Capacity of generalized diamond networks

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Abstract

We consider the problem of error correction in a network where the errors can occur only on a proper subset of the network edges. For a generalization of the so-called Diamond Network we consider lower and upper bounds for the network's (1-shot) capacity.

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1 Introduction

The correction of errors introduced by noise or adversaries in networks have been studied in a number of papers, see e.g. [8, 17]. Here we assume that the adversary can manipulate a subset of the edges of a given network. If the adversary is unrestricted, i.e., all edges can be manipulated, then it is well known that the 1-shot capacity of the network can be achieved by combining a rank-metric code with a linear network code, see e.g. [11]. In [2, 3] specific networks are considered where the actual 1-shot capacity cannot be achieved using linear operations at intermediate nodes, so that network decoding is required in order to achieve the capacity. For a unified combinatorial treatment of adversarial network channels we refer to [15].

The aim of this paper is to study the capacity of generalized diamond networks, see [2, 3] for the diamond network and Figure 1 for the generalized case. This family of networks is one of five infinite families studied in [1], more precisely family B. The dashed edges $e_0, \dots, e_{s+1}$ can be manipulated by the adversary and the solid edges $f_0, \dots, f_s$ are those that cannot be manipulated. Node S is the source, T the terminal, and V_1, V_2 are the intermediate nodes. We assume that the adversary can manipulate exactly one edge and we are interested in the 1-shot capacity, i.e., where the edges of the network are just used once.

The possible mappings from the input to the output space of the nodes are numerous. So, after stating the necessary preliminaries in Section 2, we study the structure of "good" network codes in the single-error correction setting for generalized diamond networks $\mathcal{N}_s$ in Section 3. Our main structure result, see Theorem 13, states that we may assume that node V_1 just forwards its received information and node V_2 performs a partial error-correction and sends sets of codeword candidates as states, so that the design of an optimal network code becomes a covering problem for sets. Based on these insights we determine lower and upper bounds for the 1-shot capacity of $\mathcal{N}$ in sections 4 and 5, respectively.

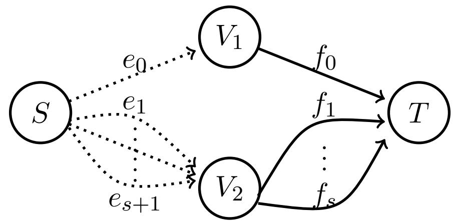

Figure 1: The generalized diamond network $\mathcal{N}_s$.

While the family of networks $\mathcal{N}_s$ seems to be rather specific and non-general, there are a few reasons to study this specific class in detail. It is one of three families of so-called 2-level networks whose capacity couldn't be determined in [1]. If we remove vertex V_1, then we are in the classical situation of coding along a single edge, so that this class is somehow the smallest possible example where we can see network (coding) effects. Even though $\mathcal{N}_s$ is very well structured, the exact determination of its 1-shot capacity for a given parameter s and a given alphabet size seems to be a combinatorial challenge. In [1] reductions for general networks to 3-level networks and then to 2-level networks were presented, so that it makes sense to study the smaller and more structured networks first. Nevertheless there is some hope that some of the presented techniques may generalize to different networks too.

2 Preliminaries

Definition 1. *A* (single-source communication) network *is a 4-tuple* $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$*, where* $(\mathcal{V}, \mathcal{E})$ *is a finite, directed, acyclic multigraph* $S \in \mathcal{V}$ *is the* source*, and* $\emptyset \neq \mathbf{T} \subseteq \mathcal{V}\backslash\{S\}$ *is the set of* terminals*. Additionally we assume that there exists a directed path from* S *to any* $T \in \mathbf{T}$ *and for every* $V \in \mathcal{V}\backslash(\{S\} \cup \mathbf{T})$ *there exist directed paths from* S *to* V *and from* V *to some* $T \in \mathbf{T}$*.*

The elements of $\mathcal{V}$ are called *nodes*, the elements of $\mathcal{E}$ are called *edges*, and the elements of $\mathcal{V}\backslash(\{S\} \cup \mathbf{T})$ are called *intermediate nodes*. The set of incoming and outgoing edges for a node $V \in \mathcal{V}$ are denoted by $in(V)$ and $out(V)$, respectively. As alphabet we choose an arbitrary set $\mathcal{A}$ with at least two elements, e.g. $\mathcal{A} = \{0, 1, \ldots, \#\mathcal{A} - 1\}$.

