_2(5,7,2) \leq 16$. For the lower bounds we have $n_2(5,3,2) \geq n_2(5,3) = 10$, $n_2(5,4,2) \geq n_2(5,4) = 11$, $n_2(5,5,2) \geq n_2(5,5) = 13$, and $n_2(5,6,2) \geq n_2(5,6) = 14$. Finally, if C is an $[15,5,7]_2$-code with locality $r = 2$, then adding a parity check bit yields an $[16,5,8]_2$-code C'. Let $\mathcal{M}$ and $\mathcal{M}'$ denote the multisets of points corresponding to C and C', respectively. Given its parameters, the code C' is unique up to isomorphism and can be obtained by the Solomon-Stifler construction of type $[1;1,0,0,0,0]$, see e.g. [KY21, Lemma 12]. Note that the maximum point multiplicity of $\mathcal{M}'$ is 1 and that $\mathcal{M}'$ does not contain a full line in its support. Since $\mathcal{M}'$ arises from $\mathcal{M}$ by increasing the point multiplicity of a (specific) point by one, also the maximum point multiplicity of $\mathcal{M}$ is 1 and $\mathcal{M}'$ does not contain a full line in its support. Thus, $\mathcal{M}$ cannot have locality 2 and we have $n_2(5,7,2) = 16$. □

Corollary 30. *We have* $n_2(5,1,2) = g_2(5,1) + 3$*,* $n_2(5,2,2) = g_2(5,2) + 2$*,* $n_2(5,2,3) = g_2(5,3) + 2$*,* $n_2(5,2,4) = g_2(5,4) + 2$*,* $n_2(5,2,5) = g_2(5,5) + 1$*,* $n_2(5,2,6) = g_2(5,6) + 1$*,* $n_2(5,2,7) = g_2(5,7) + 1$*,* $n_2(5,2,8) = g_2(5,8) + 1$*, and* $n_2(5,d,2) = g_2(5,d)$ *for all* $d \geq 9$*.*

We remark that we have $n_2(5,d) = g_2(5,d)$ for all $d \in \mathbb{N}\backslash\{3,4,5,6\}$. Note that the code C' used in the proof of Theorem 29 to show $n_2(5,7,2) > 15$ also has to be even, i.e., all of its weights are divisible by 2. So, for some parameters we might be able to show that the weight distribution of an even $[n,k,d]_2$-code C' (where also d is even) is unique and can be determined using theoretical methods. So, for each $[n-1,k,d-1]_2$-code C adding a parity check bit yields such an even $[n,k,d]_2$-code with known weight distribution. Applying the MacWilliams transform we then compute also the dual weight distribution of C'. To this end let us slightly generalize Lemma 8 and the proof idea for $n_2(5,7,2) > 15$.

Lemma 31. *Let C be an $[n,k,d]_2$-code with odd minimum distance d and C' be the $[n+1,k,d+1]_2$-code obtained from C by adding a parity check bit. If C' has dual minimum distance $d^\perp = 3$ and at less than $n/3$ dual codewords of weight 3, then the locality of C is larger than 2.*

Proof. Let $\mathcal{M}$ and $\mathcal{M}'$ be the multisets of points corresponding to C and C', respectively. Note that $\mathcal{M}'$ arises from $\mathcal{M}$ by increasing the point multiplicity of a certain point P by 1. Using the fact that the dual minimum distance of C' is 3 we conclude that both $\mathcal{M}'$ and $\mathcal{M}$ have a maximum point multiplicity of 1. Since C' contains less than $n/3$ dual codewords of weight 3, at most $n-1$ points with positive multiplicity in $\mathcal{M}'$ can be contained in a line that is fully contained in the support of $\mathcal{M}'$. Thus, there exists a point $Q \neq P$ with $\mathcal{M}(Q) = 1$ that is not contained in line L that is fully contained in the support of $\mathcal{M}$. Using Lemma 7 we conclude that the locality of $\mathcal{M}$ and C is at least 3. □

Lemma 32. *For each $t \in \mathbb{N}_{\ge 2}$ we have $n_q(2t,4,2) \le 3t+3$ and $n_2(2t+1,4,2) \le 3t+5$.*

Proof. Consider the t triples of points $L_i = \{e_{2i-1}, e_{2i}, e_{2i-1}+e_{2i}\}$ for $1 \le i \le t$, the triple of points $L' = \left\{\sum_{i=1}^{t} e_{2i-1}, \sum_{i=1}^{t} e_{2i}, \sum_{i=1}^{2t} e_i\right\}$, and the triple of points $L'' = \left\{\sum_{i=1}^{t+1} e_{2i-1}, \sum_{i=1}^{t} e_{2i}, \sum_{i=1}^{2t+1} e_i\right\}$. (Over $\mathbb{F}_2$ these triples are full lines.) With this let $\mathcal{M} = \chi_{L'} + \sum_{i=1}^{t} \chi_{L_i}$ and $\mathcal{M}' = 2\cdot\chi_P + \chi_{L''} + \sum_{i=1}^{t} \chi_{L_i}$, where $P = \langle e_{2t+1}\rangle$. Note that $\mathcal{M}$ spans $\mathrm{PG}(2t-1,q)$, $\mathcal{M}'$ spans $\mathrm{PG}(2t,q)$, $|\mathcal{M}| = 3t+3$, $|\mathcal{M}'| = 3t+5$, and both multisets of points have locality $r=2$. So, it remains to upper bound the multiplicities of the hyperplanes of the respective ambient spaces. By construction, the multiplicities of $H\cap L'$, $H\cap L''$, and $H\cap L_i$, where $1\le i\le t$, are not equal to 2 for each hyperplane H.

Let us first consider $\mathcal{M}$ in $\mathrm{PG}(2t-1,q)$. Note that a hyperplane H cannot fully contain all L_i for $1\le i\le t$. Due to symmetry we assume that L_1 is not fully contained in H. If also another triple L_i with $2\le i\le t$ is not fully contained in H, then we have $\mathcal{M}(H) \le 3t+3-2\cdot 2$. So, let us assume that H fully contains all triples L_i for $2\le i\le t$. If $|H\cap L_1| = 1$, then due to symmetry we assume $\langle e_1\rangle \le H$, so that H is uniquely determined and we have $\mathcal{M}(H) \le 3t+3-2\cdot 2$ since L' is not fully contained in H. If $|H\cap L_1| = 0$ then L' cannot be fully contained in H, so that $\mathcal{M}(H) \le 3t+3-5$.

Next consider $\mathcal{M}'$ in $\mathrm{PG}(2t,q)$. There is a unique hyperplane H that fully contains L_i for $1\le i\le t$. Here we have $\mathcal{M}'(H) \le 3t+5-4$ since H does not contain P and also does not fully contain L''. In the remaining cases we assume due to symmetry that H does not fully contain L_1. If also another triple L_i with $2\le i\le t$ is not fully contained in H, then we have $\mathcal{M}(H) \le 3t+5-2\cdot 2$. Similarly, if H does not fully contain L'', then we have $\mathcal{M}(H) \le 3t+5-2\cdot 2$. So, let us assume that H fully contains all triples L_i for $2\le i\le t$ and also fully contains L''. However, then there is a unique possibility for H and we can easily check that P is not contained in H and we also have $\mathcal{M}(H) \le 3t+5-2\cdot 2$. □

Note that the upper bounds in Lemma 32 are valid for $t<2$ also while there are better constructions for dimension $k=1$ and dimension $k=3$, when $q=2$. In Table 2 we will see that there are 41 $[12,6,4]_2$-codes while only the example from Lemma 32 has locality $r=2$.

