Trifferent codes with small lengths

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Abstract

A code $C \subseteq \{0,1,2\}^n$ of length n is called trifferent if for any three distinct elements of C there exists a coordinate in which they all differ. By $T(n)$ we denote the maximum cardinality of trifferent codes with length n. The values $T(5) = 10$ and $T(6) = 13$ were recently determined [DFGP22]. Here we determine $T(7) = 16$, $T(8) = 20$, and $T(9) = 27$. For the latter case $n = 9$ there also exist linear codes attaining the maximum possible cardinality 27.

1 Introduction

For integers $k \geq 3$ and $n \geq 1$ any set $C \subseteq \{0,1,\ldots,k-1\}^n$ is called a k-ary *code* of length n. If C has the property that for any k distinct elements there exists a coordinate in which they all differ then C is called *perfect k-hash code*. The problem of determining the largest possible size of a perfect k-hash code is a natural combinatorial problem with connections to topics in cryptography, information theory, and computer science, see e.g. [DFCD22, GR22, KM88, LS06, SZ08, WX01, XYar].

Given a perfect k-hash code C of length n we can define the subcode $C_{\neg i:j} := \{c \in C \mid c_i \neq j\}$ for each integer $1 \leq i \leq n$ and $0 \leq j \leq k-1$, i.e., the set of all codewords that do not have symbol j in position i. Since removing the ith entries from the codewords of $C_{\neg i:j}$ gives a perfect k-hash code of length $n-1$, we have $\#C \leq \left\lfloor \frac{k}{k-1} \cdot \#C_{\neg i:j} \right\rfloor$ and

$$\#C \leq (k-1) \cdot \left(\frac{k}{k-1}\right)^n \tag{1}$$

for every $k \geq 3$, c.f. [KM88].

The simple upper bound (1) was improved by Fredman and Komlós back in 1984, see [FK84], to

$$\#C \leq \left(2^{k!/k^{k-1}}\right)^n \tag{2}$$

for every $k \geq 4$ and sufficiently large n. Since then this bound was improved over the years with various restrictions on k, see e.g. [Ari94, CD21, DFCD22, GR22, KM88]. However, no “proper” improvement was obtained for the case $k = 3$.

Perfect 3-hash codes are also called *trifferent codes* and we denote the maximum possible cardinality for length n by $T(n)$. Clearly the recursion underlying the upper bound (1) can be slightly improved by downrounding to integers in each iteration:

$$T(n) \leq \left\lfloor \frac{3 \cdot T(n-1)}{2} \right\rfloor \tag{3}$$

for $n \geq 2$. Starting from $T(6) = 13$ an iterative application of (3) gives $T(11) \leq 94$, so that $T(n) \leq 1.0868 \cdot \left(\frac{3}{2}\right)^n$ for $n \geq 11$, see [DFGP22, Theorem 2.5], while (1) gives $T(n) \leq 2 \cdot \left(\frac{3}{2}\right)^n$.

Assuming that the trifferent code C has to be linear, i.e., a linear subspace over $\mathbb{F}_3$ improved upper bounds on $\#C$ can be shown [PZ22]. Currently the best upper bound is $\#C \le 1.2731^n$ for sufficiently large n [BDGP23] (assuming that C is linear).

Here we determine $T(7) = 16$, $T(8) = 20$, and $T(9) = 27$ – noting that there exists an up to symmetry unique linear trifferent code of dimension 3 and length 9. So, it remains an interesting open question whether the lower bounds $T(14) \ge 81$ and $T(19) \ge 243$, based on linear codes, from [BDGP23] can be beaten.

2 Results obtained by exhaustive enumeration

One way to represent trifferent codes is to write the codewords as columns of a matrix. As an example we state the three trifferent codes of length $n = 6$ and cardinality $T(6) = 13$ from [DFGP22]:

$$\begin{pmatrix} 0&0&2&2&2&2&2&0&1&1&1&1&1 \\ 0&1&0&2&2&2&1&2&0&1&1&1&2 \\ 0&1&1&0&2&1&2&2&2&0&1&2&1 \\ 0&1&1&1&0&2&2&2&2&2&0&1&1 \\ 0&1&1&2&1&0&2&2&2&1&2&0&1 \\ 0&1&2&1&1&1&0&2&1&2&2&2&0 \end{pmatrix}, \begin{pmatrix} 0&0&2&2&2&2&2&0&1&1&1&1&1 \\ 0&1&0&2&2&1&2&2&0&1&1&2&1 \\ 0&1&1&0&2&2&1&2&2&0&1&1&2 \\ 0&1&1&1&0&2&2&2&2&2&0&1&1 \\ 0&1&2&1&1&0&2&2&1&2&2&0&1 \\ 0&1&1&2&1&1&0&2&2&1&2&2&0 \end{pmatrix}, \begin{pmatrix} 0&0&2&2&2&2&2&0&1&1&1&1&1 \\ 0&1&0&2&2&1&2&2&0&1&1&2&1 \\ 0&1&1&0&2&2&2&2&2&0&1&1&1 \\ 0&1&1&1&0&2&1&2&2&2&0&1&2 \\ 0&1&2&1&1&0&2&2&1&2&2&0&1 \\ 0&1&1&1&2&1&0&2&2&2&1&2&0 \end{pmatrix}.$$

Obviously, applying permutations to the columns, the rows, or the symbols $\{0, 1, 2\}$ does not change the property of being trifferent, so that we speak of equivalent matrices if they arise by a sequence of the mentioned operations. With this, two trifferent codes are called equivalent, if their representing matrices are (where the ordering of the columns is arbitrary). Here we are only interested in non-equivalent codes and choose the lexicographic minimal matrix, when reading columnwise from left to right, as a unique representative of an equivalence class of codes. For our example the representatives are given by:

$$\begin{pmatrix} 0&0&0&0&0&1&1&1&1&1&2&2&2 \\ 0&0&0&1&2&0&1&1&1&2&0&1&2 \\ 0&0&0&1&2&1&2&2&2&0&2&0&1 \\ 0&1&2&0&1&1&0&1&2&0&0&1&2 \\ 0&1&2&1&2&2&2&0&1&1&1&2&0 \\ 0&1&2&2&0&0&1&2&0&2&2&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1&1&1&2&2&2&2&2 \\ 0&0&0&1&2&0&1&2&0&1&2&2&2 \\ 0&0&1&1&2&1&2&0&2&0&0&1&1 \\ 0&1&2&0&0&1&2&0&1&1&2&0&1 \\ 0&1&2&1&2&2&0&1&1&2&1&2&0 \\ 0&1&2&2&0&0&1&2&2&0&0&1&2 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1&1&1&1&1&2&2&2 \\ 0&0&0&1&2&0&1&2&2&2&0&1&2 \\ 0&0&1&0&2&0&2&1&2&2&2&1&0 \\ 0&1&0&1&2&2&0&1&0&1&1&2&0 \\ 0&1&1&2&1&2&1&2&2&0&1&0&2 \\ 0&1&2&0&0&2&2&0&1&2&2&1&0 \end{pmatrix}.$$