Definition 2. *Let* $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$ *be a network. A* network code F *for* $\mathcal{N}$ *is a family of functions* $\{F_V : V \in \mathcal{V}\backslash\{S\}\}$*, where* $F_V \colon \mathcal{A}^{in(V)} \to \mathcal{A}^{out(V)}$ *for all* $V \notin \mathbf{T}$ *and* $F_V \colon \mathcal{A}^{in(V)} \to \mathcal{A}^{out(S)}$ *for all* $V \in \mathbf{T}$*. For each* $c \in \mathcal{A}^{out(S)}$ *and each* $T \in \mathbf{T}$ *we write* $\mathcal{N}_T(F, c)$ *for the output of* F_T *propagating* c *starting from the source* S *over the network. If* $\#\mathbf{T} = 1$ *we write* $\mathcal{N}(F, c)$ *instead of* $\mathcal{N}_T(F, c)$*.*

Definition 3. *An* (outer) code *for a network* $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$ *is a subset* $C \subseteq \mathcal{A}^{out(S)}$ *with* $\#C \geq 1$*.*

For any two $c, c' \in \mathcal{A}^E$, where E is an arbitrary set, we define the Hamming distance $d(c, c')$ as the number of differing function values. For each subset $C \subseteq \mathcal{A}^E$ we write $d(C)$ for the minimum Hamming distance $d(c, c')$ for any pair of different elements $c, c' \in C$, where we formally set $d(C) := \infty$ if $\#C \leq 1$.

Definition 4. *For each positive integer* s *the generalized diamond network* $\mathcal{N}_s$ *has one source* S*, one terminal* T*, and two intermediate nodes* V_1*,* V_2*. There is exactly one directed edge from* S *to*

V_1, one directed edge from V_1 to T, $s+1$ parallel edges from S to V_2, and s parallel edges from V_2 to T, see Figure 1.

We remark that $\mathcal{N}_s$ is denoted by $\mathcal{B}_s$ in [1], where $\mathcal{N}_1 = \mathcal{B}_1$ is the original diamond network, see [3].

In the general model of Koetter and Ravagnani, see [15], an adversary can modify t edges from a given subset $\mathcal{U}$ of all edges. Here we restrict ourselves to the situation where $\mathcal{U} = out(S)$, i.e., the set of all outgoing edges of the unqiue source.

Definition 5. *We say that a given pair of a network code F and an outer code C for a network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, S, \mathbf{T})$ is t-error correcting if $\mathcal{N}_T(F, c') = c$ for all $T \in \mathbf{T}$, all $c' \in \mathcal{A}^{out(S)}$ and all $c \in C$ with $d(c', c) \le t$. We say that C can be t-error corrected on $\mathcal{N}$ if a network code F exists such that (C, F) is t-error correcting for $\mathcal{N}$.*

Here we will consider 1-error correcting codes and a unique terminal only. I.e. the adversary $\mathbf{A}$ turns a codeword $c \in C$ into $c' = \mathbf{A}(c)$ with $d(c, c') \le 1$ and all choices for c' are indeed possible. As an abbreviation we set $\overline{C} := \left\{c' \in \mathcal{A}^{out(S)} \mid \exists c \in C : d(c, c') \le 1\right\}$.

In $\mathcal{N}_s$ we label the edge from S to V_1 as e_0 and the $s+1$ edges from S to V_2 by $e_1, \ldots, e_{s+1}$. The unique edge from V_1 to T is denoted by f_0 and the s edges from V_2 to T are labeled by $f_1, \ldots, f_s$. C.f. Figure 1.

3 Structure of network codes for generalized diamond networks $\mathcal{N}_s$

The aim of this section is to obtain some structure results on network codes that are 1-error correcting for a given outer code C on the generalized diamond network $\mathcal{N}_s$ as well as on outer codes C that can be 1-error corrected.

We start with properties of the network code F. W.l.o.g. we can assume that the inner node V_1 just forwards the information symbol received via edge e_0 from S. The idea is to incorporate any modification inside V_1 directly in F_T.

Lemma 6. *If C can be 1-error corrected for $\mathcal{N}_s$, then there exists a corresponding network code F such that F_{V_1} is the identity mapping.*

Proof. Let F' be a network code for $\mathcal{N}_S$ such that (F', C) is 1-error correcting for $\mathcal{N}_s$. We choose F_{V_1} as the identity mapping and set $F_{V_2} = F'_{V_2}$. Consider the mapping $G\colon \mathcal{A}^{in(T)} \to \mathcal{A}^{in(T)}$ defined by $(f_0, \ldots, f_s) \mapsto \left(F'_{V_1}(f_0), f_1, \ldots, f_s\right)$ and set $F_T = F'_T \circ G$. □

In other words, we can assume that the intermediate node V_1 just forwards each received symbol.

For each $c \in \mathcal{A}^{out(S)}$ we write $c = (c_1, c_2)$, where c_1 corresponds to e_0 and c_2 corresponds to $e_1, \ldots, e_{s+1}$, i.e., we decompose $out(S)$ into $in(V_1)$ and $in(V_2)$. As an abbreviation we set $C_2 := \{c_2 \mid (c_1, c_2) \in C\}$.

