Symmetric Functions in MAGMA

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Overview

- Symmetric Functions
- Multiplication
- Plethysm
- Transition Matrices

Symmetric Functions

symmetric polynomial f

- multivariate polynomial: $f \in \mathbb{Q}[x_1, x_2, x_3, \ldots, x_n]$
- fixpoint of Sym_n action on the variables: $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \forall \pi \in Sym_n$

symmetric function

- infinite number of variables but finite degree

Symmetric Functions

monomial symmetric function

- orbit of a monomial $x^I := x_1^{I_1} x_2^{I_2} \dots$ for $I \in \mathbb{N}^{\mathbb{N}}$of finite weight $= \Sigma I_i$
- $m_I := Sym_{\mathbb{N}}(x^I)$
- $m_{0,2,0.1}(a,b,c,d) = a^2b + a^2c + a^2d + ab^2 \dots = Sym_4(b^2d)$

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Symmetric Functions

$m_I = m_J \iff J$ is a permutation of I

- $\{m_I | I$ is a partition of weight $\leq n\}$ is a basis of the vectorspace of symmetric functions of degree $\leq n$
- $\{m_I | I$ is a partition$\}$ is a basis of the vectorspace of symmetric functions
- $(1 + a + b + c + \ldots)(a^2 + b^2 + c^2 + \ldots) = m_2 + m_3 + m_{2,1}$

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Symmetric Functions

elementary symmetric function

- $e_k := m_{1^k}$ for $k \in \mathbb{N}$
- $e_3 = abc + abd + \ldots + bcd + bce + \ldots$
- $e_I := e_{I_1} \times e_{I_2} \times \ldots$ for $I \in \mathbb{N}^{\mathbb{N}}$ of finite weight
- $\{e_I | I$ is a partition$\}$ is a basis of the vectorspace of symmetric functions

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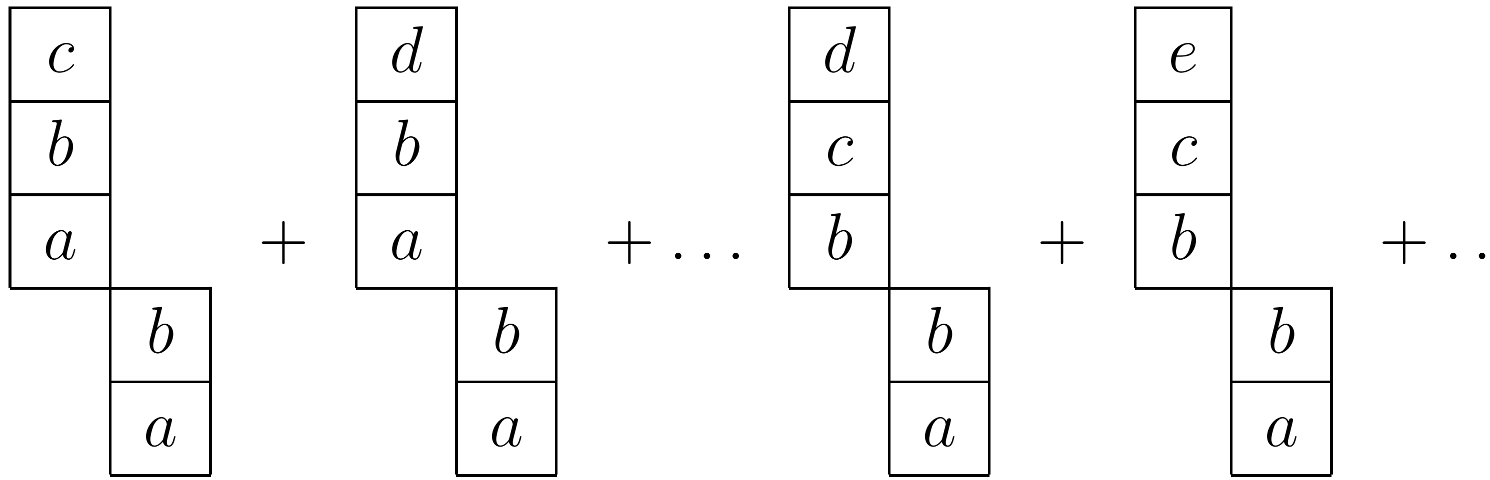
generating function

