Capacity of an infinite family of networks related to the diamond network for fixed alphabet sizes

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Abstract

We consider the problem of error correction in a network where the errors can occur only on a proper subset of the network edges. For a generalization of the so-called Diamond Network we consider lower and upper bounds for the network's (1-shot) capacity for fixed alphabet sizes.

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1 Introduction

The correction of errors introduced by noise or adversaries in networks have been studied in a number of papers, see e.g. [8, 17]. Here we assume that the adversary can manipulate a subset of the edges of a given network. If the adversary is unrestricted, i.e., all edges can be manipulated, then it is well known that the 1-shot capacity of the network can be achieved, see e.g. [11]. In [1, 3] specific networks are considered where the actual 1-shot capacity cannot be achieved using linear operations at intermediate nodes, so that network decoding is required in order to achieve the capacity. For a unified combinatorial treatment of adversarial network channels we refer to [15].

The aim of this paper is to study the capacity of a generalization of the diamond network, see [1, 3] for the diamond network and Figure 1 for the generalized case. This family of networks is one of five infinite families studied in [2], more precisely family B. The dashed edges $e_0, \ldots, e_{s+1}$ can be manipulated by the adversary and the solid edges $f_0, \ldots, f_s$ are those that cannot be manipulated. Node S is the source, T the terminal, and V_1, V_2 are the intermediate nodes. We assume that the adversary can manipulate exactly one dashed edge and we are interested in the 1-shot capacity, i.e., where the edges of the network are just used once.

The possible mappings from the input to the output space of the nodes are numerous. So, after stating the necessary preliminaries in Section 2, we study the structure of "good" network codes in the single-error correction setting for generalized diamond networks $\mathcal{N}_s$ in Section 3. Our main structure result, see Theorem 12, states that we may assume that node V_1 just forwards its received information and node V_2 performs a partial error-correction and sends sets of codeword candidates as states, so that the design of an optimal network code becomes a covering problem for sets. Based on these insights we determine lower and upper bounds for the 1-shot capacity of $\mathcal{N}_s$ in sections 4 and 5, respectively. We close with a brief conclusion in Section 6.

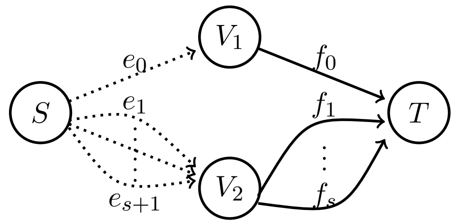

Figure 1: The network $\mathcal{N}_s$.

While the family of networks $\mathcal{N}_s$ seems to be rather specific and non-general, there are a few reasons to study this specific class in detail. It is one of three families of so-called 2-level networks whose capacity couldn't be determined in [2]. If we remove vertex V_1, then we are in the classical situation of coding along a single edge, so that this class is somehow the smallest possible example where we can see network (coding) effects. Even though $\mathcal{N}_s$ is very well structured, the exact determination of its 1-shot capacity for a given parameter s and a given alphabet size seems to be a combinatorial challenge. In [2] reductions for general networks to 3-level networks and then to 2-level networks were presented, so that it makes sense to study the smaller and more structured networks first. Nevertheless there is some hope that some of the presented techniques may generalize to different networks too.

2 Preliminaries

In this section we formally define communication networks, network codes, and the channels they induce c.f. [2, 15]. Due to a more restricted setting we use slightly different notation and refer the interested reader to [15] for more details.

Definition 1. *A* (single-source communication) network *is a 4-tuple* $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$*, where* $(\mathcal{V}, \mathcal{E})$ *is a finite, directed, acyclic multigraph* $S \in \mathcal{V}$ *is the* source*, and* $\emptyset \neq \mathbf{T} \subseteq \mathcal{V} \backslash \{S\}$ *is the set of* terminals*. Additionally we assume that there exists a directed path from* S *to any* $T \in \mathbf{T}$ *and for every* $V \in \mathcal{V} \backslash (\{S\} \cup \mathbf{T})$ *there exist directed paths from* S *to* V *and from* V *to some* $T \in \mathbf{T}$.

The elements of $\mathcal{V}$ are called *nodes*, the elements of $\mathcal{E}$ are called *edges*, and the elements of $\mathcal{V} \backslash (\{S\} \cup \mathbf{T})$ are called *intermediate nodes*. The set of incoming and outgoing edges for a node $V \in \mathcal{V}$ are denoted by $in(V)$ and $out(V)$, respectively. As alphabet we choose an arbitrary finite set $\mathcal{A}$ with at least two elements, e.g. $\mathcal{A} = \{0, 1, \ldots, |\mathcal{A}| - 1\}$.

In order to describe the transmission across the network we would actually need functions $F_V : \mathcal{A}^{in(V)} \to \mathcal{A}^{out(V)}$ for each intermediate node V, where we write $A^B := \{\varphi : B \to A\}$ for the set of all functions from B to A, For the ease of notation we associate the elements in $\varphi \in \mathcal{A}^B$, for arbitrary subsets $B \subseteq \mathcal{E}$, with the vectors in $\mathcal{A}^{|B|}$. This is made precise by fixing an ordering "$<$" of the elements of $\mathcal{E}$ and replacing φ by the vector of images $\big(\varphi(b_1), \ldots, \varphi(b_{|B|})\big) \in \mathcal{A}^{|B|}$, where $\big\{b_1, \ldots, b_{|B|}\big\} = B$ and $b_1 < \cdots < b_{|B|}$. For our network depicted in Figure 1, see Definition 6 for a formalization, we assume the order $e_0 < \cdots < e_{s+1} < f_0 < \cdots < f_s$ for the edges. E.g., if $B \subseteq \mathcal{E}$ contains e_0, then the corresponding symbol on that edge is the first coordinate of each vector $x \in \mathcal{A}^{|B|}$.

Definition 2. *Let $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$ be a network. A* network code *F for $\mathcal{N}$ is a family of functions $\{F_V : V \in \mathcal{V}(\backslash\{S\} \cup \mathbf{T})\}$, where $F_V : \mathcal{A}^{|in(V)|} \to \mathcal{A}^{|out(V)|}$ for all intermediate nodes V.*

Definition 3. *An* (outer) code *for a network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$ is a subset $C \subseteq \mathcal{A}^{|out(S)|}$ with $|C| \geq 1$.*

The elements of an outer code C are called *codewords*. For any two $c, c' \in \mathcal{A}^{|B|}$, where $B \subseteq \mathcal{E}$ is arbitrary, we define the *Hamming distance* $d(c, c')$ as the number of differing coordinates, i.e., $d(c, c') := \{1 \leq i \leq |B| : c_i \neq c'_i\}$. Note that the Hamming distance is a metric. For each subset $C \subseteq \mathcal{A}^{|out(S)|}$ we write $d(C)$ for the minimum Hamming distance $d(c, c')$ for any pair of different elements $c, c' \in C$, where we formally set $d(C) := \infty$ if $|C| \leq 1$.