Lemma 7. *Let C be an outer code and F be a network code for $\mathcal{N}_s$ such that (F, C) is 1-error correcting and F_{V_1} is the identity mapping. For each pair of different codewords $(c_1, c_2), (c'_1, c'_2) \in C$ we have $F_{V_2}(c_2) \neq F_{V_2}(c'_2)$.*

Proof. For an arbitrary symbol $a \in \mathcal{A}$ we have

$$d\big((a, c_2), (c_1, c_2)\big), d\big((a, c'_2), (c'_1, c'_2)\big) \le 1,$$

so that otherwise $in(T)$ would be given by $(a, F_{V_2}(c_2))$ for (a, c_2) as well as (a, c_2'). This contradicts the property of being 1-error correcting. $\square$

Lemma 8. *Let (F, C) be 1-error correcting for $\mathcal{N}_s$. Then, we have $d(C) \geq 3$, $\#C = \#C_2$, and $d(C_2) \geq 2$. Moreover, if $c_2, c_2' \in C_2$ with $d(c_2, c_2')$, then we have $a \neq a'$ for all $a, a' \in \mathcal{A}$ with $(a, c_2), (a', c_2') \in C$.*

Proof. Assume that there exist two different elements $c, c' \in C$ with $d(c, c') \leq 2$. Then there exist an element $\overline{c} \in \mathcal{A}^{out(S)}$ with $d(\overline{c}, c), d(\overline{c}, c') \leq 1$, i.e., $\overline{c} \in \overline{C}$. Since (F, C) is 1-error correcting, we have $\mathcal{N}(F, \overline{c}) = c$ and $\mathcal{N}(F, \overline{c}) = c'$, which contradicts $c \neq c'$. Let $c = (a, c_2) \in C$ and $c' = (a', c_2) \in C$, where $a, a' \in \mathcal{A}$ and $c_2 \in C_2$, then we have $c = c'$ since otherwise $d(c, c') \leq 1$. Thus, we have $\#C = \#C_2$. Let c_2, c_2' be two different elements in C_2 and $a, a' \in \mathcal{A}$ so that $c := (a, c_2) \in C$ and $c' := (a', c_2') \in C$. From $d(c, c') \geq 3$ we conclude $d(c_2, c_2') \geq 2$. Also the last statement follows directly. $\square$

Similarly as $\overline{C}$ we define

$$\overline{C_2} := \left\{ c_2 \in \mathcal{A}^{out(S)\backslash\{e_0\}} \mid \exists c \in C, a \in \mathcal{A} : d\big(c, (a, c_2)\big) \leq 1 \right\}.$$

For each $c_2' \in \mathcal{A}^{out(S)\backslash\{e_0\}}$ we define

$$\tau(c_2') := \left\{ (a, c_2) \in C \mid d(c_2, c_2') \leq 1 \right\}.$$

In words, if T_{V_2} receives the input c_2', then the originally sent codeword $c \in C$ has to be contained in $\tau(c_2')$. Moreover, the adversary $\mathbf{A}$ can ensure that T_{V_2} receives the input c_2' for all codewords $c \in \tau(c_2')$. Clearly, we have $\tau(c_2') = \emptyset$ if $c_2' \notin \overline{C_2}$ and $\#\tau(c_2') = 1$ if $c_2' \in C_2$. Note that no two different elements $(a, c_2), (\hat{a}, \acute{c}_2) \in \tau(c_2')$ can coincide in the e_0-component, i.e. $a \neq \hat{a}$, so that $\tau(c_2') \leq \#\mathcal{A}$.

Lemma 9. *Let (F, C) be 1-error correcting for $\mathcal{N}_s$. For each $c_2' \in \overline{C_2}$ we have $\#\tau(c_2') \leq s + 1$.*

Proof. Assume that there exist pairwise different elements $\left(a^{(i)}, c_2^{(i)}\right) \in \tau(c_2')$ for $1 \leq i \leq s + 2$. By definition we have $d\left(c_2^{(i)}, c_2'\right) \leq 1$ for all $1 \leq i \leq s + 2$. If there exists an index $1 \leq i \leq s + 2$ with $c_2^{(i)} = c_2'$, then

$$d\Big(\left(a^{(i)}, c_2^{(i)}\right), \left(a^{(j)}, c_2^{(j)}\right) \Big) \leq 1 + d\left(c_2^{(i)}, c_2^{(j)}\right) = 1 + d\left(c_2', c_2^{(j)}\right) \leq 2,$$

where $j \neq i$ is arbitrary. Since $\tau(c_2') \subseteq C$, this contradicts $d(C) \geq 3$. Thus, $\#in(V_2) = \#out(S)\backslash\{e_0\} = s + 1$ implies the existence of indices $1 \leq i < j \leq s + 1$ that $c_2^{(i)}$ and $c_2^{(j)}$ differ on the e_h component only, where $1 \leq h \leq s + 1$ is a suitable index. However, then we can compute

$$d\Big(\left(a^{(i)}, c_2^{(i)}\right), \left(a^{(j)}, c_2^{(j)}\right) \Big) \leq 1 + d\left(c_2^{(i)}, c_2^{(j)}\right)) \leq 2,$$

which again contradicts $d(C) \geq 3$ (using $\tau(c_2') \subseteq C$). $\square$