- e_k g.f. of standard tableaux of shape 1^k
- $$e_3 = \begin{array}{|c|}\hline c \\ \hline b \\ \hline a \\ \hline\end{array} + \begin{array}{|c|}\hline d \\ \hline b \\ \hline a \\ \hline\end{array} + \dots \begin{array}{|c|}\hline d \\ \hline c \\ \hline b \\ \hline\end{array} + \begin{array}{|c|}\hline e \\ \hline c \\ \hline b \\ \hline\end{array} + \dots$$

generating function

- e_I g.f. of standard tableaux of skew-shape $1^{I_1} \times 1^{I_2} \times \ldots$

- $e_{3,2} =$ 

Symmetric Functions

complete symmetric function

- $h_k := \Sigma_{I \vdash k} m_I$ for $k \in \mathbb{N}$
- $h_3 = m_3 + m_{2,1} + m_{111}$
- $h_I := h_{I_1} \times h_{I_2} \times \ldots$ for $I \in \mathbb{N}^{\mathbb{N}}$ of finite weight
- $\{h_I | I$ is a partition$\}$ is a basis of the vectorspace of symmetric functions

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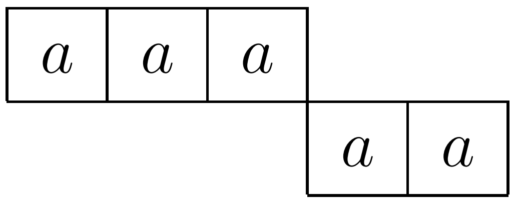
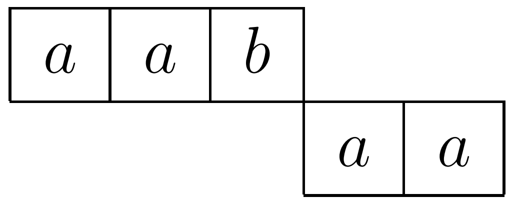
Symmetric Functions

complete symmetric function

- h_I g.f. of standard tableaux of skew-shape $I_1 \times I_2 \times \ldots$

- $h_{3,2} =$ 

$+$ 

$+ \ldots$

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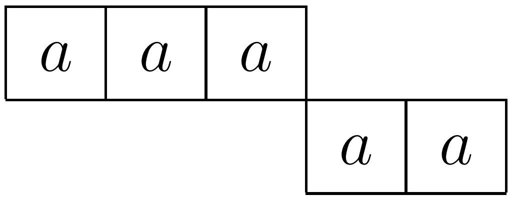
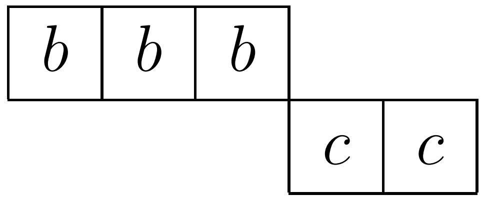
Symmetric Functions

power sum

- $p_k = m_k$ for all $k \in \mathbb{N}$
- $p_I := p_{I_1} \times p_{I_2} \times \dots$ for $I \in \mathbb{N}^{\mathbb{N}}$ of finite weight
- $p_{3,2} =$  $+ \dots$  $+ \dots$
- $\{p_I | I$ is a partition$\}$ is a basis of the vectorspace of symmetric functions

Schur function

- partition $I=\left(I_1\geq\ldots\geq I_k>I_{k+1}=0,0,\ldots\right)\in\mathbb{N}^{\mathbb{N}}$:
- S_I is the g.f. of standard tableaux of shape I
- $S_{2,1}(a,b,c) = S_{21}(A_3) =$

b	c	b	c	b	c	c	c
$a\ a$	$a\ a$	$a\ b$	$a\ b$	$a\ c$	$a\ c$	$b\ b$	$b\ c$
a^2b	a^2c	ab^2	abc	abc	ac^2	b^2c	bc^2

$=a^2b+a^2c+ab^2+2abc+ac^2+b^2c+bc^2$

Multiplication

algebra Λ of symmetric function

- elementary symmetric $e_I \times e_J = \Sigma_K \ldots e_K \;\; = e_{I \cup J}$ trivial
- same for powersum, complete
- monomial symmetric $m_I \times m_J = \Sigma_K \ldots m_K$ simple