In this paper we consider networks that are affected by potentially restricted adversarial noise. More precisely, we assume that at most t of the alphabet symbols on a given edge set $\mathcal{U} \subseteq \mathcal{E}$ can be changed into any other alphabet symbols.

Definition 4. *Let $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$ be a network, $\mathcal{A}$ an alphabet, $T \in \mathbf{T}$ a terminal, F a network code for $(\mathcal{N}, \mathcal{A})$, $\mathcal{U} \subseteq \mathcal{E}$ a subset of the edges, and $t \geq 0$ an integer. We denote by $\Omega[\mathcal{N}, \mathcal{A}, F, S \to T, \mathcal{U}, t] : \mathcal{A}^{|out(S)|} \to \mathcal{A}^{|in(T)|}$ the (adversarial)* channel *representing the transfer from S to terminal $T \in \mathbf{T}$, when the network code F is used by the vertices and at most t edges in $\mathcal{U}$ are manipulated. In this context, we call t the* adversarial power.

Let $\Omega[\mathcal{N}, \mathcal{A}, F, S \to T, \mathcal{U}, t]$ be an adversarial channel and $C \subseteq \mathcal{A}^{|out(S)|}$ be an outer code for $\mathcal{N}$. We say that C is *good* for Ω if $\Omega(c) \neq \Omega(c')$ for all codewords $c, c' \in C$ with $c \neq c'$. The *(1-shot) capacity* of $(\mathcal{N}, \mathcal{A}, \mathcal{U}, t)$ is the largest real number κ for which there exists an outer code $C \subseteq \mathcal{A}^{|out(S)|}$ and and a network code F for $(\mathcal{N}, \mathcal{A})$ with $\kappa = \log_{|\mathcal{A}|}(|C|)$ such that C is good for each adversarial channel $\Omega[\mathcal{N}, \mathcal{A}, F, S \to T, \mathcal{U}, t]$ for all $T \in \mathbf{T}$. The notation for this largest κ is $C_1(\mathcal{N}, \mathcal{A}, \mathcal{U}, t)$.

As an abbreviation we set

$$\overline{C} := \left\{ c' \in \mathcal{A}^{s+2} \mid \exists c \in C : d(c, c') \leq 1 \right\} \tag{1}$$

for the set of all possible manipulations of a codeword and the codewords themselves.

In the general model of Kschischang and Ravagnani, see [15], an adversary can modify t edges from a given subset $\mathcal{U}$ of all edges. Here we restrict ourselves to the situation where $\mathcal{U} = out(S)$, i.e., the set of all outgoing edges of the unique source. With this, given a network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$ and an alphabet $\mathcal{A}$ we say that a pair of an outer code $C \subseteq \mathcal{A}^{|out(S)|}$ and a network code F for $\mathcal{N}$ is t-error correcting if C is good for each adversarial channel $\Omega[\mathcal{N}, \mathcal{A}, F, S \to T, out(S), t]$ for all $T \in \mathbf{T}$. We also say that C can be t-error corrected on $\mathcal{N}$ if a network code F exists such that (C, F) is t-error correcting for $\mathcal{N}$.

Lemma 5. *If an outer code C can be t-error corrected for a network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$ and an arbitrary alphabet $\mathcal{A}$, then we have $d(C) \geq 2t + 1$.*

Proof. Assume that $d(C) \leq 2t$ and let c, c' be two codewords with $d(c, c') \leq 2t$. Then there exists a vector $x \in \mathcal{A}^{|outS|}$ with $d(c, x) \leq t$ and $d(x, c') \leq t$. So, for each of the codewords c and c' the adversary can modify the values of the edges in $out(S)$ to x, so that $\Omega(c) = \Omega(c')$ while $c \neq c'$ – contradiction. □

If $\mathcal{U} \neq out(S)$ then Lemma 5 does not need to be valid in general.

In this paper we will consider a specific parameterized family of networks:

Definition 6. *For each positive integer s the network $\mathcal{N}_s$ has one source S, one terminal T, and two intermediate nodes V_1, V_2. There is exactly one directed edge from S to V_1, one directed edge from V_1 to T, $s+1$ parallel edges from S to V_2, and s parallel edges from V_2 to T, see Figure 1.*

We remark that the notion B_s was used in in [2] for $\mathcal{N}_s$ and that $\mathcal{N}_1 = B_1$ is the original diamond network, see [3]. In $\mathcal{N}_s$ we label the edge from S to V_1 as e_0 and the $s+1$ edges from S to V_2 by $e_1, \ldots, e_{s+1}$. The unique edge from V_1 to T is denoted by f_0 and the s edges from V_2 to T are labeled by $f_1, \ldots, f_s$. C.f. Figure 1.

Example 7. *For alphabet $\mathcal{A} = \{0, 1, 2\}$ and $s = 2$ a code of size 5 for $\mathcal{N}_s$ is given by $C = \{0000, 0111, 1012, 1120, 2021\}$. We easily determine $d(C) = 3$ matching the lower bound in Lemma 5.*

Since we are interested in fixed alphabet sizes in the following we will directly consider the maximum cardinality $\sigma(\mathcal{N}_s, |\mathcal{A}|)$ of an outer code C that can be 1-error corrected for $(\mathcal{N}_s, \mathcal{A})$, i.e., we have $\log_{|\mathcal{A}|} \sigma(\mathcal{N}_s, |\mathcal{A}|) = C_1(\mathcal{N}_s, \mathcal{A}, out(S), 1)$. In Example 13 we will continue Example 7 and show that the stated code can be 1-error corrected, i.e., we have $\sigma(\mathcal{N}_s, 3) \geq 5$.

3 Structure of network codes for generalized diamond networks $\mathcal{N}_s$

The aim of this section is to obtain some structure results on network codes F that are 1-error correcting for a given outer code C on the network $\mathcal{N}_s$ as well as on outer codes C that can be 1-error corrected. Lemma 5 gives the lower bound $d(C) \geq 3$ for the minimum Hamming distance.