Theorem 33. *For each $d \in \mathbb{N}_{\ge 21} \cup \{17,18\}$ we have $n_2(6,d,2) = g_2(6,d)$. Moreover, we have $n_2(6,1,2) = n_2(6,2,2) = 9$, $n_2(6,3,2) = n_2(6,4,2) = 12$, $n_2(6,5,2) = n_2(6,6,2) = 15$, $n_2(6,7,2) = n_2(6,8,2) = 18$, $n_2(6,9,2) = 22$, $n_2(6,10,2) = 23$, $n_2(6,11,2) = 25$, $n_2(6,12,2) = 26$, $n_2(6,13,2) = 30$, $n_2(6,14,2) = 31$, $n_2(6,15,2) = n_2(6,16,2) = 33$, $n_2(6,19,2) = 41$, and $n_2(6,19,2) = 42$.*

Proof. For $d \ge 21$ we can consider the Solomon-Stifler construction and use Lemma 23 to check that all those examples have locality $r=2$ (and in some cases even locality $r=1$). Proposition 22 yields $n_2(6,16,2) = 33$ and Corollary 15 yields $n_2(6,1,2) = n_2(6,2,2) = 9$. Lemma 32 gives $n_2(6,3,2) \le n_2(6,4,2) \le 12$.

The generator matrices

$$\begin{pmatrix}111111100100000\\000111110010000\\011001101001000\\100011101000100\\101110010000010\\001110101000001\end{pmatrix},\quad \begin{pmatrix}111111111110100000\\000001111111010000\\001110001111001000\\010110110011000100\\111000010111000010\\011011100101000001\end{pmatrix},\quad \begin{pmatrix}1111111111111110100000\\0000000111111111010000\\0001111000011111001000\\0110011001100111000100\\1000101011111000000010\\0011100110101010000001\end{pmatrix},$$

$$\begin{pmatrix}11111111111000000100000\\00000111111111110010000\\00111001111001111001000\\01011010001010111000100\\11101100111111011000010\\11110011011111101000001\end{pmatrix},\quad \begin{pmatrix}1111111111111110000100000\\0000000111111111110010000\\0001111000011110111001000\\0110011001100111011000100\\1010111110101011111000010\\0100101010111101001000001\end{pmatrix},\quad \begin{pmatrix}11111111111111100000100000\\00000001111111111100010000\\00011110000111101110001000\\01100110011001101101000100\\10001010101000111111000010\\00111101001010110101000001\end{pmatrix},$$

$$\begin{pmatrix}111111111111111111111110100000\\000000000011111111111111010000\\000001111110000000111111001000\\001110001110001110001111000100\\010110110010110010010110000010\\100011010110010110101010000001\end{pmatrix},\quad \begin{pmatrix}1111111111111111111110000100000\\0000000001111111111111110010000\\0000011110000001111110111001000\\0011100110001110001111111000100\\0101111000110010010111001000010\\0110001011001110110010011000001\end{pmatrix},$$

$$\begin{pmatrix}1111111111111111111000000000000100000\\0000000001111111111111111110000010000\\0000011111000011111000011111110001000\\0011100011001100111001100110111000100\\0100101101010101001010101011011000010\\1001100110100101010100101101101000001\end{pmatrix},\quad \begin{pmatrix}11111111111111111110000000000000100000\\00000000011111111111111111110000010000\\00001111100001111110000111111110001000\\00110001100110001110011001110111000100\\01010010101010110010101010111011000010\\10010111001101010111110100011101000001\end{pmatrix},$$

$$\begin{pmatrix}11111111111111111111111000000000000100000\\00000000000011111111111111111111000010000\\00000001111110000111111000000111111001000\\00011110001110011000111001110011011000100\\01100110110010101001011010110101101000010\\10101010010100110010001111011001110000001\end{pmatrix},\quad \begin{pmatrix}111111111111111111111110000000000000100000\\000000000001111111111111111111111100010000\\000001111110000001111110000011111110001000\\001110001110000110011110011100011101000100\\010110010010011011100110100100101111000010\\111010110100101110101011101101010111000001\end{pmatrix}$$

give $[15,6,6]_2$-, $[18,6,8]_2$-, $[22,6,9]_2$-, $[23,6,10]_2$-, $[25,6,11]_2$-, $[26,6,12]_2$-, $[30,6,13]_2$-, $[31,6,14]_2$-, $[37,6,17]_2$-, $[38,6,18]_2$-, $[41,6,19]_2$-, and $[42,6,20]_2$-codes with locality $r=2$, so that we have $n_2(6,5,2) \le n_2(6,6,2) \le 15$, $n_2(6,7,2) \le n_2(6,8,2) \le 18$, $n_2(6,9,2) \le 22$, $n_2(6,10,2) \le 23$, $n_2(6,11,2) \le 25$, $n_2(6,12,2) \le 26$, $n_2(6,13,2) \le 30$, $n_2(6,14,2) \le 31$, $n_2(6,17,2) \le 37$, $n_2(6,18,2) \le 38$, $n_2(6,19,2) \le 41$, and $n_2(6,20,2) \le 42$. For the lower bounds we have $n_2(6,6,2) \ge n_2(6,6) = 15$, $n_2(6,8,2) \ge n_2(6,8) = 18$, $n_2(6,9,2) \ge n_2(6,9) = 22$, $n_2(6,10,2) \ge n_2(6,10) = 23$, $n_2(6,11,2) \ge n_2(6,11) = 25$, $n_2(6,12,2) \ge n_2(6,12) = 26$, $n_2(6,17,2) \ge n_2(6,17) = 37$, $n_2(6,18,2) \ge n_2(6,18) = 38$, $n_2(6,19,2) \ge n_2(6,19) = 41$, and $n_2(6,20,2) \ge n_2(6,20) = 42$. Due to length restrictions we deduce the lower bounds $n_2(6,4,2) \ge n_2(6,3,2) \ge 12$, $n_2(6,5,2) \ge 15$, $n_2(6,7,2) \ge 18$, $n_2(6,13,2) \ge 30$, $n_2(6,14,2) \ge 31$, and $n_2(6,15,2) \ge 33$ from ILP computations.

□

We remark that our stated $[18,6,8]_2$-code is a projective two-weight code. It belongs to the family of BY codes [BE97]. A $[26,6,12]_2$-code with locality $r=2$ can be obtained by shortening the unique $[27,7,12]_2$-code.[2] There is a unique $[38,6,18]_2$-code, see e.g. [BJ01], shortening gives $[37,6,17]_2$-codes with locality $r=2$. Note that both codes attain the Griesmer bound, see e.g. [Hel83] for constructions of binary codes

[2]Uniqueness was e.g. computationally verified in [BJ01]. For a purely theoretic argument see "The uniqueness of the binary linear $[27,7,12]$ code" by A. E. Brouwer from April 1992, available at https://www.win.tue.nl/∼aeb/preprints.html.

attaining the Griesmer bound when $d > 2^{k-1}$. We remark that there are 7 non-isomorphic $[43, 7, 20]_2$ codes, see e.g. [BJ01], and all their shortenings yield $[42, 6, 20]_2$-codes with locality $r = 2$.