If we interprete every column as the base-3-representation of an integer we get an even more compact representation:

- $\{0, 13, 26, 113, 231, 285, 385, 389, 399, 410, 545, 582, 694\}$,
- $\{0, 13, 53, 113, 222, 285, 397, 410, 554, 582, 669, 682, 686\}$,
- $\{0, 13, 32, 96, 237, 269, 383, 447, 466, 470, 554, 613, 654\}$.

Using the lexicographic ordering we can apply *orderly generation* [Rea78], i.e., starting from an empty list of codewords we iteratively add one codeword at each step such that the code C remains lexicographic minimal and trifferent. In Table 1 we have listed the number $a(n, l)$ of non-equivalent trifferent codes C with length n and cardinality l for all $n \le 6$. Clearly, we have $a(n, 1) = 1$ and $a(n, 2) = n$ – counting the number of possibilities of the Hamming distance $d_H(c, c') := \#\{1 \le i \le n \mid c_i \ne c'_i\}$ between two different codewords of length n.[1] The count $a(6, 13) = 3$ was also obtained in [DFGP22].

2.1 Upper bounds involving the minimum Hamming distance

Observing that the three non-equivalent trifferent codes with length 6 and cardinality 13 have minimum Hamming distance $d_H(C) := \min\{d_H(c, c') \mid c, c' \in C, c \ne c'\} = 3$ we define $T(n, d)$ as the maximum cardinality of a trifferent code with length n and minimum Hamming distance d. From our classification of trifferent codes with length $n \le 6$ we conclude the values listed in Table 2. We observe that for a relatively large minimum Hamming distance $T(n, d)$ cannot be too large due to coding theoretic upper bounds on the maximum cardinality of a 3-ary code with minimum Hamming distance at least d, see e.g. [BHOS98]. Especially, we have $T(n, n) = 3$ and $T(n, d) = 1$ for $d > n$. However, if the minimum Hamming distance d is rather small, then also cardinality $T(n)$ (marked in bold face) cannot be attained.

Our next aim is to determine $T(n, 1)$.

[1] It is an easy exercise to verify $a(n, 3) = n \cdot (n^2 + 5) / 6$.

$n \setminus \#C = l$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	1										
2	1	2	3	1									
3	1	3	7	7	2	1							
4	1	4	14	35	38	25	3	2	1				
5	1	5	25	141	613	1410	944	269	55	5			
6	1	6	41	499	8038	99612	486122	727128	339695	63781	4832	93	3

Table 1: Number $a(n, l)$ of non-equivalent trifferent codes C with length n and cardinality l.

$n \setminus d$	1	2	3	4	5	6
1	**3**	1	1	1	1	1
2	**4**	3	1	1	1	1
3	5	**6**	3	1	1	1
4	7	8	**9**	3	1	1
5	**10**	**10**	9	6	3	1
6	11	12	**13**	10	4	3

Table 2: Maximum cardinality $T(n, d)$ of a trifferent code with length n and minimum Hamming distance d.

Lemma 1. *Let C be a trifferent code with length n and $c \in C$ be arbitrary. Then,*

$$C' := \{(c, 0), (c, 1)\} \cup \{(c', 2) \mid c' \in C \backslash \{c\}\}$$

is a trifferent code with length $n + 1$, minimum Hamming distance 1, and cardinality $\#C + 1$.

Proof. Let x', y', z' be three different elements from C' and x, y, z be the corresponding codewords after removing the last coordinate. If the last coordinate of at most one element of x', y', z' is not equal to 2, then x, y, z are different elements in C and $\{x, y, z\}$ is trifferent, by assumption, and so is $\{x', y', z'\}$. If the last coordinate of at least two elements of x', y', z' is not equal to 2, then x', y', z' differ in the last coordinate. Thus, C' is trifferent. The other two statements on the cardinality and minimum Hamming distance of C' directly follow from the construction. □

Corollary 2. *For each $n \geq 2$ we have $T(n) \geq T(n, 1) \geq T(n - 1) + 1$.*

Theorem 3. *For each $n \geq 2$ we have $T(n, 1) = T(n - 1) + 1$.*

Proof. It remains to show $\#C \leq T(n-1)+1$ for each trifferent code with length n and minimum Hamming distance 1. W.l.o.g. we assume $c' := (0, \ldots, 0) \in C$ and $c'' := (1, 0, \ldots, 0) \in C$, so that every other codeword c has a 2 in the first coordinate (since $\{c', c'', c\}$ is trifferent). Thus, we have

$$\#C \leq \#C_{\neg 1:0} + 1 \leq T(n - 1) + 1. \tag{4}$$

□

Lemma 4. *Let $d \geq 2$ and let $\pi \colon C \to \{0, 1, 2\}^d$ be the projection on the first d coordinates. Suppose that π is not surjective. Then $\#C \leq T(n - d) \cdot \left(3^d - 1\right) / 2^d$.*