For each $c \in \mathcal{A}^{s+2}$ we write $c = (c_1, c_2)$, where $c_1 \in \mathcal{A}$. Given our ordering of the edges, the symbol sent along edge e_0 (before the possible modification of the adversary) is c_1. As an abbreviation we set

$$C_2 := \{c_2 \mid (c_1, c_2) \in C\}. \tag{2}$$

Since C_2 is just a truncation of C we have $d(C_2) \geq 2$ in our setting.

Definition 8. *For each $c_2 \in \mathcal{A}^{s+1}$ we define*

$$\tau(c_2) := \left\{ \left(a, c_2'\right) \in C \, : \, a \in \mathcal{A}, d(c_2, c_2') \leq 1 \right\}. \tag{3}$$

In words, if F_{V_2} receives the input c_2, then the originally sent codeword $c' \in C$ has to be contained in $\tau(c_2)$. Moreover, the adversary can ensure that F_{V_2} receives the input c_2 for all codewords $c' \in \tau(c_2)$. Clearly, we have $|\tau(c_2)| = 1$ if $c_2 \in C_2$.

Lemma 9. *Let C be a code that can be 1-error corrected for the network $\mathcal{N}_s$ and alphabet $\mathcal{A}$. For each $c_2 \in \mathcal{A}^{s+1}$ we have $|\tau(c_2)| \leq \min\{|\mathcal{A}|, s+1\}$.*

Proof. For $|\tau(c_2)| \leq 1$ the statement is true due to our assumptions $s \geq 1$ and $|\mathcal{A}| \geq 2$, so that we assume $|\tau(c_2)| \geq 2$ in the following. Since $d(C_2) \geq 2$ and $d(c_2, c_2') \leq 1$ for each codeword $(c_1', c_2') \in C$ with $c_1' \in \mathcal{A}$ we have $d(c_2, c_2') = 1$ (using $|\tau(c_2)| > 1$). I.e. c_2' differs from c_2 in exactly one of the $s+1$ coordinates. Since no two codewords can differ in the same coordinate due to $d(C_2) \geq 2$ we have $|\tau(c_2)| \leq s+1$. Moreover, $d(C) \geq 3$ implies that all codewords in $\tau(c_2)$ pairwise differ in their first coordinate, so that $|\tau(c_2)| \leq |\mathcal{A}|$. □ □

Lemma 10. *Let (C, F) be 1-error correcting for $\mathcal{N}_s$. For each different elements $c_2, c_2' \in \mathcal{A}^{s+1}$ with $F_{V_2}(c_2) = F_{V_2}(c_2')$ we have that $\tau(c_2) \cup \tau(c_2')$ does not contain two different elements with coinciding first coordinate.*

Proof. Let $c_2, c_2' \in \mathcal{A}^{s+1}$ with $F_{V_2}(c_2) = F_{V_2}(c_2')$. Assume that there exist $a \in \mathcal{A}$, $b, b' \in C_2$ with $b \neq b'$, and $(a, b), (a, b') \in \tau(c_2) \cup \tau(c_2')$. So, there exists an adversary channel Ω that changes the codeword $x := (a, b)$ to either (a, c_2) or (a, c_2') and also the codeword $x' := (a, b')$ to either (a, c_2) or (a, c_2'). Using $F_{V_2}(c_2) = F_{V_2}(c_2')$ we compute $\Omega(x) = (F_{V_1}(a), F_{V_2}(c_2)) = \Omega(x')$ while $x \neq x'$ – contradiction. □ □

Lemma 11. *Let (C, F) be 1-error correcting for $\mathcal{N}_s$. Let $c := (c_1, c_2) \in C$ be a codeword and $c' := (c_1', c_2') \in \overline{C}$ with $c_1, c_1' \in \mathcal{A}$, $c \neq c'$ and $F_{V_2}(c_2) = F_{V_2}(c_2')$. Then, we have $\tau(c_2') = \{c\}$.*

Proof. Assume that there exists a codeword $\tilde{c} := (\tilde{c}_1, \tilde{c}_2) \in \tau(c_2')$ with $\tilde{c} \neq c$ and $\tilde{c}_1 \in \mathcal{A}$. So, there exists an adversary channel Ω that changes the codeword c to $(\tilde{c}_1, c_2)$ and the codeword $\tilde{c}$ to $(\tilde{c}_1, c_2')$. Using $F_{V_2}(c_2) = F_{V_2}(c_2')$ we compute $\Omega(c) = (F_{V_1}(\tilde{c}_1), F_{V_2}(c_2)) = (F_{V_1}(\tilde{c}_1), F_{V_2}(c_2')) = \Omega(\tilde{c})$ while $c \neq \tilde{c}$ – contradiction. □ □

Based on Lemma 10 and Lemma 11 we can state a characterization of all outer codes that can be 1-error corrected for $\mathcal{N}_s$:

Theorem 12. *An outer code $\emptyset \neq C \subseteq \mathcal{A}^s$ can be 1-error corrected for $\mathcal{N}_s$ iff there exists a set $\mathcal{B}$ consisting of subsets of C with pairwise different first coordinates of cardinality $|\mathcal{B}| \leq |\mathcal{A}|^s - |C|$ such that for each $c_2 \in \mathcal{A}^{s+1}$ with $|\tau(c_2)| > 1$ there exists an element $B \in \mathcal{B}$ with $\tau(c_2) \subseteq B$.*

Proof. Let (C, F) be 1-error correcting for $\mathcal{N}_s$ and define

$$S_b = \cup_{c_2 \in \mathcal{A}^{s+1} : F_{v_2}(c_2) = b} \tau(c_2) \subseteq C \tag{4}$$

for all $b \in \mathcal{A}^s$. From Lemma 10 we conclude that the elements in S_b have pairwise different first coordinates for each $b \in \mathcal{A}^s$. For

$$\mathcal{B} := \{S_b \,:\, b \in \mathcal{A}^s, |S_b| \geq 2\} \tag{5}$$

we have that for each $c_2 \in \mathcal{A}^{s+1}$ with $|\tau(c_2)| > 1$ there exists an element $B \in \mathcal{B}$ with $\tau(c_2) \subseteq B$. For each codeword $c = (c_1, c_2)$ with $c_1 \in \mathcal{A}$ we have $S_{F_{V_2}(c_2)} = \{c\}$ due to Lemma 11. So using Lemma 10, at least $|C|$ of the sets S_b have cardinality 1 and $|\mathcal{B}| \leq |\mathcal{A}|^s - |C|$.