Theorem 34. *We have* $n_2(7,1,2) = n_2(7,2,2) = 11$, $n_2(7,3,2) = n_2(7,4,2) = 14$, $n_2(7,5,2) = 17$, $n_2(7,6,2) = 18$, $n_2(7,7,2) = n_2(7,8,2) = 20$, $n_2(7,9,2) = 24$, $n_2(7,10,2) = 25$, $n_2(7,11,2) = 27$, $n_2(7,12,2) = 28$, $n_2(7,29,2) = 62$, $n_2(7,30,2) = 63$, $n_2(7,31,2) = n_2(7,32,2) = 65$ *and* $n_2(7,d,2) = n_2(7,d)$ *for all other values of* d.

Proof. Corollary 15 yields $n_2(7,1,2) = n_2(7,2,2) = 11$, Lemma 32 yields $n_2(7,3,2) \le n_2(7,4,2) \le 14$, and Proposition 22 yields $n_2(6,32,2) = 65$. For $45 \le d \le 64$ and for $d \ge 73$ we can consider the Solomon-Stifler construction and use Lemma 23 to check that all those examples have locality $r = 2$ (and in some cases even locality $r = 1$). The Magma function BKLC(q, n, k) (Best Known Linear Codes) yields for small parameters q, k, and n an $[n, k, d]_q$-code that maximizes the minimum distance d [BCP97]. Using $q = 2$, $n = n_2(7, d)$, and $k = 7$ we obtained a series of codes, one for each value of d, that we can check for locality $r \le 2$. This check was successful for $d \in \{14, \ldots, 16\} \cup \{20, \ldots, 24\} \cup \{27, 28\} \cup \{33, \ldots, 44\} \cup \{65, \ldots, 72\}$.

The generator matrices

$$\begin{pmatrix} 11111110001000000 \\ 00011111100100000 \\ 01100110010010000 \\ 10100110100001000 \\ 11101111100000100 \\ 10011011000000010 \\ 01111111110000001 \end{pmatrix}, \begin{pmatrix} 111111111001000000 \\ 000000111110100000 \\ 000111111010010000 \\ 011001011000001000 \\ 011111101110000100 \\ 001111100000000010 \\ 011011000010000001 \end{pmatrix}, \begin{pmatrix} 1111111111001000000 \\ 0000011111100100000 \\ 0001100111101001000 0 \\ 0010111001101000100 0 \\ 0011101010110000010 0 \\ 0110111100010000001 0 \\ 1010100101110000000 1 \end{pmatrix},$$

$$\begin{pmatrix} 11111111111000000100000 0 \\ 00000111111111110010000 0 \\ 00111000011001111001000 0 \\ 01001001111010011000100 0 \\ 11011110011110111000010 0 \\ 00110110100101111000001 0 \\ 00111001101100100000000 1 \end{pmatrix}, \begin{pmatrix} 1111111111111100001000000 \\ 0000000111111111100100000 \\ 0001111000111101110010000 \\ 0010011011000111110001000 \\ 0110100101011100110000100 \\ 0111001000101010110000010 \\ 1100111000110010010000001 \end{pmatrix},$$

$$\begin{pmatrix} 111111111111111000001000000 \\ 000000000111111111110100000 \\ 000011111000111001110010000 \\ 001100111011001010110001000 \\ 110001011001110111000000100 \\ 110110011000011010010000010 \\ 011110101110110011110000001 \end{pmatrix}, \begin{pmatrix} 1111111111111111000000100000 0 \\ 0000000001111111111100100000 \\ 0000111110001110011110010000 \\ 0111000110010110101110001000 \\ 1011011001100011110000000100 \\ 0101101011100100110100000010 \\ 1001001110111000111000000001 \end{pmatrix},$$

$$\begin{pmatrix} 1111111111111110000000001000000 \\ 0000000111111111111111000100000 \\ 0000111000111110000111110010000 \\ 0011011011000110011011010001000 \\ 0111101101001010101100110000100 \\ 1001011101011000011101000000010 \\ 0010111011010011100101000000001 \end{pmatrix}, \begin{pmatrix} 111111111111111111110000000000001000000 \\ 000000000111111111111111111100000100000 \\ 000011111000000111110000111111100010000 \\ 001100111001110001100110011101110001000 \\ 010101011010010110101110100110110000100 \\ 011110001000111011011000101110000000010 \\ 101010011100010101111011000110100000001 \end{pmatrix},$$

$$\begin{pmatrix}
1111111111111111111100000000000000001000000\\
0000000001111111111111111111000000100000\\
0000111110000011111000011111111100010000\\
0011001110011100011001100011001110001000\\
0101010110101101100010100101010110000100\\
1001101010010100111111001000110010000010\\
1100001011001111001101010000101110000001
\end{pmatrix},\quad
\begin{pmatrix}
111111111111111111111100000000000001000000\\
000000000001111111111111111111100000100000\\
000000111111000011111100000111111100010000\\
000111001111001100001100011001101110001000\\
011011110011011101110111101011111110000100\\
101111010101100100111101110110110110000010\\
110101100011001110011010101100110100000001
\end{pmatrix},$$

give $n_2(7,5,2) \le 17$, $n_2(7,6,2) \le 18$, $n_2(7,7,2) \le n_2(7,8,2) \le 20$, $n_2(7,9,2) \le 24$, $n_2(7,10,2) = 25$, $n_2(7,11,2) \le 27$, $n_2(7,12,2) \le 28$, $n_2(7,13,2) \le 31$, $n_2(7,17,2) \le 39$, $n_2(7,18,2) \le 40$, and $n_2(7,19,2) \le 42$. For the other cases we will state explicit examples by listing sets of points for the occurring non-zero multiplicities, where the points are stated as integers whose base-2-representation is a generator of the point.

- $d = 55$: $1 \to \{1, 2, 4, 7, 8, 9, 12, 13, 16, 19, 21, 22, 25, 27, 32, 35, 37, 38, 39, 41, 44, 47, 49, 50, 55, 56, 59, 61, 64, 67, 73, 76, 81, 84, 90, 91, 94, 98, 100, 103, 104, 105, 107, 108, 110, 112, 115, 117, \ldots, 122\}$, $2 \to \{70\}$;
- $d = 56$: $1 \to \{1, 2, 4, 8, 10, \ldots, 13, 16, 19, 21, 22, 25, 31, 32, 35, 37, 38, 40, 41, 42, 44, 47, 50, 52, 55, 56, 59, 61, 62, 64, 67, 69, 74, 76, 80, 82, 84, 87, 91, 93, 94, 98, 100, 103, 110, 111, 112, 115, 117, 121, 122, 125\}$, $3 \to \{97\}$;
- $d = 62$: $1 \to \{1, 2, 4, 5, 8, \ldots, 11, 13, 14, 16, \ldots, 19, 21, 22, 28, 30, 32, 35, 36, 37, 40, 43, 44, 46, 49, 51, 52, 55, 59, 60, 61, 64, 67, 69, 72, 74, 76, 79, 81, 82, 85, 87, 90, \ldots, 93, 98, 99, 103, 105, 106, 112, 115, 120, 122, 125, 126, 127\}$, $2 \to \{101\}$;
- $d = 63$: $1 \to \{1, 3, 4, 7, 8, 11, 13, 15, 16, 17, 20, 21, 26, 27, 29, 30, 32, 33, 40, 42, 45, 46, 49, 50, 54, 55, 56, 59, 60, 64, 65, 68, 73, 74, 76, 77, 81, 83, 85, 86, 88, 91, 93, 95, 98, 99, 101, 102, 106, 107, 108, 111, 114, 115, 120, 125, 126\}$, $2 \to \{2, 36, 116\}$;
- $d = 65$: $1 \to \{1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31, 32, 35, 37, 38, 41, 42, 44, 47, 49, 50, 52, 55, 56, 59, 61, 62, 64, 67, 69, 70, 73, 74, 76, 79, 81, 82, 84, 87, 88, 91, 93, 94, 96, 97, 98, 100, 103, 104, 107, 109, 110, 112, 115, 117, 118, 121, 122, 124, 127\}$.

