The Geometry of $(t \mod q)$-arcs

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Abstract

In this paper, we give a geometric construction of the three strong non-lifted $(3 \mod 5)$-arcs in $\mathrm{PG}(3,5)$ of respective sizes 128, 143, and 168, and construct an infinite family of non-lifted, strong $(t \mod q)$-arcs in $\mathrm{PG}(r,q)$ with $t = (q+1)/2$ for all $r \geq 3$ and all odd prime powers q.

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1 Introduction

The strong $(t \mod q)$-arcs were introduced and investigated in [2, 4, 6, 7] in connection with the extendability problem for Griesmer arcs. This problem is related in turn to the problem of the existence and extendability of arcs associated with Griesmer codes. In [2] the classification of the strong $(3 \mod 5)$-arcs was used to rule out the existence of the hypothetical $[104,4,82]_5$-code, one of the four undecided cases for codes of dimension 4 over $\mathbb{F}_5$. It turns out that apart from the many strong $(3 \mod 5)$-arcs obtained from the canonical lifting construction, there exist three non-lifted strong $(3 \mod 5)$-arcs of respective sizes 128, 143, and 168. This is a counterexample to the conjectured impossibility of strong $(3 \mod 5)$-arcs in geometries over $\mathbb{F}_5$ in dimensions larger than 2. The three arcs are constructed by a computer search, but display regularities which suggest a nice geometric structure.

In this paper, we give a geometric, computer-free construction of the three non-lifted strong $(3 \mod 5)$-arcs in $\mathrm{PG}(3,5)$. Two of them are related to the non-degenerate quadrics of $\mathrm{PG}(3,5)$. Their construction can be generalized further to larger fields and larger dimensions.

2 Preliminaries

We define an arc in $\mathrm{PG}(r,q)$ as a mapping from the point set $\mathcal{P}$ of the geometry to the non-negative integers: $\mathcal{K}\colon \mathcal{P} \to \mathbb{N}_0$. An arc $\mathcal{K}$ in $\mathrm{PG}(r,q)$ is called a $(t \mod q)$-arc if $\mathcal{K}(L) \equiv t \pmod q$ for every line L. It is immediate that $\mathcal{K}(S) \equiv t \pmod{q}$ for every subspace S with $\dim S \geq 1$. Increasing the multiplicity of an arbitrary point by q preserves the property of being a $(t \mod q)$-arc. So, we can assume that the point multiplicities are integers contained in the interval $[0, q-1]$. If the maximal point multiplicity is at most t we call $\mathcal{K}$ a *strong* $(t \mod q)$-arc.

The extendability of the so-called t-quasidivisible arcs is related to structure properties of $(t \mod q)$-arcs. In particular, an (n, s)-arc $\mathcal{K}$ in $\mathrm{PG}(r, q)$ with spectrum (a_i) is called *t-quasidivisible* with divisor Δ if $s \equiv n + t \pmod{\Delta}$ and $a_i = 0$ for all $i \not\equiv n, n+1, \ldots, n+t \pmod{\Delta}$. It is quite common in coding theory that hypothetical Griesmer codes are associated with arcs that turn out to be t-quasidivisible with divisor q for some t. The extendability of t-quasidivisible arcs is related to the structure of particular strong $(t \mod q)$-arcs associated with them.

Let $\mathcal{K}$ be an arc in $\mathrm{PG}(r, q)$ and let $\sigma : \mathbb{N}_0 \to \mathbb{Q}$ be a function satisfying $\sigma(\mathcal{K}(H)) \in \mathbb{N}_0$ for every hyperplane H in $\mathcal{H}$, where $\mathcal{H}$ is the set of all hyperplanes in $\mathrm{PG}(r, q)$. The arc $\mathcal{K}^\sigma : \mathcal{H} \to \mathbb{N}_0$, $H \to \sigma(\mathcal{K}(H))$ is called the σ-dual of $\mathcal{K}$. For a t-quasidivisible arc with divisor q, we consider the σ-dual arc obtained for $\mathcal{K}^\sigma(H) = n + t - \mathcal{K}(H) \pmod{q}$. It turns out that with this σ the σ-dual to a t-quasidivisible arc $\mathcal{K}$ is a strong $(t \mod q)$-arc. Moreover, if $\mathcal{K}^\sigma$ contains a hyperplane in its support then $\mathcal{K}$ is extendable [4, 7].

There exist several straightforward constructions of $(t \mod q)$-arcs [4, 6, 7]. The first is the so-called sum-of-arcs construction.