For the other direction let a set $\mathcal{B}$ satisfying the stated properties be given. We set $\mathcal{B}' := \big\{\{c\} : c \in C\big\}$ and $\mathcal{B}'' := \mathcal{B} \cup \mathcal{B}'$, so that $|\mathcal{B}''| \leq |\mathcal{A}|^s$. Let $\pi \colon \mathcal{B}'' \to \mathcal{A}^s$ be an arbitrary injection. With this we define the network code F for $\mathcal{N}_s$ as follows. We choose F_{V_1} as the identity mapping, i.e., $F_{V_1}(x) = x$ for all $x \in \mathcal{A}$. For each $c_2 \in \mathcal{A}^{s+1}$ with $|\tau(c_2)| > 1$ we set $F_{V_2}(c_2)$ to the lexicographically minimal element in $\{\pi(B) \,:\, B \in \mathcal{B}, \tau(c_2) \subseteq B\}$. If $|\tau(c_2)| = 1$ we set $F_{V_2}(c_2)$ to $\pi(B)$ for the unique element $B \in \tau(c_2) \subseteq \mathcal{B}'$ and to $\pi(B)$ for an arbitrary element $B \in \mathcal{B}'$ if $\tau(c_2) = \emptyset$. With this F is well defined. In order to show that (C, F) is 1-error correcting for $\mathcal{N}_s$ we construct a "decoder" $F_T \colon \mathcal{A}^{|in(T)|} \to \mathcal{A}^{|out(S)|}$ with $F_T(\Omega(c)) = c$ for every codeword $c \in C$ and every adversary channel Ω. For a given $y := (y_1, y_2) \in \mathcal{A}^{s+1}$ with $y_1 \in \mathcal{A}$ let $B \in \mathcal{B}''$ be the unique element with $\pi(B) = y_2$ or $B = \emptyset$ if no such element B exists. If $|B| = 0$, then we set $F_T(y)$ to an arbitrary element in $\mathcal{A}^{s+2}$ since this case cannot occur if a codeword is sent across the network. If $|B| = 1$ we set $F_T(y)$ to the unique codeword contained in B. If $|B| \geq 2$, then the adversary has not modified the value on edge e_0, so that we can set $F_T(y)$ to the unique codeword in B whose first coordinate equals $y_1 = F_{V_1}(y_1)$. □ □

Example 13. *As in Example 7 we consider the network $\mathcal{N}_2$, the alphabet $\mathcal{A} = \{0,1,2\}$, and the outer code C given by the codewords $c_1 = (0,0,0,0)$, $c_2 = (0,1,1,1)$, $c_3 = (1,0,1,2)$, $c_4 = (1,1,2,0)$, and $c_5 = (2,0,2,1)$. It can be easily checked that the different values of $\tau(c_2)$ for all $c_2 \in \mathcal{A}^{s+2}$ are given by*

$$\Big\{\emptyset, \{c_1\}, \{c_2\}, \{c_3\}, \{c_4\}, \{c_5\}, \{c_2, c_3, c_5\}, \{c_1, c_3\}, \{c_1, c_4, c_5\}, \{c_2, c_4, c_5\}\Big\}.$$

For the set $\mathcal{B}$ in Theorem 12 we may choose

$$\mathcal{B} = \big\{\{c_2, c_3, c_5\}, \{c_1, c_3\}, \{c_1, c_4, c_5\}, \{c_2, c_4, c_5\}\big\}.$$

Since the elements of $\mathcal{B}$ have cardinality at most $|\mathcal{A}| = 3$ we cannot join two elements of $\mathcal{B}$ and all $3^2 = 9$ different output vectors of the intermediate node V_2 have to be used. As injection π we may choose e.g. $\pi(\{c_1\}) = (0,0)$, $\pi(\{c_2\}) = (0,1)$, $\pi(\{c_3\}) = (0,2)$, $\pi(\{c_4\}) = (1,0)$, $\pi(\{c_5\}) = (1,1)$, $\pi(\{c_2, c_3, c_5\}) = (1,2)$, $\pi(\{c_1, c_3\}) = (2,0)$, $\pi(\{c_1, c_4, c_5\}) = (2,1)$, and $\pi(\{c_2, c_4, c_5\}) = (2,2)$. With this we e.g. have $F_{V_2}\big((0,1,2)\big) = (0,2)$ and $F_{V_2}\big((0,1,1)\big) = (1,2)$.

Example 14. *Again we consider the network $\mathcal{N}_2$ and the alphabet $\mathcal{A} = \{0,1,2\}$. Now let the outer code C be given by the codewords $c_1 = (0,0,0,0)$, $c_2 = (0,1,1,1)$, $c_3 = (0,2,2,2)$, $c_4 = (1,0,1,2)$, and $c_5 = (2,0,2,1)$. Here the different values of $\tau(c_2)$ for all $c_2 \in \mathcal{A}^{s+2}$ are given by*

$$\Big\{\emptyset, \{c_1\}, \{c_2\}, \{c_3\}, \{c_4\}, \{c_5\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_4, c_5\}, \{c_3, c_4, c_5\}\Big\}.$$

The sets $\{c_1, c_4\}$ and $\{c_1, c_5\}$ can be joined to $\{c_1, c_4, c_5\}$, so that we set $\mathcal{B} = \big\{\{c_1, c_4, c_5\}, \{c_2, c_4, c_5\}, \{c_3, c_4, c_5\}\big\}$ in Theorem 12 and only $|C| + |\mathcal{B}| = 8 < 3^2$ different output states are used in the corresponding function F_{v_2}.

As a byproduct of the proof of Theorem 12 we see that it does not make a difference in terms of the capacity if the adversary can manipulate edge e_0 or e_0 and f_0. We may even remove the intermediate vertex V_1 and replace the two edges e_0 and f_0 by a direct edge between S and T that can be manipulated by the adversary. Going over the proofs of lemmas 5-11 we see that the number of outgoing edges of V_2 does not play a role. So, denoting the network where we replace the s outgoing edges of V_2 to T by s' such edges by $\mathcal{N}_{s,s'}$ we can prove a variant of Theorem 18 where we replace the upper bound on the cardinality by $|\mathcal{B}| \le |\mathcal{A}|^{s'} - |C|$. C.f. the code of Example 14 in the situation where we replace the two outgoing edges of V_2 in $\mathcal{N}_2$ by three outgoing edges, which however can only transmit symbols from the restricted alphabet $\{0,1\}$. i.e., a rather specific case of the more general situation where the allowed alphabets may differ on the different edges.