The lower bounds in those cases where $n_2(7,d,2) > n_2(7,d)$ are verified using ILP computations. □

Lemma 35. *We have* $n_q(k, 2d', 1) \le 2 \cdot n_q(k, d')$, $n_q(k, d' + 2 \cdot q^{k-1}, 1) \le 2 \cdot \left(q^k - 1\right)/(q-1) + n_q(k, d')$, *and* $n_q(k, 3d' - 1, 1) \le 3 \cdot n_q(k, d') - 1$.

Proof. If $\mathcal{M}$ is the multiset of points corresponding to an $[n', k, d']_q$-code, then $2 \cdot \mathcal{M}$ corresponds to an $[2n', k, 2d']_q$-code with locality 1 since each point has an even multiplicity. Denoting the ambient space by A, we have that $\mathcal{M} + 2 \cdot \chi_A$ corresponds to an $\left[2 \cdot \left(q^k - 1\right)/(q-1) + n', k, 2 \cdot q^{k-1} + d'\right]_q$-code with locality $r = 1$ since each point has multiplicity at least 2. Observing that $3 \cdot \mathcal{M}$ corresponds to an $[3n', k, 3d']_q$-code with locality 1 since each point has an multiplicity that is divisible, we can decrease the point multiplicity of one point with positive multiplicity by one to obtain an $[3n' - 1, k, 3d' - 1]_q$-code. □

Theorem 36. *We have* $n_2(3,1,1) = n_2(3,2,1) = 6$, $n_2(3,3,1) = n_2(3,4,1) = 8$, $n_2(3,5,1) = 11$, $n_2(3,6,1) = 12$, $n_2(3,7,1) = 14$, *and* $n_2(3,d,1) = n_2(3,d)$ *for all* $d \ge 8$.

Proof. The lower bounds for $d \le 7$ can be obtained by ILP computations. The necessary upper bounds can be deduced from Lemma 35 and $n_2(k, d-1, 1) \le n_2(k, d, 1)$. □

Theorem 37. *We have* $n_2(4,1,1) = n_2(4,2,1) = 8$, $n_2(4,3,1) = n_2(4,4,1) = 10$, $n_2(4,5,1) = n_2(4,6,1) = 14$, $n_2(4,7,1) = n_2(4,8,1) = 16$, $n_2(4,9,1) = 20$, $n_2(4,10,1) = 22$, $n_2(4,11,1) = 23$, $n_2(4,12,1) = 24$, $n_2(4,13,1) = 27$, $n_2(4,14,1) = 28$, $n_2(4,15,1) = 30$, $n_2(4,d,1) = n_2(4,d)$ *for all* $d \ge 16$.

Proof. The lower bounds for $d \le 15$ can be obtained by ILP computations. Except $d = 9$ and $d = 13$, the necessary upper bounds can be deduced from Lemma 35 and $n_2(k, d-1, 1) \le n_2(k, d, 1)$. For $d = 9$ let A denote the ambient space and $E := \langle e_1, e_2, e_3 \rangle$. A suitable $[19, 4, 9]_2$-code with locality $r = 1$ can be obtained from the multiset of points $2 \cdot \chi_{A \setminus E} + \chi_{\langle e_4 \rangle} + \chi_{\langle e_4 + e_1 \rangle} + \chi_{\langle e_4 + e_2 \rangle} + \chi_{\langle e_4 + e_3 \rangle}$. For $d = 13$ let A denote the ambient space and $E := \langle e_1, e_2 \rangle$. A suitable $[27, 4, 13]_2$-code with locality $r = 1$ can be obtained from the multiset of points $2 \cdot \chi_{A \setminus L} + \chi_{\langle e_3 \rangle} + \chi_{\langle e_3 + e_1 \rangle} + \chi_{\langle e_3 + e_2 \rangle}$. □

Theorem 38. *We have*

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$n_2(5, d, 1)$	10	10	12	12	17	18	19	20	24	25	27	28	30	30	32	32	37

d	18	19	20	21	22	23	24	25	26	27	28	29	30	31	36	38
$n_2(5, d, 1)$	38	41	42	45	46	47	48	52	54	55	56	59	60	62	72	76

for small values of d *and* $n_2(5, d, 1) = n_2(5, d)$ *in all other cases.*

Proof. The lower bounds for $d \le 38$ can be obtained by ILP computations. The necessary upper bounds can almost always be deduced from the constructions in Lemma 35 and $n_2(k, d-1, 1) \le n_2(k, d, 1)$. For the other cases we will state explicit examples by listing sets of points for the occurring non-zero multiplicities, where the points are stated as integers whose base-2-representation is a generator of the point.

- $d = 5$: $2 \to \{1, 2, 8, 16, 23, 29, 30\}$, $3 \to \{4\}$;
- $d = 7$: $2 \to \{2, 4, 8, 14, 16, 23, 26, 29\}$, $3 \to \{1\}$;
- $d = 9$: $2 \to \{1, 2, 15, 16, 23, 27\}$, $3 \to \{4, 8, 29, 30\}$;
- $d = 10$: $2 \to \{1, 2, 4, 8, 16\}$, $3 \to \{15, 23, 27, 29, 30\}$;
- $d = 11$: $2 \to \{1, 2, 4, 5, 8, 15, 16, 17, 27, 28, 29, 30\}$, $3 \to \{23\}$;
- $d = 17$: $2 \to \{2, 4, 7, 8, 11, 14, 16, 21, 22, 25, 31\}$, $3 \to \{1, 13, 19, 26, 28\}$;
- $d = 18$: $2 \to \{7, 11, 13, 14, 19, 21, 22, 25, 26, 28\}$, $3 \to \{1, 2, 4, 8, 16, 31\}$;
- $d = 19$: $2 \to \{1, 2, 4, 8, 11, 16, 28\}$, $3 \to \{7, 13, 14, 19, 21, 22, 25, 26, 31\}$;
- $d = 21$: $2 \to \{8, 13, 31\}$, $3 \to \{1, 2, 4, 7, 11, 14, 16, 19, 21, 22, 25, 26, 28\}$;
- $d = 23$: $2 \to \{8\}$, $3 \to \{1, 2, 4, 7, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31\}$;
- $d = 25$: $2 \to \{0, 1, 2, 3, 4, 6, 7, 8, 10, 13, 14, 15, 16, 21, 22, 25, 26, 27, 28, 30\}$, $3 \to \{13, 19, 21, 25\}$;
- $d = 27$: $2 \to \{1, 2, 4, 7, 8, 11, 13, 14, 16, 18, 19, 21, 22, 25, 26, 28, 31\}$, $3 \to \{3, 5, 9, 15, 20, 24, 30\}$;
- $d = 29$: $2 \to \{1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 20, 22, 23, 24, 25, 27, 28, 29, 30, 31\}$, $3 \to \{6, 9, 19\}$;
- $d = 35$: $2 \to \{3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 24, 25, 26, 28\}$, $3 \to \{1, 2, 4, 8, 15, 16, 23, 27, 29, 30\}$;
- $d = 37$: $2 \to \{1, 4, 5, 6, 8, 9, 11, 12, 18, 19, 22, 23, 26, 27, 30, 31\}$, $3 \to \{2, 3, 7, 10, 14, 15, 16, 17, 20, 21, 24, 25, 28, 29\}$.