The fact that $\tau(c_2')$ does not contain two different elements with the same e_0-component is crucial since the terminal T gets only the outputs from V_2 and V_1, where the latter equals $F_{V_1}(a) = a$ if (a, c_2') is the output of the source S after the modification of the adversary $\mathbf{A}$. So, if F_{V_2} maps two different elements $c_2, c_2' \in \overline{C_2}$ to the same output, then the same property should hold for $\tau(c_2) \cup \tau(c_2')$.

Lemma 10. *Let (F, C) be 1-error correcting for $\mathcal{N}_s$. For each different elements $c_2, c_2' \in \overline{C_2}$ with $F_{V_2}(c_2) = F_{V_2}(c_2')$ we have that $\tau(c_2) \cup \tau(c_2')$ does not contain two different elements with coinciding e_0-component.*

Proof. Let $c_2, c_2' \in \overline{C_2}$ with $F_{V_2}(c_2) = F_{V_2}(c_2')$. Assume that there exist $a \in \mathcal{A}$, $b, b' \in C_2$ with $b \neq b'$, and $(a, b), (a, b') \in \tau(c_2) \cup \tau(c_2')$. W.l.o.g. we can assume that F_{V_1} is the identity mapping, see Lemma 6. Thus, the adversary can enforce that for both different codewords (a, b), (a, b') the input for the terminal T is given by a and $F_{V_2}(c_2) = F_{V_2}(c_2')$, so that no correct decoding is possible. □

Lemma 11. *Let (F, C) be 1-error correcting for $\mathcal{N}_s$. For each pair $c := (c_1, c_2) \in C$, $c' := (c_1', c_2') \in \overline{C}$ with $c \neq c'$ and $F_{V_2}(c_2) = F_{V_2}(c_2')$ we have $\tau(c_2') = \{c\}$.*

Proof. Assume $\tilde{c} := (\tilde{c}_1, \tilde{c}_2) \in \tau(c_2')$ with $\tilde{c} \neq c$ to the contrary. W.l.o.g. we can assume that F_{V_1} is the identity mapping, see Lemma 6. From Lemma 7 we conclude $\tilde{c}_2 \neq c_2'$. The codeword $\tilde{c}$ can be modified to $(\tilde{c}_1, c_2')$ by the adversary, so that the terminal T receives the input $F_{V_1}(\tilde{c}_1) = \tilde{c}_1$, $F_{V_2}(c_2') = F_{V_2}(c_2)$ which then has to be decoded to $\tilde{c}$. However, the codeword c can be modified to $(\tilde{c}_1, c_2)$ by the adversary, so that the terminal T receives the very same input, which contradicts the assumption that (F, C) be 1-error correcting for $\mathcal{N}_s$. □

So, for each $b \in \mathcal{A}^{out(V_2)}$ such that there exists a $c_2 \in \overline{C}_2$ with $\mathbb{F}_{V_2}(c_2) = b$ there exists a set $S_b \subseteq C$ with $\tau(c_2') \subseteq S_b$ for all $c_2' \in \overline{C}_2$ satisfying $\mathbb{F}_{V_2}(c_2') = b$. For those $b \in \mathcal{A}^{out(V_2)}$ that cannot be reached by $c_2 \in \overline{C}_2$, we set $S_b := \emptyset$. With this, we can uniquely define the S_b by assuming their minimality with respect to the above property. From Lemma 10 and Lemma 11 we can further conclude that no set S_b contains two elements with the same e_0-component. By Lemma 6 we can assume that the intermediate node V_1 just forwards its input. With this a suitable decoder F_T can be described easily. Let a, b with $a \in \mathcal{A}^{out(V_1)}$ and $b \in \mathcal{A}^{out(V_2)}$ be the input. If $S_b = \{c\}$ then set $F_T(a, b) = c$ and otherwise choose the unique element $(a, c_2) \in S_b$ as the output. To sum up, from Lemma 6, Lemma 10 and Lemma 11 we can conclude:

Proposition 12. *Let (F, C) be 1-error correcting for $\mathcal{N}_s$. Then there exists sets $S_b \subseteq C$ for all $b \in \mathcal{A}^{out(V_2)}$ such that $S_b = \cup_{c_2 \in \overline{C}_2 \,:\, F_{v_2}(c_2) = b} \tau(c_2)$. Let the network code F' be defined by choosing F'_{V_1} as the identity mapping and F'_{V_2}, F'_T as follows. For each $c_2 \in \overline{C}_2$ let $b \in \mathcal{A}^{out(V_2)}$ be the lexicographical smallest element with $\tau(c_2) \subseteq S_b$ and $\#S_b = 1$ iff $\#\tau(c_2) = 1$. With this we set $F'_{V_2}(c_2) = b$, $F'_T(a, b) = c$ if $S_b = \{c\}$, and $F'_T(a, b) = c'$ if $\#S_b > 1$, where $c' \in S_b$ is the unique element with e_0-component a. Then, (F', C) is also 1-error correcting for $\mathcal{N}_s$.*

Starting from a network code F' as in Proposition 12 we can perform further modifications to "optimize" the network code. It may happen that there exist different elements $b, b' \in \mathcal{A}^{out(V_2)}$ such that $S_b \cup S_{b'}$ also contains to two elements with the same e_0-element, including the case $S_b = S_{b'}$. In such a situation we can redefine all instances $c_2 \in \overline{C}_2$ where $F'_{V_2}(c_2) = b'$ by $F'_{V_2}(c_2) = b$ and free the output vector b'. Since for each codeword $c \in C$ we have a set $S_b = \{c\}$ with $b \in \mathcal{A}^{out(V_2)}$

n	3	4	5	6	7	8	9	10	11	12	13	14	15
max $\#C$	2	2	4	8	16	20	40	72	144	256	512	1024	2048
$\#$	1	2	1	1	1	5	1	562	7398	237610	117832	38408	5983

Table 1: Number of non-isomorphic optimal binary $(n, \#C, 3)$ codes.

anyway, we may assume by convention that we set $F'_{V_2}(c'_2) = b$ for all $c'_2 \in \overline{C}_2$ with $\tau(c'_2) = \{c\}$. With this the determination of the minimum number of used output symbols in $\mathcal{A}^{out(V_2)}$ becomes a so-called cover problem:

Theorem 13. *Let $C \subseteq \mathcal{A}^{out(S)}$ be a code with $d(C) \geq 3$, then C can be 1-error corrected for $\mathcal{N}_s$ iff there exists a set $\mathcal{B}$ consisting of subsets of C with pairwise different e_0-component that cover all $\tau(c_2)$ with $c_2 \in \overline{C}_2$ and $\#\tau(c_2) > 1$, i.e., for each $c_2 \in \overline{C}_2$ with $\#\tau(c_2) > 1$ there exists an element $B \in \mathcal{B}$ with $\tau(c_2) \subseteq B$, that satisfies $\#C + \#\mathcal{B} \leq \#\mathcal{A}^s$.*

So, if $\#C$ and $\#\mathcal{B}$ are relatively small we may even save an outgoing edge from V_2 or only use a subset of the alphabet $\mathcal{A}$ on some outgoing edges. In any case Theorem 13 gives us a first algorithm to determine the largest possible size $\sigma(s, \#\mathcal{A})$ of a code that can be 1-error corrected for $\mathcal{N}_s$ for any parameter $s \in \mathbb{N}_{\geq 1}$ and any alphabet size $\#\mathcal{A}$. To this end we just have to loop over all codes with minimum Hamming distance at least 3 and use Theorem 13 to check whether C can be 1-error corrected for $\mathcal{N}_s$. We remark that the underlying cover problem can be solved by a straight forward integer linear programming (ILP) formulation, as it is usually the case for cover problems. We have applied this approach to the list of classified optimal binary one-error-correcting codes. These are denoted as $(n, \#C, 3)$ codes, where the length is given by $n = s + 2$ in our situation and $\#C$ is the maximum possible code size. For lengths $4, \dots, 11$ the codes were classified in [13], for lengths $12, 13$ in [12], and for lengths $14, 15$ in [14]. The corresponding numbers are summarized in Table 1 and the codes can be downloaded at `http://pottonen.kapsi.fi/codes`. If we only consider one-error correcting codes up to symmetry we have to loop over the coordinates and decide which one should be the one corresponding to e_0. Interestingly enough, for all $n \in \{3, \dots, 15\} \backslash \{3, 7, 15\}$ all choices for e_0 and all choices for C lead to a code that can be 1-error corrected for $\mathcal{N}_{n-2}$. Note that for $n \in \{3, 7, 15\}$ we deal with the parameters of the Hamming codes. Here no choice of C or the coordinate of e_0 leads to a a code that can be 1-error corrected for $\mathcal{N}_{n-2}$. Thus, we have $\sigma(1,2) \leq 1$, $\sigma(2,2) = 2$, $\sigma(3,2) = 4$, $\sigma(4,2) = 8$, $\sigma(5,2) \leq 15$, $\sigma(6,2) = 20$, $\sigma(7,2) = 40$, $\sigma(8,2) = 72$, $\sigma(9,2) = 144$, $\sigma(10,2) = 256$, $\sigma(11,2) = 512$, $\sigma(12,2) = 1024$, and $\sigma(13,2) \leq 2047$.

