The Tensor Product Multilevel Method for High-dimensional Meshfree Approximation

Von der Universität Bayreuth
zur Erlangung des Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
genehmigte Abhandlung

von

Rüdiger Kempf
aus Bayreuth

1. Gutachter: Prof. Dr. Holger Wendland
2. Gutachter: Prof. Dr. Peter Giesl

Tag der Einreichung: 05.10.2022
Tag des Kolloquiums: 02.02.2023

Abstract

High-dimensional approximation problems appear naturally in many applications and they all suffer from the curse of dimensionality. The Smolyak algorithm gives a deterministic and easily analyzable way to alleviate that curse. Its construction allows the use of well-known low-dimensional approximation operators and combines them in a predefined way to obtain a reconstruction of an unknown, high-dimensional target function.

In the context of mesh-free approximation, rescaled kernel-based methods are proven to have desirable properties, e.g., they are fast and stable as long as the set of sites satisfies minimal requirements. However, they do not converge if the scaling parameter is coupled linearly to the fill distance of the set of sites. A way to circumvent this trade-off principle is to thin out the point set in a controlled way and solve a stationary approximation problem on every level. This gives rise to the kernel-based multilevel method.

This thesis combines the Smolyak algorithm with the ideas of kernel-based multilevel approach to obtain the tensor product multilevel method. In contrast to reconstruction approaches built upon polynomials or splines, this is a new approximation method for moderately high-dimensional target functions that is capable of combining arbitrary low-dimensional domains. This new method is introduced for different settings, its convergence is analyzed and numerical examples are given to support the theoretical results.

Zusammenfassung

Hochdimensionale Approximationsprobleme treten in vielen Anwendungen auf und sie leiden alle unter dem Fluch der Dimensionalität. Der Smolyak-Algorithmus bietet eine deterministische und leicht zu analysierende Möglichkeit, diesen Fluch zu mildern. Seine Konstruktion ermöglicht die Verwendung bekannter niedrigdimensionaler Approximationsoperatoren und kombiniert diese in einer vordefinierten Weise, um eine Rekonstruktion einer unbekannten, hochdimensionalen Zielfunktion zu erhalten.

Im Zusammenhang mit der netzfreien Approximation haben reskalierte kernelbasierte Verfahren nachweislich wünschenswerte Eigenschaften, z. B. sind sie schnell und stabil, solange die Menge der Stützstellen minimale Anforderungen erfüllt. Sie konvergieren jedoch nicht, wenn der Skalierungsparameter linear an die Fülldichte der Menge der Stützstellen gekoppelt ist. Eine Möglichkeit zur Umgehung dieses *trade-off principle*s besteht darin, die Punktmenge kontrolliert auszudünnen und auf jedem Level ein stationäres Approximationsproblem zu lösen. Daraus ergibt sich die kernbasierte Multilevel-Methode.

In dieser Arbeit wird der Smolyak-Algorithmus mit den Ideen der kernbasierten Multilevel-Methode kombiniert, um die Tensorprodukt-Multilevel-Methode zu erhalten. Im Gegensatz zu Rekonstruktionsansätzen, die auf Polynomen oder Splines aufbauen, handelt es sich hierbei um eine neue Approximationsmethode, die in der Lage ist, höherdimensionale Zielfunktionen auf Gebieten zu rekonstruieren, die Kombinationen von beliebigen niedrigdimensionalen Gebieten sind. Diese neue Methode wird für verschiedene Situationen vorgestellt, ihre Konvergenz wird analysiert und numerische Beispiele zur Unterstützung der theoretischen Ergebnisse gegeben.

Contents

CHAPTER 1

Introduction

The numerical simulation of complex systems is gaining more and more importance. One of the main reasons for this is the continuous improvement of numerical tools, which make it possible to simulate large and thus practically relevant processes. Nowadays, numerical simulations are not only used in classical technical or engineering applications, but also in social sciences or finance. Furthermore, the field of life sciences also offers manifold applications of computer-aided simulations.

The first fundamental step towards a simulation is mathematical modeling. Here, it is important to capture the underlying processes in formulas. Usually, the result is a system of partial differential equations. One example is the subsurface flows in porous media, such as rocks, which is modeled as an elliptic boundary value problem, see, e.g., [19, 22, 56]. A good model to describe this subsurface flow is Darcy's law, c.f., [17]. The key ingredient is the hydraulic conductivity a that enters the system of equations as a coefficient function. This parameter measures the transmissivity of a fluid through an aquifer. It depends on the permeability of the heterogeneous media and on the dynamic viscosity of the fluid. Darcy's law states that the flow velocity $\boldsymbol{v}$ is proportional to the gradient of the hydraulic head u times the hydraulic conductivity parameter a. The subsurface flow can then be modeled by the following system of partial differential equations:

$$\begin{aligned} \boldsymbol{v} + a\nabla u &= \boldsymbol{F}, \quad \text{in } \mathcal{D}, \\ \operatorname{div} \boldsymbol{v} &= 0, \quad \text{in } \mathcal{D}, \end{aligned}$$

where $\mathcal{D} \subseteq \mathbb{R}^n$ denotes an $n \in \mathbb{N}$-dimensional spatial domain and the vector field $\boldsymbol{F}$ describes a forcing term, i.e., sources and sinks in the domain $\mathcal{D}$. The second equation is simply the law of mass conservation, see, e.g., [14]. To get a well-defined problem this system has to be equipped with appropriate boundary conditions which are usually, for simplicity, homogeneous Dirichlet conditions of the hydraulic head u.

The system above can now be transformed into an elliptic boundary value problem for u by applying the divergence operator and setting $f = -\operatorname{div} \boldsymbol{F}$. This leads to the system

$$\begin{aligned} -\nabla \cdot (a\nabla u) &= f, \qquad \text{in } \mathcal{D}, \\ u &= 0, \qquad \text{in } \partial\mathcal{D}. \end{aligned} \tag{1.0.1}$$

These kinds of elliptic boundary value problems are well understood and there are different black-box solvers, see, e.g., [10, 77], which solve them with high accuracy, but only if the input data is known exactly. Unfortunately, in most cases, the coefficient function a is not given exactly and has to be determined from measurements. This leads to a limited accuracy of the

model since we have to assume that we only have a discrete number of measurements of the hydraulic conductivity.

Assuming that the porous media is heterogeneous we can follow a different approach. We can model the coefficient function a as a random field over a probability space $(\Omega, \mathcal{A}, P)$, see, e.g., [22, 60], where Ω denotes a set, $\mathcal{A}$ a σ-algebra on Ω and P a probability measure on $\mathcal{A}$. Following this idea further the equations in (1.0.1) become an elliptic boundary value problem with random diffusion coefficient. It is given by

$$\begin{aligned} -\nabla \cdot (a(\boldsymbol{x}, \omega) \nabla u(\boldsymbol{x}, \omega)) &= f(\boldsymbol{x}), && \text{in } \mathcal{D} \times \Omega, \\ u(\boldsymbol{x}, \omega) &= 0, && \text{in } \partial\mathcal{D} \times \Omega, \end{aligned} \tag{1.0.2}$$

where the function u is now also a random field.

In recent years this problem has been studied extensively, see, e.g., [4, 5, 14, 15, 79]. The usual method to simplify the problem in (1.0.2) is to assume that $(\boldsymbol{x}, \omega) \mapsto a(\boldsymbol{x}, \omega)$ represents a certain stochastic process, e.g., a Gaussian one. In these cases we can express the coefficient function as an infinite sum, i.e., $a(\boldsymbol{x}, \omega)$ allows for each $\boldsymbol{x} \in \mathcal{D}$ and $\omega \in \Omega$ the expansion

$$a(\boldsymbol{x}, \omega) = \sum_{j=0}^{\infty} \sqrt{\lambda_j} e_j(\boldsymbol{x}) Y_j(\omega). \tag{1.0.3}$$

This is known as the Karhunen-Loève expansion of a stochastic process, see, e.g., [47, 59, 80]. With this ansatz we already achieved a separation of the spatial variable $\boldsymbol{x}$ and the stochastic variable ω. Without going into more detail this expansion is achieved by spectral expansion of the covariance operator of the stochastic process. Then λ_j denotes the j-th eigenvalue, e_j the j-th eigenvector of this operator and the $(Y_j)_{j \in \mathbb{N}_0}$ are certain independent random variables, whose distributions are given by density functions.

The next step is now to replace the random variables $(Y_j)_{j \in \mathbb{N}_0}$ by random parameters $(y_j)_{j \in \mathbb{N}_0}$ which are defined on the image of the corresponding random variables, see, e.g., [24], usually we assume that $y_j \in [-1, 1]$, $j \in \mathbb{N}_0$. Additionally we cut the expansion in (1.0.3) after $d \in \mathbb{N}$ summands. Usually d is large to give a good approximation to $a(\boldsymbol{x}, \omega)$. This then yields a new, parameter-dependent coefficient function $a : \mathcal{D} \times [-1, 1]^d \to \mathbb{R}$ given by

$$a(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=0}^{d-1} \sqrt{\lambda_j} e_j(\boldsymbol{x}) y_j$$

and the problem in (1.0.2) becomes a parametric partial differential equation

$$\begin{aligned} -\nabla \cdot (a(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) &= f(\boldsymbol{x}), \quad \text{in } \mathcal{D} \times [-1, 1]^d, \\ u(\boldsymbol{x}, \boldsymbol{y}) &= 0, \quad \text{in } \partial\mathcal{D} \times [-1, 1]^d. \end{aligned} \tag{1.0.4}$$

Although it is known that the solution u is smooth as a function of the parameter $\boldsymbol{y}$, see, e.g., [15], the computational cost to solve the problem (1.0.4) by, e.g., a polynomial chaos expansion, c.f., [28, 34], or stochastic collocation methods, c.f., [67, 68], suffers from the curse of dimensionality. This term was coined by Bellman in [9] and can be interpreted that in order to approximate a d-variate continuous function to accuracy $\varepsilon < 1$ we need $\mathcal{O}(\varepsilon^{-d})$ samples.

We choose a set of samples $(\boldsymbol{y}_k)_{1\leq k\leq N}$ in the d-dimensional parameter space $[-1,1]^d$ and solve for each sample $\boldsymbol{y}_k$ the now deterministic problem

$$\begin{aligned} -\nabla\cdot(a(\boldsymbol{x},\boldsymbol{y}_k)\nabla u(\boldsymbol{x},\boldsymbol{y}_k)) &= f(\boldsymbol{x}), && \text{in } \mathcal{D},\\ u(\boldsymbol{x},\boldsymbol{y}_k) &= 0, && \text{in } \partial\mathcal{D}. \end{aligned} \tag{1.0.5}$$

Often, we are not only interested in the solution $u(\boldsymbol{x},\boldsymbol{y})$, $\boldsymbol{x}\in\mathcal{D}$, $\boldsymbol{y}\in[-1,1]^d$ of (1.0.4) but rather in a derived value, the quantity of interest, which is modeled as a function on the parameter space, i.e., it is given as mapping $Q:[-1,1]^d\to\mathbb{R}$ defined by $Q(\boldsymbol{y})=q(u(\cdot,\boldsymbol{y}))$ with a linear functional q which operates on the solution of (1.0.5) seen as a function of the spatial variable. The main task is to recover the mapping Q from sampled values $Q(\boldsymbol{y}_k)=q(u(\cdot,\boldsymbol{y}_k))$, $1\leq k\leq N$, see, e.g., [41, 55]. This is a high-dimensional scattered data reconstruction problem. Additionally, the data may contain noise since we obtain it by a numerical solution of (1.0.5).

The setting discribed above, the parametric partial differential equation (1.0.4) and the possible reconstruction of a high-dimensional quantity of interest of the solution of this equation, is the starting point of many papers, see, e.g., [4, 15, 39, 41, 55, 62, 67, 68], among others. In this thesis, we leave this motivational example and possible applications behind and focus only on methods to solve high-dimensional reconstruction problems with possibly noisy data. There are several approaches to solve this problem. In recent years the method that garnered the most attention is approximation by neural networks, see, e.g., [21, 49, 52, 96]. Neural networks yield remarkable results in some applications, such as image or speech recognition, some even claiming that they beat the curse of dimensionality, e.g., [37]. Once they are fully trained the approximations can be computed very fast, however, the training needs a lot of training data which may be expensive to obtain. Furthermore, their black-box like nature makes it hard to derive error estimates or even to understand how the approximation was obtained.

Another approach to high-dimensional reconstruction processes, in particular in the context of quadrature rules, are Monte Carlo methods, see, e.g., [44]. Once we distributed an arbitrary number of points according to a given probability distribution the method simply averages the evaluations of the integrand at these sample points. This method has the advantage that it convergences independently of d if the integrand satisfies low regularity requirements. However, the convergence rate is of low algebraic order and there are no deterministic error bounds available since the Monte Carlo estimator itself is a random variable. Nevertheless, the law of large numbers states that the estimator converges with high probability and several other convergence properties are known, c.f., [13, 87].

Following the same idea as Monte Carlo methods but, instead of distributing the sample points randomly, using certain deterministic sequences of points yields the quasi-Monte Carlo method, c.f., [25, 57]. Clearly, the method now depends on the chosen sequence of points and there are known quasi-Monte Carlo methods that exhibit dimension-independent convergence rates in certain weighted settings, see, e.g., [48, 51].

In this thesis however, we use a fourth approach. The idea is attributed to Smolyak [83]. This method uses sequences of low-dimensional approximation operators and combines them in an orderly fashion to obtain a high-dimensional reconstruction operator which exhibits nearly the same approximation features as the low-dimensional schemes. The main tools for this construction are tensor products which are very well understood, see, e.g., [42, 76, 81], and allow deterministic, a-priori error estimates, c.f., [71, 72]. The main advantage of the Smolyak method is that we have a lot of freedom to choose the low-dimensional approximation operators. The best studied reconstruction schemes use polynomial or spline interpolation, see, e.g., [6, 11, 30], but it is also possible to use low-dimensional quadrature rules, c.f., [69]. This was the original motivation in [83].

We combine Smolyak's algorithm with kernel-based multilevel operators with compactly supported radial basis functions, see, e.g., [27, 91, 92]. These methods allow stable and fast high-order approximations of functions using scattered data in general low-dimensional domains. The construction of the Smolyak method is perfectly suited for use with multilevel methods. This was already discussed in [11, 30] for the context of spline interpolation. The idea to use kernel-based multilevel methods is not entirely new, e.g., in [32, 89] Gaussians are used in this context. However, these papers are purely numerical. We derive the method, provide several representations of the multivariate approximation operator and give rigorous error estimates.

This new tensor product multilevel method has several advantages compared to the methods using polynomials or splines. While polynomials become more and more expensive if larger and larger data sets are used, they provide easily even spectral convergence. However, the possible point sets are usually restricted to Chebyshev or Clenshaw-Curtis points. In contrast, splines are computationally more efficient but usually only produce low approximation orders. Both have in common that a generalisation to other low-dimensional domains than intervals are not straight forward. This, however, might be desirable. Using the kernel-based multilevel method has none of these drawbacks since they can be used for arbitrary sets of sites in nearly any domain.

The outline of this thesis is structured as follows. In Chapter 2 we repeat the most important results from functional analytics and give a short introduction into reproducing kernels where we also briefly discuss the construction of Wendland functions, the class of compactly supported radial basis functions we have in mind throughout this thesis. We also review results concerning the approximation power of these kernels in their original and rescaled form. To prepare for Chapter 3 we also study different properties of the resulting kernel matrix.

We use the properties of the kernel matrix to derive an alternative basis of the approximation space. Its members are known as Lagrange functions. The introduction of these functions is necessary to derive the representation of the tensor product multilevel method in its most general form. Additionally, we modify these Lagrange functions such that they satisfy the Lagrange condition on a subset of the sites. The construction and study of these localized Lagrange functions in the context of Wendland functions is original work, however, follows closely known ideas.

Then, in Chapter 4, we introduce the kernel-based multilevel method and provide error estimates for different approximation schemes. We will use these error bounds later in Chapter 7. One caveat in those error estimates is that there is an unknown constant which can potentially spoil the convergence. We give numerical estimates of this constant in different parameter settings. Additionally, we introduce two adaptive versions of the multilevel method and derive an alternative representation using the Lagrange functions. To finish this chapter we also derive a convergence result in the case that we use the localized Lagrange functions.

Next, we give a thorough introduction into tensor products in Chapter 5. Starting with the purely algebraic construction of the tensor product spaces then we study tensor product norms and tensor products of operators. The results obtained here are the main ingredients to understand Smolyak's algorithm and to derive needed, preliminary error estimates.

We give a general introduction into the Smolyak method in Chapter 6. We study the resulting operator in its most general form before restricting ourselves to a special class of index set and refine some of the general statements for this special case. To end the chapter, we introduce the idea of an adaptive version of the Smolyak algorithm.

Finally, we combine Smolyak's method and the kernel-based multilevel method in Chapter 7 to obtain the tensor product multilevel method. We discuss this new approximation operator in detail and give several representations. Then we use the error bounds of Chapter 4 and combine them with estimates obtained in Chapter 6 to obtain a thorough error analysis. We also provide numerical examples to support these theoretical results. To finish, we discuss the possibility to combine the adaptive version of the Smolyak method with the adaptive version of the multilevel method which yields the double-adaptive tensor product multilevel method.

Notation and Terminology

In the most part the notation in this work is standard and chosen in a self-explanatory way. In addition, the less frequently used notation is introduced or repeated at the respective point. Nevertheless, we will give a small overview of the notation here, in case there should be any ambiguities at some point.

As usual $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural and real numbers. $\mathbb{R}_+$ denotes the set of all positive real numbers. We denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Cartesian product of two (or more) sets X and Y will be denoted by $X \times Y$, while, as usual, a superscript of a respective set denotes the dimension of the Cartesian power, for example, $\mathbb{R}^n$ is the n-dimensional real space and $\mathbb{R}^{m\times n}$ is the space of real values matrices with m rows and n columns. In general, the letters n and d are fixed as natural numbers.

For better readability elements of higher-dimensional objects are written in bold. The i-th entry of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is denoted by $x_i \in \mathbb{R}$ for $1 \leq i \leq n$. For $1 \leq p \leq \infty$, $\|\boldsymbol{x}\|_p$ denotes the p-norm of a vector. For a multiindex $\boldsymbol{\lambda} \in \mathbb{N}^d$ $\|\boldsymbol{\lambda}\|_1$ denotes the 1-norm, i.e., $\|\boldsymbol{\lambda}\|_1 := \sum_{j=1}^d |\lambda_j|$.

For a set $\Omega \subseteq \mathbb{R}^n$ and a $k \in \mathbb{N}_0$, $C^k(\Omega)$ denotes the space of all k-times continuously differentiable functions $f : \Omega \to \mathbb{R}$. If $k = 0$ we simply write $C(\Omega)$. For a $1 \leq p \leq \infty$ and measurable $\Omega \subseteq \mathbb{R}^n$, $L_p(\Omega)$ denotes the Lebesgue space.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously or weakly differentiable. The i-th partial (weak) derivative of f is denoted by $\partial_i f$. For a multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^n$, the $\boldsymbol{\alpha}$-th (weak) derivative of f is denoted by $D^{\boldsymbol{\alpha}} f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$. The Laplacian is denoted by $\Delta f := \sum_{i=1}^n \partial_i^2 f$.

We also encounter mappings where we keep a certain argument fixed. As an example, given a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for some fixed $x \in \mathbb{R}$, $f(x, \cdot)$ is seen as a mapping $\mathbb{R} \to \mathbb{R}$, $y \mapsto f(x, y)$. Here, the dot symbol is a free argument.

If $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are normed spaces we denote the norm of an operator $A : V \to W$ by

$$\|A\|_{V\to W} := \sup_{\substack{\boldsymbol{x}\in V \\ \|\boldsymbol{x}\|_V=1}} \|A\boldsymbol{x}\|_W.$$

We will use the Fourier transform, denoted by $\widehat{\cdot}$, as an operator $L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$. For $f \in L_1(\mathbb{R}^n)$ the transform is defined by

$$\widehat{f}(\boldsymbol{\xi}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-i\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\xi}} \, d\boldsymbol{x}$$

and then extended to elements of $L_2(\mathbb{R}^n)$ in the usual way.

In the later chapters of this thesis we have to distinguish between dimensions and directions. We will denote the dimensions with n and the number of directions by d. It will occur that we encounter sets of the kind $\Omega = \Omega^{(1)} \times \cdots \times \Omega^{(d)} \subseteq \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$, where each $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$ is a n_j-dimensional set and Ω is of dimension $n = \sum_{j=1}^{d} n_j$. If we have to denote objects in different directions we use the superscript (j) for the respective object in direction j.

Finally, the letters $c > 0$ and $C > 0$ denote arbitrary constants which can vary from line to line within each calculation.

CHAPTER 2

Kernel-based Approximation

This chapter repeats important results in the context of kernel-based approximation. After recalling well-known tools from analysis in Section 2.1 we investigate the relation between positive definite kernels and native spaces in Section 2.2. There we also repeat the basic construction of the class of radial basis functions we are most interested in, the Wendland functions. Next, we study different approximation schemes in Section 2.3. We start by looking at interpolation of functions that lie in the native space, but also show that we can interpolate continuous functions that are rougher. The second approximation scheme we introduce is the penalized least squares method that allows us to cope with noise on the data. Although we see that those schemes converge with $h_{X,\Omega} \to 0$ they become numerically more and more expensive, hence, in Section 2.4 we study a way to rescale kernels such that the resulting kernel matrix is well-behaved. There are several other classes of basis functions, such as Gaussians, (inverse) multiquadrics or Matérn kernels, to name a few, that can be used in these approximation methods and that may fit the assumptions of some theoretical results, but we are have mainly the Wendland functions in mind. To end this chapter, in Section 2.5, we study the kernel matrix of these compactly supported rescaled kernels in greater detail. The results obtained there lead directly into the next chapter.

2.1. Some Fundamental Information on Sobolev Spaces

We start this chapter with providing some general functional analytical results we will use throughout this text. Although most statements can be made using more general assumptions, we restrict ourselves on domains, i.e., subsets of $\mathbb{R}^n$ that satisfy the following definition.

Definition 2.1.1. *We call $\Omega \subseteq \mathbb{R}^n$* domain *if Ω is non-empty, open and connected.*

First, we give the definition of certain subspaces of $L_p(\Omega)$-spaces.

Definition 2.1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. We define the* Sobolev space $W_p^k(\Omega)$ *by*

$$W_p^k(\Omega) := \{f \in L_p(\Omega) \ : \ D^{\boldsymbol{\alpha}} f \in L_p(\Omega) \text{ for all } 0 \leq \|\boldsymbol{\alpha}\|_1 \leq k\}$$

and its norm by

$$\|f\|_{W_p^k(\Omega)} := \left(\sum_{0 \leq \|\boldsymbol{\alpha}\|_1 \leq k} \|D^{\boldsymbol{\alpha}} f\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. *For* $p = \infty$ *the norm is defined by*

$$\|f\|_{W_\infty^k(\Omega)} := \max_{0 \leq \|\boldsymbol{\alpha}\|_1 \leq k} \|D^{\boldsymbol{\alpha}} f\|_{L_\infty(\Omega)}.$$

Usually, $k \in \mathbb{N}$ is called the *smoothness* of $f \in W_p^k(\Omega)$. It is well known that Sobolev spaces are complete, see, e.g., [10, Theorem 1.3.2].

Lemma 2.1.3. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a domain,* $k \in \mathbb{N}_0$ *and* $1 \leq p \leq \infty$. *Then* $\left(W_p^k(\Omega), \|\cdot\|_{W_p^k(\Omega)}\right)$ *is a Banach space.*

If $p = 2$, *the space* $\left(W_2^k(\Omega), \|\cdot\|_{W_2^k(\Omega)}\right)$ *is a Hilbert space and its norm is induced by the scalar product*

$$\langle f, g \rangle_{W_2^k(\Omega)} := \left(\sum_{0 \leq \|\boldsymbol{\alpha}\|_1 \leq k} \langle D^{\boldsymbol{\alpha}} f, D^{\boldsymbol{\alpha}} g \rangle_{L_2(\Omega)} \right)^{\frac{1}{2}}.$$

There are several approaches to generalize Sobolev spaces to non-integer smoothnesses, see, e.g., [1, Chapter VII]. We are most interested in a construction using the Fourier transform of the Bessel potential, i.e., the Fourier transform of functions of the kind $(\iota - \Delta)^s f$ with a real number $s \geq 0$, where ι denotes the identity operator and Δ the Laplace operator. This construction requires us to first consider only $\Omega = \mathbb{R}^n$. Furthermore, we restrict ourselves to the Hilbert space case, i.e., Sobolev spaces with $p = 2$.

Definition 2.1.4. *Let* $0 \leq s < \infty$. *Then the space* $H^s(\mathbb{R}^n)$ *is defined as*

$$H^s(\mathbb{R}^n) := \left\{ f \in L_2(\mathbb{R}^n) \,:\, \boldsymbol{\xi} \mapsto \left(1 + \|\boldsymbol{\xi}\|_2^2\right)^{\frac{s}{2}} |\widehat{f}(\boldsymbol{\xi})| \in L_2(\mathbb{R}^n) \right\}.$$

Its norm is given by

$$\|f\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(1 + \|\boldsymbol{\xi}\|_2^2\right)^s |\widehat{f}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \right)^{\frac{1}{2}}. \tag{2.1.1}$$

For integer smoothness, i.e., $s = k \in \mathbb{N}_0$, the spaces $H^k(\mathbb{R}^n)$ and $W_2^k(\mathbb{R}^n)$ coincide, however the associated norms are only equivalent.

Definition 2.1.5. *Let* V *be a linear space. Then two norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *on* V *are called* equivalent *if there are constants* c_1, $c_2 > 0$ *such that*

$$c_1 \|v\|_1 \leq \|v\|_2 \leq c_2 \|v\|_1, \quad v \in V. \tag{2.1.2}$$

Since we used the Fourier transform for the definition of the Sobolev Hilbert space of fractional smoothness it was necessary to set $\Omega = \mathbb{R}^n$. However, we can canonically restrict the elements of $H^s(\mathbb{R}^n)$ to Ω and obtain the Sobolev space $H^s(\Omega)$ this way. This restriction can be done without any assumptions on Ω.

It is also possible to extend elements from $H^s(\Omega)$ to functions in $H^s(\mathbb{R}^n)$ in a controlled way. But in contrast to the restriction, the extension requires the boundary of Ω to be smooth in the sense that it can be parameterized by countably many Lipschitz maps. We quote the precise definition from [10, Definition 1.4.4].

Definition 2.1.6. *We say a domain Ω has a* Lipschitz boundary *$\partial\Omega$ provided there exists a collection of open sets O_i, a positive parameter ε, an integer N and a finite number M, such that for all $\boldsymbol{x} \in \partial\Omega$ the ball of radius ε centered at $\boldsymbol{x}$ is contained in some O_i, no more than N of the sets O_i intersect nontrivially, and each domain $O_i \cap \Omega$ satisfies $O_i \cap \Omega = O_i \cap \Omega_i$, where Ω_i is a domain whose boundary is a graph of a Lipschitz function ϕ_i, i.e., $\Omega_i = \{(\boldsymbol{x}, y) \in \mathbb{R}^n \ : \ \boldsymbol{x} \in \mathbb{R}^{n-1}, y < \phi_i(\boldsymbol{x})\}$, satisfying $\|\phi_i\|_{Lip(\mathbb{R}^{n-1})} \leq M$, where $\|\cdot\|_{Lip(\mathbb{R}^{n-1})}$ denotes the Lipschitz norm, defined by*

$$\|f\|_{Lip(\mathbb{R}^{n-1})} = \|f\|_{L_\infty(\mathbb{R}^{n-1})} + \sup\left\{ \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|} \ : \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n-1}, \boldsymbol{x} \neq \boldsymbol{y} \right\}.$$

We often call domains Ω which have a Lipschitz boundary *Lipschitz domains*. The simplest and, in the context of approximation, most often used Lipschitz domains are polytopal domains, such as the hypercube $[-1, 1]^n$.

With this, we are now able to discuss a way to extend functions from $H^s(\Omega)$ to functions in $H^s(\mathbb{R}^n)$. The *extension operator* was first introduced in [85] for the case of integer smoothness k and later, e.g., in [10, Section 14.2] and references therein, generalized to real $s \geq 0$.

Theorem 2.1.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain. Let $s \geq 0$. Then there exists a linear operator $E : H^s(\Omega) \to H^s(\mathbb{R}^n)$ such that*

(1) $Ef|_\Omega = f$, and
(2) $\|Ef\|_{H^s(\mathbb{R}^n)} \leq C_E \|f\|_{H^s(\Omega)}$

holds for all $f \in H^s(\Omega)$.

The constant $C_E = C_E(\Omega, s)$ depends only on the domain Ω and the smoothness s.

Property (2) implies that the extension operator E is bounded. Furthermore, we can use the same operator E for every $s \geq 0$, however the constant C_E grows with s. Unfortunately, the norm of the extension operator is in most cases unknown. We give an example where we can derive bounds if Ω is a one-dimensional interval, that is, $\Omega = (a, b)$. We have to distinguish two cases, depending on the length of the interval.

To obtain bounds for the norms of extension operators on $H^s(\Omega)$ we introduce another useful tool, the *operator interpolation*. The next result is a special version of, e.g., [10, Proposition 14.1.5] and uses, in its original form, a norm-equivalent version of $H^s(\Omega)$.

Theorem 2.1.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain. Let $s_1 \geq s_0$ and $t_1 \geq t_0$. For $0 \leq \theta \leq 1$ let*

$$s_\theta = (1 - \theta)s_0 + \theta s_1 \quad \text{and} \quad t_\theta = (1 - \theta)t_0 + \theta t_1.$$

If T is linear and bounded as an operator mapping $H^{s_0}(\Omega) \to H^{t_0}(\Omega)$ and $H^{s_1}(\Omega) \to H^{t_1}(\Omega)$ simultaneously, then it is also bounded as a linear operator mapping $H^{s_\theta}(\Omega) \to H^{t_\theta}(\Omega)$ with

$$\|T\|_{H^{s_\theta}(\Omega) \to H^{t_\theta}(\Omega)} \leq \|T\|^{1-\theta}_{H^{s_0}(\Omega) \to H^{t_0}(\Omega)} \, \|T\|^{\theta}_{H^{s_1}(\Omega) \to H^{t_1}(\Omega)}. \tag{2.1.3}$$

Theorem 2.1.8 allows us to generalize the statements of [12, Section 6.1, Lemma 9, Lemma 10]. There, only Sobolev spaces of integer smoothness

are considered. Again, we are only interested in the Sobolev Hilbert space $H^s((a,b))$.

Lemma 2.1.9. *Let $s \geq 1$ and $-\infty < a < b < \infty$. Then there exists a linear extension operator $E : H^s((a,b)) \to H^s(\mathbb{R})$, such that we have for $b-a \leq 1$,*

$$\|E\|_{H^s((a,b))\to H^s(\mathbb{R})} \leq c4(64(1+256e))^s \frac{s^s}{(b-a)^{s-\frac{1}{2}}} \tag{2.1.4}$$

and for $b-a \geq 1$,

(2.1.5)

$$\|E\|_{H^s((a,b))\to H^s(\mathbb{R})} \leq C \max\left(b-a, 2^{s+2}\right)(80+1024e)^s \left(1 + \frac{s^s}{(b-a)^{s-\frac{1}{2}}}\right),$$

where $c, C > 0$ depend on the equivalence constants of Definition 2.1.5.

Proof. We only give a proof for $b-a \geq 1$ as the same ideas apply to the other case $b-a \leq 1$. For $b-a \geq 1$ we have by [12, Section 9.1, Lemma 10] the bound for the norm of the extension operator for $k \in \mathbb{N}$,

$$\|E\|_{H^k((a,b))\to H^k(\mathbb{R})} \leq c_2 c^k \left(1 + \frac{k^k}{(b-a)^{k-\frac{1}{2}}}\right), \tag{2.1.6}$$

with the constant $c = 80 + 1024e$. For any $s \in \mathbb{R}$ with $s \geq 1$ there is a $k \in \mathbb{N}$ such that $k \leq s \leq k+1$. Hence we use (2.1.6) for k and $k+1$ and apply the operator interpolation of Theorem 2.1.8 with $\theta = s - k \geq 0$. This yields

$$\begin{aligned}
\frac{1}{C}\|E\|_{H^s((a,b))\to H^s(\mathbb{R})} &\leq \|E\|^{1-\theta}_{H^k((a,b))\to H^k(\mathbb{R})} \|E\|^{\theta}_{H^{k+1}((a,b))\to H^{k+1}(\mathbb{R})} \\
&\leq c^{k(1-\theta)} \left(1 + \frac{k^k}{(b-a)^{k-\frac{1}{2}}}\right)^{1-\theta} \cdot c^{(k+1)\theta} \left(1 + \frac{(k+1)^{(k+1)}}{(b-a)^{(k+1)-\frac{1}{2}}}\right)^{\theta} \\
&= c^{k+\theta}(b-a)^{-(k-\frac{1}{2})(1-\theta)}(b-a)^{-(k+\frac{1}{2})\theta} \left((b-a)^{k-\frac{1}{2}} + k^k\right)^{1-\theta} \cdot \\
&\quad \cdot \left((b-a)^{k+\frac{1}{2}} + (k+1)^{(k+1)}\right)^{\theta} \\
&\leq c^s (b-a)^{-(s-\frac{1}{2})} \left((b-a)^{s-\frac{1}{2}} + s^s\right)^{1-\theta} \left((b-a)^{k+\frac{1}{2}} + (k+1)^{k+1}\right)^{\theta}.
\end{aligned}$$

We now investigate the term

$$\left((b-a)^{k+\frac{1}{2}} + (k+1)^{k+1}\right)^{\theta}$$

separately. We first find that

$$\begin{aligned}
(b-a)^{k+\frac{1}{2}} &= (b-a)(b-a)^{k-\frac{1}{2}} \\
&\leq (b-a)(b-a)^{k+\theta-\frac{1}{2}},
\end{aligned}$$

Additionally, we have with $0 \leq \theta \leq 1$

$$\begin{aligned}(k+1)^{k+1} &= (k+\theta+(1-\theta))^{k+1} \leq (k+\theta)^{k+1}\left(1+\frac{1-\theta}{k+\theta}\right)^{k+1} \\ &\leq (k+\theta)^{k+\theta+(1-\theta)}\left(1+\frac{1-\theta}{k+\theta}\right)^{k+\theta+(1-\theta)} \\ &= (k+\theta)^{k+\theta}\cdot(k+\theta)^{1-\theta}\cdot\left(1+\frac{1-\theta}{k+\theta}\right)^{k+\theta}\cdot\left(1+\frac{1-\theta}{k+\theta}\right)^{1-\theta} \\ &\leq s^s\cdot s\cdot 2^s\cdot 2 \leq 2^{s+2}s^s.\end{aligned}$$

Putting these two estimates back yields

$$\begin{aligned}\frac{1}{C}\|E\|_{H^s((a,b))\to H^s(\mathbb{R})} &\leq c^s(b-a)^{-(s-\frac{1}{2})}\left((b-a)^{s-\frac{1}{2}}+s^s\right)^{1-\theta}\cdot \\ &\quad\cdot\left((b-a)^{k+\frac{1}{2}}+(k+1)^{(k+1)}\right)^{\theta} \\ &\leq c^s(b-a)^{-(s-\frac{1}{2})}\left((b-a)^{s-\frac{1}{2}}+s^s\right)^{1-\theta}\cdot \\ &\quad\cdot\left((b-a)(b-a)^{s-\frac{1}{2}}+2^{s+2}s^s\right)^{\theta} \\ &\leq c^s\max(b-a,2^{s+2})(b-a)^{-(s-\frac{1}{2})}\left((b-a)^{s-\frac{1}{2}}+s^s\right).\end{aligned}$$

That is (2.1.5). □

We also quote [12, Section 9.1, Lemma 11] which gives a lower bound for the norm of the extension operator, however only for integer smoothness. Again, we are only interested in the case $p = 2$.

Lemma 2.1.10. *Let $k \in \mathbb{N}$ and $a, b \in \mathbb{R}$ such that $-\infty < a < b < \infty$. Then the norm of any extension operator $E : H^k((a,b)) \to H^k(\mathbb{R})$ can be bounded from below by*

$$\|E\|_{H^k((a,b))\to H^k(\mathbb{R})} \geq c_1\frac{1}{8\sqrt{k}}\left(\frac{4}{e}\right)^k k^k(b-a)^{-k+\frac{1}{2}}, \tag{2.1.7}$$

where $c_1 > 0$ is the equivalence constant from Definition 2.1.5.

The interval $\Omega = (-1, 1)$ will be of special interest later. Hence, we give the specific bounds.

Proposition 2.1.11. *For $a = -1$, $b = 1$ and $s > 1$ the norm of the extension operator $E : H^s((-1,1)) \to H^s(\mathbb{R})$ satisfies*

$$\|E\|_{H^s((-1,1))\to H^s(\mathbb{R})} \leq 4c_2(80+1024e)^s\left(2^s+\sqrt{2}s^s\right). \tag{2.1.8}$$

If $s = k \in \mathbb{N}$, the norm of the extension operator $E : H^k((-1,1)) \to H^k(\mathbb{R})$ satisfies the lower bound

$$\|E\|_{H^s((-1,1))\to H^s(\mathbb{R})} \geq c_1 2^{k-1}\frac{1}{e^k\sqrt{k}}k^k. \tag{2.1.9}$$

Extension in the pure operator setting is much easier. The following theorem is well-known, see, e.g., [42, Chapter 4].

Theorem 2.1.12. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Let $V_0 \subseteq V$ be a dense subspace. Let $A_0 : V_0 \to W$ be a bounded linear operator. Then there exists a unique bounded linear operator $A : V \to W$ such that*

(1) $A_0(v_0) = A(v_0)$ holds for all $v_0 \in V$, and
(2) $\|A_0\|_{V\to W} = \|A\|_{V\to W}$.

The operator A in Theorem 2.1.12 is called the *extension* of A_0. Often, we use the same symbol for both operators.

To end this section, we give a special case of the Sobolev embedding theorem. For more general cases see, e.g., [10, Theorem 1.4.6]. This theorem gives us an easy criterion to check whether a given Sobolev space can be imbedded into the space of continuous functions and hence allows us to use point-evaluations of Sobolev functions. We note that we use the same notation for the embedding operator as for the identity operator.

Theorem 2.1.13. *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain. Let $s > n/2$. Then there is a linear and bounded embedding operator $\iota : H^s(\Omega) \to C(\Omega)$.*

We emphasize that the condition $s > n/2$ means that for large dimensions n only very smooth Sobolev spaces can be imbedded into the space of continuous functions.

2.2. Kernel Functions and Native Spaces

We now give a short survey over the wide field of kernel functions and their native spaces. This theory originated early, see [2], and is the fundamental basis for meshfree or scattered data approximation.

We start by giving the definition of special functions in Hilbert spaces.

Definition 2.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$. A function $K : \Omega\times\Omega \to \mathbb{R}$ is called* reproducing kernel *for $\mathcal{H}$ if*

(1) $K(\cdot, \boldsymbol{y}) \in \mathcal{H}$ for all $\boldsymbol{y} \in \Omega$, and
(2) $f(\boldsymbol{y}) = \langle f, K(\cdot, \boldsymbol{y})\rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$ and all $\boldsymbol{y} \in \Omega$.

We call a Hilbert space $\mathcal{H}$ with such a kernel K a *reproducing kernel Hilbert space*. In the next theorem, we collect some important properties of these spaces and their reproducing kernel.

Theorem 2.2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a Hilbert space of functions $f : \Omega \to \mathbb{R}$. Then $\mathcal{H}$ is a reproducing kernel Hilbert space if and only if the point-evaluation functionals $\delta_{\boldsymbol{x}}$ are continuous for every $\boldsymbol{x} \in \Omega$. In this case, the following properties hold:*

(1) The reproducing kernel K is unique.
(2) $K(\boldsymbol{x}, \boldsymbol{y}) = \langle K(\cdot, \boldsymbol{x}), K(\cdot, \boldsymbol{y})\rangle_{\mathcal{H}} = \langle \delta_{\boldsymbol{x}}, \delta_{\boldsymbol{y}}\rangle_{\mathcal{H}^*}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.
(3) $K(\boldsymbol{x}, \boldsymbol{y}) = K(\boldsymbol{y}, \boldsymbol{x})$ for $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.

Proof. The proof is a combination of the proofs of [94, Theorem 10.2 and Theorem 10.3]. □

There is a strong connection between reproducing kernel Hilbert spaces and positive definite kernels. We define this property first, before we specify this connection.

Definition 2.2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. A continuous function $K : \Omega \times \Omega \to \mathbb{R}$ is called* positive semi-definite *if for all $N \in \mathbb{N}$, all sets of pairwise distinct centers $X := \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ and all $\boldsymbol{\alpha} \in \mathbb{R}^N$, the quadratic form*

$$\sum_{i,j=1}^{N} \alpha_i \alpha_j K(\boldsymbol{x}_i, \boldsymbol{x}_j)$$

is non-negative. The kernel K is called positive definite *if the quadratic form is positive for all $\boldsymbol{\alpha} \in \mathbb{R}^N \setminus \{\boldsymbol{0}\}$.*

The connection between reproducing kernel Hilbert spaces and positive definite kernels is easily proven, see, e.g., [94, Theorem 10.4].

Theorem 2.2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a reproducing kernel Hilbert space with reproducing kernel $K : \Omega \times \Omega \to \mathbb{R}$. Then K is positive semi-definite. Furthermore, K is positive definite if and only if the point-evaluation functionals are linearly independent in $\mathcal{H}^*$.*

The first property of Definition 2.2.1 means in particular that $\mathcal{H}$ contains functions of the form $f = \sum_{j=1}^{N} \alpha_j K(\cdot, \boldsymbol{x}_j)$ with arbitrary, pairwise distinct points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_N \in \Omega$, $N \in \mathbb{N}$. The norm of these functions is given as

$$\|f\|_{\mathcal{H}}^2 = \sum_{i,j=1}^{N} \alpha_i \alpha_j \langle K(\cdot, \boldsymbol{x}_i), K(\cdot, \boldsymbol{x}_j) \rangle_{\mathcal{H}} = \sum_{i,j=1}^{N} \alpha_i \alpha_j K(\boldsymbol{x}_i, \boldsymbol{x}_j).$$

Theorem 2.2.4 yields, in particular, that positive definite kernels appear naturally in the context of reproducing kernel Hilbert spaces. However, the other direction is also true. We can construct a Hilbert space from a given positive definite kernel.

We assume that $K : \Omega \times \Omega \to \mathbb{R}$ is a symmetric, positive definite kernel. Motivated by the consideration above, we define a linear space

$$F_K(\Omega) := \operatorname{span}\{K(\cdot, \boldsymbol{y}) \ : \ \boldsymbol{y} \in \Omega\}$$

and equip it with the bilinear form $\langle \cdot, \cdot \rangle_K : F_K(\Omega) \times F_K(\Omega) \to \mathbb{R}$ which is defined as

$$\left\langle \sum_{i=1}^{N} \alpha_i K(\cdot, \boldsymbol{x}_i), \sum_{j=1}^{M} \beta_j K(\cdot, \boldsymbol{x}_j) \right\rangle_K := \sum_{i=1}^{N} \sum_{j=1}^{M} \alpha_i \beta_j K(\boldsymbol{x}_i, \boldsymbol{x}_j). \tag{2.2.1}$$

This bilinear mapping is a candidate for an inner product on $F_K(\Omega)$. For a proof of the next theorem, we refer to, e.g., [94, Theorem 10.7].

Theorem 2.2.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. If $K : \Omega \times \Omega \to \mathbb{R}$ is a symmetric, positive definite kernel then $\langle \cdot, \cdot \rangle_K$ as in (2.2.1) defines an inner product on $F_K(\Omega)$. Furthermore, $(F_K(\Omega), \langle \cdot, \cdot \rangle_K)$ is a pre-Hilbert space with reproducing kernel K.*

The next step is then to complete $F_K(\Omega)$.

Definition 2.2.6. *The completion of F_K with respect to the norm induced by $\langle \cdot, \cdot \rangle_K$ of (2.2.1) is called* native space of K *and is denoted by $\mathcal{N}_K(\Omega)$.*

The technical nuances how this completion is done can be found in, e.g., [94, pp. 137]. There, we also see that the native space is a reproducing kernel Hilbert space.

Theorem 2.2.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $K : \Omega \times \Omega \to \mathbb{R}$ be a symmetric, positive definite kernel. Then its associated native space $\mathcal{N}_K(\Omega)$ is a Hilbert space with reproducing kernel K.*

In conclusion, there is a one-to-one relation between positive definite kernels and reproducing kernels of Hilbert spaces. We can start with either the space or the kernel and obtain the other one as described above. Additionally, the native space is unique. For a proof we refer to [94].

Theorem 2.2.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $K : \Omega \times \Omega \to \mathbb{R}$ be a symmetric, positive definite kernel. Let $\mathcal{G}$ be a Hilbert space of functions $f : \Omega \to \mathbb{R}$ with reproducing kernel K. Then $\mathcal{G}$ is the native space $\mathcal{N}_K(\Omega)$ and the inner products are the same.*

From a numerical point of view it is preferable to use as easy a kernel as possible.

Definition 2.2.9. *Let $\Omega = \mathbb{R}^n$. We call the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$* translation invariant *if there is a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ such that $K(\boldsymbol{x}, \boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. If K is additionally symmetric and positive definite, we denote its native space by $\mathcal{N}_\Phi(\mathbb{R}^n)$.*

Note that we initially have to use $\Omega = \mathbb{R}^n$. We will discuss extensions and restrictions of native space functions to domains $\Omega \subsetneq \mathbb{R}^n$ later in this section.

Characterizations and properties of translation invariant kernels in relation to positive definiteness are well known, see, e.g., [94, Chapter 6], and will not be addressed further in this text. We only give one possible characterization of the native space of a translation invariant kernel, see, e.g., [94, Theorem 10.12].

Theorem 2.2.10. *Let $\Phi \in C(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$ be a real-valued, positive definite function. Define*

$$\mathcal{G} := \left\{ f \in L_2(\mathbb{R}^n) \cap C(\mathbb{R}^n) \ : \ \frac{\widehat{f}}{\sqrt{\widehat{\Phi}}} \in L_2(\mathbb{R}^n) \right\}$$

and equip this space with the bilinear form

$$\langle f, g \rangle_{\mathcal{G}} := \left\langle \frac{\widehat{f}}{\sqrt{\widehat{\Phi}}}, \frac{\widehat{g}}{\sqrt{\widehat{\Phi}}} \right\rangle_{L_2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \frac{\widehat{f}(\boldsymbol{\xi})\overline{\widehat{g}(\boldsymbol{\xi})}}{\widehat{\Phi}(\boldsymbol{\xi})} \, d\boldsymbol{\xi}.$$

Then $\mathcal{G}$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ and reproducing kernel Φ. Hence, $\mathcal{G}$ is the native space $\mathcal{N}_\Phi(\mathbb{R}^n)$ and the inner products are the same.

If the Fourier transform of the kernel Φ decays sufficiently fast we can even connect its native space to a Sobolev Hilbert space. The proof of the following corollary can also be found in [94].

Corollary 2.2.11. *Let $\Phi \in L_1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and assume that its Fourier transform satisfies*

$$c_1(1 + \|\boldsymbol{\xi}\|_2^2)^{-s} \leq \widehat{\Phi}(\boldsymbol{\xi}) \leq c_2(1 + \|\boldsymbol{\xi}\|_2^2)^{-s}, \quad \boldsymbol{\xi} \in \mathbb{R}^n, \tag{2.2.2}$$

with $s > n/2$ and two positive constants $c_1 \leq c_2$. Then the native space $\mathcal{N}_\Phi(\mathbb{R}^n)$ of Φ coincides with the Sobolev space $H^s(\mathbb{R}^n)$, and the native space norm and the Sobolev norm are equivalent.

Next we want to transfer the result of the preceding corollary to native spaces which are defined only on a domain $\Omega \subset \mathbb{R}^n$. We will see that this is possible with a weak assumption on Ω. To do this we first need to discuss the restriction of functions in a native space $\mathcal{N}_\Phi(\Omega_2)$ to a domain Ω_1, where $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n$. We have the following theorem whose proof is given in the discussion before [94, Theorem 10.47].

Theorem 2.2.12. *Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n$. Then the restriction $f|_{\Omega_1}$ of any function $f \in \mathcal{N}_\Phi(\Omega_2)$ is contained in $\mathcal{N}_\Phi(\Omega_1)$ with a norm that is less than or equal to the norm of f.*

This theorem can be used to prove the analogue of Corollary 2.2.11. However first, we introduce a result on the extension of native space functions, taken from [94, Theorem 10.46].

Theorem 2.2.13. *Each function $f \in \mathcal{N}_\Phi(\Omega_1)$ has a natural extension to a function $Ef \in \mathcal{N}_\Phi(\Omega_2)$. Furthermore, $\|Ef\|_{\mathcal{N}_\Phi(\Omega_2)} = \|f\|_{\mathcal{N}_\Phi(\Omega_1)}$.*

We are now in the position to give the desired analogue of Corollary 2.2.11, see, e.g., [94, Corollary 10.48].

Corollary 2.2.14. *Suppose that $\Phi \in L_1(\mathbb{R}^n)$ has a Fourier transform that satisfies (2.2.2), i.e., for $s > n/2$ there are constants $0 < c_1 \leq c_2$ such that*

$$c_1(1 + \|\boldsymbol{\xi}\|_2^2)^{-s} \leq \widehat{\Phi}(\boldsymbol{\xi}) \leq c_2(1 + \|\boldsymbol{\xi}\|_2^2)^{-s}, \quad \boldsymbol{\xi} \in \mathbb{R}^n.$$

Suppose further that $\Omega \subseteq \mathbb{R}^n$ has a Lipschitz boundary. Then $\mathcal{N}_\Phi(\Omega)$ coincides with $H^s(\Omega)$ and the norms are equivalent.

To conclude this section, we construct the specific family of positive definite kernels whose elements not only satisfy (2.2.2) but also have compact support and are radial.

Definition 2.2.15. *We call a function $\Phi : \mathbb{R}^n \to \mathbb{R}$* radial *if there exists a function $\phi : [0, \infty) \to \mathbb{R}$ such that $\Phi(\boldsymbol{x}) = \phi(\|\boldsymbol{x}\|_2)$ for all $\boldsymbol{x} \in \mathbb{R}^n$.*

For more details, especially in the context of positive definiteness, on radial functions we refer to [94, Chapter 6.3]. We only remark that we always assume the univariate function $\phi : [0, \infty) \to \mathbb{R}$ to be defined on all of $\mathbb{R}$ by even extension, i.e., by setting $\phi(r) := \phi(-r)$ if $r < 0$.

The starting point of the construction of the desired kernels is the following compactly supported univariate function.

Definition 2.2.16. *For $\nu \in \mathbb{N}$ the* truncated power function *$\phi_\nu : \mathbb{R} \to \mathbb{R}$ is defined as*

$$\phi_\nu(r) = (1 - r)_+^\nu, \quad r \geq 0 \tag{2.2.3}$$

and even extension to $\mathbb{R}$. We used the notation $(x)_+ = \max(0, x)$.

It is known that these functions are positive definite on $\mathbb{R}^n$, if $\nu \geq \lfloor n/2 \rfloor + 1$ [94, Theorem 6.20]. However, they are even for large ν only continuous. To construct positive definite functions of given smoothness and spatial dimension n, we need to define an operator which transports properties between dimensions.

Definition 2.2.17. *Let $\phi : [0, \infty) \to \mathbb{R}$ be given so that the mapping $t \mapsto \phi(t)t$ is in $L_1([0, \infty))$, then we define for $r \geq 0$ the* dimension walk operator *$\mathcal{I}$ as*

$$\mathcal{I}\phi(r) := \int_r^\infty t\phi(t)\, dt.$$

Again, we obtain by even extension a function $\mathcal{I}\phi$ defined on $\mathbb{R}$. Now we apply this integral operator k-times to the truncated power function ϕ_ν for a specifically chosen ν and obtain the functions of our choice.

Definition 2.2.18. *The* Wendland function *of smoothness $2k$ for space dimension n is defined as*

$$\phi_{n,k}(r) := \mathcal{I}^k \phi_{\lfloor \frac{n}{2} \rfloor + k + 1}(r), \quad 0 \leq r \leq 1. \tag{2.2.4}$$

These Wendland functions have several remarkable properties. We collect some of these in the next theorem, c.f., [94].

Theorem 2.2.19. *The functions $\phi_{n,k}$ are positive definite on every $\mathbb{R}^m$, with $m \leq n$ and are of the form*

$$\phi_{n,k}(r) = \begin{cases} p_{n,k}(r), & 0 \leq r \leq 1, \\ 0, & r > 1, \end{cases}$$

with a univariate polynomial $p_{n,k}$ of degree $\lfloor n/2 \rfloor + 3k + 1$. They possess continuous derivatives up to order $2k$. The degree of $p_{n,k}$ is minimal for a given space dimension n and smoothness $2k$ and are up to a constant factor uniquely determined by this setting.

We give a selection of Wendland functions $\phi_{n,k}$ for different space dimensions n in Table 1. We used the notation $\doteq$ to denote equality up to a multiplicative constant. Additionally, we give the explicit smoothness following Theorem 2.2.19.

Finally, we transfer these results to the kernel $\Phi_{n,k} : \mathbb{R}^n \to \mathbb{R}$ and find that the Fourier transform of this *(compactly supported) radial basis function*, or in short *(C)RBF*, exhibits an algebraic decay of certain order. We collect the results in the following theorem, see [94, Theorem 10.35].

Theorem 2.2.20. *Let $\Phi_{n,k} = \phi_{n,k}(\| \cdot \|_2)$ denote the compactly supported RBF of minimal degree that is positive definite on $\mathbb{R}^n$ and in $C^{2k}(\mathbb{R}^n)$. Let $n \geq 3$ if $k = 0$. Then there exist constants $c_1, c_2 > 0$ depending only on n and k such that*

$$c_1(1 + \|\boldsymbol{\xi}\|_2)^{-n-2k-1} \leq \widehat{\Phi_{n,k}}(\boldsymbol{\xi}) \leq c_2(1 + \|\boldsymbol{\xi}\|_2)^{-n-2k-1}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$. This means in particular that

$$\mathcal{N}_{\Phi_{n,k}}(\mathbb{R}^n) \cong H^{\frac{n}{2}+k+\frac{1}{2}}(\mathbb{R}^n),$$

Space dimension	Function	Smoothness
$n = 1$	$\phi_{1,0}(r) = (1-r)_+$	C^0
	$\phi_{1,1}(r) \doteq (1-r)_+^3(3r+1)$	C^2
	$\phi_{1,2}(r) \doteq (1-r)_+^5(8r^2+5r+1)$	C^4
$n \leq 3$	$\phi_{3,0}(r) = (1-r)_+^2$	C^0
	$\phi_{3,1}(r) \doteq (1-r)_+^4(4r+1)$	C^2
	$\phi_{3,2}(r) \doteq (1-r)_+^6(35r^2+18r+3)$	C^4
	$\phi_{3,3}(r) \doteq (1-r)_+^8(32r^3+25r^2+8r+1)$	C^6
$n \leq 5$	$\phi_{5,0}(r) = (1-r)_+^3$	C^0
	$\phi_{5,1}(r) \doteq (1-r)_+^5(5r+1)$	C^2
	$\phi_{5,2}(r) \doteq (1-r)_+^7(16r^2+7r+1)$	C^4

TABLE 1. Specific Wendland functions $\phi_{n,k}$ for different spatial dimensions n

i.e., the native space of $\Phi_{n,k}$ coincides with a classical Sobolev space of smoothness $s = \frac{n}{2} + k + \frac{1}{2}$ and the native space norm and the Sobolev norm are equivalent.

Because of this connection of the native space $\mathcal{N}_\Phi(\mathbb{R}^n)$ of the Wendland function and the Sobolev space we often call $\Phi_{n,k}$ the reproducing kernel of $H^s(\mathbb{R}^n)$.

The theorem above directly implies that we can use Wendland functions as reproducing kernels for Sobolev spaces of order $n/2+k+1/2$ in dimension n. However, this means that we do not cover Sobolev spaces of integer smoothness if the spacial dimension n is even. Although there is an explicit construction for these *missing Wendland functions* given in [75], we can and will simply take the traditional Wendland functions for larger n.

2.3. Approximation with Compactly Supported RBFs

We introduce now the basics of approximation by compactly supported RBFs. This method allows us to reconstruct unknown functions from pointwise data. In comparison to other popular approximation methods we require no additional information like meshes.

2.3.1. General Setup for Scattered Data Approximation. We start by giving the general setup for scattered data approximation and introduce the specific notation we use for the rest of this thesis.

The main goal is to approximate an unknown function $f : \Omega \to \mathbb{R}$, defined on a domain $\Omega \subseteq \mathbb{R}^n$, from only unstructured point data. To this end, we fix a discrete set of sites $X := \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subset \Omega$. For this point set X we define two characteristic quantities.

Definition 2.3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. For a discrete set of sites $X := \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subset \Omega$ the* mesh size *or* fill distance *$h_{X,\Omega}$ is given by*

$$h_{X,\Omega} := \sup_{\boldsymbol{x}\in\Omega} \min_{1\leq j\leq N} \|\boldsymbol{x} - \boldsymbol{x}_j\|_2. \tag{2.3.1}$$

This is the radius of the largest ball whose center is an element of Ω and which does not contain an element of X.

Definition 2.3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. For a discrete set of sites $X := \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \Omega$ the* separation radius q_X *is given by*

$$q_X := \frac{1}{2} \min_{i \neq j} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|_2. \tag{2.3.2}$$

The separation radius gives the largest radius for two balls centered at different elements of X to be disjoint.

In order to obtain a good approximation to the function f, we have to assume that the set of sites X is well distributed in Ω.

Definition 2.3.3. *We call a set of sites $X \subset \Omega$* quasi-uniform *if there is a constant $c_{qu} > 0$ such that*

$$q_X \leq h_{X,\Omega} \leq c_{qu} q_X. \tag{2.3.3}$$

We need to discuss this definition in more detail. We see that the first inequality always holds with the definitions of the fill distance and the separation radius. Also, for a single set X, we can always find a constant $c_{qu} > 0$ such that $h_{X,\Omega} \leq c_{qu} q_X$. Hence, in applications, the definition of quasi-uniformity has to be understood in the context of more than one set of sites. In the error analysis the idea is that we have a sequence of sample points which fills Ω more and more. Then it is important that every member of this sequence satisfies (2.3.3) with the same constant c_{qu}.

Next, we give an error estimate established in [95], which is a generalization of the results in [65]. These estimates are called *sampling inequalities* and allow us to bound a weak norm by a weighted sum of a Sobolev norm and a discrete ℓ_∞-norm on the set of sites.

Theorem 2.3.4. *Let $s > n/2$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Then there are constants $C > 0$ and $h_0 > 0$ such that for all $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \Omega$ with $h_{X,\Omega} \leq h_0$ and all $u \in H^s(\Omega)$ the inequalities*

$$\|u\|_{L_\infty(\Omega)} \leq C \left(h_{X,\Omega}^{s-\frac{n}{2}} \|u\|_{H^s(\Omega)} + \|u\|_{\ell_\infty(X)} \right), \tag{2.3.4}$$

and

$$\|u\|_{L_2(\Omega)} \leq C \left(h_{X,\Omega}^{s} \|u\|_{H^s(\Omega)} + \|u\|_{\ell_\infty(X)} \right) \tag{2.3.5}$$

hold. If, in addition, $u|_X = 0$, then

$$\|u\|_{H^t(\Omega)} \leq C h_{X,\Omega}^{s-t} \|u\|_{H^s(\Omega)} \tag{2.3.6}$$

holds for all $0 \leq t \leq s$.

Next, we introduce the notation for the function space in which we search for the approximation to the unknown function f. In the theory of approximation with RBFs, it is usual to use finite-dimensional subspaces of Sobolev Hilbert spaces.

Definition 2.3.5. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies* (2.2.2). *Let $X :=$*

$\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ *be a discrete set of sites. Then we define the* (kernel-based) approximation space *by*

$$V_N := \operatorname{span}\{\Phi(\cdot - \boldsymbol{x}_i) \ : \ \boldsymbol{x}_i \in X\}. \tag{2.3.7}$$

Clearly, the assumption on the smoothness s is necessary since the approximation methods will rely on point-evaluations and we know by Theorem 2.1.13, that for $s > n/2$ we can embed $H^s(\Omega)$ into $C(\Omega)$.

We now introduce three examples of approximation methods. First, we give the basics of interpolation of a function $f \in H^s(\Omega)$, $s > n/2$. Then, we discuss how to interpolate a function $f \in H^t(\Omega)$ with $n/2 < t < s$, i.e., a function which does not lie in the native space of Φ. And finally, we introduce a method to approximate a function without forcing interpolation. This method is particularly useful if we know that the data is only close to $f(\boldsymbol{x}_j)$ and only disturbed by relatively small noise from, e.g., numerical errors.

2.3.2. Interpolation. The easiest example for an approximation process is interpolation, where we want to find an element of V_N which attains the values $f(\boldsymbol{x}_j)$ of the target function f in every $\boldsymbol{x}_j \in X$.

Definition 2.3.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subset \Omega$ be a set of sites. Let V_N be the kernel-based approximation space defined in Definition 2.3.5. For $f \in H^s(\Omega)$, $s > n/2$, the* interpolation problem *is given as:*

$$\textit{Find } s_{f,X} \in V_N \textit{ such that } s_{f,X}(\boldsymbol{x}_j) = f(\boldsymbol{x}_j) \textit{ for every } \boldsymbol{x}_j \in X.$$

In our setting the interpolation problem is well-posed. This allows us to define the corresponding operator, which maps the function f to the solution of the interpolation problem $s_{f,X}$.

Definition 2.3.7. *With the notation and assumptions of Definition 2.3.6 we define the* interpolation operator $I_{X,\Phi} : H^s(\Omega) \to V_N$ *by*

$$I_{X,\Phi} f := s_{f,X}.$$

$I_{X,\Phi} f$ *is called the* interpolant of f in X.

Clearly, we can express the interpolant as a linear combination of shifts of Φ, i.e., we have

$$I_{X,\Phi} f = s_{f,X} = \sum_{j=1}^{N} \alpha_j \Phi(\cdot - \boldsymbol{x}_j).$$

To compute the coefficient vector $\boldsymbol{\alpha} \in \mathbb{R}^N$ we enforce the interpolation condition $s_{f,X}(\boldsymbol{x}_j) = f(\boldsymbol{x}_j)$, $1 \le j \le N$, and arrive at the linear system

$$M_{X,\Phi} \boldsymbol{\alpha} = \boldsymbol{f}.$$

The $N \times N$-matrix is called the *kernel matrix* of the kernel Φ on X. Its entries are given by $(M_{X,\Phi})_{ij} := \Phi(\boldsymbol{x}_i - \boldsymbol{x}_j)$. Clearly, if Φ is a positive definite kernel then the corresponding kernel matrix is positive definite and there exists a unique solution $\boldsymbol{\alpha}$ for every right-hand side $\boldsymbol{f} = (f(\boldsymbol{x}_j))_{1 \le j \le N}$.

Indeed, we see that the interpolant $I_{X,\Phi} f$ is the best-approximation of f from V_N with respect to the native space norm, see, e.g., [94, Theorem 13.1].

Theorem 2.3.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be the reproducing kernel of $\mathcal{N}_\Phi(\Omega)$. Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ be a set of sites and $f \in \mathcal{N}_\Phi(\Omega)$. Then the interpolant $I_{X,\Phi} f \in V_N$ is the best-approximation to f with respect to the native space norm, i.e., the estimate*

$$\|f - I_{X,\Phi} f\|_{\mathcal{N}_\Phi(\Omega)} \leq \|f - s\|_{\mathcal{N}_\Phi(\Omega)}$$

holds for every $s \in V_N$.

Hence, $I_{X,\Phi} f$ is the orthogonal projection of f on V_N.

We can choose $s = 0$ in the theorem above and obtain the estimate

$$\|f - I_{X,\Phi} f\|_{\mathcal{N}_\Phi(\Omega)} \leq \|f\|_{\mathcal{N}_\Phi(\Omega)}. \tag{2.3.8}$$

The next lemma transfers this estimate to Sobolev norms, see, e.g., [94, Corollary 11.33].

Lemma 2.3.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subset \Omega$ be a set of sites. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Then there exists a constant $C = C(\Omega, \Phi, s) > 0$, depending only on Ω, Φ and s, such that*

$$\|f - I_{X,\Phi} f\|_{H^s(\Omega)} \leq C \|f\|_{H^s(\Omega)}$$

holds for all $f \in H^s(\Omega)$.

Furthermore, the interpolant $I_{X,\Phi} f$ is norm-minimal. The proof can be found in, e.g., [94, Proof of Theorem 13.2].

Theorem 2.3.10. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be the reproducing kernel of $\mathcal{N}_\Phi(\Omega)$. Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ be a set of sites and $f \in \mathcal{N}_\Phi(\Omega)$. Then the interpolant $I_{X,\Phi} f$ has minimal $\mathcal{N}_\Phi(\Omega)$-norm of all functions $s \in \mathcal{N}_\Phi(\Omega)$ that interpolate the data $\{f(\boldsymbol{x}_j)\}$, i.e.,*

$$\|I_{X,\Phi} f\|_{\mathcal{N}_\Phi(\Omega)} = \min \left\{ \|s\|_{\mathcal{N}_\Phi(\Omega)} \ : \ s \in \mathcal{N}_\Phi(\Omega) \text{ with } s(\boldsymbol{x}_j) = f_j, 1 \leq j \leq N \right\}.$$

We now use these general results, together with the sampling inequalities of Theorem 2.3.4, to derive error bounds for the interpolation process in terms of the fill distance $h_{X,\Omega}$.

Corollary 2.3.11. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ be a set of sites with sufficiently small fill distance $h_{X,\Omega}$. Let $I_{X,\Phi} : H^s(\Omega) \to V_N$ be the interpolation operator of Definition 2.3.7. Then there is a constant $C > 0$ such that the estimates*

$$\|f - I_{X,\Phi} f\|_{L_\infty(\Omega)} \leq C h_{X,\Omega}^{s - \frac{n}{2}} \|f\|_{H^s(\Omega)}, \tag{2.3.9}$$

and

$$\|f - I_{X,\Phi} f\|_{H^t(\Omega)} \leq C h_{X,\Omega}^{s-t} \|f\|_{H^s(\Omega)}, \quad 0 \leq t \leq s, \tag{2.3.10}$$

hold for every $f \in H^s(\Omega)$.

Proof. The claim follows from the inequalities of Theorem 2.3.4. We use that $I_{X,\Phi} f$ interpolates f in the points of X, which leads to $\|f - I_{X,\Phi} f\|_{\ell_\infty(X)} = 0$. Finally, we use the estimate of Lemma 2.3.9, which finishes the proof. □

The second estimate yields for $t = 0$ the $L_2(\Omega)$-error estimate of the interpolation process

$$\|f - I_{X,\Phi} f\|_{L_2(\Omega)} \leq C h_{X,\Omega}^s \|f\|_{H^s(\Omega)}. \tag{2.3.11}$$

2.3.3. Interpolating Rougher Functions. At first glance it seems that we can only recover functions which lie in the native space of the chosen kernel Φ, which is norm-equivalent to the Sobolev space $H^s(\mathbb{R}^n)$, $s > n/2$. In fact however, the authors of [65] showed that we can even recover functions which are rougher and lie only in a Sobolev space $H^t(\Omega)$ with $n/2 < t \leq s$. This method is sometimes refered to as *escaping the native space*. It turns out that the essential tool are so-called *band-limited functions*, i.e., functions whose Fourier transforms have compact support. Again, we have to use $\Omega = \mathbb{R}^n$.

Definition 2.3.12. *For $\sigma > 0$ the space of band-limited functions $\mathcal{B}_\sigma \subset L_2(\mathbb{R}^n)$ is defined as*

$$\mathcal{B}_\sigma := \left\{ f \in L_2(\mathbb{R}^n) \ : \ \operatorname{supp} \widehat{f} \subseteq \overline{B_\sigma(\mathbf{0})} \right\}.$$

This space $\mathcal{B}_\sigma$ is a closed subspace of $L_2(\mathbb{R}^n)$, see, e.g., [18], and this closedness allows us to measure the distance of a function $f \in H^t(\mathbb{R}^n)$ to $\mathcal{B}_\sigma$.

Definition 2.3.13. *We define the distance of a function $f \in H^t(\mathbb{R}^n)$ to the space $\mathcal{B}_\sigma$ measured in the $H^t(\mathbb{R}^n)$-norm by*

$$\operatorname{dist}_{H^t(\mathbb{R}^n)}(f, \mathcal{B}_\sigma) := \inf_{g \in \mathcal{B}_\sigma} \|f - g\|_{H^t(\mathbb{R}^n)}.$$

The following auxiliary result, taken from [65, Theorem 3.4], gives a first error estimate.

Theorem 2.3.14. *Let $X := \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \mathbb{R}^n$ be a set of sites. Let $s, t \in \mathbb{R}$ with $n/2 < t \leq s$. If $f \in H^s(\mathbb{R}^n)$ then there exists an $f_\sigma \in \mathcal{B}_\sigma$ such that $f_\sigma|_X = f|_X$ and*

$$\|f - f_\sigma\|_{H^t(\mathbb{R}^n)} \leq 5 \operatorname{dist}_{H^t(\mathbb{R}^n)}(f, \mathcal{B}_\sigma) \leq 5\kappa^{-s+t} q_X^{s-t} \|f\|_{H^s(\mathbb{R}^n)},$$

with $\sigma = \kappa / q_X$, where $\kappa \geq 1$ depends only on n and t.

With the extension operator from Theorem 2.1.7 we can also use this theorem for functions $f \in H^s(\Omega)$, if $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain. First we extend this f to $Ef \in H^s(\mathbb{R}^n)$ and find an $f_\sigma \in \mathcal{B}_\sigma$. Then, by the preceding theorem, we have $f_\sigma|_X = Ef|_X = f|_X$ and the estimate

$$\begin{aligned} \|f - f_\sigma\|_{H^t(\Omega)} &= \|Ef - f_\sigma\|_{H^t(\Omega)} \\ &\leq \|Ef - f_\sigma\|_{H^t(\mathbb{R}^n)} \leq C q_X^{s-t} \|Ef\|_{H^s(\mathbb{R}^n)} \\ &\leq C q_X^{s-t} \|f\|_{H^t(\Omega)}. \end{aligned} \tag{2.3.12}$$

The inequality in (2.3.12) has an interesting consequence. If we set $s = t$ we obtain the estimate

$$\begin{aligned} \|f_\sigma\|_{H^t(\mathbb{R}^n)} &\leq \|Ef - f_\sigma\|_{H^t(\mathbb{R}^n)} + \|Ef\|_{H^t(\mathbb{R}^n)} \leq C \|Ef\|_{H^t(\mathbb{R}^n)} \\ &\leq C \|f\|_{H^t(\Omega)}. \end{aligned}$$

We collect these remarks in the next corollary.

Corollary 2.3.15. *With the assumptions and notation of Theorem 2.3.14, there exists an $f_\sigma \in \mathcal{B}_\sigma$ such that*

$$\|f - f_\sigma\|_{H^t(\Omega)} \leq C q_X^{s-t} \|f\|_{H^s(\Omega)}$$

holds for a constant $C > 0$.

In addition, we have

$$\|f_\sigma\|_{H^t(\mathbb{R}^n)} \leq C \|f\|_{H^t(\Omega)}.$$

This allows us to show that we can interpolate functions $f \in H^t(\Omega)$ although the RBF we choose is the reproducing kernel of $H^s(\mathbb{R}^n)$, $n/2 < t \leq s$. The proof of the next theorem is taken from [65] and repeated for the sake of completeness.

Theorem 2.3.16. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \Omega$ be a set of sites with sufficiently small fill distance $h_{X,\Omega}$. Then there is a constant $C > 0$ such that*

$$\|f - I_{X,\Phi} f\|_{H^r(\Omega)} \leq C h_{X,\Omega}^{t-r} \left(\frac{h_{X,\Omega}}{q_X} \right)^{s-r} \|f\|_{H^t(\Omega)}, \quad 0 \leq r \leq t$$

holds for all $f \in H^t(\Omega)$ with $n/2 < t \leq s$.

Proof. We start by using the sampling inequality (2.3.6) for $f - I_{X,\Phi} f$ with $0 \leq r \leq t$. This yields

$$\|f - I_{X,\Phi} f\|_{H^r(\Omega)} \leq C h_{X,\Omega}^{t-r} \|f - I_{X,\Phi} f\|_{H^t(\Omega)}. \tag{2.3.13}$$

Hence, we need to bound $\|f - I_{X,\Phi} f\|_{H^t(\Omega)}$. We first use Theorem 2.3.14 which gives a $\kappa \geq 1$ such that, with $\sigma = \kappa / q_X$, we find an $f_\sigma \in \mathcal{B}_\sigma$ such that $f_\sigma|_X = f|_X$ and the estimate

$$\|f - f_\sigma\|_{H^t(\Omega)} \leq C \|f\|_{H^t(\Omega)} \tag{2.3.14}$$

holds with a constant $C > 0$. Since $f_\sigma(\boldsymbol{x}_j) = f(\boldsymbol{x}_j)$ for all $\boldsymbol{x}_j \in X$ we have $I_{X,\Phi} f_\sigma = I_{X,\Phi} f$. This, together with (2.3.14), yields

$$\begin{aligned} \|f - I_{X,\Phi} f\|_{H^t(\Omega)} &\leq \|f - f_\sigma\|_{H^t(\Omega)} + \|f_\sigma - I_{X,\Phi} f\|_{H^t(\Omega)} \\ &\leq C \|f\|_{H^t(\Omega)} + \|f_\sigma - I_{X,\Phi} f_\sigma\|_{H^t(\Omega)}. \end{aligned} \tag{2.3.15}$$

Next, we use Corollary 2.3.11 and estimate

$$\|f_\sigma - I_{X,\Phi} f_\sigma\|_{H^t(\Omega)} \leq C h_{X,\Omega}^{s-t} \|f_\sigma\|_{H^s(\Omega)}. \tag{2.3.16}$$

Additionally, we have by the Bernstein theorem for band-limited functions, see, e.g., [66], with $\sigma = \kappa / q_X$, the bound

$$\|f_\sigma\|_{H^s(\mathbb{R}^n)} \leq C q_X^{t-s} \|f_\sigma\|_{H^t(\mathbb{R}^n)}.$$

Inserting this into (2.3.16) yields

$$\|f_\sigma - I_{X,\Phi} f_\sigma\|_{H^s(\Omega)} \leq C h_{X,\Omega}^{s-t} q_X^{t-s} \|f_\sigma\|_{H^t(\mathbb{R}^n)}. \tag{2.3.17}$$

From Corollary 2.3.15, we have $\|f_\sigma\|_{H^t(\mathbb{R}^n)} \leq C \|f\|_{H^t(\Omega)}$, with a constant $C > 0$ independent of f. Using this, together with (2.3.17), in (2.3.15) yields the bound

$$\|f - I_{X,\Phi} f\|_{H^t(\Omega)} \leq C \left(\frac{h_{X,\Omega}}{q_X} \right)^{s-t} \|f\|_{H^t(\Omega)}.$$

Putting this back into (2.3.13) finishes the proof. □

Setting $r = 0$ in Theorem 2.3.16 and using a quasi-uniform set of sites yields the following interesting corollary.

Corollary 2.3.17. *With the notation and assumptions of Theorem 2.3.16, where we additionally assume that X is quasi-uniform, there is a constant $C > 0$ such that the estimate*

$$\|f - I_{X,\Phi} f\|_{L_2(\Omega)} \leq C h_{X,\Omega}^t \|f\|_{H^t(\Omega)}$$

holds for every $f \in H^t(\Omega)$, $n/2 < t \leq s$.

The estimate in Corollary 2.3.17 means that the interpolation error converges with the expected, optimal order, the smoothness of the target function $f \in H^t(\Omega)$. Hence, from a theoretical point of view, there is no harm in choosing a kernel that is the reproducing kernel of a smoother Sobolev Hilbert space.

2.3.4. Penalized Least-Squares Approximation. Usually, we want to use interpolation if we can be sure that the data is exact. But if the data contains noise, because, e.g., it is generated by real-life measurement or as a consequence of numerical errors, we do not want to force an exact fit to the data but rather give the reconstruction some leeway to cope with the noise. This leads to the *penalized least-squares method* which is a special case of the Tikhonov regularization and also plays a role in kernel-based learning theory. For further information we refer to, e.g., [86]. Instead of searching a solution to the interpolation problem Definition 2.3.6, i.e., a function $s_{f,X} \in V_N$, which satisfies $s_{f,X}(\boldsymbol{x}_j) = f(\boldsymbol{x}_j)$ for every $\boldsymbol{x}_j \in X$, we look for the solution of the following problem.

Definition 2.3.18. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \Omega$ be a set of sites. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be the reproducing kernel of $\mathcal{N}_\Phi(\Omega)$. Define the functional $J_{\lambda,f} : \mathcal{N}_\Phi(\Omega) \to \mathbb{R}$ by*

$$J_{\lambda,f}(s) = \sum_{j=1}^{N} |s(\boldsymbol{x}_j) - f(\boldsymbol{x}_j)|^2 + \lambda \|s\|_{\mathcal{N}_\Phi(\Omega)}^2. \tag{2.3.18}$$

Then the penalized least-squares problem *is defined as:*

$$\text{Find } s_{\lambda,f} \in \mathcal{N}_\Phi(\Omega) \text{ such that } s_{\lambda,f} = \operatorname*{argmin}_{s \in \mathcal{N}_\Phi(\Omega)} J_{\lambda,f}(s). \tag{2.3.19}$$

The functional in (2.3.18) consists of two terms. In addition to the usual least-squares term $\sum_{j=1}^{N} |s(\boldsymbol{x}_j) - f(\boldsymbol{x}_j)|^2$, which penalizes the lack of fit to the data, we introduce another term, $\|s\|_{\mathcal{N}_\Phi(\Omega)}$, which penalizes a large native space norm of the approximation. This term ensures a smoother approximation. The *smoothing parameter* $\lambda > 0$ serves as a moderator between these two conflicting terms. The best choice of λ is still part of ongoing research, see, e.g., [50, 95], but is of no concern for this text.

At first glance we have to solve an optimization problem on the whole, infinite dimensional Hilbert space $\mathcal{N}_\Phi(\Omega)$. As it turns out, we can limit the optimization space to the kernel-based approximation space V_N to find the unique solution. This general result is known as *representer theorem*.

Theorem 2.3.19. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be the reproducing kernel of $\mathcal{N}_\Phi(\Omega)$ and $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ be a set of sites. Then, for all $\lambda > 0$, there exists a unique solution $s_{\lambda,f}$ of the penalized least-squares problem* (2.3.19).

In addition, there exists a coefficient vector $\boldsymbol{\alpha} \in \mathbb{R}^N$ such that

$$s_{\lambda,f} = \sum_{j=1}^{N} \alpha_j \Phi(\cdot - \boldsymbol{x}_j), \tag{2.3.20}$$

i.e., $s_{\lambda,f} \in \operatorname{span}\{\Phi(\cdot - \boldsymbol{x}_j) \,:\, \boldsymbol{x}_j \in X\}$.

Proof. Existence and uniqueness follows primarily from the convexity of the functional $J_{\lambda,f}$ and the strict convexity of the native space norm. The representation of the minimizer can be proved by splitting $\mathcal{N}_\Phi(\Omega)$ into $\operatorname{span}\{\Phi(\cdot - \boldsymbol{x}_j) \,:\, \boldsymbol{x}_j \in X\}$ and its orthogonal complement.

Details of the proof can be found in, e.g., [86]. □

The representation of the minimizer directly leads to the way to compute the coefficient vector $\boldsymbol{\alpha}$. Inserting the representation (2.3.20) into the single parts of $J_{\lambda,f}$ yields on the one hand

$$\begin{aligned}
\|s_{\lambda,f}\|^2_{\mathcal{N}_\Phi(\Omega)} &= \left\langle \sum_{j=1}^{N} \alpha_j \Phi(\cdot - \boldsymbol{x}_j), \sum_{i=1}^{N} \alpha_i \Phi(\cdot - \boldsymbol{x}_i) \right\rangle_{\mathcal{N}_\Phi(\Omega)} \\
&= \sum_{j=1}^{N} \sum_{i=1}^{N} \alpha_i \alpha_j \left\langle \Phi(\cdot - \boldsymbol{x}_j), \Phi(\cdot - \boldsymbol{x}_i) \right\rangle_{\mathcal{N}_\Phi(\Omega)} \\
&= \sum_{i,j=1}^{N} \alpha_i \alpha_j \Phi(\boldsymbol{x}_j - \boldsymbol{x}_i) = \boldsymbol{\alpha}^{\mathrm{T}} M_{X,\Phi} \boldsymbol{\alpha},
\end{aligned}$$

and on the other hand

$$\sum_{j=1}^{N} |s_{\lambda,f}(\boldsymbol{x}_j) - f(\boldsymbol{x}_j)|^2 = (\boldsymbol{f} - M_{X,\Phi}\boldsymbol{\alpha})^{\mathrm{T}} (\boldsymbol{f} - M_{X,\Phi}\boldsymbol{\alpha}),$$

where we again used the kernel matrix $M_{X,\Phi} = (\Phi(\boldsymbol{x}_i - \boldsymbol{x}_j))$. Combining these two terms and taking the derivative with respect to $\boldsymbol{\alpha}$ leads to the linear system

$$(M_{X,\Phi} + \lambda I)\boldsymbol{\alpha} = \boldsymbol{f},$$

where I denotes the $N \times N$ identity matrix and $\boldsymbol{f}$ is again given componentwise by $\boldsymbol{f}_j = f(\boldsymbol{x}_j)$, $1 \le j \le N$.

Next, we consider kernel functions Φ that are reproducing kernels of $H^s(\mathbb{R}^n)$ and derive error estimates similar to those in Corollary 2.3.11 and Theorem 2.3.16. Again, we use the sampling inequalities of Theorem 2.3.4, but this time, we do not have $(f - s_{\lambda,f})|_X = 0$. However, we can still control this error with the help of the following two preliminary estimates.

Proposition 2.3.20. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies* (2.2.2). *Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ be a set of sites with sufficiently small fill distance $h_{X,\Omega}$. Assume that $f \in H^s(\Omega)$. Then the*

following two estimates hold for the unique minimizer $s_{\lambda,f}$ of the penalized least-squares problem in Definition 2.3.18:

$$\|s_{\lambda,f}\|_{\mathcal{N}_\Phi(\Omega)} \leq \|f\|_{\mathcal{N}_\Phi(\Omega)}$$

and

$$|f(\boldsymbol{x}_i) - s_{\lambda,f}(\boldsymbol{x}_i)| \leq \sqrt{\lambda}\|f\|_{\mathcal{N}_\Phi(\Omega)}, \quad \boldsymbol{x}_i \in X.$$

PROOF. We know that the interpolant $I_{X,\Phi}f$ of f satisfies

$$\sum_{j=1}^{N} |f(\boldsymbol{x}_j) - I_{X,\Phi}f(\boldsymbol{x}_j)|^2 = 0 \quad \text{and} \quad \|I_{X,\Phi}f\|_{\mathcal{N}_\Phi(\Omega)} \leq \|f\|_{\mathcal{N}_\Phi(\Omega)}.$$

This leads directly to

$$\begin{aligned}
\max\left\{|f(\boldsymbol{x}_i) - s_{\lambda,f}(\boldsymbol{x}_i)|^2, \lambda\|s_{\lambda,f}\|^2_{\mathcal{N}_\Phi(\Omega)}\right\} \leq \\
&\leq \sum_{j=1}^{N} |f(\boldsymbol{x}_j) - s_{\lambda,f}(\boldsymbol{x}_j)|^2 + \lambda\|s_{\lambda,f}\|^2_{\mathcal{N}_\Phi(\Omega)} \\
&\leq \sum_{j=1}^{N} |f(\boldsymbol{x}_j) - I_{X,\Phi}f(\boldsymbol{x}_j)|^2 + \lambda\|I_{X,\Phi}f\|^2_{\mathcal{N}_\Phi(\Omega)} \\
&\leq \lambda\|f\|^2_{\mathcal{N}_\Phi(\Omega)}.
\end{aligned}$$

□

With these estimates the proof of the next theorem is straight forward.

Theorem 2.3.21. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \Omega$ be a set of sites with sufficiently small fill distance $h_{X,\Omega}$. Assume that $f \in H^s(\Omega)$. Then there is a constant $C > 0$ such that the error estimates for the unique minimizer $s_{\lambda,f}$ of the penalized least-squares problem*

$$\|f - s_{\lambda,f}\|_{L_\infty(\Omega)} \leq C\left(h_{X,\Omega}^{s-\frac{n}{2}} + \sqrt{\lambda}\right)\|f\|_{H^s(\Omega)},$$

and

$$\|f - s_{\lambda,f}\|_{L_2(\Omega)} \leq C\left(h_{X,\Omega}^{s} + \sqrt{\lambda}\right)\|f\|_{H^s(\Omega)},$$

hold.