□

Theorem 39. *We have*

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$n_2(6,d,1)$	12	12	14	14	20	20	22	22	27	28	30	30	33	34	35	36	41

d	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
$n_2(6,d,1)$	42	45	46	49	50	51	52	57	58	59	60	62	62	64	64	70	72

d	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
$n_2(6,d,1)$	74	76	78	80	81	82	86	88	90	90	93	94	95	96	101	102	105

d	52	53	54	55	56	57	58	59	60	61	62	63	64	71
$n_2(6,d,1)$	106	109	110	111	112	116	118	119	120	123	124	126	126	142

d	72	75	76	78	84
$n_2(6,d,1)$	143	150	151	156	168

for small values of d and $n_2(6,d,1) = n_2(6,d)$ in all other cases.

Proof. The lower bounds for $d \le 84$ can be obtained by ILP computations. The necessary upper bounds can almost always be deduced from the constructions in Lemma 35 and $n_2(k,d-1,1) \le n_2(k,d,1)$. For the other cases we will state explicit examples by listing sets of points for the occurring non-zero multiplicities, where the points are stated as integers whose base-2-representation is a generator of the point.

- $d = 9$: $2\rightarrow \{8,13,16,42,52,55\}$, $3\rightarrow \{1,2,4,27,32\}$;
- $d = 13$: $2\rightarrow \{1,2,4,7,8,16,19,32,37,41,47,50,52,56,61\}$, $3\rightarrow \{29\}$;
- $d = 15$: $2\rightarrow \{1,2,4,8,23,27,29,30,32,47,51,53,54,57,58,60\}$, $3\rightarrow \{16\}$;
- $d = 17$: $2\rightarrow \{4,5,15,16,18,25,30,32,41,42,53,61,62\}$, $3\rightarrow \{1,2,8,38,51\}$;
- $d = 18$: $2\rightarrow \{1,4,8,16,32,34,39,43,44,49,52,58\}$, $3\rightarrow \{2,31,45,55,57,62\}$;
- $d = 19$: $2\rightarrow \{1,2,4,8,16,23,27,32,33,34,36,39,40,43,46,47,48,51,54,58,62\}$, $3\rightarrow \{61\}$;
- $d = 21$: $2\rightarrow \{1,2,4,8,11,16,17,19,22,42,46,50,51,53,55,57,62\}$, $3\rightarrow \{15,32,41,44,52\}$;
- $d = 23$: $2\rightarrow \{1,2,3,4,7,8,9,15,16,25,27,31,32,35,37,40,45,46,50,51,52,57,62,63\}$, $3\rightarrow \{21\}$;
- $d = 25$: $2\rightarrow \{4,8,11,13,14,16,19,21,22,32,37,41,42,44,47,49,50,52,55,59,62\}$, $3\rightarrow \{1,2,25,35,61\}$;
- $d = 27$: $2\rightarrow \{1,4,7,8,13,14,16,19,21,22,25,26,28,31,32,35,37,38,41,42,44,47,49,50,55,56,59,62\}$, $3\rightarrow \{2\}$;
- $d = 33$: $2\rightarrow \{1,2,4,7,8,11,16,19,22,25,26,28,31,32,35,37,38,41,42,44,47,50,52,55,56,62\}$, $3\rightarrow \{13,14,21,49,59,61\}$;
- $d = 34$: $2\rightarrow \{1,4,7,13,14,16,19,21,22,28,31,32,41,44,47,49,50,56,59,61,62\}$, $3\rightarrow \{2,8,25,26,35,37,38,42,52,55\}$;
- $d = 35$: $2\rightarrow \{4,7,14,16,21,22,25,26,28,31,35,37,38,41,42,47,50,55,56,59,61,62\}$, $3\rightarrow \{1,2,8,11,13,19,32,44,49,52\}$;
- $d = 37$: $2\rightarrow \{1,11,16,19,21,26,28,32,35,38,42,47,49,50,52,55,61,62\}$, $3\rightarrow \{2,4,7,8,13,14,22,25,31,37,41,44,56,59\}$;
- $d = 38$: $2\rightarrow \{2,4,8,11,16,22,37,41,42,47,50,52,55,56,61,62\}$, $3\rightarrow \{1,7,13,14,19,21,25,26,28,31,32,35,37,38,44,49,59\}$;
- $d = 39$: $2\rightarrow \{1,4,7,8,11,13,14,19,21,23,37,38,41,44,52,55,56,61\}$, $3\rightarrow \{2,16,22,25,26,28,31,32,35,42,47,49,50,59,62\}$;