In the next section we will determine $\sigma(1,2)$, $\sigma(5,2)$, and $\sigma(s,a)$ for small parameters s and $a \geq 3$. We remark that $\sigma(1,a) = a - 1$ was determined in [2, 3].

4 Exact values and lower bounds for the maximum code sizes

As mentioned in the previous section, $\sigma(1,a) = a - 1$ was determined in [2, 3]. Here an element $\tilde{a} \in \mathcal{A}$ is chosen as s special symbol. With this the code C is a threefold repetition code of the symbols in $\mathcal{A} \backslash \{\tilde{a}\}$. If $\#\tau(c'_2) > 1$, then V_2 outputs $\tilde{a}$ with the corresponding set $B = C$. Here we will determine a few exact values and lower bounds for $\sigma(s,a)$ for small parameters. Upper bounds are discussed in Section 5.

One of the most simple algorithmic approaches is to loop of all $\#\mathcal{A}$-ary codes of length $s+2$ and minimum Hamming distance at least three with increasing sizes. For each candidate code C we can use Theorem 13 and a small ILP computation to decide whether C can be 1-error corrected. For alphabet size $\#\mathcal{A}=3$ and $s=2$ a code of size 5 is given by $C=\{0000,0111,1012,1120,2021\}$. If we number the codewords by $0,\dots,4$ and choose e_0 as the first component, then the different values of $\tau(c_2)$ for all $c_2\in\overline{C}_2$ are given by $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,2,4\}$, $\{0,2\}$, $\{0,3,4\}$, and $\{1,3,4\}$. Since $\#S_b\le\#\mathcal{A}$ we cannot join two of these sets and all $3^2=9$ different output vectors of the intermediate node V_2 have to be used. Another code for these parameters is given by $C=\{0000,0111,0222,1012,2021\}$. Here the different values of $\tau(c_2)$ for all $c_2\in\overline{C}_2$ are given by $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0,3\}$, $\{0,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$. The sets $\{0,3\}$ and $\{0,4\}$ can be joined to $\{0,3,4\}$, so that only 8 different output states are used. By exhaustive search we verified that no 3-ary code with length 4, minimum Hamming distance at least 3, and size 6 can be 1-error corrected, so that $\sigma(2,3)=5$.

In the following we will only state the codes C since suitable sets S_b can be easily computed using Theorem 13. For $\#\mathcal{A}=3$ and $s=3$ a code of size 14 is given by

$$C=\{00000,00111,00222,01012,01120,02201,10021,11102,11211,12010,20210,21001,22022,22100\}.$$

For $\#\mathcal{A}=3$ and $s=4$ a code of size 35 is given by

$$\begin{aligned} C \;=\; \{&000000,212000,102021,122210,012122,120121,202201,211220,110211,121022,001120,\\ &202110,002212,020012,021102,011001,111100,200122,200011,121201,022111,102102,\\ &010110,211112,000221,101111,220200,110020,201002,010202,210101,022020,221010,\\ &112012,222222\}.\end{aligned}$$

By exhaustive search we have verified $\sigma(3,3)=14$. For $\#\mathcal{A}=4$ and $s=2$ a code of size 9 is given by $C=\{0000,0111,0222,1012,1103,2021,2130,3201,3310\}$.

For $\#\mathcal{A}=2$ we have that $\#\tau(c_2')>1$ is equivalent to $\#\tau(c_2')$ and the two codewords in $\tau(c_2')$ are at Hamming distance 2. Moreover, for each pair of codewords $a,b\in C_2$ with $d(a,b)=2$ there are exactly two vectors $x\in\mathcal{A}^{out(S)\backslash\{e_0\}}$ with $d(a,x)=d(x,b)=1$. This characterization allows to determine $\sigma(s,2)$ via the following ILP formulation. For each $v\in\{0,1\}^{s+2}$ we introduce binary variables x_v and binary variables $y_{v'}$ for each $v'\in\{0,1\}^{s+1}$. The meaning of the x_v's is given by $x_v=1$ iff $v\in C$. The meaning of the $y_{v'}$'s is given by $y_{v'}=1$ iff $v'\in C_2'$ with $\#\tau(v')>1$. We minimize the code size $\#C=\sum_{v\in\{0,1\}^{s+2}}x_v$ subject to the constraints

$$\sum_{v\in\{0,1\}^{s+1}} x_v+\frac{1}{2}\cdot\sum_{v'\in\{0,1\}^{s+1}} y_{v'}\le 2^s \tag{1}$$