PROOF. The proof is an easy application of the sampling inequalities of Theorem 2.3.4 with the help of the estimates of Proposition 2.3.20. □

2.4. Rescaled Compactly Supported RBFs

The preceding subsections show that we can expect convergence of the approximation error as $h_{X,\Omega}$ approaches zero in different settings. Although similar results hold for other radial basis functions we are mainly interested in the compactly supported Wendland functions introduced in Definition 2.2.18. The support is, by definition, the closed unit ball in $\mathbb{R}^n$, i.e.,

$$\operatorname{supp}\Phi_{n,k} = \overline{B_1(\mathbf{0})}.$$

From a numerical point of view, the main difficulty in all approximation methods introduced above is the solution of a linear system, either with matrix $M_{X,\Phi} = (\Phi(\boldsymbol{x}_i - \boldsymbol{x}_j))_{1\le i,j\le N}$, or with matrix $(M_{X,\Phi} + \lambda I)$. In both cases the matrix is sparse, however the number of non-zero entries in the j-th row depends on how many sampling points $\boldsymbol{x}_i$ are in the support of $\Phi_{n,k}(\cdot - \boldsymbol{x}_j)$. As $h_{X,\Omega}$ gets small more and more points will be in this support and the matrix fills more and more. This makes working with a fixed support radius unattractive and we now introduce a way to rescale the RBF such that only a constant number of sites are in the support of each translate of the kernel. This number can even be made independent of $h_{X,\Omega}$. The idea of rescaling can also be applied to other RBFs.

Definition 2.4.1. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a radial basis function and let $\delta > 0$ be a* scaling parameter. *Then we define the* rescaled radial basis function *$\Phi_\delta : \mathbb{R}^n \to \mathbb{R}$ by*

$$\Phi_\delta(\boldsymbol{x}) = \delta^{-n}\Phi\left(\frac{\boldsymbol{x}}{\delta}\right), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

If the support of the kernel Φ is the closed unit ball $\overline{B_1(\mathbf{0})}$, then the support of the rescaled kernel satisfies

$$\operatorname{supp}\Phi_\delta = \overline{B_\delta(\mathbf{0})}.$$

Furthermore, if the Fourier transform of the basis kernel Φ exhibits algebraic decay, i.e., Φ satisfies (2.2.2) with constants c_1, c_2 and exponent $-s$, then the Fourier transform of the rescaled RBF satisfies

$$c_1(1+\delta^2\|\boldsymbol{\xi}\|_2^2)^{-s} \le \widehat{\Phi_\delta}(\boldsymbol{\xi}) \le c_2(1+\delta^2\|\boldsymbol{\xi}\|_2^2)^{-s}, \quad \boldsymbol{\xi} \in \mathbb{R}^n, \tag{2.4.1}$$

with the same constants c_1 and c_2.

This allows us to show the following norm-equivalence of the native space $\mathcal{N}_{\Phi_\delta}(\mathbb{R}^n)$ of the rescaled RBF and the Sobolev space $H^s(\mathbb{R}^n)$. To avoid cluttered notation, we will from here on write $\|\cdot\|_{\Phi_\delta}$ for the norm of the native space $\mathcal{N}_{\Phi_\delta}(\mathbb{R}^n)$. The proof of the next lemma can be found in, e.g., [91].

Lemma 2.4.2. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies* (2.2.2). *For $\delta \in (0,1]$ let Φ_δ be defined by $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$. Then Φ_δ is also a reproducing kernel of $H^s(\mathbb{R}^n)$ and the norm equivalence*

$$\sqrt{c_1}\|g\|_{\Phi_\delta} \le \|g\|_{H^s(\mathbb{R}^n)} \le \sqrt{c_2}\delta^{-s}\|g\|_{\Phi_\delta} \tag{2.4.2}$$

holds for every $g \in H^s(\mathbb{R}^n)$.

We see that Φ_δ is also a reproducing kernel of $H^s(\mathbb{R}^n)$. That means that the space itself is independent of the scaling parameter $\delta \in (0,1]$. This is no contradiction to the uniqueness of the reproducing kernel since the norm of the native space $\mathcal{N}_{\Phi_\delta}(\mathbb{R}^n)$ depends on δ. In (2.4.2) we also see that the equivalence constants depend on the scaling parameter.

While we always get the lower bound $\|g\|_{\Phi_\delta} \le C\|g\|_{H^s(\mathbb{R}^n)}$ with $C > 0$ independent of δ, we have for the upper estimate $\|g\|_{H^s(\mathbb{R}^n)} \le c\delta^{-s}\|g\|_{\Phi_\delta}$. Here the equivalence constant $c\delta^{-s}$ tends to infinity as $\delta \to 0$. This can be

explained by the fact that $\|\cdot\|_{\Phi_\delta} \to \|\cdot\|_{L_2(\mathbb{R}^n)}$ as δ approaches zero and the constant has to balance this loss of derivative in the norm [92].

Additionally, we immediately have the following estimates for the norms of the scaled kernel.

Lemma 2.4.3. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2), and let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel with scaling parameter $\delta \in (0, 1]$. Then the following estimate holds for $0 \le t \le s$*

$$\|\Phi_\delta\|_{H^t(\mathbb{R}^n)} \le \delta^{-\frac{n}{2}-t}\|\Phi\|_{H^t(\mathbb{R}^n)}. \tag{2.4.3}$$

Additionally, we have

$$\|\Phi_\delta\|_{L_\infty(\mathbb{R}^n)} \le \delta^{-n}\|\Phi\|_{L_\infty(\mathbb{R}^n)}. \tag{2.4.4}$$

and, for $0 < \delta' < \delta$,

$$\|\Phi_\delta\|_{\Phi_{\delta'}} \le C(\Phi)\delta^{-\frac{n}{2}}. \tag{2.4.5}$$

Proof. We only show (2.4.5). Using (2.4.1) for Φ_δ and $\Phi_{\delta'}$, respectively, we have

$$\widehat{\Phi_\delta}(\boldsymbol{\xi}) \le c_2(1 + \delta^2\|\boldsymbol{\xi}\|_2^2)^{-s},, \quad \boldsymbol{\xi} \in \mathbb{R}^n,$$

and

$$\widehat{\Phi_{\delta'}}(\boldsymbol{\xi}) \ge c_1(1 + (\delta')^2\|\boldsymbol{\xi}\|_2^2)^{-s}, \quad \boldsymbol{\xi} \in \mathbb{R}^n.$$

These estimates give for $\delta' < \delta$ the upper bound

$$\frac{\widehat{\Phi_\delta}(\boldsymbol{\xi})}{\widehat{\Phi_{\delta'}}(\boldsymbol{\xi})} \le \frac{c_2}{c_1}\frac{(1 + \delta^2\|\boldsymbol{\xi}\|_2^2)^{-s}}{(1 + (\delta')^2\|\boldsymbol{\xi}\|_2^2)^{-s}} \le \frac{c_2}{c_1}.$$

This, together with the definition of the native space norm in Theorem 2.2.10, yields

$$\begin{aligned}\|\Phi_\delta\|_{\Phi_{\delta'}} &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \left(\frac{|\widehat{\Phi_\delta}(\boldsymbol{\xi})|^2}{\widehat{\Phi_{\delta'}}(\boldsymbol{\xi})}\, d\boldsymbol{\xi}\right)^{\frac{1}{2}} \\ &\le \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} (2\pi)^{-\frac{n}{2}} \left(\int_{\mathbb{R}^n} \widehat{\Phi_\delta}(\boldsymbol{\xi})\, d\boldsymbol{\xi}\right)^{\frac{1}{2}} = C(\Phi)\delta^{-\frac{n}{2}}.\end{aligned}$$

□

The results of Lemma 2.4.3 hold also for translated kernels $\Phi(\cdot - \boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^n$.

We motivated the use of rescaled kernels by a loss of sparseness of the kernel matrix $M_{X,\Phi}$ for data sites $X \subset \Omega$ with small $h_{X,\Omega}$. To avoid this we now couple the scaling parameter δ to the fill distance $h_{X,\Omega}$ in the following way: For fixed *overlap parameter* $\nu > 1$ and constant $c_\nu \in (0, 1)$ we choose the scaling parameter δ such that

$$c_\nu \nu h_{X,\Omega} \le \delta \le \nu h_{X,\Omega} \tag{2.4.6}$$

holds. This setting is generally called *stationary*. By varying ν we can control how many sites $\boldsymbol{x}_i$ are in the support of the rescaled kernel $\Phi_\delta(\cdot - \boldsymbol{x}_j)$, centered at $\boldsymbol{x}_j$. This number can be bounded by a constant independent of N, the number of sites.

Lemma 2.4.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Let $X \subseteq \Omega$ be a quasi-uniform set of sites with fill distance $h_{X,\Omega}$ and separation radius q_X. Let $\delta > 0$ be coupled to $h_{X,\Omega}$ as in (2.4.6) with overlap parameter $\nu > 1$. Then the estimate*

$$\# \left(X \cap \overline{B_\delta(\boldsymbol{x}_j)} \right) \leq (1 + \nu c_{qu})^n \tag{2.4.7}$$

holds for every $\boldsymbol{x}_j \in X$.

Proof. The claim follows from a standard comparison of volumes. For fixed $\boldsymbol{x}_j \in X$ we have the inclusion

$$\bigcup_{\boldsymbol{x}_i \in X \cap \overline{B_\delta(\boldsymbol{x}_j)}} B_{q_X}(\boldsymbol{x}_i) \subseteq B_{\delta + q_X}(\boldsymbol{x}_j).$$

Next we compare the volumes of the two sets. We see that, by the definition of the separation radius, the balls $B_{q_X}(\boldsymbol{x}_i)$ with radius q_X centered in every $\boldsymbol{x}_i \in X \cap \overline{B_\delta(\boldsymbol{x}_j)}$ are disjoint. This yields

$$\begin{aligned} \operatorname{vol}\left(\bigcup_{\boldsymbol{x}_i \in X \cap \overline{B_\delta(\boldsymbol{x}_j)}} B_{q_X}(\boldsymbol{x}_i) \right) &= \sum_{\boldsymbol{x}_i \in X \cap \overline{B_\delta(\boldsymbol{x}_j)}} \operatorname{vol}(B_{q_X}(\boldsymbol{x}_i)) \\ &= \# \left(X \cap \overline{B_\delta(\boldsymbol{x}_j)} \right) \operatorname{vol}(B_1(\mathbf{0})) q_X^n \\ &\leq \operatorname{vol}(B_{\delta + q_X}(\boldsymbol{x}_j)) = \operatorname{vol}(B_1(\mathbf{0}))(\delta + q_X)^n. \end{aligned}$$

Using the upper bound on δ in (2.4.6) and the assumption that X is quasi-uniform yields

$$\# \left(X \cap \overline{B_\delta(\boldsymbol{x}_j)} \right) \leq \frac{(\delta + q_X)^n}{q_X^n} \leq (1 + \nu c_{qu})^n.$$

□

From a numerical point of view using the rescaled kernel Φ_δ is advantageous. First, we have seen in the previous sections that we have to solve a linear system in order to compute either the interpolant or the penalized least-squares approximation. If we use the rescaled kernel the matrix now becomes $M_{X,\Phi_\delta} = (\Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_j))_{1 \leq i,j \leq N}$, which is, by Lemma 2.4.4, a sparse matrix whose sparseness we can control by varying the overlap parameter ν. We will study this matrix in more detail in Section 2.5. There we will be particularly focused on its condition number and norms of its inverse.

The second advantage is that, for example in the case of interpolation, the solution of the approximation problem can be written as

$$I_{X,\Phi_\delta} f(\boldsymbol{x}) = \sum_{i=1}^{N} \alpha_i \Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_i) = \sum_{i \,:\, \|\boldsymbol{x} - \boldsymbol{x}_i\| \leq \delta} \alpha_i \Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_i), \quad \boldsymbol{x} \in \Omega.$$

This means that, if the relevant indices are known, point-evaluations of the interpolant can be done in constant time, independent of the number of sites.

However, a negative consequence is that in the stationary setting we cannot expect convergence of the approximation process. The following theorem, taken from [92], handles the case of interpolation, however the same holds true for the other approximation schemes.

Theorem 2.4.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel with scaling parameter $\delta < 1$. Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ be a set of sites. Then there exists a constant $C > 0$ such that*

$$\|f - I_{X,\Phi_\delta} f\|_{L_2(\Omega)} \leq C \left(\frac{h_{X,\Omega}}{\delta}\right)^s \|f\|_{H^s(\Omega)}$$

holds for all $f \in H^s(\Omega)$.

If we couple δ and $h_{X,\Omega}$ as in (2.4.6), we see that we can not expect convergence of the method for $h_{X,\Omega} \to 0$.

We will introduce a method to overcome this *trade-off principle* in Chapter 4.

2.5. The Kernel Matrix M_{X,Φ_δ}

We now study the properties of the kernel matrix M_{X,Φ_δ} of the rescaled, compactly supported RBF Φ_δ in more detail. The results given here are mostly taken from [88].

We assume that Φ_δ is a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$. Additionally, we assume that X is quasi-uniform, i.e., the fill distance $h_{X,\Omega}$ and the separation radius q_X are coupled according to (2.3.3). Furthermore, we assume that the scaling parameter δ is coupled to $h_{X,\Omega}$ according to (2.4.6) with overlap parameter $\nu > 1$. For this subsection we omit the indices of the matrix and use the notation

$$M := M_{X,\Phi_\delta}.$$

The first properties are straight forward and follow directly from the properties of the kernel Φ_δ.

Corollary 2.5.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $X \subseteq \Omega$ be a quasi-uniform set of sites with fill distance $h_{X,\Omega}$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ as in (2.4.6) with overlap parameter $\nu > 1$. Then the following statements hold:*

1. *The kernel matrix $M = (\Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_j))$ is a symmetric, positive definite and sparse matrix.*
2. *The number of non-zero entries per row can be bounded from above by $(1 + \nu c_{qu})^n$.*

We emphasize that M is invertible and the next goal is to investigate this inverse M^{-1}. We start by estimating the to the 2-norm associated matrix norm. To do this we give bounds on the smallest eigenvalue λ_{min} and the largest eigenvalue λ_{max} of M separately. The proofs of these classical results can be found in, e.g., [64, 74] for the estimate on λ_{min} and in [88] for the estimate on λ_{max}.

Theorem 2.5.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $X \subseteq \Omega$ be a quasi-uniform set of sites with fill distance $h_{X,\Omega}$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform*

of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ as in (2.4.6) with overlap parameter $\nu > 1$. Let the matrix M be defined by $M_{ik} = \Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_k)$, $1 \le i,k \le N$. Then there is a constant $C = C(\Phi, n, s)$ such that the bound for the smallest eigenvalue λ_{min} of M

$$\lambda_{min} \ge C(\nu c_{qu})^{n-2s}\delta^n$$

holds. The largest eigenvalue λ_{max} of M can be bounded from above by

$$\lambda_{max} \le (1 + \nu c_{qu})^n \delta^{-n}\Phi(\mathbf{0}).$$

These bounds directly yield estimates on the 2-norm of M^{-1} and on the associated condition number $\mathrm{cond}_2(M)$ of M.

Corollary 2.5.3. *With the notation and assumptions of Theorem 2.5.2 we have*

$$\|M^{-1}\|_2 \le C(\nu c_{qu})^{2s-n}\delta^{-n}.$$

We can then estimate the corresponding condition number $\mathrm{cond}_2(M)$ *by*

$$\mathrm{cond}_2(M) \le C(\nu c_{qu})^{2s-n}(1 + \nu c_{qu})^n. \tag{2.5.1}$$

The bound in (2.5.1) shows in particular that the condition number of M is independent of number of sampling points N and the scaling parameter δ.

Next, we derive an estimate on the absolute value of the entries of M^{-1}, denoted by M^{-1}_{ik}. To do so, we have to interpret M as a special kind of banded matrix. The kernel matrix M is in general not a banded matrix in the traditional sense and the structure of non-zero entries is highly dependent on the order of the centers in X. That can be seen with an easy one-dimensional example. Suppose that $X = \{x_1, x_2, x_3, x_4\} = \{0, 1/3, 2/3, 1\}$ and that $\widetilde{X} = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4\} = \{0, 1, 1/3, 2/3\}$. Note that X and $\widetilde{X}$ contain the same sites, we only changed the order. Take $\Phi = \phi_{1,0}$, the one-dimensional Wendland function with $k = 0$, see Table 1 for the precise form of the function. We rescale Φ with $\delta = 0.4$. This means that $\Phi_{0.4}$ is a hat function with at most three sites in the support of $\Phi_{0.4}(\cdot - x_i)$ and $\Phi_{0.4}(\cdot - \widetilde{x}_i)$, $1 \le i \le 4$, respectively. We can easily see that the kernel matrices for X and $\widetilde{X}$ then have the form

$$M_{X,\Phi_{0.4}} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \quad \text{and} \quad M_{\widetilde{X},\Phi_{0.4}} = \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \end{pmatrix},$$

where $*$ denotes an non-zero entry. This means that a simple permutation in the set of sites leads to a completely different structure of the kernel matrix.

However, if we change the indexing of M from pairs $(i,k) \in \mathbb{N}^2$ to pairs of multi-indices $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Z}^n \times \mathbb{Z}^n$, where n denotes the dimension of the ambient space $\Omega \subseteq \mathbb{R}^n$, we are able to use the geometric structure of the points in Ω, rather than the order of indexing in X.

Definition 2.5.4. *We call a mapping $\mathcal{M} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}$ with finite support a* multivariate matrix.

Additionally, we have the following natural extensions of terms for classical to multivariate matrices, see [7, Definition 3.2].

Definition 2.5.5. *We call a multivariate matrix* $\mathcal{M} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}$

(1) symmetric*, if* $\mathcal{M}(\boldsymbol{i}, \boldsymbol{k}) = \mathcal{M}(\boldsymbol{k}, \boldsymbol{i})$*, for all* $\boldsymbol{i}, \boldsymbol{k} \in \mathbb{Z}^n$*,*

(2) positive definite*, if*

$$\sum_{\boldsymbol{i},\boldsymbol{k}\in\mathbb{Z}^n} \alpha_{\boldsymbol{i}}\alpha_{\boldsymbol{k}}\mathcal{M}(\boldsymbol{i}, \boldsymbol{k}) > 0$$

for every finitely supported, non-zero sequence $(\alpha_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n} \subset \mathbb{R}$ *such that the support of the multivariate matrix* $(\alpha_{\boldsymbol{i}}\alpha_{\boldsymbol{k}})_{\boldsymbol{i},\boldsymbol{k}\in\mathbb{Z}^n}$ *is contained in the support of* $\mathcal{M}$*.*

Before we study these multivariate matrices in more detail we demonstrate how we transform the kernel matrix $M = (\Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_k))$ into a multivariate matrix $\mathcal{M}$. The idea is to assign to each $\boldsymbol{x}_i \in X$ a unique multi-index $\boldsymbol{y}_i \in \mathbb{Z}^n$ such that

$$\mathcal{M}(\boldsymbol{y}_i, \boldsymbol{y}_k) = \Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_k), \quad 1 \le i, k \le N.$$

It is easy to see that, if such an identification is possible, the multivariate matrix is symmetric and positive definite.

To achieve this transformation we will make use of the following lemma, taken from [7, Lemma 3.7].

Lemma 2.5.6. *Let* $\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_N\}$ *be any subset of* $\mathbb{R}^n$ *such that* $\|\boldsymbol{z}_i - \boldsymbol{z}_k\|_2 \ge \sqrt{n}$ *for* $i \ne k$*. Define for* $1 \le i \le N$*,*

$$\boldsymbol{y}_i := \left(\lfloor z_{i,1} \rfloor, \dots, \lfloor z_{i,n} \rfloor\right)^T, \tag{2.5.2}$$

where $z_{i,m}$ *denotes the* m*-th component of* $\boldsymbol{z}_i$*.*

Then the points $\{\boldsymbol{y}_i\}_{1\le i\le N}$ *are all different. Further, if* $\|\boldsymbol{y}_i - \boldsymbol{y}_k\|_2 \ge R$ *and* $R \ge 4\sqrt{n}$*, then*

$$\|\boldsymbol{z}_i - \boldsymbol{z}_k\|_2 \ge \frac{R}{2}.$$

From the definition of the separation radius of X we have $\frac{1}{2}\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2 \ge q_X$, for every $i \ne k$. If we set

$$\boldsymbol{z}_i := \frac{\sqrt{n}}{2q_X}\boldsymbol{x}_i, \quad 1 \le i \le N, \tag{2.5.3}$$

we have $\|\boldsymbol{z}_i - \boldsymbol{z}_k\|_2 \ge \sqrt{n}$. Defining $\boldsymbol{y}_i$ as in (2.5.2) then yields that these $\boldsymbol{y}_i$ are pairwise distinct. These are indeed the multi-indices required for the identification of M as a multivariate matrix $\mathcal{M}$.

Next we show that this $\mathcal{M}$ satisfies the following definition.

Definition 2.5.7. *A multivariate matrix* $\mathcal{M} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}$ *is called* R-banded *with* $R > 0$*, if* $\mathcal{M}(\boldsymbol{y}_i, \boldsymbol{y}_k) = 0$ *if* $\|\boldsymbol{y}_i - \boldsymbol{y}_k\|_2 > R$*.*

We reiterate that in our case R-bandedness is preferable to classical bandedness of matrices since it describes the geometry of the points in Ω rather than the order in X. We now show that the multivariate matrix $\mathcal{M}$ which is derived from the kernel matrix M is indeed R-banded, see also [88].

Theorem 2.5.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $X \subseteq \Omega$ be a quasi-uniform set of sites with fill distance $h_{X,\Omega}$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ as in (2.4.6) with overlap parameter $\nu > 1$. Let the matrix M be defined by $M_{ik} = \Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_k)$, $1 \le i, k \le N$.*

Then the multivariate matrix $\mathcal{M}$ derived from the kernel matrix M is R-banded with

$$R = \max(\nu c_{qu}, 4)\sqrt{n}.$$

PROOF. For fixed $\boldsymbol{x}_i \in X$ and associated $\boldsymbol{z}_i \in \mathbb{R}^n$ as in (2.5.3) we set $\boldsymbol{y}_i \in \mathbb{Z}^n$ as in (2.5.2).

We need to show that if $\|\boldsymbol{y}_i - \boldsymbol{y}_k\|_2 \ge R$ we have $\mathcal{M}(\boldsymbol{y}_i, \boldsymbol{y}_k) = 0$. With the relations of $\boldsymbol{y}_i$ and $\boldsymbol{x}_i$ we see that it suffices to show that $\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2 \ge \delta$. Then the compact support of Φ_δ implies

$$0 = \Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_k) = \mathcal{M}(\boldsymbol{y}_i, \boldsymbol{y}_k).$$

Clearly, with Lemma 2.5.6, (2.5.3), (2.5.2) and the choice of R we have

$$\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2 = \frac{2q_X}{\sqrt{n}}\|\boldsymbol{z}_i - \boldsymbol{z}_k\|_2 \ge \frac{2q_X}{\sqrt{n}}\frac{R}{2} \ge \frac{2q_X}{\sqrt{n}}\frac{c_\gamma c_{qu}\sqrt{n}}{2} \ge \delta.$$

This finishes the proof. □

This allows us to use the following theorem and derive an estimate on $|M_{ik}^{-1}|$. This result is taken from [7] and is a direct extension of the respective result for symmetric, positive definite matrices to symmetric, positive definite multivariate matrices.

Theorem 2.5.9. *Let $\mathcal{M} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}$ be a symmetric, positive definite multivariate matrix. Assume that $\mathcal{M}$ is R-banded with $R > 0$. Then*

$$|\mathcal{M}^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})| \le 2\|\mathcal{M}^{-1}\|_2 \widetilde{\eta}^{\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|_2}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}^n, \tag{2.5.4}$$

where

$$\widetilde{\eta} = \left(\frac{\sqrt{\mathrm{cond}_2(\mathcal{M})} - 1}{\sqrt{\mathrm{cond}_2(\mathcal{M})} + 1}\right)^{\frac{1}{R}}. \tag{2.5.5}$$

For the kernel matrix M we then have the following estimate.

Theorem 2.5.10. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $X \subseteq \Omega$ be a quasi-uniform set of sites with fill distance $h_{X,\Omega}$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ as in (2.4.6) with overlap parameter $\nu > 1$. Let the matrix M be defined by $M_{ik} = \Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_k)$, $1 \le i, k \le N$. Then there exists a constant $C > 0$ such that estimate*

$$|M_{ik}^{-1}| \le C q_X^n e^{-\eta \frac{\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2}{q_X}} \tag{2.5.6}$$

holds with

$$\eta = -\frac{1}{2\max(\nu c_{qu}, 4)\sqrt{n}} \log\left(\frac{\sqrt{\mathrm{cond}_2(M)} - 1}{\sqrt{\mathrm{cond}_2(M)} + 1}\right). \tag{2.5.7}$$

PROOF. With $\boldsymbol{y}_i$ and $\boldsymbol{z}_i$ as in (2.5.2) and (2.5.3), respectively, we have, together with the relation $||\lfloor a \rfloor - \lfloor b \rfloor| \geq |a-b| - 1$ for $a, b \in \mathbb{R}$ the estimate

$$\begin{aligned}\|\boldsymbol{y}_i - \boldsymbol{y}_k\|_2 &\geq \frac{1}{\sqrt{n}}\|\boldsymbol{y}_i - \boldsymbol{y}_k\|_1 \geq \frac{1}{\sqrt{n}}\|\boldsymbol{z}_i - \boldsymbol{z}_k\|_2 - \sqrt{n} \\ &= \frac{1}{2q_X}\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2 - \sqrt{n}\end{aligned}$$

for $i \neq k$. Furthermore, we have with Theorem 2.5.8

$$\begin{aligned}|M_{ik}^{-1}| = |\mathcal{M}^{-1}(\boldsymbol{y}_i, \boldsymbol{y}_k)| &\leq 2\|M^{-1}\|_2 \widetilde{\eta}^{\|\boldsymbol{y}_i - \boldsymbol{y}_k\|_2} \\ &\leq C2(c_\gamma c_{qu})^{2s-n}\delta^{-n}\widetilde{\eta}^{\frac{1}{2}\frac{\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2}{q_X}}\widetilde{\eta}^{-\sqrt{n}} \\ &\leq Cq_X^n e^{-\eta\frac{\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2}{q_X}}\end{aligned}$$

with $\eta = -\log\widetilde{\eta} > 0$. □

This theorem will prove to be an essential tool in the next chapter. Another result we need in the next chapter is the following, uniform bound on the entries of the inverse of M. The proof follows the same ideas as the ones that lead to an analguous result in [46, Section 3.1].

Lemma 2.5.11. *With the notation and assumptions of Theorem 2.5.10 there is a constant $C > 0$ such that the estimate*

$$\left|M_{ik}^{-1}\right| \leq Cq_X^{n-2s} \tag{2.5.8}$$

holds for all $1 \leq i, k \leq N$.

We remark that the bound in (2.5.8) also holds for kernel matrices on subsets of X, i.e., if $\widetilde{X} \subseteq X$ and we set $\widetilde{M} = (\Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_k))_{\boldsymbol{x}_i, \boldsymbol{x}_k \in \widetilde{X}}$ then

$$\left|\widetilde{M}_{i,k}^{-1}\right| \leq Cq_X^{n-2s} \tag{2.5.9}$$

holds with a potentially different constant $C > 0$.

We note that the estimates in (2.5.8) and (2.5.9) still depend on the scaling parameter δ, although this dependence is implicit since we assume that the set of sites X is quasi-uniform and we couple δ to the fill distance $h_{X,\Omega}$ and hence, the occurring terms that involve the scaling parameter can be rephrased into terms of the separation radius q_X.

CHAPTER 3

Lagrange Functions and their Localization

In Section 2.3 we introduced the kernel based approximation space as the set of linear combinations of the translates of a chosen (rescaled) RBF. This yields the most obvious basis for this space, however, the question arises if it is the best basis. The question has been the topic of several papers, see, e.g., [8, 26]. In this chapter we derive different bases for the approximation spaces of rescaled Wendland kernels. We start with studying full Lagrange functions in Section 3.1 and prove that they are a stable basis. Then we introduce intermediate functions in Section 3.2. These, too, are an alternative basis for the approximation space. Finally, in Section 3.3, we introduce and study localized Lagrange functions. These are functions that satisfy the Lagrange condition only on a subset of the set of sites but can be shown to be also a basis. This chapter is motivated by [29, 46], where similar results are derived for different kernel functions. The proofs in Section 3.1 are taken from [20] and [93], an internal communication paper.

3.1. Full Lagrange Functions

We start by deriving and investigating a new basis for the kernel-based approximation space

$$V_N = \operatorname{span}\{\Phi_\delta(\cdot - \boldsymbol{x}) \ : \ \boldsymbol{x} \in X\}$$

of the rescaled, compactly supported RBF introduced in Section 2.4. The first corollary we obtain is straight-forward, since the kernel matrix is positive definite.

Corollary 3.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ be a quasi-uniform set of sites in Ω with fill distance $h_{X,\Omega}$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ as in (2.4.6). Let $M = (\Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_k))_{1\le i,k\le N} \in \mathbb{R}^{N\times N}$ be the kernel matrix. Then the linear system*

$$M \begin{pmatrix} \chi_1(\boldsymbol{x}) \\ \vdots \\ \chi_N(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_1) \\ \vdots \\ \Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_N) \end{pmatrix} \tag{3.1.1}$$

has for every $\boldsymbol{x} \in \mathbb{R}^n$ a unique solution $(\chi_1(\boldsymbol{x}), \dots, \chi_N(\boldsymbol{x}))^T \in \mathbb{R}^N$.

The solution vector $(\chi_1(\boldsymbol{x}),\dots,\chi_N(\boldsymbol{x}))^{\mathrm{T}}$ is given point-wise. We can use this to define for $1 \le k \le N$ functions $\chi_k : \mathbb{R}^n \to \mathbb{R}$ by

$$\chi_k = \sum_{i=1}^{N} M_{ki}^{-1} \Phi_\delta(\cdot - \boldsymbol{x}_i), \tag{3.1.2}$$

where M_{ik}^{-1} denotes the entries of the inverse of the kernel matrix M. This already yields that $\chi_k \in V_N$. Additionally, if we investigate the system in (3.1.1) closer we see that, if we set $\boldsymbol{x} = \boldsymbol{x}_1$, an immediate solution is the vector

$$\begin{pmatrix} \chi_1(\boldsymbol{x}_1) \\ \chi_2(\boldsymbol{x}_1) \\ \vdots \\ \chi_N(\boldsymbol{x}_1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and since the solution to (3.1.1) is unique this allows us to derive the defining condition of the functions χ_k.

Corollary 3.1.2. *The functions $\chi_k \in V_N$ defined by* (3.1.2) *satisfy for all $\boldsymbol{x}_i \in X$ the* Lagrange condition

$$\chi_k(\boldsymbol{x}_i) = \delta_{ki} := \begin{cases} 1, & k = i, \\ 0, & k \neq i. \end{cases}$$

Indeed, the set of these functions is our candidate for the desired alternative basis of V_N.

Definition 3.1.3. *We call the function $\chi_k \in V_N$, $1 \le k \le N$, defined by* (3.1.2) Lagrange *or* cardinal function. *We call the point $\boldsymbol{x}_k \in X$ the* anchor *of χ_k.*

We now need to check that the Lagrange functions are a basis.

Lemma 3.1.4. *With the notation and assumptions of Corollary 3.1.1 the set of Lagrange functions $\{\chi_k\}_{1\le k\le N}$ is a basis of V_N.*

Proof. We only have to show linear independence, i.e.,

$$\sum_{k=1}^{N} \alpha_k \chi_k(\boldsymbol{x}) = 0 \tag{3.1.3}$$

for every $\boldsymbol{x} \in \mathbb{R}^n$ implies $\alpha_k = 0$ for every $1 \le k \le N$. This can be seen by setting $\boldsymbol{x} = \boldsymbol{x}_i$ and using the Lagrange condition of Corollary 3.1.2 in (3.1.3). This leads to $\alpha_i = 0$. Repeating this for every $1 \le i \le N$ leads to the linear independence of the set $\{\chi_k\}_{1\le k\le N}$. □

Often we call $\{\chi_k\}_{1\le k\le N}$ the *Lagrange basis* of V_N. We can now use the properties of the kernel matrix M, in particular the exponential decay of the entries of its inverse M^{-1}, (2.5.6), to derive further properties of the Lagrange functions. First, we see that the Lagrange functions, too, decay exponentially away from their anchor. This behavior is often called *local*, see, e.g., [29, 45, 46]. The proof of the following theorem is taken from [20]. However, since this is one of the key estimates that enables us to derive the results in Sections 3.2 and 3.3, we repeat the proof here.

Theorem 3.1.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ be a quasi-uniform set of sites in Ω with fill distance $h_{X,\Omega}$ and separation radius q_X. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ as in (2.4.6). Let $M = (\Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_k))_{1\le i,k\le N} \in \mathbb{R}^{N\times N}$ be the kernel matrix. For $1 \le k \le N$ let χ_k be given by*

$$\chi_k = \sum_{i=1}^{N} M^{-1}_{ik}\Phi_\delta(\cdot - \boldsymbol{x}_i).$$

Then there is a constant $C = C(\Phi, n, s, \eta) > 0$ such that, for every $1 \le k \le N$,

$$|\chi_k(\boldsymbol{x})| \le Ce^{-\eta\frac{\|\boldsymbol{x}-\boldsymbol{x}_k\|_2}{q_X}}, \quad \boldsymbol{x} \in \mathbb{R}^n, \tag{3.1.4}$$

where

$$\eta = -\frac{1}{2}\frac{1}{\max(\nu c_{qu}, 4)\sqrt{n}} \log\left(\frac{\sqrt{\operatorname{cond}_2(M)} - 1}{\sqrt{\operatorname{cond}_2(M)} + 1}\right). \tag{3.1.5}$$

Proof. We recall from Theorem 2.5.10 that the entries of the inverse of the kernel matrix satisfy

$$\left|M^{-1}_{ik}\right| \le Cq_X^n e^{-\eta\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_k\|_2}{q_X}}$$

with η as in (3.1.5). Additionally, we have, for $1 \le i \ne k \le N$ and $\boldsymbol{x} \in B_\delta(\boldsymbol{x}_i)$,

$$\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2 \ge \|\boldsymbol{x} - \boldsymbol{x}_k\|_2 - \|\boldsymbol{x} - \boldsymbol{x}_i\|_2 \ge \|\boldsymbol{x} - \boldsymbol{x}_k\|_2 - \delta.$$

Using these two estimates in the representation of χ_k, (3.1.2), yields for $\boldsymbol{x} \in \mathbb{R}^n$ the chain of inequalities

$$\begin{aligned}
|\chi_k(\boldsymbol{x})| &\le \sum_{i=1}^{N} \left|M^{-1}_{ik}\right| |\Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_i)| \\
&= \sum_{i\,:\,\|\boldsymbol{x}-\boldsymbol{x}_i\|_2\le\delta} \left|M^{-1}_{ik}\right| |\Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_i)| \\
&\le C(\Phi, n, s)\|\Phi\|_{L_\infty(\mathbb{R}^n)} \sum_{i\,:\,\|\boldsymbol{x}-\boldsymbol{x}_i\|_2\le\delta} e^{-\eta\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_k\|_2}{q_X}} \\
&\le C(\Phi, n, s) \sum_{i\,:\,\|\boldsymbol{x}-\boldsymbol{x}_i\|_2\le\delta} e^{-\eta\frac{\|\boldsymbol{x}-\boldsymbol{x}_k\|_2}{q_X}} e^{\eta\frac{\delta}{q_X}} \\
&\le C(\Phi, n, s)(1 + \nu c_{qu})^n e^{\nu c_{qu}\eta} e^{-\eta\frac{\|\boldsymbol{x}-\boldsymbol{x}_k\|_2}{q_X}}.
\end{aligned}$$

In the last estimate we used Corollary 2.5.1 and that for quasi-uniform sets X the inequalities

$$c_\nu \nu q_X \le \delta \le \nu c_{qu} q_X$$

hold, if δ is coupled to $h_{X,\Omega}$ as in (2.4.6). □

We can use this local behavior of the Lagrange functions to show the next result. For a proof we refer to [20, Corollary 2.4].

Corollary 3.1.6. *With the notation and assumptions of Theorem 3.1.5 the Lagrange function χ_k, $1 \le k \le N$, is Lipschitz continuous, i.e., there is a constant $C_L > 0$ such that*

$$|\chi_k(\boldsymbol{x}) - \chi_k(\boldsymbol{y})| \le C_L \frac{\|\boldsymbol{x} - \boldsymbol{y}\|_2}{q_X}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

Furthermore, the Lagrange functions are uniformly bounded. This follows from a more general result, taken from [20].

Corollary 3.1.7. *With the notation and assumptions of Theorem 3.1.5 there is a constant $C > 0$ such that for $\ell \in \mathbb{N}_0$ the estimate*

$$\sum_{k=1}^{N} \|\boldsymbol{x} - \boldsymbol{x}_k\|_2^\ell |\chi_k(\boldsymbol{x})| \le C h_{X,\Omega}^\ell$$

holds for all $\boldsymbol{x} \in \mathbb{R}^n$.

Setting $\ell = 0$ shows that

$$\sum_{k=1}^{N} |\chi_k(\boldsymbol{x})| \le C, \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{3.1.6}$$

This also implies the boundedness of the Lebesgue constant.

Next, we follow [45] and use the Lagrange condition, Corollary 3.1.2, the local behavior, Theorem 3.1.5, and the Lipschitz continuity, Corollary 3.1.6, to show that the Lagrange basis is stable. The proof given here is taken from the internal communication paper [93].

Theorem 3.1.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ be a quasi-uniform set of sites with fill distance $h_{X,\Omega}$ and separation radius q_X. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ as in (2.4.6) with overlap parameter $\nu > 1$. For $1 \le k \le N$ let the Lagrange function χ_k be defined as in (3.1.2). Then, for $1 \le p \le \infty$, there exist constants $c_1, c_2 > 0$ such that the bounds*

$$c_1 q_X^{\frac{n}{p}} \|\boldsymbol{a}\|_p \le \left\| \sum_{k=1}^{N} a_k \chi_k \right\|_{L_p(\Omega)} \le c_2 \left(\frac{q_X}{\eta} \right)^{\frac{n}{p}} \|\boldsymbol{a}\|_p \tag{3.1.7}$$

hold for $\boldsymbol{a} \in \mathbb{R}^N$, where η is given by (3.1.5).

Proof. We start by deriving the easier, upper bound. For $p = \infty$ we have immediately

$$\left| \sum_{k=1}^{N} a_k \chi_k(\boldsymbol{x}) \right| \le \|\boldsymbol{a}\|_\infty \sum_{k=1}^{N} |\chi_k(\boldsymbol{x})| \le C_\infty \|\boldsymbol{a}\|_\infty,$$

where C_∞ is the constant from (3.1.6). For $p = 1$ we see that

$$\int_\Omega \left| \sum_{k=1}^{N} a_k \chi_k(\boldsymbol{x}) \right| \, d\boldsymbol{x} \le \sum_{k=1}^{N} |a_k| \max_{1 \le k \le N} \int_\Omega |\chi_k(\boldsymbol{x})| \, d\boldsymbol{x}.$$

Although the integrals on the right-hand side are already bounded we can do better. Using (3.1.4) yields

$$\begin{aligned}\int_\Omega |\chi_k(\boldsymbol{x})|\ d\boldsymbol{x} &\leq \int_{\mathbb{R}^n} |\chi_k(\boldsymbol{x})|\ d\boldsymbol{x} \leq C \int_{\mathbb{R}^n} e^{-\eta \frac{\|\boldsymbol{x}-\boldsymbol{x}_k\|_2}{q_X}}\ d\boldsymbol{x} \\ &= C \int_0^\infty e^{-\eta \frac{\|\boldsymbol{x}\|_2}{q_X}}\ d\boldsymbol{x} = C \int_0^\infty e^{-\eta \frac{r}{q_X}} r^{n-1}\ dr \\ &= C \left(\frac{q_X}{\eta}\right)^n \int_0^\infty e^{-t} t^{n-1}\ dt,\end{aligned}$$

where we increased the integration domain and introduced polar coordinates. We can compute the last integral analytically and arrive at

$$\int_\Omega \left| \sum_{k=1}^N a_k \chi_k(\boldsymbol{x}) \right|\ d\boldsymbol{x} \leq C\Gamma(n) \left(\frac{q_X}{\eta}\right)^n \|\boldsymbol{a}\|_1.$$

The result for general p then follows by operator interpolation, see Theorem 2.1.8. To be precise, we have

$$\left\| \sum_{k=1}^N a_k \chi_k \right\|_{L_p(\Omega)} \leq \left(C \left(\frac{q_X}{\eta}\right)^n \right)^{\frac{1}{p}} C_\infty^{1-\frac{1}{p}} \|\boldsymbol{a}\|_p.$$

Next, we prove the lower bound in (3.1.7). For $p = \infty$, the estimate follows directly from the Lagrange condition of χ_k. We have

$$\left\| \sum_{k=1}^N a_k \chi_k \right\|_{L_\infty(\Omega)} \geq \left\| \sum_{k=1}^N a_k \chi_k \right\|_{\ell_\infty(X)} = \max_{1\leq k\leq N} |a_k| = \|\boldsymbol{a}\|_\infty.$$

For $1 \leq p < \infty$, the bound is significantly harder to obtain. The ideas of the proof follow mainly those of [45, Section 3.2]. However they are modified to fit our setting.

First, we note that we assume that Ω has a Lipschitz boundary. That means it also satisfies an interior cone condition, see, e.g., [1, Section 4.11]. Hence, we find for each $\boldsymbol{x} \in \Omega$ a cone $C_r(\boldsymbol{x}) \subseteq \Omega$ with a fixed opening angle θ and a radius $r \leq r_0$, where $r_0 > 0$ is a fixed radius. This allows us to find a constant $\gamma > 0$ such that for all $\boldsymbol{x} \in \Omega$ and all $r \leq r_0$ we have

$$\operatorname{vol}(C_r(\boldsymbol{x})) \geq \gamma \operatorname{vol}(B_r(\boldsymbol{x})) = \gamma \operatorname{vol}(B_1(\boldsymbol{0})) r^n. \tag{3.1.8}$$

Furthermore, if we set $r = \varepsilon q_X$, with a free constant $\varepsilon \in (0,1)$, all cones $C_{\varepsilon q_X}(\boldsymbol{x}_i)$, $\boldsymbol{x}_i \in X$, are disjoint. This immediately yields the lower bound

$$\begin{aligned}\left\| \sum_{k=1}^N a_k \chi_k \right\|_{L_p(\Omega)}^p &= \int_\Omega \left| \sum_{k=1}^N a_k \chi_k(\boldsymbol{x}) \right|^p\ d\boldsymbol{x} \\ &\geq \sum_{i=1}^N \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{k=1}^N a_k \chi_k(\boldsymbol{x}) \right|^p\ d\boldsymbol{x}. \qquad (3.1.9)\end{aligned}$$

One main tool in the following will be the inequality

$$\left| \sum_{i=1}^M \alpha_i \right|^p \leq M^{p-1} \sum_{i=1}^M |\alpha_i|^p, \tag{3.1.10}$$

which holds for $1 \leq p < \infty$ and all $\alpha_1, \ldots, \alpha_M \in \mathbb{R}$, $M \in \mathbb{N}$. For $M = 2$ we have in particular

$$|\alpha_1|^p \geq 2^{1-p}|\alpha_1 + \alpha_2|^p - |\alpha_2|^p. \tag{3.1.11}$$

We introduce the shorthand notation, for fixed $1 \leq k \leq N$,

$$\sum_{i \neq k} := \sum_{\substack{i=1 \\ i \neq k}}^{N}.$$

Using (3.1.11) with $\alpha_1 = \sum_{k=1}^{N} a_k \chi_k(\boldsymbol{x})$ and $\alpha_2 = -\sum_{i \neq k} a_i \chi_i(\boldsymbol{x})$, we obtain for all $\boldsymbol{x} \in \Omega$

$$\left| \sum_{k=1}^{N} a_k \chi_k(\boldsymbol{x}) \right|^p \geq 2^{1-p} |a_i \chi_i(\boldsymbol{x})|^p - \left| \sum_{m \neq i} a_m \chi_m(\boldsymbol{x}) \right|^p. \tag{3.1.12}$$

Inserting (3.1.12) into (3.1.9) yields

$$\begin{aligned}
&\left\| \sum_{k=1}^{N} a_k \chi_k \right\|_{L_p(\Omega)}^p \geq \sum_{i=1}^{N} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{k=1}^{N} a_k \chi_k(\boldsymbol{x}) \right|^p \, d\boldsymbol{x} \\
&\geq \sum_{i=1}^{N} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left(2^{1-p} |a_i \chi_i(\boldsymbol{x})|^p - \left| \sum_{m \neq i} a_m \chi_m(\boldsymbol{x}) \right|^p \right) \, d\boldsymbol{x} \\
&= 2^{1-p} \sum_{i=1}^{N} |a_i|^p \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} |\chi_i(\boldsymbol{x})|^p \, d\boldsymbol{x} - \sum_{i=1}^{N} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{m \neq i} a_m \chi_m(\boldsymbol{x}) \right|^p \, d\boldsymbol{x}.
\end{aligned}$$

Now, we bound first term

$$2^{1-p} \sum_{i=1}^{N} |a_i|^p \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} |\chi_i(\boldsymbol{x})|^p \, d\boldsymbol{x}$$

from below. We use the Lagrange condition, Corollary 3.1.2, and the Lipschitz continuity, Corollary 3.1.6, to derive for $\boldsymbol{x} \in C_{\varepsilon q_X}(\boldsymbol{x}_i)$

$$|\chi_i(\boldsymbol{x}) - \chi_i(\boldsymbol{x}_i)| = |\chi_i(\boldsymbol{x}) - 1| \leq C_L \frac{\|\boldsymbol{x} - \boldsymbol{x}_i\|_2}{q_X} \leq C_L \varepsilon.$$

Hence, for such $\boldsymbol{x}$ we have $\chi_i(\boldsymbol{x}) \geq 2/3$ if we choose $\varepsilon < \frac{1}{3C_L}$. This leads to

$$\begin{aligned}
&\left\| \sum_{k=1}^{N} a_k \chi_k \right\|_{L_p(\Omega)}^p \geq \\
&\geq 2^{1-p} \sum_{i=1}^{N} |a_i|^p \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} |\chi_i(\boldsymbol{x})|^p \, d\boldsymbol{x} - \sum_{i=1}^{N} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{m \neq i} a_m \chi_m(\boldsymbol{x}) \right|^p d\boldsymbol{x} \\
&\geq 2^{1-p} \sum_{i=1}^{N} |a_i|^p \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left(\frac{2}{3} \right)^p d\boldsymbol{x} - \sum_{i=1}^{N} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{m \neq i} a_m \chi_m(\boldsymbol{x}) \right|^p d\boldsymbol{x} \\
&= \frac{2}{3^p} \sum_{i=1}^{N} |a_i|^p \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} 1 \, d\boldsymbol{x} - \sum_{i=1}^{N} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{m \neq i} a_m \chi_m(\boldsymbol{x}) \right|^p d\boldsymbol{x} \\
&\geq \frac{2\gamma \operatorname{vol}(B_1(\mathbf{0}))}{3^p} (\varepsilon q_X)^n \sum_{i=1}^{N} |a_i|^p - \sum_{i=1}^{N} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{m \neq i} a_m \chi_m(\boldsymbol{x}) \right|^p d\boldsymbol{x}, \qquad (3.1.13)
\end{aligned}$$

where we used (3.1.8) to arrive at the last inequality. It remains to bound

$$\sum_{i=1}^{N} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{m \neq i} a_m \chi_m(\boldsymbol{x}) \right|^p d\boldsymbol{x}$$

from above. We show that the term

$$\frac{1}{2} \left(\frac{2\gamma \operatorname{vol}(B_1(\mathbf{0}))}{3^p} (\varepsilon q_X)^n \sum_{i=1}^{N} |a_i|^p \right)$$

is a valid bound. We look at each summand of the outer sum and employ (3.1.10) with $M = 2$, some constant $\Gamma > 1$, which is independent of ε and q_X, and

$$\alpha_1 = \sum_{\substack{m \neq i \\ \|\boldsymbol{x}_m - \boldsymbol{x}_i\|_2 \leq \Gamma q_X}} a_m \chi_m(\boldsymbol{x}) \quad \text{and} \quad \alpha_2 = \sum_{\substack{m \neq i \\ \|\boldsymbol{x}_m - \boldsymbol{x}_i\|_2 > \Gamma q_X}} a_m \chi_m(\boldsymbol{x}).$$

This gives

$$\begin{aligned}
&\int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{m \neq i} a_m \chi_m(\boldsymbol{x}) \right|^p d\boldsymbol{x} \\
&\leq 2^{p-1} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{\substack{m \neq i \\ \|\boldsymbol{x}_m - \boldsymbol{x}_i\|_2 \leq \Gamma q_X}} a_m \chi_m(\boldsymbol{x}) \right|^p + \left| \sum_{\substack{m \neq i \\ \|\boldsymbol{x}_m - \boldsymbol{x}_i\|_2 > \Gamma q_X}} a_m \chi_m(\boldsymbol{x}) \right|^p d\boldsymbol{x} \\
&=: 2^{p-1}(I_i + II_i).
\end{aligned}$$

We deal with both terms, I_i and II_i, separately. We start by estimating II_i and define the annuli $E_j(\boldsymbol{x}_j)$ to be

$$E_j(\boldsymbol{x}_i) = \{\boldsymbol{x} \in \mathbb{R}^n \ : \ 2^j \Gamma q_X < \|\boldsymbol{x} - \boldsymbol{x}_i\|_2 \leq 2^{j+1}\Gamma q_X\}$$

for $\boldsymbol{x}_i \in X$ and $1 \leq j \leq N$. A standard comparison of volumes shows that the number of centers in such an annulus is bounded by

$$\#(X \cap E_j(\boldsymbol{x}_i)) \leq 2^n n \left(\Gamma 2^{j+1}\right)^n. \tag{3.1.14}$$

Next, we note that inductively applying (3.1.11) yields the estimate

$$\left|\sum_{i=1}^{M} \alpha_i\right|^p \leq \sum_{i=1}^{M} 2^{i(p-1)} |\alpha_i|^p. \tag{3.1.15}$$

This, together with the definition of the annuli, allows us to bound II_i. We have

$$\begin{aligned}
II_i &= \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{\substack{m \neq i \\ \|\boldsymbol{x}_m - \boldsymbol{x}_i\|_2 > \Gamma q_X}} a_m \chi_m(\boldsymbol{x}) \right|^p \, d\boldsymbol{x} \\
&= \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{j=0}^{\infty} \sum_{\boldsymbol{x}_m \in E_j(\boldsymbol{x}_i)} a_m \chi_m(\boldsymbol{x}) \right|^p \, d\boldsymbol{x} \\
&\leq \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \sum_{j=0}^{\infty} 2^{(j+1)(p-1)} \left| \sum_{\boldsymbol{x}_m \in E_j(\boldsymbol{x}_i)} a_m \chi_m(\boldsymbol{x}) \right|^p \, d\boldsymbol{x} \\
&\leq \sum_{j=0}^{\infty} 2^{(j+1)(p-1)} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} (\#(X \cap E_j(\boldsymbol{x}_i)))^{p-1} \sum_{\boldsymbol{x}_m \in E_j(\boldsymbol{x}_i)} |a_m \chi_m(\boldsymbol{x})|^p \, d\boldsymbol{x} \\
&\leq \sum_{j=0}^{\infty} 2^{(j+1)(p-1)} \left(2^n n \left(\Gamma 2^{j+1}\right)^n\right)^{p-1} \sum_{\boldsymbol{x}_m \in E_j(\boldsymbol{x}_i)} |a_m|^p \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} |\chi_m(\boldsymbol{x})|^p \, d\boldsymbol{x},
\end{aligned}$$

where we used (3.1.10) with $M = \#(X \cap E_j(\boldsymbol{x}_i))$ in the second to last estimate.

Next, we use the exponential decay of the Lagrange functions. To be more precise, we have for $\boldsymbol{x} \in C_{\varepsilon q_X}(\boldsymbol{x}_i)$

$$\|\boldsymbol{x} - \boldsymbol{x}_m\|_2 \geq \|\boldsymbol{x}_m - \boldsymbol{x}_i\|_2 - \|\boldsymbol{x}_i - \boldsymbol{x}\|_2 \geq \Gamma 2^j q_X - \varepsilon q_X$$

and hence, by (3.1.4) and for $\varepsilon \in (0,1)$,

$$\begin{aligned}
|\chi_m(\boldsymbol{x})| &\leq C_E e^{-\eta \frac{\|\boldsymbol{x} - \boldsymbol{x}_m\|_2}{q_X}} \leq C_E e^{-\eta \Gamma 2^j} e^{\eta \varepsilon} \\
&\leq C_E e^{\eta} e^{-\eta \Gamma 2^j}
\end{aligned}$$

for all $\boldsymbol{x} \in C_{\varepsilon q_X}(\boldsymbol{x}_i)$. This, together with (3.1.8), yields

$$\begin{aligned} II_i \leq & \\ \leq & \sum_{j=0}^{\infty} 2^{(j+1)(p-1)} \left(2^n n \left(\Gamma 2^{j+1}\right)^n\right)^{p-1} \gamma (\varepsilon q_X)^n C_E^p e^{\eta p} e^{-\eta p \Gamma 2^j} \sum_{\boldsymbol{x}_m \in E_j(\boldsymbol{x}_i)} |a_m|^p \\ = & C_{II} (\varepsilon q_X)^n \Gamma^{n(p-1)} \sum_{j=0}^{\infty} 2^{j(n+1)(p-1)} e^{-\eta p \Gamma 2^j} \sum_{\boldsymbol{x}_m \in E_j(\boldsymbol{x}_i)} |a_m|^p \end{aligned}$$

with $C_{II} = C_E^p \gamma e^{\eta p} n^{p-1} 2^{(2n+1)(p-1)}$. Now, we can sum up over all i, exchange the summation and use (3.1.14) to derive

$$\begin{aligned} \sum_{i=1}^{N} II_i &\leq C_{II} (\varepsilon q_X)^n \Gamma^{n(p-1)} \sum_{j=0}^{\infty} 2^{j(n+1)(p-1)} e^{-\eta p \Gamma 2^j} \sum_{i=1}^{N} \sum_{\boldsymbol{x}_m \in E_j(\boldsymbol{x}_i)} |a_m|^p \\ &= C_{II} (\varepsilon q_X)^n \Gamma^{n(p-1)} \sum_{j=0}^{\infty} 2^{j(n+1)(p-1)} e^{-\eta p \Gamma 2^j} \sum_{m=1}^{N} \sum_{\boldsymbol{x}_i \in E_j(\boldsymbol{x}_m)} |a_m|^p \\ &\leq C_{II} (\varepsilon q_X)^n \Gamma^{n(p-1)} \sum_{j=0}^{\infty} 2^{j(n+1)(p-1)} e^{-\eta p \Gamma 2^j} 2^n n \left(\Gamma 2^{j+1}\right)^n \sum_{m=1}^{N} |a_m|^p \\ &= C_{II} 2^{2n} n (\varepsilon q_X)^n \Gamma^{np} \sum_{j=1}^{\infty} 2^{j(p(n+1)-1)} e^{-\eta p \Gamma 2^j} \sum_{m=1}^{N} |a_m|^p. \end{aligned}$$

To simplify the term in the last line, we can use that

$$e^{-\eta p \Gamma 2^j} = e^{\eta p \Gamma 2^{j-1}} e^{-\eta p \Gamma 2^{j-1}} \leq e^{\eta p \frac{\Gamma}{2}} e^{-\eta p 2^{j-1}}$$

holds for $\Gamma \geq 1$. This yields the bound

$$\Gamma^{np} \sum_{j=0}^{\infty} 2^{j(p(n+1)-1)} e^{-\eta p \Gamma 2^j} \leq \left(\Gamma^{np} e^{-\eta p \frac{\Gamma}{2}}\right) \left(\sum_{j=0}^{\infty} 2^{j(p(n+1)-1)} e^{-\eta p 2^{j-1}}\right).$$

The infinite sum in the second bracket is independent of Γ and using, e.g., the quotient criterion, can easily be seen to converge. The first factor converges to zero with $\Gamma \to \infty$. This means that we can find a $\Gamma^* > 1$, which only depends on n, η, p, C_E and γ but not on ε and q_X such that

$$2^{p-1} \sum_{i=1}^{N} II_i \leq \frac{1}{2} \frac{\gamma}{3^p} (\varepsilon q_X)^n \sum_{i=1}^{N} |a_i|^p \tag{3.1.16}$$

holds for all $\Gamma > \Gamma^*$, q_X, ε and a_i.

After fixing Γ^* and bounding the sum over all II_i terms, we turn now to the sum over all I_i terms. We can use Corollary 3.1.6 to bound the Lagrange functions by

$$|\chi_m(\boldsymbol{x})| = |\chi_m(\boldsymbol{x}) - \chi_m(\boldsymbol{x}_i)| \leq C_L \frac{\|\boldsymbol{x} - \boldsymbol{x}_i\|_2}{q_X} \leq C_L \varepsilon,$$

for all $\boldsymbol{x} \in C_{\varepsilon q_X}(\boldsymbol{x}_i)$, since $\boldsymbol{x}_i \neq \boldsymbol{x}_m$. Then, (3.1.8), (3.1.10) and

$$\#(X \cap B_{\Gamma q_X}(\boldsymbol{x}_i)) \leq \mathrm{vol}(B_1(\boldsymbol{0}))(\Gamma + 1)^n \tag{3.1.17}$$

yield

$$
\begin{aligned}
I_i &= \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \left| \sum_{\substack{m\neq i \\ \|\boldsymbol{x}_i-\boldsymbol{x}_m\|_2\leq \Gamma q_X}} a_m \chi_m(\boldsymbol{x}) \right|^p \, d\boldsymbol{x} \\
&\leq \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} (\#(X \cap B_{\Gamma q_X}(\boldsymbol{x}_i)))^{p-1} \sum_{\substack{m\neq i \\ \|\boldsymbol{x}_i-\boldsymbol{x}_m\|_2\leq \Gamma q_X}} |a_m|^p |\chi_m(\boldsymbol{x})|^p \, d\boldsymbol{x} \\
&\leq C_L^p \varepsilon^p \left(\operatorname{vol}(B_1(\mathbf{0}))\right)^{p-1} (\Gamma+1)^{n(p-1)} \int_{C_{\varepsilon q_X}(\boldsymbol{x}_i)} \sum_{\substack{m\neq i \\ \|\boldsymbol{x}_i-\boldsymbol{x}_m\|_2\leq \Gamma q_X}} |a_m|^p d\boldsymbol{x} \\
&\leq C_L^p \varepsilon^p \left(\operatorname{vol}(B_1(\mathbf{0}))\right)^{p-1} (\Gamma+1)^{n(p-1)} \gamma \operatorname{vol}(B_1(\mathbf{0}))(\varepsilon q_X)^n \sum_{\substack{m\neq i \\ \|\boldsymbol{x}_i-\boldsymbol{x}_m\|_2\leq \Gamma q_X}} |a_m|^p \\
&= C_I \varepsilon^p (\varepsilon q_X)^n \sum_{\substack{m\neq i \\ \|\boldsymbol{x}_i-\boldsymbol{x}_m\|_2\leq \Gamma q_X}} |a_m|^p
\end{aligned}
$$

with $C_I = C_L^p \left(\operatorname{vol}(B_1(\mathbf{0}))\right)^p \gamma(\Gamma+1)^{n(p-1)}$. Now, we can sum up over all i, exchange the summation and use (3.1.17) to obtain

$$
\begin{aligned}
\sum_{i=1}^N I_i &\leq C_I \varepsilon^p (\varepsilon q_X)^n \sum_{i=1}^N \sum_{\substack{m\neq i \\ \|\boldsymbol{x}_i-\boldsymbol{x}_m\|_2\leq \Gamma q_X}} |a_m|^p \\
&\leq C_I \varepsilon^p (\varepsilon q_X)^n \operatorname{vol}(B_1(\mathbf{0}))(\Gamma+1)^n \sum_{i=1}^N |a_i|^p.
\end{aligned}
$$

We can now choose $\varepsilon > 0$ so small that this expression satisfies

$$
2^{p-1} \sum_{i=1}^N I_i \leq \frac{1}{2} \frac{\gamma}{3^p} (\varepsilon q_X)^n \sum_{i=1}^N |a_i|^p
$$

for all q_X and a_i. Together with (3.1.13) and (3.1.16) this finally gives the bound

$$
\left\| \sum_{k=1}^N a_k \chi_k \right\|_{L_p(\Omega)} \geq \frac{\gamma \varepsilon^n}{3^p} q_X^n \sum_{k=1}^N |a_k|^p
$$

with a fixed, sufficiently small ε which does not depend on X nor $\boldsymbol{a}$. □

This stability result for the Lagrange basis of V_N allows us to prove a Nikolskii inequality. This allows us to bound the L_p-norm of elements of V_N by their L_q-norm.

Corollary 3.1.9. *With the notation and assumptions of Theorem 3.1.8 and $1 \leq p \leq q \leq \infty$, there is a constant $C > 0$ such that the estimate*

$$
\|s\|_{L_p(\Omega)} \leq C q_X^{-n\left(\frac{1}{q}-\frac{1}{p}\right)_+} \|s\|_{L_q(\Omega)}
$$

holds for all $s \in V_N$, where $(x)_+ = \max(x, 0)$.

PROOF. Using the Lagrange basis, we can write $s \in V_N$ as

$$s = \sum_{k=1}^{N} a_k \chi_k,$$

where, for the coefficient vector $\boldsymbol{a} \in \mathbb{R}^N$, we have the estimate

$$\|\boldsymbol{a}\|_p \leq N^{\left(\frac{1}{p}-\frac{1}{q}\right)_+} \|\boldsymbol{a}\|_q.$$

Applying (3.1.7) twice, once for p and once for q, yields

$$\begin{aligned} \|s\|_{L_p(\Omega)} &\leq c_2 q_X^{\frac{n}{p}} \|\boldsymbol{a}\|_p \leq c_2 q_X^{\frac{n}{p}} N^{\left(\frac{1}{p}-\frac{1}{q}\right)_+} \|\boldsymbol{a}\|_q \\ &\leq \frac{c_2}{c_1} C(\Omega) q_X^{\frac{n}{p}} q_X^{-n\left(\frac{1}{p}-\frac{1}{q}\right)_+} q_X^{-\frac{n}{q}} \|s\|_{L_q(\Omega)}, \end{aligned}$$

where the assumption that X is quasi-uniform allowed us to estimate $N \leq C(\Omega) q_X^{-n}$. Collecting the q_X-terms and using $x - (x)_+ = -(-x)_+$ finishes the proof. □

Using the Lagrange functions $\{\chi_k\}_{1 \leq k \leq N}$ instead of $\{\Phi_\delta(\cdot - \boldsymbol{x}_k)\}_{1 \leq k \leq N}$ as a basis of V_N has several advantages. First, we see that we can express the interpolant in an explicit way.

Proposition 3.1.10. *We can express the interpolant $I_{X,\Phi_\delta} f$ to $f \in H^s(\Omega)$, $s > n/2$, in V_N as*

$$I_{X,\Phi_\delta} f(\boldsymbol{x}) = \sum_{k=1}^{N} f(\boldsymbol{x}_k) \chi_k(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{3.1.18}$$

Second, we do not have to use the point-wise definition of χ_k as in (3.1.1). The Lagrange condition $\chi_k(\boldsymbol{x}_i) = \delta_{ki}$, $1 \leq k, i \leq N$, allows us to compute for every $1 \leq k \leq N$ the coefficients $\alpha_i^{(k)}$, $1 \leq i \leq N$, in the expansion

$$\chi_k = \sum_{i=1}^{N} \alpha_i^{(k)} \Phi_\delta(\cdot - \boldsymbol{x}_i)$$

by solving the linear system

$$M \boldsymbol{\alpha}^{(k)} = \boldsymbol{e}_k, \tag{3.1.19}$$

where $\boldsymbol{e}_k$ denotes the k-th unit vector in $\mathbb{R}^n$. This means that we have to solve N-many $N \times N$ sparse linear systems in order to obtain the Lagrange basis of V_N. This is clearly more expensive than computing the interpolant

$$I_{X,\Phi_\delta} f(\boldsymbol{x}) = \sum_{i \,:\, \|\boldsymbol{x}-\boldsymbol{x}_i\| \leq \delta} \alpha_i \Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_i), \quad \boldsymbol{x} \in \Omega, \tag{3.1.20}$$

where $\boldsymbol{\alpha}$ is the unique solution of the linear system

$$M \boldsymbol{\alpha} = \boldsymbol{f}.$$

However, the Lagrange functions are independent of the data $f(\boldsymbol{x}_i)$, $1 \leq i \leq N$, and only depend on the set of sites $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$. This separation of data and sites means that we can compute the solutions to (3.1.19) a-priori, store them and use them for several data sets, as long as X stays the same.

In the case of (3.1.20) we have to solve a linear system every time we change the data.

Using the Lagrange basis has a major downside. Numerical tests show that evaluations of the interpolant (3.1.18) are expensive since the support of Lagrange functions χ_k is too large. Nevertheless, in the next sections, using the local behavior of χ_k in (3.1.4), we derive a modification of the Lagrange functions which have a smaller, compact support. Additionally, the size of this support can be chosen by us.

For the sake of completeness we briefly discuss Lagrange functions for penalized least-squares approximation. We recall that we can write the interpolant as

$$I_{X,\Phi_\delta} f(\boldsymbol{x}) = \sum_{k=1}^{N} \alpha_k \Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_k), \quad \boldsymbol{x} \in \Omega,$$

where the coefficient vector is given as $\boldsymbol{\alpha} = M^{-1}\boldsymbol{f}$ with $f_k = f(\boldsymbol{x}_k)$. Introducing the notation $\boldsymbol{r}(\boldsymbol{x})^{\mathrm{T}} \in \mathbb{R}^N$ with $(r(\boldsymbol{x}))_k = \Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_k)$, $1 \le k \le N$, this yields for $\boldsymbol{x} \in \Omega$

$$\begin{aligned} I_{X,\Phi_\delta} f(\boldsymbol{x}) &= \sum_{k=1}^{N} \alpha_k \Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_k) = \boldsymbol{r}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{\alpha} \\ &= \boldsymbol{r}(\boldsymbol{x})^{\mathrm{T}} M^{-1} \boldsymbol{f} = \sum_{k=1}^{N} f(\boldsymbol{x}_k) \chi_k(\boldsymbol{x}). \end{aligned}$$

Hence, we can express the Lagrange function χ_k point-wise as

$$\chi_k(\boldsymbol{x}) = \boldsymbol{r}(\boldsymbol{x})^{\mathrm{T}} M^{-1} \boldsymbol{e}_k, \quad \boldsymbol{x} \in \mathbb{R}^n.$$

This idea also carries over to the penalized least-squares setting. In Section 2.3.4 we saw that the coefficient vector $\boldsymbol{\alpha}$ is the unique solution of $(M + \lambda I)\boldsymbol{\alpha} = \boldsymbol{f}$, with the smoothing parameter $\lambda > 0$ and $I \in \mathbb{R}^{N \times N}$ the identity matrix. Following the same ideas as above we can write the approximant as

$$s_{\lambda,f}(\boldsymbol{x}) = \boldsymbol{r}(\boldsymbol{x})^{\mathrm{T}} (M + \lambda \boldsymbol{I})^{-1} \boldsymbol{f} =: \sum_{k=1}^{N} f(\boldsymbol{x}_k) \chi_k^{LS,\lambda}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

This motivates the following definition.

Definition 3.1.11. *For $\lambda > 0$ and $1 \le k \le N$ we define the* modified Lagrange *or* modified cardinal functions $\chi_k^{LS,\lambda} \in V_N$ *by*

$$\chi_k^{LS,\lambda}(\boldsymbol{x}) = \boldsymbol{r}(\boldsymbol{x})^T (M + \lambda I)^{-1} \boldsymbol{e}_k, \quad \boldsymbol{x} \in \mathbb{R}^n, \tag{3.1.21}$$

with $(r(\boldsymbol{x}))_i = \Phi_\delta(\boldsymbol{x} - \boldsymbol{x}_i)$.

Although the functions $\{\chi_k^{LS,\lambda}\}_{1 \le k \le N}$ do not satisfy the Lagrange condition of Corollary 3.1.2, we choose the name modified Lagrange functions to emphasize the connection to the full Lagrange functions $\{\chi_k\}_{1 \le k \le N}$.

Definition 3.1.11 allows us to use the Lagrange representation of the penalized least-squares approximant. However, for the rest of the chapter we will focus on the Lagrange functions χ_k.

3.2. Cut-Off Lagrange Functions

On the way to Lagrange functions with smaller compact support we have to introduce an intermediate function. The idea is to use the exponential decay of the coefficients M_{ki}^{-1} and to not use the indices $1 \leq i \leq N$ in the expansion

$$\chi_k = \sum_{i=1}^{N} M_{ki}^{-1} \Phi_\delta(\cdot - \boldsymbol{x}_i)$$

for which

$$\left|M_{ki}^{-1}\right| \leq C q_X^n e^{-\eta \frac{\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2}{q_X}} < \varepsilon$$

for a given threshold $\varepsilon > 0$. Omitting the constant $C > 0$ and rearranging the second inequality means that we choose to ignore those $\boldsymbol{x}_i \in X$ which satisfy

$$\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2 > \frac{q_X}{\eta} \left|\log\left(\frac{\varepsilon}{q_X^n}\right)\right|.$$

This gives the *cut-off radius*

$$r(\varepsilon) := \frac{q_X}{\eta} \left|\log\left(\frac{\varepsilon}{q_X^n}\right)\right| \tag{3.2.1}$$

With this in mind we have the next definition.

Definition 3.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $X := \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \Omega$ be a discrete set of sites. Let $\chi_k \in V_N$ be the Lagrange function anchored in $\boldsymbol{x}_k \in X$. Then the set $X_r(\boldsymbol{x}_k) \subseteq X$, defined by*

$$X_r(\boldsymbol{x}_k) := X \cap \overline{B_r(\boldsymbol{x}_k)}, \tag{3.2.2}$$

is called the footprint *of χ_k with radius $r > 0$.*

We can use the same proof ideas which led to Lemma 2.4.4 to show that the following estimate for the cardinality of the footprint holds.

Corollary 3.2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \Omega$ be a discrete set of sites with separation radius q_X and let $r(\varepsilon)$ be as in (3.2.1) with sufficiently small $\varepsilon > 0$. Then the cardinality of the footprint $X_{r(\varepsilon)}(\boldsymbol{x}_k)$ satisfies, for every $1 \leq k \leq N$,*

$$\#X_{r(\varepsilon)}(\boldsymbol{x}_k) \leq \left(\frac{2}{\eta}\right)^n \left|\log\left(\frac{\varepsilon}{q_X^n}\right)\right|^n. \tag{3.2.3}$$

Definition 3.2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \Omega$ be a discrete set of sites. For $1 \leq k \leq N$ let $X_r(\boldsymbol{x}_k)$ be the footprint of the Lagrange function $\chi_k \in V_N$ with cut-off radius $r > 0$. We define the* cut-off Lagrange function $\widetilde{\chi}_k : \mathbb{R}^n \to \mathbb{R}$ *by*

$$\widetilde{\chi}_k = \sum_{i \,:\, \boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)} M_{ki}^{-1} \Phi_\delta(\cdot - \boldsymbol{x}_i). \tag{3.2.4}$$

Clearly, we have still $\widetilde{\chi}_k \in V_N$ for all $1 \leq k \leq N$. We emphasize that we use the same coefficients M_{ki}^{-1} in the expansion (3.2.4) as in the definition of the full Lagrange functions χ_k in (3.1.2). We only change the number

of summands. This has several important consequences. First, the cut-off Lagrange functions do not satisfy the Lagrange condition $\widetilde{\chi}_k(\boldsymbol{x}_i) = \delta_{ki}$ for $\boldsymbol{x}_i \in X$ and not even for $\boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)$. However, the Lagrange condition holds for points of X in a smaller ball centered in $\boldsymbol{x}_k$, i.e., we have $\widetilde{\chi}_k(\boldsymbol{x}_i) = \delta_{ki}$ for $\boldsymbol{x}_i \in X \cap \overline{B_{r-\delta}(\boldsymbol{x}_k)}$, if we assume that $r \geq \delta$.

Furthermore, the cut-off Lagrange functions $\widetilde{\chi}_k$ has compact support in $\overline{B_{r+\delta}(\boldsymbol{x}_k)}$, which follows simply from the expansion (3.2.4) and the fact that $\Phi_\delta(\cdot - \boldsymbol{x}_i)$ has compact support in $\overline{B_\delta(\boldsymbol{x}_i)}$.

It turns out that the set of cut-off Lagrange functions $\{\widetilde{\chi}_k\}_{1\leq k\leq N}$ is also a stable basis of the approximation space V_N if the cut-off radius is sufficiently large. First, we have to show linear independence of the functions, which is, again, equivalent to invertibility of the matrix $T = (\widetilde{\chi}_k(\boldsymbol{x}_i))$. To prove this we quote a general perturbation result from [7].

Lemma 3.2.4. *Let $(V, \|\cdot\|)$ be a normed space and let $S : V \to V$ be an invertible, bounded and linear operator whose inverse is also bounded. If $T : V \to V$ is a linear operator such that*

$$\|S - T\|_{V\to V} \leq \frac{1}{C\|S\|_{V\to V}}$$

with a constant $C > 1$. Then T is invertible and

$$\|T^{-1}\|_{V\to V} \leq \frac{C}{C-1}\|S^{-1}\|_{V\to V}.$$

In our setting we have $V = \mathbb{R}^N$. We set $\|\cdot\| = \|\cdot\|_\infty$ and use for the operator norm the matrix norm associated to the ∞-norm. Finally, we use $S = I = (\chi_k(\boldsymbol{x}_i))_{ki}$ the identity. Then we have

$$\begin{aligned} \|S - T\|_\infty &= \max_{1\leq k\leq N} \sum_{i=1}^{N} |\chi_k(\boldsymbol{x}_i) - \widetilde{\chi}_k(\boldsymbol{x}_i)| \\ &\leq N \max_{1\leq k\leq N} \|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)}. \end{aligned} \tag{3.2.5}$$

Hence, if we can bound the error $\|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)}$ then $T = (\widetilde{\chi}_k(\boldsymbol{x}_i))$ is invertible. Additionally, if this error is small enough we can carry over the stability of $\{\chi_k\}_{1\leq k\leq N}$ to the cut-off Lagrange functions. We summarize our findings in the following theorem.

Theorem 3.2.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \Omega$ be a quasi-uniform set of sites with fill distance $h_{X,\Omega}$ and with sufficiently small separation radius q_X. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies* (2.2.2)*. Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ according to* (2.4.6)*. Let $\{\chi_k\}_{1\leq k\leq N}$ be the Lagrange basis of V_N with χ_k defined as is* (3.1.2)*. For $1 \leq k \leq N$ let $\widetilde{\chi}_k$ be the cut-off Lagrange function defined in* (3.2.4) *with general footprint $X_r(\boldsymbol{x}_k)$. Assume that there are constants $C > 0$ and $J > n$ such that*

$$\|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)} \leq C q_X^J, \quad 1 \leq k \leq N. \tag{3.2.6}$$

Then $\{\widetilde{\chi}_k\}_{1\le k\le N}$ is a stable basis of V_N and the stability estimate

$$C_1 q_X^{\frac{n}{p}} \|\boldsymbol{a}\|_p \le \left\| \sum_{k=1}^N a_k \widetilde{\chi}_k \right\|_{L_p(\Omega)} \le C_2 q_X^{\frac{n}{p}} \|\boldsymbol{a}\|_p \tag{3.2.7}$$

holds for all $1 \le p \le \infty$ with constants $C_1, C_2 > 0$.

Proof. We assume that the separation radius q_X of X is sufficiently small, i.e., there is a constant $q_0 < 1$ such that $q_X < q_0$. Then we have with Lemma 3.2.4, in particular (3.2.5) and the assumption(3.2.6), that

$$\begin{aligned} \|S - T\|_\infty &\le N \max_{1\le k\le N} \|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)} \\ &\le C q_X^{-n} q_X^J \le C q_0^{J-n} \\ &\le \frac{1}{\widetilde{C}\|I^{-1}\|_\infty} \end{aligned}$$

with $\widetilde{C} = \frac{1}{C q_0^{J-n}} > 1$ for sufficiently small q_0 and sufficiently large J.

To derive the stability estimate (3.2.7) we first fix an $\boldsymbol{a} \in \mathbb{R}^N$ and set $s = \sum_{k=1}^N a_k \chi_k$ and $\widetilde{s} = \sum_{k=1}^N a_k \widetilde{\chi}_k$. To obtain the upper bound in (3.2.7) we bound $s - \widetilde{s}$ in the $L_1(\Omega)$- and $L_\infty(\Omega)$-norm and use again the operator interpolation Theorem 2.1.8. For the $L_1(\Omega)$-estimate we use the Hölder inequality and (3.2.6). We have

$$\begin{aligned} \|s - \widetilde{s}\|_{L_1(\Omega)} &= \left\| \sum_{k=1}^N a_k (\chi_k - \widetilde{\chi}_k) \right\| \\ &\le \|\boldsymbol{a}\|_1 \|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)} \le C q_X^J \|\boldsymbol{a}\|_1. \end{aligned}$$

We can bound the $L_\infty(\Omega)$-norm directly.

$$\begin{aligned} \|s - \widetilde{s}\|_{L_\infty(\Omega)} &\le \|\boldsymbol{a}\|_\infty \max_{\boldsymbol{x}\in\Omega} \sum_{k=1}^N |\chi_k(\boldsymbol{x}) - \widetilde{\chi}_k(\boldsymbol{x})| \\ &\le \|\boldsymbol{a}\|_\infty N \max_{1\le k\le N} \|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)} \\ &\le C q_X^{J-n} \|\boldsymbol{a}\|_\infty, \end{aligned}$$

where we used that X is assumed to be quasi-uniform, i.e., $N \le C q_X^{-n}$. Using operator interpolation yields finally

$$\|s - \widetilde{s}\|_{L_p(\Omega)} \le C q_X^{J-n} q_X^{\frac{n}{p}} \|\boldsymbol{a}\|_p. \tag{3.2.8}$$

The inverse triangle inequality for $\|s - \widetilde{s}\|_{L_p(\Omega)}$ yields the chain of inequalities

$$\|s\|_{L_p(\Omega)} - \|s - \widetilde{s}\|_{L_p(\Omega)} \le \|\widetilde{s}\|_{L_p(\Omega)} \le \|s\|_{L_p(\Omega)} + \|s - \widetilde{s}\|_{L_p(\Omega)},$$

which gives with the stability estimate for the full Lagrange basis (3.1.7) and (3.2.8)

$$c_1 q_X^{\frac{n}{p}} \|\boldsymbol{a}\|_p (1 - C q_X^{J-n}) \le \|\widetilde{s}\|_{L_p(\Omega)} \le c_2 q_X^{\frac{n}{p}} \|\boldsymbol{a}\|_p (1 + C q_X^{J-n}).$$

This is (3.2.7) if q_X is small enough such that $C q_X^{J-n} < 1/2$ and if we set $C_1 = c_1/2$ und $C_2 = (3c_2)/2$. □

Now we have to make sure that the assumption (3.2.6) holds if we construct the cut-off Lagrange functions with the cut-off radius $r(\varepsilon) = \frac{q_X}{\eta}\left|\log\left(\frac{\varepsilon}{q_X^n}\right)\right|$. We will see that the main tool is an upper bound for the sum of the tail of coefficients,

$$\sum_{i\,:\,\boldsymbol{x}_i\in X\setminus X_{r(\varepsilon)}(\boldsymbol{x}_k)} M_{ki}^{-1}, \quad 1\le k\le N.$$

Hence, we give the estimate before we bound the error $\chi_k - \widetilde{\chi}_k$ in different norms.

Lemma 3.2.6. *Let M_{ki}^{-1} be the i-th coefficient in the expansion of χ_k as in* (3.1.2) *Then, after cutting with $r(\varepsilon)$, there exists a constant $C>0$ such that the estimate*

$$\sum_{i\,:\,\boldsymbol{x}_i\in X\setminus X_{r(\varepsilon)}(\boldsymbol{x}_k)} \left|M_{ki}^{-1}\right| \le C\frac{e^\eta}{\eta^n} q_X^{\frac{n}{2}}\sqrt{\varepsilon}, \tag{3.2.9}$$

holds for all $1\le k\le N$.

Proof. We begin the proof with considering a general cut-off radius $r>0$ and associated footprint $X_r(\boldsymbol{x}_k)$. The exponential decay of the coefficients M_{ki}^{-1} in (2.5.6) yields

$$\sum_{i\,:\,\boldsymbol{x}_i\in X\setminus X_r(\boldsymbol{x}_k)} \left|M_{ki}^{-1}\right| \le C \sum_{i\,:\,\|\boldsymbol{x}_i-\boldsymbol{x}_k\|_2>r} q_X^n e^{-\eta\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_k\|_2}{q_X}}.$$

We now want to control the sum by an integral. Therefore, we note that for $\boldsymbol{x}_i\in X\setminus X_r(\boldsymbol{x}_k)$ the estimate $q_X^n \le C(n)\,\mathrm{vol}(B_{q_X}(\boldsymbol{x}_i)\setminus B_r(\boldsymbol{x}_k))$ holds. With this we have for $1\le k\le N$

$$\begin{aligned}
\sum_{i\,:\,\boldsymbol{x}_i\in X\setminus X_r(\boldsymbol{x}_k)} \left|M_{ki}^{-1}\right| &\le C \sum_{i\,:\,\|\boldsymbol{x}_i-\boldsymbol{x}_k\|_2>r} q_X^n e^{-\eta\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_k\|_2}{q_X}} \\
&\le C \sum_{i\,:\,\|\boldsymbol{x}_i-\boldsymbol{x}_k\|_2>r} e^{-\eta\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_k\|_2}{q_X}} \int_{B_{q_X}(\boldsymbol{x}_i)\setminus B_r(\boldsymbol{x}_k)} 1\; d\boldsymbol{y} \\
&\le Ce^\eta \int_{\mathbb{R}^n\setminus B_r(\boldsymbol{x}_k)} e^{-\eta\frac{\|\boldsymbol{y}-\boldsymbol{x}_k\|_2}{q_X}}\; d\boldsymbol{y}.
\end{aligned}$$

In the last estimate we have used that the sets $(B_{q_X}(\boldsymbol{x}_i)\setminus B_r(\boldsymbol{x}_k))$ are disjoint since we assumed that $\boldsymbol{x}_i\in X\setminus X_r(\boldsymbol{x}_k)$, and that for $\boldsymbol{y}\in B_{q_X}(\boldsymbol{x}_i)$ the estimate

$$\|\boldsymbol{x}_k-\boldsymbol{y}\|_2 \le \|\boldsymbol{x}_k-\boldsymbol{x}_i\|_2 + \|\boldsymbol{x}_i-\boldsymbol{y}\|_2 \le \|\boldsymbol{x}_k-\boldsymbol{x}_i\|_2 + q_X,$$

and consequently

$$-\|\boldsymbol{x}_k-\boldsymbol{x}_i\|_2 \le -\|\boldsymbol{x}_k-\boldsymbol{y}\|_2 + q_X$$

holds. This leads to the constant factor e^η.

Now, changing to polar coordinates and splitting $\eta = \eta/2 + \eta/2$ yields

$$\begin{aligned}\sum_{i\,:\,\boldsymbol{x}_i \in X \setminus X_r(\boldsymbol{x}_k)} \left|M_{ki}^{-1}\right| &\leq Ce^{\eta} \int_{\mathbb{R}^n \setminus B_r(\boldsymbol{x}_k)} e^{-\eta \frac{\|\boldsymbol{y}-\boldsymbol{x}_k\|_2}{q_X}}\, d\boldsymbol{y} \\ &\leq Ce^{\eta} \int_r^{\infty} e^{-\eta \frac{\rho}{q_X}} \rho^{n-1}\, d\rho \\ &\leq Ce^{\eta} e^{-\frac{\eta}{2}\frac{r}{q_X}} \int_r^{\infty} e^{-\frac{\eta}{2}\frac{\rho}{q_X}} \rho^{n-1}\, d\rho.\end{aligned}$$

Next, we bound the integral by increasing the integration domain from $[r, \infty)$ to $[0, \infty)$ and then compute the integral explicitly. We have

$$\int_0^{\infty} e^{-\frac{\eta}{2}\frac{\rho}{q_X}} \rho^{n-1}\, d\rho = 2^n (n-1)! \left(\frac{q_X}{\eta}\right)^n.$$

This then yields the general bound

$$\sum_{i\,:\,\boldsymbol{x}_i \in X \setminus X_r(\boldsymbol{x}_k)} \left|M_{ki}^{-1}\right| \leq Ce^{\eta} \left(\frac{q_X}{\eta}\right)^n e^{-\frac{\eta}{2}\frac{r}{q_X}}.$$

Finally, inserting the specific cut-off radius $r(\varepsilon) = \frac{q_X}{\eta} \left|\log\left(\frac{\varepsilon}{q_X^n}\right)\right|$ gives

$$\sum_{i\,:\,\boldsymbol{x}_i \in X \setminus X_{r(\varepsilon)}(\boldsymbol{x}_k)} \left|M_{ki}^{-1}\right| \leq C \frac{e^{\eta}}{\eta^n} q_X^{\frac{n}{2}} \varepsilon^{\frac{1}{2}}.$$

□

This estimate allows us now to bound the error between the full Lagrange functions and their cut-off versions.