- $d = 40$: $2\to \{8, 14, 16, 19, 26, 31, 32, 35, 42, 47, 49, 50, 52, 54, 55, 59, 61\}$, $3\to \{1, 2, 4, 7, 11, 13, 21, 22, 25, 28, 37, 38, 41, 44, 56, 62\}$;
- $d = 41$: $2\to \{2, 21, 22, 38, 47, 49, 55, 59, 61, 62\}$, $3\to \{1, 4, 7, 8, 11, 13, 14, 16, 19, 25, 26, 28, 31, 32, 35, 37, 41, 42, 44, 50, 52, 56\}$;
- $d = 45$: $2\to \{1, 2, 4, 7, 8, 9, 10, 11, 13, 14, 15, 16, 21, 22, 24, 25, 26, 27, 28, 29, 30, 32, 35, 37, 38, 41, 42, 43, 44, 45, 46, 47, 49, 50, 52, 56, 57, 58, 60, 61, 62, 63\}$, $3\to \{12, 31, 55\}$;
- $d = 47$: $2\to \{8\}$, $3\to \{1, 2, 4, 7, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31, 32, 35, 37, 38, 41, 42, 44, 47, 49, 50, 52, 55, 56, 59, 61, 62\}$;
- $d = 49$: $2\to \{1, 3, 4, 5, 6, 8, 10, 13, 14, 15, 16, 17, 18, 21, 22, 23, 25, 27, 28, 30, 32, 33, 35, 36, 38, 39, 40, 42, 43, 44, 45, 47, 48, 50, 51, 52, 53, 55, 57, 59, 60, 62, 63\}$, $3\to \{2, 9, 26, 29, 56\}$;
- $d = 50$: $2\to \{1, 3, 4, 6, 8, 10, 13, 14, 15, 16, 18, 19, 21, 23, 25, 26, 27, 28, 30, 31, 32, 34, 35, 37, 38, 39, 41, 42, 43, 44, 46, 49, 50, 51, 52, 54, 56, 58, 59, 61, 62, 63\}$, $3\to \{2, 7, 11, 22, 47, 55\}$;
- $d = 51$: $2\to \{1, 4, 5, 6, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 22, 24, 25, 26, 27, 28, 32, 33, 34, 35, 40, 41, 42, 43, 51, 52, 53, 54, 55, 57, 58, 60, 61, 62, 63\}$, $3\to \{2, 11, 21, 31, 36, 39, 45, 46, 48\}$;
- $d = 53$: $2\to \{1, 3, 4, 6, 8, 9, 11, 12, 13, 14, 15, 16, 18, 20, 21, 23, 24, 26, 29, 30, 31, 32, 33, 35, 36, 37, 38, 39, 40, 41, 43, 44, 46, 48, 49, 50, 51, 52, 53, 55, 56, 57, 58, 59, 61, 62, 63\}$, $3\to \{2, 5, 19, 25, 47\}$;
- $d = 55$: $2\to \{1, 3, 4, 6, 8, 10, 13, 15, 16, 18, 21, 23, 25, 26, 27, 28, 30, 33, 35, 36, 38, 40, 42, 45, 47, 48, 50, 53, 55, 57, 59, 60, 62\}$, $3\to \{2, 7, 11, 14, 19, 22, 31, 32, 37, 41, 44, 49, 52, 56, 61\}$;
- $d = 57$: $2\to \{1, 3, \ldots, 8, 10, \ldots, 19, 21, 23, \ldots, 28, 30, \ldots, 37, 39, \ldots, 44, 46, 47, 49, \ldots, 56, 58, 60, 61, 62, 63\}$, $3\to \{2, 22, 38, 59\}$;
- $d = 59$: $2\to \{1, 3, ;11, 13, 15, 16, 18, 20, \ldots, 28, 30, 32, \ldots, 36, 38, 40, 42, 44, \ldots, 51, 53, 55, 56, 57, 59, \ldots, , 63\}$, $3\to \{2, 14, 19, 31, 37, 41, 52\}$;
- $d = 61$: $2\to \{1, 3, \ldots, 23, 25, 27, \ldots, 39, 41, \ldots, 47, 49, 51, \ldots, 63\}$, $3\to \{2, 26, 50\}$;
- $d = 65$: $2\to \{1, 3, 4, 6, \ldots, 14, 16, \ldots, 20, 22, 25, \ldots, 36, 38, 39, 41, 43, 44, 45, 48, 49, 51, \ldots, 60, 62\}$, $3\to \{2, 5, 15, 21, 24, 37, 40, 46, 47, 50, 56, 63\}$;
- $d = 66$: $2\to \{1, 3, 4, 7, \ldots, 12, 14, \ldots, 19, 21, \ldots, 37, 39, \ldots, 42, 44, 47, \ldots, 60, 62, 63\}$, $3\to \{2, 5, 6, 13, 20, 38, 45, 46, 61\}$;
- $d = 67$: $2\to \{1, 3, 4, 6, 8, 9, 10, 12, 13, 15, 16, 17, 19, 21, 22, 24, \ldots, 32, 35, 37, 38, 40, \ldots, 47, 49, 51, 52, 54, 56, 57, 58, 60, 61, 63\}$, $3\to \{2, 7, 11, 14, 18, 20, 23, 33, 34, 36, 39, 50, 55, 59, 62\}$;
- $d = 68$: $2\to \{3, 6, 10, 15, 16, 17, 19, \ldots, 22, 24, 25, 26, 28, 29, 31, 33, \ldots, 36, 38, 39, 40, 42, 43, 45, 46, 47, 49, 52, 56, 61\}$, $3\to \{1, 2, 4, 7, 8, 11, 13, 14, 18, 23, 27, 30, 32, 37, 41, 44, 48, 51, 53, 54, 57, 58, 60, 63\}$;
- $d = 69$: $2\to \{1, 5, \ldots, 10, 12, 13, 15, \ldots, 18, 20, 21, 23, 25, 29, \ldots, 35, 37, 40, 41, 43, \ldots, 46, 48, 49, 51, \ldots, 54, 57, 58, 59, 61\}$, $3\to \{2, 3, 4, 11, 14, 19, 22, 26, 27, 28, 38, 39, 42, 47, 50, 55, 56, 62, 63\}$;
- $d = 70$: $2\to \{1, 3, 5, 6, 8, 11, 12, 14, 16, 19, 21, 23, 24, 26, 29, 30, 32, 33, 36, \ldots, 45, 48, \ldots, 51, 54, 55, 58, \ldots, 63\}$, $3\to \{2, 4, 7, 9, 10, 15, 17, 18, 20, 25, 28, 31, 34, 35, 46, 47, 52, 53, 56, 57\}$;
- $d = 73$: $2\to \{1, 4, 8, 9, 12, 13, 16, 17, 20, 21, 24, 25, 26, 28, 29, 31, \ldots, 34, 36, 37, 39, \ldots, 42, 44, 45, 48, 49, 52, 53, 57, 60\}$, $3\to \{2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 27, 30, 35, 38, 43, 46, 47, 50, 51, 54, 55, 58, 59, 62, 63\}$;

- $d = 74$: $2 \to \{1, 4, 5, 8, 9, 12, 13, 15, 16, 17, 18, 20, 21, 24, 25, 28, 32, 33, 34, 36, 37, 40, 41, 44, 49, 52, 53, 56, 57, 60, 61, 63\}$, $3 \to \{2, 3, 6, 7, 10, 11, 14, 19, 22, 23, 26, 27, 30, 31, 35, 38, 39, 42, 43, 46, 47, 50, 51, 54, 55, 58, 59, 62\}$;
- $d = 77$: $2 \to \{1, 4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24, 25, 28, 29, 30, 34, 35, 38, 39, 42, 43, 44, 46, 47, 51, 54, 55, 58, 59, 62, 63\}$, $3 \to \{2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 26, 27, 31, 32, 33, 36, 37, 40, 41, 45, 48, 49, 52, 53, 56, 57, 60, 61\}$;
- $d = 83$: $2 \to \{1, 8, 9, 12, 18, 26, 27, 31, 32, 33, 36, 40, 44, 45, 50, 51, 55, 59, 62, 63\}$, $3 \to \{2, \ldots, 7, 10, 11, 13, \ldots, 17, 20, \ldots, 25, 28, 29, 30, 34, 35, 37, 38, 39, 41, 42, 43, 46, \ldots, 49, 52, 53, 54, 56, 57, 58, 60, 61\}$;

□

Theorem 40. *We have*

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	17	21
$n_3(3, d, 1)$	6	6	8	8	11	12	14	14	16	16	18	18	21	22	24	26	32

for small values of d and $n_3(3, d, 1) = n_3(3, d)$ in all other cases.

Proof. The lower bounds for $d \le 21$ can be obtained by ILP computations. The necessary upper bounds can almost always be deduced from the constructions in Lemma 35 and $n_3(k, d-1, 1) \le n_3(k, d, 1)$. For the other cases we will state explicit examples by listing sets of points for the occurring non-zero multiplicities, where the points are stated as integers denoting the position in the list of lexicographical minimal vectors that are generators of a point (starting to count from zero).

- $d = 5$: $2 \to \{0, 1, 11, 12\}$, $3 \to \{4\}$;
- $d = 13$: $2 \to \{4, 5, 7, 11, 12\}$, $3 \to \{0, 1, 2, 9\}$.

□

Theorem 41. *We have*

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$n_3(4, d, 1)$	8	8	10	10	14	14	16	16	18	18	20	20	24	25	27	28	29	30

d	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	37
$n_3(4, d, 1)$	33	34	36	36	38	38	41	42	44	46	47	48	50	50	52	52	54	58

d	38	39	40	41	42	43	44	45	47	49	50	51	53	57	61	62	63	69
$n_3(4, d, 1)$	59	61	62	63	64	67	68	69	72	75	76	78	80	87	93	94	95	104

for small values of d and $n_3(4, d, 1) = n_3(4, d)$ in all other cases.