Proof. W.l.o.g. the vector $(1, \ldots, 1)$ is not in the image of π. We call $z \in \{0, 1, 2\}^d$ *odd* if $z_1 + \cdots + z_d$ is odd and *even* if this sum is even. For z in $\{0, 1, 2\}^d$ define $B_z := \left\{w \in \{0, 1, 2\}^d \,:\, w_i \neq z_i \,\forall i = 1, \ldots, d\right\}$ and $C_z := \{c \in C \,:\, \pi(c) \in B_z\}$. Observe that for any $w \in \{0, 1, 2\}^d$ unequal to $(1, \ldots, 1)$, there are 2^{d-1} sets B_z that contain w for which z is odd (and equally many for z even). Since $\#C_z \leq T(n - d)$ for all z, we obtain

$$2^{d-1} \cdot \#C = \sum_{z \text{ odd}} \#C_z \leq T(n - d) \cdot \left(3^d - 1\right) / 2$$

since there are $\left(3^d-1\right)/2$ vectors z that are odd. $\square$

Corollary 5. *If a trifferent code C of length n has a pair at Hamming distance $d \ge 2$, then we have $\#C \le T(n-d)\cdot\left(3^d-1\right)/2^d$.*

Proof. W.l.o.g. the two words at distance d are $0\ldots00\ldots0$ and $1\ldots10\ldots0$ in $\{0,1,2\}^{d+(n-d)}$. Then every other codeword $c \in C$ must have a 2 in the first d positions because of the trifferent property. So π is not surjective and we can apply Lemma 4. $\square$

Since the convergence of $\frac{3^d-1}{2^d}$ towards $\frac{3^d}{2^d}$ is too fast we even cannot deduce an improved upper bound of the form $T(n) \le c\cdot\left(\frac{3}{2}\right)^n/\log\log n$ for some constant $c>0$ and sufficiently large n.

2.2 Upper bounds involving the number of occurrences of symbols

For a given trifferent code C with length $n \ge 2$ let $s_{i,j}$ be the number of codewords $c \in C$ with $c_i = j$, where $1 \le i \le n$ and $j \in \{0,1,2\}$. For each triple of pairwise different symbols $j_1, j_2, j_3 \in \{0,1,2\}$ we have $s_{i,j_1} + s_{i,j_2} = \#C_{\neg i:j_3} \le T(n-1)$, so that $\#C = s_{i,j_1} + s_{i,j_2} + s_{i,j_3}$ implies

$$\#C \le 2\cdot T(n-1) - s_{i,j_1}. \tag{5}$$

Let $\tau = \tau(C)$ denote the maximum value of the sums $s_{i,j_1}+s_{i,j_2}$ and $i^\star, j_1^\star, j_2^\star, j_3^\star$ be such that $\tau = s_{i^\star,j_1^\star}+s_{i^\star,j_2^\star}$, $\{j_1^\star, j_2^\star, j_3^\star\} = \{0,1,2\}$. With this, let C' arise from $C_{\neg i^\star:j_3^\star}$ by removing the $i^\star$th coordinates from the codewords, i.e., C' is a trifferent code with length $n-1$ and cardinality τ.

As the numbers $a(n,l)$ of non-equivalent trifferent codes with length n and cardinality l grow very quickly, see Table 1, we want restrict our exhaustive search to rather large values of l. To this end we start from a trifferent code C' with length $n-1$ and cardinality τ, fill the nth coordinate with all 2^τ choices of symbols in $\{0,1\}$, and iteratively extend by codewords have a 2 in the nth coordinate such that the code remains trifferent. Starting from an arbitrary trifferent code C of length n to choose C' as described above, the list of constructed trifferent codes of length n contains a code that is equivalent to C. Assuming that $\#C' = \tau(C)$ we can upper bound the cardinality of C by

$$\#C = \frac{1}{2}\cdot\left(s_{1,0}+s_{1,1} \;+\; s_{1,0}+s_{1,2} \;+\; s_{1,1}+s_{1,2}\right) \le \left\lfloor\frac{3\tau}{2}\right\rfloor = \left\lfloor\frac{3\#C'}{2}\right\rfloor. \tag{6}$$

So, if we want to construct all trifferent codes with length 9 and cardinality at least 28, then it suffices to start with all trifferent codes of length 8 and cardinality at least 19. The latter can be constructed from trifferent codes with length 7 and cardinality at least 13 and those can in turn be constructed from all trifferent codes of length 6 and cardinality at least 9.

Given the parameter $\tau = \#C'$ we can slightly improve Inequality (5) to

$$\#C \le 2\tau - s_{i,j_1}. \tag{7}$$

While we do not know the exact value of s_{i,j_1} in the intermediate steps of the generation of the codes C, we always have lower bounds for s_{i,j_1} which we use to cut the search trees. As computational results we have obtained

- $a(7,13) = 22736583$, $a(7,14) = 429602$, $a(7,15) = 2164$, $a(7,16) = 3$, $a(7,17) = 0$;
- $a(8,19) = 38581$, $a(8,20) = 57$, $a(8,21) = 0$;
- $a(9,28) = 0$,

which implies:

Theorem 6. *$T(7) = 16$, $T(8) = 20$, $T(9) \le 27$, and $T(n) \le 0.6937\cdot\left(\frac{3}{2}\right)^n$ for $n \ge 10$.*

We remark that extending the 38638 non-equivalent trifferent codes with length 8 and cardinality at least 19 yields just 44 non-equivalent trifferent codes with length 9, cardinality 25 and none with cardinality 26. However, one can easily construct a linear trifferent code of length 9 and cardinality 27, as we will explain in the next section, so that $T(9) = 27$:

$$C = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&2&2&2&2&2&2&2&2&2 \\ 0&0&0&1&1&1&2&2&2&0&0&0&1&1&1&2&2&2&0&0&0&1&1&1&2&2&2 \\ 0&0&0&1&1&1&2&2&2&1&1&1&2&2&2&0&0&0&2&2&2&0&0&0&1&1&1 \\ 0&0&0&1&1&1&2&2&2&2&2&2&0&0&0&1&1&1&1&1&1&2&2&2&0&0&0 \\ 0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0&1&2 \\ 0&1&2&0&1&2&0&1&2&1&2&0&1&2&0&1&2&0&2&0&1&2&0&1&2&0&1 \\ 0&1&2&0&1&2&0&1&2&2&0&1&2&0&1&2&0&1&1&2&0&1&2&0&1&2&0 \\ 0&1&2&1&2&0&2&0&1&0&1&2&1&2&0&2&0&1&0&1&2&1&2&0&2&0&1 \\ 0&1&2&2&0&1&1&2&0&0&1&2&2&0&1&1&2&0&0&1&2&2&0&1&1&2&0 \end{pmatrix}$$

We can computationally check that $C_{\neg i:j}$ has cardinality 18 for all $1 \le i \le 9$, $j \in \{0, 1, 2\}$ (which can be deduced from the fact that C is linear with dual minimum distance of at least 2). Repeating the process a second time we can computationally check that all resulting trifferent codes of length 7 have cardinality 12 (which can also be deduced from the fact that C is linear with dual minimum distance of at least 3, i.e., that the code is projective). So, clearly the search described above could not find a code equivalent to C. However, we can apply the extension search to all $C_{\neg i:j}$ after removing the ith coordinates. As a result we have obtained 166, 39, and 11 non-equivalent trifferent codes with lengths 9 and cardinalities 25, 26, and 27, respectively.