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Multiplication

product of Schur functions

- $S_I \times S_J = \Sigma_K c_{I,J,K} S_K$
- Littlewood-Richardson Rule:

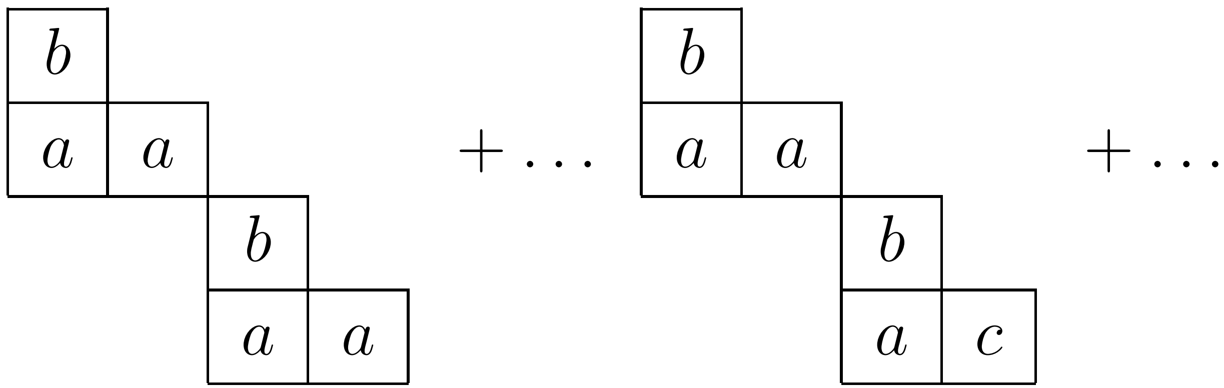
$c_{I,J,K} =$ number of some combinatorial objects

- useful for a single coefficient $c_{I,J,K}$

Multiplication

Littlewood Richardson rule

- $S_I \times S_J$ is the g.f. of standard tableaux of skew-shape $I \times J$
- $S_{2,1} \times S_{2,1} =$



jeu de taquin

- $S_{2,1} \times S_{2,1} =$
-

$$\begin{array}{|c|c|c|c|} \hline b & & & \\ \hline a & a & & \\ \hline & & b & \\ \hline & & a & a \\ \hline \end{array} + \ldots \quad \begin{array}{|c|c|c|c|} \hline d & & & \\ \hline c & c & & \\ \hline & & b & \\ \hline & & a & c \\ \hline \end{array} + \ldots$$

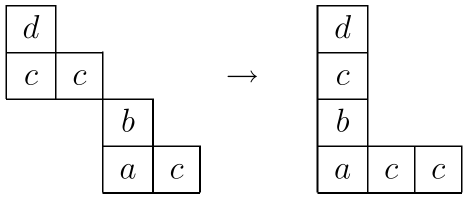
$$\begin{array}{|c|c|c|c|} \hline b & & & \\ \hline a & & & \\ \hline & a & b & \\ \hline & & a & a \\ \hline \end{array} + \ldots \quad \begin{array}{|c|c|c|c|} \hline d & & & \\ \hline c & c & & \\ \hline & b & & \\ \hline & & a & c \\ \hline \end{array} + \ldots$$

$$\begin{array}{|c|} \hline d \\ \hline \end{array}$$

Multiplication

Littlewood Richardson rule

- combinatorial LR (tableaux, non-commutative)



- polynomial LR (commutative)
$S_{21} \times S_{21} = \ \ldots + S_{3111} + \ldots$

- polynomial LR (fixed number of variables)
$S_{21}(A_3) \times S_{21}(A_3) = \ \ldots + (S_{3111}(A_3) = 0) + \ldots$