Proof. The lower bounds for $d \le 69$ can be obtained by ILP computations. The necessary upper bounds can almost always be deduced from the constructions in Lemma 35 and $n_3(k, d-1, 1) \le n_3(k, d, 1)$. For the other cases we will state explicit examples by listing sets of points for the occurring non-zero multiplicities, where the points are stated as integers denoting the position in the list of lexicographical minimal vectors that are generators of a point (starting to count from zero).

- $d = 13$: $2 \to \{0, 1, 9, 13, 26, 33\}$, $3 \to \{4, 20, 30, 34\}$;
- $d = 14$: $2 \to \{1, 13, 21, 26, 38\}$, $3 \to \{0, 4, 11, 24, 34\}$;
- $d = 15$: $2 \to \{0, 1, 2, 4, 11, 12, 13, 18, 21, 24, 27, 32\}$, $3 \to \{34\}$;
- $d = 17$: $2 \to \{0\}$, $3 \to \{1, 4, 11, 13, 18, 26, 30, 32, 37\}$;
- $d = 19$: $2 \to \{0, 1, 4, 5, 7, 13, 16, 27, 29, 35, 36, 39\}$, $3 \to \{2, 14, 22\}$;

- $d = 25$: $2\to \{0, 1, 4, 5, 9, 11, 13, 20, 21, 24, 25, 26, 30, 33, 35, 38\}$, $3\to \{2, 14, 36\}$;
- $d = 26$: $2\to \{1, 3, 4, 5, 7, 9, 13, 15, 16, 24, 28, 29, 35, 36, 39\}$, $3\to \{0, 20, 25, 32\}$;
- $d = 27$: $2\to \{0, 1, 2, 3, 4, 5, 6, 13, 16, 19, 22, 26, 30, 31, 36, 38\}$, $3\to \{10, 14, 27, 39\}$;
- $d = 29$: $2\to \{0, 1, 2, 4, 6, 9, 10, 13, 15, 16, 17, 18, 19, 22, 23, 24, 25, 26, 29, 32, 35, 37\}$, $3\to \{8\}$;
- $d = 37$: $2\to \{1, 3, 4, 6, 7, 8, 11, 12, 13, 14, 18, 19, 21, 23, 24, 25, 27, 28, 31, 33, 34, 35, 39\}$, $3\to \{0, 17, 29, 38\}$;
- $d = 38$: $2\to \{0, 1, 3, 4, 6, 7, 8, 11, 13, 15, 16, 20, 21, 22, 23, 26, 27, 30, 32, 33, 36, 38\}$, $3\to \{12, 17, 28, 34, 37\}$;
- $d = 39$: $2\to \{0, 1, 3, 4, 5, 8, 9, 12, 13, 16, 17, 21, 23, 24, 31, 32, 35, 36, 37, 39\}$, $3\to \{10, 15, 20, 25, 27, 28, 29\}$;
- $d = 41$: $2\to \{0, 1, 2, 4, 12, 13, 14, 18, 20, 21, 22, 24, 26, 27, 28, 32, 33, 39\}$, $3\to \{5, 7, 9, 11, 16, 29, 34, 35, 37\}$;
- $d = 43$: $2\to \{0, 1, 4, 5, 8, 9, 10, 12, 17, 29, 30, 33, 34, 36\}$, $3\to \{3, 13, 14, 18, 19, 21, 22, 24, 25, 26, 32, 37, 38\}$;
- $d = 45$: $2\to \{1, 4, 10, 12, 15, 20, 23, 25, 27, 28, 29, 31, 36\}$, $3\to \{0, 3, 8, 9, 13, 16, 17, 21, 24, 32, 35, 37, 39\}$, $4\to \{5\}$;
- $d = 49$: $2\to \{1, \dots, 11, 13, \dots, 17, 19, \dots, 22, 25, \dots, 36, 38\}$, $3\to \{0, 24, 39\}$;
- $d = 67$: $2\to \{0, 5, 7, 12, 14, \dots, 17, 19, 21, 22, 27, 29, 32, 34, 39\}$, $3\to \{1, 2, 4, 6, 8, 9, 10, 11, 13, 18, 20, 23, \dots, 26, 28, 30, 31, 33, 35, \dots, 38\}$;
- $d = 68$: $2\to \{0, 1, 3, 4, 8, 12, 31, \dots, 39\}$, $3\to \{5, 6, 7, 9, 10, 11, 13, \dots, 30\}$;

□

We remark that all $[69, 4, 45]_3$-codes that have locality $r = 1$, when considered as a multiset of points in PG(3, 3), have maximum point multiplicity of at least 4 (and not 3 as in all other stated examples for the other parameters).

5 Enumeration results

Instead of solving ILPs to determine locally recoverable codes with the minimum possible length $n_q(k, d, r)$ one may also enumerate all $[n, k, d]_q$-codes with minimum possible length $n = n_q(k, d)$ or slightly larger length and check whether those codes have relatively small locality constants r. For those enumerations we have applied the software `LinCode` [BBK21]. We present our computational results in Tables 1, 2, and 3. Locality $r = 1$ did not occur in any of these cases.

References

[ABH+18] Abhishek Agarwal, Alexander Barg, Sihuang Hu, Arya Mazumdar, and Itzhak Tamo. Combinatorial alphabet-dependent bounds for locally recoverable codes. *IEEE Transactions on Information Theory*, 64(5):3481–3492, 2018.

[APD+13] Megasthenis Asteris, Dimitris Papailiopoulos, Alexandros G Dimakis, Ramkumar Vadali, Scott Chen, and Dhruba Borthakur. XORing elephants: Novel erasure codes for big data. *Proceedings of the VLDB Endowment*, 6(5):325–336, 2013.

[BBK21] Iliya Bouyukliev, Stefka Bouyuklieva, and Sascha Kurz. Computer classification of linear codes. *IEEE Transactions on Information Theory*, 67(12):7807–7814, 2021.

n	8	9	10	13	14	15	16	17	20	21	23	24	27	28	30	31	36	37	39	40	43	44	46	47	51	52	54	55
d	2	3	4	5	6	7	8	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$\#r = 2$	2	0	0	5	3	0	0	3	3	2	1	1	1	1	1	1	70	13	7	3	5	2	2	1	12	4	2	1
$\#r \geq 3$	25	5	4	10	3	1	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\#$	27	5	4	15	6	1	1	4	3	2	1	1	1	1	1	1	70	14	7	3	5	2	2	1	12	4	2	1

Table 1: Locality constants for 5-dimensional binary codes with small lengths.

n	7	10	11	12	14	15	17	18	22	23	25	26	29	30	30	31	31	32	37	38	41	42
d	2	3	4	4	5	6	7	8	9	10	11	12	13	13	14	14	15	16	17	18	19	20
$\#r = 2$	0	0	0	1	0	1	0	1	93	12	4	2	0	4786	0	295	0	0	2	1	235	35
$\#r \geq 3$	1	4	2	40	11	4	3	1	155	17	1	0	2	133	2	18	1	1	0	0	0	0
$\#$	1	4	2	41	11	5	3	2	248	29	5	2	2	4919	2	313	1	1	2	1	235	35

Table 2: Locality constants for 6-dimensional binary codes with small lengths.

n	15	16	17	16	17	18	18	19	20	19	20	23	24	24	25	26	27	27	28
d	5	5	5	6	6	6	7	7	7	8	8	9	9	10	10	11	11	12	12
$\#r = 2$	0	0	48	0	0	25	0	0	23	0	1	0	≥ 411	0	502	0	40	0	7
$\#r \geq 3$	6	4013	459067	3	377	76922	2	82	207117	1	25	29	≥ 3572275	6	188064	2	1598	1	122
$\#$	6	4013	459115	3	377	76947	2	82	207140	1	26	29	≥ 3572686	6	188566	2	1638	1	129

Table 3: Locality constants for 7-dimensional binary codes with small lengths.