3 Linear trifferent codes

We call a q-ary code C that forms a linear subspace over $\mathbb{F}_q$ a *linear* code. If C has length n and cardinality q^k, then we speak of an $[n, k]_q$-code. A non-zero codeword $c \in C$ is called *minimal* if the *support* $\operatorname{supp}(c) := \{i \mid c_i \neq 0\}$ of c is minimal with respect to inclusion in the set $\{\operatorname{supp}(u) \mid u \in C \backslash \mathbf{0}\}$. The code C is a *minimal code* if all its non-zero codewords are minimal. In [BDGP23] it was shown that a linear 3-ary code is minimal iff it is trifferent.

In contrast to Theorem 3 and Lemma 3 the minimum Hamming distance of a linear trifferent code needs to increase with the length of the codewords, i.e., for every minimal $[n, k]_q$-code we have $d \ge (k-1)(q-1)+1$ for the minimum Hamming distance d, see e.g. [ABNR22, Theorem 2.8], [HN21, Theorem 23]. Using the software `LinCode` for the enumeration of linear codes [BBK21] we have classified the minimal $[n, k]_3$-codes with the minimum possible length n for small k up to symmetry. For dimension $k = 1$ the smallest example is the trivial $[1, 1]_3$ code with minimum distance 1 and for $k = 2$ the unique smallest example is a $[4, 2]_3$ code with minimum distance 3. Also for dimension $k = 3$ there exists a unique smallest example[2] given e.g. by the generator matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 & 0 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

The code has weight enumerator $1 + 6x^5 + 8x^6 + 12x^7$, i.e., its minimum distance is 5, and its automorphism group has order 48. All of these codes attain the maximum cardinality $T(n) = 3^k$ of trifferent codes for their respective length n and for $k = 1, 2$ even all such examples are linear while there are others for $k = 3$. For dimension $k = 4$ the minimum length n of a minimal $[n, k]_3$-code is 14 and there are exactly three non-equivalent such codes given e.g. by the generator matrices

$$\begin{pmatrix} 11111111101000 \\ 00011122210100 \\ 11101201220010 \\ 01221211200001 \end{pmatrix}, \begin{pmatrix} 11111111101000 \\ 00011112210100 \\ 01211121220010 \\ 10001220110001 \end{pmatrix}, \text{ and } \begin{pmatrix} 11111111101000 \\ 00011112210100 \\ 11202221220010 \\ 02010121020001 \end{pmatrix}.$$

The corresponding weight enumerators are given by $1 + 14x^7 + 4x^8 + 20x^9 + 16x^{10} + 26x^{11}$, $1 + 12x^7 + 12x^8 + 8x^9 + 24x^{10} + 24x^{11}$, $1 + 12x^7 + 12x^8 + 8x^9 + 24x^{10} + 24x^{11}$ and the orders of the corresponding automorphism

[2]We remark that there are nine $[9, 3]_3$-codes and one $[8, 3]_3$-code with minimum distance at least 5 – only one of these codes is minimal.

groups are 16, 12, 48, respectively. For larger values of n the number of non-equivalent minimal $[n,4]_3$-codes grows quickly, i.e., there are 53 minimal $[15,4]_3$-, 818 minimal $[16,4]_3$-, 9266 minimal $[17,4]_3$-, and 80999 minimal $[18,4]_3$-codes. We remark that the three minimal $[14,4]_3$-codes all violate the *Ashikhmin–Barg condition* $w_{\max} < w_{\min} \cdot \frac{q}{q-1}$ which is a sufficient condition on the maximum and minimum weight of a linear q-ary code to be minimal [AB98].

For dimension $k=5$ the minimum length n of a minimal $[n,k]_3$-code is 19 and up to equivalence there is a unique such code given e.g. by the generator matrix

$$\begin{pmatrix}1111111111100010000\\0001111222211101000\\1110111022201200100\\0122000122220000010\\1220012101212100001\end{pmatrix}.$$

The corresponding weight enumerator is given by $1+26x^9+132x^{12}+84x^{15}$, i.e., the code is a 3-divisible 3-weight code, and the automorphism group has order 8. (The support contains two full lines intersecting in a common point.) One dimension less, there are 27282 minimal $[18,4,9]_3$-codes. For the next case we report 263 minimal $[19,4,11]_3$-, 49104 minimal $[20,4,11]_3$-, and 1615534 minimal $[21,4,11]_3$ codes. A minimal $[n,5,11]_3$-code with maximum column multiplicity has length at least 21 and we have found 101 non-isomorphic such $[21,5,11]_3$-codes. None of them can be extended to a minimal $[22,6,11]_3$-code, so that $m(6,3) \ge 23$. Extending roughly 2% of the 1615534 minimal $[21,4,11]_3$ codes, we constructed 2165971 minimal $[22,5,11]_3$ codes. Extending 6784 of the latter unfortunately has not resulted in a minimal $[23,6,11]_3$ code, while one might conjecture that such a code exist. We remark that the minimum lengths of minimal $[n,k]_3$-codes for $k \le 5$ have also been determined in [BDGP23]. Using integer linear programming computations it was also shown that a minimal $[n,6]_3$-code has length at least 22 and an example with length 24 was found.

To conclude this section we consider the well-known correspondence between (non-degenerated) $[n,k]_q$-codes and multisets of points in the projective space $\mathrm{PG}(k-1,q)$ of cardinality n, i.e., the columns of a generator matrix each generate a point, see e.g. [DS98].