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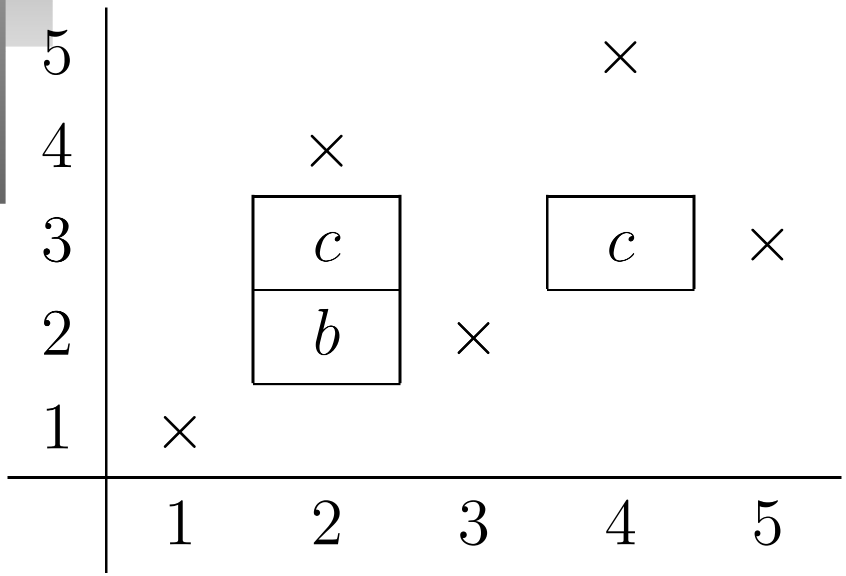
Multiplication

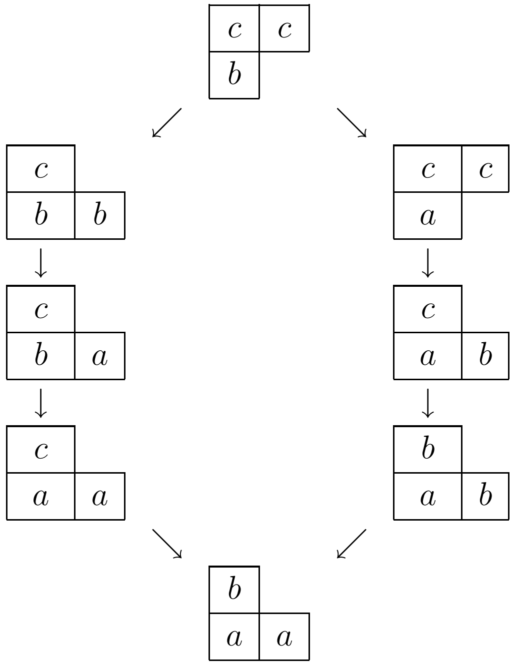
Schubert polynomial

- multivariate $\in \mathbb{N}[x_1, x_2, \ldots, x_n]$
- labeled by permutations: X_w for $w \in Sym_n$
- homogeneous, degree $= l(w)$
- non-symmetric
- generalizes Schur polynomials

Multiplication

Schubert polynomial X_{13524}

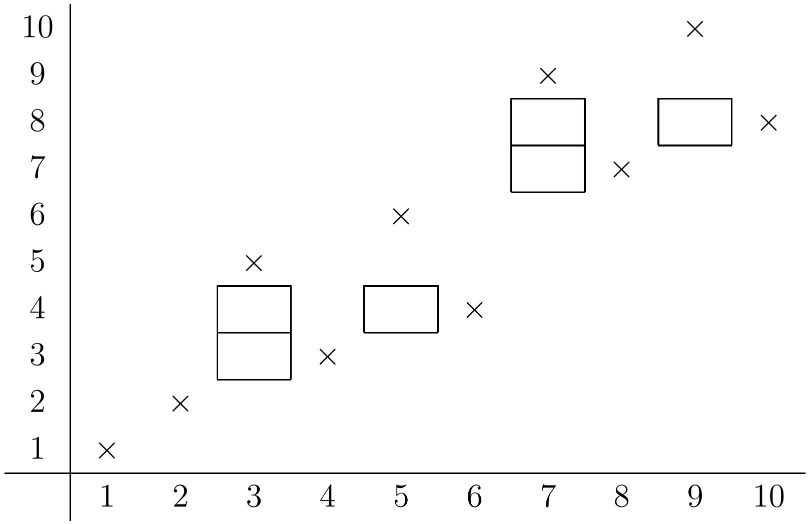




$= S_{21}(A_3)$

Multiplication

Schubert polynomial $X_{1246358\,10\,79}$



- $X_{1246358\,10\,79} = S_{21}(\mathbb{A}_4) \times S_{21}(\mathbb{A}_8)$
- $X_{1246358\,10\,79} \downarrow \mathbb{A}_4 = S_{21}(\mathbb{A}_4) \times S_{21}(\mathbb{A}_4)$