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993). URL: `http://dx.doi.org/10.1006/jsco.1996.0125`, `doi:10.1006/jsco.1996.0125`.

[BE97] Jürgen Bierbrauer and Yves Edel. A family of 2-weight codes related to BCH-codes. *Journal of Combinatorial Designs*, 5(5):391–396, 1997.

[BJ01] Iliya Bouyukliev and David B. Jaffe. Optimal binary linear codes of dimension at most seven. *Discrete Mathematics*, 226(1-3):51–70, 2001.

[BJV00] Iliya Bouyukliev, David B. Jaffe, and Vesselin Vavrek. The smallest length of eight-dimensional binary linear codes with prescribed minimum distance. *IEEE Transactions on Information Theory*, 46(4):1539–1544, 2000.

[BM73] L.D. Baumert and R.J. McEliece. A note on the Griesmer bound. *IEEE Transactions on Information Theory*, pages 134–135, 1973.

[CHL07] Minghua Chen, Cheng Huang, and Jin Li. On the maximally recoverable property for multi-protection group codes. In *2007 IEEE International Symposium on Information Theory*, pages 486–490. IEEE, 2007.

[CM15] Viveck R Cadambe and Arya Mazumdar. Bounds on the size of locally recoverable codes. *IEEE Transactions on Information Theory*, 61(11):5787–5794, 2015.

[GC14] Sreechakra Goparaju and Robert Calderbank. Binary cyclic codes that are locally repairable. In *2014 IEEE International Symposium on Information Theory*, pages 676–680. IEEE, 2014.

[GFHWH17] Matthias Grezet, Ragnar Freij-Hollanti, Thomas Westerbäck, and Camilla Hollanti. On binary matroid minors and applications to data storage over small fields. In *International Castle Meeting on Coding Theory and Applications*, pages 139–153. Springer, 2017.

[GFHWH19] Matthias Grezet, Ragnar Freij-Hollanti, Thomas Westerbäck, and Camilla Hollanti. Alphabet-dependent bounds for linear locally repairable codes based on residual codes. *IEEE Transactions on Information Theory*, 65(10):6089–6100, 2019.

[GHSY12] Parikshit Gopalan, Cheng Huang, Huseyin Simitci, and Sergey Yekhanin. On the locality of codeword symbols. *IEEE Transactions on Information Theory*, 58(11):6925–6934, 2012.

[GJR23] Anina Gruica, Benjamin Jany, and Alberto Ravagnani. LRCs: Duality, LP bounds, and field size. *arXiv preprint 2309.03676*, 2023.

[Gri60] James H. Griesmer. A bound for error-correcting codes. *IBM Journal of Research and Development*, 4(5):532–542, 1960.

[GWFHH19] Matthias Grezet, Thomas Westerbäck, Ragnar Freij-Hollanti, and Camilla Hollanti. Uniform minors in maximally recoverable codes. *IEEE Communications Letters*, 23(8):1297–1300, 2019.

[Hel83] Tor Helleseth. New constructions of codes meeting the Griesmer bound. *IEEE Transactions on Information Theory*, 29(3):434–439, 1983.

[HSX+12] Cheng Huang, Huseyin Simitci, Yikang Xu, Aaron Ogus, Brad Calder, Parikshit Gopalan, Jin Li, and Sergey Yekhanin. Erasure coding in windows azure storage. In *Proceedings of the 2012 USENIX conference on Annual Technical Conference*, pages 15–26, 2012.

[HXC16] Jie Hao, Shu-Tao Xia, and Bin Chen. Some results on optimal locally repairable codes. In *2016 IEEE International Symposium on Information Theory (ISIT)*, pages 440–444. IEEE, 2016.

[HXC17] Jie Hao, Shu-Tao Xia, and Bin Chen. On optimal ternary locally repairable codes. In *2017 IEEE International Symposium on Information Theory (ISIT)*, pages 171–175. IEEE, 2017.

[HXS$^+$20] Jie Hao, Shu-Tao Xia, Kenneth W Shum, Bin Chen, Fang-Wei Fu, and Yixian Yang. Bounds and constructions of locally repairable codes: parity-check matrix approach. *IEEE Transactions on Information Theory*, 66(12):7465–7474, 2020.

[HYUS15] Pengfei Huang, Eitan Yaakobi, Hironori Uchikawa, and Paul H. Siegel. Cyclic linear binary locally repairable codes. In *2015 IEEE Information Theory Workshop (ITW)*, pages 1–5. IEEE, 2015.

[HYUS16] Pengfei Huang, Eitan Yaakobi, Hironori Uchikawa, and Paul H. Siegel. Binary linear locally repairable codes. *IEEE Transactions on Information Theory*, 62(11):6268–6283, 2016.

[KY21] Sascha Kurz and Eitan Yaakobi. PIR codes with short block length. *Designs, Codes and Cryptography*, 89:559–587, 2021.

[LXY18] Yuan Luo, Chaoping Xing, and Chen Yuan. Optimal locally repairable codes of distance 3 and 4 via cyclic codes. *IEEE Transactions on Information Theory*, 65(2):1048–1053, 2018.

[Mar97] Tatsuya Maruta. On the achievement of the griesmer bound. *Designs, Codes and Cryptography*, 12:83–87, 1997.

[PKLK12] N. Prakash, Govinda M. Kamath, V. Lalitha, and P. Vijay Kumar. Optimal linear codes with a local-error-correction property. In *2012 IEEE International Symposium on Information Theory Proceedings*, pages 2776–2780. IEEE, 2012.

[SS65] Gustave Solomon and Jack J. Stiffler. Algebraically punctured cyclic codes. *Information and Control*, 8(2):170–179, 1965.

[TB14] Itzhak Tamo and Alexander Barg. A family of optimal locally recoverable codes. *IEEE Transactions on Information Theory*, 60(8):4661–4676, 2014.

[TBGC15] Itzhak Tamo, Alexander Barg, Sreechakra Goparaju, and Robert Calderbank. Cyclic LRC codes and their subfield subcodes. In *2015 IEEE International Symposium on Information Theory (ISIT)*, pages 1262–1266. IEEE, 2015.

[TPD16] Itzhak Tamo, Dimitris S. Papailiopoulos, and Alexandros G. Dimakis. Optimal locally repairable codes and connections to matroid theory. *IEEE Transactions on Information Theory*, 62(12):6661–6671, 2016.

[vT81] Henk C.A. van Tilborg. The smallest length of binary 7–dimensional linear codes with prescribed minimum distance. *Discrete Mathematics*, 33(2):197–207, 1981.

[WFHEH16] Thomas Westerbäck, Ragnar Freij-Hollanti, Toni Ernvall, and Camilla Hollanti. On the combinatorics of locally repairable codes via matroid theory. *IEEE Transactions on Information Theory*, 62(10):5296–5315, 2016.

[XKG22] Yuanxiao Xi, Xiangliang Kong, and Gennian Ge. Optimal quaternary locally repairable codes attaining the Singleton-like bound. *arXiv preprint 2206.05805*, 2022.