Theorem 1. *Let $\mathcal{K}$ and $\mathcal{K}'$ be a $(t_1 \mod q)$- and a $(t_2 \mod q)$-arc in $\mathrm{PG}(r, q)$, respectively. Then $\mathcal{K} + \mathcal{K}'$ is a $(t \mod q)$-arc with $t \equiv t_1 + t_2 \pmod{q}$. Similarly, $\alpha\mathcal{K}$, where $\alpha \in \{0, \ldots, \ldots, p-1\}$ and p is the characteristic of $\mathbb{F}_q$, is a $(t \mod q)$-arc with $t \equiv \alpha t_1 \pmod{q}$.*

For the special case of $t = 0$, and $q = p$ we have that the sum of two $(0 \mod p)$-arcs and the scalar multiple of a $(0 \mod p)$-arc are again $(0 \mod p)$-arcs. Hence the set of all $(0 \mod p)$-arcs is a vector space over $\mathbb{F}_p$, cf. [6].

The second construction is the so-called *lifting construction*, see [6, p. 230].

Theorem 2. *Let $\mathcal{K}_0$ be a (strong) $(t \mod q)$-arc in a projective s-space Σ of $\mathrm{PG}(r, q)$, where $1 \leq s < r$. For a fixed projective $(r-s-1)$-space Γ of $\mathrm{PG}(r, q)$, disjoint from Σ, let $\mathcal{K}$ be the arc in $\mathrm{PG}(r, q)$ defined as follows:*

- *for each point P of Γ, set $\mathcal{K}(P) = t$;*
- *for each point $Q \in \mathrm{PG}(r, q) \setminus \Gamma$, set $\mathcal{K}(Q) = \mathcal{K}_0(R)$, where $R = \langle \Gamma, Q \rangle \cap \Sigma$.*

Then $\mathcal{K}$ is a (strong) $(t \mod q)$-arc in $\mathrm{PG}(r, q)$ of cardinality $q^{r-s} \cdot |\mathcal{K}_0| + t\frac{q^{r-s}-1}{q-1}$.

Arcs obtained by the lifting construction are called *lifted arcs*. If Σ is a point, then we speak of a *lifting point*. The iterative application of the lifting constructions gives the more general version stated above. In the other direction, in [6, Lemma 1] i was shown that the set of all lifting points forms a subspace.

The classification of strong $(t \mod q)$ arcs in $\mathrm{PG}(2, q)$ is equivalent to that of certain plane blocking sets [5].

Theorem 3. *A strong $(t \mod q)$-arc $\mathcal{K}$ in $\mathrm{PG}(2, q)$ of cardinality $mq + t$ exists if and only if there exists an $((m-t)q + m, \geq m-t)$-blocking set $\mathcal{B}$ with line multiplicities contained in the set $\{m-t, m-t+1, \ldots, m\}$.*

The condition that the multiplicity of each point is at most t turns out to be very strong. For $t = 0$, we have that the only strong $(0 \mod q)$-arc is the trivial zero-arc. For $t = 1$ the strong $(1 \mod q)$-arcs are the hyperplanes. For $t = 2$ all strong $(2 \mod q)$ arcs in $\mathrm{PG}(r, q)$, for $r \geq 3, q \geq 5$, turn out to be lifted [6]. In $\mathrm{PG}(2, q)$, all $(2 \mod q)$-arcs are also known (cf. [2, Lemma 3.7]). Apart from one sporadic example, all such arcs are again lifted. It was conjectured in [5] that all strong $(3 \mod 5)$-arcs in $\mathrm{PG}(r, 5)$, $r \geq 3$, are lifted. The computer classification reported in [2] shows that this conjecture is wrong: there exist $(3 \mod 5)$-arcs of respective sizes 128, 143, and 168 that are not lifted. In the next sections we give a geometric (computer-free) description of these arcs and define an infinite class of strong $(t \mod q)$-arcs in $\mathrm{PG}(r, q)$, $r \geq 3$, that are not lifted.

3 The arc of size 128

We shall need the classification of all strong $(3 \mod 5)$-arcs in $\mathrm{PG}(2, 5)$ of sizes 18, 23, 28 and 33. It is obtained easily from Theorem 3 and can be found in [2, 6].

Theorem 4. *Let $\mathcal{K}$ be a strong $(3 \mod 5)$-arc in $\mathrm{PG}(2, 5)$. Let λ_i, $i = 0, 1, 2, 3$, denote the number of i points of $\mathcal{K}$.*

(a) If $|\mathcal{K}| = 18$ then $\mathcal{K}$ is the sum of three lines.