Definition 7. *A multiset M of points in a projective space is called a* strong blocking multiset *if for every hyperplane H, we have $\langle S \cap H\rangle = H$.*

If M is the multiset of points associated to a linear code C, then C is minimal iff M is a strong blocking multiset, see e.g. [ABN22, TQLZ21]. Directly from the definition of a strong blocking multiset we can read off that a multiset of points in $\mathrm{PG}(1,\mathrm{q})$ is a strong blocking multiset iff it contains every point of the entire projective space. Clearly adding points to a multiset does not destroy the property of being a strong blocking multiset, so that we consider *minimal strong blocking sets* in the following, i.e., set of points that are a strong blocking multiset but such that every proper subset is not a strong blocking multiset. So, in $\mathrm{PG}(1,q)$ the unique minimal strong blocking set is a line.[3] As e.g. observed in [ABN22] a set of points in $\mathrm{PG}(2,\mathrm{q})$ is a strong blocking set iff every line contains at least 2 points, i.e., a 2-fold blocking set (with respect to lines). In $\mathrm{PG}(2,3)$ the complement of a minimal strong blocking set is an *oval*, which, for $q=3$, coincides with a (projective) frame, i.e., a set of 4 points each three of them spanning the entire space.[4] While there is an unique minimal strong blocking set in $\mathrm{PG}(2,3)$, in $\mathrm{PG}(3,3)$ we have several non-equivalent such sets, see Table 3. So, if we start from the set of 40 points of the projective solid $\mathrm{PG}(3,3)$ and iteratively remove points without destroying the property of being a strong blocking set, we can delete between 26 and 20 points.

[3] For completeness we remark that the corresponding minimal $[4,2]_3$-code has an automorphism group of order 48.

[4] We said that it is "easy" to construct a minimal (or trifferent) $[9,3]_3$-code, since we can start with the point set of the projective plane $\mathrm{PG}(2,3)$ and iteratively remove arbitrary points that are not contained on any secant. This process always stops after 4 removals and we end up with a unique set of points (up to equivalence).

n	14	15	16	17	18	19	20
#	3	39	363	1517	3736	5791	5110

Table 3: Number of non-equivalent minimal strong blocking sets in PG$(3,3)$ per cardinality n.

Acknowledgments

The author thanks Gianira Alfarano, Anurag Bishnoi, Jozefien D'haeseleer, Dion Gijswijt, Alessandro Neri, Sven Polak, and Martin Scotti for many helpful remarks on an earlier version of this paper. Especially, Lemma 4 and Corollary 5 are stated with courtesy of Dion Gijswijt (simplifying and generalizing partial results of the author).

A Trifferent codes with small lengths and relatively large cardinalities

There exist $a(4,7)=3$ non-equivalent trifferent codes with length 4 and cardinality 7:

- $\{0,1,14,35,50,71,74\}$;
- $\{0,13,26,32,42,46,61\}$;
- $\{0,4,11,34,47,70,77\}$.

There exist $a(4,8)=2$ non-equivalent trifferent codes with length 4 and cardinality 8:

- $\{0,13,26,32,42,46,61,65\}$;
- $\{0,4,11,34,47,70,77,78\}$.

There exists an up to equivalence unique trifferent code with length 4 and maximum cardinality $T(4)=9$:

- $\{0,13,26,32,42,46,61,65,75\}$.

The code is linear.

There exist $a(5,10)=5$ non-equivalent trifferent codes with length 5 and maximum cardinality $T(5)=10$:

- $\{0,1,41,80,98,128,140,185,197,227\}$;
- $\{0,4,11,34,101,142,209,232,239,240\}$;
- $\{0,4,38,79,97,131,132,137,182,196\}$;
- $\{0,4,38,79,97,131,137,182,196,230\}$;
- $\{0,4,38,79,97,131,137,182,196,231\}$.

There exist $a(6,12)=93$ non-equivalent trifferent codes with length 6 and cardinality 12:

- $\{0,13,113,123,223,227,289,399,431,566,586,696\}$;
- $\{0,13,26,113,123,223,289,308,399,586,659,696\}$;
- $\{0,13,26,113,231,285,385,389,399,410,545,582\}$;
- $\{0,13,26,113,231,285,385,389,399,410,545,586\}$;
- $\{0,13,26,113,231,285,385,389,399,410,545,694\}$;
- $\{0,13,26,113,231,285,385,389,410,545,586,696\}$;
- $\{0,13,26,32,123,289,385,389,399,420,586,707\}$;
- $\{0,13,26,32,123,289,466,470,480,640,663,707\}$;
- $\{0,13,26,32,96,223,227,237,289,447,586,626\}$;
- $\{0,13,26,32,96,223,227,289,447,480,586,626\}$;
- $\{0,13,26,32,96,223,289,447,470,480,586,626\}$;
- $\{0,13,26,32,96,223,289,447,470,586,626,723\}$;
- $\{0,13,26,32,96,223,289,447,480,586,626,713\}$;
- $\{0,13,26,32,96,262,383,447,613,709,713,723\}$;
- $\{0,13,26,32,96,262,464,640,690,709,713,723\}$;
- $\{0,13,26,32,96,289,447,466,470,480,586,626\}$;
- $\{0,13,26,32,96,289,447,466,470,480,586,707\}$;
- $\{0,13,26,32,96,289,447,466,470,586,626,723\}$;
- $\{0,13,32,107,123,289,304,389,399,586,654,697\}$;