Theorem 3.2.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \Omega$ be a quasi-uniform set of sites with fill distance $h_{X,\Omega}$ and separation radius q_X. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ according to (2.4.6). For $1 \leq k \leq N$ let χ_k be the Lagrange function anchored in $\boldsymbol{x}_k \in X$ and $\widetilde{\chi}_k$ be the cut-off Lagrange function with footprint $X_{r(\varepsilon)}(\boldsymbol{x}_k)$ with $r(\varepsilon) = \frac{q_X}{\eta} \left|\log\left(\frac{\varepsilon}{q_X^n}\right)\right|$. Then there exists a constant $C = C(\Omega, \Phi, n, s)$ such that for $0 \leq t \leq s$ the estimate*

$$\|\chi_k - \widetilde{\chi}_k\|_{H^t(\Omega)} \leq C \frac{e^{\eta}}{\eta^n} q_X^{-t} \sqrt{\varepsilon} \tag{3.2.10}$$

holds for all $1 \leq k \leq N$. Furthermore, there exists a constant $C = C(\Omega, \Phi, n)$ such that the estimate

$$\|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)} \leq C \frac{e^{\eta}}{\eta^n} q_X^{-\frac{n}{2}} \sqrt{\varepsilon} \tag{3.2.11}$$

holds for all $1 \leq k \leq N$.

PROOF. We begin with the $H^t(\Omega)$-norm estimate. For $1 \le k \le N$ we have

$$\begin{aligned}\|\chi_k - \widetilde{\chi}_k\|_{H^t(\Omega)} &= \\ &= \left\| \sum_{i=1}^{N} M_{ki}^{-1} \Phi_\delta(\cdot - \boldsymbol{x}_i) - \sum_{i\,:\,\boldsymbol{x}_i \in X_{r(\varepsilon)}(\boldsymbol{x}_k)} M_{ki}^{-1} \Phi_\delta(\cdot - \boldsymbol{x}_i) \right\|_{H^t(\Omega)} \\ &= \left\| \sum_{i\,:\,\boldsymbol{x}_i \in X \setminus X_{r(\varepsilon)}(\boldsymbol{x}_k)} M_{ki}^{-1} \Phi_\delta(\cdot - \boldsymbol{x}_i) \right\|_{H^t(\Omega)},\end{aligned}$$

since we use the same coefficients M_{ki}^{-1} for both functions. Using the estimate on $\|\Phi_\delta(\cdot - \boldsymbol{x}_i)\|_{H^t(\Omega)}$ in Lemma 2.4.3 we obtain, with the help of the triangle inequality,

$$\begin{aligned}\|\chi_k - \widetilde{\chi}_k\|_{H^t(\Omega)} &= \left\| \sum_{i\,:\,\boldsymbol{x}_i \in X \setminus X_{r(\varepsilon)}(\boldsymbol{x}_k)} M_{ki}^{-1} \Phi_\delta(\cdot - \boldsymbol{x}_i) \right\|_{H^t(\Omega)} \\ &\le \sum_{i\,:\,\boldsymbol{x}_i \in X \setminus X_{r(\varepsilon)}(\boldsymbol{x}_k)} \left| M_{ki}^{-1} \right| \|\Phi_\delta(\cdot - \boldsymbol{x}_i)\|_{H^t(\Omega)} \\ &\le \delta^{-\frac{n}{2}-t} \|\Phi\|_{H^t(\mathbb{R}^n)} \sum_{i\,:\,\boldsymbol{x}_i \in X \setminus X_{r(\varepsilon)}(\boldsymbol{x}_k)} \left| M_{ki}^{-1} \right|.\end{aligned}$$

Next, we use (3.2.9) and the assumption that X is quasi-uniform. This yields

$$\begin{aligned}\|\chi_k - \widetilde{\chi}_k\|_{H^t(\mathbb{R}^n)} &\le \delta^{-\frac{n}{2}-t} \|\Phi\|_{H^t(\Omega)} \sum_{i\,:\,\boldsymbol{x}_i \in X \setminus X_{r(\varepsilon)}(\boldsymbol{x}_k)} \left| M_{ki}^{-1} \right| \\ &\le C(\Omega, \Phi, n, s) \frac{e^\eta}{\eta^n} q_X^{-\frac{n}{2}-t} q_X^{\frac{n}{2}} \sqrt{\varepsilon}.\end{aligned}$$

The same ideas lead to the estimate in (3.2.11). □

This result, in particular the error estimate in (3.2.11), together with Theorem 3.2.5 proves the next lemma.

Corollary 3.2.8. *With the notation and assumptions of Theorem 3.2.7, the set of cut-off Lagrange functions $\{\widetilde{\chi}_k\}_{1 \le k \le N}$ is a stable basis of V_N if $\varepsilon \le q_X^{2J+n}$ with a free constant $J > n$.*

Inserting $\varepsilon = q_X^{2J+n}$ into the cut-off radius $r(\varepsilon)$ yields a radius

$$r(q_X^{2J+n}) = 2J \frac{q_X}{\eta} \left| \log(q_X) \right|.$$

We note that for quasi-uniform X this is similar to the radius the authors of [29, 46] use.

Although we know from Corollary 3.2.8 that the cut-off Lagrange functions are, with the right choice of ε, a stable basis of V_N their construction still requires to set up an $N \times N$ linear system and to solve it N-times.

3.3. Localized Lagrange Functions

We now construct yet another basis of V_N by enforcing the Lagrange condition as in Corollary 3.1.2 only on the footprint $X_{r(\varepsilon)}(\boldsymbol{x}_k)$.

Definition 3.3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ be a quasi-uniform set of sites in Ω with fill distance $h_{X,\Omega}$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., its Fourier transform satisfies* (2.2.2)*. Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ the rescaled kernel, where δ is coupled to $h_{X,\Omega}$ according to* (2.4.6)*. For $1 \le k \le N$ let $X_r(\boldsymbol{x}_k)$ be a general footprint defined in* (3.2.2) *with cut-off radius $r > 0$. For $1 \le k \le N$ we define the* localized Lagrange function $\chi_k^{loc} : \mathbb{R}^n \to \mathbb{R}$ *by*

$$\chi_k^{loc} := \sum_{i \,:\, \boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)} \alpha_i^{(k)} \Phi_\delta(\cdot - \boldsymbol{x}_i), \tag{3.3.1}$$

where the coefficient vector $\boldsymbol{\alpha}^{(k)}$ is the unique solution of the linear system

$$M|_{X_r(\boldsymbol{x}_k)} \boldsymbol{\alpha}^{(k)} = \boldsymbol{e}_k.$$

The matrix $M|_{X_r(\boldsymbol{x}_k)} \in \mathbb{R}^{\#X_r(\boldsymbol{x}_k) \times \#X_r(\boldsymbol{x}_k)}$ is given by

$$M|_{X_r(\boldsymbol{x}_k)} = (\Phi_\delta(\boldsymbol{x}_i - \boldsymbol{x}_j))_{\boldsymbol{x}_i, \boldsymbol{x}_j \in X_r(\boldsymbol{x}_k)}$$

and $\boldsymbol{e}_k \in \mathbb{R}^{\#X_r(\boldsymbol{x}_k)}$ denotes the k-th unit vector.

Again, we see that $\chi_k^{loc} \in V_N$, $1 \le k \le N$ and, with the same reasoning as in Corollary 3.1.2, we see that the localized Lagrange functions satisfy the Lagrange condition $\chi_k^{loc}(\boldsymbol{x}_i) = \delta_{ki}$ for $\boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)$, $1 \le k \le N$.

We now work towards error estimates for $\chi_k - \chi_k^{loc}$. Using these allows us, similar to Section 3.2, to show that the set $\{\chi_k^{loc}\}_{1 \le k \le N}$ is a stable basis of V_N, depending on the right choice for the cut-off radius r. To this end, we fix $\boldsymbol{x}_k \in X$ and introduce for $\boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)$ the vector $\boldsymbol{y} \in \mathbb{R}^{\#X_r(\boldsymbol{x}_k)}$, which is given component-wise as

$$y_i := \chi_k^{loc}(\boldsymbol{x}_i) - \widetilde{\chi}_k(\boldsymbol{x}_i) = \delta_{ki} - \widetilde{\chi}_k(\boldsymbol{x}_i). \tag{3.3.2}$$

Additionally, we observe that

$$\begin{aligned} \chi_k^{loc} - \widetilde{\chi}_k &= \sum_{i \,:\, \boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)} (\alpha_i^{(k)} - M_{ki}^{-1}) \Phi_\delta(\cdot - \boldsymbol{x}_i) \\ &=: \sum_{i \,:\, \boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)} \beta_i \Phi_\delta(\cdot - \boldsymbol{x}_i). \end{aligned} \tag{3.3.3}$$

We see that the vector $\boldsymbol{\beta} = (\beta_i)_i \in \mathbb{R}^{\#X_r(\boldsymbol{x}_k)}$ is the solution of the linear system

$$M|_{X_r(\boldsymbol{x}_k)} \boldsymbol{\beta} = \boldsymbol{y}.$$

Next, we derive a bound for the matrix norm associated to the 1-norm of $M|^{-1}_{X_r(\boldsymbol{x}_k)}$. We recall Lemma 2.5.11 which gives us the following uniform bound on the entries of this matrix,

$$\left| \left(M|^{-1}_{X_r(\boldsymbol{x}_k)} \right)_{ik} \right| \le C q_X^{n-2s},$$

for all $1 \le i,k \le \#X_r(\boldsymbol{x}_k)$ with a constant $C > 0$. With this, we obtain the bound

$$\left\| M|^{-1}_{X_r(\boldsymbol{x}_k)} \right\|_1 \le C(\#X_r(\boldsymbol{x}_k)) q_X^{n-2s}. \tag{3.3.4}$$

Recalling the component-wise definition of $\boldsymbol{y}$ in (3.3.2), $y_i = \delta_{ki} - \widetilde{\chi}_k(\boldsymbol{x}_i)$, and using the fact that the full Lagrange function satisfies the Lagrange condition $\chi_k(\boldsymbol{x}_i) = \delta_{ki}$ for $\boldsymbol{x}_i \in X_r(\boldsymbol{x}_k) \subseteq X$, we can estimate

$$\begin{aligned} \|\boldsymbol{y}\|_\infty &= \|\chi_k^{loc} - \widetilde{\chi}_k\|_{\ell_\infty(X_r(\boldsymbol{x}_k))} = \|\chi_k - \widetilde{\chi}_k\|_{\ell_\infty(X_r(\boldsymbol{x}_k))} \\ &\le \|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)}. \end{aligned} \tag{3.3.5}$$

Combining (3.3.4) and (3.3.5) with $\|\boldsymbol{y}\|_1 \le (\#X_r(\boldsymbol{x}_k))\|\boldsymbol{y}\|_\infty$ yields finally the estimate

$$\begin{aligned} \sum_{i\,:\,\boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)} |\beta_i| &\le \left\| M|^{-1}_{X_r(\boldsymbol{x}_k)} \right\|_1 \|\boldsymbol{y}\|_1 \\ &\le C(\#X_r(\boldsymbol{x}_k))^2 q_X^{n-2s} \|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)}. \end{aligned} \tag{3.3.6}$$

With this we can now obtain bounds for the error $\chi_k^{loc} - \chi_k$ for general cut-off radius $r > 0$. After that we refine the estimate by using the specific cut-off radius $r(\varepsilon) = \frac{q_X}{\eta} \left| \log\left(\frac{\varepsilon}{q_X^n} \right) \right|$.

Theorem 3.3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subseteq \Omega$ be a quasi-uniform set of sites with fill distance $h_{X,\Omega}$ and separation radius q_X. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ be the rescaled kernel where δ is coupled to $h_{X,\Omega}$ according to (2.4.6). For $1 \le k \le N$ let $\chi_k \in V_N$ be the full Lagrange function anchored in $\boldsymbol{x}_k \in X$ and χ_k^{loc} be the localized Lagrange function as in (3.3.1) with general footprint $X_r(\boldsymbol{x}_k)$ and cut-off radius $r > 0$. Then there exists a constant $C = C(\Omega, \Phi, n) > 0$ such that the estimate*

$$\|\chi_k^{loc} - \chi_k\|_{L_\infty(\Omega)} \le \left(1 + C(\#X_r(\boldsymbol{x}_k))^2 q_X^{-2s}\right) \|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)} \tag{3.3.7}$$

holds for all $1 \le k \le N$. Furthermore, for $0 \le t \le s$ there is a constant $C = C(\Omega, \Phi, n, t) > 0$ such that the estimate

$$\|\chi_k^{loc} - \chi_k\|_{H^t(\Omega)} \le \left(1 + C(\#X_r(\boldsymbol{x}_k))^2 q_X^{\frac{n}{2}-2s-t}\right) \|\chi_k - \widetilde{\chi}_k\|_{H^t(\Omega)} \tag{3.3.8}$$

holds for all $1 \le k \le N$.

Proof. With the triangle inequality we have

$$\|\chi_k^{loc} - \chi_k\|_{L_\infty(\Omega)} \le \|\chi_k^{loc} - \widetilde{\chi}_k\|_{L_\infty(\Omega)} + \|\widetilde{\chi}_k - \chi_k\|_{L_\infty(\Omega)}.$$

We bound the first term by

$$\begin{aligned} \|\chi_k^{loc} - \widetilde{\chi}_k\|_{L_\infty(\Omega)} &\le \sum_{i\,:\,\boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)} |\beta_i| \|\Phi_\delta(\cdot - \boldsymbol{x}_i)\|_{L_\infty(\Omega)} \\ &\le C(\Omega, n)\|\Phi\|_{L_\infty(\mathbb{R}^n)} q_X^{-n} \sum_{i\,:\,\boldsymbol{x}_i \in X_r(\boldsymbol{x}_k)} |\beta_i| \\ &\le C(\Omega, \Phi, n) q_X^{-n} (\#X_r(\boldsymbol{x}_k))^2 q_X^{n-2s} \|\chi_k - \widetilde{\chi}_k\|_{L_\infty(\Omega)}. \end{aligned}$$

We employed the help of the estimate for the norm of the rescaled kernel in Lemma 2.4.3, (3.3.3) and (3.3.6).

The estimate in (3.3.8) can be obtained with the same approach. □

Next, we derive a refined bound for the specific cut-off radius $r(\varepsilon)$ in (3.2.1). The proof uses the estimate on the cardinality of the footprint in (3.2.3) and the error estimates of Theorem 3.2.7.

Theorem 3.3.3. *With the notation and assumptions of Theorem 3.3.2, where we additionally assume that the localized Lagrange functions use the footprint $X_{r(\varepsilon)}(\boldsymbol{x}_k)$ with cut-off radius $r(\varepsilon) = \frac{q_X}{\eta}\left|\log\left(\frac{\varepsilon}{q_X}\right)\right|$ with an $\varepsilon > 0$, there is a constant $C = C(\Omega, \Phi, n) > 0$ such that the estimate*

$$\|\chi_k^{loc} - \chi_k\|_{L_\infty(\Omega)} \leq C\frac{e^\eta}{\eta^{3n}}\left|\log\left(\frac{\varepsilon}{q_X^n}\right)\right|^{2n} q_X^{-\frac{n}{2}-2s}\sqrt{\varepsilon} \tag{3.3.9}$$

holds for all $1 \leq k \leq N$. Furthermore, for $0 \leq t \leq s$ there is a constant $C = C(\Omega, \Phi, n, t) > 0$ such that the estimate

$$\|\chi_k^{loc} - \chi_k\|_{H^t(\Omega)} \leq C\frac{e^\eta}{\eta^{3n}}\left|\log\left(\frac{\varepsilon}{q_X^n}\right)\right|^{2n} q_X^{\frac{n}{2}-2s-2t}\sqrt{\varepsilon} \tag{3.3.10}$$

holds for all $1 \leq k \leq N$.

These error estimates, in particular (3.3.9), allow us to prove that the set $\{\chi_k^{loc}\}_{1\leq k\leq N}$ is a stable basis of the kernel-based approximation space V_N. It is easy to see that we can apply Theorem 3.2.5 in this setting and obtain the stability of the localized Lagrange basis.

Lemma 3.3.4. *With the notation and assumptions of Theorem 3.3.3 the set of localized Lagrange functions $\{\chi_k^{loc}\}_{1\leq k\leq N}$ is a stable basis of V_N if $\varepsilon > 0$ is chosen sufficiently small such that the inequality*

$$C\frac{e^\eta}{\eta^{3n}}\left|\log\left(\frac{\varepsilon}{q_X^n}\right)\right|^{2n} q_X^{-\frac{n}{2}-2s}\sqrt{\varepsilon} \leq q_X^J,$$

holds with a constant $C > 0$ and a free $J > n$.

Lemma 3.3.4 implies that, assuming we chose the cut-off radius $r(\varepsilon)$ correctly, we can express elements of V_N as a linear combination of $\{\chi_k^{loc}\}_{1\leq k\leq N}$. In particular, for a given $f \in H^s(\Omega)$, $s > n/2$, we have that

$$\mathcal{I}^{loc}(f) := \sum_{k=1}^{N} f(\boldsymbol{x}_k)\chi_k^{loc} \in V_N. \tag{3.3.11}$$

This motivates the next definition.

Definition 3.3.5. *With the notation and assumptions of Theorem 3.3.3, where we choose $\varepsilon > 0$ such that Lemma 3.3.4 holds, we define the* local Lagrange approximation operator $\mathcal{I}^{loc} : H^s(\Omega) \to V_N$ *by* (3.3.11).

Clearly, the operator in Definition 3.3.5 is not an interpolation operator. Approximations of the form (3.3.11) are often called quasi-interpolation, see, e.g., [94, Chapter 3]. However, we usually require quasi-interpolants to reproduce polynomials up to a certain degree. We can not expect the local Lagrange approximation to have this property. Furthermore, with the

same reasoning as in Section 2.4, we do not expect convergence of $\mathcal{I}^{loc}(f)$ towards f for $h_{X,\Omega} \to 0$. However, in the next chapter we will work towards a convergent method using local Lagrange approximation operators.

Nevertheless, using the localized Lagrange basis has an advantage. To compute the basis we need to solve only an $\#X_{r(\varepsilon)}(\boldsymbol{x}_i) \times \#X_{r(\varepsilon)}(\boldsymbol{x}_i)$ linear system for every $\boldsymbol{x}_i \in X$. Each of these N-many systems will be sparse if the cut-off radius $r(\varepsilon)$ is sufficiently larger than δ and depending on this cut-off radius the computational cost will be lower than the solution of N-many $N \times N$ sparse systems to compute the full Lagrange basis introduced in Section 3.1.

CHAPTER 4

The Kernel-based Multilevel Method

We use this chapter to study a way to beat the trade-off principle discussed at the end of Section 2.4. The idea of the kernel-based multilevel method can be traced back to [73] and became publicly known through [27]. However, these early contributions only observe numerical convergence. The first theoretical error analysis was given in [63], but for a different multilevel scheme, where the basis functions for each level are not obtained by scaling but rather by convolution of the RBF for the finest level. A first convergence theorem for the multilevel method in the setting we use in this thesis was provided in [36] if the domain is the sphere. This result was then generalized to general bounded domains in [91]. We also refer to the survey paper [92].

We first introduce the setting of multiscale analysis in Section 4.1. This can be seen as the transfer of the multiresolution analysis to the theory of RBFs. In Section 4.2 we formally introduce the kernel-based multilevel method before, in Section 4.3, we derive convergence results for the different approximation methods introduced in Section 2.3. In all those error estimates there appears a constant that potentially can spoil the convergence. In Section 4.4 we give the results of numerical simulations and show that for the, in the context of this thesis important, cases the analytically unknown constant has no negative effects. Next, we briefly discuss two adaptive versions of the multilevel method in Section 4.5. To close this chapter we use the functions introduced in Section 3.1 to first derive an alternative representation of the approximation operator and then the functions of Section 3.3 to potentially speed up point-evaluations of approximations using this representation. Additionally, we provide a convergence estimate.

4.1. Multiscale Analysis

We start by fixing a sequence of sets of sites $X_1, X_2, X_3, \cdots \subseteq \Omega$, with

$$X_i := \{\boldsymbol{x}_{i,1}, \dots, \boldsymbol{x}_{i,N_i}\},$$

in a bounded domain $\Omega \subseteq \mathbb{R}^n$. In many applications these sets will be nested, that is, $X_i \subseteq X_{i+1}$, however for the analysis of the multilevel method this is not necessary. Nevertheless, we require that the sets become denser in Ω, which we quantify in the following way: We assume that the associated fill distances $h_i := h_{X_i,\Omega}$, $i = 1, 2, 3, \dots$, decrease in a uniform way, i.e., we fix a *refinement parameter* $\mu \in (0, 1)$ independent of the *level* i and assume that there is a constant $c \in (0, 1]$ such that

$$c\mu h_i \leq h_{i+1} \leq \mu h_i, \quad i \in \mathbb{N}. \tag{4.1.1}$$

Although it is useful to have the leeway the constant c gives in the coupling of h_{i+1} and h_i it makes some discussions unnecessarily cumbersome.

Hence, for the following we assume that $c = 1$ and we have $h_{i+1} = \mu h_i$. In some way the parameter μ determines the growth of the cardinality of the X_i. If we recall that for quasi-uniform X_i we have $N_i^{-n} \sim h_i$ we see that $\mu \approx (N_i/N_{i+1})^n$. Hence the choice of a small value of μ, close to 0, leads to a fast growth of the cardinality of X_i and a value close to 1 ensures that the X_i fill out Ω slowly. For practical considerations both of these extremes is not preferable, we would rather choose a moderate value, e.g., $\mu = 1/2$.

Next, we follow the ideas of Chapter 2 and fix a radial basis function $\Phi : \mathbb{R}^n \to \mathbb{R}$ and scale this *mother kernel* with a level dependent scaling parameter $\delta_i > 0$, i.e., we define level-dependent rescaled RBFs by

$$\Phi_i := \Phi_{\delta_i} := \delta_i^{-n} \Phi\left(\frac{\cdot}{\delta_i}\right). \tag{4.1.2}$$

This way, we obtain a basis function for every level $i \in \mathbb{N}$. In principle, the fixing and rescaling of one single mother kernel Φ is not necessary and we can choose a different kernel for each level. However, we want to follow [91, 92] and use the setup above.

On each level $i \in \mathbb{N}$, we use the stationary setting of Section 2.4, i.e., we couple δ_i to h_i. To be precise and recalling (2.4.6), we have constants $c_\nu \in (0, 1]$ and $\nu > 1$ such that

$$c_\nu \nu h_i \leq \delta_i \leq \nu h_i, \quad i \in \mathbb{N}.$$

Note that neither c_ν nor ν depend on the level i. In applications we often couple the refinement parameter μ to the overlap parameter ν.

We define now the approximation spaces of the multilevel method.

Definition 4.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Let $X_1, X_2, X_3, \cdots \subseteq \Omega$ be a sequence of sets of sites. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a radial basis function. For $i \in \mathbb{N}$ let Φ_i be the rescaled kernel with level-dependent scaling parameter δ_i. The* local approximation *or* detail space *on level i is given by*

$$W_i = \operatorname{span}\{\Phi_i(\cdot - \boldsymbol{x}) \ : \ \boldsymbol{x} \in X_i\}$$

and the global approximation space *on level $L \in \mathbb{N}$ is defined as*

$$V_L = W_1 + \cdots + W_L.$$

In the error analysis we are interested in the approximation power of the space V_L as $L \to \infty$, but in the numerical reality we fix a level $L \in \mathbb{N}$ and compute the approximation in V_L.

If we assume that the mother kernel Φ, and hence the level-dependently rescaled kernel Φ_i, is a reproducing kernel of $H^s(\mathbb{R}^n)$ we have the following proposition for the local and global approximation spaces.

Proposition 4.1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. For $i \in \mathbb{N}$ let $X_i \subseteq \Omega$ be a sets of sites with fill distance h_i. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies* (2.2.2)*. For $i \in \mathbb{N}$ let $\Phi_i = \delta_i^{-n} \Phi(\cdot/\delta_i)$ be the rescaled kernel where the scaling parameter δ_i is coupled to h_i as in* (2.4.6)*. Set $V_0 := \{0\}$.*

Then, for every $i \in \mathbb{N}_0$, the global approximation space V_i is a closed and finite dimensional subspace of $H^s(\Omega)$ and therefore of $L_2(\Omega)$. Furthermore, the following statements hold:

(1) $V_i \subseteq V_{i+1}$, $i \in \mathbb{N}_0$,
(2) $\overline{\bigcup_{i=0}^{\infty} V_i} = L_2(\Omega)$ *if we take the closure with respect to the* $L_2(\Omega)$*-norm.*

Additionally, for every $i \in \mathbb{N}$, *the sum*

$$V_i = V_{i-1} \oplus W_i$$

is direct, i.e., we have

$$\bigoplus_{i=1}^{\infty} W_i = L_2(\Omega).$$

PROOF. The proof for the case that $\Omega = S^{n-1}$ is the unit sphere can be found in [36]. The ideas there can be directly transferred to the case of general bounded domains $\Omega \subseteq \mathbb{R}^n$. □

The properties of the local and global approximation spaces in Proposition 4.1.2 are the usual setup for the multiresolution analysis in the context of, e.g., wavelets [18]. However, the setting for kernel-based spaces misses any kind of refinement property, i.e., we can not expect that $f \in V_i$ if and only if $f(\cdot/\delta_i) \in V_1$ which would mean that the spaces V_i are scaled versions of V_1.

4.2. Description of the Method

We now discuss the algorithmic approach to the kernel-based multilevel method. The main idea is similar to the one of wavelets: We compute a rough approximation on a coarse point set and then fill in the details which we compute on finer and finer sets of sites. This section follows closely [91, Section 2].

The general procedure is given in Algorithm 1 which we now discuss in greater detail. We assume that we already know the family of point sets $(X_i)_{1 \le i \le L}$. How to determine this set itself is part of ongoing research in computer science and of no further concern in this thesis.

As input we expect the maximum number of levels L and the tuples $\{(\boldsymbol{x}_{i,k}, f(\boldsymbol{x}_{i,k})\} \subseteq X_i \times \mathbb{R}$, $1 \le i \le L$, $1 \le k \le N_i$. If the sites are nested, i.e., $X_i \subseteq X_{i+1}$ it suffices to know the data on the highest level L.

For every level $0 \le i \le L$ we keep track of two functions. The *global approximation* $f_i \in V_i$ and the residual e_i, which represents the remaining error.

We initialize the global approximation on level zero as the zero function, $f_0 = 0$, and the residual of the zeroth level as the target function $e_0 = f$. In the i-th step we compute the *local approximation* $s_i \in W_i$ as the approximation to the residual e_{i-1} of the preceding level using the sites X_i of level i, i.e., we compute a coefficient vector $\boldsymbol{\alpha}^{(i)} \in \mathbb{R}^{N_i}$ according to the chosen approximation method. The local approximation s_i then has the representation

$$s_i = \sum_{k=1}^{N_i} \alpha_k^{(i)} \Phi_i(\cdot - \boldsymbol{x}_{i,k}).$$

Next, we update the global approximation, $f_i = f_{i-1} + s_i$, and the residual, $e_i = e_{i-1} - s_i$. To be more precise we have for the i-th level

$$f_i = s_1 + \cdots + s_i, \tag{4.2.1}$$

$$e_i = f - (s_1 + \cdots + s_i). \tag{4.2.2}$$

We follow this procedure until we arrive at level L and return the global approximation f_L.

Algorithm 1: Multilevel approximation

Data: Number of levels L, right-hand side f
Result: Approximate solution $f_L \in V_L = W_1 + \cdots + W_L$
Set $f_0 = 0, e_0 = f$;
for $i = 1, 2, \ldots, L$ **do**
  Determine a local approximant $s_i \in W_i$ to e_{i-1} on X_i;
  Set $f_i = f_{i-1} + s_i$;
  Set $e_i = e_{i-1} - s_i$;
end

The algorithm can also be put in an alternative, matrix form. If we assume that the approximation methods we use for computing the local approximations s_i are linear, we can find matrices $M_i \in \mathbb{R}^{N_i \times N_i}$, $i \in \mathbb{N}$, such that the coefficients $\boldsymbol{\alpha}^{(i)}$ can be expressed as

$$\boldsymbol{\alpha}^{(i)} = M_i^{-1} \boldsymbol{e}_{i-1},$$

where $\boldsymbol{e}_{i-1} = e_{i-1}|_{X_i}$. For instance, in the case of interpolation we have $M_i = M_{X_i,\Phi_i}$ and in the case of penalized least-squares approximation we have $M_i = M_{X_i,\Phi_i} + \lambda_i I$. Note that we can, and we will see later that we have to choose a level-dependent penalization parameter λ_i. In addition to M_i we introduce the evaluation matrix

$$B_{ik} = (\Phi_k(\boldsymbol{x}_{i,m} - \boldsymbol{x}_{k,p})) \in \mathbb{R}^{N_i \times N_k}$$

which represents the evaluation of the kernel of the k-th level, centered in points of X_k at points of X_i. With these matrices we can express $e_i|_{X_i}$ as in (4.2.2) by

$$M_i \boldsymbol{\alpha}^{(i)} = \boldsymbol{f}^{(i)} - \sum_{k=1}^{i-1} B_{ik} \boldsymbol{\alpha}^{(k)}$$

with $\boldsymbol{f}^{(i)} = f|_{X_i} \in \mathbb{R}^{N_i}$, $1 \le i \le L$. Collecting these equations in one large $\left(\sum_{i=1}^{L} N_i\right) \times \left(\sum_{i=1}^{L} N_i\right)$ linear system yields a block-triangular system of the form

$$\begin{pmatrix} M_1 & & & & \\ B_{21} & M_2 & & & \\ B_{31} & B_{32} & M_3 & & \\ \vdots & \vdots & \cdots & \ddots & \\ B_{L1} & B_{L2} & \cdots & B_{L(L-1)} & M_L \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}^{(1)} \\ \boldsymbol{\alpha}^{(2)} \\ \boldsymbol{\alpha}^{(3)} \\ \vdots \\ \boldsymbol{\alpha}^{(L)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}^{(1)} \\ \boldsymbol{f}^{(2)} \\ \boldsymbol{f}^{(3)} \\ \vdots \\ \boldsymbol{f}^{(L)} \end{pmatrix}.$$

The method described in Algorithm 1 simply solves this system by mimicking forward substitution, however since the entries are matrices, we have to solve a linear system with a growing number of unknowns in every step. At first glance this seems expensive, however from a numerical point of view this scheme and the whole multilevel method is extremely efficient as the matrices M_i, $1 \leq i \leq L$, are essentially the kernel matrices discussed in Section 2.5. With Corollary 2.5.1 we immediately obtain the next corollary.

Corollary 4.2.1. *With the notation and assumptions of Proposition 4.1.2 let* $M_i \in \mathbb{R}^{N_i \times N_i}$ *be defined by either* $M_i = (\Phi_i(\boldsymbol{x}_{i,j} - \boldsymbol{x}_{i,k}))$ *or* $M_i = (\Phi_i(\boldsymbol{x}_{i,j} - \boldsymbol{x}_{i,k}) + \lambda_i I)$. *Then*

(1) The number of non-zero entries per row of M_i *is independent of the level* i.
(2) The number of steps required by a non-preconditioned CG method is independent of the level i.
(3) The computational cost to solve the approximation problem on level i *is* $\mathcal{O}(N_i)$.

From an operator point of view we can express the multilevel method with the help of the following lemma.

Lemma 4.2.2. *With the notation and assumptions of Proposition 4.1.2 let the local approximations* $s_i \in W_i$ *be computed with specific operators* $\mathcal{I}_i : H^s(\Omega) \to W_i$, *i.e., we have* $s_i = \mathcal{I}_i(e_{i-1})$. *Then, for each* $L \in \mathbb{N}$, *there is an operator* $A_L : H^s(\Omega) \to V_L$ *defined by*

$$A_L(f) := \sum_{i=1}^{L} \mathcal{I}_i(e_{i-1}) \tag{4.2.3}$$

such that $f_L = A_L(f)$.

Proof. This is a direct consequence of (4.2.1) and (4.2.2). □

Definition 4.2.3. *For* $i \in \mathbb{N}$ *the operator* $\mathcal{I}_i$ *of Lemma 4.2.2 is called* local approximation operator *and the operator* A_L *is called* global approximation *or* multilevel operator.

Later, we will call the multilevel operator A_L in the form (4.2.3) *residual correction scheme*.

4.3. Convergence of the Method

Now, we investigate the kernel-based multilevel method further and derive convergence results for the three different approximation methods introduced in Section 2.3, interpolation, interpolation of rougher function and penalized least-squares approximation. We will see that the key for the convergence results is a recursion formula of the form

$$\|Ee_{i+1}\|_{\Phi_i} \leq \alpha \|Ee_i\|_{\Phi_{i-1}}$$

with a constant $\alpha < 1$. With the right choice of the occurring parameters this will lead to linear convergence of the method in all three cases.

The ideas of the proofs for all three approximation methods are taken from [91], however, we keep meticulous track of the occuring constants.

4.3.1. Interpolation. We start by investigating interpolation, i.e., the local approximation operator $\mathcal{I}_i$ on level i is the interpolation operator I_{X_i,Φ_i}. This means we have

$$\mathcal{I}_i(e_{i-1}) = s_i = I_{X_i,\Phi_i}(e_{i-1}).$$

First, we have to make sure that $A_L(f) = \sum_{i=1}^{L} I_{X_i,\Phi_i}(e_{i-1})$ is indeed an interpolant of f on X_L.

Lemma 4.3.1. *For every $1 \leq i \leq L$ the function $f_i = \sum_{k=1}^{i} I_{X_k,\Phi_k}(e_{k-1})$ interpolates f on X_i.*

Proof. We have by induction that $e_k|_{X_k} \equiv 0$ for $1 \leq k \leq i$. Furthermore, for these k we have by (4.2.2) the relation $f - f_k = e_k$ between the residuals and the interpolants. Hence, $f_i|_{X_i} = f|_{X_i} - e_i|_{X_i} = f|_{X_i}$ holds. □

We find the following recursion inequality which allows us to bound the Φ_{i+1}-norm of the residual e_i of level i by the Φ_i-norm of e_{i-1}.

Theorem 4.3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. For $i \in \mathbb{N}$ let $X_i \subseteq \Omega$ be a set of sites with fill distance h_i. Assume that $c\mu h_i \leq h_{i+1} \leq \mu h_i$ holds for $i \in \mathbb{N}$ with fixed constants $\mu \in (0,1)$, $c \in (0,1]$ and h_1 sufficiently small. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_i = \delta_i^{-n}\Phi(\cdot/\delta_i)$ be the rescaled kernel where the scaling parameter δ_i is coupled to h_i according to (2.4.6). Assume that the overlap parameter ν satisfies $\frac{1}{h_1} \geq \nu$. Assume that $f \in H^s(\Omega)$. Then there exists a constant $C_1 = C_1(\Omega, \Phi, s) > 0$ such that*

$$\|Ee_i\|_{\Phi_{i+1}} \leq C_1(\mu^s + \nu^{-s})\|Ee_{i-1}\|_{\Phi_i}, \quad i \in \mathbb{N}, \tag{4.3.1}$$

where $E : H^s(\Omega) \to H^s(\mathbb{R}^n)$ is the extension operator introduced in Theorem 2.1.7.

Proof. We start by estimating

$$\delta_{i+1} = \nu h_{i+1} \leq \nu\mu h_i = \mu\delta_i < \delta_i.$$

This means that $\delta_{i+1} < \delta_1 = \nu h_1$ and with the assumption $\frac{1}{h_1} \geq \nu$ we have $\delta_{i+1} < 1$. Hence, we can use the norm equivalence for the rescaled RBFs in (2.4.2) and obtain the bound

$$\begin{aligned}
\|Ee_i\|_{\Phi_{i+1}}^2 &= \int_{\mathbb{R}^n} \frac{|\widehat{Ee_i}(\boldsymbol{\xi})|^2}{\widehat{\Phi_{i+1}}(\boldsymbol{\xi})}\, d\boldsymbol{\xi} \\
&\leq \frac{1}{c_1}\int_{\mathbb{R}^n} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \left(1 + \delta_{i+1}^2\|\boldsymbol{\xi}\|_2^2\right)^s\, d\boldsymbol{\xi} \\
&=: \frac{1}{c_1}(I_1 + I_2)
\end{aligned}$$

with

$$\begin{aligned}
I_1 &:= \int_{\|\boldsymbol{\xi}\|_2 \leq \frac{1}{\delta_{i+1}}} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \left(1 + \delta_{i+1}^2\|\boldsymbol{\xi}\|_2^2\right)^s\, d\boldsymbol{\xi}, \\
I_2 &:= \int_{\|\boldsymbol{\xi}\|_2 \geq \frac{1}{\delta_{i+1}}} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \left(1 + \delta_{i+1}^2\|\boldsymbol{\xi}\|_2^2\right)^s\, d\boldsymbol{\xi}.
\end{aligned}$$

We bound these two integrals separately.

First, we observe that the interpolants $I_{X_i,\Phi_i} e_{i-1}$ and $I_{X_i,\Phi_i}(Ee_{i-1})$ coincide, since the interpolant is uniquely determined by the values on $X_i \subseteq \Omega$. Then we have with the properties of the extension operator and Lemma 2.4.2

$$\begin{aligned}
\|e_i\|_{H^s(\Omega)} &= \|e_{i-1} - I_{X_i,\Phi_i}(e_{i-1})\|_{H^s(\Omega)} \\
&= \|Ee_{i-1} - I_{X_i,\Phi_i}(Ee_{i-1})\|_{H^s(\Omega)} \\
&\leq \|Ee_{i-1} - I_{X_i,\Phi_i}(Ee_{i-1})\|_{H^s(\mathbb{R}^n)} \\
&\leq c_2^{\frac{1}{2}} \delta_i^{-s} \|Ee_{i-1} - I_{X_i,\Phi_i}(Ee_{i-1})\|_{\Phi_i} \\
&\leq c_2^{\frac{1}{2}} \delta_i^{-s} \|Ee_{i-1}\|_{\Phi_i}. \qquad (4.3.2)
\end{aligned}$$

To arrive at the last inequality we used the Φ_i-norm minimality of the interpolant (2.3.8).

To bound I_1 we use that $\delta_{i+1}\|\boldsymbol{\xi}\|_2 \leq 1$ and then the sampling inequality (2.3.11), since e_i vanishes on X_i. This yields

$$\begin{aligned}
I_1 &= \int_{\|\boldsymbol{\xi}\|_2 \leq \frac{1}{\delta_{i+1}}} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \left(1 + \delta_{i+1}^2 \|\boldsymbol{\xi}\|_2^2\right)^s \, d\boldsymbol{\xi} \\
&\leq 2^s \int_{\|\boldsymbol{\xi}\|_2 \leq \frac{1}{\delta_{i+1}}} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\
&\leq 2^s \int_{\mathbb{R}^n} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\
&= 2^s \|Ee_i\|_{L_2(\mathbb{R}^n)}^2 \\
&\leq 2^s C_E^2 \|e_i\|_{L_2(\Omega)}^2 \\
&= 2^s C_E^2 \|e_{i-1} - I_{X_i,\Phi_i}(e_{i-1})\|_{L_2(\Omega)}^2 \\
&\leq \widetilde{c}(\Omega, s) 2^s C_E^2 h_i^{2s} \|e_{i-1}\|_{H^s(\Omega)}^2 \\
&\leq \widetilde{c}(\Omega, s) 2^s C_E^2 c_2 \left(\frac{h_i}{\delta_i}\right)^{2s} \|Ee_{i-1}\|_{\Phi_i}^2 \\
&\leq c(\Omega, s) 2^s C_E^2 c_2 \nu^{-2s} \|Ee_{i-1}\|_{\Phi_i}^2,
\end{aligned}$$

where $\widetilde{c}(\Omega, s)$ is the constant appearing in the sampling inequality (2.3.11) and $c(\Omega, s) := \widetilde{c}(\Omega, s) c_\nu$. The constant c_ν is the constant from (2.4.6). The application of the sampling inequality is possible since h_1 assumed to be sufficiently small.

To bound I_2 we first observe that $\delta_{i+1}\|\boldsymbol{\xi}\|_2 \geq 1$. This yields

$$\left(1 + \delta_{i+1}^2 \|\boldsymbol{\xi}\|_2^2\right)^s \leq 2^s \delta_{i+1}^{2s} \|\boldsymbol{\xi}\|_2^{2s} \leq 2^s \delta_{i+1}^{2s} \left(1 + \|\boldsymbol{\xi}\|_2^2\right)^s.$$

Together with (4.3.2), we have

$$\begin{aligned} I_2 &= \int_{\|\boldsymbol{\xi}\|_2 \geq \frac{1}{\delta_{i+1}}} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \left(1 + \delta_{i+1}^2 \|\boldsymbol{\xi}\|_2^2\right)^s \, d\boldsymbol{\xi} \\ &\leq 2^s \delta_{i+1}^{2s} \int_{\mathbb{R}^n} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \left(1 + \|\boldsymbol{\xi}\|_2^2\right)^s \, d\boldsymbol{\xi} \\ &= 2^s \delta_{i+1}^{2s} \|Ee_i\|_{H^s(\mathbb{R}^n)}^2 \\ &\leq 2^s \delta_{i+1}^{2s} C_E^2 \|e_i\|_{H^s(\Omega)}^2 \\ &\leq 2^s C_E^2 c_2 \left(\frac{\delta_{i+1}}{\delta_i}\right)^{2s} \|Ee_{i-1}\|_{\Phi_i}^2 \\ &= 2^s C_E^2 c_2 \mu^{2s} \|Ee_{i-1}\|_{\Phi_i}^2. \end{aligned}$$

Putting these bounds together gives the estimate (4.3.1) with

$$C_1 = \max(c(\Omega, s), 1) 2^{\frac{s}{2}} C_E c_2^{\frac{1}{2}}. \tag{4.3.3}$$

□

In the proof of Theorem 4.3.2 we made an effort to keep track of the constants appearing in the estimates. Unfortunately, although we can track the individual factors contributing to C_1 quite precisely, sharp bounds are not known. However, we will give numerical estimates of this constant in Section 4.4.

We now use the estimate in (4.3.1) to derive convergence estimates for the error $f - f_L$ in the $L_2(\Omega)$-norm. We use again the sampling inequalities of Theorem 2.3.4 and recall that $e_L|_{X_L} = 0$. This then yields

$$\begin{aligned} \|f - f_L\|_{L_2(\Omega)} = \|e_L\|_{L_2(\Omega)} &\leq C h_L^s \|e_L\|_{H^s(\Omega)} \\ &\leq C h_L^s \|Ee_L\|_{H^s(\mathbb{R}^n)} \\ &\leq C h_L^s \delta_{L+1}^{-s} \|Ee_L\|_{\Phi_{L+1}} \\ &= C \left(\frac{1}{\nu\mu}\right)^s \|Ee_L\|_{\Phi_{L+1}}, \end{aligned}$$

and since we assume that $c\mu h_i \leq h_{i+1}$, with $c \in (0, 1]$, we obtain

$$\frac{h_L}{\delta_{L+1}} = \frac{h_L}{\nu h_{L+1}} \leq \frac{1}{c\nu\mu}.$$

Applying (4.3.1) L times, we obtain

$$\begin{aligned} \|f - f_L\|_{L_2(\Omega)} &\leq C \left(\frac{1}{\nu\mu}\right)^s \left[C_1 \mu^s + C_1 \nu^{-s}\right]^L \|Ef\|_{\Phi_1} \\ &\leq C \left(\frac{1}{\nu\mu}\right)^s \left[C_1 \mu^s + C_1 \nu^{-s}\right]^L \|Ef\|_{H^s(\mathbb{R}^n)} \\ &\leq C \left(\frac{1}{\nu\mu}\right)^s \left[C_1 \mu^s + C_1 \nu^{-s}\right]^L \|f\|_{H^s(\Omega)}. \end{aligned}$$

This proves the following lemma.

Lemma 4.3.3. *With the notation and assumptions of Theorem 4.3.2 there exists a constant $C > 0$ such that*

$$\|f - f_L\|_{L_2(\Omega)} \leq C \left(\frac{1}{\nu\mu}\right)^s \left[C_1\mu^s + C_1\nu^{-s}\right]^L \|f\|_{H^s(\Omega)} \tag{4.3.4}$$

holds for $L \in \mathbb{N}$.

The bounds in Theorem 4.3.2 and Lemma 4.3.3 are in the most general form. However, we usually couple the two critical parameters μ and ν, i.e., we assume that there is a fixed $\gamma > 0$ such that

$$\frac{1}{h_1} \geq \nu \geq \frac{\gamma}{\mu}. \tag{4.3.5}$$

With this, we can restate Theorem 4.3.2 and combine it with the specific result of Lemma 4.3.3, see also [91, Theorem 1].

Theorem 4.3.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. For $i \in \mathbb{N}$ let $X_i \subseteq \Omega$ be a set of sites with fill distance h_i. Assume that $c\mu h_i \leq h_{i+1} \leq \mu h_i$ holds for $i \in \mathbb{N}$ with fixed constants $\mu \in (0,1)$, $c \in (0,1]$ and h_1 sufficiently small. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_i = \delta_i^{-n}\Phi(\cdot/\delta_i)$ be the rescaled kernel where the scaling parameter δ_i is coupled to h_i according to (2.4.6). Assume that the overlap parameter ν satisfies $\frac{1}{h_1} \geq \nu$. Assume that there is a constant $\gamma > 0$ such that μ is coupled to ν according to (4.3.5). Assume that $f \in H^s(\Omega)$. Then there exists a constant $C_1 = C_1(\Omega, \Phi, s, \gamma) > 0$ such that, with $\alpha = C_1\mu^s$, the estimate*

$$\|Ee_i\|_{\Phi_{i+1}} \leq \alpha \|Ee_{i-1}\|_{\Phi_i}, \tag{4.3.6}$$

holds for $i \in \mathbb{N}$. $E : H^s(\Omega) \to H^s(\mathbb{R}^n)$ denotes the extension operator introduced in Theorem 2.1.7.

Let $f_L = A_L(f)$ denote the multilevel interpolant. Then there exists a constant $C > 0$ such that the error bound

$$\|f - f_L\|_{L_2(\Omega)} \leq C(C_1\mu^s)^L \|f\|_{H^s(\Omega)} \tag{4.3.7}$$

holds for all $L \in \mathbb{N}$. Thus f_L converges linearly to f in the L_2-norm if $\alpha = C_1\mu^s < 1$.

Proof. We use the notation of the proof of Theorem 4.3.2. With the assumptions on μ and ν we can bound I_1 by

$$\begin{aligned} I_1 &= \int_{\|\boldsymbol{\xi}\|_2 \leq \frac{1}{\delta_{i+1}}} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \left(1 + \delta_{i+1}^2 \|\boldsymbol{\xi}\|_2^2\right)^s \, d\boldsymbol{\xi} \\ &\leq c(\Omega, s) 2^s C_E^2 c_2 \nu^{-2s} \|Ee_{i-1}\|_{\Phi_i}^2 \\ &\leq c(\Omega, s) 2^s C_E^2 c_2 \left(\frac{\mu}{\gamma}\right)^{2s} \|Ee_{i-1}\|_{\Phi_i}^2 \end{aligned}$$

The bound for

$$\begin{aligned} I_2 &= \int_{\|\boldsymbol{\xi}\|_2 \geq \frac{1}{\delta_{i+1}}} |\widehat{Ee_i}(\boldsymbol{\xi})|^2 \left(1 + \delta_{i+1}^2 \|\boldsymbol{\xi}\|_2^2\right)^s \, d\boldsymbol{\xi} \\ &\leq 2^s C_E^2 c_2 \mu^{2s} \|Ee_{i-1}\|_{\Phi_i}^2 \end{aligned}$$

is unchanged. Putting these estimates into

$$\|Ee_i\|_{\Phi_{i+1}}^2 \leq \frac{1}{c_1}(I_1 + I_2)$$

yields (4.3.6). Keeping track of the occurring constants means that we can express $C_1 = C_1(\Omega, \Phi, s, \gamma)$ as

$$C_1 = 2^{\frac{s}{2}} C_E c_2^{\frac{1}{2}} \left(1 + c(\Omega, s)\gamma^{-2s}\right)^{\frac{1}{2}}. \tag{4.3.8}$$

We obtain the estimate in (4.3.7) the same way we proved Lemma 4.3.3. □

Similarly, we get the $L_\infty(\Omega)$-norm error estimate if we use (2.3.4).

Lemma 4.3.5. *With the notation and assumptions of Theorem 4.3.4 there exists a constant $C > 0$ such that the error bound*

$$\|f - f_L\|_{L_\infty(\Omega)} \leq C \left(C_1 \mu^{s-\frac{n}{2}}\right)^L \|f\|_{H^s(\Omega)} \tag{4.3.9}$$

holds for all $L \in \mathbb{N}$. The constant $C_1 = C_1(\Omega, \Phi, s, \gamma)$ can be expressed as in (4.3.8).

4.3.2. Interpolating Rougher Functions. Next we investigate the multilevel scheme applied to the escaping the native space method introduced in Section 2.3.3. We recall that we derived error estimates for interpolation when the target function is not in the native space of the kernel we use.

For briefness we omit the technical preparations for the proof of the following convergence result. They can be found in [91, Section 5]. The main idea is to use a recursion inequality as in (4.3.1), but not in the native space norm of the rescaled kernel used to compute the interpolant but rather use the rescaled kernel of the space the target function is an element of. The rest of the proof is very much the same as the proof of Theorem 4.3.2

Theorem 4.3.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. For $i \in \mathbb{N}$ let $X_i \subseteq \Omega$ be a quasi-uniform set of sites with fill distance h_i. Assume that $c\mu h_{i+1} \leq h_i \leq \mu h_{i+1}$ holds for $i \in \mathbb{N}$ with fixed constants $\mu \in (0, 1)$, $c \in (0, 1]$ and h_1 sufficiently small. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies* (2.2.2). *Let $\Phi_i = \delta_i^{-n}\Phi(\cdot/\delta_i)$ be the rescaled kernel where the scaling parameter δ_i is coupled to h_i according to* (2.4.6). *Assume that the overlap parameter ν satisfies $\frac{1}{h_1} \geq \nu$. Assume that there is a constant $\gamma > 0$ such that μ is coupled to ν according to* (4.3.5). *Let Ψ be a reproducing kernel of $H^t(\mathbb{R}^n)$, $n/2 < t \leq s$, and let $\Psi_i = \delta_i^{-n}\Psi(\cdot/\delta_i)$ be the rescaled RBF with the same scaling parameter δ_i. Assume that $f \in H^t(\Omega)$. Then there exists a constant $C_1 = C_1(\Omega, \Phi, s, \gamma)$ such that, with $\alpha = C_1\mu^t$, the estimate*

$$\|Ee_i\|_{\Psi_{i+1}} \leq \alpha \|Ee_{i-1}\|_{\Psi_i}$$

holds for $i \in \mathbb{N}$.

Let $f_L = A_L(f)$ denote the multilevel interpolant. Then there exists a constant $C > 0$ such that the error bounds

$$\|f - f_L\|_{L_2(\Omega)} \leq C(C_1\mu^t)^L \|f\|_{H^t(\Omega)},$$

and

$$\|f - f_L\|_{L_\infty(\Omega)} \leq C \left(C_1 \mu^{t-\frac{n}{2}}\right)^L \|f\|_{H^t(\Omega)},$$

hold for all $L \in \mathbb{N}$.

4.3.3. Penalized Least-Squares Approximation. To close this section we now give a convergence result for the multilevel method using penalized least-squares approximation in every level. That means that the local approximation operators $\mathcal{I}_i : H^s(\Omega) \to W_i$ are given by

$$\mathcal{I}_i(e_{i-1}) = \operatorname*{argmin}_{s \in H^s(\Omega)} \left\{ \sum_{\boldsymbol{x} \in X_i} |e_{i-1}(\boldsymbol{x}) - s(\boldsymbol{x})|^2 + \lambda_i \|s\|_{\Phi_i}^2 \right\}. \tag{4.3.10}$$

Again, the proof of the convergence result is very much the same as in Section 4.3.1. However, we have to use the error estimates of Theorem 2.3.21 which in turn leads to the assumption on λ_i. We omit the proof and only give the following theorem.

Theorem 4.3.7. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded Lipschitz domain. For* $i \in \mathbb{N}$ *let* $X_i \subseteq \Omega$ *be a set of sites with fill distance* h_i*. Assume that* $c\mu h_{i+1} \leq h_i \leq \mu h_{i+1}$ *holds for* $i \in \mathbb{N}$ *with fixed constants* $\mu \in (0,1)$*,* $c \in (0,1]$ *and* h_1 *sufficiently small. Let* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *be a reproducing kernel of* $H^s(\mathbb{R}^n)$*,* $s > n/2$*, i.e., the Fourier transform of* Φ *satisfies* (2.2.2)*. Let* $\Phi_i = \delta_i^{-n} \Phi(\cdot/\delta_i)$ *be the rescaled kernel where the scaling parameter* δ_i *is coupled to* h_i *according to* (2.4.6)*. Assume that the overlap parameter* ν *satisfies* $\frac{1}{h_1} \geq \nu$*. Assume that there is a constant* $\gamma > 0$ *such that* μ *is coupled to* ν *according to* (4.3.5)*. Assume that* $f \in H^s(\Omega)$*. Let* s_i *be the unique solution of* (4.3.10) *with smoothing parameters* λ_i *such that* $\lambda_i \leq \kappa \left(h_i/\delta_i\right)^{2s}$ *with a fixed constant* $\kappa > 0$*. Then there exists a constant* $C_1 = C_1(\Omega, \Phi, s, \gamma) > 0$ *such that, with* $\alpha = C_1 \mu^s$ *the estimate*

$$\|Ee_i\|_{\Phi_{i+1}} \leq \alpha \|Ee_{i-1}\|_{\Phi_i}$$

holds for $i \in \mathbb{N}$.

Let $f_L = A_L(f)$ *denote the multilevel penalized least-squares approximation. Then there exists a constant* $C > 0$ *such that the error bounds*

$$\|f - f_L\|_{L_2(\Omega)} \leq C(C_1 \mu^s)^L \|f\|_{H^s(\Omega)} \tag{4.3.11}$$

and

$$\|f - f_L\|_{L_\infty(\Omega)} \leq C \left(C_1 \mu^{s-\frac{n}{2}}\right)^L \|f\|_{H^s(\Omega)} \tag{4.3.12}$$

holds for all $L \in \mathbb{N}$.

We note that in Theorem 4.3.7 we have to choose the smoothing parameter level-dependent and sufficiently small.

4.4. Numerical Estimate of C_1

In the convergence results of the previous section, most importantly Theorem 4.3.4, we see that the multilevel method converges if $\alpha = C_1 \mu^s$

is less than one. This can be achieved by choosing a suitable refinement parameter $\mu \in (0,1)$ such that

$$\mu < C_1^{-\frac{1}{s}}.$$

Unfortunately, we cannot derive any analytic bounds on $C_1 = C_1(\Omega, \Phi, s, \gamma)$ because of the unknown constants c_2 and $c(\Omega, s)$ appearing in (4.3.8). Furthermore, we know bounds for C_E only in very specific settings, see, e.g., Lemma 2.1.9. However, in this section, we will give numerical estimates for C_1.

Before we come to the results we first discuss the setup of the experiment. Following (4.3.7) we have the error at level L

$$Err(L) := \|f - f_L\|_{L_2(\Omega)} \leq C(C_1\mu^s)^L \|f\|_{H^s(\Omega)}.$$

Assuming equality and taking the quotient of the errors of two consecutive levels yields

$$\frac{Err(L+1)}{Err(L)} = \frac{(C_1\mu^s)^{L+1}}{(C_1\mu^s)^L} = C_1\mu^s.$$

Hence, the estimated constant can be computed by the formula given in the next definition.

Definition 4.4.1. *For $f \in H^s(\Omega)$, $s > n/2$, let $Err(L) = \|f - f_L\|_{L_2(\Omega)}$ denote the multilevel approximation error on level L. We define the* estimated multilevel constant *as*

$$C_1^{est} := \frac{Err(L+1)}{Err(L)}\mu^{-s}, \tag{4.4.1}$$

where $\mu \in (0,1)$ is the refinement parameter used to compute the multilevel approximation.

Although the theoretical constant C_1 as in (4.3.8) is clearly independent of μ, the estimated constant C_1^{est} in (4.4.1) depends explicitly on μ^{-s}. However, this dependence seems to be very mild as the numerical tests we give next show. We suspect that this explicit dependence is partly compensated by the implicit dependence of the error $Err(L+1)$ and $Err(L)$ on μ.

Looking at (4.3.8) we see that C_1 depends on the domain Ω, the smoothness of the target function s and the parameter γ, which couples the refinement parameter μ to the overlap parameter ν, see (4.3.5). Hence, our numerical tests should cover different values for γ and s on several domains. Furthermore, to obtain as good of an estimate on C_1 as possible it is important to choose settings that yield approximations as close to the theoretical bound $C(C_1\mu^s)^L$ as possible.

In the following we focus on one-dimensional domains, i.e., Ω is an interval, since those will be the main examples for the numerical tests done in Section 7.3. Furthermore, we use target functions $f_s : \Omega \to \mathbb{R}$ given by $f(x) = |x - x_0|^{s-0.4}$, where x_0 is an offset chosen such that the kink is the midpoint of the interval. We can control the smoothness of f_s with the parameter $s \in \mathbb{N}$. We see that $f_s \in H^s(\Omega)$ and $f \notin H^{s+1}(\Omega)$. To each of the values of s we pick a reproducing kernel of the corresponding Sobolev Hilbert space $H^s(\mathbb{R})$, to be more precise we choose the parameter k in the definition of the Wendland functions $\Phi_{1,k}$, defined in Definition 2.2.18, to be $k = s - 1$.

For a given refinement parameter μ we construct the sets of sites $X_i \subseteq \Omega$ as equidistant grids such that, for level $0 \leq i \leq L$, the cardinality of X_i is given as $\#X_i = \left\lceil \frac{1}{\mu} \right\rceil^{i+2}$.

4.4.1. $\Omega = [-1, 1]$. We start by looking at the domain $\Omega = [-1, 1]$ and test two refinement parameter, $\mu = 0.4$ and $\mu = 0.5$. The first choice leads to a tripling where the second one leads to a doubling of the number of points per level. To keep this section brief we provide here only the values of the estimated multilevel constant C_1^{est}. The corresponding errors can be found in Appendix A.1.

In Tables 2 to 4 we fix $\mu = 0.4$ and vary the parameter $k = 1, 2, 3$ in the Wendland function $\Phi_{1,k}$, and hence the smoothness of the target function. In each of the Tables 2 to 4 we give a selection of values of the parameter γ, which couples the overlap parameter ν to the refinement parameter μ.

Similarly, we fix for Tables 5 to 7 the refinement parameter to be $\mu = 0.5$ and again vary k and γ.

Taking a closer look at each of the tables we can see the dependencies of C_1^{est} on the parameters γ and k. First, we see that in most rows of most tables the value of the estimated multilevel constant decreases with growing γ. We attribute the occuring exceptions to numerical artifacts we can not explain. To check if this decrease is indeed with the right speed we would have to conduct more thorough simulations. We limit ourselves to values of γ between 1.5 and 4, since these values are used in applications and follow [91].

Second, comparing the same entries in Tables 2 to 4 or in Tables 5 to 7 we can see the dependence on the smoothness. It is does not matter that we approximate different functions since the constant C_1^{est} is independent of the target function.

Third, comparing the entries in Table 2 with those in Table 5, and similarly those in Table 3 with Table 6 and Table 4 with Table 7, we see that the estimated multilevel constant C_1^{est} exhibits a dependence on the refinement parameter μ, however it is less than an order of magnitude going from $\mu = 0.4$ to $\mu = 0.5$.

And finally, we remark on the magnitude of C_1^{est}. We started this section with the observation that a large value of C_1 can potentially spoil the convergence of the multilevel method or forces us to use a very small refinement parameter μ. However, all of the estimated values in Tables 2 to 7 suggest that C_1 is reasonably small. We even see that in Tables 2 and 5 $C_1^{est} < 1$ which would lead to a speed up in the convergence of the method.

4.4.2. $\Omega = [0, 1]$. We now consider the domain $\Omega = [0, 1]$. Partly, because we will use this interval later in this text and partly because we want to test how much the estimated multilevel constant C_1^{est} depends on the domain. Again, we provide the error tables separately in Appendix A.2.

We give the tables for the C_1^{est} for $\mu = 0.4$ in Tables 8 to 10 and for $\mu = 0.5$ in Tables 11 to 13, where we again use $k = 1, 2, 3$ for each table respectively and vary in each table the parameter γ.

Comparing the respective tables for $\Omega = [-1, 1]$ and $\Omega = [0, 1]$ we can see the influence of the domain on the estimated multilevel constant and then

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	0.31607	0.32947	0.23046	0.18635	0.15303	0.13762
2	0.95800	0.58130	0.54252	0.49704	0.48660	0.43240
3	0.65534	0.57147	0.51464	0.47583	0.46317	0.44478
4	0.62960	0.55811	0.50894	0.47424	0.46142	0.44817
5	0.62923	0.55711	0.50835	0.47418	0.46262	0.44779
6	0.62779	0.55705	0.50817	0.47452	0.46337	0.45011
7	0.62748	0.55706	0.50812	0.47484	0.46421	0.45207
8	0.62717	0.55685	0.50791	0.47505	0.46520	0.45569
9	0.60645	0.53746	0.48984	0.46070	0.45721	0.46031
10	0.20988	0.26422	0.24527	0.29639	0.38110	0.50423

TABLE 2. Table of C_1^{est} with $\Omega = [-1, 1]$, $\mu = 0.4$ and $k = 1$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	0.82799	1.63520	1.52119	1.16660	0.85756	0.68003
2	5.68928	2.08084	1.55917	1.30715	1.22586	1.13442
3	1.92980	1.80891	1.57349	1.37658	1.24928	1.15059
4	2.17771	1.81606	1.58209	1.38714	1.25858	1.15707
5	2.11381	1.81312	1.58387	1.38871	1.25999	1.15758
6	2.12176	1.81196	1.58421	1.38885	1.26016	1.15747
7	2.12110	1.81155	1.58427	1.38882	1.26016	1.15738
8	2.12081	1.81071	1.58366	1.38821	1.25959	1.15682
9	2.04177	1.75246	1.53297	1.34048	1.21471	1.11519
10	0.26095	0.91406	0.92893	0.80302	0.72147	0.67056

TABLE 3. Table of C_1^{est} with $\Omega = [-1, 1]$, $\mu = 0.4$ and $k = 2$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	2.26544	2.48169	3.87432	4.43881	4.28929	3.76967
2	7.83229	8.72575	4.75761	3.44157	2.98691	2.75450
3	5.78067	3.46119	2.92972	2.77065	2.60023	2.42737
4	4.45609	3.51350	3.02509	2.76465	2.55332	2.35799
5	4.06979	3.27020	2.96055	2.74283	2.54067	2.34963
6	3.95914	3.29986	2.96274	2.73522	2.53661	2.34830
7	3.91723	3.27702	2.95957	2.73194	2.53525	2.34825
8	3.90170	3.28426	2.95929	2.73048	2.53478	2.34840
9	3.89133	3.28456	2.95908	2.72981	2.53460	2.34853
10	3.88397	3.28690	2.95904	2.72949	2.53453	2.34861

TABLE 4. Table of C_1^{est} with $\Omega = [-1, 1]$, $\mu = 0.4$ and $k = 3$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	0.22279	0.34330	0.39828	0.30307	0.26954	0.29154
2	2.29454	1.10363	0.78488	0.72231	0.64636	0.46929
3	0.79579	0.76777	0.64880	0.62576	0.63196	0.66408
4	0.83834	0.71410	0.63988	0.61081	0.60489	0.61174
5	0.79146	0.70560	0.64169	0.61404	0.61107	0.59566
6	0.79081	0.70187	0.64063	0.61406	0.60853	0.59918
7	0.78879	0.70163	0.64120	0.61270	0.60898	0.60103
8	0.78870	0.70092	0.64135	0.61322	0.60986	0.59971
9	0.78868	0.70119	0.64145	0.61243	0.61050	0.60109
10	0.78873	0.70101	0.64150	0.61282	0.61102	0.60135

TABLE 5. Table of C_1^{est} with $\Omega = [-1, 1]$, $\mu = 0.5$ and $k = 1$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	0.93686	0.94794	1.36523	1.37125	1.14448	0.91606
2	3.86615	3.50889	1.91807	1.52262	1.37418	1.28471
3	2.13099	1.49701	1.39912	1.30027	1.20697	1.13139
4	1.97838	1.58321	1.40092	1.27868	1.17609	1.10645
5	1.79629	1.52290	1.39165	1.27627	1.17535	1.10492
6	1.80196	1.52998	1.38873	1.27559	1.17566	1.10383
7	1.77513	1.52780	1.38729	1.27530	1.17571	1.10317
8	1.77863	1.52851	1.38658	1.27518	1.17567	1.10276
9	1.77427	1.52865	1.38623	1.27511	1.17564	1.10255
10	1.77535	1.52881	1.38606	1.27508	1.17561	1.10243

TABLE 6. Table of C_1^{est} with $\Omega = [-1, 1]$, $\mu = 0.5$ and $k = 2$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	2.26544	2.48169	3.87432	4.43881	4.28929	3.76967
2	7.83229	8.72575	4.75761	3.44157	2.98691	2.75450
3	5.78067	3.46119	2.92972	2.77065	2.60023	2.42737
4	4.45609	3.51350	3.02509	2.76465	2.55332	2.35799
5	4.06979	3.27020	2.96055	2.74283	2.54067	2.34963
6	3.95914	3.29986	2.96274	2.73522	2.53661	2.34830
7	3.91723	3.27702	2.95957	2.73194	2.53525	2.34825
8	3.90170	3.28426	2.95929	2.73048	2.53478	2.34840
9	3.89133	3.28456	2.95908	2.72981	2.53460	2.34853
10	3.88397	3.28690	2.95904	2.72949	2.53453	2.34861

TABLE 7. Table of C_1^{est} with $\Omega = [-1, 1]$, $\mu = 0.5$ and $k = 3$.

make assumptions for the constant C_1. Clearly, we see that there is only a slight influence of the domain, since halving the length of the interval leads to less than halving of C_1^{est}. However, we do not expect to be able to derive any analytical connection other than the one in (4.3.8).

We conclude from the results discussed in this section that even choices of $0.5 < \mu < 1$ are numerically justifiable.

4.5. Adaptive Versions of the Multilevel Method

We now briefly introduce two variations of the multilevel method, taken from [92]. We focus on the case where the local approximation operators $\mathcal{I}_i$ are interpolation operators although it is also possible to use the ideas presented here for the penalized least-squares approximation.

The first version we want to investigate was first used in [35] in the context of spheres and can be understood as a data compression scheme. The main idea is that on each level $i \in \mathbb{N}$ we discard all coefficients $\alpha_k^{(i)}$ of the local interpolation

$$s_i = \sum_{k=1}^{N_i} \alpha_k^{(i)} \Phi_i(\cdot - \boldsymbol{x}_{i,k})$$

whose absolute value is less than a given level-dependent threshold $\varepsilon_i \in \mathbb{R}$. That means, in particular, that we have to compute these coefficients first. Hence, we do not reduce any computational cost, only the cost for storing the coefficients is lower since we discard those with small absolute value. Furthermore, potential evaluations of the global approximation $f_L = s_1 + \cdots + s_L$ can be significantly cheaper. Since the discarding of the unnecessary coefficients is done during each level this method is named *discarding dynamically*. We give the formal algorithm in Algorithm 2.

Algorithm 2: Multilevel approximation with dynamical discarding

Data: Number of levels L, right-hand side f, thresholds ε_i
Result: Approximate solution $\widetilde{f}_L \in V_L = W_1 + \cdots + W_L$
Set $\widetilde{f}_0 = 0, \widetilde{e}_0 = f$;
for $i = 1, 2, \ldots, L$ **do**
  Determine a local interpolant $s_i \in W_i$ to $\widetilde{e}_{i-1}$ on X_i;
  Drop all coefficients $|\alpha_k^{(i)}| \leq \varepsilon_i$ to define $\widetilde{s}_i$;
  Set $\widetilde{f}_i = \widetilde{f}_{i-1} + \widetilde{s}_i$;
  Set $\widetilde{e}_i = \widetilde{e}_{i-1} - \widetilde{s}_i$;
end

Clearly, we cannot expect the global approximation $\widetilde{f}_L$ to be an interpolant to f in the centers X_L. However, it is still possible to show convergence if the thresholds ε_i are chosen in a specific way. The proof of the next theorem, which we quote from [92, Theorem 11], is similar to those in Section 4.3. For details we refer to [35].

Theorem 4.5.1. *With the notation and assumptions of Theorem 4.3.2 let $\varepsilon > 0$ be given. For $i \in \mathbb{N}$ let $\varepsilon_i > 0$ such that $\varepsilon_i \leq c\varepsilon\delta_i^{\frac{n}{2}}$ holds with a constant*

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	0.18267	0.11634	0.08871	0.07532	0.05430	0.05973
2	0.46695	0.45887	0.44441	0.42353	0.42907	0.41261
3	0.47645	0.45255	0.44859	0.44910	0.46948	0.44107
4	0.47135	0.43988	0.42583	0.41590	0.39829	0.39123
5	0.47671	0.45508	0.46042	0.47035	0.49797	0.49390
6	0.47042	0.43478	0.41858	0.41115	0.38654	0.37682
7	0.47844	0.46522	0.48514	0.50647	0.56504	0.57141
8	0.46913	0.42831	0.40816	0.40034	0.37095	0.36109
9	0.46265	0.47031	0.51618	0.55219	0.64077	0.66191
10	0.20057	0.19736	0.20151	0.19976	0.19048	0.18353

TABLE 8. Table of C_1^{est} with $\Omega = [0, 1]$, $\mu = 0.4$ and $k = 1$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	1.02056	0.59261	0.42169	0.29271	0.24651	0.18069
2	1.30313	1.12281	0.93785	0.86576	0.77084	0.73705
3	1.37591	1.14995	0.99473	0.88906	0.80929	0.76183
4	1.38703	1.15695	1.00137	0.89424	0.81778	0.76471
5	1.38869	1.15756	1.00230	0.89542	0.81889	0.76497
6	1.38885	1.15747	1.00239	0.89559	0.81895	0.76468
7	1.38882	1.15738	1.00238	0.89562	0.81896	0.76465
8	1.38821	1.15682	1.00189	0.89518	0.81854	0.76420
9	1.34048	1.11519	0.96440	0.86089	0.78668	0.73435
10	0.80302	0.67056	0.57374	0.51060	0.47695	0.50754

TABLE 9. Table of C_1^{est} with $\Omega = [0, 1]$, $\mu = 0.4$ and $k = 2$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	5.17231	3.10048	1.94825	1.36621	0.97020	0.76782
2	4.42539	3.35325	2.78487	2.34507	2.05810	1.80285
3	4.55747	3.55093	2.91917	2.46844	2.14527	1.89765
4	4.58447	3.57766	2.93735	2.48278	2.15452	1.90677
5	4.58956	3.58171	2.94010	2.48436	2.15522	1.90731
6	4.59063	3.58231	2.94052	2.48442	2.15511	1.90714
7	4.59085	3.58237	2.94056	2.48435	2.15500	1.90702
8	4.58911	3.58068	2.93903	2.48292	2.15370	1.90592
9	4.44359	3.44865	2.82131	2.37720	2.05975	1.84432
10	2.83234	2.12802	1.70690	1.42198	2.18957	7.56504

TABLE 10. Table of C_1^{est} with $\Omega = [0, 1]$, $\mu = 0.4$ and $k = 3$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	0.30307	0.29154	0.21912	0.16619	0.14380	0.13096
2	0.72231	0.46929	0.60096	0.66218	0.50144	0.52124
3	0.62576	0.66408	0.62556	0.48702	0.70708	0.49862
4	0.61081	0.61174	0.58110	0.63628	0.55657	0.66738
5	0.61404	0.59566	0.59461	0.59785	0.56521	0.58586
6	0.61406	0.59918	0.59823	0.59551	0.59465	0.58734
7	0.61270	0.60103	0.59942	0.59982	0.59560	0.60164
8	0.61322	0.59971	0.59918	0.60015	0.60193	0.61039
9	0.61243	0.60109	0.60136	0.60355	0.60693	0.61664
10	0.61282	0.60135	0.60218	0.60645	0.61278	0.62447

TABLE 11. Table of C_1^{est} with $\Omega = [0, 1]$, $\mu = 0.5$ and $k = 1$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	1.37125	0.91606	0.71151	0.59491	0.48975	0.40856
2	1.52262	1.28471	1.18565	0.99618	1.07347	1.06061
3	1.30027	1.13139	0.98395	0.98509	0.87005	0.86797
4	1.27868	1.10645	1.00642	0.94166	0.88644	0.86079
5	1.27627	1.10492	1.00652	0.93744	0.89111	0.85664
6	1.27559	1.10383	1.00565	0.93620	0.89105	0.85535
7	1.27530	1.10317	1.00504	0.93557	0.89070	0.85476
8	1.27518	1.10276	1.00466	0.93515	0.89047	0.85448
9	1.27511	1.10255	1.00446	0.93492	0.89033	0.85433
10	1.27508	1.10243	1.00435	0.93479	0.89025	0.85421

TABLE 12. Table of C_1^{est} with $\Omega = [0, 1]$, $\mu = 0.5$ and $k = 2$.

	Constant C_1^{est}					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	4.43881	3.76967	2.71233	2.08448	1.70679	1.44576
2	3.44157	2.75450	2.40305	2.24953	1.97158	1.83393
3	2.77065	2.42737	2.12771	1.86877	1.74893	1.60265
4	2.76465	2.35799	2.05424	1.84096	1.68030	1.54839
5	2.74283	2.34963	2.04634	1.83343	1.66832	1.54206
6	2.73522	2.34830	2.04341	1.83065	1.66477	1.53859
7	2.73194	2.34825	2.04227	1.82954	1.66332	1.53714
8	2.73048	2.34840	2.04178	1.82908	1.66268	1.53651
9	2.72981	2.34853	2.04156	1.82888	1.66238	1.53622
10	2.72949	2.34861	2.04146	1.82879	1.66224	1.53608

TABLE 13. Table of C_1^{est} with $\Omega = [0, 1]$, $\mu = 0.5$ and $k = 3$.

$c > 0$. Then there is a constant $C > 0$ such that, with $\alpha = C_1(\mu^\sigma + \nu^{-\sigma})$, the estimate

$$\|\widetilde{e}_i\|_{\Phi_{i+1}} \leq \alpha \|\widetilde{e}_{i-1}\|_{\Phi_i} + C\varepsilon$$

holds for $i \in \mathbb{N}$.

Let $\widetilde{f}_L$ denote the approximant from Algorithm 2. Then there exists a constant $C > 0$ such that

$$\|f - \widetilde{f}_L\|_{L_2(\Omega)} \leq C\alpha^L \|f\|_{H^\sigma(\Omega)} + C\varepsilon \frac{1-\alpha^L}{1-\alpha}$$

holds for all $L \in \mathbb{N}$.

The second version we discuss is a truly adaptive one, again taken from [92]. It checks if the absolute value of the residual on level i evaluated on the points of the upcoming level X_{i+1} is larger than a given threshold. Only these points will be used for the local interpolation problem on level $i+1$.

We now go into more detail. We deal with two point sets. First, we assume that the family of sets of sites $(X_i)_{i\in\mathbb{N}}$ are given and satisfy the usual assumptions for the multilevel method, i.e., there is a uniform refinement parameter $\mu \in (0,1)$ such that $c\mu h_i \leq h_{i+1} \leq \mu h_i$ with a constant $c \in (0,1]$. The other parameters of the multilevel method, in particular the overlap parameter ν and therefore also the scaling parameter δ_i, are all fixed relative to these sets. The second family of sets are the adaptive point sets which are actually used for the local interpolation. We denote these sets by $(\mathcal{X}_i)_{i\in\mathbb{N}}$. We now discuss the general approach which is given in Algorithm 3. After computing the local interpolant s_i on $\mathcal{X}_i$ to e_{i-1} we check the error $e_i = e_{i-1} - s_i$ on the upcoming point set X_{i+1}. However, instead of interpolating e_i on all of X_{i+1} we use only those points of X_{i+1} in which e_i has an absolute value larger than a given level-dependent threshold $\varepsilon_i > 0$. These points define the adaptive set of sites $\mathcal{X}_{i+1}$ on level $i+1$. Clearly, for all $i \in \mathbb{N}$ we have the inclusion $\mathcal{X}_i \subseteq X_i$. However, even if the original sets of sites (X_i) are nested we can not expect the adaptive point sets $(\mathcal{X}_i)$ to satisfy $\mathcal{X}_i \subseteq \mathcal{X}_{i+1}$, $i \in \mathbb{N}$.

An error analysis of the adaptive method in this setting is problematic since the techniques of Section 4.3 require us to control $\|e_i\|_{\ell_\infty(X_i)}$. However, we only know that

$$e_i(\boldsymbol{x}) = e_{i-1}(\boldsymbol{x}) - I_{\mathcal{X}_i,\Phi_i}(e_{i-1})(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathcal{X}_i,$$

and we have no information about e_i on $X_i \setminus \mathcal{X}_i$.

A way to remedy this situation is to create another loop in which we check e_i on X_i and add those $\boldsymbol{x}$ to $\mathcal{X}_i$ for which $|e_i(\boldsymbol{x})|$ is still too large. We then compute the new local interpolant and residual using this new $\mathcal{X}_i$ [92]. This way we can potentially control the residual e_i on the whole set of sites X_i, which potentially enables us to derive a convergence result. A more thourough exploration of this idea has to be done in the future.

Algorithm 3: Adaptive multilevel approximation

Data: Number of levels L, right-hand side f, thresholds ε_i, point sets $X_1, \dots, X_L$
Result: Approximate solution $f_L \in V_L = W_1 + \cdots + W_L$
Set $f_0 = 0, e_0 = f$, $\mathcal{X}_1 = X_1$;
for $i = 1, 2, \dots, L$ **do**
  Determine a local interpolant $s_i = I_{\mathcal{X}_i,\Phi_i}(e_{i-1}) \in W_i$;
  Set $f_i = f_{i-1} + s_i$;
  Set $e_i = e_{i-1} - s_i$;
  for $\boldsymbol{x} \in X_{i+1}$ **do**
    if $|e_i(\boldsymbol{x})| > \varepsilon_i$ **then**
      $\mathcal{X}_{i+1} = \mathcal{X}_{i+1} \cup \{\boldsymbol{x}\}$;
    end
  end
end

4.6. Multilevel Method Using Lagrange Functions

In Section 4.2 we introduced the multilevel operator $A_L : H^s(\Omega) \to V_L$ in its residual correction representation

$$A_L(f) = \sum_{i=1}^{L} \mathcal{I}_i(e_{i-1}), \quad f \in H^s(\Omega),$$

where the $\mathcal{I}_i : H^s(\Omega) \to W_i$ are the local approximation operators of level i applied to the residuals of the preceding level. For the later application in Chapter 7 however, we need a better understanding of the operator A_L in general and how it depends on the target function f in particular. This is the goal of this section. First, we derive a different representation, eliminating the residuals of the representation of A_L. And then, we show one possible way to deal with the downside of this representation using the localized Lagrange functions of Section 3.3.

4.6.1. A Different Representation. We first derive a general, new view on the multilevel method. First, we introduce a special kind of set of integers.

Definition 4.6.1. *We call a finite set* $\mathfrak{u} = \{u_1, \dots, u_{\#\mathfrak{u}}\} \subset \mathbb{N}$ *an* ordered set *if* $u_1 < u_2 < \cdots < u_{\#\mathfrak{u}}$.

With this we can define a special combination of operators.

Definition 4.6.2. *Let V be a linear space and $L \in \mathbb{N}$. Let $W_1, \dots, W_L \subseteq V$ be subspaces. For $1 \le i \le L$ let $\mathcal{I}_i : V \to W_i$ be given operators. Let $\mathfrak{u} = \{u_1, \dots, u_k\} \subseteq \{1, \dots, L\}$ be an ordered set. Then the* combined operator *$\mathcal{I}_\mathfrak{u} : V \to W_{u_k}$ is defined by*

$$\mathcal{I}_\mathfrak{u} := \mathcal{I}_{u_k} \mathcal{I}_{u_{k-1}} \cdots \mathcal{I}_{u_1}. \tag{4.6.1}$$

We note that is crucial that the elements of $\mathfrak{u}$ in the definition of $\mathcal{I}_\mathfrak{u}$ are ordered since the operators $\mathcal{I}_i$ usually do not commute. Additionally,

we stress that in the definition of $\mathcal{I}_{\mathfrak{u}}$ we first apply the operator with the smallest index then the one with next bigger index and so on.

We can now express the multilevel operator A_L with the help of the combined operators.

Theorem 4.6.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, let $s > n/2$ and let $L \in \mathbb{N}$. Let $W_1, \dots, W_L$ be the local and $V_L = W_1 + \cdots + W_L$ be the global approximation spaces defined in Definition 4.1.1. For $1 \leq i \leq L$ let $\mathcal{I}_i : H^s(\Omega) \to W_i$ be the local approximation operator. Let $\mathfrak{u} = \{u_1, \dots, u_k\} \subseteq \{1, \dots, L\}$ be an ordered set. Let $\mathcal{I}_{\mathfrak{u}} : H^s(\Omega) \to W_{u_k}$ be the combined operator defined in Definition 4.6.2. Then the multilevel operator $A_L : H^s(\Omega) \to V_L$ has the general representation*

$$A_L = \sum_{\substack{\mathfrak{u} \subseteq \{1,\dots,L\} \\ 1 \leq \#\mathfrak{u} \leq L}} (-1)^{\#\mathfrak{u}+1} \mathcal{I}_{\mathfrak{u}} = \sum_{i=1}^{L} (-1)^{i+1} \sum_{\substack{\mathfrak{u} \subseteq \{1,\dots,L\} \\ \#\mathfrak{u} = i}} \mathcal{I}_{\mathfrak{u}}. \tag{4.6.2}$$

Proof. We recall that we denote the identity operator by ι. The definition of the residual e_i in level i, see (4.2.2), yields the recursion $e_i = (\iota - \mathcal{I}_i)e_{i-1}$. This means that the identity

$$e_L = (\iota - \mathcal{I}_L)(\iota - \mathcal{I}_{L-1}) \cdots (\iota - \mathcal{I}_1) f$$

holds, since we set $e_0 = f$. By induction we see that we can express the operator on the right-hand side as

$$(\iota - \mathcal{I}_L)(\iota - \mathcal{I}_{L-1}) \cdots (\iota - \mathcal{I}_1) = \iota + \sum_{\substack{\mathfrak{u} \subseteq \{1,\dots,L\} \\ \mathfrak{u} \neq \emptyset}} (-1)^{\#\mathfrak{u}} \mathcal{I}_{\mathfrak{u}}.$$

Finally, together with the identity $A_L(f) = f_L = f - e_L$, this yields the stated representation. □

This general representation allows us now to derive specific representations for the multilevel approximation operator for both, the interpolation and the penalized least-squares approximation. We use the general setup of the kernel-based multilevel method in Section 4.2.

We start by investigating interpolation. Following the ideas of Section 3.1 we can express the local approximation operator $\mathcal{I}_i$, $i \in \mathbb{N}$, as

$$\mathcal{I}_i(f) = \sum_{k=1}^{N_i} f(\boldsymbol{x}_{i,k}) \chi_{i,k}, \quad f \in H^s(\Omega), s > n/2, \tag{4.6.3}$$

where for each i and $1 \leq k \leq N_i$ the function $\chi_{i,k} \in W_i$ denotes the Lagrange function anchored in $\boldsymbol{x}_{i,k} \in X_i$, i.e., $\chi_{i,k}$ satisfies the Lagrange condition $\chi_{i,k}(\boldsymbol{x}_{i,m}) = \delta_{km}$, $1 \leq k, m \leq N_i$.

Using this representation in (4.6.2) yields the following result.