(b) If $|\mathcal{K}| = 23$ then it has $\lambda_3 = 3, \lambda_2 = 4, \lambda_1 = 6$. The four 2-points form a quadrangle, the three 3-points are the diagonal points of the quadrangle, and the 1-points are the intersections of the diagonals with the sides of the quadrangle.

(c) If $|\mathcal{K}| = 28$ then it has $\lambda_3 = 6, \lambda_1 = 10$. The 3-points form an oval, and the 1-points are the internal points to this oval.

(d) There exist ten non-isomorphic arcs with $|\mathcal{K}| = 33$. These are:

(i) the duals of the complements of the seven $(10, 3)$-arcs in $\mathrm{PG}(2, 5)$ (cf. [3]);

(ii) the dual of the multiset which is complement of the $(11, 3)$-arc with four external lines plus one point which is not on a 6-line ($\lambda_3 = 6, \lambda_2 = 5, \lambda_1 = 5$);

(iii) the dual of a blocking set in which one double point forms an oval with five of the 0-points; the tangent to the oval in the 2-point is a 3-line ($\lambda_3 = 6, \lambda_2 = 5, \lambda_1 = 5$);

(iv) the modulo 5 sum of three non-concurrent lines: two of them are lines of 3-points and one is a line of 2-points ($\lambda_3 = 8, \lambda_2 = 4, \lambda_1 = 1$).

Let us note that one of the strong $(3 \mod 5)$-arcs in case $(d(i))$ is obtained by taking as 3-points the points of an oval and as 1-points the external points to the oval.

Consider a $(3 \mod 5)$-arc $\mathcal{K}$ in $\mathrm{PG}(3, 5)$ which is of multiplicity 128. Let φ be a projection from an arbitrary 0-point P to a plane π not incident with P:

$$\varphi\colon \left\{ \begin{array}{lcl} \mathcal{P} \setminus \{P\} & \to & \pi \\ Q & \to & \pi \cap \langle P, Q \rangle. \end{array} \right. \tag{1}$$

Here $\mathcal{P}$ is again the set of points of $\mathrm{PG}(3, 5)$. Note that φ maps the lines through P into points from π, and the planes through P into lines in π. For every set of points $\mathcal{F} \subset \pi$, define the induced arc $\mathcal{K}^\varphi$ by

$$\mathcal{K}^\varphi(\mathcal{F}) = \sum_{\varphi(P)\in\mathcal{F}} \mathcal{K}(P).$$

It is clear that P is incident with 3- and 8-lines, only. If there exists a 13-line L through P then all planes through L have multiplicity at least 33 (Theorem 4) and $|\mathcal{K}| \geq 6 \cdot 33 - 5 \cdot 13 = 133$, a contradiction.

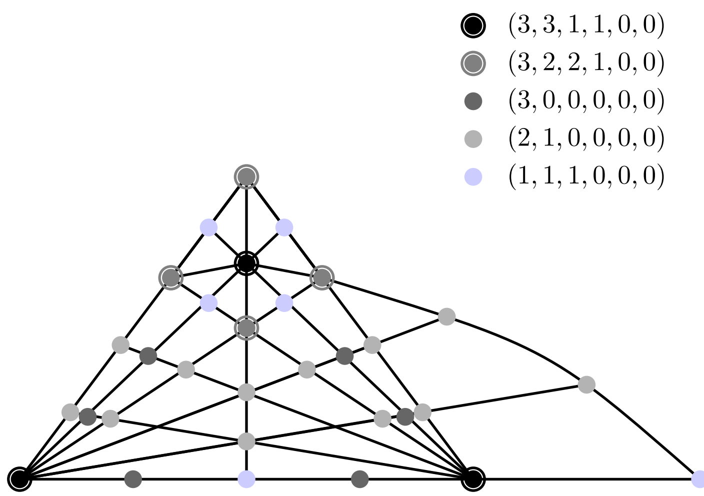
An 8-line through P is either of type (3,3,1,1,0,0) (type (α), or of type (3,2,2,1,0,0) (type (β)). Other types for an 8-line are impossible by the same counting argument as above: a plane through such a line has to be of multiplicity at least 33 (18-planes are impossible since P is a 0-point), and we get a contradiction by the same counting argument as above. A 3-line through P is of type (γ_1) (3,0,0,0,0,0), (γ_2) (2,1,0,0,0,0), or (γ_3) (1,1,1,0,0,0). A point in the projection plane is said to be of type (α), (β), or (γ_i) if it is the image of a line of the same type. Let us note that type (α) and (β) are the same as types (B_2) and (B_3) from [2]; similarly type (γ_i) coincides with type (A_i), $i = 1, 2, 3$.