In other words, the conditions of Theorem 12 say that the elements of $\mathcal{B}$ cover all $\tau(c_2)$ with $c_2 \in \mathcal{A}^{s+1}$ and $|\tau(c_2)| > 1$. The underlying cover problem can be solved by a straight forward integer linear programming (ILP) formulation, as it is usually the case for cover problems. This yields a first algorithm to determine the largest possible size $\sigma(\mathcal{N}_s, |\mathcal{A}|)$ of a code that can be 1-error corrected for $\mathcal{N}_s$ and alphabet $\mathcal{A}$ for any parameter $s \in \mathbb{N}_{\ge 1}$. Due to Lemma 5 we just have to loop over all codes with minimum Hamming distance at least 3 and use Theorem 12 to check whether C can be 1-error corrected for $\mathcal{N}_s$ via an ILP computation.

We have applied this approach to the list of classified optimal[1] binary one-error-correcting codes. These are denoted as $(n, |C|, 3)$ codes, where the length is given by $n = s + 2$ in our situation and

[1]Here by an optimal code we understand a code of the maximum possible size given block length, alphabet size, and minimum distance.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$\max \lvert C\rvert$	2	2	4	8	16	20	40	72	144	256	512	1024	2048
number	1	2	1	1	1	5	1	562	7398	237610	117832	38408	5983

Table 1: Number of non-isomorphic optimal binary $(n, |C|, 3)$ codes.

$|C|$ is the maximum possible code size. For lengths $4, \ldots, 11$ the codes were classified in [13], for lengths $12, 13$ in [12], and for lengths $14, 15$ in [14]. The corresponding numbers are summarized in Table 1 and the codes can be downloaded at `http://pottonen.kapsi.fi/codes`. Since those enumerations only consider one-error correcting codes up to symmetry we have to loop over the coordinates and decide which one should be the one corresponding to e_0. Interestingly enough, for all $n \in \{3, \ldots, 15\} \backslash \{3, 7, 15\}$ all choices for e_0 and all choices for C lead to a code that can be 1-error corrected for $\mathcal{N}_{n-2}$. Note that for $n \in \{3, 7, 15\}$ we deal with the parameters of the Hamming codes. Here no choice of C or the coordinate of e_0 leads to a code that can be 1-error corrected for $\mathcal{N}_{n-2}$. Thus, we have $\sigma(\mathcal{N}_1, 2) \le 1$, $\sigma(\mathcal{N}_2, 2) = 2$, $\sigma(\mathcal{N}_3, 2) = 4$, $\sigma(\mathcal{N}_4, 2) = 8$, $\sigma(\mathcal{N}_5, 2) \le 15$, $\sigma(\mathcal{N}_6, 2) = 20$, $\sigma(\mathcal{N}_7, 2) = 40$, $\sigma(\mathcal{N}_8, 2) = 72$, $\sigma(\mathcal{N}_9, 2) = 144$, $\sigma(\mathcal{N}_{10}, 2) = 256$, $\sigma(\mathcal{N}_{11}, 2) = 512$, $\sigma(\mathcal{N}_{12}, 2) = 1024$, and $\sigma(\mathcal{N}_{13}, 2) \le 2047$.

In the next section we will determine $\sigma(\mathcal{N}_1, 2) = 1$, $\sigma(\mathcal{N}_5, 2) = 14$, and $\sigma(\mathcal{N}_s, a)$ for other small parameters s and $a \ge 3$. We remark that $\sigma(\mathcal{N}_1, a) = a - 1$ was determined in [1, 3].

4 Exact values and lower bounds for the maximum code sizes

As mentioned in the previous section, $\sigma(\mathcal{N}_1, a) = a - 1$ was determined in [1, 3]. Here an element $\tilde{a} \in \mathcal{A}$ is chosen as s special symbol. With this a suitable code C is a threefold repetition code of the symbols in $\mathcal{A} \backslash \{\tilde{a}\}$. If $|\tau(c_2)| > 1$, then V_2 outputs $\tilde{a}$ with the corresponding set $B = C$. Here we will determine a few exact values and lower bounds for $\sigma(\mathcal{N}_s, a)$ for small parameters. Upper bounds are discussed in Section 5.

One of the most simple algorithmic approaches is to loop over all $|\mathcal{A}|$-ary codes of length $s + 2$ and minimum Hamming distance at least three with increasing sizes. For each candidate code C we can use Theorem 12 and a small ILP computation to decide whether C can be 1-error corrected. For alphabet size $|\mathcal{A}| = 3$ and $s = 2$ two codes that can be 1-error corrected and have cardinality 5 are given in Example 13 and Example 14. By exhaustive search we have verified that no 3-ary code with length 4, minimum Hamming distance at least 3, and size 6 can be 1-error corrected, so that $\sigma(\mathcal{N}_2, 3) = 5$.

In the following we will only state the codes C since a suitable collection of sets $\mathcal{B}$ can be easily computed using Theorem 12 and a small ILP computation. For $|\mathcal{A}| = 3$ and $s = 3$ a code of size 14 is given by

$$\begin{aligned} C = \{ & 00000, 00111, 00222, 01012, 01120, 02201, 10021, \\ & 11102, 11211, 12010, 20210, 21001, 22022, 22100\}. \end{aligned}$$

For $|\mathcal{A}| = 3$ and $s = 4$ a code of size 35 is given by

$$\begin{aligned} C \;=\; \{&000000, 212000, 102021, 122210, 012122, 120121, 202201, 211220, 110211,\\ &121022, 001120, 202110, 002212, 020012, 021102, 011001, 111100, 200122,\\ &200011, 121201, 022111, 102102, 010110, 211112, 000221, 101111, 220200,\\ &110020, 201002, 010202, 210101, 022020, 221010, 112012, 222222\}. \end{aligned}$$

By exhaustive search we have verified $\sigma(\mathcal{N}_3, 3) = 14$. For $|\mathcal{A}| = 4$ and $s = 2$ a code of size 9 is given by

$$C \;=\; \{0000, 0111, 0222, 1012, 1103, 2021, 2130, 3201, 3310\}.$$

For $|\mathcal{A}| = 4$ and $s = 3$ a code of size 31 is given by

$$\begin{aligned} C \;=\; \{&00000, 32222, 21202, 31010, 31133, 22311, 20323, 01320, 03131, 30120,\\ &11100, 00232, 12021, 23233, 11331, 32101, 22132, 13201, 30302, 20210,\\ &13113, 23020, 02303, 33330, 01213, 10033, 02012, 21121, 33003, 12230,\\ &13322\}. \end{aligned}$$