Theorem 4.6.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. For $i \in \mathbb{N}$ let $X_i \subseteq \Omega$ be a quasi-uniform set of sites with fill distance h_i. Assume that $c\mu h_i \leq h_{i+1} \leq \mu h_i$ holds for $i \in \mathbb{N}$ with fixed constants $\mu \in (0,1)$, $c \in (0,1]$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (2.2.2). Let $\Phi_i = \delta_i^{-n} \Phi(\cdot/\delta_i)$ be the rescaled kernel where the scaling parameter δ_i is coupled to h_i according to (2.4.6). For every $1 \leq k \leq N_i$ let*

$\chi_{i,k} \in W_i := \operatorname{span}\{\Phi_i(\cdot - \boldsymbol{x}) \,:\, \boldsymbol{x} \in X_i\}$ *be the Lagrange function anchored in* $\boldsymbol{x}_{i,k}$ *as in* (3.1.2). *Let* $\mathcal{I}_i : H^s(\Omega) \to W_i$ *be the local interpolation operator of the form* (4.6.3). *Let* $\mathfrak{u} = \{u_1, \dots, u_{\#\mathfrak{u}}\}$ *be an ordered set. Then, using the notation* $\boldsymbol{k} = (k_1, \dots, k_{\#\mathfrak{u}})^T \in \mathbb{N}^{\#\mathfrak{u}}$ *and* $\boldsymbol{N}_\mathfrak{u} = (N_{u_1}, \dots, N_{\#\mathfrak{u}})^T \in \mathbb{N}^{\#\mathfrak{u}}$, *the combined operator* $\mathcal{I}_\mathfrak{u} : H^s(\Omega) \to W_{\#\mathfrak{u}}$ *has the representation*

$$\mathcal{I}_\mathfrak{u}(f) = \sum_{\boldsymbol{k} \le \boldsymbol{N}_\mathfrak{u}} a(\mathfrak{u}, \boldsymbol{k}) f(\boldsymbol{x}_{u_1,k_1}) \chi_{u_{\#\mathfrak{u}},k_{\#\mathfrak{u}}},$$

where the coefficients are given by $a(\mathfrak{u}, \boldsymbol{k}) = 1$ *if* $\#\mathfrak{u} = 1$ *and*

$$a(\mathfrak{u}, \boldsymbol{k}) = \prod_{m=1}^{\#\mathfrak{u}-1} \chi_{u_m,k_m}(\boldsymbol{x}_{u_{m+1},k_{m+1}})$$

if $\#\mathfrak{u} > 1$.

Moreover, the multilevel interpolation operator $A_L : H^s(\Omega) \to V_L$ *has the representation*

$$A_L(f) = \sum_{\substack{\mathfrak{u} \subseteq \{1,\dots,L\} \\ 1 \le \#\mathfrak{u} \le L}} \sum_{\boldsymbol{k} \le \boldsymbol{N}_\mathfrak{u}} (-1)^{\#\mathfrak{u}+1} a(\mathfrak{u}, \boldsymbol{k}) f(\boldsymbol{x}_{u_1,k_1}) \chi_{u_{\#\mathfrak{u}},k_{\#\mathfrak{u}}}. \tag{4.6.4}$$

Proof. We want to use the second identity of the general representation of A_L from (4.6.2). Hence, we fix an ordered set $\mathfrak{u} = \{u_1, \dots, u_i\}$ with $\#\mathfrak{u} = i$ and derive for the combined operator $\mathcal{I}_\mathfrak{u}$

$$\begin{aligned}
\mathcal{I}_\mathfrak{u} f &= \mathcal{I}_{u_i} \cdots \mathcal{I}_1 f \\
&= \mathcal{I}_{u_i} \cdots \mathcal{I}_2 \sum_{k_1=1}^{N_{u_1}} f(\boldsymbol{x}_{u_1,k_1}) \chi_{u_1,k_1} \\
&= \mathcal{I}_{u_i} \cdots \mathcal{I}_3 \sum_{k_2=1}^{N_{u_2}} \sum_{k_1=1}^{N_{u_1}} \chi_{u_1,k_1}(\boldsymbol{x}_{u_2,k_2}) f(\boldsymbol{x}_{u_1,k_1}) \chi_{u_2,k_2} \\
&= \sum_{k_{u_i}=1}^{N_{u_i}} \cdots \sum_{k_1=1}^{N_{u_1}} \chi_{u_{i-1},k_{i-1}}(\boldsymbol{x}_{u_i,k_i}) \cdots \chi_{u_1,k_1}(\boldsymbol{x}_{u_2,k_2}) f(\boldsymbol{x}_{u_1,k_1}) \chi_{u_i,k_i} \\
&= \sum_{\boldsymbol{k} \le \boldsymbol{N}_\mathfrak{u}} \left[\prod_{m=1}^{i-1} \chi_{u_m,k_m}(\boldsymbol{x}_{u_{m+1},k_{m+1}}) \right] f(\boldsymbol{x}_{u_1,k_1}) \chi_{u_i,k_i}.
\end{aligned}$$

Inserting this into (4.6.2) and some easy manipulation gives the representation for the multilevel operator. □

At first glance it seems strange that only the function values $f(\boldsymbol{x}_{u_1,k_1})$ appear in the representation of A_L in (4.6.4). These are the function values of points of the coarsest point set X_{u_1}. However, in (4.6.3) we sum over all possible ordered sets $\mathfrak{u} \subseteq \{1, \dots, L\}$ and therefore, one specific $\mathfrak{u}$ is $\{L\}$, the last level with the associated finest point set X_L. Hence, we indeed use point data from every level.

We get a similar result for penalized least-squares approximation. To find a representation for the multilevel approximation operator like the one in

Theorem 4.6.4, we have to replace the Lagrange function $\chi_{i,k}$ by the modified Lagrange functions $\chi_{i,k}^{LS}$. We recall their point-wise definition, (3.1.21), to be

$$\chi_{i,k}^{LS}(\boldsymbol{x}) = \boldsymbol{r}_i(\boldsymbol{x})^{\mathrm{T}}(M_i + \lambda_i I)^{-1}\boldsymbol{e}_k, \quad \boldsymbol{x} \in \Omega, 1 \leq k \leq N_i,$$

where $\boldsymbol{r}_i(\boldsymbol{x}) = (\Phi_i(\boldsymbol{x}, \boldsymbol{x}_{i,1}), \ldots, \Phi_i(\boldsymbol{x}, \boldsymbol{x}_{i,N_i}))^{\mathrm{T}} \in \mathbb{R}^{N_i}$.

This, together with the representation (4.6.2), gives the following result.

Corollary 4.6.5. *With the assumptions and notation of Theorem 4.6.4, where we replace the full Lagrange functions $\chi_{i,k}$ with the modified Lagrange functions $\chi_{i,k}^{LS}$ defined in Definition 3.1.11 and use the local approximation operator at level i*

$$\mathcal{I}_i(f) = \sum_{k=1}^{N_i} f(\boldsymbol{x}_{i,k})\chi_{i,k}^{LS}, \quad f \in H^s(\Omega).$$

The multilevel penalized least-squares operator $A_L : H^s(\Omega) \to V_L$ has the representation (4.6.4)*, where we replace the full Lagrange functions with the modified Lagrange functions.*

Although Theorem 4.6.4 and Corollary 4.6.5 provide a valid alternative representation of the multilevel operator A_L, and they are essential for some results in Chapter 7, we stress that using them should be avoided unless it is absolutely necessary. Even though all major computations for the approximation with this representation can be done beforehand, as soon as the point sets $X_1, \ldots, X_L$ are fixed, numerical tests show that point evaluations of the approximation $A_L(f)$ to f are very expensive. We can identify two reasons. On the one hand in (4.6.4) we have to sum over all ordered sets $\mathfrak{u} \subseteq \{1, \ldots, L\}$ and on the other hand, for every point-evaluation of $A_L(f)$, we have to evaluate every $\chi_{i,k}$, $1 \leq i \leq L$, $1 \leq k \leq N_i$, since the Lagrange functions have global support.

In the next subsection we will replace the full with the localized Lagrange functions derived in Section 3.3 to deal with the second downside described above.

4.6.2. Convergence of the Multilevel Method using Localized Lagrange Functions. We already discussed at the end of Chapter 3 the possibility to use the localized Lagrange functions in the approximation operator defined in Definition 3.3.5. We recall that the defining identity is given by

$$\mathcal{I}^{loc}(f) = \sum_{k=1}^{N} f(\boldsymbol{x}_k)\chi_k^{loc}.$$

Similar to the discussion at the end of Section 2.4 we can not expect convergence of $\mathcal{I}^{loc}(f)$ towards f if $N \to \infty$. However, we can, similar to the ideas presented in Section 4.2 use these local approximation operators in the residual correction scheme, i.e., we set $f_0 = 0$ and $e_0^{loc} = f$ and on the i-th level we have

$$s_i^{loc} = \mathcal{I}_i^{loc}(e_{i-1}^{loc})$$

and we update

$$f_i = s_1^{loc} + \cdots + s_i^{loc} \tag{4.6.5}$$

$$e_i^{loc} = f - \left(s_1^{loc} + \cdots + s_i^{loc} \right). \tag{4.6.6}$$

Again, we define the corresponding multilevel operator.

Definition 4.6.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. For $i \in \mathbb{N}$ let $X_i = \{\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,N_i}\} \subseteq \Omega$ be a quasi-uniform set of sites with fill distance h_i. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies* (2.2.2)*. For $i \in \mathbb{N}$ let $\Phi_i = \delta_i^{-n}\Phi(\cdot/\delta_i)$ be the rescaled kernel where the scaling parameter δ_i is coupled to h_i as in* (2.4.6)*. For every $1 \leq k \leq N_i$ let $\chi_{i,k}^{loc} \in W_i$ be the localized Lagrange function anchored in $\boldsymbol{x}_{i,k}$ defined as in* (3.3.1) *with general footprint $X_{i,r_i}(\boldsymbol{x}_{i,k})$ as in* (3.2.2) *and level-dependent cut-off radius $r_i > 0$. Let $\mathcal{I}_i^{loc} : H^s(\Omega) \to W_i$ be the local approximation operators defined in Definition 3.3.5.*

Then we define the local Lagrange multilevel operator *$Q_L^{loc} : H^s(\Omega) \to V_L$ by*

$$Q_L^{loc}(f) = \sum_{i=1}^{L} \mathcal{I}_i^{loc} \left(e_{i-1}^{loc} \right).$$

We remark that we have to use a set of localized Lagrange functions for every level. Clearly, we can also use Theorem 4.6.4 to obtain an equivalent definition of the local Lagrange multilevel operator.

Also, we do not expect that the operators $\mathcal{I}_i^{loc}$, $1 \leq i \leq L$, or the operator Q_L^{loc} to be interpolation operators. However, we can still prove convergence of local Lagrange multilevel method, for the level-dependent footprints $X_{r(\varepsilon_i)}(\boldsymbol{x}_{i,k})$, $1 \leq i \leq L$, $1 \leq k \leq N_i$, if we choose ε_i sufficiently small. But we can not employ the techniques we used in the proofs of Section 4.3 because we can not use estimates of the form (2.3.11).

For the following proof we need an equivalent formulation of the algebraic decay of the Fourier decay of the chosen kernel function (2.2.2). The proof of the lemma is straight-forward, see, e.g., [92].

Lemma 4.6.7. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$. Then its Fourier transform $\widehat{\Phi}$ satisfies* (2.2.2) *with constants $c_1, c_2 > 0$ if and only if it satisfies*

$$\widetilde{c}_1(1 + \|\boldsymbol{\xi}\|_2^{2s})^{-1} \leq \widehat{\Phi}(\boldsymbol{\xi}) \leq \widetilde{c}_2(1 + \|\boldsymbol{\xi}\|_2^{2s})^{-1}, \quad \boldsymbol{\xi} \in \mathbb{R}^n, \tag{4.6.7}$$

with constants $\widetilde{c}_1 = c_1 2^{-s}$ and $\widetilde{c}_2 = c_2$.

That also means that, if we assume that the Fourier decay of the kernel Φ satisfies (4.6.7), the norm equivalence (2.4.2) between the Sobolev space $H^s(\mathbb{R}^n)$ and the native space $\mathcal{N}_{\Phi_\delta}$ of the rescaled kernel $\Phi_\delta = \delta^{-n}\Phi(\cdot/\delta)$ holds with constants $\widetilde{c}_1$ and $\widetilde{c}_2$.

We are now in the position to state the convergence theorem for the multilevel method using localized Lagrange functions.

Theorem 4.6.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. For $i \in \mathbb{N}$ let $X_i = \{\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,N_i}\} \subseteq \Omega$ be a quasi-uniform set of sites with fill distance h_i and separation radius q_i. Assume that $c\mu h_i \leq h_{i+1} \leq \mu h_i$ holds for all*

$i \in \mathbb{N}$ with fixed constants $\mu \in (0,1)$, $c \in (0,1]$ and h_1 sufficiently small such that $q_1 < 1$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported reproducing kernel of $H^s(\mathbb{R}^n)$, $s > n/2$, i.e., the Fourier transform of Φ satisfies (4.6.7). *For $i \in \mathbb{N}$ let $\Phi_i = \delta_i^{-n}\Phi(\cdot/\delta_i)$ be the rescaled kernel where the scaling parameter δ_i is coupled to h_i as in* (2.4.6). *Assume that the overlap parameter ν satisfies $\frac{1}{h_1} \geq \nu$. Assume that there is a constant $\gamma > 0$ such that μ is coupled to ν according to* (4.3.5).

For $i \in \mathbb{N}$ and for every $1 \leq k \leq N_i$ let $\chi_{i,k}^{loc} \in W_i$ be the localized Lagrange function anchored in $\boldsymbol{x}_{i,k}$ defined as in (3.3.1) *with footprint $X_{i,r(\varepsilon_i)}(\boldsymbol{x}_{i,k})$ as in* (3.2.2) *and cut-off radius $r(\varepsilon_i) > 0$ and $\varepsilon_i > 0$. For target a function $f \in H^s(\Omega)$ let $e_0^{loc} = f$ and the residual of level $i \in \mathbb{N}$ be given by $e_i^{loc} = e_{i-1}^{loc} - \mathcal{I}_i^{loc}(e_{i-1}^{loc})$.*

If ε_i is chosen small enough such that

$$C(\Omega,\Phi,n)\frac{e^{\eta_i}}{\eta_i^{3n}}\left|\log\left(\frac{\varepsilon_i}{q_i^n}\right)\right|^{2n} q_i^{-\frac{n}{2}-4s}\sqrt{\varepsilon_i} \leq 1 \tag{4.6.8}$$

then the recursion estimate

$$\|Ee_i^{loc}\|_{\Phi_{i+1}} \leq \alpha^{loc}\|Ee_{i-1}^{loc}\|_{\Phi_i} \tag{4.6.9}$$

holds for all $i \in \mathbb{N}$ with

$$\alpha^{loc} := C_1^{loc}\mu^s,$$

where $C_1^{loc} = C(\Omega,\Phi,n,s,\gamma) > 0$ is a constant.

Proof. Similar to the proof of Theorem 4.3.2 we split $\|Ee_i^{loc}\|_{\Phi_{i+1}}$ into two separate parts. However, here, we follow the approach of [92]. We have with the norm equivalence, which follows from (4.6.7),

$$\|Ee_i^{loc}\|_{\Phi_{i+1}}^2 \leq \frac{1}{\widetilde{c}_1}\int_{\mathbb{R}^n} |\widehat{Ee_i^{loc}}(\boldsymbol{\xi})|^2 \left(1 + (\delta_{i+1}\|\boldsymbol{\xi}\|_2)^{2s}\right)\, d\boldsymbol{\xi} =: \frac{1}{\widetilde{c}_1}(I_1 + I_2), \tag{4.6.10}$$

with

$$I_1 = \int_{\mathbb{R}^n} |\widehat{Ee_i^{loc}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \quad \text{and} \quad I_2 = \delta_{i+1}^{2s}\int_{\mathbb{R}^n} |\widehat{Ee_i^{loc}}(\boldsymbol{\xi})|^2\|\boldsymbol{\xi}\|_2^{2s}\, d\boldsymbol{\xi}.$$

We start with bounding I_1. With Plancharel's theorem and the properties of the extension operator, Theorem 2.1.7, we have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} |\widehat{Ee_i^{loc}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} = \|Ee_i^{loc}\|_{L_2(\mathbb{R}^n)}^2 \\ &\leq C_E^2\|e_i^{loc}\|_{L_2(\Omega)} = C_E^2\|e_{i-1}^{loc} - \mathcal{I}_i^{loc}(e_{i-1}^{loc})\|_{L_2(\Omega)}. \end{aligned}$$

We introduce the interpolation operator I_{X_i,Φ_i} which uses the full Lagrange functions, and obtain

$$\begin{aligned} I_1 &\leq C_E^2\|e_{i-1}^{loc} - \mathcal{I}_i^{loc}(e_{i-1}^{loc})\|_{L_2(\Omega)} \\ &\leq C_E^2\left(\|e_{i-1}^{loc} - I_{X_i,\Phi_i}(e_{i-1}^{loc})\|_{L_2(\Omega)} + \|I_{X_i,\Phi_i}(e_{i-1}^{loc}) - \mathcal{I}_i^{loc}(e_{i-1}^{loc})\|_{L_2(\Omega)}\right)^2. \end{aligned} \tag{4.6.11}$$

We bound the two norms in the bracket of (4.6.11) separately. For the first term we have, with the sampling inequality (2.3.11), since $I_{X_i,\Phi_i}(e_{i-1}^{loc})$ interpolates e_{i-1}^{loc} on X_i, and (2.4.2), the estimate

$$\begin{aligned}\|e_{i-1}^{loc} - I_{X_i,\Phi_i}(e_{i-1}^{loc})\|_{L_2(\Omega)} &\le c(\Omega,s)h_i^s\|e_{i-1}^{loc}\|_{H^s(\mathbb{R}^n)} \le c(\Omega,s)\left(\frac{h_i}{\delta_i}\right)^s\|e_{i-1}^{loc}\|_{\Phi_i}\\ &\le c(\Omega,s)\nu^{-s}\|e_{i-1}^{loc}\|_{\Phi_i}.\end{aligned}$$

To bound the second norm in the bracket in (4.6.11) we use that

$$I_{X_i,\Phi_i}(e_{i-1}^{loc}) = I_{X_i,\Phi_i}(Ee_{i-1}^{loc}) \quad \text{and} \quad \mathcal{I}_i^{loc}(e_{i-1}^{loc}) = \mathcal{I}_i^{loc}(Ee_{i-1}^{loc}),$$

since both approximation processes only use $X_i \subseteq \Omega$. This yields

$$\begin{aligned}&\|I_{X_i,\Phi_i}(e_{i-1}^{loc}) - \mathcal{I}_i^{loc}(e_{i-1}^{loc})\|_{L_2(\Omega)} = \|I_{X_i,\Phi_i}(Ee_{i-1}^{loc}) - \mathcal{I}_i^{loc}(Ee_{i-1}^{loc})\|_{L_2(\Omega)}\\ &= \left\|\sum_{k=1}^{N_i} Ee_{i-1}^{loc}(\boldsymbol{x}_{i,k})\chi_{i,k} - \sum_{k=1}^{N_i} Ee_{i-1}^{loc}(\boldsymbol{x}_{i,k})\chi_{i,k}^{loc}\right\|_{L_2(\Omega)}\\ &\le \|Ee_{i-1}^{loc}\|_{\ell_\infty(X_i)}\sum_{k=1}^{N_i}\|\chi_{i,k} - \chi_{i,k}^{loc}\|_{L_2(\Omega)}.\end{aligned} \tag{4.6.12}$$

To bound this further we see that the estimate

$$\begin{aligned}\|Ee_{i-1}^{loc}\|_{\ell_\infty(X_i)} &\le \|Ee_{i-1}^{loc}\|_{L_\infty(\Omega)} \le c(\Omega,s)\|Ee_{i-1}^{loc}\|_{H^s(\Omega)}\\ &\le c(\Omega,\Phi,s)\delta_i^{-s}\|Ee_{i-1}^{loc}\|_{\Phi_i}\end{aligned} \tag{4.6.13}$$

holds.

Using (4.6.13) in (4.6.12) leads to

$$\begin{aligned}&\|I_{X_i,\Phi_i}(e_{i-1}^{loc}) - \mathcal{I}_i^{loc}(e_{i-1}^{loc})\|_{L_2(\Omega)} \le \|Ee_{i-1}^{loc}\|_{\ell_\infty(X_i)}\sum_{k=1}^{N_i}\|\chi_{i,k} - \chi_{i,k}^{loc}\|_{L_2(\Omega)}\\ &\le c(\Omega,\Phi,s)\delta_i^{-s}\|Ee_{i-1}^{loc}\|_{\Phi_i}\sum_{k=1}^{N_i}\|\chi_{i,k} - \chi_{i,k}^{loc}\|_{L_2(\Omega)}.\end{aligned}$$

This yields for I_1

$$\begin{aligned}I_1 &\le C_E^2\left(\|e_{i-1}^{loc} - I_{X_i,\Phi_i}(e_{i-1}^{loc})\|_{L_2(\Omega)} + \|I_{X_i,\Phi_i}(e_{i-1}^{loc}) - \mathcal{I}_i^{loc}(e_{i-1}^{loc})\|_{L_2(\Omega)}\right)^2\\ &\le c(\Omega,\Phi,s)\left(\nu^{-s} + \delta_i^{-s}\sum_{k=1}^{N_i}\|\chi_{i,k} - \chi_{i,k}^{loc}\|_{L_2(\Omega)}\right)^2\|Ee_{i-1}^{loc}\|_{\Phi_i}^2\\ &\le c(\Omega,\Phi,s)\left(1 + h_i^{-s}\sum_{k=1}^{N_i}\|\chi_{i,k} - \chi_{i,k}^{loc}\|_{L_2(\Omega)}\right)^2\nu^{-2s}\|Ee_{i-1}^{loc}\|_{\Phi_i}^2\\ &\le c(\Omega,\Phi,s,\gamma)\left(1 + h_i^{-s}\sum_{k=1}^{N_i}\|\chi_{i,k} - \chi_{i,k}^{loc}\|_{L_2(\Omega)}\right)^2\mu^{2s}\|Ee_{i-1}^{loc}\|_{\Phi_i}^2\\ &\le c(\Omega,\Phi,s,\gamma)\mu^{2s}\|Ee_{i-1}^{loc}\|_{\Phi_i}^2,\end{aligned} \tag{4.6.14}$$

where we have used that μ is coupled to ν according to (4.3.5). To arrive at (4.6.14) we used that we have, with (3.3.10),

$$\begin{aligned} h_i^{-s}\sum_{k=1}^{N_i}\|\chi_{i,k}-\chi_{i,k}^{loc}\|_{L_2(\Omega)} &\le C(\Phi,\Omega,n)h_i^{-s}N_i\frac{e^{\eta_i}}{\eta_i^{3n}}\left|\log\left(\frac{\varepsilon_i}{q_i^n}\right)\right|^{2n} q_i^{\frac{n}{2}-2s}\sqrt{\varepsilon_i} \\ &\le C(\Phi,\Omega,n)\frac{e^{\eta_i}}{\eta_i^{3n}}\left|\log\left(\frac{\varepsilon_i}{q_i^n}\right)\right|^{2n} q_i^{-\frac{n}{2}-3s}\sqrt{\varepsilon_i} \\ &\le 1, \end{aligned}$$

since we assumed that X_i is quasi-uniform, that ε_i is chosen such that (4.6.8) holds and $q_i^{-n/2-3s}\le q_i^{-n/2-4s}$, since $q_i<1$.

Using similar ideas for I_2 leads to

$$\begin{aligned} I_2 &= \delta_{i+1}^{2s}\int_{\mathbb{R}^n}|\widehat{Ee_{i-1}^{loc}}(\boldsymbol{\xi})|^2\|\boldsymbol{\xi}\|_2^{2s}\,d\boldsymbol{\xi} \;\le\; \delta_{i+1}^{2s}\int_{\mathbb{R}^n}|\widehat{Ee_{i-1}^{loc}}(\boldsymbol{\xi})|^2(1+\|\boldsymbol{\xi}\|_2^2)^s\,d\boldsymbol{\xi} \\ &= \delta_{i+1}^{2s}\|Ee_{i-1}^{loc}\|_{H^s(\mathbb{R}^n)}^2 \;\le\; C_E^2\delta_{i+1}^{2s}\|e_{i-1}^{loc}\|_{H^s(\Omega)}^2 \end{aligned}$$

(4.6.15)

$$\begin{aligned} &\le C_E^2\delta_{i+1}^{2s}\Big(\|e_{i-1}^{loc}-I_{X_i,\Phi_i}(e_{i-1}^{loc})\|_{H^s(\Omega)}+ \\ &\qquad +\|I_{X_i,\Phi_i}(e_{i-1}^{loc})-\mathcal{I}_i^{loc}(e_{i-1}^{loc})\|_{H^s(\Omega)}\Big)^2. \end{aligned}$$

Again, we bound the two norms in (4.6.15) separately. The first norm can be estimated as in (4.3.2). We have

$$\|e_{i-1}^{loc}-I_{X_i,\Phi_i}(e_{i-1}^{loc})\|_{H^s(\Omega)}\le c(\Omega,\Phi,s)^{\frac12}\delta_i^{-s}\|Ee_{i-1}^{loc}\|_{\Phi_i}.$$

To bound the second norm we use the same ideas that led to (4.6.13), using (4.6.12). We have

$$\begin{aligned} &\|I_{X_i,\Phi_i}(e_{i-1}^{loc})-\mathcal{I}_i^{loc}(e_{i-1}^{loc})\|_{H^s(\Omega)}\le \\ &\qquad\le C(\Phi,\Omega,s)\delta_i^{-s}\|Ee_{i-1}^{loc}\|_{\Phi_i}\sum_{k=1}^{N_i}\|\chi_{i,k}-\chi_{i,k}^{loc}\|_{H^s(\Omega)}. \end{aligned}$$

Putting these two estimates back into (4.6.15) yields the bound for I_2

$$\begin{aligned} I_2 &\le C_E^2\delta_{i+1}^{2s}\Big(\|e_{i-1}^{loc}-I_{X_i,\Phi_i}(e_{i-1}^{loc})\|_{H^s(\Omega)}+ \\ &\qquad +\|I_{X_i,\Phi_i}(e_{i-1}^{loc})-\mathcal{I}_i^{loc}(e_{i-1}^{loc})\|_{H^s(\Omega)}\Big)^2 \\ &\le C_E^2\left(c(\Omega,\Phi,s)\left(\frac{\delta_{i+1}}{\delta_i}\right)^s + c(\Phi,\Omega,s)\left(\frac{\delta_{i+1}}{\delta_i}\right)^s\sum_{k=1}^{N_i}\|\chi_{i,k}-\chi_{i,k}^{loc}\|_{H^s(\Omega)}\right)^2\cdot \\ &\quad\cdot\|Ee_{i-1}^{loc}\|_{\Phi_i}^2 \\ &\le C(\Omega,\Phi,s)\left(1+\sum_{k=1}^{N_i}\|\chi_{i,k}-\chi_{i,k}^{loc}\|_{H^s(\Omega)}\right)^2\mu^{2s}\|Ee_{i-1}^{loc}\|_{\Phi_i}^2 \end{aligned}$$

(4.6.16)

$$\le C(\Omega,\Phi,s)\mu^{2s}\|Ee_{i-1}^{loc}\|_{\Phi_i}^2.$$

Again, we used that we can bound

$$\begin{aligned}\sum_{k=1}^{N_i} \|\chi_{i,k} - \chi_{i,k}^{loc}\|_{H^s(\Omega)} &\leq C(\Omega,\Phi,n,s) N_i \frac{e^{\eta_i}}{\eta_i^{3n}} \left|\log\left(\frac{\varepsilon_i}{q_i^n}\right)\right|^{2n} q_i^{\frac{n}{2}-4s}\sqrt{\varepsilon_i} \\ &\leq C(\Omega,\Phi,n,s) \frac{e^{\eta_i}}{\eta_i^{3n}} \left|\log\left(\frac{\varepsilon_i}{q_i^n}\right)\right|^{2n} q_i^{-\frac{n}{2}-4s}\sqrt{\varepsilon_i} \\ &\leq 1,\end{aligned}$$

since we assumed that X_i is quasi-uniform and with the help of (3.3.10) and our choice of ε_i such that (4.6.8) holds.

Putting the bound on I_1, (4.6.14), and the bound on I_2, (4.6.16), back into (4.6.10) yields the claim with a constant $C_1^{loc} := C(\Omega,\Phi,n,s,\gamma) > 0$. □

Clearly, we need to discuss this result in greater detail. We see that the recursion estimate (4.6.9) has the same form as the one in Theorem 4.3.4, only with a different constant C_1^{loc}. Again, this will allow us to show the convergence of the local Lagrange multilevel method. In order for the recursion estimate to hold, we have to choose ε_i for every $i \in \mathbb{N}$ small enough. On first sight, that is a restriction. However, recalling Lemma 3.3.4, there is already an upper bound for ε_i such that $\{\chi_{i,k}^{loc}\}_{1\leq k\leq N_i}$ is a stable basis of W_i. It reads as

$$C\frac{e^{\eta}}{\eta^{3n}} \left|\log\left(\frac{\varepsilon_i}{q_i^n}\right)\right|^{2n} q_i^{-\frac{n}{2}-2s-J}\sqrt{\varepsilon_i} \leq 1,$$

with a free $J > n$. Comparing this to (4.6.8) we see that these are the same conditions for ε_i if $J = 2s > n$, since $s > n/2$. Hence, (4.6.8) does not seem to be a restriction at all, since we want to work with a stable basis $\{\chi_{i,k}^{loc}\}_{1\leq k\leq N_i} \subseteq W_i$ anyway.

To end this chapter, we use Theorem 4.6.8 to obtain the analogue to Theorem 4.3.4 for the local Lagrange multilevel method.

Corollary 4.6.9. *With the assumptions and notation of Theorem 4.6.8 there is a constant $C > 0$ such that the estimate*

$$\|f - Q_L^{loc}(f)\|_{L_2(\Omega)} \leq C\left(\alpha^{loc}\right)^L \|f\|_{H^s(\Omega)} \tag{4.6.17}$$

holds for all $L \in \mathbb{N}$. Thus, $Q_L^{loc}(f)$ converges linearly to f if $\alpha^{loc} < 1$.

The idea to use the localized Lagrange functions in the multilevel scheme came to mind when we studied the alternative representation of the multilevel operator in Section 4.6.1. Numerically, there is no advantage using this new representation compared to the standard one in Section 4.2. However, in our application in Chapter 7, it is necessary to use Lagrange functions. Unfortunately, numerical tests show that this new representation is much more costly. Although further research has to be done, we hope that the local Lagrange functions will help in reducing the costs.

CHAPTER 5

Tensor Products

We give now an introduction into tensor products, an essential tool for the rest of this thesis. This introduction is, in parts, more thorough than necessary, however it provides a solid, theoretical basis for the applications in Chapters 6 and 7. The groundwork provided here is mainly a synthesis of [42], on the one hand, and [58, 76] on the other hand.

The chapter is organized as follows. In Section 5.1 we introduce the tensor product as a bilinear mapping and study its codomain, the algebraic tensor product space, and its elements, tensors. Then, in Section 5.2, we equip the algebraic tensor product spaces with norms, where we are particularly interested in crossnorms and their properties. In this section we provide several examples of tensor Banach spaces which we will use in chapters to come. Finally, we define the tensor product of operators in Section 5.3. This will open the door to the main topic of this thesis in Chapters 6 and 7, the construction of high-directional approximation operators by taking the tensor product of approximation operators defined for each direction.

5.1. Algebraic Tensor Product Space

We start by giving the definition of a special property of bilinear mappings.

Definition 5.1.1. *Let V, W and T be linear spaces. The bilinear mapping $\varphi : V \times W \to T$ has the* universal property *if*

(1) φ generates T, i.e.,

$$\operatorname{Im}(\varphi) := \operatorname{span}\{\varphi(v, w) \ : \ v \in V, w \in W\} = T,$$

(2) for every other bilinear mapping ψ from $V \times W$ into another linear space S, there exists a linear map $f : T \to S$ such that the diagram

(5.1.1)
$$\begin{array}{ccc} V \times W & \xrightarrow{\psi} & S \\ {\scriptstyle\varphi}\downarrow & \nearrow_{f} & \\ T & & \end{array}$$

commutes. This means we have $f \circ \varphi = \psi$.

We can combine (1) and (2) in Definition 5.1.1 into one single condition. The proof can be found in [38].

Theorem 5.1.2. *Let V, W, T and S be linear spaces. The bilinear mapping φ has the universal property, if for every bilinear mapping $\psi : V \times W \to S$ there exists a unique linear map $f : T \to S$ such that* (5.1.1) *commutes.*

From this we can immediately show that the linear space T is uniquely determined up to an isomorphism.

Theorem 5.1.3. *Let V, W, T and $\widetilde{T}$ be linear spaces. If $\varphi : V \times W \to T$ and $\widetilde{\varphi} : V \times W \to \widetilde{T}$ are bilinear maps that have the universal property then there is a linear isomorphism $f : T \to \widetilde{T}$.*

Proof. By using Theorem 5.1.2 for φ and $\widetilde{\varphi}$, respectively, we have the existence of unique linear maps $f : T \to \widetilde{T}$ and $\widetilde{f} : \widetilde{T} \to T$ such that $\widetilde{\varphi} = f \circ \varphi$ and $\varphi = \widetilde{f} \circ \widetilde{\varphi}$. Here, we set $S := \widetilde{T}$ and $\psi := \widetilde{\varphi}$, when using (5.1.1) for φ and $S := T$ and $\psi := \varphi$ for $\widetilde{\varphi}$.

This shows that $\varphi = \widetilde{f} \circ f \circ \varphi$ and that $\widetilde{\varphi} = f \circ \widetilde{f} \circ \widetilde{\varphi}$. This means indeed that f and $\widetilde{f}$ are bijective and the inverse of each other. □

We can now use the result of Theorem 5.1.3 to define the algebraic tensor product space.

Definition 5.1.4. *Let V, W and T be linear spaces. Let $\varphi : V \times W \to T$ be a bilinear map that has the universal property. Then T is the* (algebraic) tensor product *or* (algebraic) tensor product space *of V and W and is denoted by $V \otimes W$.*

The mapping φ is called the tensor product *and denoted by $\otimes$. The elements of T are called* (algebraic) tensors. *Tensors of the form $v \otimes w := \otimes(v, w) = \varphi(v, w)$ are called* elementary tensors *or* dyads.

We have only shown that if an algebraic tensor product space exists it is, up to an isomorphism, unique without checking if there even exists a linear map that has the universal property. This, however, is always true for arbitrary linear spaces V and W. For details we refer to, e.g., [38, Chapter 1.7].

We introduced the algebraic tensor product space as the tensor product of two linear spaces. However, we can generalize the statements above to the d-fold tensor product, $d \in \mathbb{N}$, in the following way: Let $V^{(1)}, \dots, V^{(d)}$ be linear spaces. Then define the *(algebraic) tensor product space of order d*, denoted by $\bigotimes_{j=1}^{d} V^{(j)} := V^{(1)} \otimes \cdots \otimes V^{(d)}$, by iteratively applying Definition 5.1.1 to $V = (V^{(1)} \otimes \cdots \otimes V^{(k)})$ and $W = V^{(k+1)}$, $k = 1, \dots, d-1$. This then yields the algebraic tensor product space

$$(\cdots (V^{(1)} \otimes V^{(2)}) \otimes V^{(3)}) \otimes \cdots \otimes V^{(d-1)}) \otimes V^{(d)}.$$

At first glance it seems that the order of the $V^{(j)}$ is important. However we see next that the tensor product is commutative and associative. This also justifies in some sense the term *product*.

Lemma 5.1.5. *Let U, V, W be linear spaces.*

(1) The tensor product is commutative in the sense that $V \otimes W$ and $W \otimes V$ are isomorphic, i.e.,

$$V \otimes W \cong W \otimes V.$$

(2) The tensor product is associative, i.e.,

$$U \otimes V \otimes W \cong (U \otimes V) \otimes W \cong U \otimes (V \otimes W).$$

PROOF. To show (1) we use the universal property. Consider two bilinear mappings

$$\psi : V \times W \to W \otimes V \quad \text{and} \quad \widetilde{\psi} : W \times V \to V \otimes W$$

given by

$$\psi(v, w) = w \otimes v \quad \text{and} \quad \widetilde{\psi}(w, v) = v \otimes w,$$

respectively. By Definition 5.1.1 (2) there are linear maps

$$f : V \otimes W \to W \otimes V \quad \text{and} \quad g : W \otimes V \to V \otimes W,$$

such that the corresponding diagrams (5.1.1)

$$\begin{array}{ccc} V \times W & \xrightarrow{\psi} & W \otimes V \\ {\scriptstyle\otimes}\downarrow & \nearrow_{f} & \\ V \otimes W & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W \times V & \xrightarrow{\widetilde{\psi}} & V \otimes W \\ {\scriptstyle\otimes}\downarrow & \nearrow_{g} & \\ W \otimes V & & \end{array}$$

commute. We now need to show, that f and g are inverse isomorphisms. For $v \in V$ and $w \in W$ we have on the left-hand side

$$w \otimes v = \varphi(v, w) = f(v \otimes w)$$

and on the right-hand side, accordingly,

$$v \otimes w = \psi(w, v) = g(w \otimes v).$$

This implies on the one hand

$$w \otimes v = f(v \otimes w) = f(g(w \otimes v)).$$

On the other hand, we have

$$v \otimes w = g(w \otimes v) = g(f(v \otimes w)).$$

This shows immediately that f and g are indeed inverse isomorphisms and therefore the commutativity of the tensor product.

The proof of (2) follows the same ideas and can be found in, e.g., [38, Proposition 1.10.1]. □

These results justify that, for this chapter, we only consider tensor product spaces of order $d = 2$. Generalizations to higher orders are mostly straightforward inductions over the number of factors d.

Until now, we studied properties of the tensor product $\otimes$ as a mapping from $V \times W$ to $V \otimes W$. Now, we want to investigate the tensor space $V \otimes W$ and its elements closer. For later references we first collect several features of elementary tensors. They all follow directly from the bilinearity of $\otimes$ and the universal property Definition 5.1.1.

Proposition 5.1.6. *Let V and W be linear spaces.*

(1) Every tensor $\boldsymbol{t} \in V \otimes W$ has a representation

$$\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i,$$

with $v_i \in V$, $w_i \in W$ and an $n \in \mathbb{N}$. Here, the order of the terms does not matter, i.e., for any permutation $\pi : \{1, \dots n\} \to \{1, \dots, n\}$ we have

$$\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i = \sum_{i=1}^{n} v_{\pi(i)} \otimes w_{\pi(i)}.$$

(2) To formally define the linear space structure of $V \otimes W$, we characterize the addition of two tensors by

$$\sum_{i=1}^{n} v_i \otimes w_i + \sum_{i=n+1}^{m} v_i \otimes w_i = \sum_{i=1}^{m} v_i \otimes w_i,$$

for $m > n$.
The multiplication by a scalar $a \in \mathbb{K}$ is described by

$$a \sum_{i=1}^{n} v_i \otimes w_i = \sum_{i=1}^{n} (a v_i) \otimes w_i = \sum_{i=1}^{n} v_i \otimes (a w_i).$$

(3) Finally, the zero element $\mathbf{0}_{V \otimes W}$ is determined by the equation

$$0_V \otimes w = v \otimes 0_W = \mathbf{0}_{V \otimes W}, \quad v \in V, w \in W.$$

Here, we used the index to clearify, which zero element of which vector space we use. We will omit it, if there is no risk of confusion.

Now, we want to construct a basis of the tensor product space $V \otimes W$ by using bases of the vector spaces V and W. To do this we need the following, preliminary corollary, taken from [38, Lemma 1.5.1].

Corollary 5.1.7. *Let V and W be linear space. If $v \neq 0_V$ and $w \neq 0_W$, then $v \otimes w \neq \mathbf{0}_{V \otimes W}$.*

Or, equivalently, if $v \otimes w = \mathbf{0}$, then at least one of the v and w has to be the zero vector of the respective space.

This allows us to state the theorem which describes how to obtain a basis of the tensor product space.

Theorem 5.1.8. *Let V and W be linear spaces with tensor product space $V \otimes W$. Let I and J be, not necessarily finite, index sets. Let $\{v_i\}_{i \in I}$ be a basis of V and $\{w_j\}_{j \in J}$ a basis of W. Then the set*

$$\mathcal{B} := \{v_i \otimes w_j \ : \ i \in I, j \in J\}$$

is a basis of $V \otimes W$.

Proof. First, we prove that $\mathcal{B}$ is linearly independent. We need show that $\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} v_i \otimes w_j = \mathbf{0}$ implies $\alpha_{ij} = 0$ for every finite n and m such that $\{1, \dots, n\} \subseteq I$ and $\{1, \dots, m\} \subseteq J$.

With Proposition 5.1.6(2), we have

$$\begin{aligned} \mathbf{0} &= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} v_i \otimes w_j = \sum_{i=1}^{n} v_i \otimes \left(\sum_{j=1}^{m} \alpha_{ij} w_j \right) \\ &=: \sum_{i=1}^{n} v_i \otimes \widetilde{w}_i. \end{aligned} \tag{5.1.2}$$

Now, we associate to every tensor $\sum_{i=1}^n x_i \otimes y_i \in V \otimes W$ an operator $A_{\sum_{i=1}^n x_i \otimes y_i} : V^* \to W$, defined by

$$A_{\sum_{i=1}^n x_i \otimes y_i}(\phi) = \sum_{i=1}^n \phi(x_i) y_i. \tag{5.1.3}$$

Clearly, such an operator is linear and $A_{\mathbf{0}}(\phi) = 0$ for all $\phi \in V^*$.

Let $\{\phi_i\}_{i\in\{1,\dots,n\}} \subset V^*$ be the dual basis to $\{v_i\}_{i\in\{1,\dots,n\}}$, i.e., $\phi_i(v_k) = \delta_{ik}$ holds for all $1 \le i,k \le n$. Using (5.1.2) we see that for $k \in \{1,\dots,n\}$ we have

$$0 = A_{\mathbf{0}}(\phi_k) = A_{\sum_{i=1}^n v_i \otimes \widetilde{w}_i}(\phi_k) = \sum_{i=1}^n \phi_k(v_i)\widetilde{w}_i = \widetilde{w}_k. \tag{5.1.4}$$

We recall that we set $\widetilde{w}_k = \sum_{j=1}^m \alpha_{kj} w_j$ and that the set $\{w_j\}_{j\in\{1,\dots,m\}}$ is, as a subset of the basis of W, linearly independent. This, together with (5.1.4), yields $\alpha_{kj} = 0$ for all $j \in \{1,\dots,m\}$. Furthermore, since $k \in \{1,\dots,n\}$ was arbitrarily chosen, we have $\alpha_{kj} = 0$ for all $k \in \{1,\dots,n\}$, $j \in \{1,\dots,m\}$, which is the linearly independence of $\mathcal{B}$.

To finish the proof, we need to show that $\operatorname{span}(\mathcal{B}) = V \otimes W$. We consider a tensor $\boldsymbol{t} \in V \otimes W$, with representation $\boldsymbol{t} = \sum_{k=1}^n v_k \otimes w_k$. Each v_k itself admits a representation in the basis, i.e., $v_k = \sum_{i\in I_k} \alpha_i^k v_i$, and similarly $w_k = \sum_{j\in J_k} \beta_j^k w_j$ with finite sets $I_k \subseteq I$ and $J_k \subseteq J$.

The representation for $\boldsymbol{t}$ becomes, by Proposition 5.1.6,

$$\begin{aligned}\boldsymbol{t} &= \sum_{k=1}^n \left(\sum_{i\in I_k} \alpha_i^k v_i\right) \otimes \left(\sum_{j\in J_k} \beta_j^k w_j\right) \\ &= \sum_{i,j} \left(\sum_{k=1}^n \alpha_i^k \beta_j^k\right) v_i \otimes w_j,\end{aligned}$$

where the length of the outer sum is finite for i and j. Hence, we have $V \otimes W = \operatorname{span}(\mathcal{B})$. □

We will later see that the operator introduced in (5.1.3) plays an important role in representing tensors of $V \otimes W$. We also remark that we can just as well define a corresponding operator as a mapping from W^* to V without any significant change in the proof.

Before we discuss these points further, we use the preceding theorem to introduce the first examples of algebraic tensor product spaces.

Theorem 5.1.9. *The tensor product space $\mathbb{R} \otimes \mathbb{R}$ is isomorphic to $\mathbb{R}$.*

Proof. $\{1\}$ is a basis of $\mathbb{R}$. This means by Theorem 5.1.8 that $\{1 \otimes 1\}$ is a basis of $\mathbb{R}\otimes\mathbb{R}$. Assume that $\boldsymbol{t} \in \mathbb{R}\otimes\mathbb{R}$ has a representation $\boldsymbol{t} = \sum_{i=1}^n a_i \otimes b_i$, with $a_i, b_i \in \mathbb{R}$, $1 \le i \le n$. Then we have

$$\begin{aligned}\boldsymbol{t} &= \sum_{i=1}^n a_i \otimes b_i = \sum_{i=1}^n a_i \otimes (1 \cdot b_i) = \sum_{i=1}^n (a_i \cdot b_i) \otimes 1 \\ &= \sum_{i=1}^n (a_i \cdot b_i \cdot 1) \otimes 1 = \sum_{i=1}^n (a_i \cdot b_i) 1 \otimes 1.\end{aligned}$$

Hence, we define the linear map $f : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ given by

$$f\left(\sum_{i=1}^{n} a_i \otimes b_i\right) = \sum_{i=1}^{n} a_i b_i.$$

It is easy to see that $\mathrm{Im}(f) = \mathbb{R}$ and $\mathrm{Kern}(f) = \mathbf{0}$. That means that f is the isomorphism between $\mathbb{R} \otimes \mathbb{R}$ and $\mathbb{R}$. □

The proof of the preceding theorem implies that we can identify tensors $\boldsymbol{t} \in \mathbb{R} \otimes \mathbb{R}$ with representation $\boldsymbol{t} = \sum_{i=1}^{n} a_i \otimes b_i$ with real numbers $\sum_{i=1}^{n} a_i b_i$. We will meet more of these kinds of identifications later on.

The next example concerns the tensor product of functions.

Theorem 5.1.10. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, be given non-empty sets. Assume that $V(S) = \{v : S \to \mathbb{R}\}$ and $W(T) = \{w : T \to \mathbb{R}\}$ are linear spaces of functions on S and T, respectively. Define the elementary tensors $v \otimes w : S \times T \to \mathbb{R}$ by*

$$(v \otimes w)(\boldsymbol{s}, \boldsymbol{t}) := v(\boldsymbol{s}) \cdot w(\boldsymbol{t}). \tag{5.1.5}$$

Then the tensor product space $V \otimes W$ is given by

$$V(S) \otimes W(T) = \mathrm{span}\{v \otimes w \ : \ v \in V(S), w \in W(T)\}. \tag{5.1.6}$$

Proof. We omit the full proof here and just summarize the main ideas. Clearly, for $V(S) \otimes W(T)$ defined in (5.1.6) we find a bilinear map $\varphi : V(S) \times W(T) \to V(S) \otimes W(T)$ defined by $\varphi(v, w) = v \otimes w$, where $v \otimes w$ is defined point-wise as in (5.1.5). This φ satisfies $V(S) \otimes W(T) = \mathrm{Im}(\varphi)$.

All that is left is to show that φ has the universal property. To this end, it is necessary to show that the set $\{v \otimes w \ : \ v \in \mathcal{B}_V, w \in \mathcal{B}_W\}$, where $\mathcal{B}_V \subseteq V$ and $\mathcal{B}_W \subseteq W$ are bases of their respective spaces, is a basis of $V(S) \otimes W(T)$. Here, we can not directly quote Theorem 5.1.8 since we have to use the specific definition of $v \otimes w$ in (5.1.5). However, the ideas of the proof of Theorem 5.1.8 carry over.

Having the basis of $V(S) \otimes W(T)$ then allows us to finish the proof since for any linear space U and any linear map $\psi : V(S) \times W(T) \to U$ we can define the linear map $f : V(S) \otimes W(T) \to U$ by its values on the basis to show that $f(v \otimes w) = \psi(v, w)$. This then shows that $\varphi(v, w) = v \otimes w$ with $v \otimes w$ defined in (5.1.5) has the universal property and $V(S) \otimes W(T)$ is indeed a tensor product space. □

This allows us to easily evaluate the elements of algebraic tensor product function spaces.

After these two examples we now study representations of tensors in more detail. From Proposition 5.1.6 we know that we can express a tensor $\boldsymbol{t} \in V \otimes W$ as $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i$. However one of the most important results is that this representation is not unique. This fact will lead to an alternative way to represent tensors, used in, e.g., [58, 76]. First, we find that for tensors $\boldsymbol{t} \neq \mathbf{0}$ we always find a representation with a minimal number of summands. This number is sometimes called the *rank* of the tensor.

Lemma 5.1.11. *Let V and W be linear spaces with algebraic tensor product space $V \otimes W$. Every tensor $\boldsymbol{t} \in V \otimes W$ is either $\mathbf{0}_{V \otimes W}$ or there is an $r \in \mathbb{N}$*

and a representation

$$\boldsymbol{t} = \sum_{i=1}^{r} v_i \otimes w_i,$$

where $\{v_1, \dots, v_r\} \subseteq V$ *and* $\{w_1, \dots, w_r\} \subseteq W$ *are linearly independent.*

PROOF. Let $\boldsymbol{t} \in V \otimes W$ with representation $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i$. If one of the sets $\{v_i\}_{1 \le i \le n}$ or $\{w_i\}_{1 \le i \le n}$ is not linearly independent, we may assume without restriction that $\{v_i\}_{1 \le i \le n}$ is not linearly independent. Then we can express at least one v_i by a linear combination of the others. Without restriction we assume that

$$v_n = \sum_{i=1}^{n-1} \alpha_i v_i.$$

Then, with the rules of Proposition 5.1.6, we have

$$v_n \otimes w_n = \left(\sum_{i=1}^{n-1} \alpha_i v_i \right) \otimes w_n = \sum_{i=1}^{n-1} v_i \otimes (\alpha_i w_n),$$

which leads to

$$\boldsymbol{t} = \left(\sum_{i=1}^{n-1} v_i \otimes w_i \right) + v_n \otimes w_n = \sum_{i=1}^{n-1} v_i \otimes \widetilde{w}_i,$$

with $\widetilde{w}_i = w_i + \alpha_i w_n \in W$. This is a shorter representation of $\boldsymbol{t}$. We can repeat this process until we either arrive at a representation in which $\{v_1, \dots, v_r\}$ and, after doing the same for the set $\{w_i\}$, $\{w_1, \dots, w_r\}$ are linearly independent or we end up at one of the representations $0 \otimes w_k$ or $v_k \otimes 0$ and so, by Proposition 5.1.6, $\boldsymbol{t} = \boldsymbol{0}$. □

Both, Theorem 5.1.8 and Lemma 5.1.11, have an immediate very important consequence. They both show that the representation of a tensor $\boldsymbol{t} \in V \otimes W$ is not unique. If we have $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i$ for some $v_i \in V$ and $w_i \in W$ and an $n \in \mathbb{N}$, we can always add a dyad $v_{n+1} \otimes w_{n+1}$, with v_{n+1} and w_{n+1} a linear combination of $\{v_i\}_{1 \le i \le n}$ and $\{w_i\}_{1 \le i \le n}$, respectively, and obtain a new representation for the same tensor $\boldsymbol{t}$.

We now work towards a better way to represent tensors. We need the following definition formalizing (5.1.3).

Definition 5.1.12. *Let* V *and* W *be linear spaces with algebraic tensor product space* $V \otimes W$*. We define a linear operator* $A_{\sum_{i=1}^{n} v_i \otimes w_i} : V^* \to W$ *associated to a representation* $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i$ *of* $\boldsymbol{t} \in V \otimes W$ *by*

$$A_{\sum_{i=1}^{n} v_i \otimes w_i}(\phi) := \sum_{i=1}^{n} \phi(v_i) w_i.$$

We will see this operator provides a way to represent tensors in a unique way. However, first we need to discuss how a linear mappings $\phi^{(1)} : V \to \mathbb{R}$ and $\phi^{(2)} : W \to \mathbb{R}$ act on the corresponding components of tensors.

Lemma 5.1.13. *Let V and W be linear spaces with algebraic tensor product $V \otimes W$. Let $\phi^{(1)} \in V^*$ and $\phi^{(2)} \in W^*$. Assume that*

$$\sum_{i=1}^{n} v_i \otimes w_i = \sum_{j=1}^{m} x_j \otimes y_j \tag{5.1.7}$$

for $v_i, x_j \in V$, $w_i, y_j \in W$ and $n, m \in \mathbb{N}$. Then the identities

$$\sum_{i=1}^{n} \phi^{(1)}(v_i) w_i = \sum_{j=1}^{m} \phi^{(1)}(x_j) y_j \tag{5.1.8}$$

and

$$\sum_{i=1}^{n} \phi^{(1)}(v_i) \cdot \phi^{(2)}(w_i) = \sum_{j=1}^{m} \phi^{(1)}(x_j) \cdot \phi^{(2)}(y_j) \tag{5.1.9}$$

hold.

Proof. We start by showing (5.1.8). A fixed $\phi^{(1)} \in V^*$ induces a bilinear map

$$\psi^{(1)} : V \times W \to W, \quad (v, w) \mapsto \phi^{(1)}(v) w.$$

By the universal property of Definition 5.1.1, there is a unique linear map $f : V \otimes W \to W$ with

$$f(v \otimes w) = \psi^{(1)}(v, w) = \phi^{(1)}(v) w.$$

Applying this to the identity in (5.1.7) we find

$$\begin{aligned} \sum_{i=1}^{n} \phi^{(1)}(v_i) w_i = f\left(\sum_{i=1}^{n} v_i \otimes w_i \right) = f\left(\sum_{j=1}^{m} x_j \otimes y_j \right) \\ = \sum_{j=1}^{m} \phi^{(1)}(x_j) y_j. \end{aligned}$$

We used the linearity and bijectivity of f.

The second equality (5.1.9) follows simply from the linearity of $\phi^{(2)}$ and (5.1.8). □

Next, we see that only the zero operator is associated to representations of the zero tensor.

Lemma 5.1.14. *Let V and W be linear spaces with algebraic tensor product space $V \otimes W$. Then $A_{\sum_{i=1}^{n} v_i \otimes w_i} = 0$ if and only if, $\sum_{i=1}^{n} v_i \otimes w_i = \mathbf{0}_{V \otimes W}$.*

Proof. Assume that $\sum_{i=1}^{n} v_i \otimes w_i \neq \mathbf{0}$. Then, by Lemma 5.1.11, there is an $r \in \mathbb{N}$ such that $\sum_{i=1}^{n} v_i \otimes w_i = \sum_{i=1}^{r} x_i \otimes y_i$ with $\{x_i\}_{1 \leq i \leq r}$ and $\{y_i\}_{1 \leq i \leq r}$ linearly independent sets and $x_1 \neq 0$. Consequently, we can find a $\phi \in V^*$ such that $\phi(x_1) \neq 0$. The linearly independence of $\{y_i\}_{1 \leq i \leq r}$ then implies that $\sum_{i=1}^{r} \phi(x_i) y_i \neq 0$. Applying Lemma 5.1.13 leads then to $\sum_{i=1}^{n} \phi(v_i) w_i \neq 0$ and therefore

$$A_{\sum_{i=1}^{n} v_i \otimes w_i} \neq 0.$$

The other implication follows from the definition of the operator. □

We combine Lemmas 5.1.13 and 5.1.14 to obtain the required theorem, which gives a way to identify if two finite sums of dyads are representations of the tensor $\boldsymbol{t} \in V \otimes W$.

Theorem 5.1.15. *Let V and W be linear spaces with algebraic tensor product space $V \otimes W$. Two expressions $\sum_{i=1}^{n} v_i \otimes w_i$ and $\sum_{k=1}^{m} x_k \otimes y_k$ are representations for the same tensor $\boldsymbol{t} \in V \otimes W$ if and only if, the associated linear operators $A_{\sum_{i=1}^{n} v_i \otimes w_i}$ and $A_{\sum_{j=1}^{m} x_j \otimes y_j}$ as in Definition 5.1.12 are equal.*

For further information on algebraic tensor product spaces and the role of the operators of Definition 5.1.12 we refer to [76].

5.2. Norms on Tensor Product Spaces

We now extend the concept of tensor product spaces form the purely algebraic setting to normed spaces. We recall that the algebraic tensor product space of the linear spaces V and W is given by

$$V \otimes W = \operatorname{span}\{v \otimes w \ : \ v \in V, w \in W\}.$$

We start by giving the definition of a norm on the tensor product space. We follow [76, Definition 2.1] and account for the non-uniqueness of the representations of an element of the algebraic tensor product space directly in the definition. This is necessary because we define the tensor product norm as a function on the representation of a tensor.

Definition 5.2.1. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces with algebraic tensor product space $V \otimes W$. We call a mapping $\alpha : V \otimes W \to [0, \infty)$ a* norm *on the tensor space $V \otimes W$ if the following conditions hold:*

(1) $\alpha(\sum_{i=1}^{n} v_i \otimes w_i) = 0$ if and only if, $\sum_{i=1}^{n} v_i \otimes w_i = \mathbf{0}$.
(2) $\alpha(a \sum_{i=1}^{n} v_i \otimes w_i) = |a| \alpha(\sum_{i=1}^{n} v_i \otimes w_i)$ for any $a \in \mathbb{R}$.
(3) $\alpha(\sum_{i=1}^{n} v_i \otimes w_i + \sum_{j=1}^{m} x_j \otimes y_j) \leq \alpha(\sum_{i=1}^{n} v_i \otimes w_i) + \alpha(\sum_{j=1}^{m} x_j \otimes y_j)$.
(4) $\alpha(\sum_{i=1}^{n} v_i \otimes w_i) = \alpha(\sum_{j=1}^{m} x_j \otimes y_j)$, if $\sum_{i=1}^{n} v_i \otimes w_i = \sum_{j=1}^{m} x_j \otimes y_j$.

For given normed spaces V and W there are several ways to construct a norm on $V \otimes W$. To make clear which normed space we investigate, we introduce the notation $V \otimes_\alpha W$ for the normed space $(V \otimes W, \alpha)$.

A first candidate for a norm as in Definition 5.2.1 uses the operator of Definition 5.1.12.

Definition 5.2.2. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces with algebraic tensor product space $V \otimes W$. The λ- or* injective norm *$\lambda : V \otimes W \to [0, \infty)$ is defined by*

$$\lambda\left(\sum_{i=1}^{n} v_i \otimes w_i\right) := \sup\left\{\left\|\sum_{i=1}^{n} \phi(v_i) w_i\right\|_W \ : \ \phi \in V^*, \|\phi\|_{V^*} = 1\right\}. \tag{5.2.1}$$

We check that this mapping satisfies Definition 5.2.1.

Theorem 5.2.3. *The λ-norm is a norm.*

Proof. We have to verify the four conditions of Definition 5.2.1. (4) follows immediately by Lemma 5.1.13, i.e., the norm is independent of the representation of the tensor. (1) follows from Lemma 5.1.14.

(2) and (3) fare a consequence of the linearity of ϕ and the fact that $\|\cdot\|_W$ is a norm. □

If we take the λ-norm of a dyad $v \otimes w \in V \otimes W$ we have

$$\begin{aligned}\lambda(v \otimes w) &= \sup\{\|\phi(v)w\|_W \ : \ \phi \in V^*, \|\phi\|_{V^*} = 1\} \\ &= \sup\{|\phi(v)|\|w\|_W \ : \ \phi \in V^*, \|\phi\|_{V^*} = 1\} \\ &= \|v\|_V\|w\|_W. \qquad (5.2.2)\end{aligned}$$

This means the norm of the dyad equals the product of the respective norms of the components of the dyad. This is a desired property of norms on tensor product spaces. We study these norms in the next subsection.

5.2.1. Crossnorms. We formalize (5.2.2) in the following definition.

Definition 5.2.4. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces with algebraic tensor product space $V \otimes W$. Let α be a norm on $V \otimes W$. We say that α is a* crossnorm *if, for all $v \in V$ and $w \in W$,*

$$\alpha(v \otimes w) = \|v\|_V\|w\|_W \qquad (5.2.3)$$

holds.

We are mostly interested in working with crossnorms. Hence, we introduce the notation $\|\cdot\|_\alpha$ if the norm of the normed space $V \otimes_\alpha W$ is a crossnorm. If we want to emphasize the linear space we will write $\|\cdot\|_{\alpha, V\otimes W}$.

We have already proven the next statement in (5.2.2).

Lemma 5.2.5. *The λ-norm of Definition 5.2.2 is a crossnorm.*

Another example for a crossnorm is the next mapping.

Definition 5.2.6. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces with algebraic tensor product space $V \otimes W$. The γ- or* projective norm *$\gamma : V \otimes W \to [0, \infty)$ is defined by*

$$\gamma(\boldsymbol{t}) := \inf\left\{\sum_{i=1}^n \|v_i\|_V\|w_i\|_W \ : \ v_i \in V, w_i \in W, \boldsymbol{t} = \sum_{i=1}^n v_i \otimes w_i\right\}. \qquad (5.2.4)$$

Again, we see that γ is indeed a norm and a crossnorm.

Theorem 5.2.7. *The γ-norm is a norm and a crossnorm.*

Proof. First we need to show that γ is a norm. To show Property (1) in Definition 5.2.1 we first see that $\gamma(\boldsymbol{t}) = 0$ if $\boldsymbol{t} = \boldsymbol{0}$. Assume that $\boldsymbol{t} \neq \boldsymbol{0}$. Then we find a representation

$$\boldsymbol{t} = \sum_{i=1}^n v_i \otimes w_i, \quad n \in \mathbb{N}, v_i \in V \setminus \{0\}, w_i \in W \setminus \{0\}.$$

By Lemma 5.1.14 the to $\boldsymbol{t}$ associated linear operator $A_{\boldsymbol{t}} : V^* \to W$ is not the zero operator. Hence, for any $\phi \in V^*$ with $\|\phi\|_{V^*} = 1$ we have

$$\begin{aligned}0 < \|A_{\boldsymbol{t}}(\phi)\|_W &= \left\|\sum_{i=1}^n \phi(v_i)w_i\right\|_W \\ &\leq \sum_{i=1}^n |\phi(v_i)|\|w_i\|_W \leq \sum_{i=1}^n \|v_i\|_V\|w_i\|_W.\end{aligned}$$

We recall Theorem 5.1.15 which states that the operator $A_{\boldsymbol{t}}$ is independent of the representation. Hence, taking the infimum of all representations of $\boldsymbol{t}$ in the inequality above, yields

$$0 < \|A_{\boldsymbol{t}}\|_{V^* \to W} \leq \gamma(\boldsymbol{t}).$$

Property (2) in Definition 5.2.1 is obviously satisfied since $\|\cdot\|_V$ and $\|\cdot\|_W$ are norms. Property (4) is also fulfilled since we take the infimum of all representations of $\boldsymbol{t}$. To show the triangle inequality we choose an $\varepsilon > 0$ and representations $\boldsymbol{t} = \sum_{i=1}^n v_i \otimes w_i$ and $\boldsymbol{u} = \sum_{j=1}^m x_j \otimes y_j$ with

$$\sum_{i=1}^n \|v_i\|_V \|w_i\|_W \leq \gamma(\boldsymbol{t}) + \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{j=1}^m \|x_j\|_V \|y_j\|_W \leq \gamma(\boldsymbol{u}) + \frac{\varepsilon}{2}$$

for tensors $\boldsymbol{t}, \boldsymbol{u} \in V \otimes W$.

For $\boldsymbol{t} + \boldsymbol{u}$ we have a representation

$$\boldsymbol{t} + \boldsymbol{u} = \sum_{i=1}^n v_i \otimes w_i + \sum_{j=1}^m x_j \otimes y_j$$

and, the definition of the γ norm gives

$$\begin{aligned} \gamma(\boldsymbol{t} + \boldsymbol{u}) &\leq \sum_{i=1}^n \|v_i\|_V \|w_i\|_W + \sum_{j=1}^m \|x_j\|_V \|y_j\|_W \\ &\leq \gamma(\boldsymbol{t}) + \gamma(\boldsymbol{u}) + \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, this yields (3) of Definition 5.2.1. Hence, γ is indeed a norm.

To complete the proof we need to show that γ satisfies Definition 5.2.4, i.e., it is a crossnorm. Suppose that $\boldsymbol{t} \in V \otimes W$ has one the hand the representation $v \otimes w$ and on the other hand the representation $\sum_{i=1}^n v_i \otimes w_i$. First, we obviously have

$$\gamma(\boldsymbol{t}) = \gamma(v \otimes w) \leq \|v\|_V \|w\|_W.$$

We have to prove the reverse inequality. We have, by Lemma 5.1.13, for all $\phi \in V^*$

$$\phi(v)w = \sum_{i=1}^n \phi(v_i) w_i.$$

This holds for a particular $\widetilde{\phi} \in V^*$ with $\widetilde{\phi}(v) = \|v\|_V$ and $\|\widetilde{\phi}\|_{V^*} = 1$. Such a functional exists as a consequence of the Hahn-Banach theorem [3]. For this $\widetilde{\phi}$, the norm properties of $\|\cdot\|_W$ give

$$\|v\|_V \|w\|_W = \left\| \widetilde{\phi}(v) w \right\|_W = \left\| \sum_{i=1}^n \widetilde{\phi}(v_i) w_i \right\|_W \leq \sum_{i=1}^n \|v_i\|_V \|w\|_W.$$

As this holds for all possible representations of $\boldsymbol{t}$, we see

$$\|v\|_V \|w\|_W \leq \gamma(\boldsymbol{t}).$$

□

We will see that the λ- and γ-norms are particularly important. However first we need to discuss tensor products of functionals on normed spaces.

Definition 5.2.8. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces with algebraic tensor product $V \otimes W$. For $\phi^{(1)} \in V^*$ and $\phi^{(2)} \in W^*$ we define a linear mapping $\phi^{(1)} \otimes \phi^{(2)} : V \otimes W \to \mathbb{R}$ by*

$$\left(\phi^{(1)} \otimes \phi^{(2)}\right)(v \otimes w) := \phi^{(1)}(v) \cdot \phi^{(2)}(w)$$

and more generally, for $\boldsymbol{t} = \sum_{i=1}^n v_i \otimes w_i$,

$$\left(\phi^{(1)} \otimes \phi^{(2)}\right)(\boldsymbol{t}) := \sum_{i=1}^n \left(\phi^{(1)} \otimes \phi^{(2)}\right)(v_i \otimes w_i) = \sum_{i=1}^n \phi^{(1)}(v_i) \cdot \phi^{(2)}(w_i)$$

In Lemma 5.1.13 we have already proven that the value $\left(\phi^{(1)} \otimes \phi^{(2)}\right)(\boldsymbol{t})$ is independent of the representation of $\boldsymbol{t} \in V \otimes W$. Furthermore, this independence also guarantees the linearity of the mapping. In turn, this means that $\phi^{(1)} \otimes \phi^{(2)}$ is an element of the algebraic dual space of $V \otimes W$. We can also equip this space with a norm.

Definition 5.2.9. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces with normed tensor product space $(V \otimes W, \alpha)$. We define the* to α dual norm *$\alpha^* : V^* \otimes W^* \to [0, \infty)$ by*

$$\alpha^*\left(\phi^{(1)} \otimes \phi^{(2)}\right) = \sup_{\substack{\boldsymbol{t} \in V \otimes W \\ \alpha(\boldsymbol{t}) = 1}} \left|\left(\phi^{(1)} \otimes \phi^{(2)}\right)(\boldsymbol{t})\right|. \tag{5.2.5}$$

We have by definition that $\phi^{(1)} \otimes \phi^{(2)} \in V^* \otimes W^*$, if $\phi^{(1)} \in V^*$ and $\phi^{(2)} \in W^*$. However, that does not mean that $\phi^{(1)} \otimes \phi^{(2)}$ is a bounded linear functional on $(V \otimes W, \alpha)$ and this will be in general not the case. The next definition gives a sufficient condition on the tensor product norm such that $\phi^{(1)} \otimes \phi^{(2)} \in (V \otimes W)^*$.

Definition 5.2.10. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces with algebraic tensor product $V \otimes W$. A norm $\alpha : V \otimes W \to \mathbb{R}$ is* reasonable *if*

$$\alpha^*\left(\phi^{(1)} \otimes \phi^{(2)}\right) = \left\|\phi^{(1)}\right\|_{V^*} \left\|\phi^{(2)}\right\|_{W^*}, \quad \phi^{(1)} \in V^*, \phi^{(2)} \in W^*.$$

This definition is given for general norms α. However, we are mainly interested in crossnorms, hence we now give a way to check if a given norm is a reasonable crossnorm.

Lemma 5.2.11. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces with algebraic tensor product $V \otimes W$. Assume that the norm $\alpha : V \otimes W \to [0, \infty)$ satisfies*

$$\alpha(v \otimes w) \leq \|v\|_V \|w\|_W, \quad v \in V, w \in W, \tag{5.2.6}$$

and its dual norm $\alpha^* : V^* \otimes W^* \to [0, \infty)$ satisfies

$$\alpha^*\left(\phi^{(1)} \otimes \phi^{(2)}\right) \leq \left\|\phi^{(1)}\right\|_{V^*} \left\|\phi^{(2)}\right\|_{W^*}, \quad \phi^{(1)} \in V^*, \phi^{(2)} \in W^*. \tag{5.2.7}$$

Then α is a reasonable crossnorm.

Proof. For any $\phi^{(1)} \in V^*$, $\phi^{(2)} \in W^*$, $v \in V$ and $w \in W$ we have

$$\begin{aligned} \phi^{(1)}(v)\phi^{(2)}(w) &= \left(\phi^{(1)} \otimes \phi^{(2)}\right)(v \otimes w) \\ &\leq \alpha^*\left(\phi^{(1)} \otimes \phi^{(2)}\right) \alpha(v \otimes w). \end{aligned} \tag{5.2.8}$$

We use this to show the reverse inequalities to (5.2.6) and (5.2.7). This proves that α is a reasonable crossnorm.

We start by deriving the reverse inequality to (5.2.6). Using (5.2.7) in (5.2.8) gives the bound

$$\left|\phi^{(1)}(v)\phi^{(2)}(w)\right| \leq \left\|\phi^{(1)}\right\|_{V^*} \left\|\phi^{(2)}\right\|_{W^*} \alpha(v \otimes w). \tag{5.2.9}$$

We recall that we can always express the norm on V by its dual norm. We have

$$\|v\|_V = \sup_{\substack{\phi \in V^* \\ \|\phi\|_{V^*}=1}} |\phi(v)|.$$

This allows us to estimate

$$\begin{aligned}
\|v\|_V \|w\|_W &= \sup_{\substack{\phi^{(1)} \in V^* \\ \|\phi^{(1)}\|_{V^*}=1}} |\phi^{(1)}(v)| \sup_{\substack{\phi^{(2)} \in W^* \\ \|\phi^{(2)}\|_{W^*}=1}} |\phi^{(2)}(w)| \\
&= \sup_{\|\phi^{(1)}\|_{V^*}=\|\phi^{(2)}\|_{W^*}=1} |\phi^{(1)}(v)||\phi^{(2)}(w)| \\
&\leq \alpha(v \otimes w),
\end{aligned}$$

where we used (5.2.9) to obtain the last bound.

Similarly, combining (5.2.8) with (5.2.6) yields the bound

$$\left|\phi^{(1)}(v)\phi^{(2)}(w)\right| \leq \|v\|_V \|w\|_W \alpha^*\left(\phi^{(1)} \otimes \phi^{(2)}\right),$$

which leads, with the definition of the dual norm α^* in (5.2.5), to

$$\begin{aligned}
\|\phi^{(1)}\|_{V^*} \|\phi^{(2)}\|_{W^*} &= \sup_{\substack{v \in V \\ \|v\|_V=1}} |\phi^{(1)}(v)| \sup_{\substack{w \in W \\ \|w\|_W=1}} |\phi^{(2)}(w)| \\
&= \sup_{\|v\|_V=\|w\|_W=1} |\phi^{(1)}(v)||\phi^{(2)}(w)| \\
&\leq \alpha^*\left(\phi^{(1)} \otimes \phi^{(2)}\right).
\end{aligned}$$

□

We now come back to the λ- and γ-norms and see that they both are reasonable crossnorms. Additionally, the λ-norm is the strongest and the γ-norm is the weakest reasonable crossnorm.

Theorem 5.2.12. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces with algebraic tensor product space $V \otimes W$.*

(1) The λ- and γ-norms are reasonable crossnorms.

(2) If $\|\cdot\| : V \otimes W \to [0, \infty)$ is a reasonable crossnorm, then

$$\|\boldsymbol{t}\|_\lambda \leq \|\boldsymbol{t}\| \leq \|\boldsymbol{t}\|_\gamma$$

holds for all $\boldsymbol{t} \in V \otimes W$.

Proof. The proof can be found in, e.g., [42]. □

We will revisit both the λ- and the γ-norm for examples later on. But first we come back to the tensor product space $\mathbb{R} \otimes \mathbb{R}$ from Theorem 5.1.9.

Theorem 5.2.13. *Let $\|\cdot\|_{\mathbb{R}\otimes\mathbb{R}}$ be a reasonable crossnorm on the tensor product space $\mathbb{R}\otimes\mathbb{R}$. Then*

$$\left\|\sum_{i=1}^{n} a_i \otimes b_i\right\|_{\mathbb{R}\otimes\mathbb{R}} = \left|\sum_{i=1}^{n} a_i b_i\right|$$

holds for all a_i, $b_i \in \mathbb{R}$, $1 \le i \le n$, $n \in \mathbb{N}$.

Proof. We show that

$$\left\|\sum_{i=1}^{n} a_i \otimes b_i\right\|_{\lambda} = \left|\sum_{i=1}^{n} a_i b_i\right| \ge \left\|\sum_{i=1}^{n} a_i \otimes b_i\right\|_{\gamma},$$

which together with Theorem 5.2.12 (2) yields the stated identity. We first note that we have for every $\phi \in \mathbb{R}^*$

$$\|\phi\|_{\mathbb{R}^*} = \sup_{|a|=1} |\phi(a)| = \sup_{|a|=1} |\phi(1)a| = |\phi(1)|.$$

This yields for $\boldsymbol{t} = \sum_{i=1}^{n} a_i \otimes b_i$ the lower bound

$$\begin{aligned}
\|\boldsymbol{t}\|_{\lambda} &= \sup_{\substack{\|\phi^{(j)}\|_{\mathbb{R}^*}=1 \\ j=1,2}} \left|\sum_{i=1}^{n} \phi^{(1)}(a_i)\phi^{(2)}(b_i)\right| \\
&= \sup_{\substack{\|\phi^{(j)}\|_{\mathbb{R}^*}=1 \\ j=1,2}} \left|\sum_{i=1}^{n} \phi^{(1)}(1)\phi^{(2)}(1)a_i b_i\right| \\
&= \sup_{\substack{\|\phi^{(j)}\|_{\mathbb{R}^*}=1 \\ j=1,2}} \left|\phi^{(1)}(1)\phi^{(2)}(1)\right| \left|\sum_{i=1}^{n} a_i b_i\right| = \left|\sum_{i=1}^{n} a_i b_i\right| \\
&\ge \inf\left\{\left|\sum_{i=1}^{n} a_i b_i\right| |1|\cdot|1| \ : \ \boldsymbol{t} = \sum_{i=1}^{n} (a_i b_i) 1 \otimes 1\right\} \\
&= \left\|\sum_{i=1}^{n} (a_i b_i) 1 \otimes 1\right\|_{\gamma} = \|\boldsymbol{t}\|_{\gamma}.
\end{aligned}$$

□

5.2.2. Tensor Product Banach Spaces. We now return to general norms α and investigate the normed space $V \otimes_\alpha W$ further, especially regarding completeness. In general, even if we start with two Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ the tensor product space $(V \otimes W, \alpha)$ will not be complete. However, we can always complete the algebraic tensor product space $V \otimes W$ with respect to the norm α.

Definition 5.2.14. *Let V and W be linear spaces with algebraic tensor product space $V \otimes W$. Let $\alpha : V \otimes W \to [0, \infty)$ be a norm on $V \otimes W$. Then the* tensor product space $(V \overline{\otimes} W, \alpha)$ *is the completion of $V \otimes W$ under the norm α.*

Again, if α is a crossnorm write $V \overline{\otimes}_{\|\cdot\|} W$ instead of $V \overline{\otimes}_\alpha W$. This is the case we are most interested in.

First we see that we do not necessarily have to take the closure of the algebraic tensor product space of the Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$. Completing the tensor product of dense subspaces $V_0 \subset V$ and $W_0 \subset W$ suffices. We obtain the same Banach space either way.

Theorem 5.2.15. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be Banach spaces with algebraic tensor product space $V \otimes W$. Let $\|\cdot\| : V \otimes W \to [0, \infty)$ be a crossnorm on $V \otimes W$. Let V_0 and W_0 be dense subspaces of $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$. Then the algebraic tensor product space $V_0 \otimes W_0$ is a dense subspace of $V \overline{\otimes}_{\|\cdot\|} W$, i.e.,*

$$\overline{V_0 \otimes_{\|\cdot\|} W_0} = V \overline{\otimes}_{\|\cdot\|} W.$$

Proof. We have to show that for any $\varepsilon > 0$ and any $\boldsymbol{t} \in V \overline{\otimes}_{\|\cdot\|} W$ there is a $\boldsymbol{t}_\varepsilon \in V_0 \otimes W_0$ such that $\|\boldsymbol{t} - \boldsymbol{t}_\varepsilon\| \leq \varepsilon$.

Since $V \overline{\otimes}_{\|\cdot\|} W$ is the closure of $V \otimes W$ with respect to the crossnorm $\|\cdot\|$ we find a $\boldsymbol{t}' \in V \otimes W$ such that $\|\boldsymbol{t} - \boldsymbol{t}'\| \leq \varepsilon/2$. By Lemma 5.1.11 it has a representation $\boldsymbol{t}' = \sum_{i=1}^n v_i' \otimes w_i'$ with $v_i' \in V$, $w_i' \in W$ and $n \in \mathbb{N}$. Next, we fix

$$C_{max} := \max_{1 \leq i \leq n} \left\{ \|v_i'\|_V, \|w_i'\|_W \right\}$$

and choose $\delta > 0$ small enough such that

$$n\delta(2C_{max} + \delta) < \frac{\varepsilon}{2}. \tag{5.2.10}$$

Because V_0 and W_0 are dense in V and W, respectively, we can find elements $v_i \in V_0$ and $w_i \in W_0$ with $\|v_i' - v_i\|_V < \delta$ and $\|w_i' - w_i\|_W < \delta$ and set

$$\boldsymbol{t}_\varepsilon := \sum_{i=1}^n v_i \otimes w_i.$$

Finally, we have, together with Proposition 5.1.6, the bound

$$\begin{aligned}
\|\boldsymbol{t}' - \boldsymbol{t}_\varepsilon\| &= \left\| \sum_{i=1}^n (v_i' \otimes w_i' - v_i \otimes w_i) \right\| \\
&= \left\| \sum_{i=1}^n [(v_i' - v_i) \otimes w_i' + v_i' \otimes (w_i' - w_i) + (v_i - v_i') \otimes (w_i' - w_i)] \right\| \\
&\leq \sum_{i=1}^n \left[\|v_i' - v_i\| \|w_i'\| + \|v_i'\| \|w_i' - w_i\| + \|v_i - v_i'\| \|w_i' - w_i\| \right] \\
&\leq n\delta(2C_{max} + \delta) \leq \frac{\varepsilon}{2},
\end{aligned}$$

where we have used the triangle inequality, the fact that $\|\cdot\|$ is a crossnorm and finally (5.2.10). This allows us to estimate

$$\|\boldsymbol{t} - \boldsymbol{t}_\varepsilon\| \leq \|\boldsymbol{t} - \boldsymbol{t}'\| + \|\boldsymbol{t}' - \boldsymbol{t}_\varepsilon\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

These statements also hold in a more general setting, see, e.g., [42, Lemma 4.40], but since we are only interested in crossnorms this version suffices.

We now come back to the two examples of crossnorms we discussed before and connect them to two important function spaces. We will see that the completion of $C(S) \otimes C(T)$ with respect to the λ-norm, defined in Definition 5.2.2, leads to the space $C(S \times T)$. Similarly we can interpret $L_1(S \times T)$ as the space $L_1(S) \otimes L_1(T)$, completed with respect to the γ-norm of Definition 5.2.6. The proofs have two steps: First, we will see that we have an isomorphism between certain Banach space valued mappings and the tensor product of the corresponding spaces. In the second step we use the right function space for the Banach space in step one and obtain the required isomorphism.

We start by looking at continuous functions. We introduce the notation $C(S, W)$ of the Banach space of all continuous maps f from a compact $S \subseteq \mathbb{R}^{n_1}$, $n_1 \in \mathbb{N}$, to a Banach space $(W, \|\cdot\|_W)$ with norm defined by

$$\|f\|_\infty := \sup_{s \in S} \|f(s)\|_W.$$

On $C(S)$ we use the $L_\infty(S)$-norm defined by

$$\|g\|_{L_\infty(S)} = \sup_{s \in S} |g(s)|, \quad g \in C(S).$$

Theorem 5.2.16. *Let $S \subseteq \mathbb{R}^{n_1}$, $n_1 \in \mathbb{N}$, be compact and non-empty. Let $(W, \|\cdot\|_W)$ be a Banach space and let λ be the norm defined in Definition 5.2.2. Then we have*

$$(C(S) \overline{\otimes} W, \|\cdot\|_\lambda) \cong (C(S, W), \|\cdot\|_\infty).$$

Proof. Consider an element $\boldsymbol{t}$ of the algebraic tensor product space $C(S) \otimes_\lambda W$ with representation $\boldsymbol{t} = \sum_{i=1}^n v_i \otimes w_i$, $v_i \in C(S)$ and $w_i \in W$. For this $\boldsymbol{t}$ we define a $F_{\boldsymbol{t}} \in C(S, W)$ by

$$F_{\boldsymbol{t}}(s) = \sum_{i=1}^n v_i(s) w_i, \quad s \in S.$$

Obviously, $F_{\boldsymbol{t}}$ is linear in $\boldsymbol{t}$.

Next, we take the λ-norm of $\boldsymbol{t}$, however we use it as a mapping $W \otimes C(S) \to [0, \infty)$. This yields

$$\begin{aligned}
\lambda(\boldsymbol{t}) &= \sup_{\substack{\psi \in W^* \\ \|\psi\|=1}} \left\| \sum_{i=1}^n \psi(w_i) v_i \right\|_\infty = \sup_{\substack{\psi \in W^* \\ \|\psi\|=1}} \sup_{s \in S} \left| \sum_{i=1}^n \psi(w_i) v_i(s) \right| \\
&= \sup_{s \in S} \sup_{\substack{\psi \in W^* \\ \|\psi\|=1}} \left| \psi \left(\sum_{i=1}^n v_i(s) w_i \right) \right| = \sup_{s \in S} \sup_{\substack{\psi \in W^* \\ \|\psi\|=1}} |\psi(F_{\boldsymbol{t}}(s))| \\
&= \sup_{s \in S} \|F_{\boldsymbol{t}}(s)\|_W = \|F_{\boldsymbol{t}}\|_\infty.
\end{aligned}$$

We used in the second to last step that we can express a norm on a linear space by

$$\|\cdot\|_W = \sup_{\substack{\psi \in W^* \\ \|\psi\|_{W^*}=1}} |\psi(\cdot)|.$$

This shows that the linear map $\boldsymbol{t} \mapsto F_{\boldsymbol{t}}$ is an isometry from the algebraic tensor product space $C(S) \otimes_\lambda W$ to $C(S, W)$ that can be continuously extended to a map from $C(S)\overline{\otimes}_\lambda W$ into $C(S, W)$.

Next, we show that the image of $C(S)\otimes_\lambda W$ under $F_{\boldsymbol{t}}$ is dense in $C(S, W)$. Completion with respect to the λ-norm then yields the claim.

Let $f \in C(S, W)$ and let $\varepsilon > 0$. The continuity of f and the compactness of S yield that $K := f(S)$ is also compact and thus totally bounded. Therefore there exist finitely many $w_1, \ldots, w_n \in K$ such that the open balls $B_\varepsilon(w_i)$ with radius ε and center w_i cover K. Hence, we can find a partition of unity on K, i.e., a family of smooth functions $g_1, \ldots, g_n : K \to [0, 1]$ such that

$$g_i(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in K \setminus B_\varepsilon(w_i) \quad \text{and} \quad \sum_{i=1}^{n} g_i = 1.$$

Next, for $1 \le i \le n$, define $v_i := g_i \circ f$. Then we have

$$v_i \in C(S), \quad 0 \le v_i \le 1 \quad \text{and} \quad \sum_{i=1}^{n} v_i(s) = 1,\ s \in S.$$

We use these v_i in the representation of $\boldsymbol{t}$, $\boldsymbol{t} := \sum_{i=1}^n v_i \otimes w_i$. If $s \in S$ we have

$$\begin{aligned}\left\| f(s) - \sum_{i=1}^{n} v_i(s) w_i \right\|_W &= \left\| \sum_{i=1}^{n} v_i(s)[f(s) - w_i] \right\|_W \\ &\le \sum_{i=1}^{n} v_i(s) \|f(s) - w_i\|_W < \varepsilon,\end{aligned}$$

because $v_i(s) = 0$, if $\|f(s) - w_i\|_W \ge \varepsilon$. Therefore, we have $\|f - F_{\boldsymbol{t}}\|_\infty < \varepsilon$, which completes the proof. □

With this result we can now show that the tensor product of spaces of continuous functions can be identified with the space of continuous functions on the Cartesian product of the domains.

Corollary 5.2.17. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, be compact and non-empty. Consider the Banach spaces $\big(C(S), \|\cdot\|_{L_\infty(S)}\big)$ and $\big(C(T), \|\cdot\|_{L_\infty(T)}\big)$. Then we have*

$$(C(S)\overline{\otimes}C(T), \|\cdot\|_\lambda) \cong \big(C(S \times T), \|\cdot\|_{L_\infty(S\times T)}\big).$$

Proof. We use Theorem 5.2.16 with $W := C(T)$ and have

$$C(S)\overline{\otimes}_\lambda C(T) \cong C(S, C(T)).$$

Hence, we need to find an isomorphism mapping $C(S, C(T))$ into $C(S \times T)$. Associate to any $f \in C(S \times T)$ a function $\widetilde{f}$ by setting

$$\widetilde{f}(s) = f(s, \cdot), \quad s \in S.$$

Then, this function satisfies $\widetilde{f} \in C(S, C(T))$ and is the required identification. □

This result makes working with elements of the tensor product of spaces of continuous functions easy as we can always interpret them as two-variate continuous functions.