By Theorem 4, if a line in the projection plane has one 8-point then it contains:

- one point of type (α), one point of type (γ_1), and four points of type (γ_2), or else

- one point of type (β), two points of type (γ_1), two points of type (γ_2) and one point of type (γ_3).

We are going to prove that if $\mathcal{K}$ is a strong (3 mod 5)-arc in PG(3,5) of cardinality 128 then the induced arc $\mathcal{K}^\varphi$ in PG(2,5) is unique (up to isomorphism). It consists of seven 8-points and 24 3-points. Three of the 8-points are of type (α), and four are of type (β). The 3-points are: six of type (γ_1), twelve of type (γ_2), and six of type (γ_3). The points of type (β) form a quadrangle, and the points of type (α) are the diagonal points. The intersections of the lines defined by the diagonal points with the sides of the quadrangle are points of type (γ_3); the six points on the lines defined by the diagonal points that are not on sides of the quadrangle are of type (γ_1); all the remaining 3-points are of type (γ_2). The induced arc $\mathcal{K}^\varphi$ is presented on the picture below.



Lemma 5. *Let $\mathcal{K}$ be a strong* (3 mod 5)*-arc in* PG(3,5) *of cardinality 128. Let φ be the projection from an arbitrary 0-point in* PG(3,5) *into a plane disjoint from that point. Then the arc $\mathcal{K}^\varphi$ is unique up to isomorphism and has the structure described above.*

Proof. We have seen that 0-points are incident only with lines of multiplicity 3 and 8. Hence $\mathcal{K}^{\varphi}$ has seven 8-points and twenty-four 3-points. Assume that six of the 8-points are collinear. Clearly, every 8-point is of type (α) since it is on a line containing two 8-points (and hence the image of a 28-plane). Every other point in the projection plane is also on a line containing two 8-points; hence all 3- points in the plane are of type (γ_1) or (γ_3). But now a line with one 8-point cannot have points of type (γ_2), which is a contradiction with the structure of the $(3 \mod 5)$-arc of size 23.

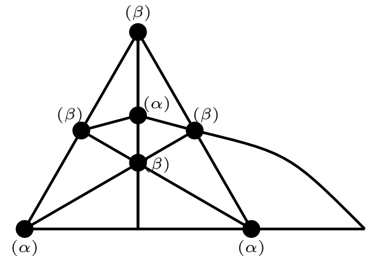
Assume that five of the 8-points are collinear. Let L be the line that contains them. If the two 8-points off L define a line meeting L in a 3-point, the proof is completed as above. Otherwise, the points off L are on four lines containing two 8-points. Now it is easily checked that there exists a line with exactly one 8-point which has at least four 3-points that are not of type (γ_2). This is a contradiction with the structure of the $(3 \mod 5)$-arc of size 23.

A similar argument rules out the possibility of four collinear 8-points. In all cases these have to be points of type (α). So are the remaining three 8-points. Now for all possible configurations of these seven points we get a 23-line without enough points of type (γ_2).

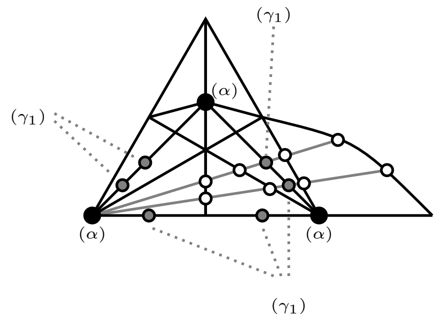
We are going to consider in full detail the case when at most three 8-points in the projection plane are collinear. Assume there exists an oval of 8-points, $X_1, \dots, X_6$, say, and let Y be the seventh 8-point. All 8-points are of type (α) and let YX_1X_2 be a secant to the oval through Y. The lines X_1X_j, $j = 3, 4, 5, 6$, are images of planes without 2-points. Now an external line to the oval through Y is a 23-line and has at most one point of type (γ_2), a contradiction. In a similar way, we rule out the case where there exist five 8-points no three of which are collinear. We have to consider the different possibilities for the line defined by the remaining two 8-points: secant, tangent, or external line to the oval formed by the former five points and one additional point which has to be a 3-point.