Multiplication

Monk's rule $X_w \in \mathbb{N}[a_1, \ldots]$

$$a_i \times X_w = \sum_{w'} X_{w'} - \sum_{w''} X_{w''}$$

- $w' = w(i,j)$ with $j > i$ and $l(w') = l(w) + 1$
- $w'' = w(i,j)$ with $j < i$ and $l(w'') = l(w) + 1$

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Multiplication

Monk's rule $X_{1246358\,10\,79}$

$a_7 \times X_{1246358\,10\,79} =$ $X_{124635\,10\,879}$ $+X_{1246359\,10\,78}$

$-X_{1246385\,10\,79}$ $-X_{1248356\,10\,79}$

$a_8 \times X_{124835697\,10} =$ $X_{1248356\,10\,79}$

$-X_{124835967\,10}$ $-X_{124935687\,10}$

- i index of last decrease in w :
  $a_i \times X_w = X_{w'} - \sum X_{w''}$

- $X_{1248356\,10\,79} = a_8 \times X_{124835697} + X_{124835967} + X_{124935687}$

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Littlewood Richardson - Lascoux Schützenberger

1246358 10 79
0012001200

↓

124635987
001200210

↙ ↘

124637958
001201200

124735968
001300200

↙ ↘

12473856
00130200

12483576
00140010

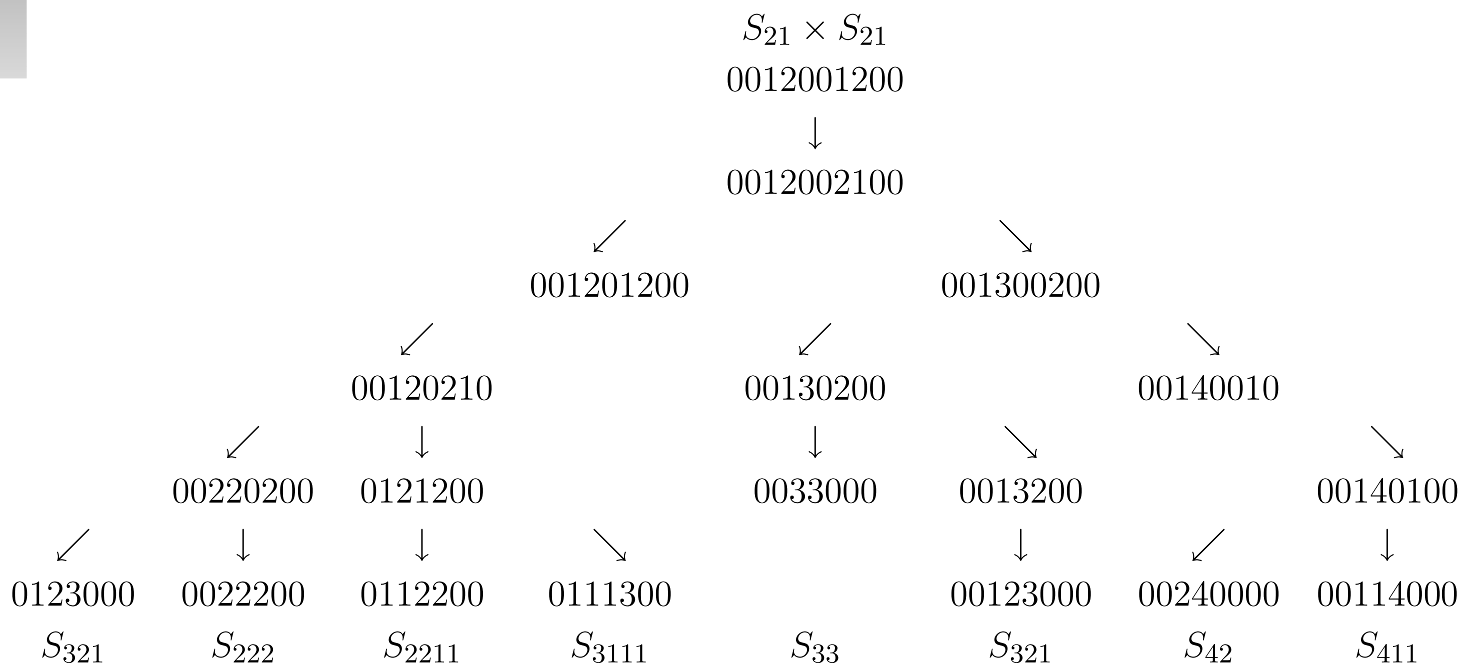
$a_8 \times X_{124635897} + X_{124635987}$