The situation is slightly more complex for differentiable functions. Consider compact, non-empty intervals $S, T \subseteq \mathbb{R}$. The standard norm on $C^1(S)$ is

$$\|f\|_{C^1(S)} = \max\{\|f\|_{L_\infty(S)}, \|f'\|_{L_\infty(S)}\}.$$

It is well-known that the normed space $(C^1(S), \|\cdot\|_{C^1(S)})$ is a Banach space. Similarly, the standard norm on $C^1(S \times T)$ is given as

$$\|\boldsymbol{f}\|_{C^1(S\times T)} := \max_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^2 \\ \|\boldsymbol{\alpha}\|_1\leq 1}} \|D^{\boldsymbol{\alpha}}\boldsymbol{f}\|_{L_\infty(S\times T)}.$$

However, by constructing a counter example, we can show that this norm is not a crossnorm on $C^1(S) \otimes C^1(T)$. This means that the Banach space $(C^1(S \times T), \|\cdot\|_{C^1(S\times T)})$ can not be identified with $(C^1(S)\overline{\otimes}C^1(T), \|\cdot\|_\lambda)$. It turns out that we have to take the mixed derivatives $D^{\boldsymbol{\alpha}}\boldsymbol{f}$ with $\boldsymbol{\alpha} = (1, 1)^{\mathrm{T}}$ into account which then yields the *mixed regularity space*. We formalize this in the next definition.

Definition 5.2.18. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, be compact and non-empty. Let $K := S \times T$. For $\boldsymbol{f} : K \to \mathbb{R}$ and $\boldsymbol{\alpha}^{(j)} := (\alpha_1^{(j)}, \ldots, \alpha_{n_j}^{(j)})^T \in \mathbb{N}_0^{n_j}$, $j = 1, 2$, we will write*

$$\begin{aligned} D^{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}\boldsymbol{f} &:= \frac{\partial^{\|\boldsymbol{\alpha}^{(1)}\|_1+\|\boldsymbol{\alpha}^{(2)}\|_1}\boldsymbol{f}}{\partial(\boldsymbol{x}^{(1)})^{\boldsymbol{\alpha}^{(1)}}\partial(\boldsymbol{x}^{(1)})^{\boldsymbol{\alpha}^{(1)}}} \\ &= \frac{\partial^{\|\boldsymbol{\alpha}^{(1)}\|_1+\|\boldsymbol{\alpha}^{(2)}\|_1}\boldsymbol{f}}{(\partial x_1^{(1)})^{\alpha_1^{(1)}}\cdots(\partial x_{n_1}^{(1)})^{\alpha_{n_1}^{(1)}}(\partial x_1^{(2)})^{\alpha_1^{(2)}}\cdots(\partial x_{n_2}^{(2)})^{\alpha_{n_2}^{(2)}}}, \end{aligned}$$

where $\boldsymbol{x}^{(j)} = (x_1^{(j)}, \ldots, x_{n_j}^{(j)})^T \in \mathbb{R}^{n_j}$, $j = 1, 2$. Then, for $\boldsymbol{m} \in \mathbb{N}_0^2$, the mixed regularity space *$C_{mix}^{\boldsymbol{m}}(K)$ is given by*

$$\begin{aligned} C_{mix}^{\boldsymbol{m}}(K) := \Big\{ &\boldsymbol{f} \in C(K) \ : \\ &D^{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}\boldsymbol{f} \in C(K) \text{ for } \boldsymbol{\alpha}^{(j)} \in \mathbb{N}_0^{n_j}, \|\boldsymbol{\alpha}^{(j)}\|_1 \leq m_j, j = 1, 2\Big\}, \end{aligned}$$

i.e., the normed space $C_{mix}^{\boldsymbol{m}}(K)$ consists of all functions $\boldsymbol{f} \in C(K)$ which, together with all their relevant derivatives, are continuous. The norm on $C_{mix}^{\boldsymbol{m}}(K)$ is defined by

$$\|\boldsymbol{f}\|_{C_{mix}^{\boldsymbol{m}}(K)} := \max_{j=1,2} \max_{\|\boldsymbol{\alpha}^{(j)}\|_1\leq m_j} \|D^{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}\boldsymbol{f}\|_{L_\infty(K)}. \tag{5.2.11}$$

If $m_1 = m_2 = m$ we also simply write $C_{mix}^m(K)$.

One can show that the space $(C_{mix}^{\boldsymbol{m}}(K), \|\cdot\|_{C_{mix}^{\boldsymbol{m}}(K)})$ is a Banach space. The special case $m_1 = m_2 = m$ and $n_1 = n_2 = 1$ is widely used. In that situation we simply have

$$C_{mix}^m(K) = \{\boldsymbol{f} \in C(K) \ : \ D^{\boldsymbol{\alpha}}\boldsymbol{f} \in C(K), \|\boldsymbol{\alpha}\|_\infty \leq m\}.$$

Here, we see also that the mixed regularity space controls far more derivatives than the classical space of continuously differentiable functions.

We will see in the next corollary that the mixed regularity space coincides with the tensor product space $C^{m_1}(S)\overline{\otimes}_\lambda C^{m_2}(T)$.

Corollary 5.2.19. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$ be compact and non-empty. Let $K := S \times T$. Let $\boldsymbol{m} \in \mathbb{N}_0^2$ be given and consider the direction-wise Banach spaces $V = C^{m_1}(S)$, with norm $\|v\|_V = \max_{\|\boldsymbol{\alpha}\| \leq m_1} \|D^{\boldsymbol{\alpha}} v\|_{L_\infty(S)}$ and $W = C^{m_2}(T)$, with norm $\|w\|_W = \max_{\|\boldsymbol{\alpha}\| \leq m_2} \|D^{\boldsymbol{\alpha}} w\|_{L_\infty(T)}$.*

Then the norm $\|\cdot\|_{C^{\boldsymbol{m}}_{mix}(K)}$ defined in (5.2.11) is a crossnorm on the algebraic tensor product space $C^{m_1}(S) \otimes C^{m_2}(T)$. The identity

$$\|\boldsymbol{f}\|_\lambda = \|\boldsymbol{f}\|_{C^{\boldsymbol{m}}_{mix}(K)}$$

holds for all $\boldsymbol{f} \in C^{m_1}(S) \otimes C^{m_2}(T)$.

Furthermore, the space $C^{\boldsymbol{m}}_{mix}(K)$ is the corresponding tensor product space, i.e.,

$$(C^{m_1}(S)\overline{\otimes}C^{m_2}(T), \|\cdot\|_\lambda) \cong \left(C^{\boldsymbol{m}}_{mix}(K), \|\cdot\|_{C^{\boldsymbol{m}}_{mix}(K)}\right).$$

Proof. The proof follows the same ideas of and the same techniques as the proofs of Theorem 5.2.16 and Corollary 5.2.17. □

We now come to the space of integrable functions. We follow the same ideas as for the space of continuous functions and will see that we can identify the Banach space $(L_1(S)\overline{\otimes}L_1(T), \|\cdot\|_\gamma)$ with the well-known space $(L_1(S\times T), \|\cdot\|_{L_1(S\times T)})$. Since the proof ideas are virtually the same, however require some technical preparation, we omit the proofs here and refer the reader to, e.g., [58]. We only state the final theorem.

Theorem 5.2.20. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, be measurable. Then*

$$(L_1(S)\overline{\otimes}L_1(T), \|\cdot\|_\gamma) \cong \left(L_1(S\times T), \|\cdot\|_{L_1(S\times T)}\right).$$

On the basis of these examples we have seen that we can interpret popular function spaces on sets, which are Cartesian products of lower-dimensional sets, as the tensor product of the functions spaces on the single sets. There are several other examples, most importantly the class of the *p-nuclear norms* α_p, $1 < p < \infty$. These norms would allow us to extend the results above to L_p-spaces. For further information on these norms we refer to [58, 1.45 Definition, ff.].

The given examples and this discussion motivates that we can interpret the γ-norm as the tensor product L_1-norm, the p-nuclear norms as tensor product L_p-norms and the λ-norm as the L_∞-norm. Consequently, we could expect an identification of $L_\infty(S\times T)$ as $L_\infty(S)\overline{\otimes}_\lambda L_\infty(T)$. However, in general this is not possible as the next theorem, taken from [58, 1.53 Theorem], shows.

Theorem 5.2.21. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, be measurable. Then the space $L_\infty(S)\overline{\otimes}_\lambda L_\infty(T)$ is a subspace of $L_\infty(S \times T)$. This subspace is usually proper.*

This circumstance will make the L_∞-error analysis in the later chapters more involved as we first have to embed this space in a suitable tensor product space, the mixed regularity space of Definition 5.2.18. The existence of such an embedding will be discussed in Theorem 5.3.9.

5.2.3. Tensor Product Hilbert Spaces. The rest of this section is devoted to the tensor product of (pre-)Hilbert spaces. We will see that in this setting things are much easier than in the case of general normed spaces. First, we see that the inner product on the algebraic tensor product space of pre-Hilbert spaces is easily constructed.

Definition 5.2.22. *Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be pre-Hilbert spaces. Then the* induced inner product *on the algebraic tensor product space $V \otimes W$ is the bilinear mapping $\langle\cdot,\cdot\rangle_{V\otimes W} : (V \otimes W) \times (V \otimes W) \to \mathbb{R}$ defined by*

$$\langle \boldsymbol{t}, \boldsymbol{s}\rangle_{V\otimes W} := \langle v, x\rangle_V \langle w, y\rangle_W, \quad \boldsymbol{t} = v \otimes w, \boldsymbol{s} = x \otimes y \tag{5.2.12}$$

and more generally, for $\boldsymbol{t} = \sum_{i=1}^n v_i \otimes w_i$ and $\boldsymbol{s} = \sum_{k=1}^m x_k \otimes y_k$

$$\langle \boldsymbol{t}, \boldsymbol{s}\rangle_{V\otimes W} := \sum_{i=1}^{n}\sum_{k=1}^{m} \langle v_i, x_k\rangle_V \langle w_i, y_k\rangle_W. \tag{5.2.13}$$

Next, we establish that the mapping in (5.2.13) is indeed an inner product and that the definition is independent of the representations of $\boldsymbol{t}$ and $\boldsymbol{s}$.

Lemma 5.2.23. *Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be pre-Hilbert spaces. Then the induced inner product of Definition 5.2.22 is well-defined and an inner product on the algebraic tensor product space $V \otimes W$.*

Proof. We start by showing that the definition is independent of the representation of the tensors. Assume that $\boldsymbol{t} \in V \otimes W$ has two representations $\boldsymbol{t} = \sum_{i=1}^n v_i \otimes w_i$ and $\boldsymbol{t} = \sum_{i=1}^{\widetilde{n}} \widetilde{v}_i \otimes \widetilde{w}_i$, with $n, \widetilde{n} \in \mathbb{N}$, $v_i, \widetilde{v}_i \in V$ and $w_i, \widetilde{w}_i \in W$. The inner product with an elementary tensor $\boldsymbol{s} = x \otimes y$ is then given by

$$\left\langle \sum_{i=1}^{n} v_i \otimes w_i, x \otimes y \right\rangle_{V\otimes W} = \sum_{i=1}^{n} \langle v_i, x\rangle_V \langle w_i, y\rangle_W$$

and

$$\left\langle \sum_{i=1}^{\widetilde{n}} \widetilde{v}_i \otimes \widetilde{w}_i, x \otimes y \right\rangle_{V\otimes W} = \sum_{i=1}^{\widetilde{n}} \langle \widetilde{v}_i, x\rangle_V \langle \widetilde{w}_i, y\rangle_W.$$

With the Riesz representer theorem we find two functionals $\phi^{(1)} \in V^*$ and $\phi^{(2)} \in W^*$ such that $\phi^{(1)} = \langle\cdot, x\rangle_V$ and $\phi^{(2)} = \langle\cdot, y\rangle_W$. This yields

$$\left\langle \sum_{i=1}^{n} v_i \otimes w_i, x \otimes y \right\rangle_{V\otimes W} = \sum_{i=1}^{n} \phi^{(1)}(v_i)\phi^{(2)}(w_i)$$

and

$$\left\langle \sum_{i=1}^{\widetilde{n}} \widetilde{v}_i \otimes \widetilde{w}_i, x \otimes y \right\rangle_{V\otimes W} = \sum_{i=1}^{\widetilde{n}} \phi^{(1)}(\widetilde{v}_i)\phi^{(2)}(\widetilde{w}_i).$$

Following Lemma 5.1.13 the right-hand sides of both equations are equal, hence we have the independence of the actual representation of $\boldsymbol{t}$. The general case, in which $\boldsymbol{s}$ is also a linear combination of elementary tensors follows accordingly. This shows that the induced inner product is well-defined.

To see that it is a proper inner product on the algebraic tensor product space $V \otimes W$ we only need show definiteness. Bilinearity and symmetry follow

directly from the definition. From Lemma 5.1.11 we know that, for a tensor $\boldsymbol{t} \neq \boldsymbol{0}$, we find a representation $\boldsymbol{t} = \sum_{i=1}^{r} v_i \otimes w_i$ with a minimal $r \in \mathbb{N}$ and linear independent $\{v_i\}_{1\leq i\leq r}$ and $\{w_i\}_{1\leq i\leq r}$. We have

$$\langle \boldsymbol{t}, \boldsymbol{t}\rangle_{V\otimes W} = \sum_{i,k=1}^{r} \langle v_i, v_k\rangle_V \langle w_i, w_k\rangle_W.$$

Hence, we define the symmetric, positive definite matrix $A = (\langle v_i, v_k\rangle_V) \in \mathbb{R}^{r\times r}$. There is a symmetric and positive definite root $B \in \mathbb{R}^{r\times r}$ such that $A = B^2$. We see that

$$\begin{aligned}\langle \boldsymbol{t}, \boldsymbol{t}\rangle_{V\otimes W} &= \sum_{i,k=1}^{r} \langle v_i, v_k\rangle_V \langle w_i, w_k\rangle_W = \sum_{i,k=1}^{r} A_{ik}\langle w_i, w_k\rangle_W \\ &= \sum_{i,k=1}^{r}\sum_{p=1}^{r} B_{ip}B_{pk}\langle w_i, w_k\rangle_W \\ &= \sum_{p=1}^{r} \left\langle \sum_{i=1}^{r} B_{ip}w_i, \sum_{k=1}^{r} B_{pk}w_k \right\rangle_W > 0,\end{aligned}$$

since $\{w_i\}_{1\leq i\leq r}$ is linearly independent. □

Hence, the induced inner product induces a norm on the algebraic tensor space $V \otimes W$.

Definition 5.2.24. *Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be pre-Hilbert spaces with algebraic tensor product space $V \otimes W$. The β- or* induced *norm $\beta : V \otimes W \to [0, \infty)$ is defined by*

$$\beta(\boldsymbol{t}) := \sqrt{\langle \boldsymbol{t}, \boldsymbol{t}\rangle_{V\otimes W}}.$$

For a $\boldsymbol{t} \in V \otimes W$ with representation $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i$ we have

$$\beta\left(\sum_{i=1}^{n} v_i \otimes w_i\right) := \left(\sum_{i,k=1}^{n} \langle v_i, v_k\rangle_V \langle w_i, w_k\rangle_W\right)^{\frac{1}{2}}.$$

For a dyad $v \otimes w$ we immediately have

$$\beta(v \otimes w) = (\langle v, v\rangle_V \langle w, w\rangle_W)^{1/2} = \|v\|_V \|w\|_W.$$

This already proves part of the next lemma.

Lemma 5.2.25. *The β-norm is a reasonable crossnorm.*

Proof. It remains only to show that β is a reasonable norm. We use Lemma 5.2.11 and only have to show that

$$\|\phi^{(1)} \otimes \phi^{(2)}\|_{\beta^*} \leq \|\phi^{(1)}\|_{V^*} \|\phi^{(2)}\|_{W^*}$$

for $\phi^{(1)} \in V^*$ and $\phi^{(2)} \in W^*$. Assume that $\boldsymbol{t} \in V \otimes W$ is given and has a representation $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i$. Assume further that $\phi^{(1)} \in V^*$ and $\phi^{(2)} \in W^*$ are also given. By the Riesz representation theorem there are

elements $v \in V$ such that $\phi^{(1)}(v_i) = \langle v, v_i\rangle_V$, $\|\phi^{(1)}\|_{V^*} = \|v\|_V$ and $w \in W$ such that $\phi^{(2)}(w_i) = \langle w, w_i\rangle_W$ and $\|\phi^{(2)}\|_{W^*} = \|w\|_W$. Thus,

$$\begin{aligned}\left|\left(\phi^{(1)} \otimes \phi^{(2)}\right)(\boldsymbol{t})\right| &= \left|\sum_{i=1}^{n} \phi^{(1)}(v_i)\phi^{(2)}(w_i)\right| = \left|\sum_{i=1}^{n} \langle v, v_i\rangle_V \langle w, w_i\rangle_W\right| \\ &= |\langle \boldsymbol{t}, v \otimes w\rangle_{V\otimes W}| \leq \|\boldsymbol{t}\|_\beta \|u \otimes w\|_\beta \\ &= \|\boldsymbol{t}\|_\beta \|v\|_V \|w\|_W = \|\boldsymbol{t}\|_\beta \left\|\phi^{(1)}\right\|_{V^*} \left\|\phi^{(2)}\right\|_{W^*},\end{aligned}$$

which shows $\|\phi^{(1)} \otimes \phi^{(2)}\|_{\beta^*} \leq \|\phi^{(1)}\|_{V^*}\|\phi^{(2)}\|_{W^*}$. □

We emphasize that, contrary to the normed space setting, induced norms on the algebraic tensor product space are always reasonable crossnorms.

We can extend Theorem 5.1.8 to the pre-Hilbert space setting, i.e., we obtain an orthogonal system on the algebraic tensor product space $V \otimes W$ by taking the tensor product of the orthogonal systems of V and W.

Lemma 5.2.26. *Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be pre-Hilbert spaces with algebraic tensor product $V \otimes W$. Let I and J be, not necessarily finite, index sets. Let $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ be orthogonal (orthonormal) systems in V and W, respectively. Then*

$$\mathcal{B} := \{v_i \otimes w_j \ : \ i \in I, j \in J\}$$

is an orthogonal (orthonormal) system in $V \otimes W$.

If $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ are orthonormal bases, $\mathcal{B}$ is an orthonormal basis of $V \otimes W$.

We can now formalize the definition of the algebraic tensor product pre-Hilbert space.

Definition 5.2.27. *Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be pre-Hilbert spaces with algebraic tensor product space $V \otimes W$. Then $(V \otimes W, \langle\cdot,\cdot\rangle_{V\otimes W})$ is called the* algebraic tensor product pre-Hilbert space*.*

We will also use the notation $V \otimes_\beta W$.

Again, in general, the algebraic tensor product pre-Hilbert space $V \otimes W$ will not be complete, even if V and W are Hilbert spaces. However, as in Section 5.2.2, we can complete the tensor product pre-Hilbert space $V \otimes_\beta W$ with respect to the β-norm to obtain a Hilbert space $V\overline{\otimes}_\beta W$.

We now give some examples of popular tensor product Hilbert spaces. We start by looking at the tensor product of L_2-spaces. First we show that we can associate a function $f : S \times T \to \mathbb{R}$ to a tensor $\boldsymbol{t} \in L_2(S) \otimes L_2(T)$.

Lemma 5.2.28. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, be measurable. Let $\boldsymbol{t} \in L_2(S) \otimes L_2(T)$ have a representation $\boldsymbol{t} = \sum_{i=1}^n v_i \otimes w_i$ and define $f : S \times T \to \mathbb{R}$ by*

$$f(s,t) = \sum_{i=1}^{n} v_i(s)w_i(t).$$

Then $f \in L_2(S \times T)$ and if $\boldsymbol{t} = \boldsymbol{0}$ then $f = 0$ almost everywhere.

PROOF. Let μ and ν be the measures on S and T, respectively. Let $\sigma = \mu \otimes \nu$ be the product measure. With $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i$ we have

$$\begin{aligned}
\|f\|^2_{L_2(S\times T)} &= \int_{S\times T} f^2(s,t)\ d\sigma(s,t) = \int_S \int_T f^2(s,t)\ d\mu(s)d\nu(t) \\
&= \int_S \int_T \sum_{i,j=1}^{n} v_i(s)w_i(t)v_j(s)w_j(t)\ d\mu(s)d\nu(t) \\
&= \sum_{i,j=1}^{n} \int_S v_i(s)v_j(s)\ d\mu(s) \int_T w_i(t)w_j(t)\ d\nu(s) \\
&= \sum_{i,j=1}^{n} \langle v_i, v_j\rangle_{L_2(S)} \langle w_i, w_j\rangle_{L_2(T)} \\
&= \langle \boldsymbol{t}, \boldsymbol{t}\rangle_{L_2(S)\otimes_\beta L_2(T)} = \beta^2(\boldsymbol{t}).
\end{aligned}$$

This already shows that $f \in L_2(S \times T)$. To show the second claim, assume that $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i = \boldsymbol{0}$, this means its norm is also zero, $\beta(\boldsymbol{t}) = 0$, and therefore, by the equations above $\|f\|_{L_2(S\times T)} = 0$. Hence, $f = 0$ almost everywhere. □

Note, that the proof above is more general than necessary. In the proof we use different measures on $L_2(S)$ and $L_2(T)$. However, in our application both spaces use the Lebesgue-measure.

Lemma 5.2.28 has several consequences. First, it implies that the function $f \in L_2(S\times T)$ associated to the tensor $\boldsymbol{t}$ does not depend on the representation of $\boldsymbol{t}$. Second, the mapping $\boldsymbol{t} \mapsto f$ is linear and by the main equation in the proof,

$$\|f\|^2_{L_2(S\times T)} = \beta^2(\boldsymbol{t}), \tag{5.2.14}$$

it is norm-preserving. We will use these facts to prove the main theorem, which states that the completion of $L_2(S) \otimes L_2(T)$ with respect to the $\beta = \|\cdot\|_{L_2(S)\otimes L_2(T)}$-norm is isomorphic to $L_2(S \times T)$.

Theorem 5.2.29. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, be measurable. Let β be the norm induced by the $L_2(S) \otimes L_2(T)$ induced inner product. Then we have*

$$(L_2(S)\overline{\otimes}L_2(T), \|\cdot\|_\beta) \cong \left(L_2(S\times T), \|\cdot\|_{L_2(S\times T)}\right).$$

PROOF. With Lemma 5.2.28 we have that $L_2(S) \otimes L_2(T) \subseteq L_2(S \times T)$ with equality of norms on the subset. Thus, the closure of the normed space $(L_2(S) \otimes L_2(T), \|\cdot\|_\beta)$ is a closed subspace of $L_2(S \times T)$.

To complete the proof we show that the only $f \in L_2(S \times T)$ that is orthogonal on $L_2(S) \otimes L_2(T)$ is $f = 0$. To this end, suppose that

$$\langle f, v \otimes w\rangle_{L_2(S\times T)} = 0, \quad v \otimes w \in L_2(S) \otimes L_2(T).$$

By Fubini's theorem, we write this as

$$\langle f, v \otimes w\rangle_{L_2(S\times T)} = \int_T w(t) \int_S v(s)f(s,t)\ dsdt = 0.$$

Since $w \in L_2(T)$ is arbitrary we conclude that for almost all $t \in T$,

$$\int_S v(s) f(s,t)\ ds = 0.$$

Since $v \in L_2(S)$ is in turn arbitrary, we conclude that for almost all t $f(\cdot,t) = 0$ in $L_2(S)$. Hence

$$\int_S f^2(s,t)\ ds = \int_T \int_S f(s,t)^2\ dsdt = 0.$$

□

Similar to the spaces of mixed regularity introduced in Definition 5.2.18 it is possible to generalize this result to Sobolev Hilbert spaces.

Definition 5.2.30. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, be measurable. Let $\Omega = S \times T$ and $\boldsymbol{m} \in \mathbb{N}_0^2$. Then the* Sobolev Hilbert space of mixed regularity *or* anisotropic Sobolev Hilbert space $H^{\boldsymbol{m}}_{mix}(\Omega)$ *is defined as*

$$H^{\boldsymbol{m}}_{mix}(\Omega) := \Big\{ \boldsymbol{f} \in L_2(\Omega)\ : \\ D^{(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)} \boldsymbol{f} \in L_2(S \times T), \boldsymbol{\alpha}_j \in \mathbb{N}^{n_j}, \|\boldsymbol{\alpha}_j\|_1 \leq m_j, j = 1,2 \Big\}$$

with norm given by

$$\|\boldsymbol{f}\|_{H^{\boldsymbol{m}}_{mix}(\Omega)} := \left(\sum_{\|\boldsymbol{\alpha}_1\|_1 \leq m_1} \sum_{\|\boldsymbol{\alpha}_2\|_1 \leq m_2} \|D^{(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)} \boldsymbol{f}\|^2_{L_2(\Omega)} \right)^{1/2}.$$

Similar to the discussion in Chapter 2 we are not restricted to integer smoothness. However we have to define the resulting space on the whole $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Definition 5.2.31. *Let $\boldsymbol{s} \in [0,\infty)^2$. Then the* Sobolev Hilbert space of mixed regularity *or* anisotropic Sobolev Hilbert space $H^{\boldsymbol{s}}_{mix}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *is defined as*

$$H^{\boldsymbol{s}}_{mix}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) := \Bigg\{ \boldsymbol{f} \in L_2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\ : \\ (\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}) \mapsto \prod_{j=1}^{2} \left(1 + \|\boldsymbol{\xi}^{(j)}\|_2^2\right)^{\frac{s_j}{2}} |\widehat{\boldsymbol{f}}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)})| \in L_2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \Bigg\}$$

with norm given by

$$\|u\|_{H^{\boldsymbol{s}}_{mix}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} := \\ = \left(\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \prod_{j=1}^{2} \left(1 + \|\boldsymbol{\xi}^{(j)}\|_2^2\right)^{s_j} |\widehat{\boldsymbol{f}}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)})|^2\ d(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}) \right)^{\frac{1}{2}}.$$

As in Chapter 2 we can restrict function in $H^{\boldsymbol{s}}_{mix}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to a measurable subset $\Omega = S \times T \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and obtain the anisotropic Sobolev Hilbert space $H^{\boldsymbol{s}}_{mix}(\Omega)$ for non-integer smoothness $\boldsymbol{s} \in [0,\infty)^2$.

Proposition 5.2.32. *Both versions of the anisotropic Sobolev Hilbert space,* $(H^{\boldsymbol{m}}_{mix}(\Omega), \|\cdot\|_{H^{\boldsymbol{m}}_{mix}(\Omega)})$ *and* $(H^{\boldsymbol{s}}_{mix}(\Omega), \|\cdot\|_{H^{\boldsymbol{s}}_{mix}(\Omega)})$*, are Hilbert spaces.*

Again, as in the situation of continuous functions, Theorem 5.2.29 generalizes to the mixed regularity Sobolev Hilbert spaces.

Corollary 5.2.33. *Let* $S \subseteq \mathbb{R}^{n_1}$ *and* $T \subseteq \mathbb{R}^{n_2}$*,* $n_1, n_2 \in \mathbb{N}$*, be measurable. Let* $\Omega = S \times T$*,* $\boldsymbol{m} \in \mathbb{N}_0^2$ *and* $\boldsymbol{s} \in [0,\infty)^2$*. Then, with the respective* β*-norms on the tensor product spaces, we have*

$$(H^{m_1}(S)\overline{\otimes}H^{m_2}(T), \|\cdot\|_\beta) \cong \left(H^{\boldsymbol{m}}_{mix}(\Omega), \|\cdot\|_{H^{\boldsymbol{m}}_{mix}(\Omega)}\right)$$

and

$$(H^{s_1}(S)\overline{\otimes}H^{s_2}(T), \|\cdot\|_\beta) \cong \left(H^{\boldsymbol{s}}_{mix}(\Omega), \|\cdot\|_{H^{\boldsymbol{s}}_{mix}(\Omega)}\right).$$

We will see in the next section that these mixed regularity Sobolev Hilbert spaces have an additional advantage over classical Sobolev Hilbert spaces on high-dimensional domains: We have much lower requirements for an analogue of the embedding as in Theorem 2.1.13. This advantage, however, is bought with a stricter norm.

Before we discuss this embedding and operators on tensor product spaces in general, we give another example. It concerns the tensor product of native spaces that were introduced in Section 2.2. The proof of the following theorem can be found in, e.g., [61, Satz V 6].

Theorem 5.2.34. *Let* $S \subseteq \mathbb{R}^{n_1}$ *and* $T \subseteq \mathbb{R}^{n_2}$ *be domains. Let the Hilbert spaces* $(\mathcal{N}_{K^{(1)}}(S), \langle\cdot,\cdot\rangle_{K^{(1)}})$ *and* $(\mathcal{N}_{K^{(2)}}(T), \langle\cdot,\cdot\rangle_{K^{(2)}})$ *be the native spaces of the continuous, symmetric and positive definite kernels* $K^{(1)}$ *and* $K^{(2)}$*. Let* β *be the norm induced by the inner products* $\langle\cdot,\cdot\rangle_{K^{(1)}}$ *and* $\langle\cdot,\cdot\rangle_{K^{(2)}}$*. Then*

$$(\mathcal{N}_{K^{(1)}}(S)\overline{\otimes}\mathcal{N}_{K^{(2)}}(T), \|\cdot\|_\beta) \cong (\mathcal{N}_{\boldsymbol{K}}(S\times T), \|\cdot\|_{\boldsymbol{K}})$$

holds with $\boldsymbol{K} : (S\times S)\times(T\times T) \to \mathbb{R}$ *given as*

$$\boldsymbol{K}(\boldsymbol{x},\boldsymbol{y}) = K^{(1)}\left(\boldsymbol{x}^{(1)},\boldsymbol{y}^{(1)}\right)\cdot K^{(2)}\left(\boldsymbol{x}^{(2)},\boldsymbol{y}^{(2)}\right)$$

for $\boldsymbol{x} = (\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})^T$*,* $\boldsymbol{y} = (\boldsymbol{y}^{(1)},\boldsymbol{y}^{(2)})^T$ *and* $\boldsymbol{x}^{(1)}$*,* $\boldsymbol{x}^{(2)} \in S$ *and* $\boldsymbol{y}^{(1)}$*,* $\boldsymbol{y}^{(2)} \in T$*.*

This means that the tensor product of native spaces $\mathcal{N}_{K^{(1)}}$ and $\mathcal{N}_{K^{(2)}}$ is again a native space with reproducing kernel that is the point-wise product of the respective kernels $K^{(1)}$ and $K^{(2)}$.

5.3. Tensor Products of Operators

To finish this chapter we study tensor products of operators. We start by defining the purely algebraic construction.

Definition 5.3.1. *Let* V*,* W*,* S *and* T *be linear spaces with algebraic tensor product spaces* $V\otimes W$ *and* $S\otimes T$*. Let* $A : V \to S$ *and* $B : W \to T$ *be linear. Then the* tensor product operator $A\otimes B : V\otimes W \to S\otimes T$ *is defined by*

$$(A\otimes B)(\boldsymbol{t}) := \sum_{i=1}^{n} A(v_i)\otimes B(w_i), \tag{5.3.1}$$

where $\boldsymbol{t} \in V\otimes W$ *has a representation* $\boldsymbol{t} = \sum_{i=1}^n v_i\otimes w_i$*.*

Theorem 5.3.2. *The tensor product operator $A \otimes B$ defined by* (5.3.1) *is independent of the representation of the argument and if there is a second tensor product operator $(A \otimes B)'$ defined by* (5.3.1)*, then*

$$A \otimes B = (A \otimes B)',$$

i.e., the tensor product operator is unique for fixed A and B.

PROOF. We start by showing that the definition of the tensor product operator is independent of the representation of the argument. Hence, assume that $\boldsymbol{t} \in V \otimes W$ has two representations

$$\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i = \sum_{j=1}^{m} \widetilde{v}_j \otimes \widetilde{w}_j. \tag{5.3.2}$$

Next, let $\{b_\ell^{(1)} \,:\, \ell \in I^{(1)}\}$ be a basis of V and $\{b_\ell^{(2)} \,:\, \ell \in I^{(2)}\}$ be a basis of W. Then we have the representations

$$v_i = \sum_{\ell \in I^{(1)}} \alpha_{i,\ell} b_\ell^{(1)}, \quad \widetilde{v}_j = \sum_{\ell \in I^{(1)}} \widetilde{\alpha}_{j,\ell} b_\ell^{(1)}$$

and

$$w_i = \sum_{\ell \in I^{(1)}} \alpha_{i,\ell} b_\ell^{(2)}, \quad \widetilde{w}_j = \sum_{\ell \in I^{(2)}} \widetilde{\alpha}_{j,\ell} b_\ell^{(2)},$$

where in each of the sums all but finitely many of the coefficients are zero. Setting $I = I^{(1)} \times I^{(2)}$, it follows from (5.3.2) and the multilinearity of the tensor product that

$$\sum_{i=1}^{n} \sum_{\boldsymbol{\ell} \in I} \alpha_{i,\ell_1} \alpha_{i,\ell_2} b_{\ell_1}^{(1)} \otimes b_{\ell_2}^{(2)} = \sum_{j=1}^{m} \sum_{\boldsymbol{\ell} \in I} \widetilde{\alpha}_{j,\ell_1} \widetilde{\alpha}_{j,\ell_2} b_{\ell_1}^{(1)} \otimes b_{\ell_2}^{(2)}.$$

Exchanging in both sides the inner and outer sum and using the linear independence of the $b_{\ell_1}^{(1)} \otimes b_{\ell_2}^{(2)}$ shows

$$\sum_{i=1}^{n} \alpha_{i,\ell_1} \alpha_{i,\ell_2} = \sum_{j=1}^{m} \widetilde{\alpha}_{j,\ell_1} \widetilde{\alpha}_{j,\ell_2}.$$

From this we have immediately that

$$\begin{aligned} \sum_{i=1}^{n} Av_1 \otimes Bw_i &= \sum_{\boldsymbol{\ell} \in I} \sum_{i=1}^{n} \alpha_{i,\ell_1} \alpha_{i,\ell_2} b_{\ell_1}^{(1)} \otimes b_{\ell_2}^{(2)} \\ &= \sum_{\boldsymbol{\ell} \in I} \sum_{j=1}^{m} \widetilde{\alpha}_{j,\ell_1} \widetilde{\alpha}_{j,\ell_2} b_{\ell_1}^{(1)} \otimes b_{\ell_2}^{(2)} \\ &= \sum_{j=1}^{m} A\widetilde{v}_j \otimes B\widetilde{w}_j, \end{aligned}$$

i.e., the independence of the representation of $\boldsymbol{t}$.

Uniqueness then follows from the fact that a linear mapping is uniquely determined by its values on a basis. □

If V, W, S and T are spaces of real-valued functions we have an especially easy way to evaluate functions in the codomain of $A \otimes B$.

Proposition 5.3.3. *Let $V(\Omega^{(1)}) = \{v : \Omega^{(1)} \to \mathbb{R}\}$, $S(\Omega^{(1)}) = \{s : \Omega^{(1)} \to \mathbb{R}\}$ and $W(\Omega^{(2)}) = \{w : \Omega^{(2)} \to \mathbb{R}\}$, $T(\Omega^{(2)}) = \{t : \Omega^{(2)} \to \mathbb{R}\}$ be linear spaces of real-valued functions on domains $\Omega^{(1)}$ and $\Omega^{(2)}$. Denote by $V(\Omega^{(1)}) \otimes W(\Omega^{(2)})$ and $S(\Omega^{(1)}) \otimes T(\Omega^{(2)})$ the respective algebraic tensor product spaces. Let $A : V(\Omega^{(1)}) \to S(\Omega^{(1)})$ and $B : W(\Omega^{(2)}) \to T(\Omega^{(2)})$ be linear. Then the tensor product operator $A \otimes B : V(\Omega^{(1)}) \otimes W(\Omega^{(2)}) \to S(\Omega^{(1)}) \otimes T(\Omega^{(2)})$ is given by*

$$((A \otimes B)(\boldsymbol{t}))(\boldsymbol{x}) := ((A \otimes B)(\boldsymbol{t}))(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = \sum_{i=1}^{n} (Av)(\boldsymbol{x}^{(1)}) \cdot (Bw)(\boldsymbol{x}^{(2)}),$$

where $\boldsymbol{x}^{(1)} \in \Omega^{(1)}$, $\boldsymbol{x}^{(2)} \in \Omega^{(2)}$ and $\boldsymbol{t} \in V(\Omega^{(1)}) \otimes W(\Omega^{(2)})$ has a representation $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i$.

We now consider normed spaces. Assume that we equip the algebraic tensor spaces $V \otimes W$ and $S \otimes T$ with the γ-norm from Definition 5.2.6 using the respective norms of V and W or S and T. We recall that, in order to avoid any confusion, we denote the norm on $V \otimes W$ by $\gamma_{V \otimes W}$ and the norm on $S \otimes T$, accordingly, by $\gamma_{S \otimes T}$. Consider a tensor $\boldsymbol{t} \in V \otimes W$ with representation $\boldsymbol{t} = \sum_{i=1}^{n} v_i \otimes w_i$. Taking the $\gamma_{S \otimes T}$-norm of its image under the tensor product operator $A \otimes B$ yields

$$\begin{aligned} \gamma_{S \otimes T}\left(\sum_{i=1}^{n} Av_i \otimes Bw_i\right) &\leq \sum_{i=1}^{n} \|Av_i\|_S \|Bw_i\|_T \\ &\leq \|A\|_{V \to S} \|B\|_{W \to T} \sum_{i=1}^{n} \|v_i\|_V \|w_i\|_W. \end{aligned}$$

Now take the infimum on both sides over all representations of $\boldsymbol{t}$. We obtain the inequality

$$\gamma_{S \otimes T}((A \otimes B)(\boldsymbol{t})) \leq \|A\|_{V \to S} \|B\|_{W \to T} \cdot \gamma_{V \otimes W}(\boldsymbol{t}).$$

These types of inequalities are important for the error analysis in the later chapters, especially if we use different norms the domain and codomain of the tensor product operator. Note that we assume in the next definition that the involved operators are linear and bounded.

Definition 5.3.4. *Let V, W, S and T be normed spaces with algebraic tensor product spaces $V \otimes W$ and $S \otimes T$. Two norms $\alpha_{V \otimes W} : V \otimes W \to [0, \infty)$ and $\alpha_{S \otimes T} : S \otimes T \to [0, \infty)$ are called* uniformly compatible *if*

$$\|A \otimes B\|_{V \otimes W \to S \otimes T} \leq \|A\|_{V \to S} \|B\|_{W \to T}, \tag{5.3.3}$$

for all $A \in \mathcal{L}(V, S)$ and $B \in \mathcal{L}(W, T)$.

If $V \otimes W = S \otimes T$, a norm which is compatible to itself, is called a uniform norm*.*

Definition 5.3.4 yields immediately that a tensor product operator of bounded operators that is a mapping between two normed algebraic tensor product space with uniformly compatible norms is bounded.

If the norms on $V \otimes W$ and $S \otimes T$ are crossnorms, the situation gets even better.

Theorem 5.3.5. *Let V, W, S and T be normed spaces with algebraic tensor product spaces $V \otimes W$ and $S \otimes T$. Let $\|\cdot\|_{V\otimes T} : V \otimes W \to [0,\infty)$ and $\|\cdot\|_{S\otimes T} : S \otimes T \to [0,\infty)$ be crossnorms. Then the norms are uniformly compatible if and only if*

$$\|A \otimes B\|_{V\otimes W\to S\otimes T} = \|A\|_{V\to S}\|B\|_{W\to T}, \tag{5.3.4}$$

for all $A \in \mathcal{L}(V,S)$ and $B \in \mathcal{L}(W,T)$.

Proof. Clearly, if (5.3.4) holds, then (5.3.3) also holds. Now assume that (5.3.3) holds. We need to show

$$\|A \otimes B\|_{V\otimes W\to S\otimes T} \geq \|A\|_{V\to S}\|B\|_{W\to T}.$$

We use the usual definition of the operator norm

$$\|A \otimes B\|_{V\otimes W\to S\otimes T} := \sup_{\substack{\boldsymbol{t}\in V\otimes W\\ \|\boldsymbol{t}\|_{V\otimes W}=1}} \|(A \otimes B)(\boldsymbol{t})\|_{S\otimes T}$$

and estimate

$$\begin{aligned}
\|A \otimes B\|_{V\otimes W\to S\otimes T} &= \sup_{\substack{\boldsymbol{t}\in V\otimes W\\ \|\boldsymbol{t}\|_{V\otimes W}=1}} \|(A \otimes B)(\boldsymbol{t})\|_{S\otimes T}\\
&\geq \sup_{\substack{v\otimes w\in V\otimes W\\ \|v\otimes w\|_{V\otimes W}=1}} \|(A \otimes B)(v \otimes w)\|_{S\otimes T}\\
&= \sup_{\substack{v\otimes w\in V\otimes W\\ \|v\otimes w\|_{V\otimes W}=1}} \|Av\|_S\|Bw\|_T\\
&\geq \sup_{\substack{v\in V\\ \|v\|_V=1}} \|Av\|_S \sup_{\substack{w\in W\\ \|w\|_W=1}} \|Bw\|_T\\
&= \|A\|_{V\to S}\|B\|_{W\to T}.
\end{aligned}$$

□

Before we show that the examples for crossnorms given in the previous section are uniformly compatible, we remark that the construction of the tensor product operator is purely algebraic. Hence, we need to discuss the case when the domain and codomain are tensor product spaces, e.g., closures of algebraic tensor spaces with respect to certain norms. This, however, is an easy corollary of Theorem 2.1.12.

Corollary 5.3.6. *Let V, W, S and T be normed spaces with algebraic tensor product spaces $V \otimes W$ and $S \otimes T$. Let $\alpha_{V\otimes W}$ and $\alpha_{S\otimes T}$ be uniformly compatible norms. Let $A \in \mathcal{L}(V,S)$ and $B \in \mathcal{L}(W,T)$. Then the tensor product operator $A \otimes B : V \otimes W \to S \otimes T$ has a unique extension $A\overline{\otimes}B : V\overline{\otimes}W \to S\overline{\otimes}T$, such that $\|A \otimes B\|_{V\otimes W\to S\otimes T} = \|A\overline{\otimes}B\|_{V\overline{\otimes}W\to S\overline{\otimes}T}$.*

We now come back to the examples of crossnorms introduced in Section 5.2. We will see that all these norms are pairwise uniformly compatible. We recall that the λ-norm is defined in Definition 5.2.2 and the γ-norm is defined in Definition 5.2.6.

Theorem 5.3.7. *Let V, W, S and T be normed spaces with algebraic tensor product spaces $V \otimes W$ and $S \otimes T$. Then the following relations hold between the λ-, γ- and any other reasonable crossnorms $\|\cdot\|_{V\otimes W}$ and $\|\cdot\|_{S\otimes T}$.*

(1) $\|\cdot\|_{\lambda,V\otimes W}$ *and* $\|\cdot\|_{\lambda,S\otimes T}$ *are uniformly compatible.*
(2) Every reasonable crossnorm $\|\cdot\|_{V\otimes W}$ *is uniformly compatible with* $\|\cdot\|_{\lambda,S\otimes T}$.
(3) $\|\cdot\|_{\gamma,V\otimes W}$ *and* $\|\cdot\|_{\gamma,S\otimes T}$ *are uniformly compatible.*
(4) $\|\cdot\|_{\gamma,V\otimes W}$ *is uniformly compatible with every crossnorm* $\|\cdot\|_{S\otimes T}$.

PROOF. The theorem is a generalization of a result given in [58], however the proof ideas there can be applied in the same way. □

This means in particular that the λ-, γ- and β-norms are uniformly compatible. Additionally, we see that all induced norms are compatible.

Theorem 5.3.8. *Let V, W, S and T be pre-Hilbert spaces with algebraic tensor product spaces $V\otimes W$ and $S\otimes T$. Then the induced norms $\|\cdot\|_{\beta,V\otimes W}$ and $\|\cdot\|_{\beta,S\otimes T}$ are uniformly compatible.*

PROOF. Assume that $A\in\mathcal{L}(V,S)$, $B\in\mathcal{L}(W,T)$. Let $\boldsymbol{t}\in V\otimes W$. Following Lemma 5.1.11 we find an $r\in\mathbb{N}$ and linearly independent sets $\{v_i\}_{1\le i\le r}\subseteq V$ and $\{w_i\}_{1\le i\le r}\subseteq W$ such that $\boldsymbol{t}$ has the representation $\boldsymbol{t}=\sum_{i=1}^r v_i\otimes w_i$.

We define an $r\times r$ matrix $C=(\langle Bw_i,Bw_k\rangle_W)$. This matrix is symmetric and positive-definite and can therefor be written as $C=DD^{\mathrm{T}}$ with a symmetric and positive-definite $r\times r$ matrix D.

With this, we can bound

$$\begin{aligned}
\left\|\sum_{i=1}^r Av_i\otimes Bw_i\right\|_{\beta,S\times T}^2 &= \sum_{i,k=1}^r \langle Av_i,Av_k\rangle_S\langle Bw_i,Bw_k\rangle_T\\
&= \sum_{i,k=1}^r \langle Av_i,Av_k\rangle_S \sum_{\ell=1}^r d_{i\ell}d_{\ell k}\\
&= \sum_{\ell=1}^r \left\langle \sum_{i=1}^r d_{i\ell}Av_i, \sum_{k=1}^r d_{\ell k}Av_k\right\rangle_S\\
&= \sum_{\ell=1}^r \left\|\sum_{i=1}^r d_{i\ell}Av_i\right\|_S^2 = \sum_{\ell=1}^r \left\|A\left(\sum_{i=1}^r d_{i\ell}v_i\right)\right\|_S^2\\
&\le \|A\|_{V\to S}^2 \sum_{\ell=1}^r \left\langle \sum_{i=1}^r d_{i\ell}v_i, \sum_{k=1}^r d_{\ell k}v_k\right\rangle_V\\
&= \|A\|_{V\to S}^2 \sum_{i,k=1}^r \langle v_i,v_k\rangle_V \sum_{\ell=1}^r d_{i\ell}d_{\ell k}\\
&= \|A\|_{V\to S}^2 \sum_{i,k=1}^r \langle v_i,v_k\rangle_V\langle Bw_i,Bw_k\rangle_W.
\end{aligned}$$

Next, we use that the $r\times r$ matrix $\widetilde{C}=(\langle v_i,v_k\rangle_W)$ is symmetric and positive definite and can hence be written as $\widetilde{C}=\widetilde{D}\widetilde{D}^{\mathrm{T}}$. Using the same trick as

before yields

$$\begin{aligned}\sum_{i,k=1}^{r} \langle v_i, v_k\rangle_V \langle Bw_i, Bw_k\rangle_W &= \sum_{\ell=1}^{r} \left\| \sum_{i=1}^{r} \widetilde{d}_{i\ell} Bw_i \right\|_T^2 \\ &\leq \|B\|_{W\to T}^2 \sum_{i,k=1}^{r} \langle v_i, v_k\rangle_V \langle w_i, w_k\rangle_W \\ &= \|B\|_{W\to T}^2 \left\| \sum_{i=1}^{r} v_i \otimes w_i \right\|_{\beta, V\otimes W}^2 .\end{aligned}$$

Hence, we have shown that

$$\left\| \sum_{i=1}^{r} Av_i \otimes Bw_i \right\|_{\beta, S\otimes T}^2 \leq \|A\|_{V\to s}^2 \|B\|_{W\to T}^2 \left\| \sum_{i=1}^{r} v_i \otimes w_i \right\|_{\beta, V\otimes W}^2 ,$$

which means that the norms are uniformly compatible. □

The whole next chapter will be dedicated to an application of the theory presented here. Before we discuss this special tensor product operator in detail, we come back to the anisotropic Sobolev Hilbert spaces $H^{\boldsymbol{m}}_{mix}(\Omega)$ defined in Definition 5.2.30 and its generalization introduced in Definition 5.2.31 $H^{\boldsymbol{s}}_{mix}(\Omega)$. For these spaces we give now an analogous result to the Sobolev embedding theorem in Theorem 2.1.13.

Theorem 5.3.9. *Let $S \subseteq \mathbb{R}^{n_1}$ and $T \subseteq \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, be Lipschitz domains. Let $\boldsymbol{s} \in \mathbb{R}^2$ such that $s_1 > n_1/2$ and $s_2 > n_2/2$. Then there exists a bounded, linear embedding operator $\boldsymbol{\iota} : H^{\boldsymbol{s}}_{mix}(S \times T) \to C(S \times T)$.*

Proof. From Theorem 2.1.13 we know, under the assumptions on s_1 and s_2, that there are continuous, linear operators $\iota^{(1)} : H^{s_1}(S) \to C(S)$ and $\iota^{(2)} : H^{s_2}(T) \to C(T)$. Taking the tensor product of these operators yields an operator $\iota^{(1)} \otimes \iota^{(2)}$ mapping the algebraic tensor product space $H^{s_1}(S) \otimes_\beta H^{s_2}(T)$ into the algebraic tensor product space $C(S) \otimes_\lambda C(T)$, i.e.,

$$\iota^{(1)} \otimes \iota^{(2)} : H^{s_1}(S) \otimes H^{s_2}(T) \to C(S) \otimes C(T).$$

By Theorem 5.3.7 the norms on $H^{s_1}(S) \otimes H^{s_2}(T)$ and $C(S) \otimes C(T)$ are compatible, hence $\iota^{(1)} \otimes \iota^{(2)}$ is bounded. This means it can be extended to an bounded and linear operator $\boldsymbol{\iota}$, which maps $H^{\boldsymbol{s}}_{mix}(S \times T)$ into $C(S \times T)$. □

Theorem 5.3.9 allows us to embed mixed regularity Sobolev spaces into spaces of continuous functions with much lower assumptions on the smoothness, in particular, if the dimension of the domain, $n_1 + n_2$ is large. This, however, is paid with a stronger norm.

CHAPTER 6

The Smolyak Method

We are now able to introduce and examine a special tensor product operator in detail. This Smolyak operator allows us to handle high-directional approximation problems by solving approximation problems in the single directions and then combining these approximations in an orderly fashion. In Section 6.1 we study this method in the most basic setting. Next, we follow the literature and restrict ourselves to a special kind of index set which we first study in more detail in Section 6.2 and then use it in the Smolyak method in Section 6.3, where we also repeat and refine the results obtained in the first section of this chapter. Finally, we briefly introduce a general way to adaptively obtain an index set for a fixed target function in Section 6.4.

6.1. Basic Smolyak Algorithm

The Smolyak method uses two tools. First, tensor products which we studied in detail in Chapter 5 and second, special subsets of $\mathbb{N}^d$. We discuss these sets in the next subsection.

6.1.1. Monotone Index Sets. We start by introducing and studying non-empty subsets of $\mathbb{N}^d$ which exhibit a special structure. The restriction to these special *index sets* is necessary for the Smolyak method to be well-defined.

Definition 6.1.1. *A non-empty index set $\Lambda(d) \subseteq \mathbb{N}^d$ is called* monotone *if $\boldsymbol{\lambda} \in \Lambda(d)$ and $\boldsymbol{\nu} \in \mathbb{N}^d$ with $\boldsymbol{\nu} \leq \boldsymbol{\lambda}$ implies that $\boldsymbol{\nu} \in \Lambda(d)$.*

Sometimes monotone index sets are also called *downwards closed*. We can reformulate this definition in a way that makes it easier to check if an index set is monotone.

Proposition 6.1.2. *The non-empty index set $\Lambda(d) \subseteq \mathbb{N}^d$ is monotone if and only if for each $\boldsymbol{\lambda} \in \Lambda(d)$ with $\lambda_j \geq 2$, $1 \leq j \leq d$, also $\boldsymbol{\lambda} - \boldsymbol{e}_j \in \Lambda(d)$, $1 \leq j \leq d$. Here, $\boldsymbol{e}_j$ denotes the j-th unit vector in $\mathbb{R}^d$.*

In Fig. 1 we give two examples of index sets. Although $\Lambda(d)$ is defined as a subset of $\mathbb{N}^d$ it has become customary to display the set as is done here. The elements of $\Lambda(d)$ are the respective right upper corners of the gray boxes. The left depicted index set is monotone. However, the right one is an example of a non-monotone set. The index $(2,4)^{\mathrm{T}}$ is an element of this index set but $(2,3)^{\mathrm{T}}$ is not. This is a violation of the condition in Proposition 6.1.2.

For every direction $1 \leq j \leq d$ we find a unique value $\lambda_{j,max}$ which is the maximum value of the j-th component of any multi-index in $\Lambda(d)$.

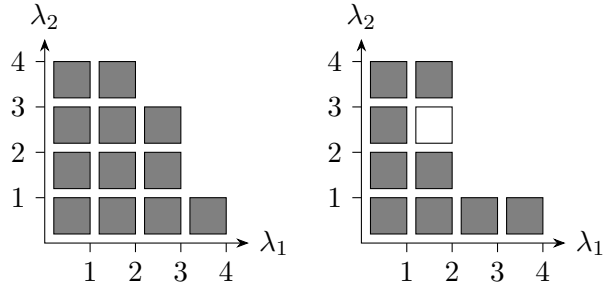

FIGURE 1. Example of a monotone index set (left) and a non-monotone index set (right).

Definition 6.1.3. *Let $\Lambda(d) \subseteq \mathbb{N}^d$ be a bounded index set. Then the maximum value of the j-th component of any multi-index $\boldsymbol{\lambda} \in \Lambda(d)$ is defined as*

$$\lambda_{j,max} := \max\{\lambda_j \ : \ \boldsymbol{\lambda} \in \Lambda(d)\}. \tag{6.1.1}$$

Clearly, for bounded index sets this value always exists and is for every $1 \leq j \leq d$ unique. However, there is not necessarily only one $\boldsymbol{\lambda} \in \Lambda(d)$ which has $\lambda_{j,max}$ as its j-th entry. Indeed, as we see in in the left set of Fig. 1 there are two multi-indices, $(1,4)^{\mathrm{T}}$ and $(2,4)^{\mathrm{T}}$, whose second components are $\lambda_{2,max} = 4$.

The special structure of the monotone index set $\Lambda(d)$ allows us to incrementally build it direction by direction. To do this, we need to introduce two further sets, which are subsets of $\Lambda(d)$.

Definition 6.1.4. *Let $\Lambda(d) \subseteq \mathbb{N}^d$, $d \in \mathbb{N}$, be a monotone index set. Define for $1 \leq i \leq d$ the set $\widetilde{\Lambda}(i) \subseteq \Lambda(d)$ by*

$$\widetilde{\Lambda}(i) := \left\{\widetilde{\boldsymbol{\lambda}} \in \mathbb{N}^i \ : \ \text{there is a } \boldsymbol{\lambda} \in \Lambda(d) \text{ such that } \widetilde{\lambda}_m = \lambda_m, 1 \leq m \leq i\right\} \tag{6.1.2}$$

and for $1 \leq i \leq d-1$ and $1 \leq k \leq \lambda_{i+1,max}$ the set $\widetilde{\Lambda}(k,i) \subseteq \Lambda(d)$ by

$$\widetilde{\Lambda}(k,i) := \left\{\widetilde{\boldsymbol{\lambda}} \in \mathbb{N}^i \ : \ \text{there is a } \boldsymbol{\lambda} \in \Lambda(d) \text{ such that } \widetilde{\lambda}_m = \lambda_m, 1 \leq m \leq i, \text{ and } \lambda_{i+1} = k\right\}. \tag{6.1.3}$$

The index set $\widetilde{\Lambda}(i) \subseteq \mathbb{N}^i$ in (6.1.2) is the restriction of the monotone index set $\Lambda(d)$ to the first i directions. The set $\widetilde{\Lambda}(k,i)$ in (6.1.3) is a special restriction to the first i directions where we only take elements of $\Lambda(d)$ whose $(i+1)$st entry is equal to k. We give examples of these sets in Fig. 2.

From Definition 6.1.4, the following proposition is straight forward.

Proposition 6.1.5. *Let $\Lambda(d) \subseteq \mathbb{N}^d$, $d \in \mathbb{N}$, be a monotone index set. Then the following statements hold.*

(1) For fixed $1 \leq i \leq d-1$, the sets $\widetilde{\Lambda}(k,i)$ are nested for $2 \leq k \leq \lambda_{i+1,max}$, i.e.,

$$\widetilde{\Lambda}(k,i) \subseteq \widetilde{\Lambda}(k-1,i).$$

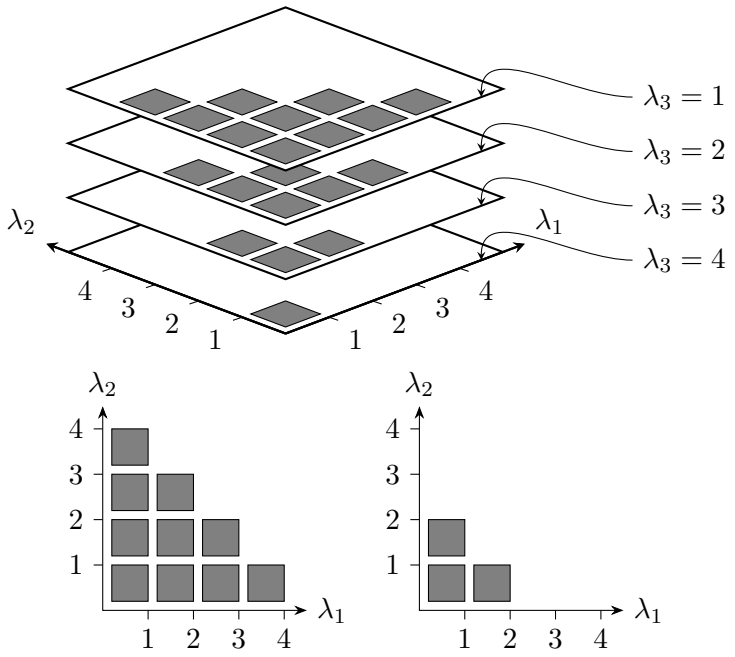

FIGURE 2. Examples of $\Lambda(3)$ (top) and the associated subsets $\widetilde{\Lambda}(2)$, the restriction of $\Lambda(3)$ to the first 2 directions defined in (6.1.2) (bottom left) and $\widetilde{\Lambda}(3,2)$, the restriction of $\Lambda(3)$ to the first 2 directions using those $\boldsymbol{\lambda} \in \Lambda(3)$ with $\lambda_3 = 3$, defined in (6.1.3) (bottom right).

(2) We can construct $\Lambda(d)$ inductively. We have $\Lambda(1) = \{1, \ldots, \lambda_{1,max}\}$ and for $i = 2, \ldots, d$ we have

$$\widetilde{\Lambda}(i) = \bigcup_{k=1}^{\lambda_{i,max}} \widetilde{\Lambda}(k, i-1) \times \{k\}. \tag{6.1.4}$$

And obviously, $\widetilde{\Lambda}(d) = \Lambda(d)$.

In Fig. 3 we give three examples of two-directional monotone index sets. The one on the left is usually called *isotropic*, i.e., it is extended in all directions uniformly, and the one in the middle is most often called *anisotropic*, i.e., the extension of the set in each direction can be different. We will see in Section 6.2 that isotropic index sets can be seen as special cases of anisotropic ones. We will study anisotropic index sets in more detail there. The right most set is an example of a *full product* set. This set is, for fixed $\lambda_{1,max}, \ldots, \lambda_{d,max}$, the index set with the most elements of all monotone sets and usually an undesired extremal case.

6.1.2. Definition of the Basic Smolyak Operator. We now give the precise definition of the Smolyak operator. We assume that we have two families of linear spaces, $V^{(1)}, \ldots, V^{(d)}$ and $S^{(1)}, \ldots, S^{(d)}$. These spaces may coincide, i.e., $V^{(i)} = V^{(j)}$, $S^{(i)} = S^{(j)}$ or even $V^{(i)} = S^{(j)}$, for some $1 \le i, j \le d$, however we do not require them to. Usually these spaces are

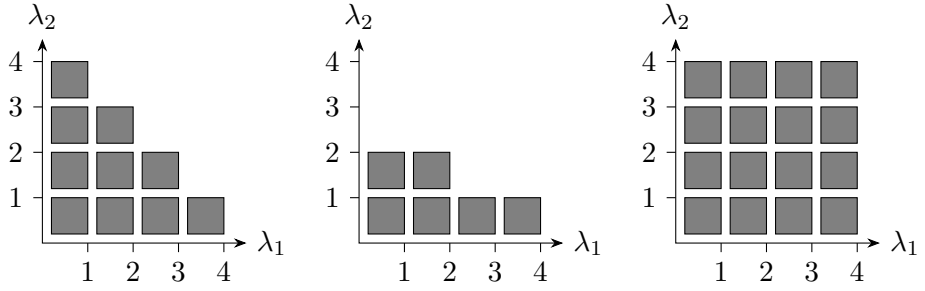

FIGURE 3. Three examples of a monotone index set: Isotropic (left), anisotropic (middle) and full product (right).

either Sobolev spaces on lower-dimensional domains $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $1 \leq j \leq d$ and $n_j = 1, 2, 3$, i.e., $V^{(j)} = W_{p_j}^{\sigma_j}(\Omega^{(j)})$ and $S^{(j)} = W_{q_j}^{\tau_j}(\Omega^{(j)})$ with $1 \leq p_j, q_j \leq \infty$ and $0 \leq \sigma_j, \tau_j \leq \infty$ for every $1 \leq j \leq d$, or spaces of continuously differentiable functions $C^{k_j}(\Omega^{(j)})$, $k_j \in \mathbb{N}_0 \cup \{\infty\}$ on $\Omega^{(j)}$. Another suitable choice is $S^{(j)} = \mathbb{R}$, if we are interested in quadrature rules, see, e.g., [83].

On these spaces we define special operators.

Definition 6.1.6. *Let $V^{(1)}, \dots, V^{(d)}$ and $S^{(1)}, \dots, S^{(d)}$ be linear spaces. For $1 \leq j \leq d$, $k \in \mathbb{N}$ let $A_k^{(j)} : V^{(j)} \to S^{(j)}$ be linear operators and set $A_0^{(j)} = 0$. For $1 \leq j \leq d$ and $k \in \mathbb{N}$ we define the* difference operators $\Delta_k^{(j)} : V^{(j)} \to S^{(j)}$ *by*

$$\Delta_k^{(j)} := A_k^{(j)} - A_{k-1}^{(j)}. \tag{6.1.5}$$

Definition 6.1.6 means that we can express $A_k^{(j)}$ as a telescoping sum

$$A_k^{(j)} = \sum_{i=1}^{k} \Delta_i^{(j)}, \quad 1 \leq j \leq d,\ k \in \mathbb{N}, \tag{6.1.6}$$

with, again, $A_0^{(j)} = 0$.

In the original construction of Smolyak in [83] the operators $A_k^{(j)}$ were univariate quadrature rules. Later on, e.g., in [11, 90], different interpolation and approximation operators have been used. In this thesis we will set $A_k^{(j)}$ to be different kernel-based approximation operators. Special interest lies in the choice of multi-level operators $A_{L^{(j)}}^{(j)}$ of certain, direction dependent levels $L^{(j)}$. We studied these operators Chapter 4 in detail.

The original idea of Smolyak was to combine tensor products of the difference operators to obtain an operator which maps the algebraic tensor product space $V^{(1)} \otimes \cdots \otimes V^{(d)}$ into the algebraic tensor product space $S^{(1)} \otimes \cdots \otimes S^{(d)}$. This combination is done in such a way that the resulting tensor product operator retains the approximation properties of the direction-wise operators $(A_k^{(j)})_{k \in \mathbb{N}}$ but is also not too expensive to compute. The idea is that this combination is done according to a given monotone index set $\Lambda(d) \subseteq \mathbb{N}^d$.

Definition 6.1.7. *Let $V^{(1)}, \dots, V^{(d)}$ and $S^{(1)}, \dots, S^{(d)}$ be linear space with algebraic tensor product spaces $\bigotimes_{j=1}^d V^{(j)}$ and $\bigotimes_{j=1}^d S^{(j)}$. Let $\Lambda(d) \subseteq \mathbb{N}^d$ be a*

monotone index set. For linear operators $A_k^{(j)} : V^{(j)} \to S^{(j)}$, $1 \leq j \leq d$, $k \in \mathbb{N}$, *let* $\Delta_k^{(j)} = A_k^{(j)} - A_{k-1}^{(j)}$ *be the difference operators defined in Definition 6.1.6.*

Then the (basic) Smolyak operator $\mathcal{A}_{\Lambda(d)} : \bigotimes_{j=1}^d V^{(j)} \to \bigotimes_{j=1}^d S^{(j)}$ *is defined as the tensor product operator*

$$\mathcal{A}_{\Lambda(d)} := \sum_{\boldsymbol{\lambda} \in \Lambda(d)} \boldsymbol{\Delta}_{\boldsymbol{\lambda}} := \sum_{\boldsymbol{\lambda} \in \Lambda(d)} \bigotimes_{j=1}^d \Delta_{\lambda_j}^{(j)}. \tag{6.1.7}$$

We will also call this operator *(basic) Smolyak method* or *(basic) Smolyak algorithm*. The definition of $\mathcal{A}_{\Lambda(d)}$ makes it obvious why we require the index set $\Lambda(d)$ to be monotone, otherwise the difference operators $\Delta_k^{(j)}$ would not be well-defined.

Following Corollary 5.3.6 we can extend $\mathcal{A}_{\Lambda(d)}$ to an operator on the closure of the algebraic tensor product spaces.

Proposition 6.1.8. *For* $1 \leq j \leq d$ *let* $(V^{(j)}, \|\cdot\|_{V^{(j)}})$ *and* $(S^{(j)}, \|\cdot\|_{S^{(j)}})$ *be normed spaces. Let* $\bigotimes_{j=1}^d V^{(j)}$ *and* $\bigotimes_{j=1}^d S^{(j)}$ *be the respective algebraic tensor product spaces. Assume that the tensor product norms* $\alpha : \bigotimes_{j=1}^d V^{(j)} \to [0, \infty)$ *and* $\widetilde{\alpha} : \bigotimes_{j=1}^d S^{(j)} \to [0, \infty)$ *are uniformly compatible and let* $V^{(1)} \overline{\otimes} \cdots \overline{\otimes} V^{(d)}$ *and* $S^{(1)} \overline{\otimes} \cdots \overline{\otimes} S^{(d)}$ *be the closures of the algebraic tensor product spaces with respect to* α *and* $\widetilde{\alpha}$. *Assume further that the operators* $A_k^{(j)} : V^{(j)} \to S^{(j)}$ *are bounded for all* $1 \leq j \leq d$ *and* $k \in \mathbb{N}$.

Then there exists a unique extension of the Smolyak operator to an operator $V^{(1)} \overline{\otimes} \cdots \overline{\otimes} V^{(d)} \to S^{(1)} \overline{\otimes} \cdots \overline{\otimes} S^{(d)}$.

We write for this extension again $\mathcal{A}_{\Lambda(d)}$.

We now derive some alternative, but equivalent, representations of the Smolyak operator $\mathcal{A}_{\Lambda(d)}$. First, we note that for $d = 1$ we have, with (6.1.6),

$$\mathcal{A}_{\Lambda(1)} = \sum_{\boldsymbol{\lambda} \in \Lambda(1)} \boldsymbol{\Delta}_{\boldsymbol{\lambda}} = \sum_{k=1}^{\lambda_{1,max}} \Delta_k^{(1)} = A_{\lambda_{1,max}}^{(1)}.$$

This is simply the one-directional operator of the highest index.

Next, we follow [90] and express $\mathcal{A}_{\Lambda(d)}$ in terms of the operators $A_k^{(j)}$. To do so, we need to resolve the tensor products of the difference operators in (6.1.7). We find that, for a fixed $\boldsymbol{\lambda} \in \Lambda(d)$, the identity

$$\boldsymbol{\Delta}_{\boldsymbol{\lambda}} = \bigotimes_{j=1}^d \Delta_{\lambda_j}^{(j)} = \sum_{\substack{\boldsymbol{\beta} \in \{0,1\}^d \\ \boldsymbol{\lambda} - \boldsymbol{\beta} \geq 1}} (-1)^{\|\boldsymbol{\beta}\|_1} \bigotimes_{j=1}^d A_{\lambda_j - \beta_j}^{(j)}$$

holds. Here, we used that we set $A_0^{(j)} = 0$, $1 \leq j \leq d$. Inserting this representation into (6.1.7) yields a formula which is universally called *combination technique*.

Proposition 6.1.9. *With the notation and assumptions of Definition 6.1.7 the Smolyak operator can be expressed as*

$$\mathcal{A}_{\Lambda(d)} = \sum_{\boldsymbol{\lambda}\in\Lambda(d)} \sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^d \\ \boldsymbol{\lambda}+\boldsymbol{\beta}\in\Lambda(d)}} (-1)^{\|\boldsymbol{\beta}\|_1} \left(A^{(1)}_{\lambda_1} \otimes \cdots \otimes A^{(d)}_{\lambda_d}\right). \tag{6.1.8}$$

This representation allows us to use the operators $A^{(j)}_k$ directly instead of having to compute the difference operators first. This makes an implementation of the Smolyak operator particularly simple. It also allows us to derive an explicit formula for the point-evaluation of the codomain of $\mathcal{A}_{\Lambda(d)}$. This is an easy conclusion of Theorem 5.1.10.

Proposition 6.1.10. *For $1 \leq j \leq d$ let $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, be a domain. Assume that $V^{(j)}(\Omega^{(j)}) = \{v^{(j)} : \Omega^{(j)} \to \mathbb{R}\}$ and $S^{(j)}(\Omega^{(j)}) = \{s^{(j)} : \Omega^{(j)} \to \mathbb{R}\}$ are linear spaces of functions defined on $\Omega^{(j)}$ with algebraic tensor product spaces $\bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)})$ and $\bigotimes_{j=1}^d S^{(j)}(\Omega^{(j)})$. For all $1 \leq j \leq d$ and $k \in \mathbb{N}$ let $A^{(j)}_k : V^{(j)}(\Omega^{(j)}) \to S^{(j)}(\Omega^{(j)})$ be linear operators. Let $\Lambda(d) \subseteq \mathbb{N}^d$ be a monotone index set.*

Let $\mathcal{A}_{\Lambda(d)} : \bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)}) \to \bigotimes_{j=1}^d S^{(j)}(\Omega^{(j)})$ be the Smolyak operator. Then the identity

$$\mathcal{A}_{\Lambda(d)}(v^{(1)} \otimes \cdots \otimes v^{(d)})(\boldsymbol{x}) = \sum_{\boldsymbol{\lambda}\in\Lambda(d)} \sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^d \\ \boldsymbol{\lambda}+\boldsymbol{\beta}\in\Lambda(d)}} (-1)^{\|\boldsymbol{\beta}\|_1} \prod_{j=1}^d \left(A^{(j)}_{\lambda_j} v^{(j)}\right)(\boldsymbol{x}^{(j)}),$$

holds for all $v^{(1)} \otimes \cdots \otimes v^{(d)} \in \bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)})$ and $\boldsymbol{x} = (\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)})^T \in \times_{j=1}^d \Omega^{(j)}$.

Another way to represent the Smolyak operator is by means of a recursion formula, i.e., expressing the d-directional operator $\mathcal{A}_{\Lambda(d)}$ using a $d-1$-directional Smolyak operator and a one-directional difference operator. The key is to use the recursive representation of the monotone index set in (6.1.4). That is, we can express $\Lambda(d)$ as a union of sets $(\widetilde{\Lambda}(k, d-1) \times \{k\})_{1\leq k\leq \lambda_{d,max}}$. We have

$$\begin{aligned}
\mathcal{A}_{\Lambda(d)} &= \sum_{\boldsymbol{\lambda}\in\Lambda(d)} \bigotimes_{j=1}^d \left(A^{(j)}_{\lambda_j} - A^{(j)}_{\lambda_j - 1}\right) \\
&= \sum_{k=1}^{\lambda_{d,max}} \sum_{\widetilde{\boldsymbol{\lambda}}\in\widetilde{\Lambda}(k,d-1)} \bigotimes_{j=1}^{d-1} \left(A^{(j)}_{\widetilde{\lambda}_j} - A^{(j)}_{\widetilde{\lambda}_{j+1}}\right) \otimes \left(A^{(d)}_k - A^{(d)}_{k-1}\right) \\
&= \sum_{k=1}^{\lambda_{d,max}} \mathcal{A}_{\widetilde{\Lambda}(k,d-1)} \otimes \left(A^{(d)}_k - A^{(d)}_{k-1}\right).
\end{aligned}$$

Hence, we have a second alternative to express the basic Smolyak operator.

Proposition 6.1.11. *With the notation and assumptions of Definition 6.1.7 the Smolyak operator can be expressed as*

$$\mathcal{A}_{\Lambda(d)} = \sum_{k=1}^{\lambda_{d,max}} \mathcal{A}_{\widetilde{\Lambda}(k,d-1)} \otimes \left(A_k^{(d)} - A_{k-1}^{(d)} \right). \tag{6.1.9}$$

This representation will prove essential when we show some of the properties of the Smolyak operator in Section 6.1.4.

The last representation we give follows the same ideas but uses recursion in a different way. It is based on the following observation.

Proposition 6.1.12. *Let $\Lambda(d) \subseteq \mathbb{N}^d$ be a monotone index set and $\widetilde{\Lambda}(i)$ be the restriction of $\Lambda(d)$ to the first i, $1 \leq i \leq d-1$, directions. For every $\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(i)$ we find a unique $\lambda_{i+1}(\widetilde{\boldsymbol{\lambda}}) \in \mathbb{N}$ with*

$$\lambda_{i+1}(\widetilde{\boldsymbol{\lambda}}) = \max \left\{ k \in \{1, \ldots, \lambda_{i+1,max}\} \ : \ (\widetilde{\boldsymbol{\lambda}}, k)^T \in \widetilde{\Lambda}(i+1) \right\}.$$

This allows us to express the Smolyak operator in a different recursive way.

Proposition 6.1.13. *With the notation and assumptions of Definition 6.1.7 the Smolyak operator can be expressed as*

$$\mathcal{A}_{\Lambda(d)} = \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(d-1)} \bigotimes_{j=1}^{d-1} \Delta_{\widetilde{\lambda}_j}^{(j)} \otimes A_{\lambda_d(\widetilde{\boldsymbol{\lambda}})}^{(d)}, \tag{6.1.10}$$

where $\lambda_d(\widetilde{\boldsymbol{\lambda}})$ is as in Proposition 6.1.12.

Proof. With $\lambda_d(\widetilde{\boldsymbol{\lambda}})$ as in Proposition 6.1.12 we have

$$\begin{aligned}
\mathcal{A}_{\Lambda(d)} &= \sum_{\boldsymbol{\lambda} \in \Lambda(d)} \bigotimes_{j=1}^{d} \Delta_{\lambda_j}^{(j)} \\
&= \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(d-1)} \sum_{k=1}^{\lambda_d(\widetilde{\boldsymbol{\lambda}})} \bigotimes_{j=1}^{d-1} \Delta_{\widetilde{\lambda}_j}^{(j)} \otimes \left(A_k^{(d)} - A_{k-1}^{(d)} \right) \\
&= \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(d-1)} \bigotimes_{j=1}^{d-1} \Delta_{\widetilde{\lambda}_j}^{(j)} \otimes \left(\sum_{k=1}^{\lambda_d(\widetilde{\boldsymbol{\lambda}})} \left(A_k^{(d)} - A_{k-1}^{(d)} \right) \right).
\end{aligned}$$

In the last step we used the multilinearity of the tensor product. Resolving the telescope sum

$$\sum_{k=1}^{\lambda_d(\widetilde{\boldsymbol{\lambda}})} \left(A_k^{(d)} - A_{k-1}^{(d)} \right) = A_{\lambda_d(\widetilde{\boldsymbol{\lambda}})}^{(d)}$$

yields the claim. □

We will use this form of the Smolyak operator to derive error representations of $\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)}$. We note that for general monotone index sets $\Lambda(d)$ we can not give an explicit formula for $\lambda_d(\widetilde{\boldsymbol{\lambda}})$. However, later, we discuss the anisotropic Smolyak operator and there we can give a closed form for this quantity.

6.1.3. Sparse Grids. Sparse grids are inseparably linked to Smolyak operators. This goes even so far that the corresponding methods are often used as synonyms, see, e.g., [70, Chapter 15]. Assume that the low-dimensional operators $(A_i^{(j)})_{i\in\mathbb{N}}$, $1 \leq j \leq d$, are approximation operators that use point data of the target functions. Then the point sets the Smolyak operator of (6.1.7) uses are called *sparse grids*. We now go into more detail.

We assume that every low-dimensional domain $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, $1 \leq j \leq d$, contains a sequence of point sets $(X_i^{(j)})_{i\in\mathbb{N}}$ with cardinality $\#X_i^{(j)} = N_i^{(j)}$, i.e., for $i \in \mathbb{N}$, we have

$$X_i^{(j)} := \left\{ \boldsymbol{x}_{i,1}^{(j)}, \dots, \boldsymbol{x}_{i,N_i^{(j)}}^{(j)} \right\} \subseteq \Omega^{(j)}.$$

We assume that these $X_i^{(j)}$ get denser and denser in $\Omega^{(j)}$, i.e., $N_i^{(j)} \leq N_{i+1}^{(j)}$, $1 \leq j \leq d$, $i \in \mathbb{N}$, and, although in applications this will often be the case, we do not require them to be nested.

We assume further that for every $1 \leq j \leq d$ we have spaces of functions $V^{(j)}(\Omega^{(j)})$ and $S^{(j)}(\Omega^{(j)})$ defined on $\Omega^{(j)}$. We assume further that for every $i \in \mathbb{N}$ we find in $S^{(j)}(\Omega^{(j)})$ a set of Lagrange functions $\{\chi_{i,m}^{(j)}\}_{1\leq m\leq N_i^{(j)}} : S^{(j)}(\Omega^{(j)}) \to \mathbb{R}$ corresponding the set $X_i^{(j)}$. We recall that Lagrange functions are mappings that satisfy $\chi_{i,k}^{(j)}(\boldsymbol{x}_{i,m}^{(j)}) = \delta_{k,m}$ for all $1 \leq k, m \leq N_i^{(j)}$.

We then define operators $A_i^{(j)} : V^{(j)}(\Omega^{(j)}) \to S^{(j)}(\Omega^{(j)})$ by

$$A_i^{(j)}(f^{(j)}) = \sum_{k=1}^{N_i^{(j)}} f^{(j)}(\boldsymbol{x}_{i,k}^{(j)})\chi_{i,k}^{(j)}, \quad f^{(j)} \in V^{(j)}(\Omega^{(j)}),\ \boldsymbol{x}_{i,k}^{(j)} \in X_i^{(j)}.$$

Examples for these operators are approximation operators using piece-wise linear splines, see, e.g., [11], Lagrange polynomials of certain degree, see, e.g., [90], or, in the case of this thesis, kernel-based (multilevel) operators using Lagrange functions, see Chapters 3 and 4.

Using these operators in the formula of the combination technique, (6.1.8), with a monotone index set $\Lambda(d) \subseteq \mathbb{N}^d$ yields for $\boldsymbol{f} \in \bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)})$

$$\mathcal{A}_{\Lambda(d)}(\boldsymbol{f}) =$$

$$= \sum_{\boldsymbol{\lambda}\in\Lambda(d)} \sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^d \\ \boldsymbol{\lambda}+\boldsymbol{\beta}\in\Lambda(d)}} (-1)^{\|\boldsymbol{\beta}\|_1} \sum_{k_1=1}^{N_{\lambda_1}^{(1)}} \cdots \sum_{k_d=1}^{N_{\lambda_d}^{(d)}} \boldsymbol{f}\left(\boldsymbol{x}_{\lambda_1,k_1}^{(1)}, \dots, \boldsymbol{x}_{\lambda_d,k_d}^{(d)}\right) \bigotimes_{j=1}^{d} \chi_{\lambda_j,k_j}^{(j)}.$$

Investigating this identity closely yields that the Smolyak operator only uses functions values of $\boldsymbol{f} \in \bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)})$ on special combinations of the point sets $X_i^{(j)}$.

Definition 6.1.14. *For $1 \leq j \leq d$ let $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, be low-dimensional domains and let $(X_i^{(j)})_{i\in\mathbb{N}} \subseteq \Omega^{(j)}$ be a sequence of sets of sites. Let $\Lambda(d) \subseteq \mathbb{N}^d$ be a monotone index set. Then the* sparse grid $H_{\Lambda(d)} \subseteq$

$\times_{j=1}^{d} \Omega^{(j)}$ *is defined by*

$$H_{\Lambda(d)} = \bigcup_{\boldsymbol{\lambda} \in \Lambda(d)} \left(X_{\lambda_1}^{(1)} \times \cdots \times X_{\lambda_d}^{(d)} \right). \tag{6.1.11}$$

In Fig. 4 we give examples of two-directional sparse grids in $[-1,1]^2$. In each column we use the corresponding index set depicted in Fig. 3 and for each row we use a different direction-independent family of one-dimensional point sets $(X_i)_{1 \le i \le 4}$. Here we omit the superscript. In each of the three families the sets have cardinality $N_1 = 1$ and $N_i = 2^{i-1} + 1$, $i = 2, 3, 4$. In any of the three cases we set $X_1 = \{0\}$.

On the top row we discretized the interval $[-1, 1]$ uniformly with fill distance $h_i = 2^{-i+1}$ for $2 \le i \le 4$, i.e., the point sets X_i^{unif} are defined as

$$X_i^{unif} := \{-1.0 + k \cdot h_i \ : \ 0 \le k \le 2^{i-1}\}, \quad 2 \le i \le 4.$$

In the middle row we use the extrema of the i-th Chebyshev polynomial. These can be computed to be

$$x_{i,k}^{CC} = -\cos\left(\frac{\pi \cdot (k-1)}{N_i - 1} \right), \quad 1 \le k \le N_i, 2 \le i \le 4$$

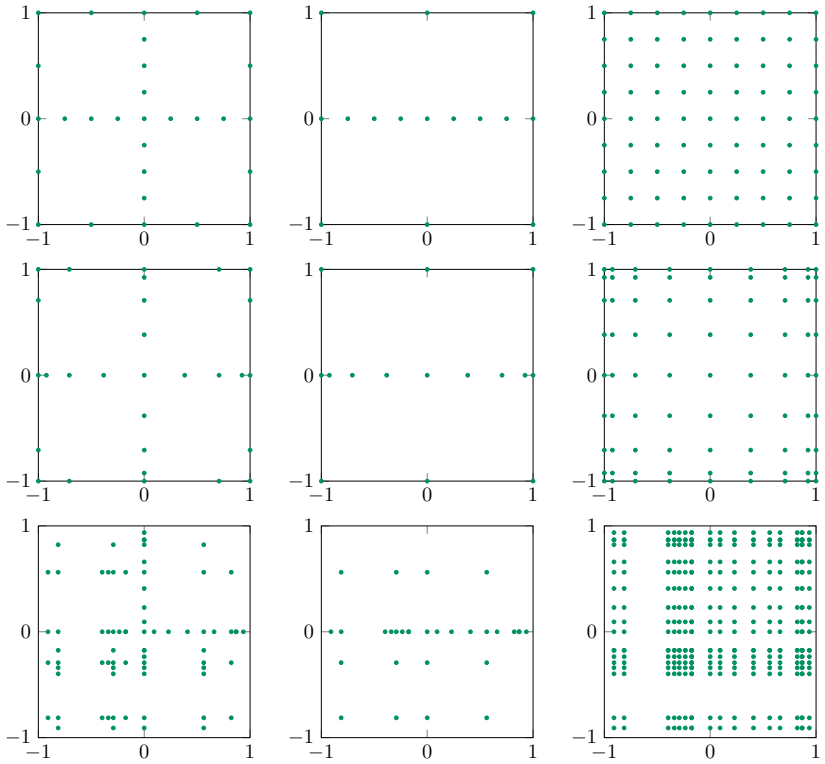

FIGURE 4. Examples of sparse grids for the index sets of Fig. 3 for uniformly distributed points (top row), Clenshaw-Curtis points (middle row) and i.i.d. drawn points (bottom row).

yielding the *Clenshaw-Curtis* points

$$X_i^{CC} := \{x_{i,k}^{CC} \ : \ 1 \leq k \leq N_i\}.$$

In the bottom row we have drawn N_i independent and identically distributed points using the C++ -class `std::uniform_real_distribution<>` to obtain the point sets (X_i^{rdm}).

The first two families, (X_i^{unif}) and (X_i^{CC}), are examples for nested point sets. Another often used example for a nested point set are the *Leja-points* defined recursively in the following way, see, e.g., [53]: Define $X_1^{Leja} = \{x_0\}$ for a given point x_0, and for $i = 2, 3, \ldots$, define the Leja point set X_i^{Leja} by

$$X_i^{Leja} = X_{i-1}^{Leja} \cup \left\{ x_i \ : \ x_i = \operatorname*{argmax}_{x \in [-1,1]} \prod_{k=0}^{i-1} |x - x_k|, x_k \in X_{i-1}^{Leja} \right\}.$$

While the cardinality of uniformly distributed and Clenshaw-Curtis point sets doubles from i to $i+1$, the growth of the cardinality of the Leja point sets is just 1.