- $\{0, 13, 32, 107, 123, 304, 399, 461, 586, 654, 697, 716\}$;
- $\{0, 13, 32, 107, 232, 285, 308, 385, 399, 582, 626, 694\}$;
- $\{0, 13, 32, 96, 127, 237, 269, 466, 470, 554, 620, 654\}$;
- $\{0, 13, 32, 96, 134, 153, 311, 457, 480, 613, 670, 707\}$;
- $\{0, 13, 32, 96, 134, 262, 461, 475, 554, 613, 654, 723\}$;
- $\{0, 13, 32, 96, 223, 269, 289, 447, 635, 639, 670, 704\}$;
- $\{0, 13, 32, 96, 223, 269, 289, 480, 635, 670, 681, 704\}$;
- $\{0, 13, 32, 96, 223, 269, 370, 480, 635, 670, 681, 704\}$;
- $\{0, 13, 32, 96, 223, 269, 383, 447, 470, 480, 613, 670\}$;
- $\{0, 13, 32, 96, 223, 269, 451, 480, 635, 670, 681, 704\}$;
- $\{0, 13, 32, 96, 237, 269, 289, 447, 466, 641, 654, 697\}$;
- $\{0, 13, 32, 96, 237, 269, 383, 447, 466, 470, 554, 613\}$;
- $\{0, 13, 32, 96, 237, 269, 383, 447, 466, 470, 554, 654\}$;
- $\{0, 13, 32, 96, 237, 311, 370, 411, 626, 690, 709, 713\}$;
- $\{0, 13, 53, 113, 222, 285, 397, 410, 554, 582, 669, 682\}$;
- $\{0, 13, 53, 113, 222, 285, 410, 554, 582, 669, 682, 686\}$;
- $\{0, 4, 11, 115, 236, 290, 394, 401, 402, 412, 547, 587\}$;
- $\{0, 4, 11, 115, 236, 290, 394, 401, 412, 547, 587, 699\}$;
- $\{0, 4, 11, 115, 236, 290, 394, 402, 412, 547, 587, 698\}$;
- $\{0, 4, 11, 124, 239, 293, 385, 398, 421, 556, 590, 699\}$;
- $\{0, 4, 11, 124, 240, 294, 385, 398, 421, 556, 591, 698\}$;
- $\{0, 4, 11, 34, 101, 304, 425, 628, 695, 718, 725, 726\}$;
- $\{0, 4, 119, 239, 290, 380, 416, 483, 554, 590, 610, 680\}$;
- $\{0, 4, 119, 240, 290, 380, 416, 482, 555, 591, 610, 681\}$;
- $\{0, 4, 38, 101, 149, 218, 322, 358, 421, 455, 636, 695\}$;
- $\{0, 4, 38, 101, 149, 322, 358, 393, 421, 461, 699, 707\}$;
- $\{0, 4, 38, 101, 150, 218, 322, 358, 421, 456, 635, 695\}$;
- $\{0, 4, 38, 101, 150, 322, 358, 392, 421, 461, 698, 708\}$;
- $\{0, 4, 38, 104, 286, 294, 384, 392, 455, 551, 655, 668\}$;
- $\{0, 4, 38, 104, 286, 303, 311, 428, 591, 632, 655, 695\}$;
- $\{0, 4, 38, 124, 241, 293, 383, 421, 461, 555, 591, 682\}$;
- $\{0, 4, 38, 124, 241, 293, 383, 421, 554, 591, 668, 682\}$;
- $\{0, 4, 38, 124, 241, 293, 383, 421, 555, 591, 668, 682\}$;
- $\{0, 4, 38, 124, 241, 293, 384, 421, 461, 555, 590, 682\}$;
- $\{0, 4, 38, 124, 241, 293, 384, 421, 555, 590, 668, 682\}$;
- $\{0, 4, 38, 124, 241, 294, 383, 421, 461, 555, 591, 682\}$;
- $\{0, 4, 38, 124, 241, 294, 383, 421, 555, 591, 668, 682\}$;
- $\{0, 4, 38, 50, 97, 205, 267, 464, 474, 537, 623, 646\}$;
- $\{0, 4, 38, 50, 97, 231, 267, 448, 464, 565, 618, 623\}$;
- $\{0, 4, 38, 51, 124, 241, 293, 421, 461, 554, 590, 627\}$;
- $\{0, 4, 38, 51, 97, 205, 266, 465, 473, 536, 623, 646\}$;
- $\{0, 4, 38, 51, 97, 230, 266, 448, 465, 565, 617, 623\}$;
- $\{0, 4, 38, 79, 131, 230, 294, 340, 380, 555, 668, 682\}$;
- $\{0, 4, 38, 79, 131, 294, 340, 380, 473, 636, 668, 682\}$;
- $\{0, 4, 38, 79, 131, 312, 425, 439, 473, 583, 618, 623\}$;
- $\{0, 4, 38, 79, 131, 312, 425, 439, 473, 583, 618, 704\}$;
- $\{0, 4, 38, 79, 131, 312, 425, 439, 473, 618, 623, 664\}$;
- $\{0, 4, 38, 79, 131, 312, 425, 439, 473, 618, 664, 704\}$;
- $\{0, 4, 38, 79, 131, 312, 425, 439, 473, 636, 664, 704\}$;
- $\{0, 4, 38, 79, 97, 131, 137, 294, 473, 618, 668, 682\}$;
- $\{0, 4, 38, 79, 97, 131, 299, 456, 623, 668, 682, 716\}$;
- $\{0, 4, 38, 79, 97, 131, 299, 456, 623, 668, 682, 717\}$;
- $\{0, 4, 38, 79, 97, 132, 299, 455, 623, 668, 682, 716\}$;
- $\{0, 4, 38, 79, 97, 132, 299, 455, 623, 668, 682, 717\}$;
- $\{0, 4, 38, 79, 97, 137, 230, 294, 374, 668, 682, 717\}$;
- $\{0, 4, 38, 79, 97, 230, 294, 374, 380, 668, 682, 717\}$;
- $\{0, 4, 38, 79, 97, 230, 294, 374, 623, 668, 682, 717\}$;
- $\{0, 4, 38, 79, 97, 231, 293, 375, 380, 668, 682, 716\}$;
- $\{0, 4, 38, 79, 97, 231, 293, 375, 623, 668, 682, 716\}$;
- $\{0, 4, 38, 79, 97, 293, 421, 618, 623, 668, 682, 716\}$;
- $\{0, 4, 38, 79, 97, 294, 421, 617, 623, 668, 682, 716\}$;
- $\{0, 4, 38, 79, 97, 299, 421, 623, 668, 682, 716, 717\}$;
- $\{0, 4, 38, 97, 131, 267, 473, 542, 618, 646, 655, 695\}$;
- $\{0, 4, 38, 97, 131, 267, 473, 565, 618, 623, 668, 682\}$;
- $\{0, 4, 38, 97, 131, 294, 299, 403, 591, 668, 682, 716\}$;
- $\{0, 4, 38, 97, 131, 294, 322, 380, 473, 591, 655, 695\}$;
- $\{0, 4, 38, 97, 146, 221, 267, 455, 474, 591, 682, 695\}$;
- $\{0, 4, 38, 97, 146, 222, 266, 456, 473, 590, 682, 695\}$;
- $\{0, 4, 38, 97, 159, 266, 448, 465, 473, 590, 655, 695\}$;
- $\{0, 4, 38, 97, 230, 266, 375, 565, 623, 668, 682, 717\}$;
- $\{0, 4, 38, 97, 230, 266, 389, 448, 465, 565, 617, 668\}$;
- $\{0, 4, 38, 97, 230, 266, 389, 448, 465, 565, 618, 668\}$;
- $\{0, 4, 38, 97, 266, 308, 421, 618, 623, 682, 707, 717\}$;
- $\{0, 4, 38, 97, 294, 299, 421, 590, 632, 682, 707, 717\}$.