modeling the condition from Theorem 13,

$$y_m\ge\sum_{i\in\mathcal{A}}x_{(i,a)}+\sum_{j\in\mathcal{A}}x_{(j,b)}-1 \tag{2}$$

for all $a,b\in\{0,1\}^{s+1}$ with $d(a,b)=2$ and all $m\in\{0,1\}^{s+1}$ with $d(a,m)=d(m,b)=1$, which ensures that the pair of codewords (i,a) and (j,b) can only be taken if $y_m=1$, and

$$\sum_{v\in\{0,1\}^{s+2}\,:\,v|_S=u} x_v\le 1 \tag{3}$$

for all $u \in \{0,1\}^s$ and all $S \subseteq \{1,\ldots,s+2\}$, which models $d(C) \ge 3$, where $v|_S$ is the restriction of the vector v to the coordinates in S.

For $s=5$ a code of size 14 is given by

$$\begin{aligned} C &= \{0001101, 0011110, 0100100, 0101011, 0110111, 0111000, 1000111, \\ &\quad 1001000, 1010100, 1011011, 1100001, 1101110, 1110010, 1111101\} \end{aligned}$$

and there is no such code of size 15, so that $\sigma(5,2)=14$. We remark that the bound $\#C+\#\mathcal{B}=\#\mathcal{A}^s$ is met with equality in our example.

5 Upper bounds for the maximum code sizes

In [1] the two bounds $\sigma(s,a) \le a^s$ and $\sigma(s,a) \ge a^{s-1}$, if a is sufficiently large, were shown. Both have an easy explanation. For the lower bound we may restrict to network codes where V_1 just forwards and the terminal T ignores the output from V_1, i.e., the entire decoding is actually performed in intermediate node V_2. Let $A_q(n,d)$ denote the maximum number of vectors in a q-ary code of word length n and with Hamming distance d, so that $\sigma(s,a) \ge A_a(s+1,3)$. Since for a fixed length and minimum distance MDS codes exist for all sufficiently large alphabet sizes a this implies $\sigma(s,a) \ge a^{s-1}$. For other known lower bounds for $A_q(n,d)$ we refer to e.g. [4, 5, 7], more recent citing papers, and especially the webpage `https://www.win.tue.nl/~aeb/`. For upper bounds we can use Lemma 8 to conclude $\sigma(s,a) \le A_a(s+2,3)$, so that the Singleton bound gives $\sigma(s,a) \le a^s$ for all parameters s and a. In Table 2 we summarize some known upper bounds for $A_a(s+2,3)$ from the mentioned website.

s	2	3	4	5	6	2	3	4	5	2	3	4	5
a	3	3	3	3	3	4	4	4	4	5	5	5	5
$\le$	9	18	38	111	333	16	64	176	596	25	125	625	2291

Table 2: Upper bounds for $A_a(s+2,3)$.

While Lemma 8 gives $d(C_2) \ge 2$ only, we can use the number of ordered pairs of codewords of C_2 at distance 2 to lower bound $\#\mathcal{B}$ in Theorem 13 and to derive the following necessary criterion:

Lemma 14. *Let $C \subseteq \mathcal{A}^{out(S)}$ be a code that can be 1-error corrected for $\mathcal{N}_s$ and C_2 its restriction to $out(S)\backslash\{e_0\}$. Then, we have*

$$\#C + \frac{\Lambda}{\#\mathcal{A}(\#\mathcal{A}-1)} \le \#\mathcal{A}^s,$$

where $\Lambda := \big\{(a,b) \in C_2^2 \mid d(a,b)=2\big\}$.

Proof. Let $\mathcal{B}$ as in Theorem 13. If $a,b \in C_2$ are two codewords with $d(a,b)=2$, then there exists a vector $x \in \mathcal{A}^{out(S)\backslash\{e_0\}}$ with $d(a,x)=d(x,b)=1$, so that $a,b \in \tau(x)$. Thus, there exists a set $B \in \mathcal{B}$ with $\tau(x) \subseteq B$. Since $\#B \le \#\mathcal{A}$ at most $\#\mathcal{A}(\#\mathcal{A}-1)$ ordered pairs counted in Λ can yield the same set $B \in \mathcal{B}$, so that $\#\mathcal{B} \ge \frac{\Lambda}{\#\mathcal{A}(\#\mathcal{A}-1)}$ and the stated inequality follows from Theorem 13. $\square$

For $s=2$ the number of pairs of codewords at distance 2 in C_2 can be lower bounded easily:

Lemma 15. *Let C a code over the alphabet $\mathcal{A}$ with length 3 and minimum Hamming distance at least two, then we have $\Lambda \ge \#C \cdot (\#C - \#\mathcal{A})$ for $\Lambda := \big\{(a,b) \in C^2 \mid d(a,b) = 2\big\}$.*

Proof. For each $c \in C$ there can be at most $\#\mathcal{A} - 1$ codewords $c' \in C$ with $d(c,c') = 3$, so that at least $\#C - \#\mathcal{A}$ codewords are at distance 2 to c. □

Proposition 16. *Let $C \subseteq \mathcal{A}^{out(S)}$ be a code that can be 1-error corrected for $\mathcal{N}_2$. Then, we have*

$$\#C \le \left\lfloor -\frac{a(a-2)}{2} + \frac{a}{2} \cdot \sqrt{5a^2 - 8a + 4} \right\rfloor, \tag{4}$$

where $a = \#\mathcal{A}$.