$$a_8 \times X_{124635978} + X_{124637958} + X_{124735968}$$

$$a_7 \times X_{124735869} + X_{124738569} + X_{124835769}$$

Multiplication

Lascoux Schützenberger



Multiplication

MAGMA

```
>Q := Rationals();
>S := SFASchur(Q);
>s21 := S.[2,1];
>s21*s21;
S.[2,2,1,1] + S.[2,2,2] + S.[3,1,1,1] + 2*S.[3,2,1]
+ S.[3,3] + S.[4,1,1] + S.[4,2]
```

Plethysm

third operation on Λ

- given f, g of degree m, n
- addition $f + g$ of degree $\sim$max(m, n)
- multiplication $f \times g$ of degree $n + m$
- plethysm $f[g]$ of degree $m \times n$

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Plethysm

easiest case: e.g. $h_2[e_2]$

- $h_2 = \begin{array}{|c|c|}\hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline a & b \\ \hline\end{array} + \begin{array}{|c|c|}\hline a & c \\ \hline\end{array} + \dots$
- $e_2 = \begin{array}{|c|}\hline b \\ \hline a \\ \hline\end{array} + \begin{array}{|c|}\hline c \\ \hline a \\ \hline\end{array} + \begin{array}{|c|}\hline d \\ \hline a \\ \hline\end{array} + \dots$
- variables in h_2 = tableaux of e_2
- $h_2[e_2] = \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & c \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} + \dots$

Plethysm

easiest case: e.g. $h_2[e_2]$

- $h_2[e_2] = \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & c \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} + \dots$

  was misleading

- $h_2[e_2] = \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array} \dots \begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline\end{array} \dots \begin{array}{|c|c|}\hline d & c \\ \hline a & b \\ \hline\end{array} \dots$

- $h_2[e_2] = S_2[S_{11}] = S_{22} + S_{1111}$

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MAGMA

```
>Q := Rationals();
>S := SFASchur(Q);
>e2 := S.[1,1];
>h2:=S.[2];
>h2~e2;
S.[1,1,1,1] + S.[2,2]
```

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Plethysm

algorithm for $S_n[S_m] = \sum \ldots S_K$

- $S_n[S_m]$= 'plethystic' fillings of $n \times m$ rectangle
- $S_3[S_2] =$

| aa | aa | aa | | aa | aa | ab | | aa | aa | ac | | aa | aa | bb | | aa | aa | bc | | aa | aa | cc | ... =

| a a | a b | a c | b b | b c | c c |
| a a | a a | a a | a a | a a | a a | ...
| a a | a a | a a | a a | a a | a a |

- $=: P_{222} = P_{2^3}.$

Plethysm

for arbitrary partitions $I = k^{m_k},\ldots,1^{m_1}$

- $P_I = \prod P_{i^{m_i}}$
- $P_{m^n}(a,b,c,\ldots) = \sum_{i=0}^{n\cdot m} a^i \sum_{|I|=nm-i, I\subseteq m^n} P_I(b,c,d,\ldots)$
-