There exist $a(6,13) = 3$ non-equivalent trifferent codes with length 6 and maximum cardinality $T(6) = 13$:

- {0, 13, 26, 113, 231, 285, 385, 389, 399, 410, 545, 582, 694};
- {0, 13, 32, 96, 237, 269, 383, 447, 466, 470, 554, 613, 654};
- {0, 13, 53, 113, 222, 285, 397, 410, 554, 582, 669, 682, 686}.

There exist $a(7,16) = 3$ non-equivalent trifferent codes with length 7 and maximum cardinality $T(7) = 16$:

- {0, 13, 110, 277, 426, 458, 797, 885, 929, 1258, 1342, 1455, 1804, 1841, 1934, 2004};
- {0, 13, 110, 277, 610, 662, 798, 887, 1202, 1419, 1599, 1723, 1865, 1924, 2101, 2124};
- {0, 13, 110, 277, 663, 801, 878, 1021, 1203, 1339, 1424, 1561, 1761, 2032, 2081, 2124}.

There exist $a(8,20) = 57$ non-equivalent trifferent codes with length 8 and maximum cardinality $T(8) = 20$:

- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3418, 3539, 3760, 3821, 3849, 5074, 5087, 5206, 5340, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3418, 3539, 3760, 3821, 3849, 5074, 5087, 5206, 5343, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3418, 3539, 3859, 3872, 4975, 5036, 5064, 5206, 5337, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3418, 3539, 3859, 3872, 4975, 5036, 5064, 5206, 5340, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3418, 3539, 3859, 3872, 4975, 5036, 5064, 5206, 5343, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3421, 3620, 3760, 3821, 3849, 5074, 5087, 5206, 5337, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3421, 3620, 3760, 3821, 3849, 5074, 5087, 5206, 5340, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3421, 3620, 3760, 3821, 3849, 5074, 5087, 5206, 5343, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3421, 3620, 3859, 3872, 4975, 5036, 5064, 5206, 5337, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3421, 3620, 3859, 3872, 4975, 5036, 5064, 5206, 5340, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2016, 2074, 2508, 2588, 3421, 3620, 3859, 3872, 4975, 5036, 5064, 5206, 5343, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2022, 2102, 2502, 2560, 3418, 3539, 3760, 3821, 3849, 5074, 5087, 5206, 5340, 6094, 6293};
- ,{0, 13, 110, 1006, 1202, 2022, 2102, 2502, 2560, 3418, 3539, 3760, 3821, 3849, 5074, 5087, 5206, 5343, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2022, 2102, 2502, 2560, 3418, 3539, 3859, 3872, 4975, 5036, 5064, 5206, 5337, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2022, 2102, 2502, 2560, 3418, 3539, 3859, 3872, 4975, 5036, 5064, 5206, 5340, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2022, 2102, 2502, 2560, 3418, 3539, 3859, 3872, 4975, 5036, 5064, 5206, 5343, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2022, 2102, 2502, 2560, 3421, 3620, 3760, 3821, 3849, 5074, 5087, 5206, 5337, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2022, 2102, 2502, 2560, 3421, 3620, 3760, 3821, 3849, 5074, 5087, 5206, 5340, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2022, 2102, 2502, 2560, 3421, 3620, 3859, 3872, 4975, 5036, 5064, 5206, 5337, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2022, 2102, 2502, 2560, 3421, 3620, 3859, 3872, 4975, 5036, 5064, 5206, 5340, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2022, 2102, 2533, 2670, 3421, 3620, 3859, 3872, 4975, 5036, 5064, 5233, 5256, 6091, 6212};
- ,{0, 13, 110, 1006, 1202, 2022, 2102, 2533, 2670, 3421, 3620, 3859, 3872, 4975, 5036, 5064, 5233, 5337, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2047, 2178, 2502, 2560, 3418, 3539, 3760, 3821, 3849, 5074, 5087, 5181, 5261, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2047, 2181, 2502, 2560, 3418, 3539, 3760, 3821, 3849, 5074, 5087, 5181, 5261, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2047, 2181, 2502, 2560, 3418, 3539, 3859, 3872, 4975, 5036, 5064, 5181, 5261, 6094, 6293};
- {0, 13, 110, 1006, 1202, 2047, 2181, 2502, 2560, 3421, 3620, 3760, 3821, 3849, 5074, 5087, 5181, 5261, 6091, 6212};
- {0, 13, 110, 1006, 1202, 2047, 2184, 2502, 2560, 3418, 3539, 3760, 3821, 3849, 5074, 5087, 5181, 5261, 6094, 6293};
- {0, 13, 110, 1015, 1202, 2004, 2102, 2499, 2610, 3409, 3539, 3760, 3821, 3849, 4993, 5060, 5260, 5310, 6097, 6239};
- {0, 13, 110, 1015, 1202, 2004, 2102, 2499, 2610, 3409, 3539, 3778, 3845, 4975, 5036, 5064, 5260, 5310, 6097, 6239};
- {0, 13, 110, 1015, 1202, 2004, 2102, 2499, 2610, 3424, 3566, 3760, 3821, 3849, 4993, 5060, 5260, 5310, 6082, 6212};
- {0, 13, 110, 1015, 1202, 2004, 2102, 2499, 2610, 3424, 3566, 3778, 3845, 4975, 5036, 5064, 5260, 5310, 6082, 6212};
- {0, 13, 110, 1015, 1202, 2004, 2151, 2499, 2588, 3424, 3566, 3760, 3821, 3849, 4993, 5060, 5260, 5283, 6082, 6212};