Proof. From Lemma 14 and Lemma 15 we conclude $\#C + \frac{(\#C - a)\#C}{a(a-1)} \le a^2$, which can be easily solved for $\#C$. □

For $a := \#\mathcal{A} \ge 3$ we have

$$-\frac{a(a-2)}{2} + \frac{a}{2} \cdot \sqrt{5a^2 - 8a + 4} \le -\frac{a(a-2)}{2} + \frac{a}{2} \cdot \sqrt{5} \cdot \left(a - \tfrac{3}{5}\right),$$

which asymptotically equals $\frac{\sqrt{5}-1}{2} \cdot a^2$, i.e., we have $\sigma(2,a) < 0.6181a^2$ for large a. For e.g. $a = 7$ we obtain $\#C \le \lfloor 32.587922696 \rfloor = 32$.

In [1, Remark 5.10] the "conjecture" $\sigma(s,a) = \frac{a^s + a}{2} - 1$ was mentioned. For

$$(s,a) \in \big\{(1,\star), (2,2), (3,2), (4,2), (2,3), (3,3), (2,4)\big\}$$

code sizes $\frac{a^s+a}{2} - 1$ can indeed be attained. However, in [16][1] the upper bound

$$A_a(n,3) \le \frac{a^n \cdot (hn - x(a-x))}{(hn + x) \cdot (hn + x - a)},$$

where $h = a - 1$, $n \equiv x \pmod{a}$, and $1 \le x \le a$, was concluded for $a > 2$ and sufficiently large n from Delsarte's linear programming method. Thus, we have $A_a(n,3) < \frac{a^n}{(a-2)n}$ for $a > 2$ and sufficiently large n, i.e., if the conjecture is true, then a has to be sufficiently large for a given parameter s. For $s = 4$ and alphabet size 3 we have $35 \le \sigma(4,3) \le A_3(6,3) = 38 < 41$ and also $\sigma(s,2) < \frac{2^s+2}{2} - 1$ for all $s \ge 5$.

We can also use Delsarte's linear programming method to derive lower bounds for Λ. To this end we mention that for integers $n \ge 1$ and $q \ge 2$ the *Krawtchouk polynomials* are defined as

$$K_i^{(n,q)}(z) := \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} q^j \binom{n-j}{n-i} \binom{z}{j} \tag{5}$$

for all $i \ge 0$, where $\binom{z}{j} := z(z-1)\cdots(z-j+1)/j!$ for all $z \in \mathbb{R}$. The vector $B(C) = (B_0, B_1, \ldots, B_n)$, where

$$B_i = \frac{1}{\#C} \cdot \left|\big\{(a,b) \in C^2 \mid d(a,b) = i\big\}\right|, \; i = 0, 1, \ldots, n \tag{6}$$

[1] C.f. https://mathscinet.ams.org/mathscinet-getitem?mr=484738

is called the *distance distribution* of C. Clearly, $B_0 = 1$ and $B_i = 0$ for all $1 \le i \le d(C) - 1$. Moreover, $\Lambda = \#C \cdot B_2$. The vector $B'(C) = (B'_0, B'_1, \dots, B'_n)$, where

$$B'_i = \frac{1}{\#C} \sum_{j=0}^{n} B_j K_i^{(n,q)}(j),\ i = 0, 1, \dots, n \tag{7}$$

is called the *dual distance distribution* of C or the *MacWilliams transform* of $B(C)$. Obviously we have $B'_0 = 1$.

Theorem 17. *([9, 10]) The dual distance distribution of C satisfies $B'_i \ge 0$ for all $0 \le i \le n$.*

In Table 3 we have listed a few explicit upper bounds for $\sigma(s, a)$ based on Lemma 14 and the linear programming method (using $B_1 = 0$ and minimizing B_2). For more details we refer to the recent survey [6].

s	2	3	4	5	2	3	4	5	2	3	4	5
a	3	3	3	3	4	4	4	4	5	5	5	5
$\le$	6	15	42	108	10	37	133	484	16	76	337	1512

Table 3: Upper bounds for $\sigma(s, a)$.

We propose the determination of tighter bounds for $\sigma(s, a)$ as an interesting open problem.

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