a	a
a	a
a	a

a	
a	a
a	a

a	
a	
a	a

a	a
a	a

a	
a	a

a	
a	
a	

a	
a	

a	a

a	

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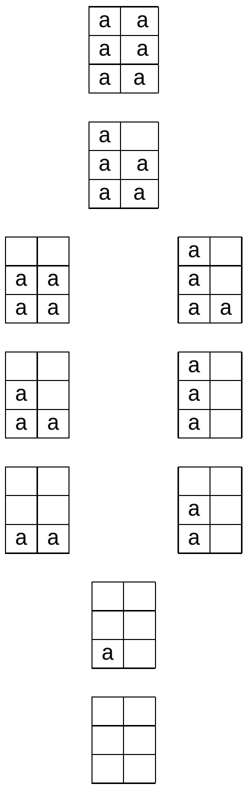
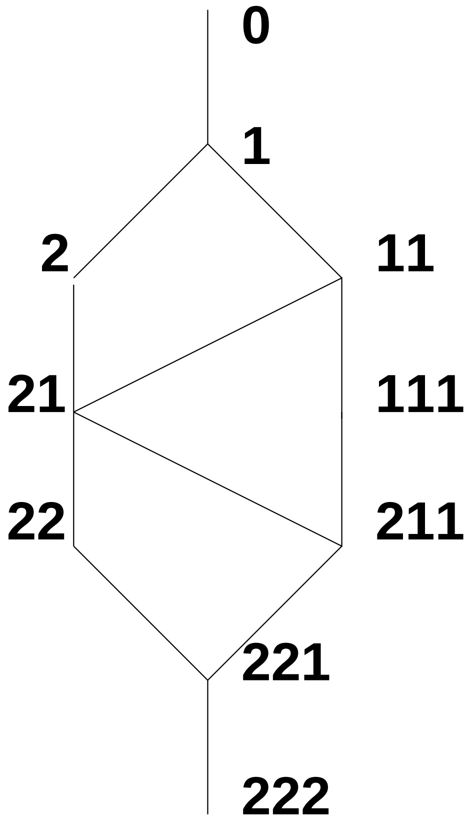
Plethysm





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Plethysm

- first step:

a	a
a	a
a	a

$\in S_3[S_2] \Rightarrow S_6 \in S_3[S_2]$

- second step:

a	P_1
a	a
a	a

$P_1 = S_1$

- S_6 without

a	a	a	a	a

= S_1

Plethysm

- third step

P_2	
a	a
a	a

$P_2 = S_2$

a	P_{11}
a	
a	a

$P_{11} = S_2$ together $2S_2$

- S_6 without

a	a	a	a

= S_2

- add to the result: S_{42}

Plethysm

- fourth step

  $\begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{P_{21}} \\ \hline a & \\ \hline a & a \\ \hline \end{array}$ $P_{21} = S_{21} + S_3$ $\begin{array}{|c|c|}\hline a & P_{111} \\ \hline a & \\ \hline a & \\ \hline \end{array}$ $P_{111} = S_3$ together

  $2S_3 + S_{21}$

- $S_6 + S_{42}$without $\begin{array}{|c|c|c|}\hline a & a & a \\ \hline \end{array}$ = $2S_3 + S_{21}$

Plethysm

$S_3[S_2]$

level	result	current	needed	new
5	S_6	$S_{6/5} = S_1$	S_1	0
4	S_6	$S_{6/4} = S_2$	$2S_2$	S_{42}
3	$S_6 + S_{42}$	$S_{6/3} + S_{42/3} =$	$2S_3 + S_{21}$	0
		$2S_3 + S_{21}$		
2	$S_6 + S_{42}$	$S_{6/2} + S_{42/2} =$	$2S_4 + 2S_{22} + S_{31}$	S_{222}
		$2S_4 + S_{22} + S_{31}$		

$S_3[S_2] = S_6 + S_{42} + S_{222}$

Transition Matrices

- 5 bases, fixed degree
- matrix size = number of partitions
- combinatorial interpretation
- power sum $\rightarrow$ Schur : character table Sym_n

MAGMA

```
>PowerSumToSchurMatrix(5);
[ 1 -1 0 1 0 -1 1]
[ 1 0 -1 0 1 0 -1]
[ 1 -1 1 0 -1 1 -1]
[ 1 1 -1 0 -1 1 1]
[ 1 0 1 -2 1 0 1]
[ 1 2 1 0 -1 -2 -1]
[ 1 4 5 6 5 4 1]
```

Transition Matrices

MAGMA

```
>S := SFASchur(Rationals());
>P:=SFAPower(Rationals());
>S!P.[1,1,1,1,1];
S.[1,1,1,1,1] + 4*S.[2,1,1,1] + 5*S.[2,2,1]
+ 6*S.[3,1,1] + 5*S.[3,2] + 4*S.[4,1] + S.[5]
```

Transition Matrices

different problems

- computation of a complete matrix
  use conjugate partition
- computation of a single row
- computation of a single value
  Murnaghan Nakayama rule

Last Page

Thank you very much for your attention.

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