Example 15. *For $|\mathcal{A}| = 5$ and $s = 2$ a code of size 15 is given by the codewords $c_0 = 0000$, $c_1 = 4341$, $c_2 = 2410$, $c_3 = 2143$, $c_4 = 3211$, $c_5 = 1421$, $c_6 = 2232$, $c_7 = 4104$, $c_8 = 3024$, $c_9 = 1244$, $c_{10} = 1012$, $c_{11} = 3302$, $c_{12} = 0433$, $c_{13} = 4220$, and $c_{14} = 1330$. A set $\mathcal{B}$ for the construction of a suitable network code is given by*

$$\begin{aligned} \Big\{&\{c_0, c_2, c_7, c_{10}, c_{11}\}, \{c_0, c_3, c_7, c_8, c_9\}, \{c_0, c_6, c_{11}, c_{13}, c_{14}\}, \{c_0, c_8, c_{10}, c_{13}\},\\ &\{c_1, c_2, c_{11}, c_{12}, c_{14}\}, \{c_1, c_3, c_4, c_9, c_{12}\}, \{c_1, c_5, c_6, c_8, c_{12}\},\\ &\{c_2, c_4, c_5, c_{13}\}, \{c_4, c_6, c_{10}, c_{12}\}, \{c_6, c_8, c_9, c_{12}, c_{13}\}\Big\}. \end{aligned}$$

For $|\mathcal{A}| = 2$ we have that $|\tau(c_2)| > 1$ is equivalent to $|\tau(c_2)| = 2$ and the two codewords in $\tau(c_2)$ are at Hamming distance 2. Moreover, for each pair of codewords $a, b \in C_2$ with $d(a, b) = 2$ there are exactly two vectors $x \in \mathcal{A}^{s+1}$ with $d(a, x) = d(x, b) = 1$. This characterization allows to determine $\sigma(\mathcal{N}_s, 2)$ via the following ILP formulation. For each $v \in \{0,1\}^{s+2}$ we introduce binary variables x_v and binary variables $y_{v'}$ for each $v' \in \{0,1\}^{s+1}$. The meaning of the x_v's is given by $x_v = 1$ iff $v \in C$. The meaning of the $y_{v'}$'s is given by $y_{v'} = 1$ iff $|\tau(v')| > 1$. We minimize the code size $|C| = \sum_{v\in\{0,1\}^{s+2}} x_v$ subject to the constraints

$$\sum_{v\in\{0,1\}^{s+2}} x_v + \frac{1}{2} \cdot \sum_{v'\in\{0,1\}^{s+1}} y_{v'} \le 2^s \tag{6}$$

modeling the condition from Theorem 12,

$$y_m \ge \sum_{i\in\mathcal{A}} x_{(i,a)} + \sum_{j\in\mathcal{A}} x_{(j,b)} - 1 \tag{7}$$

for all $a, b \in \{0,1\}^{s+1}$ with $d(a,b) = 2$ and all $m \in \{0,1\}^{s+1}$ with $d(a,m) = d(m,b) = 1$, which ensures that the pair of codewords (i,a) and (j,b) can only be taken if $y_m = 1$, and

$$\sum_{v \in \{0,1\}^{s+2} \,:\, v|_S = u} x_v \le 1 \tag{8}$$

for all $u \in \{0,1\}^s$ and all $S \subseteq \{1,\ldots,s+2\}$ with $|S| = s$, which models $d(C) \ge 3$, where $v|_S$ is the restriction of the vector v to the coordinates in S.

For $s = 5$ a code of size 14 is given by

$$\begin{aligned} C \;=\; & \{0001101, 0011110, 0100100, 0101011, 0110111, 0111000, 1000111, \\ & 1001000, 1010100, 1011011, 1100001, 1101110, 1110010, 1111101\} \end{aligned}$$

and using the above ILP we could verify that there is no such code of size 15, so that $\sigma(\mathcal{N}_5, 2) = 14$. We remark that the bound $|C| + |\mathcal{B}| = |\mathcal{A}|^s$ is met with equality in our example.

5 Upper bounds for the maximum code sizes

In [2] the two bounds $\sigma(\mathcal{N}_s, a) \le a^s$ and $\sigma(\mathcal{N}_s, a) \ge a^{s-1}$, if a is sufficiently large, were shown. Both have an easy explanation. For the lower bound we may restrict to network codes where V_1 just forwards and the terminal T ignores the output from V_1, i.e., the entire decoding is actually performed in intermediate node V_2. Let $A_q(n,d)$ denote the maximum number of vectors in a q-ary code of word length n and with Hamming distance d, so that $\sigma(\mathcal{N}_s, a) \ge A_a(s+1,3)$. Since for a fixed length and minimum distance MDS codes exist for all sufficiently large alphabet sizes a this implies $\sigma(\mathcal{N}_s, a) \ge a^{s-1}$. For other known lower bounds for $A_q(n,d)$ we refer to e.g. [4, 5, 7] and especially the webpage `https://www.win.tue.nl/~aeb/`. For upper bounds we can use Lemma 5 to conclude $\sigma(\mathcal{N}_s, a) \le A_a(s+2,3)$, so that the Singleton bound gives $\sigma(\mathcal{N}_s, a) \le a^s$ for all parameters s and a. In Table 2 we summarize some known upper bounds for $A_a(s+2,3)$ from the mentioned website.

s	2	3	4	5	6	2	3	4	5	2	3	4	5
a	3	3	3	3	3	4	4	4	4	5	5	5	5
$\le$	9	18	38	111	333	16	64	176	596	25	125	625	2291

Table 2: Upper bounds for $A_a(s+2,3)$.