The point sets (X_i^{rdm}) are with high probability not nested. This leads to sparse grids with higher numbers of points.

For each if these families of point sets we see the influence of the index set $\Lambda(2)$ on the sparse grid. In the first column of Fig. 4 we used the isotropic index set which leads to the *isotropic sparse grid*, which in turn is a special case of the *anisotropic sparse grid* in the second column. The grids in the third column do not exhibit any sparseness and are *full tensor product grids*. Although the underlying index sets fit the theory presented so far we do not study these kinds of product grids further and focus in Section 6.2 on anisotropic index sets.

In early works on Smolyak methods, see, e.g., [11, 30], give error estimates in terms of the fill distance of the sparse grid, $h_{H_{\Lambda(d)}, \times_{j=1}^d \Omega^{(j)}}$, to allow a comparison to already known, classical results in approximation theory. However, the development went away from it, as this quantity is not suitable to represent sparse grids. In more recent works error estimates are given in the total number of points $\#H_{\Lambda(d)}$ of the sparse grid, see, e.g., [16, 43, 69]. This has also drawbacks. Primarily, we lose information about the potential anisotropy of the problem as there are many sparse grids with the same number of points but completely different families of direction-wise point sets $(X_k^{(j)})$ or with different index sets $\Lambda(d)$.

Additionally, for given $X_i^{(j)}$, $1 \leq j \leq d$, $i \in \mathbb{N}$, and monotone index set $\Lambda(d)$ we know only for very few exceptions the exact number of distinct points in the corresponding sparse grid $H_{\Lambda(d)}$. This makes finding an answer to the question which sparse grid to use to achieve a certain error bound hard.

Nevertheless, we give now two examples where we do know the exact value of $\#H_{\Lambda(d)}$.

First, if $\Lambda(d)$ is the full product index set and therefore $H_{\Lambda(d)}$ the full tensor product grid, the cardinality is simply

$$\#H_{\Lambda(d)} = \prod_{j=1}^{d} \sum_{\lambda_j=1}^{\lambda_{j,max}} \left(\#X_{\lambda_j}^{(j)} - \#X_{\lambda_j-1}^{(j)} \right).$$

We used again the convention that $X_0^{(j)} = \emptyset$. This means that the cardinality of $H_{\Lambda(d)}$ grows exponentially with the number of directions d. We reiterate that this set plays a subordinate role.

The second example is more interesting. If $\Lambda(d)$ is the isotropic index set and the families of low-dimensional point sets $(X_i^{(j)})_{i\in\mathbb{N}}$ are independent of the directions $1 \le j \le d$ and nested, i.e., $X_i \subset X_{i+1}$, we know by [78, Section 3.3] that the cardinality of the corresponding isotropic sparse grid can be computed according to the following theorem.

Theorem 6.1.15. *Let $\Lambda(d)$ be an isotropic index set with uniform largest possible entry in any component of any multi-index λ_{max}. Then the cardinality of the sparse grid $H_{\Lambda(d)}$ constructed with a nested, uniform family of low-dimensional point sets $(X_i)_{i\in\mathbb{N}}$ satisfies*

$$\#H_{\Lambda(d)} = \sum_{(h_1,\dots,h_{\lambda_{max}})\in J_{\lambda_{max},d}} \frac{d!}{h_1! \cdot \dots \cdot h_{\lambda_{max}}!} \prod_{\lambda=1}^{\lambda_{max}} (\#X_\lambda - \#X_{\lambda-1})^{h_\lambda},$$

where we set $\#X_0 = 0$ and used the index set

$$J_{m,d} = \left\{ \boldsymbol{h} \in \{0,\dots,d\}^m \,:\, \|\boldsymbol{h}\|_1 = d \text{ and } \sum_{n=2}^{m} (n-1)h_n \le m-1 \right\}.$$

Although this theorem gives a formula how to compute the exact number of $\#H_{\Lambda(d)}$ it is only valid for a very restricted case. We note that in the setting Theorem 6.1.15 and even in the case of non-nested, direction independent X_i there is a more easily computable upper bound on $\#H_{\Lambda(d)}$, see, e.g., [70, Lemma 15.5]. For more general monotone index sets or non-uniform point sets we know of no analogue.

6.1.4. Selected Properties of $\mathcal{A}_{\Lambda(d)}$. One important aspect of the construction of the Smolyak algorithm is that it carries over certain nice properties of the low-dimensional approximation operators.

The first of these properties we want to study in more detail is exactness. We recall that we denote the embedding operator by ι.

Definition 6.1.16. *Let V and S be linear spaces. Let $\Pi \subseteq V$ be a subspace. We call the operator $A : V \to S$* exact on Π, *if*

$$A(f) = \iota(f), \quad f \in \Pi.$$

We see that if the family of operators $(A_i^{(j)})_{i\in\mathbb{N}}$, $1 \le j \le d$, is exact then the Smolyak operator is also exact in a specific subspace of $\bigotimes_{j=1}^{d} V^{(j)}$.

Theorem 6.1.17. *Let $V^{(1)},\dots,V^{(d)}$ and $S^{(1)},\dots,S^{(d)}$ be linear spaces with algebraic tensor product spaces $\bigotimes_{j=1}^{d} V^{(j)}$ and $\bigotimes_{j=1}^{d} S^{(j)}$. Let $(\Pi_i^{(j)})_{i\in\mathbb{N}} \subseteq$*

$V^{(j)}$, $1 \leq j \leq d$, be a sequence of nested subspaces, i.e.,

$$\emptyset := \Pi_0^{(j)} \subseteq \Pi_1^{(j)} \subseteq \Pi_2^{(j)} \subseteq \cdots \subseteq V^{(j)} \quad 1 \leq j \leq d.$$

Assume that, for $1 \leq j \leq d$ and $i \in \mathbb{N}$, the operator $A_i^{(j)} : V^{(j)} \to S^{(j)}$ is exact on $\Pi_i^{(j)}$. Then the Smolyak operator $\mathcal{A}_{\Lambda(d)} : \bigotimes_{j=1}^d V^{(j)} \to \bigotimes_{j=1}^d S^{(j)}$ is exact on $\sum_{\boldsymbol{\lambda} \in \Lambda(d)} \Pi_{\lambda_1}^{(1)} \otimes \cdots \otimes \Pi_{\lambda_d}^{(d)}$, i.e.,

$$\mathcal{A}_{\Lambda(d)}(\boldsymbol{f}) = \boldsymbol{\iota}(\boldsymbol{f}), \quad \boldsymbol{f} \in \sum_{\boldsymbol{\lambda} \in \Lambda(d)} \Pi_{\lambda_1}^{(1)} \otimes \cdots \otimes \Pi_{\lambda_d}^{(d)}.$$

The operator $\boldsymbol{\iota} = \bigotimes_{j=1}^d \iota^{(j)} : \bigotimes_{j=1}^d V^{(j)} \to \bigotimes_{j=1}^d S^{(j)}$ denotes the tensor product embedding operator.

Proof. The main idea of the proof is to use the recursion identity (6.1.9) and to do an induction over the number of directions d. Since all operators involved are linear it suffices to show exactness for elementary tensors $\boldsymbol{f} = f^{(1)} \otimes \cdots \otimes f^{(d)}$. Assume there is a multi-index $\boldsymbol{\lambda}^*$ such that $\boldsymbol{f} \in \bigotimes_{j=1}^d \Pi_{\lambda_j^*}^{(j)}$. Hence, for $1 \leq j \leq d$, we have

$$A_{\lambda_j^*}^{(j)}(f^{(j)}) = \iota^{(j)}(f^{(j)}), \quad f^{(j)} \in \Pi_{\lambda_j^*}^{(j)},$$

since we assume that $A_{\lambda_j^*}^{(j)}$ is exact on $\Pi_{\lambda_j^*}^{(j)}$. We now need to show that if $\boldsymbol{\lambda}^* \in \Lambda(d)$ then the Smolyak operator $\mathcal{A}_{\Lambda(d)}$ is exact on $\bigotimes_{j=1}^d \Pi_{\lambda_j^*}^{(j)}$.

First, consider $d = 1$. We have $\mathcal{A}_{\Lambda(1)} = A_{\lambda_{1,max}}^{(1)}$ and $\lambda_{1,max} \geq \lambda_1^*$. The nestedness of $\Pi_k^{(1)}$. in particular $\Pi_{\lambda_1^*}^{(1)} \subseteq \Pi_{\lambda_{1,max}}^{(1)}$, yields for $f^{(1)} \in \Pi_{\lambda_1^*}^{(1)}$

$$\mathcal{A}_{\Lambda(1)}(f^{(1)}) = A_{\lambda_{1,max}}^{(1)}(f^{(1)}) = A_{\lambda_1^*}^{(1)}(f^{(1)}) = \iota^{(1)}(f^{(1)}).$$

For the induction step $d - 1 \rightsquigarrow d$ we recall the recursion formula of the Smolyak operator

$$\mathcal{A}_{\Lambda(d)} = \sum_{k=1}^{\lambda_{d,max}} \mathcal{A}_{\widetilde{\Lambda}(k,d-1)} \otimes (A_k^{(d)} - A_{k-1}^{(d)}). \tag{6.1.12}$$

In direction d we know that $A_{\lambda_d^*}^{(d)}$ is exact on $\Pi_{\lambda_d^*}^{(d)}$ and by the nestedness of the $\Pi_k^{(d)}$ we know that as long as $k - 1 \geq \lambda_d^*$ we have, for $f^{(d)} \in \Pi_{\lambda_d^*}^{(d)}$,

$$A_k^{(d)}(f^{(d)}) = A_{k-1}^{(d)}(f^{(d)}) = \iota^{(d)}(f^{(d)}).$$

This means that

$$A_k^{(d)}(f^{(d)}) - A_{k-1}^{(d)}(f^{(d)}) = 0$$

holds for $k - 1 \geq \lambda_d^*$. Hence, we can truncate the sum in (6.1.12) at λ_d^* and obtain

$$\mathcal{A}_{\Lambda(d)} = \sum_{k=1}^{\lambda_d^*} \mathcal{A}_{\widetilde{\Lambda}(k,d-1)} \otimes \left(A_k^{(d)} - A_{k-1}^{(d)}\right). \tag{6.1.13}$$

We assume that $\boldsymbol{\lambda}^* \in \Lambda(d)$. This means that $(\lambda_1^*, \dots, \lambda_{d-1}^*)^{\mathrm{T}} \in \widetilde{\Lambda}(\lambda_d^*, d-1)$. And by Proposition 6.1.5(1) this also implies that $(\lambda_1^*, \dots, \lambda_{d-1}^*)^{\mathrm{T}} \in \widetilde{\Lambda}(k, d-1)$ for every $k \leq \lambda_d^*$. Hence, the induction hypothesis guarantees that

$$\mathcal{A}_{\widetilde{\Lambda}(k,d-1)}(f^{(1)} \otimes \cdots \otimes f^{(d-1)}) = \left(\bigotimes_{j=1}^{d-1} \iota^{(j)} \right) (f^{(1)} \otimes \cdots \otimes f^{(d-1)}).$$

Putting this in the recursive representation of the Smolyak operator of (6.1.13) and applying it to $\boldsymbol{f} \in \bigotimes_{j=1}^{d} \Pi_{\lambda_j^*}^{(j)}$ yields

$$\begin{aligned}
\mathcal{A}_{\Lambda(d)} \boldsymbol{f} &= \sum_{k=1}^{\lambda_d^*} \mathcal{A}_{\widetilde{\Lambda}(k,d-1)}(f^{(1)} \otimes \cdots \otimes f^{(d-1)}) \otimes \left(A_k^{(d)} - A_{k-1}^{(d)} \right) (f^{(d)}) \\
&= \left(\bigotimes_{j=1}^{d-1} \iota^{(j)} \right) (f^{(1)} \otimes \cdots \otimes f^{(d-1)}) \otimes A_{\lambda_d^*}^{(d)}(f^{(d)}) \\
&= \left(\bigotimes_{j=1}^{d-1} \iota^{(j)} \right) (f^{(1)} \otimes \cdots \otimes f^{(d-1)}) \otimes \iota^{(d)}(f^{(d)}) \\
&= \boldsymbol{\iota}(\boldsymbol{f}).
\end{aligned}$$

□

The proof has been given for different contexts in several publications, see, e.g., [23, 69]. The proof given above follows [16], which also investigates the case of general index sets and operators.

Next we see that if the operators $A_i^{(j)}$ are interpolation operators on sets $X_i^{(j)}$, $1 \leq j \leq d$, $i \in \mathbb{N}$, the Smolyak method yields an interpolation operator on the corresponding sparse grid.

Theorem 6.1.18. *For $1 \leq j \leq d$ let $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, be a domain. Let $V^{(1)}(\Omega^{(1)}), \dots, V^{(d)}(\Omega^{(d)})$ and $S^{(1)}(\Omega^{(1)}), \dots, S^{(d)}(\Omega^{(d)})$ be linear spaces of functions such that for every $1 \leq j \leq d$ the inclusion $V^{(j)}, S^{(j)} \subseteq C(\Omega^{(j)})$ holds. Let $\bigotimes_{j=1}^{d} V^{(j)}(\Omega^{(j)})$ and $\bigotimes_{j=1}^{d} S^{(j)}(\Omega^{(j)})$ be the respective algebraic tensor product spaces. For $1 \leq j \leq d$ let $\iota^{(j)} : V^{(j)} \to S^{(j)}$ and $\boldsymbol{\iota} : \bigotimes_{j=1}^{d} V^{(j)} \to \bigotimes_{j=1}^{d} S^{(j)}$ be the respective embedding operators.*

For every $1 \leq j \leq d$ let $(X_i^{(j)})_{i \in \mathbb{N}} \subseteq \Omega^{(j)}$ be a nested sequence of sets of sites, i.e.,

$$\emptyset =: X_0^{(j)} \subseteq X_1^{(j)} \subseteq X_2^{(j)} \subseteq \cdots \subseteq \Omega^{(j)}.$$

Assume further that for every $1 \leq j \leq d$ and $i \in \mathbb{N}$ the operators $A_i^{(j)} : V^{(j)}(\Omega^{(j)}) \to S^{(j)}(\Omega^{(j)})$ are interpolation operators on $X_i^{(j)}$, i.e., they satisfy

$$A_i^{(j)}(f^{(j)})(\boldsymbol{x}^{(j)}) = \iota^{(j)}(f^{(j)})(\boldsymbol{x}^{(j)}), \quad f^{(j)} \in V^{(j)}(\Omega^{(j)}),\ \boldsymbol{x}^{(j)} \in X_i^{(j)}$$

Let $\Lambda(d) \subseteq \mathbb{N}^d$ be a monotone index set.

Then the Smolyak operator $\mathcal{A}_{\Lambda(d)} : \bigotimes_{j=1}^{d} V^{(j)}(\Omega^{(j)}) \to \bigotimes_{j=1}^{d} S^{(j)}(\Omega^{(j)})$ *satisfies*

$$\mathcal{A}_{\Lambda(d)}(\boldsymbol{f})(\boldsymbol{x}) = \boldsymbol{\iota}(\boldsymbol{f})(\boldsymbol{x}), \quad \boldsymbol{f} \in \bigotimes_{j=1}^{d} V^{(j)}(\Omega^{(j)})$$

for all $\boldsymbol{x} \in H_{\Lambda(d)}$*, the sparse grid defined in* (6.1.11)*, i.e.,* $\mathcal{A}_{\Lambda(d)}$ *is an interpolation operator on* $H_{\Lambda(d)}$*.*

Proof. The idea of the proof is similar to the one of Theorem 6.1.17. We again use the recursion identity of the Smolyak operator in (6.1.9) for an induction over the number of directions d.

For $d = 1$ we again have $\mathcal{A}_{\Lambda(1)} = A^{(1)}_{\lambda_{1,max}}$. This is by assumption an interpolation operator on the point set $X^{(1)}_{\lambda_{1,max}}$. Since we assumed the $X^{(j)}_k$ to be nested, we have $H_{\Lambda(1)} = X^{(1)}_{\lambda_{1,max}}$ and therefore the claim for $d = 1$.

For the induction step $d - 1 \rightsquigarrow d$ we use the recursion formula

$$\mathcal{A}_{\Lambda(d)} = \sum_{k=1}^{\lambda_{d,max}} \mathcal{A}_{\widetilde{\Lambda}(k,d-1)} \otimes \left(A^{(d)}_k - A^{(d)}_{k-1} \right).$$

And again, since all occurring operators are linear, we assume without restriction of generality that $\boldsymbol{f} = f^{(1)} \otimes \cdots \otimes f^{(d)} \in \bigotimes_{j=1}^{d} V^{(j)}(\Omega^{(j)})$ is an elementary tensor. We fix an $\boldsymbol{x}_* = \left(\boldsymbol{x}^{(1)}_*, \ldots, \boldsymbol{x}^{(d)}_* \right)^{\mathrm{T}} \in H_{\Lambda(d)}$ and find a $1 \leq \lambda^*_d \leq \lambda_{d,max}$ such that $\boldsymbol{x}^{(d)}_* \in X^{(d)}_{\lambda^*_d} \setminus X^{(d)}_{\lambda^*_d - 1}$. This λ^*_d is the index for which the point $\boldsymbol{x}^{(d)}_*$ occurs for the first time in the sequence $(X^{(d)}_k)_{k \in \mathbb{N}}$. The remaining $d - 1$ components of $\boldsymbol{x}_*$ satisfy

$$\left(\boldsymbol{x}^{(1)}_*, \ldots, \boldsymbol{x}^{(d-1)}_* \right)^{\mathrm{T}} \in H_{\widetilde{\Lambda}(\lambda^*_d, d-1)}.$$

Additionally, we have for all $\lambda^*_d \leq k \leq \lambda_{d,max} - 1$

$$\left(A^{(d)}_k - A^{(d)}_{k-1} \right) (f^{(d)})(\boldsymbol{x}^{(d)}_*) = 0,$$

since the operators are assumed to be interpolation operators.

Together with the induction hypothesis this then yields for the Smolyak operator

$$\begin{aligned}
&\mathcal{A}_{\Lambda(d)}(\boldsymbol{f})(\boldsymbol{x}_*) = \\
&= \sum_{k=\lambda^*_d}^{\lambda_{d,max}} \mathcal{A}_{\widetilde{\Lambda}(k,d-1)}(f^{(1)} \otimes \cdots \otimes f^{(d-1)})(\boldsymbol{x}^{(1)}_*, \ldots, \boldsymbol{x}^{(d-1)}_*) \cdot \\
&\qquad\qquad \cdot \left(A^{(d)}_k - A^{(d)}_{k-1} \right) (f^{(d)})(\boldsymbol{x}^{(d)}_*) \\
&= \left(\bigotimes_{j=1}^{d} \iota^{(j)} f^{(j)} \right) (\boldsymbol{x}^{(1)}_*, \ldots, \boldsymbol{x}^{(d-1)}_*) \cdot \sum_{k=1}^{\lambda^*_d} \left(A^{(d)}_k - A^{(d)}_{k-1} \right) (f^{(d)})(\boldsymbol{x}^{(d)}_*).
\end{aligned}$$

Resolving the telescope sum yields the claim. □

6.1.5. Approximation Error Estimates. We now derive a representation of the approximation error for the Smolyak operator. We can use this representation, together with the theory introduced in Section 5.3 and minimal assumptions, to obtain basic approximation error estimates.

Theorem 6.1.19. *Let $V^{(1)}, \dots, V^{(d)}$ and $S^{(1)}, \dots, S^{(d)}$ be linear spaces with algebraic tensor product spaces $\bigotimes_{j=1}^d V^{(j)}$ and $\bigotimes_{j=1}^d S^{(j)}$. For $1 \leq j \leq d$ let $\iota^{(j)} : V^{(j)} \to S^{(j)}$ and $\boldsymbol{\iota} : \bigotimes_{j=1}^d V^{(j)} \to \bigotimes_{j=1}^d S^{(j)}$ be the respective embedding operators. For every $1 \leq j \leq d$ and $k \in \mathbb{N}$ let $A_k^{(j)} : V^{(j)} \to S^{(j)}$ be linear operators. Let $\Lambda(d) \subseteq \mathbb{N}^d$ be a monotone index set. Then the error operator $\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)} : \bigotimes_{j=1}^d V^{(j)} \to \bigotimes_{j=1}^d S^{(j)}$ can be expressed as*

(6.1.14)
$$\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)} = \left(\iota^{(1)} - A^{(1)}_{\lambda_{1,max}}\right) \otimes \bigotimes_{j=2}^{d} \iota^{(j)} + \sum_{j=2}^{d} R(j) \otimes \iota^{(j+1)} \otimes \cdots \otimes \iota^{(d)}$$

where

$$R(j) = \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(j-1)} \bigotimes_{k=2}^{j-1} \Delta^{(k)}_{\widetilde{\lambda}_k} \otimes \left(\iota^{(j)} - A^{(j)}_{\lambda_j(\widetilde{\boldsymbol{\lambda}})}\right). \tag{6.1.15}$$

Proof. We use the representation of the Smolyak operator (6.1.10) and recall Proposition 6.1.12. We find for every $\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(i)$ a $\lambda_{i+1}(\widetilde{\lambda}) \in \{1, \dots, \lambda_{i+1,max}\}$ such that $\left(\widetilde{\boldsymbol{\lambda}}, \lambda_{i+1}(\widetilde{\lambda})\right)^{\mathrm{T}} \in \widetilde{\Lambda}(i+1)$.

We have with the multilinearity of the tensor product

$$\begin{aligned}
\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)} &= \boldsymbol{\iota} - \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(d-1)} \bigotimes_{j=1}^{d-1} \Delta^{(j)}_{\widetilde{\lambda}_j} \otimes A^{(d)}_{\lambda_d(\widetilde{\boldsymbol{\lambda}})} \\
&= \bigotimes_{j=1}^{d} \iota^{(j)} - \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(d-1)} \bigotimes_{j=1}^{d-1} \Delta^{(j)}_{\widetilde{\lambda}_j} \otimes \left(A^{(d)}_{\lambda_d(\widetilde{\boldsymbol{\lambda}})} - \iota^{(d)}\right) - \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(d-1)} \bigotimes_{j=1}^{d-1} \Delta^{(j)}_{\widetilde{\lambda}_j} \otimes \iota^{(d)} \\
&= \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(d-1)} \bigotimes_{j=1}^{d-1} \Delta^{(j)}_{\widetilde{\lambda}_j} \otimes \left(\iota^{(d)} - A^{(d)}_{\lambda_d(\widetilde{\boldsymbol{\lambda}})}\right) + \\
&\quad + \left(\bigotimes_{j=1}^{d-1} \iota^{(j)} - \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(d-1)} \bigotimes_{j=1}^{d-1} \Delta^{(j)}_{\widetilde{\lambda}_j}\right) \otimes \iota^{(d)} \\
&= \left(\iota^{(1)} - A^{(1)}_{\lambda_{1,max}}\right) \otimes \bigotimes_{j=2}^{d} \iota^{(j)} + \sum_{j=2}^{d} R(j) \otimes \iota^{(j+1)} \otimes \cdots \otimes \iota^{(d)},
\end{aligned}$$

with $R(j)$ as in (6.1.15). □

This representation of the error operator $\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)}$ is remarkable. It allows us to express the operator by a combination of the low-dimensional error operators $\iota^{(j)} - A_i^{(j)}$ and the difference operators $\Delta_i^{(j)}$ for $1 \leq j \leq d$ and suitable $1 \leq i \leq \lambda_{d,max}$. Additionally, we see the exact influence of each direction on the tensor product error operator.

With the theory introduced in Section 5.3 we obtain the following general error estimate.

Theorem 6.1.20. *For $1 \leq j \leq d$ let $(V^{(j)}, \|\cdot\|_{V^{(j)}})$ and $(S^{(j)}, \|\cdot\|_{S^{(j)}})$ be normed spaces. Let $\bigotimes_{j=1}^{d} V^{(j)}$ and $\bigotimes_{j=1}^{d} S^{(j)}$ be the respective algebraic tensor product spaces. Assume that these spaces are equipped with uniformly compatible crossnorms $\|\cdot\|_{V^{(1)}\otimes\cdots\otimes V^{(d)}}$ and $\|\cdot\|_{S^{(1)}\otimes\cdots\otimes S^{(d)}}$. For $1 \leq j \leq d$ let $\iota^{(j)} : V^{(j)} \to S^{(j)}$ and $\boldsymbol{\iota} : \bigotimes_{j=1}^{d} V^{(j)} \to \bigotimes_{j=1}^{d} S^{(j)}$ be the respective embedding operators. For $1 \leq j \leq d$ and $i \in \mathbb{N}$ let $A_i^{(j)} : V^{(j)} \to S^{(j)}$ be linear operators. Assume that the following estimates hold for all $1 \leq j \leq d$ with constants $B^{(j)}$, $C_1^{(j)}$, $C_2^{(j)}$, $D^{(j)} > 0$:*

$$\|\iota^{(j)}\|_{V^{(j)}\to S^{(j)}} \leq B^{(j)} \tag{6.1.16}$$

$$\|\iota^{(j)} - A_i^{(j)}\|_{V^{(j)}\to S^{(j)}} \leq C_1^{(j)} {D^{(j)}}^i, \quad i \geq 0 \tag{6.1.17}$$

$$\|\Delta_i^{(j)}\|_{V^{(j)}\to S^{(j)}} \leq C_2^{(j)} {D^{(j)}}^i, \quad i \geq 1. \tag{6.1.18}$$

Let $\Lambda(d) \subseteq \mathbb{N}^d$ be a monotone index set and $\mathcal{A}_{\Lambda(d)} : \bigotimes_{j=1}^{d} V^{(j)} \to \bigotimes_{j=1}^{d} S^{(j)}$ be the Smolyak operator.

Then estimate for the tensor product error operator

$$\|\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)}\|_{\bigotimes_{j=1}^{d} V^{(j)}\to\bigotimes_{j=1}^{d} S^{(j)}} \leq \left(C_1^{(1)} \prod_{j=2}^{d} B^{(j)}\right) \left(D^{(1)}\right)^{\lambda_{1,max}} +$$

$$+ \sum_{j=2}^{d} \left(\left(\prod_{k=2}^{j-1} C_2^{(k)}\right) C_1^{(j)} \prod_{k=j+1}^{d} B^{(j)}\right) \sum_{\widetilde{\boldsymbol{\lambda}}\in\widetilde{\Lambda}(j-1)} \left(\prod_{k=2}^{j-1} \left(D^{(k)}\right)^{\widetilde{\lambda}_k}\right) \left(D^{(j)}\right)^{\lambda_j(\widetilde{\boldsymbol{\lambda}})}$$

holds.

Proof. In this proof we omit the indices of the occurring norms. We use the representation of the tensor product error $\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)}$ in (6.1.14). By assumption the crossnorms on $\bigotimes_{j=1}^{d} V^{(j)}$ and $\bigotimes_{j=1}^{d} S^{(j)}$ are uniformly compatible. This means that we have, with the help of the triangle inequality

$$\|\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)}\| \leq \|\iota^{(1)} - A^{(1)}_{\lambda_{1,max}}\| \cdot \prod_{j=2}^{d} \|\iota^{(j)}\| + \sum_{j=2}^{d} \|R(j)\| \cdot \prod_{k=j+1}^{d} \|\iota^{(k)}\|.$$

To bound $\|R(j)\|$ we use the representation of $R(j)$ in (6.1.15) to obtain

$$\|R(j)\| \leq \sum_{\widetilde{\boldsymbol{\lambda}}\in\widetilde{\Lambda}(j-1)} \prod_{k=2}^{j-1} \|\Delta^{(k)}_{\widetilde{\lambda}_k}\| \cdot \left\|\iota^{(j)} - A^{(j)}_{\lambda_j(\widetilde{\boldsymbol{\lambda}})}\right\|.$$

Hence, we have with the assumptions (6.1.16), (6.1.17) and (6.1.18)

$$\|\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)}\| \leq C_1^{(1)} \left(D^{(1)}\right)^{\lambda_{1,max}} \cdot \prod_{j=2}^{d} B^{(j)} +$$

$$+ \sum_{j=2}^{d} \sum_{\widetilde{\boldsymbol{\lambda}}\in\widetilde{\Lambda}(j-1)} \left(\prod_{k=2}^{j-1} C_2^{(k)} \left(D^{(k)}\right)^{\widetilde{\lambda}_k}\right) C_1^{(j)} \left(D^{(j)}\right)^{\widetilde{\lambda}_j(\widetilde{\boldsymbol{\lambda}})} \prod_{k=j+1}^{d} B^{(k)}.$$

Rearranging the right-hand side yields the claim. □

We note that the assumptions in (6.1.16), (6.1.17) and (6.1.18) are reasonable. (6.1.16) means that for all $1 \leq j \leq d$ the embedding operators $\iota^{(j)}$ are continuous. The assumption (6.1.17) means that the operators $A_i^{(j)}$, $1 \leq j \leq d$, $i \in \mathbb{N}$, approximate the corresponding embeddings. Clearly, we are only interested in operators $A_i^{(j)}$ with $D^{(j)} < 1$.

In this most general setting the error estimate of the preceding theorem is hard to interpret. However, depending on the application the constants $C_1^{(j)}$, $C_2^{(j)}$ and $D^{(j)}$ do not explicitly depend on the direction j. Furthermore, we often work with function spaces which have a canonical embedding, i.e., $B^{(j)} = B = 1$. In this case, the estimate of Theorem 6.1.20 reduces to the following corollary.

Corollary 6.1.21. *With the assumptions and notation of Theorem 6.1.20 we additionally assume that the constants $C_1^{(j)}$, $C_2^{(j)}$ and $D^{(j)}$ to be independent of the direction j, that $B^{(j)} = 1$, $1 \leq j \leq d$ and that $D < 1$. Then the error estimate*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)}\| \leq C_1 \max(1, C_2^{d-2}) d D^{d \min(\lambda_{1,max}, \ldots, \lambda_{d,max})} \#\Lambda(d).$$

holds.

Proof. We use the error estimate of Theorem 6.1.20 and obtain

$$\begin{aligned}
&\|\boldsymbol{\iota} - \mathcal{A}_{\Lambda(d)}\| \leq \\
&\leq C_1 D^{\lambda_{1,max}} + \sum_{j=2}^{d} C_2^{j-2} C_1 \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(j-1)} D^{\widetilde{\lambda}_2 + \cdots + \widetilde{\lambda}_{j-1}} D^{\lambda_j(\widetilde{\boldsymbol{\lambda}})} \\
&\leq C_1 D^{\lambda_{1,max}} + \sum_{j=2}^{d} C_2^{j-2} C_1 \sum_{\widetilde{\boldsymbol{\lambda}} \in \widetilde{\Lambda}(j-1)} D^{(j-1) \min(\lambda_{2,max}, \ldots, \lambda_{j,max})} \\
&\leq C_1 D^{\lambda_{1,max}} + C_1 \max(1, C_2^{d-2}) D^{(d-1) \min(\lambda_{2,max}, \ldots, \lambda_{d,max})} (d-1) \#\Lambda(d) \\
&\leq C_1 \max(1, C_2^{d-2}) d D^{d \min(\lambda_{1,max}, \ldots, \lambda_{d,max})} \#\Lambda(d).
\end{aligned}$$

□

In contrast to the error bound obtained in Theorem 6.1.20 the estimate in Corollary 6.1.21 is far easier to interpret. However, we only obtain a convergence order of $d \min(\lambda_{1,max}, \ldots, \lambda_{d,max})$. This can be understood in the sense that only the worst low-dimensional approximation determines the convergence order and the approximations in the other directions do not matter at all.

The convergence results in Theorem 6.1.20 and Corollary 6.1.21 can be seen as two extremes. The former retains as much information of the problem as possible but is hard to interpret while the latter gives a closed form but loses generality. We have to keep this in mind in the convergence results we give later and find a way to moderate these two extremal cases.

6.2. Anisotropic Index Set

We now study the anisotropic index set in more detail. This special set got a lot of attention in the most recent publications, see, e.g., [40, 43, 71] and is the index set we want to use for the rest of this thesis. We already mentioned this set as an example of monotone sets in Section 6.1.2 and showed an example in Fig. 3.

6.2.1. Definition and First Properties. We start with giving the definition of the anisotropic index set and then derive alternative representations.

Definition 6.2.1. *For given* weight vector $\boldsymbol{\omega} \in \mathbb{R}_+^d$ *and* threshold $\ell \in \mathbb{N}$ *we define the* anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \subseteq \mathbb{N}^d$ *by*

$$\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) := \left\{ \boldsymbol{\lambda} \in \mathbb{N}^d \ : \ \sum_{j=1}^{d} (\lambda_j - 1)\omega_j \leq \ell \min_{1 \leq j \leq d} \omega_j \right\}. \tag{6.2.1}$$

The vector $\boldsymbol{\omega}$ allows us to assign different importance to different directions. The larger ω_j the less important direction j is for us and $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ is less extended in this direction. In the special case that $\omega_1 = \omega_2 = \cdots = \omega_d$ the set is called *isotropic*. Without loss of generality we can assume that $\boldsymbol{\omega}$ is ascendingly ordered, i.e., $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_d$. We can always achieve this by permuting the single directions. It is also possible to normalize the weight vector such that, if we assume an ascendingly ordered $\boldsymbol{\omega}$, we have $\min_{1 \leq j \leq d} \omega_j = \omega_1 = 1$, This will not change the anisotropy, i.e., the quotient of the weight in direction j and direction k, but it will change the geometric shape of $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$.

We will use the following recursive representation of the anisotropic index set.

Proposition 6.2.2. *Let $\boldsymbol{\omega} \in \mathbb{R}_+^d$ be a weight vector and $\ell \in \mathbb{N}$ a threshold. Recall that $\lambda_{d,max}$ is defined by* (6.1.1)*. Set $\omega_{min} := \min_{1 \leq j \leq d} \omega_j$. Then we can express the anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ as*

$$\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) = \bigcup_{k=1}^{\lambda_{d,max}} \mathcal{I}_{\widetilde{\boldsymbol{\omega}}} \left(\ell - (k-1)\frac{\omega_d}{\omega_{min}}, d-1 \right) \times \{k\}, \tag{6.2.2}$$

with $\widetilde{\boldsymbol{\omega}} \in \mathbb{R}^{d-1}$ such that $\widetilde{\omega}_j = \omega_j$, $1 \leq j \leq d-1$.

Proof. We have

$$\begin{aligned}
\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) &= \left\{ \boldsymbol{\lambda} \in \mathbb{N}^d \ : \ \sum_{j=1}^{d} (\lambda_j - 1)\omega_j \leq \ell \omega_{min} \right\} \\
&= \left\{ \boldsymbol{\lambda} \in \mathbb{N}^d \ : \ \sum_{j=1}^{d-1} (\lambda_j - 1)\omega_j \leq \ell \omega_{min} - (\lambda_d - 1)\omega_d \right\} \\
&= \left\{ \boldsymbol{\lambda} \in \mathbb{N}^d \ : \ \sum_{j=1}^{d-1} (\lambda_j - 1)\omega_j \leq \left(\ell - (\lambda_d - 1)\frac{\omega_d}{\omega_{min}} \right) \omega_{min} \right\}.
\end{aligned}$$

We see that λ_d can only take values in $\{1, \dots, \lambda_{d,max}\}$. This then leads to

$$\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) = \left\{ \boldsymbol{\lambda} \in \mathbb{N}^d \ : \ \sum_{j=1}^{d} (\lambda_j - 1)\omega_j \leq \left(\ell - (\lambda_d - 1)\frac{\omega_d}{\omega_{min}} \right) \omega_{min} \right\}$$
$$= \bigcup_{k=1}^{\lambda_{d,max}} \left\{ \boldsymbol{\lambda} \in \mathbb{N}^{d-1} \ : \ \sum_{j=1}^{d-1} (\lambda_j - 1)\omega_j \leq \left(\ell - (k-1)\frac{\omega_d}{\omega_{min}} \right) \omega_{min} \right\} \times \{k\}.$$

□

To use the anisotropic index set of Definition 6.2.1 in the Smolyak operator we need to show that it is indeed a monotone set.

Proposition 6.2.3. *For every weight vector $\boldsymbol{\omega} \in \mathbb{R}^d_+$ and threshold $\ell \in \mathbb{N}$ the anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ is a monotone index set.*

Proof. We use Proposition 6.1.2 to check that for $\boldsymbol{\lambda} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ the multi-index $\boldsymbol{\lambda} - \boldsymbol{e}_j$, $1 \leq j \leq d$, is also an element of $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$. Take $\boldsymbol{\lambda} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ such that $\lambda_{j^*} \geq 2$ for a $1 \leq j^* \leq d$. Then $\boldsymbol{\lambda} - \boldsymbol{e}_{j^*}$ satisfies, with $\omega_{min} := \min_{1 \leq j \leq d} \omega_j$ and $\mathbf{1} := (1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^d$,

$$\sum_{k=1}^{d} (\boldsymbol{\lambda} - \boldsymbol{e}_{j^*} - \mathbf{1})_k \omega_k = \sum_{k=1}^{d} (\lambda_k - \delta_{kj^*} - 1)\omega_k \quad \leq \sum_{k=1}^{d} (\lambda_k - 1)\omega_k \leq \ell \omega_{min}.$$

Hence, $\boldsymbol{\lambda} - \boldsymbol{e}_{j^*} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$. □

Next, we can explicitly compute the largest possible j-th component $\lambda_{j,max}$.

Proposition 6.2.4. *For any weight vector $\boldsymbol{\omega} \in \mathbb{R}^d_+$ and threshold $\ell \in \mathbb{N}$ the largest possible value in the j-th component of any multi-index in the anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ can be expressed as*

$$\lambda_{j,max} = \left\lfloor \frac{\ell \omega_{min}}{\omega_j} \right\rfloor + 1, \tag{6.2.3}$$

with $\omega_{min} := \min_{1 \leq j \leq d} \omega_j$.

We see that if the weights in direction j and $j+1$, ω_j and ω_{j+1} differ not too much we can still have $\lambda_{j,max} = \lambda_{j+1,max}$. This means that those two directions are isotropic although the weights are not. This can lead to numerical artifacts. We will quantify how much ω_j and ω_{j+1} have to differ such that $\lambda_{j,max} \neq \lambda_{j+1,max}$ in Section 6.2.3.

6.2.2. Cardinality of $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$. The cardinality of the index set occurs in some error estimates for the Smolyak operator, e.g., in the estimate given in Corollary 6.1.21. We now give an estimate on $\#\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ for every $\boldsymbol{\omega} \in \mathbb{R}^d_+$ and $\ell \in \mathbb{N}$.

There are several papers, see, e.g., [40, 43, 71], which give bounds on $\#\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$. They all exhibit the similar behavior of ℓ^d as $\ell \to \infty$. However, the upper bound given in [43, Lemma 5.4] seems to be the sharpest one yet obtained. For this result it is essential that the weight vector is ordered.

Lemma 6.2.5. *Let $\boldsymbol{\omega} \in \mathbb{R}_+^d$ be an ascendingly ordered weight vector and let $\ell \in \mathbb{N}$. Then the cardinality of the anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ can be bounded by*

$$\#\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \leq \prod_{j=1}^{d} \left(\frac{\ell\omega_1}{j\omega_j} + 1 \right). \tag{6.2.4}$$

Proof. We use an induction over the number of directions d. We start with $d = 1$. Here, we have with (6.2.3)

$$\#\mathcal{I}_{\boldsymbol{\omega}}(\ell, 1) = \sum_{k=1}^{\lambda_{1,max}} 1 = \lambda_{1,max} = \ell + 1.$$

For the induction step $d - 1 \rightsquigarrow d$ we recall (6.2.2) and use that we can express the set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ by

$$\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) = \bigcup_{k=1}^{\lambda_{d,max}} \mathcal{I}_{\widetilde{\boldsymbol{\omega}}} \left(\ell - (k-1)\frac{\omega_d}{\omega_1}, d - 1 \right) \times \{k\},$$

with $\widetilde{\boldsymbol{\omega}} \in \mathbb{R}^{d-1}$ such that $\widetilde{\omega}_j = \omega_j$, $1 \leq j \leq d-1$. Hence, we have

$$\#\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) = \sum_{k=1}^{\lambda_{d,max}} \#\mathcal{I}_{\widetilde{\boldsymbol{\omega}}} \left(\ell - (k-1)\frac{\omega_d}{\omega_1}, d - 1 \right).$$

Inserting the induction hypothesis for $d - 1$ and some easy manipulation leads to

$$\begin{aligned}
\#\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) &\leq \sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} \left(\frac{(\ell - (k-1)\frac{\omega_d}{\omega_1})\omega_1}{j\omega_j} + 1 \right) \\
&= \prod_{j=1}^{d-1} \left(\frac{\ell\omega_1}{j\omega_j} + 1 \right) \sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} \frac{\frac{\ell\omega_1 - (k-1)\omega_d}{j\omega_j} + 1}{\frac{\ell\omega_1}{j\omega_j} + 1} \\
&= \prod_{j=1}^{d-1} \left(\frac{\ell\omega_1}{j\omega_j} + 1 \right) \sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} 1 - (k-1)\frac{\omega_d}{\ell\omega_1 + j\omega_j}. \qquad (6.2.5)
\end{aligned}$$

By assumption, the vector $\boldsymbol{\omega}$ is ascendingly ordered, i.e., $\omega_d \geq \omega_j$ for every $1 \leq j \leq d$. This yields for the sum in (6.2.5)

$$\begin{aligned}
&\sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} 1 - \frac{(k-1)\omega_d}{\ell\omega_1 + j\omega_j} \leq \\
&\qquad \leq \sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} 1 - (k-1)\frac{\omega_d}{\ell\omega_1 + j\omega_d} \\
&\qquad = \prod_{j=1}^{d-1} \left(\frac{\omega_d}{\ell\omega_1 + j\omega_d}\right) \sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} \left(\frac{\ell\omega_1 + j\omega_d}{\omega_d} - (k-1)\right) \\
&\qquad = \prod_{j=1}^{d-1} \left(\frac{\omega_d}{\ell\omega_1 + j\omega_d}\right) \sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} \left(j + \frac{\ell\omega_1}{\omega_d} - k + 1\right). \qquad (6.2.6)
\end{aligned}$$

Again, we need to bound the sum in (6.2.6). Here we use [43, Lemma 5.3], which states that for all $L \in \mathbb{N}$, $d \in \mathbb{N}$ and $\delta \in \mathbb{R}_+$ the estimate

$$\sum_{k=0}^{L-1} \prod_{j=1}^{d-1} (j + \delta + k) \leq \frac{1}{d} \prod_{j=0}^{d-1} (L + \delta + j)$$

holds. In our case we set $L := \lambda_{d,max}$ and $\delta := \frac{\ell\omega_1}{\omega_d} + 1 - L \geq 0$. Substituting $\widetilde{k} := L - k$ and changing the direction of summation yields

$$\begin{aligned}
\sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} \left(j + \frac{\ell\omega_1}{\omega_d} - (k-1)\right) &= \sum_{\widetilde{k}=0}^{L-1} \prod_{j=1}^{d-1} (j + \delta + \widetilde{k}) \\
&\leq \frac{1}{d} \prod_{j=0}^{d-1} (L + \delta + j).
\end{aligned}$$

Hence, we have in (6.2.6)

$$\begin{aligned}
\sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} 1 - \frac{(k-1)\omega_d}{\ell\omega_1 + j\omega_j} &\leq \prod_{j=1}^{d-1} \left(\frac{\omega_d}{\ell\omega_1 + j\omega_d}\right) \sum_{k=1}^{\lambda_{d,max}} \prod_{j=1}^{d-1} \left(j + \frac{\ell\omega_1}{\omega_d} - k + 1\right) \\
&\leq \prod_{j=1}^{d-1} \left(\frac{\omega_d}{\ell\omega_1 + j\omega_d}\right) \frac{1}{d} \prod_{j=0}^{d-1} (L + \delta + j) \\
&= \frac{1}{d} \prod_{j=1}^{d-1} \left(j + \frac{\ell\omega_1}{\omega_d}\right)^{-1} \prod_{j=0}^{d-1} (L + \delta + j) \\
&\leq \frac{1}{d} \prod_{j=1}^{d-1} (L + \delta + j)^{-1} \prod_{j=0}^{d-1} (L + \delta + j) \\
&= \frac{1}{d}(L + \delta) = \frac{\ell\omega_1}{d\omega_d} + \frac{1}{d} \\
&\leq \frac{\ell\omega_1}{d\omega_d} + 1.
\end{aligned}$$

Inserting this in (6.2.5) yields

$$\begin{aligned}\#\mathcal{I}_{\boldsymbol{\omega}}(\ell,d) &\leq \prod_{j=1}^{d-1}\left(\frac{\ell\omega_1}{j\omega_j}+1\right)\sum_{k=1}^{\lambda_{d,max}}\prod_{j=1}^{d-1}1-(k-1)\frac{\omega_d}{\ell\omega_1+j\omega_j}\\ &\leq \prod_{j=1}^{d-1}\left(\frac{\ell\omega_1}{j\omega_j}+1\right)\left(\frac{\ell\omega_1}{d\omega_d}+1\right)=\prod_{j=1}^{d}\left(\frac{\ell\omega_1}{j\omega_j}+1\right).\end{aligned}$$

□

For the isotropic case, i.e., $\omega_1 = \omega_2 = \cdots = \omega_d$ the bound in (6.2.4) becomes

$$\begin{aligned}\prod_{j=1}^{d}\left(\frac{\ell\omega_1}{j\omega_j}+1\right) &= \prod_{j=1}^{d}\left(\frac{\ell}{j}+1\right)=\frac{1}{d!}\prod_{j=1}^{d}(\ell+j)\\ &= \frac{(\ell+d)!}{d!\;\ell!}=\binom{\ell+d}{d},\end{aligned}$$

which is the exact number of elements in the isotropic index set, see, [43]. This means that the estimate (6.2.4) is sharp for isotropic index sets. The question arises naturally how much the given bound overestimates $\#\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ in the general, anisotropic setting.

We give the results of numerical tests for different choices of the weight vector $\boldsymbol{\omega}$. For every given d and choice for the weight vector we can generate the index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ and compare its cardinality to the bound (6.2.4). We tested three different kinds of weights.

For the first test we chose $\boldsymbol{\omega}\in\mathbb{R}_+^d$ such that $\omega_1 = 1$ and $\omega_{j+1} = \omega_j + 1$, $1 \leq j \leq d-1$. We plot the resulting value of the bound and the actual cardinality of $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ in Fig. 5 for $d = 2, 5, 7, 10$. We see that for small d the estimate is quite sharp however for growing d the bound starts to overestimate the actual number of multi-indices in $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ significantly. This can be explained by the fact that for small d the anisotropy in $\boldsymbol{\omega}$ is not yet important and also not large enough. However, for, e.g., $d = 10$, the weight for the first direction and the last differ a lot.

For the second example we chose $\boldsymbol{\omega}\in\mathbb{R}_+^d$ such that $\omega_1 = \cdots = \omega_{d-1} = 1$ and $\omega_d = d + 5$. This means the corresponding index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ is in nearly all directions isotropic. Investigating Fig. 6 we see that we have we an effect similar to the one in the first case but with the opposite result. For small d the anisotropy in $\boldsymbol{\omega}$ is large and hence the actual cardinality of $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ and the bound differ. As the number of directions grows the index set is nearly isotropic and therefore the bound given in (6.2.4) is almost sharp.

In Fig. 7 we give the plots for the third test case where we chose the weight vector $\boldsymbol{\omega}\in\mathbb{R}_+^d$ such that $\omega_1 = 1$ and $\omega_j = d + 5$ for $2 \leq j \leq d$. Here we can see that the estimate in (6.2.4) overestimates $\#\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ more and more as d grows.

Additionally, we have an interesting effect in the upper left plot in Fig. 6 and all four cases in Fig. 7. Apparently the growth of cardinality of $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ exhibits in some cases significant kinks. These can be explained if we recall

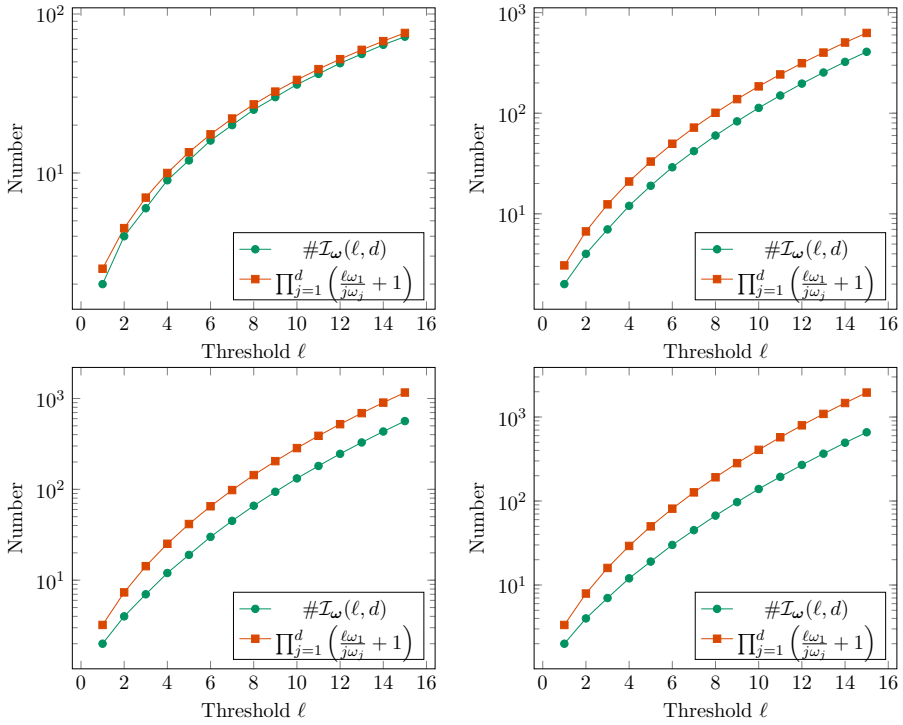

FIGURE 5. Plots of the actual number of elements in $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ and the upper bound given in (6.2.4) for $\boldsymbol{\omega}=(1,2,3,...,d)^{\mathrm{T}}$ as a function of ℓ and for different numbers of directions: $d=2$ (upper left), $d=5$ (upper right), $d=7$ (lower left) and $d=10$ (lower right).

that we can compute $\lambda_{j,max}$ by

$$\lambda_{j,max}=\left\lfloor\frac{\ell\omega_1}{\omega_j}\right\rfloor+1.$$

This term itself does not grow linearly in ℓ and most often the growth curve will be piece-wise constant. This then affects $\#\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ as we can see in the mentioned plots. For example, looking at the upper right diagram in Fig. 7, this is the case $d=5$, we see that the kink appears from $\ell=9$ going to $\ell=10$. We see that for $\ell=9$ and $\omega=1$ and $\omega_j=d+5=10$, $2\leq j\leq d$ we have

$$\lambda_{j,max}=\left\lfloor\frac{\ell\omega_1}{\omega_j}\right\rfloor+1=\left\lfloor\frac{9}{10}\right\rfloor+1=1$$

and for $\ell=10$

$$\lambda_{j,max}=2.$$

While $\lambda_{1,max}=\ell+1$ grows linearly with ℓ we see that for $2\leq j\leq d$ the corresponding $\lambda_{j,max}$ changes abruptly. The weight is chosen in such a way that this change happens in all directions simultaneously which leads to the described kinks.

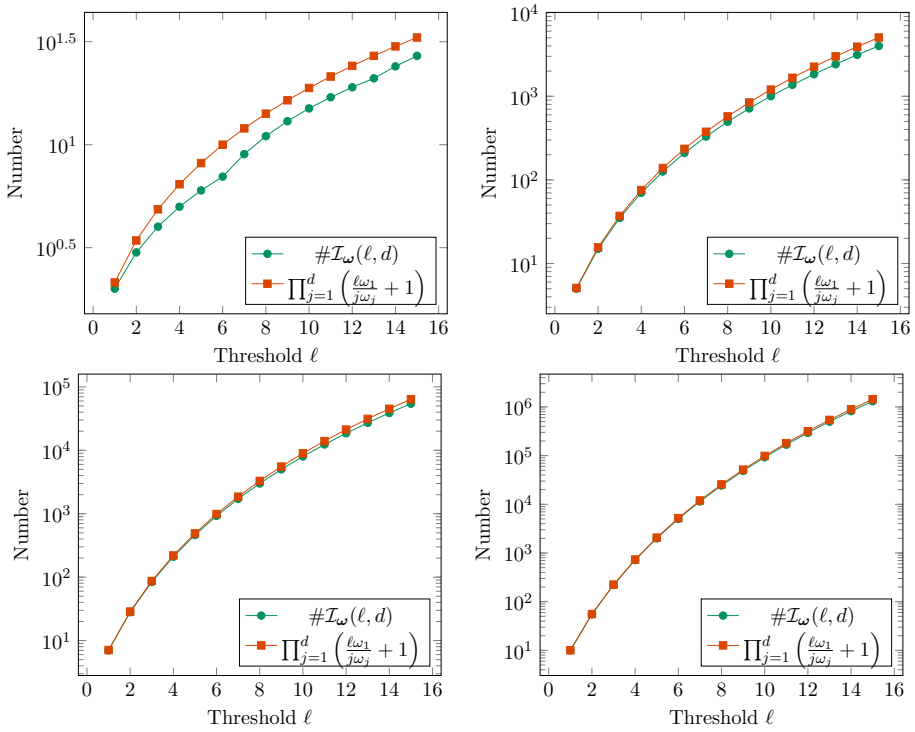

FIGURE 6. Plots of the actual number of elements in $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ and the upper bound given in (6.2.4) for $\boldsymbol{\omega}=(1,1,1,...,d+5)^{\mathrm{T}}$ as a function of ℓ and for different numbers of directions: $d=2$ (upper left), $d=5$ (upper right), $d=7$ (lower left) and $d=10$ (lower right).

6.2.3. Robustness towards Perturbation. From the discussion towards the end of the preceding subsection we see that the anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ exhibits some erratic behavior. This is mainly caused by $\lambda_{j,max}$, $1\leq j\leq d$. From

$$\lambda_{j,max}=\left\lfloor\frac{\ell\omega_1}{\omega_j}\right\rfloor+1$$

we see that small perturbations in the weight in direction j, ω_j, have no influence on the value of $\lambda_{j,max}$. We now derive bounds on these perturbations.

We fix a direction j and the weight $\omega_j\in\mathbb{R}_+$. For a perturbation $\tau\in\mathbb{R}$ set $\widetilde{\omega}_j:=\omega_j+\tau$. To derive bounds on τ such that

$$\lambda_{j,max}=\left\lfloor\frac{\ell\omega_1}{\omega_j}\right\rfloor+1=\left\lfloor\frac{\ell\omega_1}{\widetilde{\omega}_j}\right\rfloor+1=\widetilde{\lambda}_{j,max} \tag{6.2.7}$$

we have to distinguish whether $\tau\geq 0$ or $\tau\leq 0$.

If $\tau\geq 0$ we have $\omega_j\leq\widetilde{\omega}_j=\omega_j+\tau$. Hence, (6.2.7) is equivalent to

$$\left\lfloor\frac{\ell\omega_1}{\omega_j}\right\rfloor\leq\frac{\ell\omega_1}{\widetilde{\omega}_j}\leq\frac{\ell\omega_1}{\omega_j}.$$

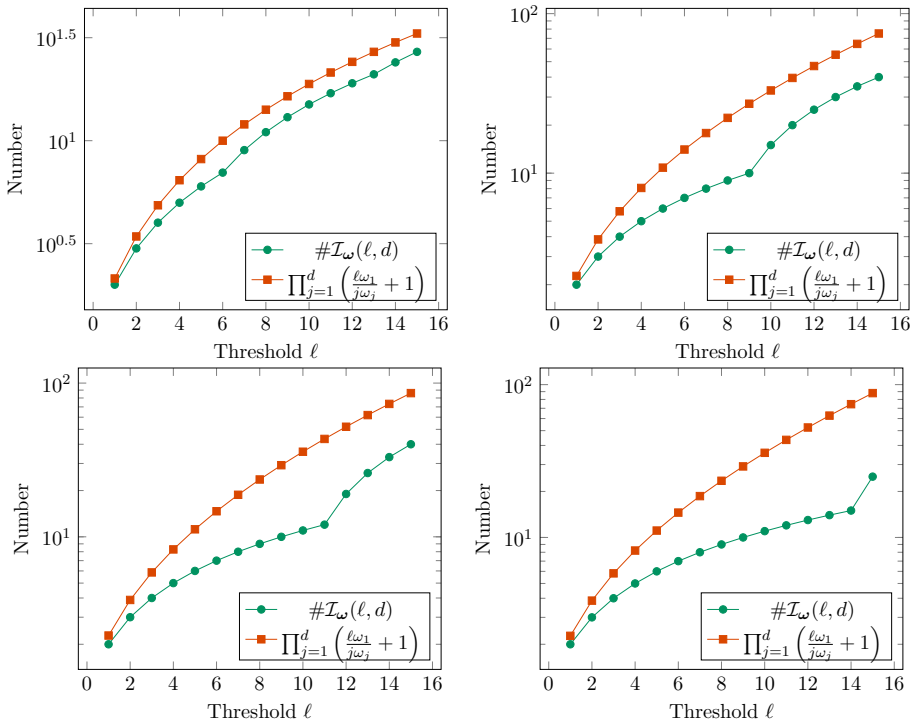

FIGURE 7. Plots of the actual number of elements in $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ and the upper bound given in (6.2.4) for $\boldsymbol{\omega}=(1,d+5,...,d+5)^{\mathrm{T}}$ as a function of ℓ and for different numbers of directions: $d=2$ (upper left), $d=5$ (upper right), $d=7$ (lower left) and $d=10$ (lower right).

This is in turn equivalent to

$$\left(\left\lfloor\frac{\ell\omega_1}{\omega_j}\right\rfloor\right)^{-1}\geq\frac{\widetilde{\omega}_j}{\ell\omega_1}\geq\frac{\omega_j}{\ell\omega_j},$$

or, in other words,

$$\begin{aligned}0\leq\tau=\widetilde{\omega}_j-\omega_j&\leq\left(\left\lfloor\frac{\ell\omega_1}{\omega_j}\right\rfloor\right)^{-1}\ell\omega_1-\omega_j\\&=\frac{\ell\omega_1}{\lambda_{j,max}-1}-\omega_j.\end{aligned}$$

Similarly, if $\tau\leq 0$ we have $\widetilde{\omega}_j=\omega_j+\tau\leq\omega_j$. In this case (6.2.7) is equivalent to

$$\frac{\ell\omega_1}{\omega_j}\leq\frac{\ell\omega_1}{\widetilde{\omega}_j}\leq\left\lfloor\frac{\ell\omega_1}{\omega_j}\right\rfloor+1.$$

With the same arguments as above we obtain

$$\begin{aligned}0\geq\tau=\widetilde{\omega}_j-\omega_j&\geq\left(\left\lfloor\frac{\ell\omega_1}{\omega_j}\right\rfloor+1\right)^{-1}\ell\omega_1-\omega_j\\&=\frac{\ell\omega_1}{\lambda_{j,max}}-\omega_j.\end{aligned}$$

We summarize this in the next lemma.

Lemma 6.2.6. *Let $\boldsymbol{\omega} \in \mathbb{R}_+^d$ be an ascendingly ordered weight vector and $\ell \in \mathbb{N}$. Let $1 \leq j^* \leq d$ be a fixed direction. Set $\widetilde{\omega}_{j^*} := \omega_{j^*} + \tau$, with a perturbation $\tau \in \mathbb{R}$. Then*

$$\lambda_{j^*,max} = \left\lfloor \frac{\ell\omega_1}{\omega_{j^*}} \right\rfloor + 1 = \left\lfloor \frac{\ell\omega_1}{\widetilde{\omega}_{j^*}} \right\rfloor + 1 = \widetilde{\lambda}_{j,max}$$

if and only if

$$\frac{\ell\omega_1}{\lambda_{j^*,max}} - \omega_{j^*} \leq \tau \leq \frac{\ell\omega_1}{\lambda_{j^*,max} - 1} - \omega_{j^*}.$$

Later we will couple the weight vector to the smoothness of the function we want to reconstruct. With this lemma we can make sure to correctly translate the anisotropy in the smoothness to the index set. Or, if a choice of weights is not possible, explain the numerical artifacts.

6.3. Anisotropic Smolyak Method

We now introduce the specific version of the Smolyak operator we want to use in Chapter 7. For this version we use the anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ of Section 6.2 in Definition 6.1.7.

Definition 6.3.1. *Let $V^{(1)}, \dots, V^{(d)}$ and $S^{(1)}, \dots, S^{(d)}$ be linear spaces with algebraic tensor product spaces $\bigotimes_{j=1}^d V^{(j)}$ and $\bigotimes_{j=1}^d S^{(j)}$. Let $\boldsymbol{\omega} \in \mathbb{R}_+^d$ be an ascendingly ordered weight vector and $\ell \in \mathbb{N}$. Let $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ be the anisotropic index set defined in Definition 6.2.1. For linear operators $A_k^{(j)} : V^{(j)} \to S^{(j)}$, $1 \leq j \leq d$, $k \in \mathbb{N}$, let $\Delta_k^{(j)} = A_k^{(j)} - A_{k-1}^{(j)}$ be the difference operators. Then the* anisotropic Smolyak operator $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : \bigotimes_{j=1}^d V^{(j)} \to \bigotimes_{j=1}^d S^{(j)}$ *is defined as the tensor product operator*

$$\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} := \sum_{\boldsymbol{\lambda} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} \boldsymbol{\Delta}_{\boldsymbol{\lambda}} := \sum_{\boldsymbol{\lambda} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} \bigotimes_{j=1}^d \Delta_{\lambda_j}^{(j)}. \tag{6.3.1}$$

In comparison to (6.1.7) we only replaced the general monotone index set $\Lambda(d)$ with the anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$. The changes are more significant if we investigate the analogue to the general combination technique respresentation in Proposition 6.1.9.

Proposition 6.3.2. *With the notation and assumptions of Definition 6.3.1 the anisotropic Smolyak operator $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$ can be expressed as*

$$\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} = \sum_{\boldsymbol{\lambda} \in \mathcal{J}_{\boldsymbol{\omega}}(\ell,d)} \sum_{\substack{\boldsymbol{\beta} \in \{0,1\}^d \\ \boldsymbol{\lambda} + \boldsymbol{\beta} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}} (-1)^{\|\boldsymbol{\beta}\|_1} \left(A_{\lambda_1}^{(1)} \otimes \cdots \otimes A_{\lambda_d}^{(d)} \right), \tag{6.3.2}$$

with the index set

$$\mathcal{J}_{\boldsymbol{\omega}}(\ell, d) := \mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \setminus \mathcal{I}_{\boldsymbol{\omega}} \left(\ell - \frac{\|\boldsymbol{\omega}\|_1}{\omega_1}, d \right). \tag{6.3.3}$$

PROOF. We can use (6.1.8) with the anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ and obtain the representation of the anisotropic Smolyak operator

$$\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} = \sum_{\boldsymbol{\lambda}\in\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} \sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^d \\ \boldsymbol{\lambda}+\boldsymbol{\beta}\in\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}} (-1)^{\|\boldsymbol{\beta}\|_1} \left(A^{(1)}_{\lambda_1} \otimes \cdots \otimes A^{(d)}_{\lambda_d}\right).$$

We now need to show that for all $\boldsymbol{\lambda} \in \mathcal{I}_{\boldsymbol{\omega}} \left(\ell - \frac{\|\boldsymbol{\omega}\|_1}{\omega_1}, d\right)$ the inner sum

$$\sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^d \\ \boldsymbol{\lambda}+\boldsymbol{\beta}\in\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}} (-1)^{\|\boldsymbol{\beta}\|_1} \left(A^{(1)}_{\lambda_1} \otimes \cdots \otimes A^{(d)}_{\lambda_d}\right) \tag{6.3.4}$$

is zero.

Hence, we assume that $\boldsymbol{\lambda}^* \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ is an element of $\mathcal{I}_{\boldsymbol{\omega}} \left(\ell - \frac{\|\boldsymbol{\omega}\|_1}{\omega_1}, d\right)$. This means that for any $\boldsymbol{\beta} \in \{0,1\}^d$ the multi-index $\boldsymbol{\lambda}^* + \boldsymbol{\beta}$ is an element of $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$. From combinatorics we see that the set $\{\boldsymbol{\beta} \in \{0,1\}^d\}$ has precisely 2^{d-1} elements with a $\|\cdot\|_1$-norm that is even and 2^{d-1} elements with an uneven $\|\cdot\|_1$-norm. Consequently,

$$\sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^d \\ \boldsymbol{\lambda}^*+\boldsymbol{\beta}\in\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}} (-1)^{\|\boldsymbol{\beta}\|_1} \left(A^{(1)}_{\lambda_1} \otimes \cdots \otimes A^{(d)}_{\lambda_d}\right) = 0,$$

which finishes the proof. □

In the case of the isotropic index set we can simplify this respresentation even further. Here, we can give an explicit expression for (6.3.4).

Proposition 6.3.3. *For the isotropic case, that is, if the weight vector $\boldsymbol{\omega} \in \mathbb{R}^d_+$ is given as $\boldsymbol{\omega} = (\omega,\dots,\omega)^T$, with $\omega > 0$, the* isotropic Smolyak operator *can be written as*

$$\mathcal{A}_{\mathcal{I}^{iso}_{\boldsymbol{\omega}}(\ell,d)} = \sum_{\boldsymbol{\lambda}\in\mathcal{J}^{iso}_{\boldsymbol{\omega}}(\ell,d)} (-1)^{\ell+d-\|\boldsymbol{\lambda}\|_1} \binom{d-1}{\ell+d-\|\boldsymbol{\lambda}\|_1} \left(A^{(1)}_{\lambda_1} \otimes \cdots \otimes A^{(d)}_{\lambda_d}\right)$$

with the index set

$$\mathcal{J}^{iso}_{\boldsymbol{\omega}}(\ell,d) = \left\{\boldsymbol{\lambda} \in \mathbb{N}^d \,:\, \ell+1 \le \|\boldsymbol{\lambda}\|_1 \le \ell+d\right\}.$$

The set $\mathcal{J}_{\boldsymbol{\omega}}(\ell,d)$ can be understood as the *surface* of the index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ in the sense that it contains all $\boldsymbol{\lambda} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ whose neighbor $\boldsymbol{\lambda}+\mathbf{1}$ is not in $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$. In Fig. 8 we give three illustrations of these surfaces.

The refinements of the results given in Section 6.1.4 are straight forward and we omit stating them here. We only restate the error representation of Theorem 6.1.19. With the special choice of $\Lambda(d) = \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ we can give an explicit representation of $\lambda_j(\widetilde{\boldsymbol{\lambda}})$. This result is well-known and can be found in, e.g., [43, 68].

Theorem 6.3.4. *Let $V^{(1)},\dots,V^{(d)}$ and $S^{(1)},\dots,S^{(d)}$ be linear spaces with algebraic tensor product spaces $\bigotimes_{j=1}^d V^{(j)}$ and $\bigotimes_{j=1}^d S^{(j)}$. For $1 \le j \le d$ let $\iota^{(j)} : V^{(j)} \to S^{(j)}$ and $\boldsymbol{\iota} : \bigotimes_{j=1}^d V^{(j)} \to \bigotimes_{j=1}^d S^{(j)}$ be the respective embedding operators. For every $1 \le j \le d$ and $k \in \mathbb{N}$ let $A^{(j)}_k : V^{(j)} \to S^{(j)}$ be linear operators. Let $\boldsymbol{\omega} \in \mathbb{R}^d_+$ be an ascendingly ordered weight vector and $\ell \in \mathbb{N}$. Let*

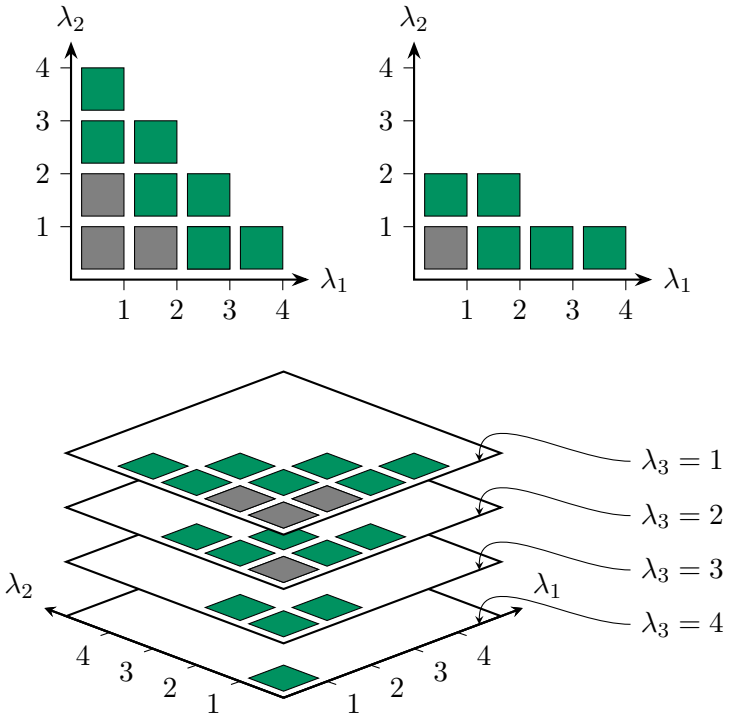

FIGURE 8. Examples of the surface $\mathcal{J}_{\boldsymbol{\omega}}(\ell, d)$ for different d and $\boldsymbol{\omega}$ in green: $\mathcal{J}_{\mathbf{1}}(3,2)$ (top left), $\mathcal{J}_{(1,2)^{\mathrm{T}}}(3,2)$ (top right) and $\mathcal{J}_{\mathbf{1}}(3,3)$ (bottom).

$\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \subseteq \mathbb{N}^d$ be the anisotropic index set defined in Definition 6.2.1. Then the error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : \bigotimes_{j=1}^{d} V^{(j)} \to \bigotimes_{j=1}^{d} S^{(j)}$ can be expressed as

$$\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} = \left(\iota^{(1)} - A^{(1)}_{\lambda_{1,max}}\right) \otimes \bigotimes_{j=2}^{d} \iota^{(j)} +$$

$$+ \sum_{j=2}^{d} \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)} \bigotimes_{k=1}^{j-1} \Delta^{(k)}_{\widetilde{\lambda}_k} \otimes \left(\iota^{(j)} - A^{(j)}_{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1}\right) \otimes \bigotimes_{k=j+1}^{d} \iota^{(k)} \tag{6.3.5}$$

where for $\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)$ the index $\lambda_j(\widetilde{\boldsymbol{\lambda}})$ is defined as

$$\lambda_j(\widetilde{\boldsymbol{\lambda}}) := \left\lfloor 2 + \ell \frac{\omega_1}{\omega_j} - \sum_{k=1}^{j-1} (\widetilde{\lambda}_k - 1) \frac{\omega_k}{\omega_j} \right\rfloor . \tag{6.3.6}$$

We will use this version of the error representation to show the convergence of the newly developed approximation method in Chapter 7.

6.4. Direction Adaptive Smolyak Method

Before we introduce this new method we discuss an adaptive version of the Smolyak method. In the construction of the anisotropic Smolyak method we have to a-priori choose the monotone index set $\Lambda(d)$ which greatly determines the performance of the corresponding Smolyak operator. A wrong choice

of this set, and especially the weight vector $\boldsymbol{\omega} \in \mathbb{R}_+^d$, leads either to a bad approximation or to an operator that is too expensive to compute numerically. In the setting of the anisotropic Smolyak operator, a way to determine the weight vector $\boldsymbol{\omega}$ is to inspect the problem, e.g., the smoothness of the target function $\boldsymbol{f}$, and choose the weights accordingly. Such a strategy has at least two drawbacks. First, we need information of $\boldsymbol{f}$ we may not have, and second, more generally, the class of anisotropic index sets itself may be inadequate for the given reconstruction problem. This can happen, e.g., if more or less multi-indices in mixed directions are required.

The basic idea of the direction adaptive Smolyak method was first introduced for sparse grid quadrature in [33]. We give a generalized version of the method in Algorithm 4 and discuss it in greater detail. To this end we define a special mapping which helps us to decide in which direction we have to adapt.

Definition 6.4.1. *Let $V^{(1)}, \dots, V^{(d)}$ be a linear space with (possibly complete) tensor product space $\bigotimes_{j=1}^d V^{(j)}$. For $\boldsymbol{f} \in \bigotimes_{j=1}^d V^{(j)}$ let* $\operatorname{error}_{\boldsymbol{f}} : \mathbb{N}^d \to \mathbb{R}$ *a decreasing and* $\operatorname{cost}_{\boldsymbol{f}} : \mathbb{N}^d \to \mathbb{R}$ *be an increasing mapping. Let $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For $\boldsymbol{f} \in \bigotimes_{j=1}^d V^{(j)}$ and* weight $\omega \in [0,1]$ *we define the* weighted error estimator $g_{\boldsymbol{f},\omega} : \mathbb{N}^d \to \mathbb{R}$ *by*

$$g_{\boldsymbol{f},\omega}(\boldsymbol{\lambda}) := q\left(\omega \operatorname{error}_{\boldsymbol{f}}(\boldsymbol{\lambda}), (1-\omega)\frac{1}{\operatorname{cost}_{\boldsymbol{f}}(\boldsymbol{\lambda})}\right). \tag{6.4.1}$$

The value $\operatorname{error}_{\boldsymbol{f}}(\boldsymbol{\lambda})$ in (6.4.1) should be chosen such that it represents the expected (relative) error of the chosen tensor product approximation method at the multi-index $\boldsymbol{\lambda} \in \mathbb{N}^d$. Similarly, $\operatorname{cost}_{\boldsymbol{f}}(\boldsymbol{\lambda})$ should expresses the (relative) cost to compute the approximation $\boldsymbol{\Delta}_{\boldsymbol{\lambda}}\boldsymbol{f}$ for $\boldsymbol{\lambda}$. The function q should moderate between these two values. A suitable choice is, e.g.,

$$q(\cdot,\cdot) = \max(\cdot,\cdot).$$

The weight $\omega \in [0,1]$ is introduced to moderate between comparatively too high work and comparatively too small error. Choosing $\omega = 1$ leads to a greedy approach, disregarding the cost completely. This should be the choice if the target function is very smooth and $\boldsymbol{f}$ can usually be approximated well. In this case the error estimate decays with increasing $\boldsymbol{\lambda}$ anyway. For $\omega = 0$ only the costs are included in the error estimate and this leads to a classical sparse grid approach [33].

The general procedure of Algorithm 4 is the following: We iteratively build a monotone index set $\mathfrak{A} \cup \mathfrak{O}$, where $\mathfrak{A}$ represents the *active indices* and $\mathfrak{O}$ the set of indices that are *offline*. In every iteration, these sets are disjoint and precisely one multi-index $\boldsymbol{\lambda}^* \in \mathfrak{A}$ goes offline, i.e., it gets removed from $\mathfrak{A}$ and is put into $\mathfrak{O}$. This is precisely the multi-index with the largest weighted error indicator $g_{\boldsymbol{f},\omega}(\boldsymbol{\lambda}^*)$. In the same iteration we add those neighbors of $\boldsymbol{\lambda}^*$, $\boldsymbol{\lambda}^* + \boldsymbol{e}_k$, $1 \le k \le d$, to $\mathfrak{A}$ such that, for all $1 \le j \le d$, the multi-index $\boldsymbol{\lambda}^* + \boldsymbol{e}_k - \boldsymbol{e}_j$ is already offline. This leads to the fact that the resulting index set $\mathcal{A} \cup \mathcal{O} \cup \{\boldsymbol{\lambda}^* + \boldsymbol{e}_k\}$ is monotone. Omitting the check if $\boldsymbol{\lambda}^* + \boldsymbol{e}_k - \boldsymbol{e}_j$ is already offline can, in the worst case, lead to non-monotone sets in later iterations.

Algorithm 4: Direction-adaptive Smolyak Algorithm

Data: Error tolerance ε, right-hand side $\boldsymbol{f}$, weighted error estimator $g_{\boldsymbol{f},\omega}$
Result: Adaptive Smolyak Approximation $\boldsymbol{f}_\varepsilon$
Initialize $\boldsymbol{\lambda} := (1,\dots,1)^{\mathrm{T}} \in \mathbb{N}^d$;
Initialize $\mathfrak{O} := \emptyset$;
Initialize $\mathfrak{A} := \{\boldsymbol{\lambda}\}$;
Initialize $\boldsymbol{f}_\varepsilon := \boldsymbol{\Delta}_{\boldsymbol{\lambda}} \boldsymbol{f}$;
Compute $g_{\boldsymbol{f},\omega}(\boldsymbol{\lambda})$;
Set $\eta := g_{\boldsymbol{f},\omega}(\boldsymbol{\lambda})$;
while $\eta > \varepsilon$ **do**
  Select $\boldsymbol{\lambda}^*$ from $\mathfrak{A}$ with largest $g_{\boldsymbol{f},\omega}(\boldsymbol{\lambda}^*)$;
  $\mathfrak{A} = \mathfrak{A} \setminus \{\boldsymbol{\lambda}^*\}$;
  $\mathfrak{O} = \mathfrak{O} \cup \{\boldsymbol{\lambda}^*\}$;
  $\eta = \eta - g_{\boldsymbol{f},\omega}(\boldsymbol{\lambda})$;
  for $j = 1,\dots,d$ **do**
    Set $\boldsymbol{\tau} := \boldsymbol{\lambda}^* + \boldsymbol{e}_j$;
    Compute $g_{\boldsymbol{f},\omega}(\boldsymbol{\tau})$;
    if $\boldsymbol{\tau} - \boldsymbol{e}_k \in \mathfrak{O}$ *for all* $k = 1,\dots,d$ **then**
      $\mathfrak{A} = \mathfrak{A} \cup \{\boldsymbol{\tau}\}$;
      $\boldsymbol{f}_\varepsilon = \boldsymbol{f}_\varepsilon + \boldsymbol{\Delta}_{\boldsymbol{\tau}} \boldsymbol{f}$;
      $\eta = \eta + g_{\boldsymbol{f},\omega}(\boldsymbol{\tau})$;
    end
  end
end

Adding and removing multi-indices increases and decreases the *global error estimate* $\eta \in \mathbb{R}$ by the value of the error indicator of the added or removed multi-index. The algorithm terminates if η is smaller than a given error tolerance ε.

Clearly, in every iteration the approximation $\boldsymbol{f}_\varepsilon$ has the form

$$\boldsymbol{f}_\varepsilon = \sum_{\boldsymbol{\lambda} \in \mathfrak{A} \cup \mathfrak{O}} \boldsymbol{\Delta}_{\boldsymbol{\lambda}} \boldsymbol{f} = \sum_{\boldsymbol{\lambda} \in \mathfrak{A} \cup \mathfrak{O}} \left(\bigotimes_{j=1}^{d} \Delta_{\lambda_j}^{(j)} \right) (\boldsymbol{f}).$$

In Fig. 9 we give a two-directional example of the iterative emergence of an monotone index set following the algorithm presented in Algorithm 4. The active indices are colored in green, the offline indices in gray and the multi-index with the largest error indicator is the one having the arrows attached, indicating the new candidates of multi-indices to be added to the active index set. We remark that in the example in the middle the index $(2,3)^{\mathrm{T}}$ will not be added to the active set since its neighbor $(2,2)^{\mathrm{T}}$ is not offline.

We emphasize that the resulting index set in the right-most picture is not an anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell,2)$. There is no combination of weight vector $\boldsymbol{\omega} \in \mathbb{R}^2_+$ and threshold $\ell \in \mathbb{N}$ such that $\mathcal{I}_{\boldsymbol{\omega}}(\ell,2)$ has this appearance.

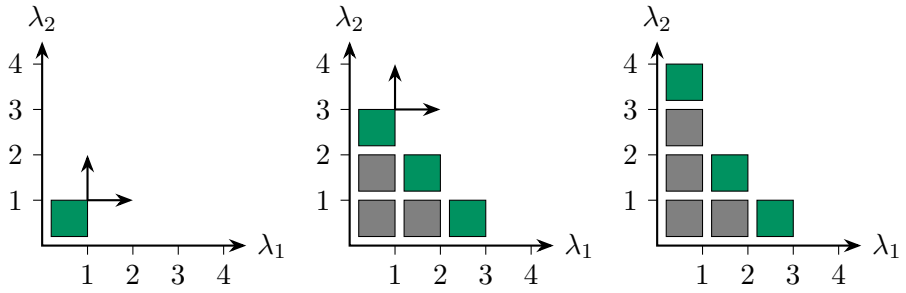

FIGURE 9. A few snapshots of the evolution of the direction-adaptive Smolyak algorithm. Active indices are green, offline indices are gray. The active indices with the arrows have the largest error indicators and are thus selected for insertion into the offline index set.

CHAPTER 7

The Tensor Product Multilevel Method

We are now in the position to introduce the tensor product multilevel method. This is a new method to compute approximations of high-directional target functions by combining the anisotropic Smolyak method of Section 6.3 with the kernel-based multilevel method introduced in Chapter 4.

In Section 7.1, after recalling the general setup, we introduce the new method in two different representations, depending on whether we have access to the direction-wise information or not. In Section 7.2 we then derive convergence estimates for different kinds of reconstruction processes, interpolation, penalized least squares and local Lagrange multilevel approximation. We give numerical examples that support these theoretical results in Section 7.3, at least for interpolation. To finish this chapter we introduce the idea of the double-adaptive tensor product multilevel method which not only constructs a monotone index set but also chooses the direction-wise point sets adaptively.

The ideas and results presented here are also part of [54] which is currently under review.

7.1. The Method and its Representations

We start by describing the new tensor product multilevel method by first repeating the setup, especially for the now direction-dependent kernel-based multilevel method, and then discussing the algorithmic structure of the new high-directional approximation method. We will see that it is advantageous to distinguish two cases. First, we assume that we have access to the direction-wise data $\left(\boldsymbol{x}_m^{(j)}, f^{(j)}(\boldsymbol{x}_m^{(j)})\right)_{1\leq m\leq N}$ and second, if we only know the high-directional data $(\boldsymbol{x}_m, \boldsymbol{f}(\boldsymbol{x}_m))_{1\leq m\leq N}$ without any further knowledge on eventually existing low-dimensional functions $f^{(j)}$ or no access to the values $f^{(j)}(\boldsymbol{x}_m^{(j)})$. We emphasize that in any case we have to have access to the components of $\boldsymbol{x}_m = (\boldsymbol{x}_m^{(1)}, \dots, \boldsymbol{x}_m^{(d)})^{\mathrm{T}}$.

First, we recall the notation introduced in the previous chapters and adapt it to fit the high-directional setting.

7.1.1. The General Setup. We have to assume that the domain $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, has cartesian product structure, i.e., there are low-dimensional domains $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, $1 \leq j \leq d$, such that

$$\Omega = \Omega^{(1)} \times \cdots \times \Omega^{(d)}.$$

The *effective dimension* of Ω is therefore $n = \sum_{j=1}^{d} n_j \geq d$. We use linear spaces of functions on these low-dimensional domains, $V^{(j)}(\Omega^{(j)}) \subseteq C(\Omega^{(j)})$,

$1 \leq j \leq d$, and denote the corresponding algebraic tensor product space as $V(\Omega) = \bigotimes_{j=1}^{d} V^{(j)}(\Omega^{(j)}) \subseteq C(\Omega)$.

The goal is to recover a function $\boldsymbol{f} \in V(\Omega)$ by only using point-wise information $(\boldsymbol{x}_m, \boldsymbol{f}(\boldsymbol{x}_m))_{1 \leq m \leq N} \subseteq \Omega \times \mathbb{R}$.

Next, we recall the setting of the kernel-based multilevel method of Chapter 4. Although the ideas stay basically the same the notation gets significantly more convoluted since the main terms are now all direction-dependent.