- {0, 13, 110, 1015, 1202, 2013, 2102, 2490, 2637, 3409, 3539, 3760, 3821, 3849, 4993, 5060, 5260, 5283, 6097, 6239};
- {0, 13, 110, 1015, 1202, 2101, 2124, 2490, 2637, 3409, 3539, 3778, 3845, 4975, 5036, 5064, 5172, 5261, 6097, 6239};
- ,{0, 13, 110, 1015, 1205, 2019, 2093, 2151, 2570, 2670, 3409, 3539, 3760, 3849, 4993, 5039, 5260, 5283, 6097, 6239};
- ,{0, 13, 110, 1015, 1205, 2019, 2093, 2151, 2570, 2670, 3409, 3539, 3778, 3824, 4975, 5064, 5260, 5283, 6097, 6239};
- {0, 13, 110, 1015, 1205, 2019, 2093, 2151, 2570, 2670, 3424, 3566, 3760, 3849, 4993, 5039, 5260, 5283, 6082, 6212};
- {0, 13, 110, 1015, 1205, 2019, 2093, 2151, 2570, 2670, 3424, 3566, 3778, 3824, 4975, 5064, 5260, 5283, 6082, 6212};
- {0, 13, 110, 1015, 1205, 2084, 2184, 2505, 2579, 2637, 3409, 3539, 3760, 3849, 4993, 5039, 5260, 5283, 6097, 6239};
- {0, 13, 110, 1015, 1205, 2084, 2184, 2505, 2579, 2637, 3424, 3566, 3760, 3849, 4993, 5039, 5260, 5283, 6082, 6212};
- {0, 13, 110, 1015, 1205, 2101, 2124, 2505, 2579, 2637, 3424, 3566, 3760, 3849, 4993, 5039, 5243, 5343, 6082, 6212};
- {0, 13, 110, 286, 878, 1113, 1938, 2151, 2603, 2806, 2991, 3578, 3606, 4191, 5083, 5149, 5267, 5932, 6155, 6468};
- {0, 13, 113, 285, 350, 937, 1290, 1403, 2410, 3224, 4015, 4153, 4268, 4326, 4925, 5245, 6012, 6231, 6307, 6521};
- ,{0, 13, 113, 285, 370, 917, 1290, 1423, 2410, 3224, 3995, 4153, 4268, 4326, 4925, 5245, 6012, 6231, 6307, 6521};
- {0, 13, 113, 285, 399, 917, 1261, 1452, 2410, 3224, 3995, 4153, 4268, 4326, 4925, 5245, 6012, 6202, 6307, 6521};
- {0, 13, 113, 296, 416, 879, 1295, 1384, 2424, 3259, 4011, 4137, 4261, 4361, 4960, 5313, 5902, 6271, 6309, 6455};
- {0, 13, 113, 296, 416, 952, 1295, 1311, 2424, 3259, 4084, 4137, 4261, 4361, 4960, 5313, 5902, 6198, 6309, 6455};
- {0, 13, 113, 308, 366, 917, 1276, 1419, 2424, 3205, 3995, 4137, 4261, 4361, 4906, 5259, 6010, 6217, 6309, 6509};
- {0, 13, 113, 308, 366, 952, 1241, 1419, 2424, 3205, 4030, 4137, 4261, 4361, 4906, 5259, 6010, 6182, 6309, 6509};
- {0, 13, 26, 113, 1014, 1384, 1415, 1884, 1990, 2491, 2576, 2769, 3043, 3153, 4352, 4717, 4935, 5567, 5955, 6055};
- {0, 13, 26, 113, 1014, 1384, 1415, 1884, 1990, 2557, 2653, 2991, 3741, 4352, 4983, 5097, 5203, 5567, 5893, 5978};
- {0, 13, 353, 466, 870, 1371, 1661, 1911, 2768, 3080, 3193, 3583, 4173, 4203, 5037, 5395, 5657, 5929, 6131, 6175};
- {0, 13, 353, 709, 878, 1365, 1905, 2394, 2560, 3335, 3501, 3950, 3963, 4219, 4907, 5098, 5389, 5510, 5929, 5987};
- {0, 13, 53, 329, 447, 961, 1425, 2046, 2449, 2761, 3077, 3413, 3636, 4038, 5056, 5564, 5892, 6313, 6356, 6467};
- {0, 13, 53, 329, 447, 961, 1425, 2046, 2684, 2761, 3077, 3178, 3636, 4038, 5056, 5564, 5892, 6121, 6232, 6548};
- {0, 13, 53, 329, 682, 829, 1068, 2067, 2613, 3382, 3705, 3845, 4118, 4288, 4930, 5070, 5276, 5426, 5823, 6455};
- {0, 13, 53, 329, 682, 902, 1068, 2067, 2613, 3382, 3705, 3772, 4045, 4361, 4930, 5070, 5203, 5426, 5823, 6455}.

Currently we know 11 non-equivalent trifferent codes with length 9 and maximum cardinality $T(9) = 27$:

- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11230,11315,11352,14886,14926,15020,15863,15900,15985,18295,18326,18417};
- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11230,11315,11352,14886,14926,15020,15865,15896,15987,18293,18330,18415};
- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11230,11315,11352,14886,14927,15019,15862,15900,15986,18296,18325,18417};
- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11230,11315,11352,14886,14927,15019,15866,15895,15987,18292,18330,18416};
- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11230,11315,11352,14886,14930,15016,15865,15897,15986,18293,18325,18420};
- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11230,11315,11352,14887,14927,15018,15861,15901,15986,18296,18324,18418};
- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11230,11315,11352,14887,14927,15018,15866,15894,15988,18291,18331,18416};
- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11230,11315,11352,14887,14930,15015,15864,15898,15986,18293,18324,18421};
- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11230,11315,11352,14890,14930,15012,15864,15895,15989,18290,18327,18421};
- {0,121,242,3164,3282,3394,6325,6437,6555,7821,7915,7955,10256,10347,10378,11241,11281,11375,14875,14960,14997,15863,15900,15985,18295,18326,18417};
- {0,121,242,3164,3282,3394,6363,6457,6497,7783,7895,8013,10276,10307,10398,11241,11281,11375,14875,14960,14997,15863,15900,15985,18275,18366,18397}

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