In [2, Remark 5.10] the "conjecture" $\sigma(\mathcal{N}_s, a) = \frac{a^s+a}{2} - 1$ was mentioned. For $(\mathcal{N}, a) \in \big\{(\mathcal{N}_1, \star), (\mathcal{N}_2, 2), (\mathcal{N}_3, 2), (\mathcal{N}_4, 2), (\mathcal{N}_2, 3), (\mathcal{N}_3, 3), (\mathcal{N}_2, 4)\big\}$ code sizes $\frac{a^s+a}{2} - 1$ can indeed be attained. However, in [16][2] the upper bound

$$A_a(n,3) \le \frac{a^n \cdot (hn - x(a-x))}{(hn+x) \cdot (hn + x - a)},$$

where $h = a-1$, $n \equiv x \pmod{a}$, and $1 \le x \le a$, was concluded for $a > 2$ and sufficiently large n from Delsarte's linear programming method. Thus, we have $A_a(n,3) < \frac{a^n}{(a-2)n}$ for $a > 2$ and sufficiently

[2]C.f. `https://mathscinet.ams.org/mathscinet-getitem?mr=484738`

large n, i.e., if the conjecture is true, then a has to be sufficiently large for a given parameter s. For $s = 4$ and alphabet size 3 we have $35 \le \sigma(\mathcal{N}_4, 3) \le A_3(6, 3) = 38 < 41$ and also $\sigma(\mathcal{N}_s, 2) < \frac{2^s+2}{2} - 1$ for all $s \ge 5$. We remark that the code of Example 15 shows $\sigma(\mathcal{N}_2, 5) \ge 15 > \frac{5^2+5}{2} - 1$.

While Lemma 5 implies $d(C_2) \ge 2$ only, we can use the number of ordered pairs of codewords of C_2 at distance 2

$$\Lambda := \left|\left\{(a, b) \in C_2^2 \;:\; d(a, b) = 2\right\}\right| \tag{9}$$

to lower bound $|\mathcal{B}|$ in Theorem 12 and to derive the following necessary criterion:

Lemma 16. *Let $C \subseteq \mathcal{A}^{s+2}$ be a code that can be 1-error corrected for $\mathcal{N}_s$. Then, we have*

$$|C| + \frac{\Lambda}{|\mathcal{A}|(|\mathcal{A}| - 1)} \le |\mathcal{A}|^s.$$

Proof. Let $\mathcal{B}$ as in Theorem 12. If $a, b \in C_2$ are two codewords with $d(a, b) = 2$, then there exists a vector $x \in \mathcal{A}^{s+1}$ with $d(a, x) = d(x, b) = 1$, so that $a, b \in \tau(x)$. Thus, there exists a set $B \in \mathcal{B}$ with $\tau(x) \subseteq B$. Since $|B| \le |\mathcal{A}|$ at most $|\mathcal{A}|(|\mathcal{A}| - 1)$ ordered pairs counted in Λ can yield the same set $B \in \mathcal{B}$, so that $|\mathcal{B}| \ge \frac{\Lambda}{|\mathcal{A}|(|\mathcal{A}|-1)}$ and the stated inequality follows from Theorem 12. □ □

A first, rather weak, lower bound for Λ can be obtained easily. For network $\mathcal{N}_s$ the Singleton bound implies $|C_2| \le |\mathcal{A}|^{s-1}$ if $d(C_2) \ge 3$, so that C_2 contains at least one pair of codewords at Hamming distance 2. Iteratively removing a codeword in such a pair, we can conclude $\Lambda \ge |C| - |\mathcal{A}|^{s-1}$ for $|C| \ge |C| - |\mathcal{A}|^{s-1}$. In Subsection 5.1 we will give a refined analysis for the special case $s = 2$ and in Subsection 5.2 we will utilize the linear programming method for codes to obtain lower bounds for Λ and state the corresponding upper bounds for $\sigma(\mathcal{N}_s, a)$ for small parameters.

5.1 Upper bounds for the code sizes for $s = 2$

For $s = 2$ the parameter Λ can be lower bounded easily:

Lemma 17. *For a code $C_2 \subseteq \mathcal{A}^3$ with $d(C_2) \ge 2$ we have $\Lambda \ge |C_2| \cdot (|C_2| - |\mathcal{A}|)$.*

Proof. For each $c \in C_2$ there can be at most $|\mathcal{A}| - 1$ codewords $c' \in C_2$ with $d(c, c') = 3$, so that at least $|C_2| - |\mathcal{A}|$ codewords are at distance 2 to c. □ □

Proposition 18. *Let $C \subseteq \mathcal{A}^4$ be a code that can be 1-error corrected for $\mathcal{N}_2$. Then, we have*

$$|C| \le \left\lfloor -\frac{a(a-2)}{2} + \frac{a}{2} \cdot \sqrt{5a^2 - 8a + 4} \right\rfloor, \tag{10}$$

where $a = |\mathcal{A}|$.

Proof. From Lemma 16 and Lemma 17 we conclude $|C| + \frac{(|C|-a)|C|}{a(a-1)} \le a^2$, which can be easily solved for $|C|$. □ □

For $a := |\mathcal{A}| \ge 3$ we have

$$-\frac{a(a-2)}{2} + \frac{a}{2} \cdot \sqrt{5a^2 - 8a + 4} \le -\frac{a(a-2)}{2} + \frac{a}{2} \cdot \sqrt{5} \cdot \left(a - \frac{3}{5}\right),$$

which asymptotically equals $\frac{\sqrt{5}-1}{2} \cdot a^2$, i.e., we have $\sigma(\mathcal{N}_2, a) < 0.6181a^2$ for large a. For e.g. $a = 7$ we obtain $|C| \le \lfloor 32.587922696 \rfloor = 32$.

5.2 Upper bounds for the code sizes for $\mathcal{N}_s$ based on the linear programming method

We can also use Delsarte's linear programming method to derive lower bounds for Λ and then apply Lemma 16 to upper bound the code size $|C|$. To this end we mention that for integers $n \geq 1$ and $q \geq 2$ the *Krawtchouk polynomials* are defined as

$$K_i^{(n,q)}(z) := \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} q^j \binom{n-j}{n-i} \binom{z}{j} \tag{11}$$

for all $i \geq 0$, where $\binom{z}{j} := z(z-1)\cdots(z-j+1)/j!$ for all $z \in \mathbb{R}$. The vector $B(C) = (B_0, B_1, \ldots, B_n)$, where

$$B_i = \frac{1}{|C|} \cdot \left| \left\{ (a,b) \in C^2 \mid d(a,b) = i \right\} \right|, \; i = 0, 1, \ldots, n \tag{12}$$

is called the *distance distribution* of C. Clearly, $B_0 = 1$ and $B_i = 0$ for all $1 \leq i \leq d(C) - 1$. Moreover, $\Lambda = |C| \cdot B_2$. The vector $B'(C) = (B'_0, B'_1, \ldots, B'_n)$, where

$$B'_i = \frac{1}{|C|} \sum_{j=0}^{n} B_j K_i^{(n,q)}(j), \; i = 0, 1, \ldots, n \tag{13}$$

is called the *dual distance distribution* of C. Obviously we have $B'_0 = 1$.