For each $1 \leq j \leq d$ and each $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$ we fix a sequence of discrete point sets $(X_i^{(j)})_{i \in \mathbb{N}} \subseteq \Omega^{(j)}$ with cardinality $\# X_i^{(j)} = N_i^{(j)}$. To be precise we assume that we can write $X_i^{(j)}$, $i \in \mathbb{N}$, as

$$X_i^{(j)} := \left\{ \boldsymbol{x}_{i,1}^{(j)}, \dots \boldsymbol{x}_{i,N_i^{(j)}}^{(j)} \right\}.$$

Associated to these point sets we have the fill distance and the separation radius, defined in (2.3.1) and (2.3.2), respectively. However, they are also direction-dependent, i.e., we write for every $1 \leq j \leq d$

$$h_i^{(j)} := h_{X_i^{(j)}, \Omega^{(j)}} \quad \text{and} \quad q_i^{(j)} := q_{X_i^{(j)}}, \quad i \in \mathbb{N}.$$

As before, we assume that the relation between the fill distances of the point sets of two consecutive levels can be expressed with a, now direction-dependent, refinement parameter $\mu^{(j)} \in (0,1)$. That means we assume that, for every $1 \leq j \leq d$, there is a constant $c^{(j)} \in (0,1]$ such that

$$c^{(j)} \mu^{(j)} h_i^{(j)} \leq h_{i+1}^{(j)} \leq \mu^{(j)} h_i^{(j)}, \quad i \in \mathbb{N}.$$

Next, we choose for every direction $1 \leq j \leq d$ a continuous, compactly supported radial basis function $\Phi^{(j)} : \mathbb{R}^{n_j} \to \mathbb{R}$. This mother kernel is then rescaled with a level- and direction-dependent scaling parameter $\delta_i^{(j)} > 0$ which is coupled to the mesh norm $h_i^{(j)}$ of the point set $X_i^{(j)}$ on level i. This means we have a direction-dependent overlap parameter $\nu^{(j)} > 1$ and a constant $c_{\nu^{(j)}} \in (0,1]$ such that

$$c_{\nu^{(j)}} \nu^{(j)} h_i^{(j)} \leq \delta_i^{(j)} \leq \nu^{(j)} h_i^{(j)}, \quad i \in \mathbb{N}.$$

This then leads to direction- and level-dependent rescaled kernels $\Phi_i^{(j)}$ defined by

$$\Phi_i^{(j)} := \left(\delta_i^{(j)}\right)^{-n_j} \Phi^{(j)} \left(\frac{\cdot}{\delta_i^{(j)}} \right), \quad i \in \mathbb{N}.$$

With these kernels we define the direction-dependent local approximation spaces as

$$W_i^{(j)} := \operatorname{span} \left\{ \Phi_i^{(j)}(\cdot - \boldsymbol{x}^{(j)}) \ : \ \boldsymbol{x}^{(j)} \in X_i^{(j)} \right\}, \quad i \in \mathbb{N}, \tag{7.1.1}$$

and the direction-dependent global approximation space up to a level $L^{(j)} \in \mathbb{N}$ by

$$V_{L^{(j)}}^{(j)} := W_1^{(j)} \oplus \cdots \oplus W_{L^{(j)}}^{(j)}. \tag{7.1.2}$$

Following the ideas of Chapter 4 further, we use local approximation operators

$$\mathcal{I}_i^{(j)} : V^{(j)}(\Omega^{(j)}) \to W_i^{(j)}, \quad i \in \mathbb{N}$$

and the multilevel approximation operator of level $L^{(j)}$ in direction j

$$A_{L^{(j)}}^{(j)} : V^{(j)}(\Omega^{(j)}) \to V_{L^{(j)}}^{(j)}. \tag{7.1.3}$$

The specific form of the local approximation operators $\mathcal{I}_i^{(j)}$ now not only depends on the approximation process we want to use but also on the type of information available. We will go into more detail in the next subsections.

Finally, for the Smolyak method introduced in Chapter 6, we use the anisotropic index set in d directions $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ with ascendingly ordered weight vector $\boldsymbol{\omega} \in \mathbb{R}_+^d$ and threshold $\ell \in \mathbb{N}$. Then the Smolyak operator is given as a mapping

$$\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : \bigotimes_{j=1}^{d} V^{(j)}(\Omega^{(j)}) \to V_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)},$$

where $V_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$ is an approximation space we have to characterize further. Following (6.1.7), the operator has the representation

$$\begin{aligned}\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} &:= \sum_{\boldsymbol{\lambda} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} \bigotimes_{j=1}^{d} \left(A_{\lambda_j}^{(j)} - A_{\lambda_j - 1}^{(j)} \right) \\ &= \sum_{\boldsymbol{\lambda} \in \mathcal{J}_{\boldsymbol{\omega}}(\ell,d)} \sum_{\substack{\boldsymbol{\beta} \in \{0,1\}^d \\ \boldsymbol{\lambda} + \boldsymbol{\beta} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}} (-1)^{\|\boldsymbol{\beta}\|_1} A_{\lambda_1}^{(1)} \otimes \cdots \otimes A_{\lambda_d}^{(d)},\end{aligned}$$

where

$$\mathcal{J}_{\boldsymbol{\omega}}(\ell, d) = \mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \setminus \mathcal{I}_{\boldsymbol{\omega}} \left(\ell - \frac{\|\boldsymbol{\omega}\|_1}{\omega_1}, d \right)$$

is the surface of $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ defined in (6.3.3).

The form of the direction-wise operators $A_i^{(j)}$ depends on the kind of information we have at our disposal. However, what we can do already is to characterize the tensor product approximation space of the method, the codomain of the operator.

Lemma 7.1.1. *For $1 \leq j \leq d$ let the local and global approximation space $W_i^{(j)}$ and $V_{L^{(j)}}^{(j)}$ be defined as in (7.1.1) and (7.1.2). Let $\boldsymbol{\omega} \in \mathbb{R}_+^d$ an ascendingly ordered weight vector and $\ell \in \mathbb{N}$. Let $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \subseteq \mathbb{N}^d$ be the anisotropic index set. Then the approximation space of the tensor product multilevel method is given by*

$$V_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} = \bigoplus_{\boldsymbol{\lambda} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} \bigotimes_{j=1}^{d} W_{\lambda_j}^{(j)}. \tag{7.1.4}$$

Proof. We have

$$V_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} = \bigotimes_{j=1}^{d} V_{\lambda_{j,max}}^{(j)} = \bigotimes_{j=1}^{d} \bigoplus_{i=1}^{\lambda_{j,max}} W_i^{(j)}.$$

The representation in (7.1.4) follows form the multilinearity of the tensor product. □

7.1.2. Simple Representation. To introduce the ideas of the tensor product multilevel method we first investigate the simple case, that is, we assume that we are given the direction-wise information $\left(\boldsymbol{x}_m^{(j)}, f^{(j)}(\boldsymbol{x}_m^{(j)})\right)$, $1 \leq m \leq N$ and $1 \leq j \leq d$, or evaluable functions $f^{(j)} \in V^{(j)}(\Omega^{(j)})$. The goal is still to reconstruct the high-directional function $\boldsymbol{f} \in \bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)})$.

Given this kind of data we can use the residual correction scheme we introduced in Chapter 4 for every direction $1 \leq j \leq d$. That means we set the residual $e_0^{(j)}$ on level 0 to $f^{(j)}$, $e_0^{(j)} = f^{(j)}$, and compute for $i = 1, 2, 3, \ldots$ the local approximation $\mathcal{I}_i^{(j)}(e_{i-1}^{(j)})$ to the residual of the previous level $e_{i-1}^{(j)}$ on the data sites on level i, $X_i^{(j)}$. The residual on level i is then given by

$$e_i^{(j)} = f^{(j)} - \sum_{k=1}^{i} \mathcal{I}_k^{(j)}(e_{k-1}^{(j)}).$$

The global approximation on level $L^{(j)} \in \mathbb{N}$ to $f^{(j)}$ is, following Section 4.2, then given as

$$A_{L^{(j)}}^{(j)}(f^{(j)}) = \sum_{i=1}^{L^{(j)}} \mathcal{I}_i^{(j)}(e_{i-1}^{(j)}). \tag{7.1.5}$$

These are the operators the Smolyak algorithm uses to build the tensor product operator $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$. To be more precise it uses the difference operators $\Delta_i^{(j)} = \left(A_i^{(j)} - A_{i-1}^{(j)}\right)$ which have, with (7.1.5), the form

$$\begin{aligned}\Delta_i^{(j)} = \left(A_i^{(j)} - A_{i-1}^{(j)}\right) &= \sum_{k=1}^{i} \mathcal{I}_k^{(j)}(e_{k-1}^{(j)}) - \sum_{k=1}^{i-1} \mathcal{I}_k^{(j)}(e_{k-1}^{(j)}) \\ &= \mathcal{I}_i^{(j)}(e_{i-1}^{(j)}). \end{aligned} \tag{7.1.6}$$

This is simply the local approximation operator of level i, applied to the residual of level $i-1$. This means that the tensor product multilevel method has the following easy representation.

Theorem 7.1.2. *For every $1 \leq j \leq d$ let $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, be a domain and let $V^{(j)}(\Omega^{(j)}) \subseteq C(\Omega^{(j)})$ be a linear space of functions on $\Omega^{(j)}$. Let $\boldsymbol{\omega} \in \mathbb{R}_+^d$ be an ascendingly ordered weight vector and $\ell \in \mathbb{N}$. Let $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ be the anisotropic index set defined in Definition 6.2.1 and let $\lambda_{j,max}$ be given by (6.2.3). For $1 \leq j \leq d$ and $1 \leq i \leq \lambda_{j,max}$ let $\mathcal{I}_i : V^{(j)}(\Omega^{(j)}) \to W_i^{(j)}$ be local approximation operators and let $A_i^{(j)} : V^{(j)}(\Omega^{(j)}) \to V_i^{(j)}$ be the multilevel operators given in (7.1.5). Assume that $\boldsymbol{f} \in \bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)})$ has a representation*

$$\boldsymbol{f} = f^{(1)} \otimes \cdots \otimes f^{(d)}.$$

Then the tensor product multilevel operator $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : \bigotimes_{j=1}^{d} V^{(j)}(\Omega^{(j)}) \to V_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$ *is given by*

$$\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}(\boldsymbol{f}) = \sum_{\boldsymbol{\lambda}\in\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} \bigotimes_{j=1}^{d} \mathcal{I}_{\lambda_j}^{(j)}(e_{\lambda_j-1}^{(j)}). \tag{7.1.7}$$

Clearly, all operators in the theorem above are linear. That means we can extend the result to functions $\boldsymbol{f} \in \bigotimes_{j=1}^{d} V^{(j)}(\Omega^{(j)})$ that have a representation

$$\boldsymbol{f} = \sum_{k=1}^{r} f_k^{(1)} \otimes \cdots \otimes f_k^{(d)}.$$

Following Theorem 5.1.10, point-evaluations of the approximation are particularly easy.

Lemma 7.1.3. *With the notation and assumptions of Theorem 7.1.2 the evaluation of* $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}(\boldsymbol{f})$ *in* $\boldsymbol{x} \in \Omega^{(1)} \times \cdots \times \Omega^{(d)}$ *with* $\boldsymbol{x} = (\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)})^T$ *is given by*

$$\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}(\boldsymbol{f})(\boldsymbol{x}) = \sum_{k=1}^{r} \sum_{\boldsymbol{\lambda}\in\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} \prod_{j=1}^{d} \mathcal{I}_{\lambda_j}^{(j)}(e_{\lambda_j-1,k}^{(j)})(\boldsymbol{x}^{(j)}). \tag{7.1.8}$$

Algorithm 5: Tensor-product Multilevel Algorithm

Data: Right-hand sides $f^{(1)}, \ldots, f^{(d)}$, weight vector $\boldsymbol{\omega}$, threshold ℓ, sets of sites $\{X_i^{(j)}\}$
Result: Smolyak approximation $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}(\boldsymbol{f})$
Generate $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ and determine $(\lambda_{1,max}, \ldots, \lambda_{d,max})$;
for $j = 1, \ldots, d$ **do**
  Compute multilevel approximation $A^{(j)}_{\lambda_{j,max}}(f^{(j)})$ to $f^{(j)}$ on $X_1^{(j)}, \ldots, X^{(j)}_{\lambda_{j,max}}$;
end
Combine the $\left(A^{(j)}_{\lambda_{j,max}}(f^{(j)})\right)_{1\le j\le d}$ to $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}(\boldsymbol{f})$ according to (7.1.7);

In Algorithm 5 we see how easy it is to compute the tensor product multilevel approximation in its simple representation. Since the direction-wise approximations $A_i^{(j)}(f^{(j)})$ are independent of each other. We note also that we do not need to construct high-directional sparse grids. Although the operator $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$ implicitly approximates $\boldsymbol{f}$ on the sparse grid $H_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$, it is a Smolyak operator after all, we only use the low-dimensional point sets $X_i^{(j)} \subseteq \Omega^{(j)}$, $1 \le j \le d$, $1 \le i \le \lambda_{j,max}$, to compute the approximation.

We remark that if we have access to the $f^{(j)}$ there is no advantage to use the combination technique representation of $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$. Indeed inserting the multilevel operators $A_i^{(j)}$ into the combination technique representation of Proposition 6.3.2 yields (7.1.7).

7.1.3. General Representation. The situation is significantly worse if we have no access to the low-dimensional functions $f^{(j)} \in V^{(j)}(\Omega^{(j)})$, $1 \le j \le d$. This is the case if, e.g., the high-dimensional target function $\boldsymbol{f}$ is not in the algebraic tensor product space $\bigotimes_{j=1}^{d} V^{(j)}(\Omega^{(j)})$ but rather in a complete tensor space. Then we know from Chapter 5 that $\boldsymbol{f}$ may not even allow a representation $\boldsymbol{f} = \sum_{k=1}^{r} f_k^{(1)} \otimes \cdots \otimes f_k^{(d)}$. Another example where we cannot use the simple representation of the previous subsection is, if we are only given information $(\boldsymbol{x}_m, \boldsymbol{f}(\boldsymbol{x}_m))_{1 \le m \le N} \subseteq \Omega \times \mathbb{R}$, and we have no further knowledge about $f^{(j)}(\boldsymbol{x}_m^{(j)})$, even if the functions $f^{(j)} \in V^{(j)}(\Omega^{(j)})$ exist.

However, we have to assume that the $\boldsymbol{x}_m$ are points of a sparse grid $H_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} = \bigcup_{\boldsymbol{\lambda} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} X_{\lambda_1}^{(1)} \times \cdots \times X_{\lambda_d}^{(d)}$ and we have access to the direction-wise point families $(X_i^{(j)})$, $1 \le j \le d$, $1 \le i \le \lambda_{j,max}$.

The fact that we can not identify the direction-wise functions $f^{(j)}$ has two consequences. First we can not use the multilevel method in its basic representation as a residual correction scheme anymore. We rather have to use the alternative representation of Section 4.6. And second, this time we have to use the combination technique representation of the Smolyak operator.

We recall the basics of Section 4.6. There, we introduced combined operators $\mathcal{I}_{\mathfrak{u}^{(j)}}^{(j)} : V^{(j)}(\Omega^{(j)}) \to W_{\#\mathfrak{u}^{(j)}}^{(j)}$ given by

$$\mathcal{I}_{\mathfrak{u}^{(j)}}^{(j)} := \mathcal{I}_{u_{\#\mathfrak{u}^{(j)}}^{(j)}}^{(j)} \mathcal{I}_{u_{\#\mathfrak{u}^{(j)}-1}^{(j)}}^{(j)} \cdots \mathcal{I}_{u_1^{(j)}}^{(j)}, \tag{7.1.9}$$

where $\mathfrak{u}^{(j)} = \{u_1^{(j)}, \ldots, u_{\#\mathfrak{u}^{(j)}}^{(j)}\}$ is an ordered set and $\mathcal{I}_i^{(j)} : V^{(j)}(\Omega^{(j)}) \to W_i^{(j)}$ are local approximation operators. These combined operators allowed us in Theorem 4.6.3 to find a new representation for the multilevel operator $A_{L^{(j)}}^{(j)} : V^{(j)}(\Omega^{(j)}) \to V_{L^{(j)}}^{(j)}$ in direction j up to level $L^{(j)}$,

$$A_{L^{(j)}}^{(j)} = \sum_{\substack{\mathfrak{u}^{(j)} \subseteq \{1,\ldots,L^{(j)}\} \\ 1 \le \#\mathfrak{u}^{(j)} \le L^{(j)}}} (-1)^{\#\mathfrak{u}^{(j)}+1} \mathcal{I}_{\mathfrak{u}^{(j)}}^{(j)}. \tag{7.1.10}$$

Inserting this in the combination technique representation of the anisotropic Smolyak operator,

$$\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} = \sum_{\boldsymbol{\lambda} \in \mathcal{J}_{\boldsymbol{\omega}}(\ell,d)} \sum_{\substack{\boldsymbol{\beta} \in \{0,1\}^d \\ \boldsymbol{\lambda}+\boldsymbol{\beta} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}} (-1)^{\|\boldsymbol{\beta}\|_1} \left(A_{\lambda_1}^{(1)} \otimes \cdots \otimes A_{\lambda_d}^{(d)} \right), \tag{7.1.11}$$

then yields the following representation of the tensor product multilevel operator.

Theorem 7.1.4. *For $1 \le j \le d$ let $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, be a domain and let $V^{(j)}(\Omega^{(j)}) \subseteq C(\Omega^{(j)})$ be a linear space of functions on $\Omega^{(j)}$. Let $\boldsymbol{\omega} \in \mathbb{R}_+^d$ be an ascendingly ordered weight vector and $\ell \in \mathbb{N}$. Let the anisotropic index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ be defined by* (6.2.1) *and let $\lambda_{j,max}$ be given by* (6.2.3).

For $1 \le j \le d$ and $1 \le i \le \lambda_{j,max}$ let $\mathcal{I}_i^{(j)} : V^{(j)}(\Omega^{(j)}) \to W_i^{(j)}$ be the local approximation operators and assume that the operators $A_i^{(j)} : V^{(j)}(\Omega^{(j)}) \to V_i^{(j)}$ are given by the multilevel operators in (7.1.10) *with the combined*

operators $\mathcal{I}^{(j)}_{\mathfrak{u}^{(j)}} : V^{(j)}(\Omega^{(j)}) \to W^{(j)}_{\#\mathfrak{u}^{(j)}}$, *where* $\mathfrak{u}^{(j)} = \{u^{(j)}_1, \dots, u^{(j)}_{\#\mathfrak{u}^{(j)}}\}$ *is an ordered set.*

Then the Smolyak operator $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : \bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)}) \to V_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$ *has the representation*

$$\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} = \sum_{\boldsymbol{\lambda}\in\mathcal{J}_{\boldsymbol{\omega}}(\ell,d)} \sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^d \\ \boldsymbol{\lambda}+\boldsymbol{\beta}\in\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}}$$

$$\sum_{\substack{\mathfrak{u}^{(1)}\subseteq\{1,\dots,\lambda_1\} \\ 1\le\#\mathfrak{u}^{(1)}\le\lambda_1}} \cdots \sum_{\substack{\mathfrak{u}^{(d)}\subseteq\{1,\dots,\lambda_d\} \\ 1\le\#\mathfrak{u}^{(d)}\le\lambda_d}} c_{\boldsymbol{\beta}}(\mathfrak{u}^{(1)},\dots,\mathfrak{u}^{(d)})\mathcal{I}^{(1)}_{\mathfrak{u}^{(1)}} \otimes \cdots \otimes \mathcal{I}^{(d)}_{\mathfrak{u}^{(d)}}, \tag{7.1.12}$$

where $\mathcal{J}_{\boldsymbol{\omega}}(\ell,d)$ *denotes the surface of* $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)$ *defined in* (6.3.3) *and*

$$c_{\boldsymbol{\beta}}(\mathfrak{u}^{(1)},\dots,\mathfrak{u}^{(d)}) = (-1)^{\|\boldsymbol{\beta}\|_1+d+\#\mathfrak{u}^{(1)}+\cdots+\#\mathfrak{u}^{(d)}}.$$

Proof. We only have to find a representation for the tensor product operator $A^{(1)}_{\lambda_1} \otimes \cdots \otimes A^{(d)}_{\lambda_d}$ for a given multi-index $\boldsymbol{\lambda} \in \mathbb{N}^d$. We use the representation of $A^{(j)}_{\lambda_j}$ in (7.1.10) for every direction $1 \le j \le d$ and have

$$A^{(1)}_{\lambda_1} \otimes \cdots \otimes A^{(d)}_{\lambda_d} = \\ = \sum_{\substack{\mathfrak{u}^{(1)}\subseteq\{1,\dots,\lambda_1\} \\ 1\le\#\mathfrak{u}^{(1)}\le\lambda_1}} \cdots \sum_{\substack{\mathfrak{u}^{(d)}\subseteq\{1,\dots,\lambda_d\} \\ 1\le\#\mathfrak{u}^{(d)}\le\lambda_d}} (-1)^{\#\mathfrak{u}^{(1)}+\cdots+\#\mathfrak{u}^{(d)}}\mathcal{I}^{(1)}_{\mathfrak{u}^{(1)}} \otimes \cdots \otimes \mathcal{I}^{(d)}_{\mathfrak{u}^{(d)}}.$$

Inserting this into (7.1.11) then yields the claimed respresentation. □

We immediately obtain the similar representation for the isotropic Smolyak operator. Again, we can explicitly compute the coefficients.

Corollary 7.1.5. *With the notation and assumptions of Theorem 7.1.4, however the weight vector* $\boldsymbol{\omega} \in \mathbb{R}^d_+$ *satisfies* $\omega_1 = \cdots = \omega_d = \omega$, *with an* $\omega > 0$, *the isotropic Smolyak operator* $\mathcal{A}_{\mathcal{I}^{iso}_{\boldsymbol{\omega}}(\ell,d)} : \bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)}) \to V_{\mathcal{I}^{iso}_{\boldsymbol{\omega}}(\ell,d)}$ *has the representation*

$$\mathcal{A}_{\mathcal{I}^{iso}_{\boldsymbol{\omega}}(\ell,d)} = \\ = \sum_{\boldsymbol{\lambda}\in\mathcal{J}^{iso}_{\boldsymbol{\omega}}(\ell,d)} \sum_{\substack{\mathfrak{u}^{(1)}\subseteq\{1,\dots,\lambda_1\} \\ 1\le\#\mathfrak{u}^{(1)}\le\lambda_1}} \cdots \sum_{\substack{\mathfrak{u}^{(d)}\subseteq\{1,\dots,\lambda_d\} \\ 1\le\#\mathfrak{u}^{(d)}\le\lambda_d}} b_{\boldsymbol{\lambda}}(\mathfrak{u}^{(1)},\dots,\mathfrak{u}^{(d)})\mathcal{I}^{(1)}_{\mathfrak{u}^{(1)}} \otimes \cdots \otimes \mathcal{I}^{(d)}_{\mathfrak{u}^{(d)}},$$

where $\mathcal{J}^{iso}_{\boldsymbol{\omega}}(\ell,d) = \{\boldsymbol{\lambda} \in \mathbb{N}^d \ : \ \ell+1 \le \|\boldsymbol{\lambda}\|_1 \le \ell+d\}$ *is the surface of the isotropic index set and*

$$b_{\boldsymbol{\lambda}}(\mathfrak{u}^{(1)},\dots,\mathfrak{u}^{(d)}) = (-1)^{\ell+d-\|\boldsymbol{\lambda}\|_1+\#\mathfrak{u}^{(1)}+\cdots+\#\mathfrak{u}^{(d)}}\binom{d-1}{\ell+d-\|\boldsymbol{\lambda}\|_1}.$$

Following Theorem 7.1.4 we only have to find a representation of

$$\left(\mathcal{I}^{(1)}_{\mathfrak{u}^{(1)}} \otimes \cdots \otimes \mathcal{I}^{(d)}_{\mathfrak{u}^{(d)}}\right)(\boldsymbol{f})$$

for a given $\boldsymbol{f} \in \bigotimes_{j=1}^d V^{(j)}(\Omega^{(j)})$ without using direction-wise functions $f^{(j)} \in V^{(j)}(\Omega^{(j)})$. The tool that allows us to do this are the Lagrange functions introduced in Chapter 3 and then in Theorem 4.6.4 used in the context of

multilevel approximation. We recall that in every local approximation space $W_i^{(j)}$, $1 \leq j \leq d$, $i \in \mathbb{N}$, we find a basis $\left\{\chi_{i,k}^{(j)}\right\}_{1 \leq k \leq N_i^{(j)}} \subseteq W_i^{(j)}$ such that the Lagrange condition $\chi_{i,k}^{(j)}(\boldsymbol{x}_{i,m}^{(j)}) = \delta_{km}$ is satisfied for all $\boldsymbol{x}_{i,m}^{(j)} \in X_i^{(j)}$.

Theorem 7.1.6. *For $1 \leq j \leq d$ let $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, be a bounded domain and let $V^{(j)}(\Omega^{(j)}) \subseteq C(\Omega^{(j)})$ be a linear space of functions on $\Omega^{(j)}$. Let $(X_i^{(j)})_{i \in \mathbb{N}}$ be sets of sites in $\Omega^{(j)}$ with fill distances h_i, satisfying $h_{i+1}^{(j)} \leq h_i^{(j)}$, $i \in \mathbb{N}$. Let $\{\chi_{i,k}^{(j)}\}_{1 \leq k \leq N_i^{(j)}} \subseteq W_i^{(j)}$ be Lagrange functions for $X_i^{(j)}$.*

For every $1 \leq j \leq d$ let $\mathfrak{u}^{(j)} = \{u_1^{(j)}, \ldots, u_{\#\mathfrak{u}^{(j)}}^{(j)}\}$ be an ordered set. Let $\mathcal{I}_i^{(j)} : V^{(j)}(\Omega^{(j)}) \to W_i^{(j)}$ be local approximation operators and let $\mathcal{I}_{\mathfrak{u}^{(j)}} : V^{(j)}(\Omega^{(j)}) \to W_{\#\mathfrak{u}^{(j)}}^{(j)}$ be the combined operator. Then the tensor product combined operator

$$\mathcal{I}_{\mathfrak{u}^{(1)}}^{(1)} \otimes \cdots \otimes \mathcal{I}_{\mathfrak{u}^{(d)}}^{(d)} : \bigotimes_{j=1}^{d} V^{(j)}(\Omega) \to \bigotimes_{j=1}^{d} W_{\#\mathfrak{u}^{(j)}}^{(j)}$$

has the form

(7.1.13)

$$\left(\mathcal{I}_{\mathfrak{u}^{(1)}}^{(1)} \otimes \cdots \otimes \mathcal{I}_{\mathfrak{u}^{(d)}}^{(d)}\right)(\boldsymbol{f}) =$$

$$= \sum_{\boldsymbol{k}^{(1)} \leq \boldsymbol{N}_{\mathfrak{u}^{(1)}}} \cdots \sum_{\boldsymbol{k}^{(d)} \leq \boldsymbol{N}_{\mathfrak{u}^{(d)}}} \left(\prod_{j=1}^{d} a^{(j)}(\mathfrak{u}^{(j)}, \boldsymbol{k}^{(j)})\right) \boldsymbol{f}\left(\boldsymbol{x}_{u_1^{(1)}, k_1^{(1)}}^{(1)}, \ldots, \boldsymbol{x}_{u_1^{(d)}, k_1^{(d)}}^{(d)}\right) \cdot$$
$$\cdot \chi_{u_{\#\mathfrak{u}^{(1)}}^{(1)}, k_{\#\mathfrak{u}^{(1)}}^{(1)}}^{(1)} \otimes \cdots \otimes \chi_{u_{\#\mathfrak{u}^{(d)}}^{(d)}, k_{\#\mathfrak{u}^{(d)}}^{(d)}}^{(d)},$$

where $\boldsymbol{N}_{\mathfrak{u}^{(j)}}^{(j)} = \left(N_{u_1^{(j)}}^{(j)}, \ldots, N_{u_{\#\mathfrak{u}^{(j)}}^{(j)}}^{(j)}\right)^T \in \mathbb{N}^{\#\mathfrak{u}^{(j)}}$ denotes the vector of cardinalities of the sets $X_{u_i^{(j)}}^{(j)}$ and the coefficients $a^{(j)}(\mathfrak{u}^{(j)}, \boldsymbol{k}^{(j)})$ are given by $a^{(j)}(\mathfrak{u}^{(j)}, \boldsymbol{k}^{(j)}) = 1$, if $\#\mathfrak{u}^{(j)} = 1$, and else by

$$a^{(j)}(\mathfrak{u}^{(j)}, \boldsymbol{k}^{(j)}) = \prod_{m=1}^{\#\mathfrak{u}^{(j)}-1} \chi_{u_m^{(j)}, k_m^{(j)}}^{(j)}\left(\boldsymbol{x}_{u_{m+1}^{(j)}, k_{m+1}^{(j)}}^{(j)}\right).$$

Proof. We know from Theorem 4.6.4 that we have for every $1 \leq j \leq d$ and ordered $\mathfrak{u}^{(j)}$

$$\mathcal{I}_{\mathfrak{u}^{(j)}}^{(j)}(f^{(j)}) = \sum_{\boldsymbol{k}^{(j)} \leq \boldsymbol{N}_{\mathfrak{u}^{(j)}}} a^{(j)}(\mathfrak{u}^{(j)}, \boldsymbol{k}^{(j)}) f^{(j)}\left(\boldsymbol{x}_{u_1^{(j)}, k_1^{(j)}}^{(j)}\right) \chi_{u_{\#\mathfrak{u}^{(j)}}^{(j)}, k_{\#\mathfrak{u}^{(j)}}^{(j)}}^{(j)}.$$

Using the multilinearity of the tensor product and then Theorem 5.1.10 leads to the claim. □

We note that, similar to the simple representation of the tensor product multilevel method in Section 7.1.2, point-evaluations of the approximation

$\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}(\boldsymbol{f})$ follows directly from Theorem 5.1.10. We only give the formula for the evaluation of the tensor product of the combined operators.

Lemma 7.1.7. *With the notation and assumptions of Theorem 7.1.6 the evaluation of* $\mathcal{I}^{(1)}_{\mathfrak{u}^{(1)}} \otimes \cdots \otimes \mathcal{I}^{(d)}_{\mathfrak{u}^{(d)}}(\boldsymbol{f})$ *in* $\boldsymbol{x} \in \Omega^{(1)} \times \cdots \times \Omega^{(d)}$ *with* $\boldsymbol{x} = (\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)})^T$ *is given by*

(7.1.14)

$$\begin{aligned}\left(\mathcal{I}^{(1)}_{\mathfrak{u}^{(1)}} \otimes \cdots \otimes \mathcal{I}^{(d)}_{\mathfrak{u}^{(d)}}\right)(\boldsymbol{f})(\boldsymbol{x}) &= \\ = \sum_{\boldsymbol{k}^{(1)} \le \boldsymbol{N}_{\mathfrak{u}^{(1)}}} \cdots \sum_{\boldsymbol{k}^{(d)} \le \boldsymbol{N}_{\mathfrak{u}^{(d)}}} \prod_{j=1}^{d} a^{(j)}(\mathfrak{u}^{(j)}, \boldsymbol{k}^{(j)}) \boldsymbol{f}\left(\boldsymbol{x}^{(1)}_{u_1^{(1)},k_1^{(1)}}, \ldots, \boldsymbol{x}^{(d)}_{u_1^{(d)},k_1^{(d)}}\right) \cdot \\ \cdot \prod_{j=1}^{d} \chi^{(j)}_{u^{(j)}_{\#\mathfrak{u}^{(j)}},k^{(j)}_{\#\mathfrak{u}^{(j)}}}(\boldsymbol{x}^{(j)}).\end{aligned}$$

We have several remarks. First, we emphasize that this representation of the tensor product multilevel method can also be used in the simple case, that is, if we have the direction-wise information $(f^{(j)}(\boldsymbol{x}^{(j)}_m))_{1 \le m \le N}$. However, since the numerical costs are much higher we would advise against it.

Second, the representation in (7.1.13) together with (7.1.12) is independent of the approximation method we use. The specific choice of reconstruction process, e.g., interpolation, penalized least-squares approximation or approximation with local Lagrange functions, only enters the tensor product multilevel method through the Lagrange functions $\chi^{(j)}_{i,k}$, $\chi^{LS,(j)}_{i,k}$ or $\chi^{loc,(j)}_{i,k}$.

And finally, although the formulas in (7.1.13) and (7.1.12) may seem complicated, the algorithmic procedure is easy to implement and most of the computations can be made in an offline phase, as soon as the input parameters, in particular the direction-wise point sets $X^{(j)}_i$, are fixed. We give a brief idea of the algorithmic approach in Algorithm 6. Once the Lagrange functions are computed we only have to combine the resulting operators.

Algorithm 6: Tensor-product Multilevel Algorithm, Version 2

Data: Right-hand side $\boldsymbol{f}$, weight vector $\boldsymbol{\omega}$, threshold ℓ, sets of sites $\{X^{(j)}_i\}$
Result: Smolyak approximation $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}(\boldsymbol{f})$
Generate $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ and determine $(\lambda_{1,max}, \ldots, \lambda_{d,max})$;
for $j = 1, \ldots, d$ **do**
  for $i = 1, \ldots, \lambda_{j,max}$ **do**
    Compute Lagrange functions $\left(\chi^{(j)}_{i,k}\right)_{1 \le k \le N^{(j)}_i}$;
  end
end
Use (7.1.13) in (7.1.12) to construct $\mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}(\boldsymbol{f})$;

7.2. Convergence Results

Now, we analyze the approximation power of the new tensor product multilevel method and give convergence results for the different approximation methods introduced in Section 2.3. We combine the representation of Theorem 6.3.4 with the error estimates of Section 4.3 and Section 4.6.2. We have to make sure that the spaces involved are equipped with uniformly compatible crossnorms. First, we have to refine the assumptions of Section 7.1.

We assume now that for every $1 \leq j \leq d$ the domains $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, are bounded Lipschitz domains. Furthermore, we switch from general linear spaces of functions on $\Omega^{(j)}$ to specific normed spaces. We set $V^{(j)}(\Omega^{(j)}) := H^{s_j}(\Omega^{(j)})$, the Sobolev Hilbert space of smoothness $s_j > n_j/2$, and equip this space with the induced norm, see Definition 2.1.4. We know from Theorem 2.1.13 that the assumption on s_j implies that there is a continuous embedding operator $\iota^{(j)} : H^{s_j}(\Omega^{(j)}) \to C(\Omega^{(j)})$. Following Theorem 5.3.9 these give rise to a continuous embedding operator $\boldsymbol{\iota} : H^{\boldsymbol{s}}_{mix}(\Omega) \to C(\Omega)$, where $\boldsymbol{s} := (s_1, \dots, s_d)^{\mathrm{T}}$ and $\Omega := \Omega^{(1)} \times \cdots \times \Omega^{(d)}$. We recall that $H^{\boldsymbol{s}}_{mix}(\Omega)$ is the anisotropic Sobolev Hilbert space of Definition 5.2.31, which we equip with the induced crossnorm $\|\cdot\|_{\beta, H^{\boldsymbol{s}}_{mix}(\Omega)}$.

Additionally, we assume that the direction-wise mother kernels $\Phi^{(j)} : \mathbb{R}^{n_j} \to \mathbb{R}$ are reproducing kernels of $H^{s_j}(\Omega^{(j)})$, $1 \leq j \leq d$. Consequently, the level-dependently scaled kernels $\Phi^{(j)}_i$ are also reproducing kernels of this space.

We are now interested in estimating different operator norms of the error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$. We distinguish again different approximation methods, however we can combine the two types of interpolation into one case.

7.2.1. Interpolation. We start with deriving error estimates in the case that the local approximation operator are kernel-based interpolation operators on $X^{(j)}_i$, i.e., we have, for $1 \leq j \leq d$,

$$\mathcal{I}^{(j)}_i = I_{X^{(j)}_i, \Phi^{(j)}_i}, \quad i \in \mathbb{N}.$$

We remark that in this case, the tensor product multilevel operator is indeed an interpolation operator. This follows directly from Theorem 6.1.18.

It turns out that we can treat both the classical interpolation, i.e., the target function is an element of the native space, and the escaping the native space case, see Section 2.3.3, simultaneously. In the former case we understand that $t_j = s_j$, the order with which the Fourier transform of the reproducing kernel $\Phi^{(j)}_i$ decays, and for the escaping the native space case we have t_j such that $s_j > t_j > n_j/2$. With this, the local approximation operators are mappings $\mathcal{I}^{(j)}_i : H^{t_j}(\Omega^{(j)}) \to W^{(j)}_i$, $1 \leq j \leq d$, $i \in \mathbb{N}$, and we can express the multilevel interpolation operator $A^{(j)}_i : H^{t_j}(\Omega^{(j)}) \to V^{(j)}_i$ as

$$A^{(j)}_i(f^{(j)}) = \sum_{k=1}^{i} \mathcal{I}^{(j)}_k(e^{(j)}_{k-1}) = \sum_{k=1}^{i} I_{X^{(j)}_k, \Phi^{(j)}_k}(e^{(j)}_{k-1}). \tag{7.2.1}$$

First, we focus on $L_2(\Omega)$-estimates. Recalling Theorem 4.3.4, we have the direction-wise error estimate

$$\|\iota^{(j)} - A_i^{(j)}\|_{H^{t_j}(\Omega^{(j)})\to L_2(\Omega^{(j)})} \leq C^{(j)} \left(C_1^{(j)} \left(\mu^{(j)}\right)^{t_j} \right)^i, \quad i \in \mathbb{N}. \tag{7.2.2}$$

We also recall the representation of the error operator of Theorem 6.3.4,

$$\begin{aligned}
\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} &= \left(\iota^{(1)} - A^{(1)}_{\lambda_{1,max}}\right) \otimes \bigotimes_{j=2}^{d} \iota^{(j)} + \\
&\quad + \sum_{j=2}^{d} \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \bigotimes_{k=1}^{j-1} \Delta^{(k)}_{\widetilde{\lambda}_k} \otimes \left(\iota^{(j)} - A^{(j)}_{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1}\right) \otimes \bigotimes_{k=j+1}^{d} \iota^{(k)} \\
&=: \left(\iota^{(1)} - A^{(1)}_{\lambda_{1,max}}\right) \otimes \bigotimes_{j=2}^{d} \iota^{(j)} + \sum_{j=2}^{d} R(\ell,j) \otimes \bigotimes_{k=j+1}^{d} \iota^{(k)},
\end{aligned}$$

where for $\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)$ the index $\lambda_j(\widetilde{\boldsymbol{\lambda}})$ is defined as

$$\lambda_j(\widetilde{\boldsymbol{\lambda}}) = \left\lfloor 2 + \ell \frac{\omega_1}{\omega_j} - \sum_{k=1}^{j-1} (\widetilde{\lambda}_k - 1) \frac{\omega_k}{\omega_j} \right\rfloor$$

and we set

$$R(\ell,j) := \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \bigotimes_{k=1}^{j-1} \Delta^{(k)}_{\lambda_k} \otimes \left(\iota^{(j)} - A^{(j)}_{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1}\right). \tag{7.2.3}$$

To simplify the notation, we will from now on suppress the index at the occurring norms. Essentially, we are only dealing with operator norms $\|\cdot\|_{H^{t_j}(\Omega^{(j)})\to L_2(\Omega^{(j)})}$ or $\|\cdot\|_{H^{\boldsymbol{t}}_{mix}(\Omega)\to L_2(\Omega)}$ and it should be clear from the context, which norm is meant.

We set $C := \max_{1\leq j\leq d} C^{(j)}$, the maximum of the constants appearing in (7.2.2). Furthermore, we use the notation

$$\alpha^{(j)} = C_1^{(j)} \left(\mu^{(j)}\right)^{t_j} < 1$$

and recall that $\|\iota^{(j)}\| = 1$, $1 \leq j \leq d$. We know from Theorem 5.3.8 that the induced crossnorms on $H^{\boldsymbol{t}}_{mix}(\Omega)$ and $L_2(\Omega)$ are uniformly compatible. This then yields

$$\left\| \left(\iota^{(1)} - A^{(1)}_{\lambda_{1,max}}\right) \otimes \bigotimes_{j=2}^{d} \iota^{(j)} \right\| = \|\iota^{(1)} - A^{(1)}_{\lambda_{1,max}}\| \leq C \left(\alpha^{(1)}\right)^{\lambda_{1,max}} \tag{7.2.4}$$

and for $2 \leq j \leq d$ also

$$\begin{aligned}\left\| R(\ell,j) \otimes \bigotimes_{k=j+1}^{d} \iota^{(j)} \right\| = \|R(\ell,j)\| &\leq \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \prod_{k=1}^{j-1} \left\| \Delta^{(k)}_{\widetilde{\lambda}_k} \right\| \left\| \iota^{(j)} - A^{(j)}_{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1} \right\| \\ &\leq \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \prod_{k=1}^{j-1} \left(2C \left(\alpha^{(k)} \right)^{\widetilde{\lambda}_k - 1} \right) C \left(\alpha^{(j)} \right)^{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1} \\ &= 2^{j-1} C^j \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \left(\prod_{k=1}^{j-1} \left(\alpha^{(k)} \right)^{\widetilde{\lambda}_k - 1} \right) \left(\alpha^{(j)} \right)^{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1}. \end{aligned} \tag{7.2.5}$$

This motivates the first, general theorem. It covers both the classical interpolation and the escaping the native space case.

Theorem 7.2.1. *For $1 \leq j \leq d$ let $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, be a bounded Lipschitz domain and let $s_j \geq t_j > n_j/2$. Set $\Omega = \Omega^{(1)} \times \cdots \times \Omega^{(d)}$. Let $\boldsymbol{\omega} \in \mathbb{R}^d_+$ be an ascendingly ordered weight vector and $\ell \in \mathbb{N}$. Let $\mathcal{I}_{\boldsymbol{\omega}}(\ell,d) \subseteq \mathbb{N}^d$ be the anisotropic index set of Definition 6.2.1.*

Assume that, for $1 \leq j \leq d$ and $i \in \mathbb{N}$, the low-dimensional reconstruction $A^{(j)}_i : H^{t_j}(\Omega^{(j)}) \to V^{(j)}_i$ is given by the multilevel interpolant (7.2.1). Assume further that this multilevel interpolant is built using a rescaled, compactly supported reproducing kernel of $H^{s_j}(\mathbb{R}^{n_j})$ with scaling parameter $\delta^{(j)}_i$ and set of sites $X^{(j)}_i$ as outlined in Section 7.1.

For $1 \leq j \leq d$ let $\iota^{(j)} : H^{t_j}(\Omega^{(j)}) \to L_2(\Omega^{(j)})$ be the canonical embedding operator and $\boldsymbol{\iota} : H^{\boldsymbol{t}}_{mix}(\Omega) \to L_2(\Omega)$ the tensor product embedding operator of Theorem 5.3.9.

Then the interpolation error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{t}}_{mix}(\Omega) \to L_2(\Omega)$ satisfies the bound

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq \sum_{j=1}^{d} 2^{j-1} C^j \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \left(\prod_{k=1}^{j-1} \left(\alpha^{(k)} \right)^{\widetilde{\lambda}_k - 1} \right) \cdot \left(\alpha^{(j)} \right)^{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1}, \tag{7.2.6}$$

where for $j = 1$ we set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1) := \{\mathbf{1}\}$ and $\lambda_1(\widetilde{\boldsymbol{\lambda}}) = \lambda_{1,max} + 1$ and for $2 \leq j \leq d$ and $\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)$ the index $\lambda_j(\widetilde{\boldsymbol{\lambda}})$ is given by

$$\lambda_j(\widetilde{\boldsymbol{\lambda}}) = \left\lfloor 2 + \ell \frac{\omega_1}{\omega_j} - \sum_{k=1}^{j-1} (\widetilde{\lambda}_k - 1) \frac{\omega_k}{\omega_j} \right\rfloor.$$

Furthermore, we use the notation $C = \max_{1 \leq j \leq d} C^{(j)}$ and

$$\alpha^{(j)} = C^{(j)}_1 \left(\mu^{(j)} \right)^{t_j}.$$

PROOF. All involved norms are uniformly compatible. Hence, we have, with (7.2.4) and (7.2.5),

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq \left\| \left(\iota^{(1)} - A^{(1)}_{\lambda_{1,max}}\right) \otimes \bigotimes_{j=2}^{d} \iota^{(j)} \right\| + \sum_{j=2}^{d} \left\| R(\ell,j) \otimes \bigotimes_{k=j+1}^{d} \iota^{(j)} \right\|$$

$$\leq C \left(\alpha^{(1)}\right)^{\lambda_{1,max}} + \sum_{j=2}^{d} 2^{j-1} C^j \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \left(\prod_{k=1}^{j-1} \left(\alpha^{(k)}\right)^{\widetilde{\lambda}_k - 1} \right) \left(\alpha^{(j)}\right)^{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1}.$$

Setting, for $j = 1$, the index set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1) = \mathcal{I}_{\boldsymbol{\omega}}(\ell, 0) := \{\mathbf{1}\}$ and defining $\lambda_1(\widetilde{\boldsymbol{\lambda}}) = \lambda_{1,max} + 1$ allows us to write

$$\left(\alpha^{(1)}\right)^{\lambda_{1,max}} = \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,0)} \left(\prod_{k=1}^{0} \left(\alpha^{(k)}\right)^{\widetilde{\lambda}_k - 1} \right) \cdot \left(\alpha^{(1)}\right)^{\lambda_1(\widetilde{\boldsymbol{\lambda}})-1}.$$

This then yields the claim. □

Similar to the error estimate for the general Smolyak method in Theorem 6.1.20 the estimate in its most general form (7.2.6) shows the contribution of each direction j and, in particular, preserves information of anisotropy, but is difficult to interpret. We will now trade this information for readability.

First, we look at the exponent in (7.2.6),

$$\lambda_j(\widetilde{\boldsymbol{\lambda}}) - 1 = \left\lfloor 2 + \ell \frac{\omega_1}{\omega_j} - \sum_{k=1}^{j-1} (\widetilde{\lambda}_k - 1) \frac{\omega_k}{\omega_j} \right\rfloor - 1,$$

for a $\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)$, $2 \leq j \leq d$. Since we assume that $\alpha^{(j)} \in (0,1)$ we want to bound it from below. We assume in Theorem 7.2.1 that the weight vector $\boldsymbol{\omega}$ is ascendingly ordered, i.e., $\omega_{j-1} \leq \omega_j$ for all $2 \leq j \leq d$. Hence, we have

$$\begin{aligned} \lambda_j(\widetilde{\boldsymbol{\lambda}}) - 1 &= \left\lfloor 2 + \ell \frac{\omega_1}{\omega_j} - \sum_{k=1}^{j-1} (\widetilde{\lambda}_k - 1) \frac{\omega_k}{\omega_j} \right\rfloor - 1 \\ &\geq \left\lfloor \ell \frac{\omega_1}{\omega_j} \right\rfloor + 1 - \sum_{k=1}^{j-1} (\widetilde{\lambda}_k - 1). \end{aligned}$$

Inserting this in (7.2.6) yields

$$\begin{aligned} \|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| &\leq \sum_{j=1}^{d} 2^{j-1} C^j \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \left(\prod_{k=1}^{j-1} \left(\alpha^{(k)}\right)^{\widetilde{\lambda}_k - 1} \right) \cdot \left(\alpha^{(j)}\right)^{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1} \\ &\leq \sum_{j=1}^{d} 2^{j-1} C^j \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \left(\prod_{k=1}^{j-1} \left(\frac{\alpha^{(k)}}{\alpha^{(j)}}\right)^{\widetilde{\lambda}_k - 1} \right) \cdot \left(\alpha^{(j)}\right)^{\left\lfloor \ell \frac{\omega_1}{\omega_j} \right\rfloor + 1} \\ &= \sum_{j=1}^{d} 2^{j-1} C^j \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell,j-1)} \left(\prod_{k=1}^{j-1} \left(\frac{\alpha^{(k)}}{\alpha^{(j)}}\right)^{\widetilde{\lambda}_k - 1} \right) \cdot \left(\alpha^{(j)}\right)^{\lambda_{j,max}}, \end{aligned} \tag{7.2.7}$$

where we used (6.1.1),

$$\lambda_{j,max} = \left\lfloor \frac{\ell\omega_1}{\omega_j} \right\rfloor + 1.$$

To simplify this further we assume that we choose the direction-wise refinement parameters $\mu^{(j)}$ in such a way that $\alpha^{(j)} = \alpha$ for all $1 \leq j \leq d$. Such a choice is reasonable since it reflects the different smoothness t_j in every direction. The larger t_j is, the easier the reconstruction problem and hence we can use a larger refinement parameter $\mu^{(j)} \in (0,1)$. With this, the error bound in (7.2.7) becomes

$$\begin{aligned}
\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| &\leq \sum_{j=1}^{d} 2^{j-1} C^j \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)} \left(\prod_{k=1}^{j-1} \left(\frac{\alpha^{(k)}}{\alpha^{(j)}} \right)^{\widetilde{\lambda}_k - 1} \right) \cdot \left(\alpha^{(j)} \right)^{\lambda_{j,max}} \\
&= \sum_{j=1}^{d} 2^{j-1} C^j \alpha^{\lambda_{j,max}} \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)} 1 \\
&\leq \sum_{j=1}^{d} 2^{j-1} C^j \# \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1) \alpha^{\lambda_{j,max}}.
\end{aligned}$$

This is the statement of the next corollary.

Corollary 7.2.2. *With the notation and assumptions of Theorem 7.2.1 we choose the direction-wise refinement parameters $\mu^{(j)} \in (0,1)$ such that there is a uniform $\alpha < 1$ such that $\alpha = \alpha^{(j)} = C_1^{(j)} \left(\mu^{(j)}\right)^{t_j}$, $1 \leq j \leq d$. Then the interpolation error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{t}}_{mix}(\Omega) \to L_2(\Omega)$ satisfies the bound*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq \# \mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \sum_{j=1}^{d} 2^{j-1} C^j \alpha^{\lambda_{j,max}}. \tag{7.2.8}$$

Going from (7.2.6) to (7.2.8), we lose some information about the anisotropy. However, we still see the contribution of every direction $1 \leq j \leq d$ to the error in form of the direction-dependent exponent $\lambda_{j,max} = \left\lfloor \frac{\ell\omega_1}{\omega_j} \right\rfloor + 1$, which is the maximum number of levels in direction j.

Going even further, with the assumption $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_d$, we can use the bound on $\# \mathcal{I}_{\boldsymbol{\omega}}(\ell, d)$ given in Lemma 6.2.5 and obtain

$$\# \mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \leq \prod_{j=1}^{d} \left(\frac{\ell\omega_1}{j\omega_j} + 1 \right) \leq \prod_{j=1}^{d} \left(\frac{\ell}{j} + 1 \right) = \binom{\ell + d}{\ell} \leq \frac{(\ell + d)^d}{d!}. \tag{7.2.9}$$

Furthermore, we can bound for every $1 \leq j \leq d$

$$\alpha^{\lambda_{j,max}} = \alpha^{\left\lfloor \frac{\ell\omega_1}{\omega_j} \right\rfloor + 1} \leq \alpha^{\left\lfloor \frac{\ell\omega_1}{\omega_d} \right\rfloor + 1}, \tag{7.2.10}$$

since we assume that $\alpha < 1$. Inserting (7.2.9) and (7.2.10) into (7.2.8) yields

$$\begin{aligned}\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| &\leq \#\mathcal{I}_{\boldsymbol{\omega}}(\ell,d) \sum_{j=1}^{d} 2^{j-1} C^j \alpha^{\lambda_{j,max}} \\ &\leq (\ell+d)^d \alpha^{\left\lfloor \frac{\ell\omega_1}{\omega_d} \right\rfloor + 1} \sum_{j=1}^{d} \frac{2^{j-1} C^j}{j!}.\end{aligned}$$

Corollary 7.2.3. *With the notation and assumptions of Corollary 7.2.2 the interpolation error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{t}}_{mix}(\Omega) \to L_2(\Omega)$ satisfies the bound*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq e^{2C} (\ell+d)^d \alpha^{\left\lfloor \frac{\ell\omega_1}{\omega_d} \right\rfloor + 1}. \tag{7.2.11}$$

The bound in (7.2.11) is much easier to interpret compared to the ones of (7.2.6) and (7.2.8). However, we lost nearly all information about the anisotropy, encoded in the weight vector $\boldsymbol{\omega}$. We have only the exponent of the convergence generating term $\alpha^{\left\lfloor \frac{\ell\omega_1}{\omega_d} \right\rfloor + 1}$, where we see the quotient of the weight in the first direction and the last, ω_1/ω_d, with no influence of the directions $2 \leq j \leq d-1$ on the convergence. However, (7.2.11) allows us to see the asymptotic behavior for $\ell \to \infty$.

Corollary 7.2.4. *With the notation and assumptions of Corollary 7.2.2 the asymptotic behavior of the error bound for the interpolation error $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}$ for $\ell \to \infty$ is given by*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq c(d) \ell^d \alpha^{\left\lfloor \frac{\ell\omega_1}{\omega_d} \right\rfloor + 1}. \tag{7.2.12}$$

Before we give analogous results for the isotropic case, we make a brief remark. So far, we have not coupled the smoothness $\boldsymbol{s}$ of the target function, which, for this discussion, we assume is ascendingly ordered, to the anisotropic index set, in particular to the weight vector $\boldsymbol{\omega}$. However, our arguments above suggest we should choose $\boldsymbol{\omega} = \boldsymbol{s}$. This leads on the one hand to $\lambda_{1,max} \geq \cdots \geq \lambda_{d,max}$, which means that we use more levels in the direction of the least smoothness. On the other hand, because $\alpha^{(j)} = C_1^{(j)} \left(\mu^{(j)}\right)^{s_j} = \alpha$ we also have $\mu^{(1)} \leq \cdots \leq \mu^{(d)}$. As the number of points $\#X_i^{(j)}$ is proportional to $\left(\mu^{(j)}\right)^{-i/n_j}$, this means that we have more points in those directions of lower smoothness.

Again, the situation is much easier if we consider the isotropic case. We consider a constant weight vector $\boldsymbol{\omega} \in \mathbb{R}^d_+$, i.e., there is a $\omega > 0$ such that $\boldsymbol{\omega} = (\omega, \ldots, \omega)^{\mathrm{T}}$. Furthermore, we assume that all other occurring terms are uniform in every direction. This then yields the following theorem.

Theorem 7.2.5. *Let $\Omega^{(1)} \subseteq \mathbb{R}^{n_1}$, $n_1 \in \mathbb{N}$, be a bounded Lipschitz domain and let $s \geq t > n_1/2$. Set $\Omega = \Omega^{(1)} \times \cdots \times \Omega^{(1)}$. Let $\omega > 0$ and $\boldsymbol{\omega} \in \mathbb{R}^d_+$ be the constant vector $\boldsymbol{\omega} = (\omega, \ldots, \omega)^T$. For a threshold $\ell \in \mathbb{N}$ let $\mathcal{I}^{iso}_{\boldsymbol{\omega}}(\ell, d) \subseteq \mathbb{N}^d$ be the isotropic index set.*

Assume that, for $1 \leq j \leq d$ and $i \in \mathbb{N}$, the low-dimensional reconstruction $A_i : H^t(\Omega^{(1)}) \to V_i$ is given by the multilevel interpolant (7.2.1). Assume further that this multilevel interpolant is built using a rescaled, compactly

supported reproducing kernel of $H^s(\mathbb{R}^{n_1})$ with scaling parameter δ_i and set of sites X_i as outlined in Section 7.1.

Let $\iota : H^t(\Omega^{(1)}) \to L_2(\Omega^{(1)})$ be the canonical embedding operator and $\boldsymbol{\iota} : H^{\boldsymbol{t}}_{mix}(\Omega) \to L_2(\Omega)$ the tensor product embedding operator of Theorem 5.3.9.

Then the interpolation error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}^{iso}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{t}}_{mix}(\Omega) \to L_2(\Omega)$ satisfies the bound

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}^{iso}_{\boldsymbol{\omega}}(\ell,d)}\| \leq 2^{d-1} \max(C^d, 1) \binom{\ell+d}{d} \alpha^{\ell+1}. \tag{7.2.13}$$

Finally, before coming to the case where the local approximation is done by penalized least-squares operators, we note that we also can derive L_∞-error estimates. We use the direction-wise error estimate of Lemma 4.3.5,

$$\|\iota^{(j)} - A_i^{(j)}\|_{H^{t_j}(\Omega^{(j)}) \to L_\infty(\Omega^{(j)})} \leq C^{(j)} \left(C_1^{(j)} \left(\mu^{(j)}\right)^{t_j - \frac{n_j}{2}} \right)^i. \tag{7.2.14}$$

We see that we can use the same techniques as in the L_2-case above, if we set $\alpha^{(j)} = C_1^{(j)} \left(\mu^{(j)}\right)^{t_j - \frac{n_j}{2}}$. That is, as long as we make sure that we use tensor spaces with uniformly compatible norms. This can be achieved if the operator $\iota^{(j)}$ is now the embedding operator $H^{t_j}(\Omega^{(j)}) \to C(\Omega^{(j)})$. Then we obtain the similar convergence estimates, however with the changed $\alpha^{(j)}$.

Theorem 7.2.6. *For $1 \leq j \leq d$ let $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, be a bounded Lipschitz domain and let $s_j \geq t_j > n_j/2$. Set $\Omega = \Omega^{(1)} \times \cdots \times \Omega^{(d)}$. Let $\boldsymbol{\omega} \in \mathbb{R}^d_+$ be an ascendingly ordered weight vector and $\ell \in \mathbb{N}$. Let $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \subseteq \mathbb{N}^d$ be the anisotropic index set of Definition 6.2.1.*

Assume that, for $1 \leq j \leq d$ and $i \in \mathbb{N}$, the low-dimensional reconstruction $A_i^{(j)} : H^{t_j}(\Omega^{(j)}) \to V_i^{(j)}$ is given by the multilevel interpolant (7.2.1). Assume further that this multilevel interpolant is built using a rescaled, compactly supported reproducing kernel of $H^{s_j}(\mathbb{R}^{n_j})$ with scaling parameter $\delta_i^{(j)}$ and set of sites $X_i^{(j)}$ as outlined in Section 7.1.

For $1 \leq j \leq d$ let $\iota^{(j)} : H^{t_j}(\Omega^{(j)}) \to C(\Omega^{(j)})$ be the bounded embedding operator of Theorem 2.1.13 and let $\boldsymbol{\iota} : H^{\boldsymbol{t}}_{mix}(\Omega) \to C(\Omega)$ be the bounded tensor product embedding operator of Theorem 5.3.9.

Then the interpolation error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{t}}_{mix}(\Omega) \to C(\Omega)$ satisfies the bound

$$\begin{aligned}\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\|_{H^{\boldsymbol{t}}_{mix}(\Omega) \to L_\infty(\Omega)} \leq \\ \leq \sum_{j=1}^{d} 2^{j-1} C^j \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)} \left(\prod_{k=1}^{j-1} \left(\alpha^{(k)}\right)^{\widetilde{\lambda}_k - 1} \right) \cdot \left(\alpha^{(j)}\right)^{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1},\end{aligned}$$

where for $j = 1$ we set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1) = \emptyset$ and $\lambda_1(\widetilde{\boldsymbol{\lambda}}) = \lambda_{1,max} + 1$ and for $2 \leq j \leq d$ the index $\lambda_j(\widetilde{\boldsymbol{\lambda}})$ is given by

$$\lambda_j(\widetilde{\boldsymbol{\lambda}}) = \left\lfloor 2 + \ell \frac{\omega_1}{\omega_j} - \sum_{k=1}^{j-1} (\widetilde{\lambda}_k - 1) \frac{\omega_k}{\omega_j} \right\rfloor,$$

Furthermore, we use the notation $C = \max_{1\leq j\leq d} C^{(j)}$ *and*

$$\alpha^{(j)} = C_1^{(j)} \left(\mu^{(j)}\right)^{s_j - \frac{n_j}{2}}.$$

With the same ideas as in the L_2-case we arrive at the analogue of Corollary 7.2.2.

Corollary 7.2.7. *With the notation and assumption of Theorem 7.2.6 we choose the direction-wise refinement parameters* $\mu^{(j)} \in (0,1)$ *such that there is a uniform* $\alpha < 1$ *such that* $\alpha = \alpha^{(j)} = C_1^{(j)} \left(\mu^{(j)}\right)^{s_j - \frac{n_j}{2}}$, $1 \leq j \leq d$. *Then the interpolation error operator* $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{t}}_{mix}(\Omega) \to C(\Omega)$ *satisfies the bound*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\|_{H^{\boldsymbol{t}}_{mix}(\Omega)\to L_\infty(\Omega)} \leq \#\mathcal{I}_{\boldsymbol{\omega}}(\ell,d) \sum_{j=1}^{d} 2^{j-1} C^j \alpha^{\lambda_{j,max}}.$$

And finally the most closed form of the error bound, the analogue of Corollary 7.2.3.

Corollary 7.2.8. *With the notation and assumption of Corollary 7.2.7 the interpolation error operator* $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{t}}_{mix}(\Omega) \to C(\Omega)$ *satisfies the bound*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\|_{H^{\boldsymbol{t}}_{mix}(\Omega)\to L_\infty(\Omega)} \leq e^{2C} (\ell + d)^d \alpha^{\left\lfloor \frac{\ell\omega_1}{\omega_d} \right\rfloor + 1}.$$

7.2.2. Penalized Least-Squares Approximation. We now give similar error estimates for the case that we do not use interpolation but rather penalized least-squares approximation operators as building blocks for the tensor product multilevel method. To be precise, we assume that the local approximation operators $\mathcal{I}_i^{(j)} : H^{s_j}(\Omega^{(j)}) \to W_i^{(j)}$ are given by

$$\mathcal{I}_i^{(j)}(e_{i-1}^{(j)}) = \operatorname*{argmin}_{s^{(j)} \in H^{s_j}(\Omega^{(j)})} \left\{ \sum_{k=1}^{N_i^{(j)}} \left| e_{i-1}^{(j)}(\boldsymbol{x}_{i,k}^{(j)}) - s^{(j)}(\boldsymbol{x}_{i,k}^{(j)}) \right|^2 + \lambda_i^{(j)} \|s^{(j)}\|^2_{\Phi_i^{(j)}} \right\}$$

where $\lambda_i^{(j)} > 0$ is the now direction- and level-dependent smoothing parameter. As pointed out in Theorem 2.3.19, this minimization problem has for every $\lambda_i^{(j)}$ a unique solution, which is an element of $W_i^{(j)}$.

Again, the direction-wise multilevel operators $A_i^{(j)} : H^{s_j}(\Omega^{(j)}) \to V_i^{(j)}$ are given as

$$A_i^{(j)}(f^{(j)}) = \sum_{k=1}^{i} \mathcal{I}_k^{(j)}(e_{k-1}^{(j)}), \quad i \in \mathbb{N} \tag{7.2.15}$$

and we have the following direction-wise L_2-error estimate, see Theorem 4.3.7,

$$\|\iota^{(j)} - A_i^{(j)}\|_{H^{s_j}(\Omega^{(j)})\to L_2(\Omega^{(j)})} \leq C^{(j)} \left(C_1^{(j)} \left(\mu^{(j)}\right)^{s_j} \right)^i,$$

where the smoothing parameter $\lambda_i^{(j)}$ has to satisfy

$$\lambda_i^{(j)} \leq \kappa^{(j)} \left(\frac{h_i^{(j)}}{\delta_i^{(j)}} \right)^{2s_j}, \quad i \in \mathbb{N}, \tag{7.2.16}$$

with a level-independent $\kappa^{(j)} > 0$.

Following the same ideas as in Section 7.2.1 we obtain the following main theorem and series of corollaries which, again, trade information about the anisotropy for readability.

Theorem 7.2.9. *For $1 \leq j \leq d$ let $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, be bounded Lipschitz domains and let $s_j > n_j/2$. Set $\Omega = \Omega^{(1)} \times \cdots \times \Omega^{(d)}$. Let $\boldsymbol{\omega} \in \mathbb{R}^d_+$ be an ascendingly ordered weight vector and $\ell \in \mathbb{N}$. Let $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \subseteq \mathbb{N}^d$ be the anisotropic index set of Definition 6.2.1.*

Assume that, for $1 \leq j \leq d$ and $i \in \mathbb{N}$, the low-dimensional reconstruction $A_i^{(j)} : H^{s_j}(\Omega^{(j)}) \to V_i^{(j)}$ is given by the multilevel approximation (7.2.15). Assume further that this multilevel approximation is built using a rescaled, compactly supported reproducing kernel of $H^{s_j}(\mathbb{R}^{n_j})$ with scaling parameter $\delta_i^{(j)}$ and set of sites $X_i^{(j)}$ as outlined in Section 7.1 and assume that the smoothing parameters $\lambda_i^{(j)}$ is chosen such that $\lambda_i^{(j)} < \kappa^{(j)} \left(\frac{h_i^{(j)}}{\delta_i^{(j)}} \right)^{2s_j}$, $1 \leq j \leq d$, $i \in \mathbb{N}$, with a fixed constant $\kappa^{(j)} > 0$.

For $1 \leq j \leq d$ let $\iota^{(j)} : H^{s_j}(\Omega^{(j)}) \to L_2(\Omega^{(j)})$ be the canonical embedding operator and $\boldsymbol{\iota} : H^{\boldsymbol{s}}_{mix}(\Omega) \to L_2(\Omega)$ the tensor product embedding operator of Theorem 5.3.9.

Then the error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{s}}_{mix}(\Omega) \to L_2(\Omega)$ satisfies the bound

(7.2.17)

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq \sum_{j=1}^{d} 2^{j-1} C^j \sum_{\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)} \left(\prod_{k=1}^{j-1} \left(\alpha^{(k)} \right)^{\widetilde{\lambda}_k - 1} \right) \cdot \left(\alpha^{(j)} \right)^{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1},$$

where, for $j = 1$, we set $\mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1) = \emptyset$ and $\lambda_1(\widetilde{\boldsymbol{\lambda}}) = \lambda_{1,max} + 1$ and, for $2 \leq j \leq d$, the index $\lambda_j(\widetilde{\boldsymbol{\lambda}})$ is given by

$$\lambda_j(\widetilde{\boldsymbol{\lambda}}) = \left\lfloor 2 + \ell \frac{\omega_1}{\omega_j} - \sum_{k=1}^{j-1} (\widetilde{\lambda}_k - 1) \frac{\omega_k}{\omega_j} \right\rfloor,$$

Furthermore, we use the notation $C = \max_{1 \leq j \leq d} C^{(j)}$ and

$$\alpha^{(j)} = C_1^{(j)} \left(\mu^{(j)} \right)^{s_j}.$$

Again, using a uniform α we obtain the next corollary.

Corollary 7.2.10. *With the notation and assumptions of Theorem 7.2.9 we choose the direction-wise refinement parameters $\mu^{(j)} \in (0,1)$ such that there is a uniform $\alpha < 1$ such that $\alpha = \alpha^{(j)} = C_1^{(j)} \left(\mu^{(j)} \right)^{s_j}$, $1 \leq j \leq d$. Then the error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{s}}_{mix}(\Omega) \to L_2(\Omega)$ satisfies the bound*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq \#\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \sum_{j=1}^{d} 2^{j-1} C^j \alpha^{\lambda_{j,max}}.$$

And finally the closest form of the error estimate.

Corollary 7.2.11. *With the notation and assumptions of Corollary 7.2.10 the error operator* $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{s}}_{mix}(\Omega) \to L_2(\Omega)$ *satisfies the bound*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq e^{2C}(\ell+d)^d \alpha^{\left\lfloor \ell \frac{\omega_1}{\omega_d} \right\rfloor + 1}.$$

We obtain the same error bounds as in Theorem 7.2.9, Corollary 7.2.10 and Corollary 7.2.10 for the case, if $\boldsymbol{\iota} : H^{\boldsymbol{s}}_{mix}(\Omega) \to C(\Omega)$ is the embedding operator and the error operator maps $H^{\boldsymbol{s}}_{mix}(\Omega) \to C(\Omega)$ and $C(\Omega)$ is equipped with the $L_\infty(\Omega)$-norm. In this case we have to use $\alpha^{(j)} = C_1^{(j)} \left(\mu^{(j)}\right)^{s_j - \frac{n_j}{2}}$. This is again completely analogous to Section 7.2.1, therefore, we do not elaborate on it further.

7.2.3. Local Lagrange Multilevel Method. Finally, we give similar convergence estimates if we use the local Lagrange multilevel operators, introduced in Definition 4.6.6, as low-dimensional approximation operators. The key to derive the error bounds is, again, to use the direction-wise estimate (4.6.17) in the error representation for the Smolyak method of Theorem 6.3.4.

We recall Section 3.3. For a fixed set of sites X, we construct the localized Lagrange function χ_i anchored in $\boldsymbol{x}_i \in X$ with cut-off radius r by enforcing the Lagrange condition Corollary 3.1.2 on those $\boldsymbol{x}_k \in X$ such that $\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2 \leq r$. Following this construction for every $X_i^{(j)}$, $1 \leq j \leq d$, $1 \leq i \leq L^{(j)}$ yields the direction-dependent local Lagrange multilevel operator $Q^{loc,(j)}_{L^{(j)}}$ given by

$$Q^{loc,(j)}_{L^{(j)}}(f^{(j)}) = \sum_{i=1}^{L^{(j)}} \mathcal{I}^{loc,(j)}(e^{loc,(j)}_{i-1}), \quad f^{(j)} \in H^{s_j}(\Omega^{(j)}), s_j > \frac{n_j}{2}, \tag{7.2.18}$$

where the local approximation operators $\mathcal{I}_i^{loc,(j)}$ are given by

$$\mathcal{I}_i^{loc,(j)}(f^{(j)}) = \sum_{k=1}^{N_i^{(j)}} f^{(j)}(\boldsymbol{x}_{i,k}^{(j)}) \chi_{i,k}^{loc,(j)}. \tag{7.2.19}$$

Using the same ideas that led to the convergence results in Sections 7.2.1 and 7.2.2 we obtain the following general theorem and series of corollaries, where we trade, again, generality for readability. Similar to Section 4.6.2 we are only interested in localized Lagrange functions that use the footprint $X^{(j)}_{i,r(\varepsilon_i^{(j)})}$ with cut-off radius $r(\varepsilon_i^{(j)})$, whose size we control with the now direction- and level-dependent parameter $\varepsilon_i^{(j)} > 0$.

Theorem 7.2.12. *For* $1 \leq j \leq d$ *let* $\Omega^{(j)} \subseteq \mathbb{R}^{n_j}$, $n_j \in \mathbb{N}$, *be a bounded Lipschitz domain and let* $s_j > n_j/2$. *Set* $\Omega = \Omega^{(1)} \times \cdots \times \Omega^{(d)}$. *Let* $\boldsymbol{\omega} \in \mathbb{R}^d_+$ *be an ascendingly ordered weight vector and* $\ell \in \mathbb{N}$. *Let* $\mathcal{I}_{\boldsymbol{\omega}}(\ell, d) \subseteq \mathbb{N}^d$ *be the anisotropic index set of Definition 6.2.1.*

For $1 \leq j \leq d$ *and* $i \in \mathbb{N}$ *let* $X_i^{(j)}$ *be a quasi-uniform set of sites with fill-distance* $h_i^{(j)}$ *sufficiently small such that* $q_i^{(j)} < 1$ *and assume that the local approximation operator* $\mathcal{I}_i^{loc,(j)}$ *in* (7.2.19) *is built using localized Lagrange functions* $\left\{\chi_{i,k}^{loc,(j)}\right\}_{1 \leq k \leq N_i^{(j)}} \subset W_i^{(j)}$ *defined as in* (3.3.1) *with*

footprint $X^{(j)}_{i,r(\varepsilon_i^{(j)})}(\boldsymbol{x}_{i,k})$ *as in* (3.2.2) *and cut-off radius* $r(\varepsilon_i^{(j)}) > 0$. *Assume that* $\varepsilon_i^{(j)} > 0$ *small enough such that* (4.6.8), *i.e.,*

$$C(\Omega,\Phi,n)\frac{e^{\eta_i^{(j)}}}{\left(\eta_i^{(j)}\right)^{3n_j}}\left|\log\left(\frac{\varepsilon_i^{(j)}}{\left(q_i^{(j)}\right)^{n_j}}\right)\right|^{2n_j}\left(q_i^{(j)}\right)^{-\frac{n_j}{2}-4s_j}\sqrt{\varepsilon_i^{(j)}} \leq 1,$$

holds.

For $1 \leq j \leq d$ *and* $i \in \mathbb{N}$, *the low-dimensional reconstruction* $Q_i^{loc,(j)} : H^{s_j}(\Omega^{(j)}) \to V_i^{(j)}$ *is given by the local Lagrange multilevel approximation* (7.2.18).

For $1 \leq j \leq d$ *let* $\iota^{(j)} : H^{s_j}(\Omega^{(j)}) \to L_2(\Omega^{(j)})$ *be the canonical embedding operator and* $\boldsymbol{\iota} : H^{\boldsymbol{s}}_{mix}(\Omega) \to L_2(\Omega)$ *the tensor product embedding operator of Theorem 5.3.9.*

Then the approximation error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{s}}_{mix}(\Omega) \to L_2(\Omega)$ *satisfies the bound*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq \sum_{j=1}^{d} 2^{j-1}C^j \sum_{\widetilde{\boldsymbol{\lambda}}\in\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\left(\prod_{k=1}^{j-1}\left(\alpha^{loc,(k)}\right)^{\widetilde{\lambda}_k-1}\right)\cdot\left(\alpha^{loc,(j)}\right)^{\lambda_j(\widetilde{\boldsymbol{\lambda}})-1},$$

where for $j = 1$ *we set* $\mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1) := \{\mathbf{1}\}$ *and* $\lambda_1(\widetilde{\boldsymbol{\lambda}}) = \lambda_{1,max} + 1$ *and for* $2 \leq j \leq d$ *and* $\widetilde{\boldsymbol{\lambda}} \in \mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)$ *the index* $\lambda_j(\widetilde{\boldsymbol{\lambda}})$ *is given by*

$$\lambda_j(\widetilde{\boldsymbol{\lambda}}) = \left\lfloor 2 + \ell\frac{\omega_1}{\omega_j} - \sum_{k=1}^{j-1}(\widetilde{\lambda}_k - 1)\frac{\omega_k}{\omega_j}\right\rfloor$$

We use the notation

$$\alpha^{loc,(j)} = C_1^{loc,(j)}\left(\mu^{(j)}\right)^{s_j}.$$

and $C = \max_{1\leq j\leq d} C^{(j)}$.

Similar to the ideas in Section 7.2.1 we can now try to obtain a more uniform estimate. Again, we have to choose the direction-dependent refinement parameter $\mu^{(j)}$ such that there is a uniform $\alpha < 1$.

Corollary 7.2.13. *With the notation and assumptions of Theorem 7.2.12, we choose the refinement parameter* $\mu^{(j)}$, $1 \leq j \leq d$, *such that there is a uniform* $\alpha^{loc} < 1$ *such that* $\alpha^{loc} = \alpha^{loc,(j)}$ *for every* $1 \leq j \leq d$.

Then the approximation error operator $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{s}}_{mix}(\Omega) \to L_2(\Omega)$ *satisfies the bound*

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq \sum_{j=1}^{d} 2^{j-1}C^j \#\mathcal{I}_{\boldsymbol{\omega}}(\ell, j-1)\left(\alpha^{loc}\right)^{\lambda_{j,max}}.$$

Finally, we use the bound on the cardinality of the anisotropic index set (7.2.9) to obtain the analogue of Corollary 7.2.3.

Corollary 7.2.14. *With the notation and assumptions of Corollary 7.2.13 approximation error operator* $\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)} : H^{\boldsymbol{s}}_{mix}(\Omega) \to L_2(\Omega)$ *satisfies the*

bound

$$\|\boldsymbol{\iota} - \mathcal{A}_{\mathcal{I}_{\boldsymbol{\omega}}(\ell,d)}\| \leq e^{2C}(\ell+d)^d \left(\alpha^{loc}\right)^{\left\lfloor \frac{\ell\omega_1}{\omega_d} \right\rfloor + 1}.$$

7.3. Numerical Examples

We now provide numerical examples to support the theoretical results of Section 7.2. We only focus on interpolation since we want to verify the convergence results and the estimates for the penalized least-squares method are virtually the same as the ones for interpolation. In all cases discussed below we will use uniformly distributed point sets $X_i^{(j)} \subseteq \Omega^{(j)}$, $1 \leq j \leq d$, $i \in \mathbb{N}$, such that, for numerical reasons,

$$\#X_i^{(j)} = \left\lceil \frac{1}{\mu^{(j)}} \right\rceil^{\text{offset}+i}. \tag{7.3.1}$$

We introduce the parameter offset $\in \mathbb{N}$ to control the cardinality of the set of sites on the first level $X_1^{(j)}$.

7.3.1. Algebraic Tensor Product Functions. We start with testing functions that are elements of algebraic tensor product spaces. This allows us to use the simple representation of the new tensor product multilevel method introduced in Section 7.1.2.

The first example is based on a well-known problem in rational approximation, see, e.g., [84] and has already been used in the context of sparse grid approximation in [72]. We set the direction-wise univariate functions $f^{(j)} : [-1,1] \to \mathbb{R}$ to be $f^{(j)}(x) = |x|^{\beta_j}$, $1 \leq j \leq d$ with $\beta_j > 0$. These functions are Hölder-continuous with exponent β_j and hence are elements of $H^{\lceil \beta_j \rceil}((-1,1))$. We also recall that we used these functions already in Section 4.4.

Following Chapter 5, the function $\boldsymbol{f} = f^{(1)} \otimes \cdots \otimes f^{(d)} : [-1,1]^d \to \mathbb{R}$ can be represented by

$$\boldsymbol{f}(\boldsymbol{x}) = \prod_{j=1}^{d} f^{(j)}(x^{(j)}) = \prod_{j=1}^{d} |x^{(j)}|^{\beta_j},$$

where $\boldsymbol{x} = (x^{(1)}, \ldots, x^{(d)})^{\mathrm{T}}$ and it is an element of the algebraic tensor product space $H^{\lceil \beta_1 \rceil}((-1,1)) \otimes \cdots \otimes H^{\lceil \beta_d \rceil}((-1,1))$.

Throughout Section 7.2 we emphasized that certain results retain more information about the anisotropy of the problem than others. However, here, we will restrict ourselves to two-directional anisotropic problems and compare the numerical results to the bound (7.2.11). Using this bound in the low-directional case is reasonable since it already incorporates the whole anisotropy.

Now, we describe the setup of the first experiment in more detail. We choose the exponent vector $\boldsymbol{\beta}$ to be $\boldsymbol{\beta} = (1.6, 3.6)^{\mathrm{T}}$. This means that $f^{(1)} \in H^2((-1,1))$ but $f^{(1)} \notin H^3((-1,1))$. Similarly, $f^{(2)} \in H^4((-1,1))$ but $f^{(2)} \notin H^5((-1,1))$. This yields that $\boldsymbol{f} \in H^{\boldsymbol{s}}_{mix}((-1,1)^2)$, with $\boldsymbol{s} = (2,4)^{\mathrm{T}}$, but not in any Sobolev space of mixed, higher-order regularity.

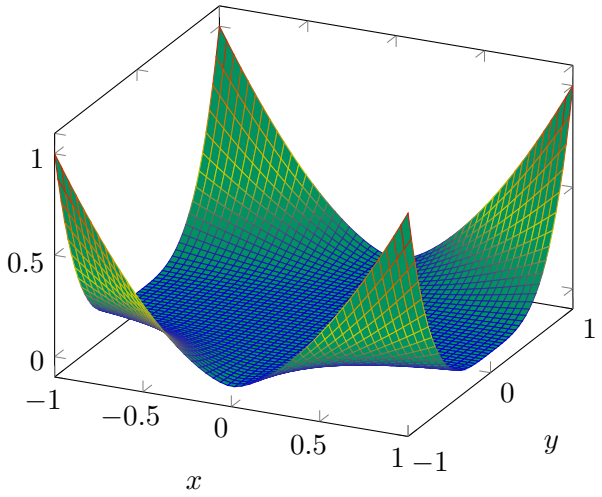

FIGURE 10. Plot of the target function $\boldsymbol{f}$.

Next, we fix the weight vector $\boldsymbol{\omega} \in \mathbb{R}^2_+$. It should represent the anisotropy in $H^{\boldsymbol{s}}_{mix}((-1,1)^2)$ and hence, we set $\boldsymbol{\omega} = \boldsymbol{s}$ and normalize it such that $\omega_1 = 1$. This yields the weight vector $\boldsymbol{\omega} = (1,2)^{\mathrm{T}}$.

To fix the parameters for the multilevel method we first look at the functions $f^{(1)}$ and $f^{(2)}$. We chose these functions such that they are elements of the native spaces of the Wendland kernels $\phi_{1,1}$ and $\phi_{1,3}$, respectively. We recall that these functions are given as

$$\begin{aligned}\phi_{1,1}(r) &= (1-r)^3_+(3r+1)\\ \phi_{1,3}(r) &= (1-r)^7_+(21r^3+19r^2+7r+1).\end{aligned}$$

We fix the refinement parameter in the first direction to be $\mu^{(1)} = 0.4$ and compute $\mu^{(2)}$ such that the assumption of Corollary 7.2.3 is satisfied. That means, the refinement parameter in the second direction has to be computed such that there is a uniform $\alpha < 1$ such that

$$\alpha = \alpha^{(1)} = C_1^{(1)} \left(\mu^{(1)}\right)^{s_1} = C_1^{(2)} \left(\mu^{(2)}\right)^{s_2} = \alpha^{(2)}. \tag{7.3.2}$$

To determine $\mu^{(2)}$ from (7.3.2) we have to know the constants $C_1^{(1)}$ and $C_1^{(2)}$. Unfortunately, these constants are unknown and we have to rely on the numerical estimates presented in Section 4.4. In the first direction we are in the setting which lead to Table 2. We choose to use $\gamma^{(1)} = 3.5$ which then leads to $C_1^{est,(1)} \approx 0.46$. However, we use a more conservative estimate and set $C_1^{num,(1)} = 1.0$.

Inserting the choice for $C_1^{num,(1)}$ and $\mu^{(1)}$ into (7.3.2) and rearranging for $\mu^{(2)}$ yields

$$\begin{aligned}\mu^{(2)} &= \left(\frac{C_1^{num,(1)}}{C_1^{num,(2)}}\left(\mu^{(1)}\right)^{s_1}\right)^{\frac{1}{s_2}} \\ &= \left(\frac{1}{C_1^{num,(2)}}\left(0.4\right)^2\right)^{\frac{1}{4}}, \end{aligned} \tag{7.3.3}$$

with $C_1^{num,(2)}$ still to determine. We recall that this constant depends strongly on $\gamma^{(2)}$, whose choice we justify next.

First, we recall that one of the remarkable properties of the tensor product multilevel method is that it has two ways to deal with anisotropy. First, by using the anisotropic index set and second, using different sets of sites in each direction. In (7.3.1) we fixed the growth of the cardinality of these sets. In this first experiment, the function in the second direction is smoother than in the first and hence, we would expect that we need less points in this direction to get a good approximation and therefore, we should be able to use a larger $\mu^{(2)}$. We fixed $\mu^{(1)} = 0.4$, which leads to $\#X_i^{(1)} = 3^{\text{offset}+i}$, a tripling of the number of points per level. Following the idea above, in direction two, it should suffice to use $\mu^{(2)} \geq 0.5$. The cardinality of the set of sites then behaves like $\#X_i^{(2)} = 2^{\text{offset}+i}$, which leads to a doubling of the number of points per level. As discussed in Section 2.5, for fixed point set $X_i^{(j)}$, this number depends on the overlap parameter $\nu^{(j)}$. In the first direction, this parameter is fixed to be

$$\nu^{(1)} \geq \frac{\gamma^{(1)}}{\mu^{(1)}} = \frac{3.5}{0.4} = 8.75.$$

In this experiment, we choose to use $\nu^{(2)} \approx \nu^{(1)} = 8.75$. This, together with the ideas above yield, that

$$\gamma^{(2)} \approx 8.75\mu^{(2)} \geq 8.75 \cdot 0.5 = 4.375.$$

Hence, the choice $\gamma^{(2)} = 4.5$ is reasonable. Using Table 7, with reservations, since we neither know $\mu^{(2)}$ nor is our choice for $\gamma^{(2)}$ in this table, suggests that $C_1^{num,(2)} \approx 2.1$ is a sensible value for the unknown constant. Inserting this choice into (7.3.3) then leads to

$$\mu^{(2)} \approx 0.52.$$

Direction j	$\gamma^{(j)}$	$\mu^{(j)}$	$\nu^{(j)}$	ω_j	s_j	β_j	$\alpha^{(j)}$
1	3.5	0.4	8.75	1.0	2	1.6	0.16
2	4.5	0.52	8.65	2.0	4	3.6	0.155

TABLE 14. Choices for the different parameters for $\boldsymbol{f}(\boldsymbol{x}) = |x_1|^{1.6} \cdot |x_2|^{3.6}$.

For the convenience of the reader we repeat our choices for the parameters in Table 14. There we also see that we achieved that $\alpha^{(1)} \approx \alpha^{(2)} \approx 0.16$ and $\nu^{(2)} \approx \nu^{(1)} = 8.75$.

After introducing and explaining our choices for the parameters in the first experiment in detail we now discuss the numerical result. In Fig. 11 we provide an ℓ_2-error plot of the tensor product multilevel interpolation error, evaluated on a uniform grid with 15000×15000 points in $[-1,1]^2$. The error is depicted as a function of the threshold ℓ, which determines the anisotropic index set $\mathcal{I}_{(1,2)}(\ell, 2)$, and therefore, the maximum number of levels per direction $\lambda_{j,max}$. As the solid green graph we give the interpolation error and as the dashed green graph we provide the bound (7.2.11), $(\ell+2)^2 0.16^{\lfloor 0.5\ell \rfloor + 1}$. We can see the solid graph exhibits plateaus where the interpolation error stays constants. These are caused by the robustness towards perturbation of $\lambda_{j,max}$ which we discussed in Section 6.2.3. This led also to the kinks seen in the examples provided in Section 6.2.2. Here, this robustness causes the anisotropic index set to change only every odd ℓ. These plateaus make it also hard to compare the error to the theoretical bound and therefore, we also provide a linear fit to the two graphs in orange. Comparing the slope of the two orange lines suggests that the tensor product multilevel interpolation converges slightly faster than expected.

Although these results look promising, the question arises if it is necessary to pay this meticulous attention that the parameters of Table 14 representing the anisotropy in $\boldsymbol{f}$. Hence, we also computed the tensor product multilevel interpolations for three other parameter settings. First, we forewent the anisotropy in the sets of sites, i.e., we used the same refinement parameters $\mu^{(1)} = \mu^{(2)} = 0.4$ in both directions but kept on using the anisotropic index set. We denote this setting as *isotropic* μ. Second, we used different sets of points for each direction but set $\boldsymbol{\omega} = (1,1)^{\mathrm{T}}$, i.e., we used the isotropic index set, denoted as *isotropic* $\boldsymbol{\omega}$. And third, we combined these two cases and used

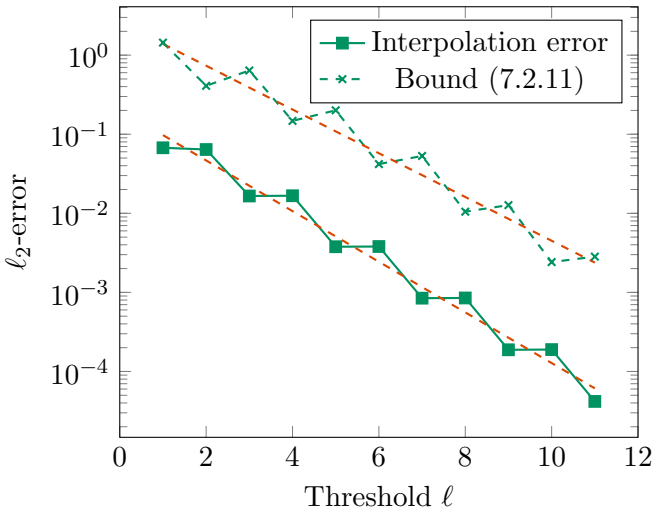

FIGURE 11. Error plot for the tensor product multilevel interpolation of $\boldsymbol{f}$ with parameters as in Table 14.

	ℓ_2-Error			
ℓ	theory	isotropic μ	isotropic ω	fully isotropic
1	$6.7421e{-}02$	$1.3593e{-}02$	$6.7421e{-}02$	$1.3593e{-}02$
2	$6.4081e{-}02$	$1.0730e{-}02$	$1.6698e{-}02$	$9.2063e{-}04$
3	$1.6558e{-}02$	$7.4758e{-}04$	$3.7867e{-}03$	$7.1249e{-}05$
4	$1.6677e{-}02$	$7.0332e{-}04$	$8.4628e{-}04$	$5.8777e{-}06$
5	$3.7791e{-}03$	$5.0826e{-}05$	$1.8833e{-}04$	$5.0076e{-}07$
6	$3.8022e{-}03$	$4.8318e{-}05$	$4.1854e{-}05$	$4.3428e{-}08$
7	$8.4518e{-}04$	$3.5061e{-}06$	$9.2975e{-}06$	$3.7381e{-}09$
8	$8.5022e{-}04$	$3.3372e{-}06$	$2.0651e{-}06$	$3.1656e{-}10$
9	$1.8810e{-}04$	$2.4226e{-}07$	$4.5865e{-}07$	$1.7636e{-}11$
10	$1.8923e{-}04$	$2.3063e{-}07$	$1.0184e{-}07$	$1.2145e{-}12$
11	$4.1803e{-}05$	$1.6736e{-}08$	$2.2522e{-}08$	$8.1582e{-}14$

TABLE 15. The ℓ_2-interpolation errors of $\boldsymbol{f}$ for different parameter settings.

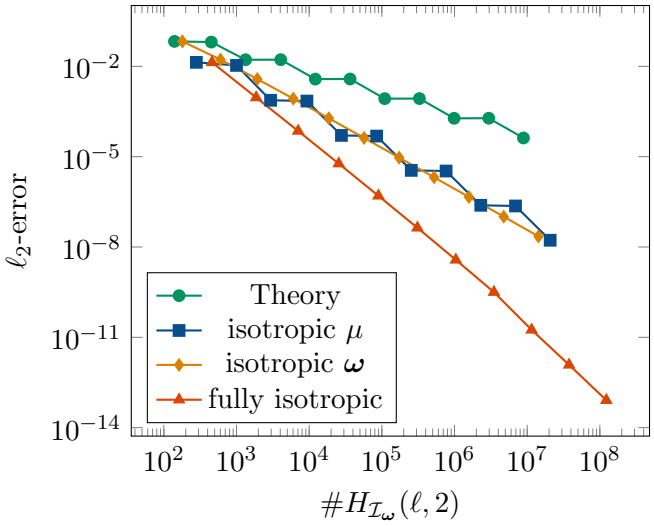

FIGURE 12. The ℓ_2-interpolation errors for different parameter settings as functions of the points of the sparse grid.

the fully isotropic setting, i.e., $\mu^{(1)} = \mu^{(2)} = 0.4$ and $\boldsymbol{\omega} = (1,1)^{\mathrm{T}}$. We call this setting *fully isotropic*. We give the corresponding errors in Table 15, where we also included the values of the interpolation in the parameter setting of Table 14 in the second column, *theory*. Not surprisingly, the fully isotropic setting performs the best since it uses the most information.

To make a fair comparison of the parameter settings we computed the corresponding sparse grids and map the error as a function of the points in the sparse grids in Fig. 12. Surprisingly, we see that even here the fully isotropic case performs much better than all others in the sense that for a fixed number of points the error is the smallest. Furthermore, even the cases were we used only one uniform parameter, either the weights or the

refinements, perform better than the case where we used the parameters according to the convergence results.

We have two reasons for this, at the first glance, negative result. First, we believe that the problem simulated here is simply not anisotropic enough which means that computing the interpolant in any of the three isotropic settings, isotropic refinement parameter, isotropic weights or fully isotropic is just not expensive enough to compensate for the lower approximation error. And indeed, adding more and more directions with parameters set as in direction 2 and computing the interpolation suggests that the theory case begins to deliver better results. However, for these high-directional problems the available memory is insufficient and we can not compute more than a few levels. This is a problem intrinsic to the method and there is no real way to avoid it, since approximation schemes derived from Smolyak's construction still need memory that grows exponential in the number of directions, albeit with a smaller base than full-grid methods, see, e.g., the discussion in Sections 6.1.3 and 6.2.2. The second justification we can think of, ties in with the discussion of Section 6.1.3. In classical numerical approximation we measure the error in terms of the fill distance of the set of sites. For higher-directional problems and in particular with Smolyak methods this is not reasonable. One way out is to use the cardinality of the used index set or, in this case, the number of sparse grid points. However, it is still unclear if these numbers are the right tools and if they lead to the best results.

7.3.2. Testing Package of Genz. After the thorough tests of Section 7.3.1 we now briefly provide numerical results for a package of test functions established in [31]. The examples presented here serve only as a proof of concept that the tensor product multilevel method works in higher dimensions and for a wide variety of target functions. The testing package was originally designed for numerical integration, see also [82] but in [6] these functions are also used for high-dimensional interpolation. The package consists of a family of six functions $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_6 : [0,1]^d \to \mathbb{R}$, in arbitrary dimension $d \in \mathbb{N}$. Each is given a name or attribute by which we will refer them as. The functions are defined as follows:

(1) OSCILLATORY:

$$\boldsymbol{f}_1(\boldsymbol{x}) = \cos\left(2\pi\theta_1 + \sum_{j=1}^{d} c^{(j)} x_j\right),$$

(2) PRODUCT PEAK:

$$\boldsymbol{f}_2(\boldsymbol{x}) = \prod_{j=1}^{d} (c_j^{-2} + (x_j - \theta_j)^2)^{-1},$$

(3) CORNER PEAK:

$$\boldsymbol{f}_3(\boldsymbol{x}) = \left(1 + \sum_{j=1}^{d} c_j x_j\right)^{-(d+1)},$$

(4) GAUSSIAN:

$$\boldsymbol{f}_4(\boldsymbol{x}) = \exp\left(-\sum_{j=1}^{d} c_j^2 (x_j - \theta_j)^2\right),$$

(5) CONTINUOUS:

$$\boldsymbol{f}_5(\boldsymbol{x}) = \exp\left(-\sum_{j=1}^{d} c_j |x_j - \theta_j|\right),$$

(6) DISCONTINUOUS:

$$\boldsymbol{f}_6(\boldsymbol{x}) = \begin{cases} 0, & \text{if } x_1 > \theta_1 \text{ or } x_2 > \theta_2, \\ \exp\left(\sum_{j=1}^{d} c_j x_j\right), & \text{otherwise.} \end{cases}$$

Each function depends on one or two parameter vectors, $\boldsymbol{c}$ and $\boldsymbol{\theta} \in \mathbb{R}_+^d$. The parameter $\boldsymbol{\theta}$ acts as a shift parameter and $\boldsymbol{c}$ determines the difficulty of the function [31]. Our examples are for $d = 7$. We follow the ideas of [6]: For each function we generate $\boldsymbol{\theta}$ and $\boldsymbol{c}'$ randomly but independently and uniformly distributed in $[0, 1]^7$. To obtain $\boldsymbol{c}$ we normalize $\boldsymbol{c}'$ such that

$$\sum_{j=1}^{7} c_j' = b_k, \quad 1 \le k \le 6,$$

where b_k depends on the function $\boldsymbol{f}_k$ and is given in Table 16.

Independently of the values of the parameters $\boldsymbol{\theta}$ and $\boldsymbol{c}$ we see that the functions PRODUCT PEAK, GAUSSIAN and CONTINUOUS are elements of a smooth algebraic tensor space and the functions OSCILLATORY and CORNER PEAK are elements of a smooth tensor space. The functions named DISCONTINUOUS is not continuous in the first two but smooth in the other directions.

The interpolations to $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_6$ were all computed in the fully uniform setting. In part, because the numerical results in [6] were obtained in this setting and in part, because the functions do not exhibit significant anisotropic properties. Hence, we use an isotropic index set, i.e., we set the weight vector to be $\boldsymbol{\omega} = \mathbf{1}$ and for the multilevel method we use in every direction the uniform refinement parameter $\mu = \mu^{(j)} = 0.5$, $1 \le j \le 7$. This leads to a doubling of points from level to level. For the overlap parameter $\nu^{(j)}$ we choose the uniform value $\nu = \nu^{(j)} = 4.0$. Additionally, we use the Wendland kernel $\phi_{1,1}$ for every direction.

Error computations for problems in higher directions is difficult itself since the set of sites used for the evaluation of the error function should be finer than the point set used for the interpolation. This leads to a potentially numerically unfeasible effort. Hence, it is custom to follow the idea of the

k	1	2	3	4	5	6
b_k	9.0	7.25	1.85	7.03	20.4	4.3

TABLE 16. Choices for the normalizations b_k, $1 \le k \le 6$

Monte Carlo method and randomly distribute an arbitrary number of sample points in the domain and average the error function at these points. We use this approach and generate one houndred random points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{100} \in [0,1]^7$ and compute the discrete ℓ_∞-error.

	ℓ_∞-Error					
ℓ	$\boldsymbol{f}_1$	$\boldsymbol{f}_2$	$\boldsymbol{f}_3$	$\boldsymbol{f}_4$	$\boldsymbol{f}_5$	$\boldsymbol{f}_6$
1	1.3465e−02	4.0097e−04	1.5206e−03	6.7757e−02	2.8715e−02	1.5620e+02
2	3.9069e−03	7.3510e−05	1.9847e−04	8.5338e−03	1.4227e−02	1.0296e+02
3	6.5404e−04	1.2211e−05	4.8816e−05	2.8317e−03	3.9765e−03	2.4964e+01
4	1.7643e−04	4.2503e−06	8.4589e−06	4.3021e−04	1.8163e−03	7.5011e+01
5	4.3791e−05	1.1884e−06	2.7708e−06	1.1511e−04	2.5722e−03	2.3303e+01
6	3.3497e−06	1.6768e−07	2.1714e−08	5.3655e−05	5.5128e−04	2.0630e+01

TABLE 17. The ℓ_∞-interpolation errors for the functions of the testing package of Genz.

In Table 17 we give the respective ℓ_∞ interpolation errors. Similar to the observations in [6], we see convergence of the method for $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_5$. For the target function DISCONTINUOUS we observe slow to no convergence at all. That is not surprising since $\boldsymbol{f}_6$ is not even continuous.

All the results in Table 17 were obtained by using an implementation of the tensor product multilevel method in general representation, see Section 7.1.3, i.e., we used only the high-directional data $(\boldsymbol{x}_i, \boldsymbol{f}_k(\boldsymbol{x}_i))_{1 \le i \le N} \subseteq [0,1]^7 \times \mathbb{R}$, $1 \le k \le 6$. In Table 18 we provide the number of information N used per threshold ℓ. These are the numbers of sparse grid points with index set $\mathcal{I}_{\mathbf{1}}(\ell, 7)$ and one-dimensional sets of sites $(X_i)_{1 \le i \le \ell+1}$. We use the sets of sites to construct the Lagrange functions.

Additionally, we provide a comparison of computation times of the two representations of the tensor product multilevel method. At the end of Section 4.6.1, we already remarked on the high cost of a point evaluation of the multilevel approximation if we use Lagrange functions. These costs are even higher for the tensor product multilevel approximation in general representation. To give an indication of this, we show in Table 19 the computation time of the one-houndred point evaluations of the approximation to the target function GAUSSIAN, $\boldsymbol{f}_5$. The times were taken on a Linux workstation with using an Intel Xeon Silver CPU with 2.2 GHz and the C++-class `std::chrono` without any parallelization. We choose this target function since the tensor product multilevel method can be used in both representations to compute the approximation to this function. We can see that the evaluation of the approximation operator in general representation takes several orders of magnitude longer than the one in simple represenation. This is a problem inherent of the representation by Lagrange functions and

ℓ	1	2	3	4	5	6
N	2187	12393	53217	198369	676161	2163969

TABLE 18. Number of information N per threshold ℓ.

	Time for 100 point evaluations (s)	
ℓ	general representation	simple representation
1	3.1581e+00	3.9870e−03
2	2.6251e+01	1.4082e−02
3	1.8862e+02	4.5716e−02
4	1.2655e+03	1.6535e−01
5	7.7446e+03	3.0210e−01
6	4.4451e+04	6.9959e−01

TABLE 19. Times for 100 point evaluations of the tensor product multilevel interpolant to the target function GAUSSIAN in seconds, for threshold $\ell = 1, \ldots, 6$, using the general and simple representations introduced in Section 7.1.

suggests that a naive implementation of (7.1.12) quickly becomes numerically infeasible. We hope that in the future we find a better way to represent the tensor product multilevel method and that the use of the localized Lagrange functions will help in reducing the computational costs.

7.4. Double-Adaptive Tensor Product Multilevel Method

To finish this chapter we introduce a double-adaptive version of the tensor product multilevel method. We restrict ourselves to the easy case, i.e., the target function is an element of an algebraic tensor product space and we have access to the direction-wise information. For the general case it would be necessary to do a thorough cost analysis of the tensor product multilevel method in its general representation.

We start by refining the ideas discussed in Section 6.4 for the new tensor product multilevel method. In Definition 6.4.1 we introduced, for a fixed target function $\boldsymbol{f}$, the weighted error estimator $g_{\boldsymbol{f},\omega} : \mathbb{N}^d \to \mathbb{R}$ given by

$$g_{\boldsymbol{f},\omega}(\boldsymbol{\lambda}) := q\left(\omega \operatorname{error}_{\boldsymbol{f}}(\boldsymbol{\lambda}), (1-\omega)\frac{1}{\operatorname{cost}_{\boldsymbol{f}}(\boldsymbol{\lambda})}\right),$$

with two mappings $\operatorname{error}_{\boldsymbol{f}} : \mathbb{N}^d \to \mathbb{R}$ and $\operatorname{cost}_{\boldsymbol{f}} : \mathbb{N}^d \to \mathbb{R}$ which should represent the expected (relative) approximation error and the expected (relative) cost to compute the approximation. The mapping $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ moderates these two values for each multi-index $\boldsymbol{\lambda} \in \mathbb{N}^d$. We fix

$$q(\cdot, \cdot) = \max(\cdot, \cdot)$$

and discuss our choices for the remaining two mappings.

Since we assume that we use the simple representation we have access to the direction-wise data $(\boldsymbol{x}_k^{(j)}, f^{(j)}(\boldsymbol{x}_k^{(j)}))$, $1 \leq j \leq d$. Hence, we can use the residual correction representation of the multilevel operator, described in Section 4.2, for every direction independently. That means in particular that we can compute the residuals $e_i^{(j)}$, $i \in \mathbb{N}$, $1 \leq j \leq d$.

First, we fix the mapping $\mathrm{error}_{\boldsymbol{f}}$. An obvious choice would be to use the recursion inequality (4.3.6),

$$\|E^{(j)}e_i^{(j)}\|_{\Phi_{i+1}^{(j)}} \leq \alpha^{(j)}\|E^{(j)}e_{i-1}^{(j)}\|_{\Phi_i^{(j)}},$$

with $\alpha^{(j)} < 1$, for every direction $1 \leq j \leq d$ and therefore we would like to set

$$\mathrm{error}_{\boldsymbol{f}}(\boldsymbol{\lambda}) := \prod_{j=1}^{d} \left\| E^{(j)}e_{\lambda_j}^{(j)} \right\|_{\Phi_{\lambda_j+1}^{(j)}}.$$

This way we guarantee monotonicity of the mapping $\mathrm{error}_{\boldsymbol{f}}$. However, in numerical reality the residual is only known point-wise and computing a continuous norm is therefore hard.

Hence, our idea is to use a discrete norm. We take a closer look at the multilevel method in Algorithm 1 and briefly omit the superscript for the dependence on the direction. We implicitly have to evaluate the residual of level i, e_i, on the data sites of the next level X_{i+1} to prepare for the approximation on level $i+1$. This is computational work that is usually done at the beginning of the for-loop with index $\widetilde{i} = i+1$, however we can also move this work to the end of the loop, which means that we obtain the data $(e_{\widetilde{i}-1}(\boldsymbol{x}_{\widetilde{i},k}))_{1\leq k\leq N_{\widetilde{i}}}$ at the end of the for-loop with index i as long as the current level is not the last. Thus, in levels $1 \leq i < L$ we can obtain the value $\|e_i\|_{\ell_\infty(X_{i+1})}$ with minimal additional computational cost, we only have to find the maximum of the values $(e_i(\boldsymbol{x}_{i+1,k}))$ which we already computed. This idea motivates our choice for $\mathrm{error}_{\boldsymbol{f}}$,

$$\mathrm{error}_{\boldsymbol{f}}(\boldsymbol{\lambda}) = \prod_{j=1}^{d} \frac{\|e_{\lambda_j-1}^{(j)}\|_{\ell_\infty(X_{\lambda_j}^{(j)})}}{\|e_0^{(j)}\|_{\ell_\infty(X_1^{(j)})}},$$

for $\boldsymbol{\lambda} \in \mathbb{N}^d$. We also introduced an optional relativation with $\|e_0^{(j)}\|_{\ell_\infty(X_1^{(j)})} = \|f^{(j)}\|_{\ell_\infty(X_1^{(j)})}$.

We already noted that this choice leads to minimal additionally computational costs, however, there is a larger downside of this choice of $\mathrm{error}_{\boldsymbol{f}}$. Unfortunately, we can not guarantee that this mapping is monotonously decreasing. Nevertheless, we will use it in applications.

Second, we have to fix the mapping $\mathrm{cost}_{\boldsymbol{f}}$, which represents the costs that arise when computing the approximation corresponding to the respective multi-index. Again, we recall Chapter 4, in particular Corollary 4.2.1. We know that the computational cost to solve the direction-wise approximation problem is linear and hence, we fix $\mathrm{cost}_{\boldsymbol{f}}$ by

$$\frac{1}{\mathrm{cost}_{\boldsymbol{f}}(\boldsymbol{\lambda})} = \prod_{j=1}^{d} \frac{N_1^{(j)}}{N_{\lambda_j}^{(j)}}.$$

This yields the following definition.

Definition 7.4.1. *In the case of the simple representation of the tensor product multilevel method Section 7.1.2 we define the weighted error estimator*

$g_{\boldsymbol{f},\omega} : \mathbb{N}^d \to \mathbb{R}$ *by*

$$g_{\boldsymbol{f},\omega}(\boldsymbol{\lambda}) = \max\left(\omega \prod_{j=1}^{d} \frac{\|e^{(j)}_{\lambda_j-1}\|_{\ell_\infty(X^{(j)}_{\lambda_j})}}{\|e^{(j)}_0\|_{\ell_\infty(X^{(j)}_1)}}, (1-\omega)\prod_{j=1}^{d}\frac{N^{(j)}_1}{N^{(j)}_{\lambda_j}}\right)$$

for fixed $\boldsymbol{f}$ *and* $\omega \in [0,1]$.

The combination with the truly adaptive version of the multilevel method of Section 4.5 is now straight forward. We recall that in this version we evaluate the residual on level i in the sites of the upcoming level and use only these points for which the absolute value of the evaluated residual is larger than a given threshold. That means that we can also use the weighted error estimator of Definition 7.4.1 in this case if we keep in mind that we do not use the original direction-wise sequence of sets of sites $X^{(j)}_i$, $i \in \mathbb{N}$, but rather the adaptive point sets $\mathcal{X}^{(j)}_i$. Clearly, we then have to also use the cardinality of $\mathcal{X}^{(j)}_i$ for $\mathrm{cost}_{\boldsymbol{f}}$.

To test these two adaptive methods we revisit the problem from Section 7.3.1, i.e., the interpolation of the two-dimensional target function

$$\boldsymbol{f} : [-1,1]^2 \to \mathbb{R}, \quad \boldsymbol{f}(\boldsymbol{x}) = |x^{(1)}|^{1.6} \cdot |x^{(2)}|^{3.6}.$$

First, we use the standard kernel-based multilevel method in both directions with parameters as in Table 14 however, we construct the index set of the Smolyak method adaptively using the error estimator of Definition 7.4.1. We test only the two extremal cases, $\omega = 0.0$ and $\omega = 1.0$.

We recall our discussion in Section 7.3.1, in particular the phenom encountered in studying Fig. 12. We saw that the tensor product multilevel method with fully isotropic parameter setting performs much better than the one in the fully anisotropic setting, in the sense that the error as a function of the number of sparse grid points decays a lot faster. In Fig. 13, we added the interpolation errors of the single-adaptive tensor product multilevel method. At the end of every iteration of the while-loop in Algorithm 4 we evaluated the interpolant on a uniform grid with 15000×15000 points and also computed the number of points in the sparse grid associated to the inductively constructed index set. We see that the single adaptive tensor product multilevel method performs as good as the fully isotropic one, independent of the value of the weight ω. Clearly, in every iteration of Algorithm 4, the change of the number of points in the sparse grid changes much less than in the non-adaptive tensor product multilevel method since at most two, the number of directions, multi-indices are added to the adaptive index set. We see also that, depending on the indices that are added, i.e., there may be not a large decay in the interpolation error.

In Fig. 14 we show snapshots of the evolution of the adaptively constructed index set for the single adaptive tensor product multilevel method with weight $\omega = 1.0$. We note that the evolution of the index set for $\omega = 0.0$ looks similar. Again, the active indices are marked in green, the offline indices in grey and the index with the largest local error estimator is the one that has the arrows attached. We remark two things: First, we see that the index sets at iterations 9, 12 and 14 are not anisotropic index sets according to Definition 6.2.1, there

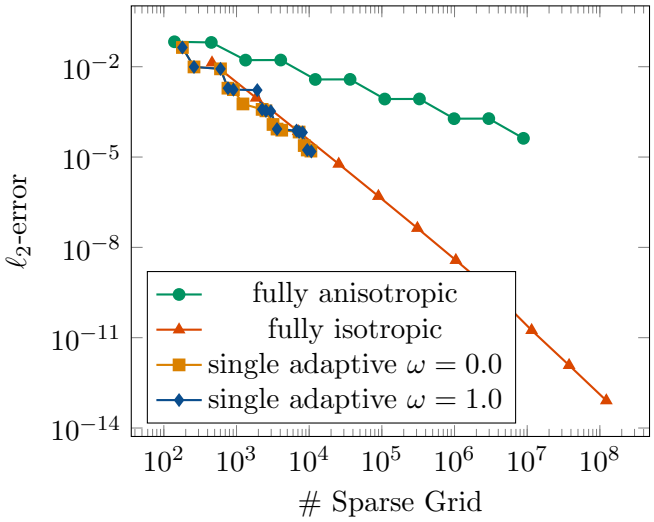

FIGURE 13. The ℓ_2-interpolation errors of $\boldsymbol{f}(\boldsymbol{x}) = |x^{(1)}|^{1.6} \cdot |x^{(2)}|^{3.6}$ for different versions of the tensor product multilevel method, as functions of the number of points in the, to the used index sets, associated sparse grids.

is no combination of weight vector $\boldsymbol{\omega}$ and threshold ℓ such that $\mathcal{I}_{\boldsymbol{\omega}}(\ell, 2)$ has this shape. And second, more importantly, in this example, the algorithm for the adaptive Smolyak method, Algorithm 4, seems to prefer the direction in which the target function $\boldsymbol{f}$ is smoother, in the sense that it chooses to extend the adaptive index set further into the second direction than the first. This is remarkable, since our intuition suggests that we need more levels in the direction where the target function is rough to obtain a better approximation. This is the reason why, in Theorem 7.2.1 and all following convergence results for the tensor product multilevel method, we assume that the weight vector $\boldsymbol{\omega}$ is ascendingly ordered. This means that the theoretical results in Section 7.2 can not be applied for the adaptive tensor product multilevel method and have to be revised to obtain convergence results for this adaptive method.

For the sake of completeness we give the values of the global error estimate η for weights $\omega = 0.0$ and 1.0 in Fig. 15. We recall that this value determines if the adaptive Smolyak method, described in Algorithm 4, terminates. Because of the relativation, η is set in the initialization phase to 1 and, in our examples, decays monotonously towards zero. This has not always to be the case and is highly dependent on the problem at hand. However, we expect that after enough iterations the value of η becomes sufficiently small for the algorithm to terminate.

Next, we demonstrate the double-adaptive version of the tensor product multilevel method. We use the same setting as before, i.e., the parameters given in Table 14 and the same target function as above. Again, we construct the index set adaptively following Algorithm 4 with the already discussed changes to the weighted error estimator of Definition 7.4.1. However, in this version of the method we use the refinement parameters $\mu^{(1)}$ and $\mu^{(2)}$ to

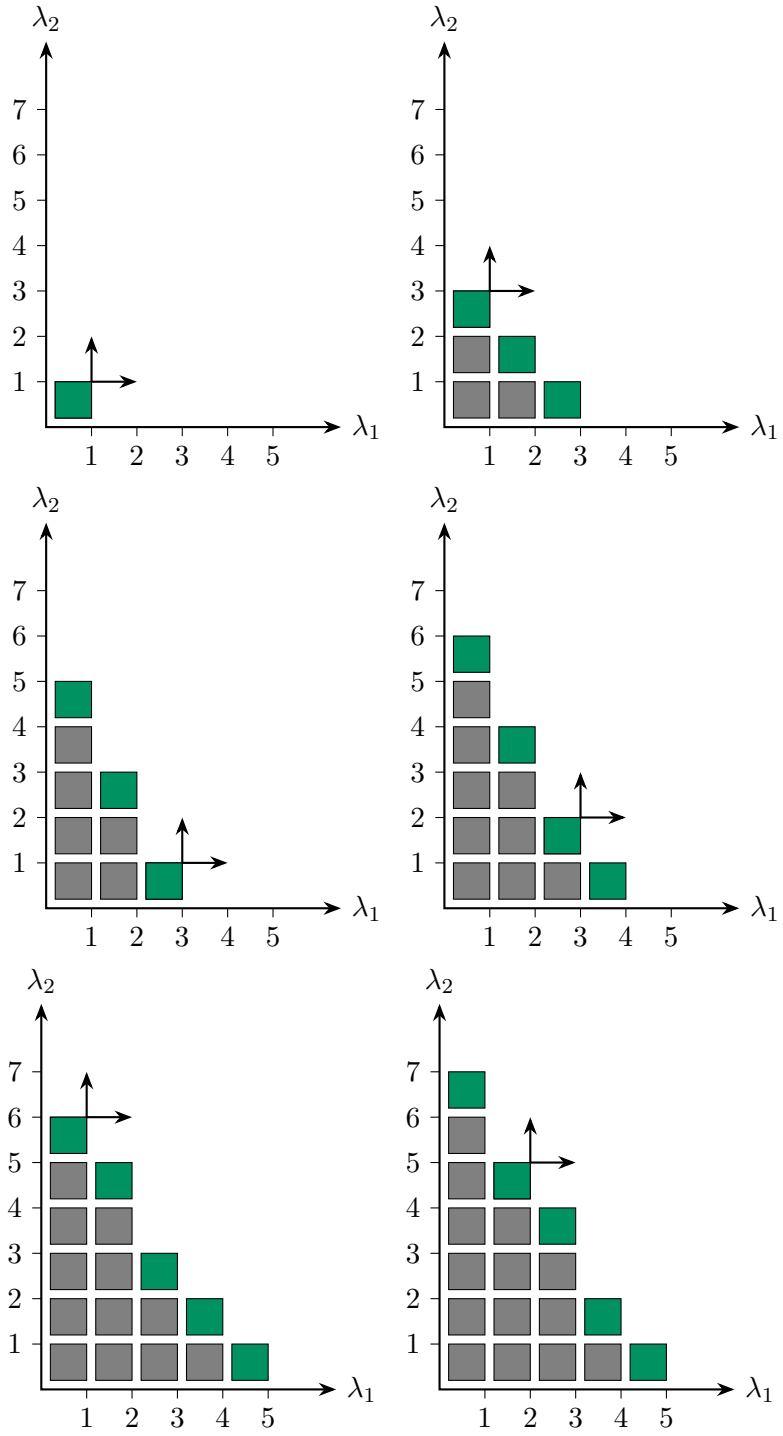

FIGURE 14. Evolution of the adaptively constructed index set for weight $\omega = 1.0$ at iterations 0, 3, 6, 9, 12 and 14 (left to right, top to bottom), for the single-adaptive tensor product multilevel method.

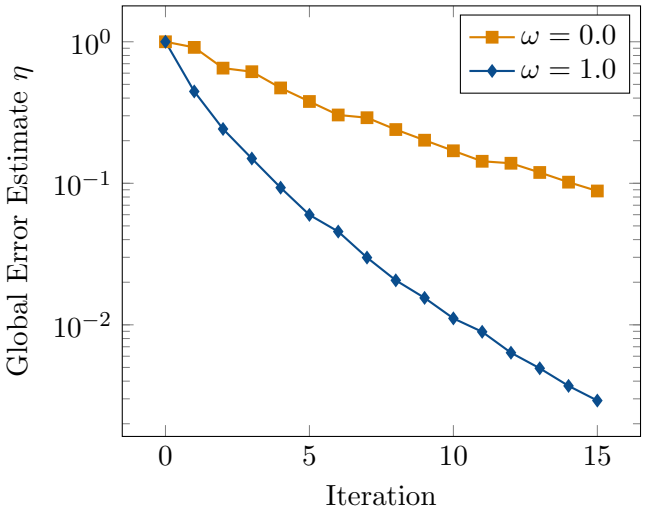

FIGURE 15. The global error estimate η as a function of the number of iterations of the adaptive Smolyak algorithm Algorithm 4, using the weighted error estimator $g_{\boldsymbol{f},\omega}$ as in Definition 7.4.1 for weights $\omega = 0.0$ and 1.0.

obtain the direction-wise sets of sites $(X_i^{(1)})$ and $(X_i^{(2)})$, respectively, and choose among their elements the adaptive point sets $(\mathcal{X}_i^{(1)})$ and $(\mathcal{X}_i^{(2)})$ as is described in Algorithm 3. We then use these as data sites for the interpolation problem. We recall that we add an $\boldsymbol{x} \in X_{i+1}^{(j)}$ to $\mathcal{X}_{i+1}^{(j)}$ if $|e_i^{(j)}(\boldsymbol{x})| > \varepsilon_i^{(j)}$ with a threshold $\varepsilon_i^{(j)}$ that we can choose. We follow the ideas of [92] and set

$$\varepsilon_i^{(j)} := 10^{-3} \cdot \max_{\boldsymbol{x} \in X_{i+1}^{(j)}} |e_i^{(j)}(\boldsymbol{x})|, \quad i \in \mathbb{N}, j = 1, 2. \tag{7.4.1}$$

We emphasize that we use different refinement parameters in the two directions but the same thresholding strategy. If this is the right strategy should be explored in the future.

Applying the double-adaptive tensor product multilevel method to the model problem of Section 7.3.1 leads to the ℓ_2-errors plotted in Fig. 16, again computed on a uniform grid with 15000×15000 points. We compare the errors to the fully isotropic and fully anisotropic setting of the standard tensor product multilevel method. We see that the double adaptive tensor product multilevel method again performs as good as the fully isotropic setting if we choose the weight $\omega = 0.0$ in the weighted error estimator. We recall that for $\omega = 0.0$ we adapt the index set on the basis of the relative costs. Surprisingly, for $\omega = 1.0$, the behavior of the error is much more erratic with a quasi constant error for a number of iterations, a sharp drop and then again quasi constant error. Looking for an explanation, we start by studying the evolution of the index set in Fig. 17. We see that that the drop in the error happens when the algorithm of the adaptive Smolyak method decides to extend the adaptive index set in the first direction and the error stagnates when the index set is extended into the second direction. Furthermore, looking at the evolution of the global error estimator η in Fig. 18, for $\omega = 1.0$, we can again

observe this behavior with two distinct peaks. These are at the number of iterations where the sharp decay of the graph in Fig. 16 happens. Again, for $\omega = 0.0$, the global error estimate decays monotonously. The final explanation for this phenom comes from Fig. 19. There, we plotted the maximum of the residual $e_i^{(j)}$ on level i, evaluated on the upcoming point set $X_{i+1}^{(j)}$ for both directions. We recall that for $\omega = 1.0$, we only refine on the basis of the residuals. We can see that the residual in the second direction grows from the start before it begins to fall monotonously from level three onward. This ties in with the beginning of the chapter where we discussed the choice for $\text{error}_{\boldsymbol{f}}$. In general and particularly for the adaptive multilevel method, we can not guarantee that $\|e_i^{(j)}\|_{\ell_\infty(X_{i+1}^{(j)})}$ or $\|e_i^{(j)}\|_{\ell_\infty(\mathcal{X}_{i+1}^{(j)})}$ and therefore our choices for $\text{error}_{\boldsymbol{f}}$ decay monotonously or even decay at all. The behavior of $e_i^{(2)}$ stems from the adaptive multilevel method. Obviously, we need a deeper theoretical understanding of this method to incorporate it into the thresholding strategy in (7.4.1) to hopefully avoid this behavior in the future.

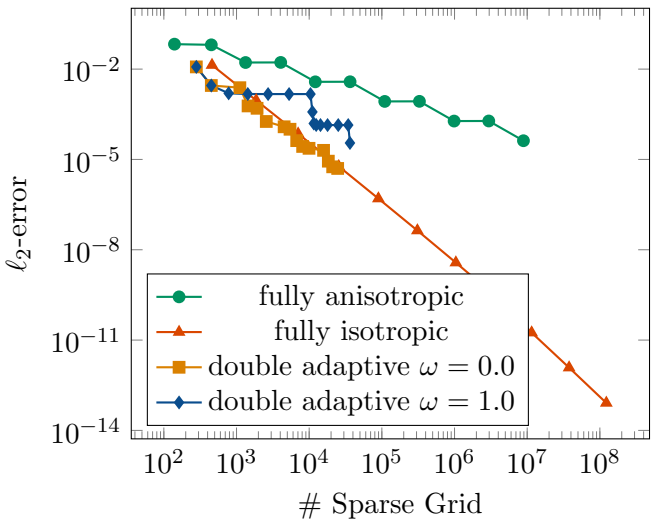

FIGURE 16. The ℓ_2-interpolation errors of $\boldsymbol{f}(\boldsymbol{x}) = |x^{(1)}|^{1.6} \cdot |x^{(2)}|^{3.6}$ for different versions of the tensor product multilevel method, as functions of the number of points in the, to the used index sets, associated sparse grids.

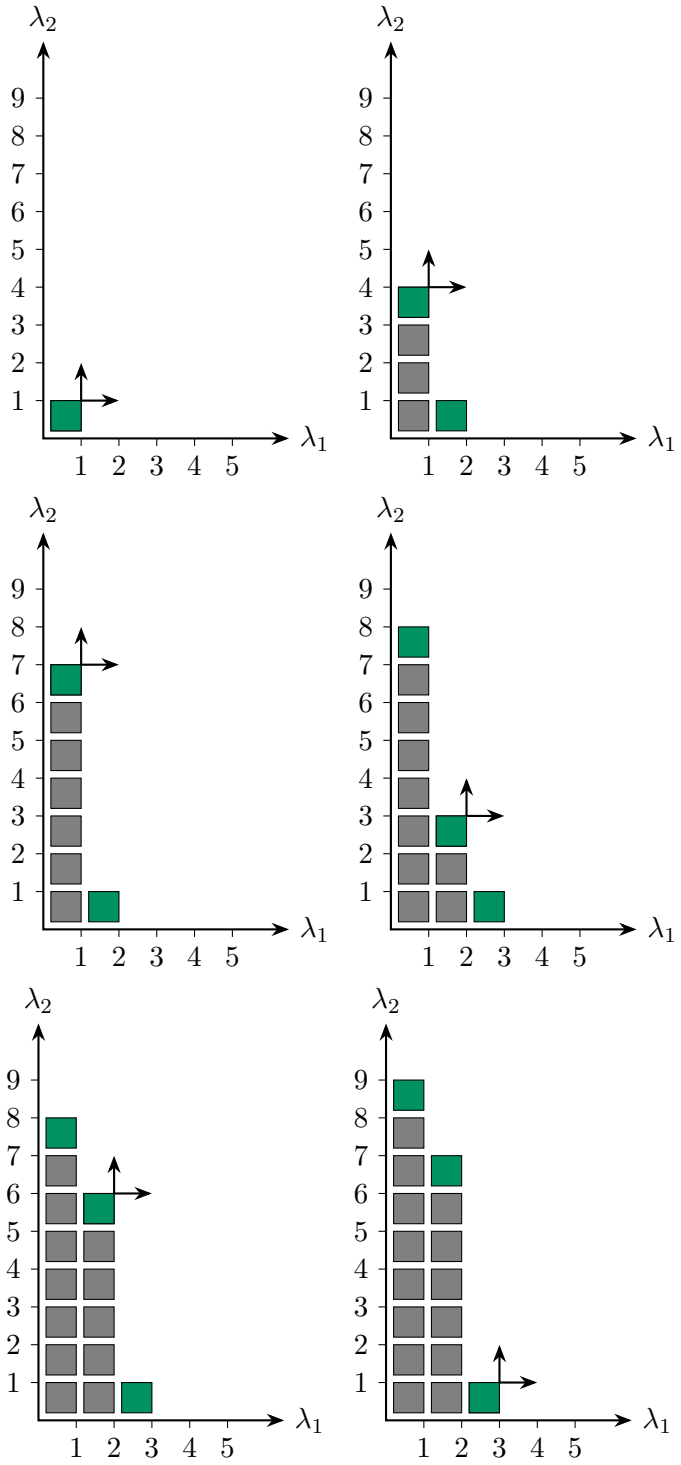

FIGURE 17. Evolution of the adaptively constructed index set for weight $\omega = 1.0$ at iterations 0, 3, 6, 9, 12 and 14 (left to right, top to bottom), for the double-adaptive tensor product multilevel method.

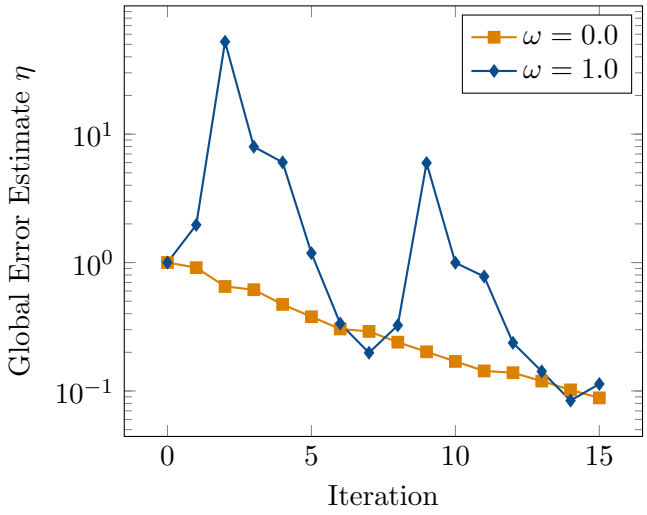

FIGURE 18. The global error estimate η as a function of the number of iterations of the adaptive Smolyak algorithm, Algorithm 4, used in the double-adaptive tensor product method, using the weighted error estimator $g_{\boldsymbol{f},\omega}$ as in Definition 7.4.1, with the necessary changes for the double-adaptive method, for weights $\omega = 0.0$ and 1.0.

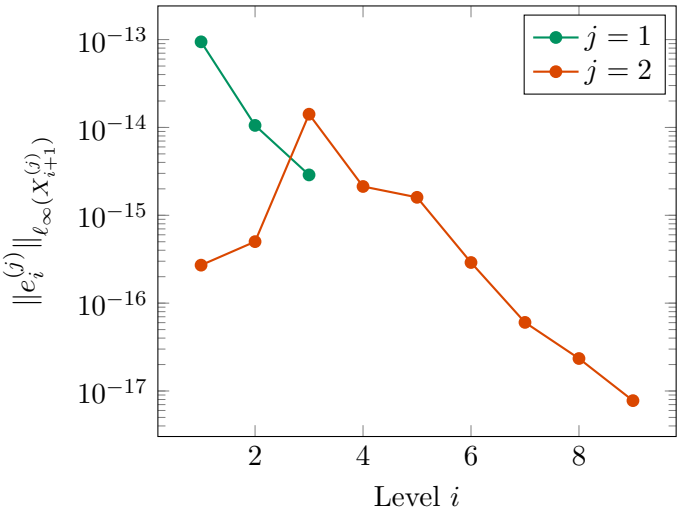

FIGURE 19. Largest values of the residuals $e_i^{(j)}$, evaluated on the upcoming point set $X_{i+1}^{(j)}$, per level i, of the adaptive multilevel method with thresholding strategy (7.4.1).

CHAPTER 8

Conclusions and Outlook

Motivated by a model problem, the reconstruction of a quantity of interest of the solution of a parameteric partial differential equation, the aim of this thesis was the development of a new method to solve high-dimensional approximation problems. We achieved this goal by combining two well-known methods. Our idea was to use the Smolyak method which enabled us to construct an high-dimensional approximation operator from families of arbitrary low-dimensional ones. This method is known to be flexible enough to use different reconstruction schemes in different directions. This is helpfull if the target function exhibits, e.g., different smoothnesses in different direction. Additionally, the construction of the Smolyak operator done such that the new reconstruction method exhibits nearly the same approximation features as the building-block operators and is perfectly suited for use with multilevel schemes.

This motivated us to use kernel-based multilevel methods as the low-dimensional approximation operators. These methods are based on a residual correction scheme and are known to be fast and stable if it uses compactly supported radial basis functions and have the advantage that we can use scattered data in nearly general domains.

The combination of these two methods, the Smolyak algorithm on the one hand and the kernel-based multilevel method on the other yielded the tensor product multilevel method. It is a scheme that allows us to recover unknown, anisotropic tensor product functions from meshless high-dimensional data in arbitrary Lipschitz domains using different reconstruction schemes such as interpolation and penalized least squares approximation. For working with tensor product functions it is reasonable to differentiate two cases, whether we have access to the low-dimensional, factor functions or not. In the former case we employed the residual correction representation of the multilevel method in each direction and obtained an easy representation of the tensor product multilevel method, see Section 7.1.2. In the latter case, it was necessary to derive a different representation of the low-dimensional approximation method in Section 4.6.1 and to get a better understanding how the reconstruction operator depends on the target function, thereby getting rid of the residuals altogether by introducing Lagrange functions. This way we obtained the tensor product multilevel method in its general representation in Section 7.1.3 which is also easy to implement.

Following the introduction of the tensor product multilevel method we gave a rigorous error analysis. The key was that the Smolyak operators allows us to use direction-wise error estimates if we use Banach spaces as domain and codomain of the operator that have uniformly compatible tensor product norms, see Section 6.1.5. With these results the error estimates of Section 7.2

were straight forward since the convergence of the multilevel method for interpolation and penalized least squares approximation are well-known, see Section 4.3. However, the results in their most general form are hard to interpret, see, e.g., Theorem 7.2.1 for interpolation, since they contain a maximum amount of information about the anisotropy of the target function. That is why we gave a series of lemmas in which we traded this information for readability of the estimate. The resulting results, e.g., in (7.2.11), are in a very closed form, however only contain information about the first and last direction.

Preliminary numerical tests indicated that the tensor product multilevel method in its general representation is very expensive but we have to use this version of the reconstruction scheme if we have no information about the target function. The cost became already numerically unfeasible for moderate dimensions and moderate numbers of levels. This can be attributed to the necessity to use the Lagrange functions and the exponential growth of the number of evaluations per level. This was the motivation to establish another basis of the approximation space, the localized Lagrange functions. The idea is to enforce the Lagrange condition only on a small subset of the set of sites and we derived conditions under which these localized Lagrange functions form a basis of the kernel-based approximation space, see Lemma 3.3.4. This allowed us to introduce a quasi-interpolation-like approximation scheme, which we also used in the multilevel scheme. This yielded the local Lagrange multilevel method. The estimates derived in Section 3.3 allowed us to obtain convergence results for this new method, which in turn were used in Section 7.2.3 to derive error bounds for the high-dimensional approximation problem.

Although inspired by previous results the introduction and study of localized Lagrange functions in the context of Wendland functions is original work and the results presented in Section 3.3 are still crude. In particular the practical use in the multilevel method should be studied in more detail in the future. We introduced them as a first way to make the tensor product multilevel method in its general representation feasible. Another possible way to speed up this method is a closer look at the new representation of the multilevel operator in Theorem 4.6.4. At least for a nested sequence of sets of sites we know that many of the coefficients $a(\mathbf{u}, \boldsymbol{k})$ are zero and we should be able to significantly reduce the number of point evaluations.

Furthermore, we want to take a more thorough look at the adaptive versions of the multilevel method in Section 4.5 and their combination with the direction-adaptive Smolyak method in Section 7.4. The former method clearly lacks a proof of convergence and the latter needs to be studied in greater depth. Obviously, we need to generalize the discussion held in Section 7.4 to the case that we do not have the direction-wise information. Again, this requires a better understanding of the tensor product multilevel method in its general representation. Additionally, a more thorough investigation of the version given here is in order. However, initial numerical results give us hope that the double-adaptive version of the tensor product multilevel method can resolve the numerical phenomenon encountered in Section 7.3.1.

APPENDIX A

Error Tables to Section 4.4

We provide here the error tables of the numerical simulation which leads to the estimated multilevel constant in Section 4.4.

A.1. $\Omega = [-1, 1]$

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	4.8796e−01	4.8796e−01	4.8796e−01	4.8796e−01	4.8796e−01	4.8796e−01
2	2.9681e−02	3.0939e−02	2.1642e−02	1.7499e−02	1.4370e−02	1.2924e−02
3	5.4722e−03	3.4612e−03	2.2596e−03	1.6739e−03	1.3457e−03	1.0754e−03
4	6.9015e−04	3.8067e−04	2.2379e−04	1.5328e−04	1.1995e−04	9.2056e−05
5	8.3623e−05	4.0886e−05	2.1919e−05	1.3990e−05	1.0652e−05	7.9398e−06
6	1.0126e−05	4.3836e−06	2.1444e−06	1.2767e−06	9.4834e−07	6.8423e−07
7	1.2234e−06	4.6994e−07	2.0972e−07	1.1659e−07	8.4569e−08	5.9270e−08
8	1.4774e−07	5.0380e−08	2.0508e−08	1.0654e−08	7.5552e−09	5.1565e−09
9	1.7832e−08	5.3990e−09	2.0045e−09	9.7401e−10	6.7640e−10	4.5221e−10
10	2.0812e−09	5.5844e−10	1.8896e−10	8.6358e−11	5.9516e−11	4.0060e−11
11	8.4063e−11	2.8396e−11	8.9196e−12	4.9259e−12	4.3650e−12	3.8873e−12

TABLE 20. Table of errors for $k = 1$ and $\mu = 0.4$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	4.0162e−01	4.0162e−01	4.0162e−01	4.0162e−01	4.0162e−01	4.0162e−01
2	2.1332e−02	4.2129e−02	3.9192e−02	3.0056e−02	2.2094e−02	1.7520e−02
3	7.7855e−03	5.6236e−03	3.9200e−03	2.5203e−03	1.7375e−03	1.2750e−03
4	9.6382e−04	6.5258e−04	3.9568e−04	2.2256e−04	1.3924e−04	9.4108e−05
5	1.3465e−04	7.6025e−05	4.0158e−05	1.9805e−05	1.1242e−05	6.9852e−06
6	1.8258e−05	8.8426e−06	4.0803e−06	1.7643e−06	9.0868e−07	5.1871e−07
7	2.4851e−06	1.0279e−06	4.1467e−07	1.5719e−07	7.3457e−08	3.8515e−08
8	3.3815e−07	1.1945e−07	4.2143e−08	1.4005e−08	5.9382e−09	2.8596e−09
9	4.6005e−08	1.3875e−08	4.2814e−09	1.2472e−09	4.7982e−10	2.1221e−10
10	6.0257e−09	1.5598e−09	4.2103e−10	1.0725e−10	3.7390e−11	1.5182e−11
11	1.0087e−10	9.1461e−11	2.5090e−11	5.5247e−12	1.7305e−12	6.5306e−13

TABLE 21. Table of errors for $k = 2$ and $\mu = 0.4$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	3.4923e−01	3.4923e−01	3.4923e−01	3.4923e−01	3.4923e−01	3.4923e−01
2	2.3306e−02	4.4288e−02	5.1376e−02	4.5944e−02	3.6961e−02	2.8228e−02
3	1.2226e−02	7.4969e−03	5.9601e−03	4.3588e−03	2.9607e−03	2.0287e−03
4	1.7556e−03	9.2141e−04	6.6342e−04	4.2522e−04	2.5243e−04	1.5420e−04
5	3.0099e−04	1.1624e−04	7.3812e−05	4.1691e−05	2.1696e−05	1.1799e−05
6	4.6721e−05	1.4586e−05	8.2025e−06	4.0917e−06	1.8676e−06	9.0367e−07
7	7.3208e−06	1.8286e−06	9.1110e−07	4.0166e−07	1.6081e−07	6.9223e−08
8	1.1311e−06	2.2916e−07	1.0118e−07	3.9430e−08	1.3848e−08	5.3027e−09
9	1.7423e−07	2.8704e−08	1.1232e−08	3.8693e−09	1.1919e−09	4.0601e−10
10	2.6359e−08	3.4635e−09	1.2080e−09	3.6765e−10	9.9048e−11	2.9941e−11
11	3.1046e−09	1.4745e−10	7.7261e−11	2.2267e−11	5.1707e−12	1.3624e−12

TABLE 22. Table of errors for $k = 3$ and $\mu = 0.4$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	4.8796e−01	4.8796e−01	4.8796e−01	4.8796e−01	4.8796e−01	4.8796e−01
2	3.8436e−02	5.9225e−02	6.8711e−02	5.2285e−02	4.6501e−02	5.0296e−02
3	3.1181e−02	2.3109e−02	1.9067e−02	1.3352e−02	1.0626e−02	8.3451e−03
4	8.7728e−03	6.2729e−03	4.3737e−03	2.9541e−03	2.3743e−03	1.9593e−03
5	2.6002e−03	1.5837e−03	9.8947e−04	6.3794e−04	5.0776e−04	4.2377e−04
6	7.2761e−04	3.9509e−04	2.2448e−04	1.3849e−04	1.0970e−04	8.9244e−05
7	2.0344e−04	9.8041e−05	5.0845e−05	3.0068e−05	2.3601e−05	1.8906e−05
8	5.6734e−05	2.4320e−05	1.1527e−05	6.5133e−06	5.0816e−06	4.0174e−06
9	1.5820e−05	6.0268e−06	2.6136e−06	1.4121e−06	1.0957e−06	8.5180e−07
10	4.4112e−06	1.4941e−06	5.9274e−07	3.0576e−07	2.3650e−07	1.8102e−07
11	1.2301e−06	3.7030e−07	1.3444e−07	6.6248e−08	5.1090e−08	3.8487e−08

TABLE 23. Table of errors for $k = 1$ and $\mu = 0.5$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	4.0162e−01	4.0162e−01	4.0162e−01	4.0162e−01	4.0162e−01	4.0162e−01
2	6.6514e−02	6.7300e−02	9.6927e−02	9.7355e−02	8.1254e−02	6.5037e−02
3	4.5459e−02	4.1746e−02	3.2865e−02	2.6204e−02	1.9739e−02	1.4770e−02
4	1.7125e−02	1.1047e−02	8.1286e−03	6.0233e−03	4.2115e−03	2.9541e−03
5	5.9891e−03	3.0919e−03	2.0130e−03	1.3615e−03	8.7560e−04	5.7781e−04
6	1.9018e−03	8.3238e−04	4.9523e−04	3.0717e−04	1.8193e−04	1.1286e−04
7	6.0580e−04	2.2513e−04	1.2158e−04	6.9266e−05	3.7810e−05	2.2022e−05
8	1.9010e−04	6.0803e−05	2.9816e−05	1.5616e−05	7.8583e−06	4.2947e−06
9	5.9771e−05	1.6429e−05	7.3082e−06	3.5201e−06	1.6332e−06	8.3722e−07
10	1.8747e−05	4.4396e−06	1.7909e−06	7.9346e−07	3.3942e−07	1.6318e−07
11	5.8837e−06	1.1998e−06	4.3881e−07	1.7885e−07	7.0538e−08	3.1801e−08

TABLE 24. Table of errors for $k = 2$ and $\mu = 0.5$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	3.4923e−01	3.4923e−01	3.4923e−01	3.4923e−01	3.4923e−01	3.4923e−01
2	6.9928e−02	7.6603e−02	1.1959e−01	1.3702e−01	1.3240e−01	1.1636e−01
3	4.8410e−02	5.9081e−02	5.0290e−02	4.1679e−02	3.4954e−02	2.8330e−02
4	2.4735e−02	1.8074e−02	1.3023e−02	1.0207e−02	8.0336e−03	6.0782e−03
5	9.7422e−03	5.6131e−03	3.4821e−03	2.4942e−03	1.8130e−03	1.2668e−03
6	3.5045e−03	1.6224e−03	9.1118e−04	6.0468e−04	4.0715e−04	2.6309e−04
7	1.2264e−03	4.7322e−04	2.3861e−04	1.4619e−04	9.1286e−05	5.4608e−05
8	4.2462e−04	1.3707e−04	6.2419e−05	3.5300e−05	2.0456e−05	1.1334e−05
9	1.4644e−04	3.9789e−05	1.6327e−05	8.5194e−06	4.5830e−06	2.3527e−06
10	5.0366e−05	1.1552e−05	4.2702e−06	2.0556e−06	1.0267e−06	4.8837e−07
11	1.7291e−05	3.3560e−06	1.1169e−06	4.9592e−07	2.3001e−07	1.0138e−07

TABLE 25. Table of errors for $k = 3$ and $\mu = 0.5$.

A.2. $\Omega = [0, 1]$

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	1.8148e−01	1.8148e−01	1.8148e−01	1.8148e−01	1.8148e−01	1.8148e−01
2	6.3800e−03	4.0653e−03	3.0983e−03	2.6309e−03	1.8963e−03	2.0862e−03
3	5.7334e−04	3.5901e−04	2.6499e−04	2.1444e−04	1.5659e−04	1.6566e−04
4	5.2571e−05	3.1268e−05	2.2876e−05	1.8534e−05	1.4148e−05	1.4062e−05
5	4.7687e−06	2.6469e−06	1.8748e−06	1.4834e−06	1.0844e−06	1.0588e−06
6	4.3750e−07	2.3182e−07	1.6612e−07	1.3428e−07	1.0393e−07	1.0064e−07
7	3.9608e−08	1.9397e−08	1.3382e−08	1.0625e−08	7.7311e−09	7.2981e−09
8	3.6469e−09	1.7366e−09	1.2494e−09	1.0356e−09	8.4069e−10	8.0255e−10
9	3.2926e−10	1.4315e−10	9.8139e−11	7.9799e−11	6.0016e−11	5.5770e−11
10	2.9317e−11	1.2957e−11	9.7491e−12	8.4801e−12	7.4010e−12	7.1044e−12
11	1.1316e−12	4.9211e−13	3.7808e−13	3.2601e−13	2.7130e−13	2.5093e−13

TABLE 26. Table of errors for $k = 1$ and $\mu = 0.4$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	8.5701e−02	8.5701e−02	8.5701e−02	8.5701e−02	8.5701e−02	8.5701e−02
2	5.6108e−03	3.2580e−03	2.3183e−03	1.6092e−03	1.3552e−03	9.9340e−04
3	4.6904e−04	2.3467e−04	1.3948e−04	8.9374e−05	6.7014e−05	4.6970e−05
4	4.1399e−05	1.7311e−05	8.9003e−06	5.0973e−06	3.4791e−06	2.2955e−06
5	3.6836e−06	1.2848e−06	5.7173e−07	2.9241e−07	1.8252e−07	1.1261e−07
6	3.2816e−07	9.5406e−08	3.6761e−08	1.6796e−08	9.5879e−09	5.5260e−09
7	2.9237e−08	7.0841e−09	2.3639e−09	9.6498e−10	5.0371e−10	2.7107e−10
8	2.6048e−09	5.2597e−10	1.5200e−10	5.5442e−11	2.6463e−11	1.3297e−11
9	2.3197e−10	3.9032e−11	9.7694e−12	3.1838e−12	1.3895e−12	6.5185e−13
10	1.9947e−11	2.7923e−12	6.0440e−13	1.7583e−13	7.0124e−14	3.0708e−14
11	1.0276e−12	1.2012e−13	2.2245e−14	5.7592e−15	2.1456e−15	9.9979e−16

TABLE 27. Table of errors for $k = 2$ and $\mu = 0.4$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	4.3772e−02	4.3772e−02	4.3772e−02	4.3772e−02	4.3772e−02	4.3772e−02
2	4.8413e−03	2.9020e−03	1.8236e−03	1.2788e−03	9.0811e−04	7.1868e−04
3	4.5813e−04	2.0809e−04	1.0859e−04	6.4125e−05	3.9965e−05	2.7706e−05
4	4.4646e−05	1.5800e−05	6.7785e−06	3.3847e−06	1.8333e−06	1.1243e−06
5	4.3767e−06	1.2088e−06	4.2576e−07	1.7970e−07	8.4462e−08	4.5839e−08
6	4.2953e−07	9.2577e−08	2.6767e−08	9.5461e−09	3.8925e−09	1.8695e−09
7	4.2164e−08	7.0915e−09	1.6831e−09	5.0714e−10	1.7938e−10	7.6241e−11
8	4.1392e−09	5.4323e−10	1.0583e−10	2.6941e−11	8.2660e−12	3.1090e−12
9	4.0618e−10	4.1594e−11	6.6510e−12	1.4304e−12	3.8068e−13	1.2671e−13
10	3.8595e−11	3.0673e−12	4.0125e−13	7.2710e−14	1.6767e−14	4.9971e−15
11	2.3375e−12	1.3957e−13	1.4645e−14	2.2109e−15	7.8502e−16	8.0835e−16

TABLE 28. Table of errors for $k = 3$ and $\mu = 0.4$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	1.6097e−01	1.6097e−01	1.6097e−01	1.6097e−01	1.6097e−01	1.6097e−01
2	1.7247e−02	1.6592e−02	1.2470e−02	9.4578e−03	8.1833e−03	7.4527e−03
3	4.4046e−03	2.7528e−03	2.6496e−03	2.2142e−03	1.4508e−03	1.3735e−03
4	9.7447e−04	6.4633e−04	5.8604e−04	3.8126e−04	3.6268e−04	2.4213e−04
5	2.1044e−04	1.3979e−04	1.2040e−04	8.5768e−05	7.1367e−05	5.7131e−05
6	4.5686e−05	2.9440e−05	2.5312e−05	1.8129e−05	1.4261e−05	1.1834e−05
7	9.9186e−06	6.2365e−06	5.3536e−06	3.8169e−06	2.9984e−06	2.4573e−06
8	2.1486e−06	1.3252e−06	1.1346e−06	8.0944e−07	6.3180e−07	5.2271e−07
9	4.6583e−07	2.8099e−07	2.4035e−07	1.7175e−07	1.3446e−07	1.1280e−07
10	1.0086e−07	5.9715e−08	5.1101e−08	3.6650e−08	2.8852e−08	2.4593e−08
11	2.1854e−08	1.2696e−08	1.0880e−08	7.8588e−09	6.2508e−09	5.4297e−09

TABLE 29. Table of errors for $k = 1$ and $\mu = 0.5$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	6.6242e−02	6.6242e−02	6.6242e−02	6.6242e−02	6.6242e−02	6.6242e−02
2	1.6058e−02	1.0727e−02	8.3318e−03	6.9664e−03	5.7351e−03	4.7843e−03
3	4.3221e−03	2.4362e−03	1.7463e−03	1.2268e−03	1.0883e−03	8.9701e−04
4	9.9347e−04	4.8724e−04	3.0375e−04	2.1364e−04	1.6739e−04	1.3764e−04
5	2.2456e−04	9.5303e−05	5.4041e−05	3.5563e−05	2.6230e−05	2.0944e−05
6	5.0665e−05	1.8615e−05	9.6155e−06	5.8934e−06	4.1319e−06	3.1716e−06
7	1.1425e−05	3.6323e−06	1.7094e−06	9.7534e−07	6.5084e−07	4.7956e−07
8	2.5756e−06	7.0836e−07	3.0370e−07	1.6131e−07	1.0248e−07	7.2463e−08
9	5.8060e−07	1.3809e−07	5.3938e−08	2.6667e−08	1.6132e−08	1.0946e−08
10	1.3087e−07	2.6915e−08	9.5775e−09	4.4072e−09	2.5389e−09	1.6531e−09
11	2.9499e−08	5.2452e−09	1.7004e−09	7.2829e−10	3.9957e−10	2.4963e−10

TABLE 30. Table of errors for $k = 2$ and $\mu = 0.5$.

	ℓ_2-Error					
level	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 4$
1	2.8800e−02	2.8800e−02	2.8800e−02	2.8800e−02	2.8800e−02	2.8800e−02
2	1.1300e−02	9.5961e−03	6.9045e−03	5.3063e−03	4.3448e−03	3.6804e−03
3	3.4372e−03	2.3363e−03	1.4665e−03	1.0551e−03	7.5715e−04	5.9658e−04
4	8.4176e−04	5.0126e−04	2.7580e−04	1.7427e−04	1.1704e−04	8.4509e−05
5	2.0569e−04	1.0447e−04	5.0078e−05	2.8357e−05	1.7383e−05	1.1566e−05
6	4.9867e−05	2.1697e−05	9.0577e−06	4.5954e−06	2.5633e−06	1.5764e−06
7	1.2056e−05	4.5035e−06	1.6360e−06	7.4358e−07	3.7718e−07	2.1438e−07
8	2.9112e−06	9.3473e−07	2.9531e−07	1.2024e−07	5.5453e−08	2.9127e−08
9	7.0259e−07	1.9402e−07	5.3295e−08	1.9440e−08	8.1494e−09	3.9558e−09
10	1.6952e−07	4.0276e−08	9.6170e−09	3.1425e−09	1.1974e−09	5.3713e−10
11	4.0899e−08	8.3608e−09	1.7353e−09	5.0796e−10	1.7593e−10	7.2927e−11

TABLE 31. Table of errors for $k = 3$ and $\mu = 0.5$.

List of Figures

List of Tables

Bibliography

[1] R. Adams and J. Fournier. *Sobolev Spaces*. ISSN. Elsevier Science, 2003 (cit. on pp. 10, 41).

[2] N. Aronszajn. "Theory of Reproducing Kernels". *Transactions of the American Mathematical Society* 68.3 (1950), pp. 337–404 (cit. on p. 14).

[3] J. Aubin. *Applied Functional Analysis*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2011 (cit. on p. 97).

[4] I. Babuška, F. Nobile, and R. Tempone. "A Stochastic Collocation Method for Elliptic Partial Differential Equations with Random Input Data". *SIAM Journal on Numerical Analysis* 45.3 (2007), pp. 1005–1034 (cit. on pp. 2, 3).

[5] I. Babuska, R. Tempone, and G. E. Zouraris. "Galerkin Finite Element Approximations of Stochastic Elliptic Partial Differential Equations". *SIAM Journal on Numerical Analysis* 42.2 (2004), pp. 800–825 (cit. on p. 2).

[6] V. Barthelmann, E. Novak, and K. Ritter. "High dimensional polynomial interpolation on sparse grids". In: vol. 12. 4. Multivariate polynomial interpolation. 2000, pp. 273–288 (cit. on pp. 4, 174–176).

[7] B. Baxter, N. Sivakumar, and J. Ward. "Regarding the p-norms of radial basis interpolation matrices". *Constructive Approximation* 10 (1994), pp. 451–468 (cit. on pp. 33, 34, 50).

[8] R. Beatson, J. Levesley, and C. Mouat. "Better bases for radial basis function interpolation problems". *Journal of Computational and Applied Mathematics* 236.4 (2011). International Workshop on Multivariate Approximation and Interpolation with Applications (MAIA 2010), pp. 434–446 (cit. on p. 37).

[9] R. Bellman, R. Corporation, and K. M. R. Collection. *Dynamic Programming*. Rand Corporation research study. Princeton University Press, 1957 (cit. on p. 2).

[10] S. Brenner and R. Scott. *The Mathematical Theory of Finite Element Methods*. Texts in Applied Mathematics. Springer New York, 2007 (cit. on pp. 1, 10, 11, 14).

[11] H.-J. Bungartz and M. Griebel. "Sparse grids". *Acta Numerica* 13 (2004), pp. 147–269 (cit. on pp. 4, 120, 124, 126).

[12] V. Burenkov. *Sobolev Spaces on Domains*. Teubner-Texte zur Mathematik. Vieweg+Teubner Verlag, 1998 (cit. on pp. 11–13).

[13] R. E. Caflisch. "Monte Carlo and quasi-Monte Carlo methods". *Acta Numerica* 7 (1998), pp. 1–49 (cit. on p. 3).

[14] K. Cliffe, I. Graham, R. Scheichl, and L. Stals. "Parallel Computation of Flow in Heterogeneous Media Modelled by Mixed Finite Elements".

Journal of Computational Physics 164.2 (2000), pp. 258–282 (cit. on pp. 1, 2).

[15] A. Cohen, R. DeVore, and C. Schwab. "Analytic Regularity and Polynomial Approximation of Parametric and Stochastic Elliptic PDE's". *Analysis and Applications* 09.01 (2011), pp. 11–47 (cit. on pp. 2, 3).

[16] P. R. Conrad and Y. M. Marzouk. "Adaptive Smolyak Pseudospectral Approximations". *SIAM Journal on Scientific Computing* 35.6 (2013), A2643–A2670 (cit. on pp. 126, 129).

[17] H. Darcy. *Les fontaines publiques de la ville de Dijon: exposition et application des principes à suivre et des formules à employer dans les questions de distribution d'eau, ouvrage terminé par un appendice relatif aux fournitures d'eau de plusieurs villes au filtrage des eaux et à la fabrication des tuyaux de fonte, de plomb, de tole et de bitume.* Les fontaines publiques de la ville de Dijon. Exposition et application des principes à suivre et des formules à employer dans les questions de distribution d'eau: ouvrage terminé par un appendice relatif aux fournitures d'eau de plusieurs villes au filtrage des eaux et à la fabrication des tuyaux de fonte, de plomb, de tole et de bitume. Victor Dalmont, éditeur, 1856 (cit. on p. 1).

[18] I. Daubechies. *Ten Lectures on Wavelets*. USA: Society for Industrial and Applied Mathematics, 1992 (cit. on pp. 23, 61).

[19] M. De Ghislain. "Quantitative hydrogeology; groundwater hydrology for engineers" (Jan. 1986) (cit. on p. 1).

[20] S. De Marchi and H. Wendland. "On the convergence of the rescaled localized radial basis function method". *Applied Mathematics Letters* 99 (2020), p. 105996 (cit. on pp. 37–40).

[21] T. De Ryck, S. Lanthaler, and S. Mishra. "On the approximation of functions by tanh neural networks". *Neural Networks* 143 (2021), pp. 732–750 (cit. on p. 3).

[22] J. P. Delhomme. "Spatial variability and uncertainty in groundwater flow parameters: A geostatistical approach". *Water Resources Research* 15.2 (1979), pp. 269–280 (cit. on pp. 1, 2).

[23] F.-J. Delvos. "d-variate Boolean interpolation". *J. Approx. Theory* 34.2 (1982), pp. 99–114 (cit. on p. 129).

[24] L. Devroye. *Non-uniform Random Variate Generation*. Springer-Verlag, 2003 (cit. on p. 2).

[25] J. Dick, F. Y. Kuo, and I. H. Sloan. "High-dimensional integration: The quasi-Monte Carlo way". *Acta Numerica* 22 (2013), pp. 133–288 (cit. on p. 3).

[26] G. E. Fasshauer and M. J. McCourt. "Stable Evaluation of Gaussian Radial Basis Function Interpolants". *SIAM Journal on Scientific Computing* 34.2 (2012), A737–A762 (cit. on p. 37).

[27] M. S. Floater and A. Iske. "Multistep scattered data interpolation using compactly supported radial basis functions". *Journal of Computational and Applied Mathematics* 73.1 (1996), pp. 65–78 (cit. on pp. 4, 59).

[28] P. Frauenfelder, C. Schwab, and R. A. Todor. "Finite elements for elliptic problems with stochastic coefficients". *Computer Methods in*

Applied Mechanics and Engineering 194 (2005), pp. 205–228 (cit. on p. 2).

[29] E. Fuselier, T. Hangelbroek, F. J. Narcowich, J. D. Ward, and G. B. Wright. "Localized Bases for Kernel Spaces on the Unit Sphere". *SIAM Journal on Numerical Analysis* 51.5 (2013), pp. 2538–2562 (cit. on pp. 37, 38, 54).

[30] J. Garcke. "Sparse Grids in a Nutshell". In: *Sparse Grids and Applications*. Ed. by J. Garcke and M. Griebel. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 57–80 (cit. on pp. 4, 126).

[31] A. Genz. "Testing Multidimensional Integration Routines". In: *Proc. of International Conference on Tools, Methods and Languages for Scientific and Engineering Computation*. Paris, France: Elsevier North-Holland, Inc., 1984, pp. 81–94 (cit. on pp. 174, 175).

[32] E. H. Georgoulis, J. Levesley, and F. Subhan. "Multilevel Sparse Kernel-Based Interpolation". *SIAM Journal on Scientific Computing* 35.2 (2013), A815–A831 (cit. on p. 4).

[33] T. Gerstner and M. Griebel. "Dimension-Adaptive Tensor-Product Quadrature". *Computing* 71 (2003), pp. 65–87 (cit. on p. 145).

[34] R. G. Ghanem and P. D. Spanos. *Stochastic Finite Elements: A Spectral Approach*. Berlin, Heidelberg: Springer-Verlag, 1991 (cit. on p. 2).

[35] Q. T. Gia and H. Wendland. "Data Compression on the Sphere Using Multiscale Radial Basis Function Approximation". *Adv. Comput. Math.* 40.4 (Aug. 2014), pp. 923–943 (cit. on p. 74).

[36] Q. T. L. Gia, I. H. Sloan, and H. Wendland. "Multiscale Analysis in Sobolev Spaces on the Sphere". *SIAM J. Numer. Anal.* 48.6 (2010), pp. 2065–2090 (cit. on pp. 59, 61).

[37] L. Gonon and C. Schwab. "Deep relu neural networks overcome the curse of dimensionality for partial integrodifferential equations". *Analysis and Applications* () (cit. on p. 3).

[38] W. Greub. *Multilinear Algebra*. Universitext. Springer New York, 2012 (cit. on pp. 87–90).

[39] M. Griebel, H. Harbrecht, and M. D. Multerer. "Multilevel Quadrature for Elliptic Parametric Partial Differential Equations in Case of Polygonal Approximations of Curved Domains". *SIAM Journal on Numerical Analysis* 58.1 (2020), pp. 684–705 (cit. on p. 3).

[40] M. Griebel and J. Oettershagen. "On Tensor Product Approximation of Analytic Functions". *J. Approx. Theory* 207.C (July 2016), pp. 348–379 (cit. on pp. 134, 135).

[41] M. Griebel and C. Rieger. "Reproducing Kernel Hilbert Spaces for Parametric Partial Differential Equations". *SIAM/ASA Journal on Uncertainty Quantification* 5.1 (2017), pp. 111–137 (cit. on p. 3).

[42] W. Hackbusch. *Tensor Spaces and Numerical Tensor Calculus*. Springer Series in Computational Mathematics. Springer Berlin Heidelberg, 2012 (cit. on pp. 4, 13, 87, 99, 101).

[43] A.-L. Haji-Ali, H. Harbrecht, M. Peters, and M. Siebenmorgen. "Novel results for the anisotropic sparse grid quadrature". *Journal of Complexity* 47 (2018), pp. 62–85 (cit. on pp. 126, 134, 135, 137, 138, 143).

[44] J. M. Hammersley and D. C. Handscomb. *Monte Carlo methods*. English. Methuen London, 1964, 178p. (Cit. on p. 3).

[45] T. Hangelbroek, F. J. Narcowich, X. Sun, and J. D. Ward. "Kernel Approximation on Manifolds II: The L_∞ Norm of the L_2 Projector". *SIAM Journal on Mathematical Analysis* 43.2 (2011), pp. 662–684 (cit. on pp. 38, 40, 41).

[46] T. Hangelbroek, F. J. Narcowich, C. Rieger, and J. D. Ward. "An inverse theorem for compact Lipschitz regions in $\mathbb{R}^d$ using localized kernel bases". *Math. Comput.* 87.312 (2018), pp. 1949–1989 (cit. on pp. 35, 37, 38, 54).

[47] H. Harbrecht, M. Peters, and M. Siebenmorgen. "Efficient approximation of random fields for numerical applications". *Numerical Linear Algebra with Applications* 22.4 (2015), pp. 596–617 (cit. on p. 2).

[48] H. Harbrecht, M. Peters, and M. Siebenmorgen. "On the quasi-Monte Carlo quadrature with Halton points for elliptic PDEs with log-normal diffusion" (2017) (cit. on p. 3).

[49] L. Herrmann, J. A. A. Opschoor, and C. Schwab. "Constructive deep ReLU neural network approximation". *J. Sci. Comput.* 90.2 (2022), Paper No. 75, 37 (cit. on p. 3).

[50] K. Hesse, I. H. Sloan, and R. S. Womersley. "Radial Basis Function Approximation of Noisy Scattered Data on the Sphere". *Numer. Math.* 137.3 (Nov. 2017), pp. 579–605 (cit. on p. 25).

[51] F. J. Hickernell and X. Wang. "The Error Bounds and Tractability of Quasi-Monte Carlo Algorithms in Infinite Dimension". *Mathematics of Computation* 71.240 (2002), pp. 1641–1661 (cit. on p. 3).

[52] K. Hornik, M. Stinchcombe, and H. White. "Multilayer feedforward networks are universal approximators". *Neural Networks* 2.5 (1989), pp. 359–366 (cit. on p. 3).

[53] P. Jantsch, C. G. Webster, and G. Zhang. "On the Lebesgue constant of weighted Leja points for Lagrange interpolation on unbounded domains". *IMA Journal of Numerical Analysis* 39.2 (2018), pp. 1039–1057 (cit. on p. 126).

[54] R. Kempf and W. Holger. "High-dimensional Approximation with Kernel-based Multilevel Methods" (cit. on p. 149).

[55] R. Kempf, H. Wendland, and C. Rieger. "Kernel-based reconstructions for parametric PDEs". In: *Meshfree Methods for Partial Differential Equations IX*. Cham: Springer, 2019, pp. 53–71 (cit. on p. 3).

[56] P. Kitanidis. *Introduction to Geostatistics: Applications in Hydrogeology*. Stanford-Cambridge program. Cambridge University Press, 1997 (cit. on p. 1).

[57] F. Y. Kuo, C. Schwab, and I. H. Sloan. "Quasi-Monte Carlo Methods for High-Dimensional Integration: The Standard (Weighted Hilbert Space) Setting and Beyond". *The ANZIAM Journal* 53.1 (2011), pp. 1–37 (cit. on p. 3).

[58] W. Light and E. Cheney. *Approximation Theory in Tensor Product Spaces*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006 (cit. on pp. 87, 92, 105, 115).

[59] M. Loeve. *Probability Theory I+II*. F.W.Gehring P.Halmos and C.Moore. Springer, 1978 (cit. on p. 2).

[60] O. Maitre and O. Knio. *Spectral Methods for Uncertainty Quantification: With Applications to Computational Fluid Dynamics*. Scientific Computation. Springer Netherlands, 2010 (cit. on p. 2).

[61] H. Meschkowski. *Hilbertsche Räume mit Kernfunktion*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 1962 (cit. on p. 111).

[62] G. Migliorati, F. Nobile, E. von Schwerin, and R. Tempone. "Approximation of Quantities of Interest in Stochastic PDEs by the Random Discrete L^2 Projection on Polynomial Spaces". *SIAM Journal on Scientific Computing* 35.3 (2013), A1440–A1460 (cit. on p. 3).

[63] F. J. Narcowich, R. Schaback, and J. D. Ward. "Multilevel interpolation and approximation". *Appl. Comput. Harmon. Anal.* 7.3 (1999), pp. 243–261 (cit. on p. 59).

[64] F. Narcowich, N. Sivakumar, and J. Ward. "On Condition Numbers Associated with Radial-Function Interpolation". *Journal of Mathematical Analysis and Applications* 186.2 (1994), pp. 457–485 (cit. on p. 31).

[65] F. J. Narcowich, J. D. Ward, and H. Wendland. "Sobolev error estimates and a Bernstein inequality for scattered data interpolation via radial basis functions". *Constr. Approx.* 24.2 (2006), pp. 175–186 (cit. on pp. 20, 23, 24).

[66] F. J. Narcowich, J. D. Ward, and H. Wendland. "Sobolev error estimates and a Bernstein inequality for scattered data interpolation via radial basis functions". *Constructive Approximation* 24.2 (Sept. 2006), pp. 175–186 (cit. on p. 24).

[67] F. Nobile, R. Tempone, and C. G. Webster. "A Sparse Grid Stochastic Collocation Method for Partial Differential Equations with Random Input Data". *SIAM Journal on Numerical Analysis* 46.5 (2008), pp. 2309–2345 (cit. on pp. 2, 3).

[68] F. Nobile, R. Tempone, and C. G. Webster. "An Anisotropic Sparse Grid Stochastic Collocation Method for Partial Differential Equations with Random Input Data". *SIAM Journal on Numerical Analysis* 46.5 (2008), pp. 2411–2442 (cit. on pp. 2, 3, 143).

[69] E. Novak and K. Ritter. "High-dimensional integration of smooth functions over cubes". *Numer. Math.* 75.1 (1996), pp. 79–97 (cit. on pp. 4, 126, 129).

[70] E. Novak and H. Woźniakowski. *Tractability of multivariate problems. Volume II: Standard information for functionals*. Vol. 12. EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2010, pp. xviii+657 (cit. on pp. 124, 127).

[71] C. Rieger and H. Wendland. "Sampling inequalities for anisotropic tensor product grids". *IMA Journal of Numerical Analysis* 40.1 (Jan. 2019), pp. 285–321 (cit. on pp. 4, 134, 135).

[72] C. Rieger and H. Wendland. "Sampling Inequalities for Sparse Grids". *Numer. Math.* 136.2 (June 2017), pp. 439–466 (cit. on pp. 4, 169).

[73] R. Schaback. "Creating surfaces from scattered data using radial basis functions". In: *Mathematical methods for curves and surfaces (Ulvik,*

1994). Vanderbilt Univ. Press, Nashville, TN, 1995, pp. 477–496 (cit. on p. 59).

[74] R. Schaback. "Error estimates and condition numbers for radial basis function interpolation". *Adv. Comput. Math.* 3.3 (1995), pp. 251–264 (cit. on p. 31).

[75] R. Schaback. "The missing Wendland functions". *Advances in Computational Mathematics* 34.1 (2011), pp. 67–81 (cit. on p. 19).

[76] R. Schatten. *A Theory of Cross-spaces*. Annals of mathematics studies. University Press, 1950 (cit. on pp. 4, 87, 92, 95).

[77] D. Schräder and H. Wendland. "A high-order, analytically divergence-free discretization method for Darcy's problem". *Mathematics of Computation* 80.273 (Jan. 2011), pp. 263–277 (cit. on p. 1).

[78] A. Schreiber. "Die Methode von Smolyak bei der multivariaten Interpolation". PhD thesis. Universität Göttingen, 2000 (cit. on p. 127).

[79] C. Schwab and C. J. Gittelson. "Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs". *Acta Numerica* 20 (2011), pp. 291–467 (cit. on p. 2).

[80] C. Schwab and R. A. Todor. "Karhunen-Loève Approximation of Random Fields by Generalized Fast Multipole Methods". *J. Comput. Phys.* 217.1 (Sept. 2006), pp. 100–122 (cit. on p. 2).

[81] W. Sickel and T. Ullrich. "Tensor products of Sobolev–Besov spaces and applications to approximation from the hyperbolic cross". *Journal of Approximation Theory* 161.2 (2009), pp. 748–786 (cit. on p. 4).

[82] I. Sloan and S. Joe. *Lattice Methods for Multiple Integration*. Oxford science publications. Clarendon Press, 1994 (cit. on p. 174).

[83] S. A. Smoljak. "Quadrature and interpolation formulae on tensor products of certain function classes". *Dokl. Akad. Nauk SSSR* 148 (1963), pp. 1042–1045 (cit. on pp. 4, 120).

[84] H. Stahl. "Best uniform rational approximation of α on [0, 1]". *Acta Mathematica* 190 (2003), pp. 241–306 (cit. on p. 169).

[85] E. Stein and P. U. Press. *Singular Integrals and Differentiability Properties of Functions*. Monographs in harmonic analysis. Princeton University Press, 1970 (cit. on p. 11).

[86] I. Steinwart and A. Christmann. *Support Vector Mashines*. Information Science and Statistics. Springer New York, 2008 (cit. on pp. 25, 26).

[87] A. Szepessy, R. Tempone, and G. E. Zouraris. "Adaptive weak approximation of stochastic differential equations". *Communications on Pure and Applied Mathematics* 54.10 (2001), pp. 1169–1214 (cit. on p. 3).

[88] A. Townsend and H. Wendland. "Multiscale analysis in Sobolev spaces on bounded domains with zero boundary values". *IMA Journal of Numerical Analysis* 33.3 (Nov. 2012), pp. 1095–1114 (cit. on pp. 31, 33).

[89] F. Usta and J. Levesley. "Multilevel quasi-interpolation on a sparse grid with the Gaussian". *Numer. Algorithms* 77.3 (2018), pp. 793–808 (cit. on p. 4).

[90] G. Wasilkowski and H. Wozniakowski. "Explicit Cost Bounds of Algorithms for Multivariate Tensor Product Problems". *J. Complex.* 11.1 (Mar. 1995), pp. 1–56 (cit. on pp. 120, 121, 124).

[91] H. Wendland. “Multiscale analysis of Sobolev spaces on bounded domains”. *Numerische Mathematik* 116.3 (Sept. 2010), pp. 493–517 (cit. on pp. 4, 28, 59–61, 63, 67, 68, 71).

[92] H. Wendland. “Multiscale radial basis functions”. In: *Frames and Other Bases in Abstract and Function Spaces : Novel Methods in Harmonic Analysis. Volume 1*. Cham: Birkhäuser, 2017, pp. 265–299 (cit. on pp. 4, 29, 30, 59, 60, 74, 77, 82, 83, 182).

[93] H. Wendland. “New Material on Radial Basis Functions”. 2020 (cit. on pp. 37, 40).

[94] H. Wendland. *Scattered Data Approximation*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, 2004 (cit. on pp. 14–18, 21, 22, 57).

[95] H. Wendland and C. Rieger. “Approximate interpolation with applications to selecting smoothing parameters”. *Numer. Math.* 101.4 (2005), pp. 729–748 (cit. on pp. 20, 25).

[96] D.-X. Zhou. “Universality of deep convolutional neural networks”. *Applied and Computational Harmonic Analysis* 48.2 (2020), pp. 787–794 (cit. on p. 3).

Publications

R. Kempf, H. Wendland, and C. Rieger. "Kernel-based reconstructions for parametric PDEs". In: *Meshfree Methods for Partial Differential Equations IX*. Cham: Springer, 2019, pp. 53–71

Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die von mir angegebenen Quellen und Hilfsmittel verwendet habe.

Weiterhin erkläre ich, dass ich die Hilfe von gewerblichen Promotionsberatern bzw. -vermittlern oder ähnlichen Dienstleistern weder bisher in Anspruch genommen habe, noch künftig in Anspruch nehmen werde.

Zusätzlich erkläre ich hiermit, dass ich keinerlei frühere Promotionsversuche unternommen habe.

Bayreuth, den 5. Oktober 2022

Rüdiger Kempf