Theorem 19. *([9, 10]) The dual distance distribution of C satisfies $B'_i \geq 0$ for all $0 \leq i \leq n$.*

In Table 3 we have listed a few explicit upper bounds for $\sigma(\mathcal{N}_s, a)$, where $2 \leq s \leq 5$ and $3 \leq a \leq 5$, based on Lemma 16 and the linear programming method (using $B_1 = 0$ and minimizing B_2). For more details we refer to the recent survey [6].

s	2	3	4	5	2	3	4	5	2	3	4	5
a	3	3	3	3	4	4	4	4	5	5	5	5
$\leq$	6	15	42	108	10	37	133	484	16	76	337	1512

Table 3: Upper bounds for $\sigma(\mathcal{N}_s, a)$ based on the linear programming method.

We remark that the upper bounds in Table 3 seem to be better than those in Tables 2 if s is rather small. E.g. the approach from this subsection implies $\sigma(\mathcal{N}_4, 3) \leq 42$, while we have $\sigma(\mathcal{N}_4, 3) \leq A_3(6,3) = 38$. On the other hand we have $\sigma(\mathcal{N}_3, 3) \leq A_3(5,3) = 18$ while Table 3 states $\sigma(\mathcal{N}_3, 3) \leq 15$ noting that we have determined $\sigma(\mathcal{N}_3, 3) = 14$ by exhaustive enumeration in Section 4. We give a summary of the best known bounds for small parameters in the subsequent conclusion.

6 Conclusion

In this paper we have considered the problem of determining the maximum size $\sigma(\mathcal{N}_s, a)$ of a code that can be 1-error corrected for the network $\mathcal{N}_s$ and a given alphabet of size a. The considered family of networks is one of five infinite families of so-called 2-level networks studied in [2], more precisely family B. Bounds for such 2-level networks can be used to conclude upper bounds for more general networks as demonstrated in [2]. Besides that they form a starting point for first exact computations there is some hope that the obtained insights might be generalized. In this context we mention that determining the maximum size of a code that can be t-error corrected for the most simple "network" consisting just of a source S, a terminal T, and n parallel edges between S and T over an alphabet of size q corresponds to the determination of $A_q(n, 2t+1)$, i.e., the maximum number of vectors in a q-ary code of word length n and with Hamming distance $2t+1$, which triggered a lot of research and turned out to be a quite intricate problem that does not seem to admit an easy closed-form solution.

Our main result is a characterization result for outer codes that can be 1-error corrected for network $\mathcal{N}_s$ in Theorem 12. On the computational side the problem of deciding whether a given code C can be 1-error corrected (and the construction of an achieving network code) is reduced to a covering problem. Via exhaustive enumeration of codes we have determined a few exact values of $\sigma(\mathcal{N}_s, a)$ in Section 3 and Section 4. For the special case of a binary alphabet we also stated a direct integer linear programming formulation for $\sigma(\mathcal{N}_s, 2)$ which was used to compute $\sigma(\mathcal{N}_5, 2) = 14$. With respect to upper bounds for the code size $|C|$ the easy observation $d(C) \geq 3$ for the minimum Hamming distance yields $\sigma(\mathcal{N}_s, a) \leq A_a(s+2, 3)$, so that known results from the literature can be applied. We e.g. remark that the "conjecture" $\sigma(\mathcal{N}_s, a) = \frac{a^s+a}{2} - 1$ from [2, Remark 5.10] is wrong if one does not assume that the alphabet size a is sufficiently large. Also based on the characterization result in Theorem 12 we have utilized the number of codewords of Hamming distance 2 to upper bound the maximum possible code size, see Lemma 16 in Section 5. This approach was pursued in Subsection 5.1 for the special case $s = 2$ and yields e.g. $\sigma(\mathcal{N}_2, a) < 0.6181a^2$ for sufficiently large a, which improves upon the previously best known upper bound $\sigma(\mathcal{N}_2, a) < a^2$. For $s > 2$ an application of the linear programming method for codes yields numerical improvements on $\sigma(\mathcal{N}_s, a) < a^s$, see Subsection 5.2. We summarize our numerical results for $\sigma(\mathcal{N}_s, a)$ and small parameters in Table 4.

s	1	2	3	4	5	6	7	8	9	10	11	12
a	2	2	2	2	2	2	2	2	2	2	2	2
$\sigma(\mathcal{N}_s, a)$	1	2	4	8	14	20	40	72	144	256	512	1024

s	1	2	3	4	1	2	3	1	2
a	3	3	3	3	4	4	4	5	5
$\sigma(\mathcal{N}_s, a)$	2	5	14	35–38	3	9–10	31–37	4	15–16

Table 4: Best known bounds for $\sigma(\mathcal{N}_s, a)$ and small parameters.

We propose the determination of tighter bounds for $\sigma(\mathcal{N}_s, a)$ as an interesting open problem. In our opinion also the two infinite families A and E from [2] deserve a tailored study for fixed alphabet sizes. For family E one can use a subset of a repetition code and a suitable network code to achieve code sizes that grow linearly in the alphabet size. For network A_t, where the adversary can manipulate t of the outgoing edges of the source S, a similar result as Theorem 12 can be shown. Without going into the details we just state that one can assume that intermediate vertex

V_1 forwards the information from its t incoming edges and for intermediate vertex V_2 we can define

$$\begin{aligned}\tau(c_2) \;&:=\; \Big(\left\{c' \in C \mid \exists c_1 \in \mathcal{A}^t; d(c', (c_1, c_2)) = 1\right\}, \dots, \\ &\qquad \left\{c' \in C \mid \exists c_1 \in \mathcal{A}^t; d(c', (c_1, c_2)) = t\right\} \Big)\end{aligned}$$

for all $c_2 \in \mathcal{A}^{2t}$. The "analog" of $\mathcal{B}$ would be a set whose a lists of length t of subsets of the outer code C. The interpretation is that those lists contain candidates for the originally sent codeword assuming that a certain number of incoming edges of V_2 was attacked. Together with the information from V_1 a unique decoding should be possible. Of course everything gets more technical and we currently do not know for which classes of 2-level networks a similar approach might work or what a good generalization would be.

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