We have shown so far that there are at most three collinear 8-points. It is also clear that there exist at least two lines that contain three 8-points. We consider the case where these lines meet in a 3-point. Denote the 8-points by X_i, Y_i, $i = 1, 2, 3$, and Z. We also assume that X_1, X_2, X_3 are collinear and so are Y_1, Y_2, Y_3. Each of the lines ZX_i, $i = 1, 2, 3$, also contains three 8-points; otherwise there exist five 8-points no three of which are collinear. Without loss of generality, the triples Z, X_i, Y_i, $i = 1, 2, 3$, are collinear. Now it is clear that all the points X_i, Y_i are of type (α). Moreover, neither of the lines X_iY_j, $i \neq j$, has 3-points of type (γ_2). Now if we consider a line through X_3 that does not have other 8-points it should contain four points of type (γ_2). On the other hand, it intersects X_1Y_2 and X_1Y_3 in points which are not of this type which gives a contradiction.

Now we are left with only one possibility for the 8-points subject to the conditions: (i) each line contains at most three 8-points, (ii) lines incident with three 8-points meet in an 8-point, (iii) every 5-tuple of 8-points contains a collinear triple. The 8-points are the vertices of a quadrangle plus the three diagonal points. Furthermore, the diagonal points have to be of type (α) while the vertices of the quadrangle are forced to be of type (β). This is due to the fact that through each of the vertices of the quadrangle there is a line with a single 8-point which meets the three lines defined by the diagonal points of type (α) in three different 3-points that are not of type (γ_2) (since a 28-plane does not have 2-points). Thus we get the picture below.



The fact that a 23-line through a point of type (α) contains four points of type (γ_2) and one point of type (γ_1) identifies the six points of type (γ_1).



Furthermore, a line with two points of type (α) must contain also two points of type (γ_1) and two points of type (γ_3). This identifies the six 3-points of type (γ_1). The remaining 3-points are all of type (γ_2). This implies the suggested structure. □

Lemma 5 implies that given a nonlifted, strong ($3 \mod 5$)-arc $\mathcal{K}$ of cardinality 128, every 0-point is incident with

- three 8-lines of type $(3, 3, 1, 1, 0, 0)$,
- four 8-lines of type $(3, 2, 2, 1, 0, 0)$,
- six 3-lines of type $(3, 0, 0, 0, 0, 0)$,
- twelve 3-lines of type $(2, 1, 0, 0, 0, 0)$,
- six 3-lines of type $(1, 1, 1, 0, 0, 0)$

Now this implies that

- #(3-points) $= 3 \cdot 2 + 4 \cdot 1 + 6 \cdot 1 = 16$,
- #(2-points) $= 4 \cdot 2 + 12 \cdot 1 = 20$,
- #(1-points) $= 3 \cdot 2 + 4 \cdot 1 + 12 \cdot 1 + 6 \cdot 3 = 40$,
- #(0-points) $= 1 + 3 \cdot 1 + 4 \cdot 1 + 6 \cdot 4 + 12 \cdot 3 + 2 \cdot 6 = 80$.

Furthermore, each 0-point is incident with six 33-planes, three 28-planes eighteen 23-planes and four 18-planes. Moreover the number of zeros in a 33-plane is 12, in a 28-plane – 15, in a 23-plane

– 18, and in an 18-plane – 16. This makes it possible to compute the spectrum of $\mathcal{K}$. We have

$$\begin{aligned} a_{33} &= \frac{80 \cdot 6}{12} = 40 \\ a_{28} &= \frac{80 \cdot 3}{15} = 16, \\ a_{23} &= \frac{80 \cdot 18}{18} = 80, \\ a_{18} &= \frac{80 \cdot 4}{16} = 20. \end{aligned}$$

Furthermore, every 33-, 28-, 23-plane in $\mathcal{K}$ is unique up to isomorphism.

From the above considerations we can deduce that no three 2-points are collinear. In other words they form a 20-cap C. Moreover, this cap has spectrum: $a_6(C) = 40, a_4(C) = 80, a_3(C) = 20, a_0(C) = 16$. It is not extendable to the elliptic quadric; in such case it would have (at least 20) tangent planes. Thus, this cap is complete and isomorphic to one of the two caps K_1 and K_2 by Abatangelo, Korchamros and Larato [1]. It is not K_2 since it has a different spectrum (cf. [1]). Hence the 20-cap on the 2-points in $\mathrm{PG}(3,5)$ is isomorphic to K_1.

Consider the complete cap K_1. The collineation group G of K_1 is a semidirect product of an elementary abelian group of order 16 and a group isomorphic to S_5 [1]. Hence $|G| = 1920$. The action of G on $\mathrm{PG}(3,5)$ splits the point set of $\mathrm{PG}(3,5)$ into four orbits on points, denoted by $O_1^P, \ldots, O_4^P$, and the set of lines into six orbits, denoted by $O_1^L, \ldots, O_6^L$. The respective sizes of these orbits are

$$|O_1^P| = 40, |O_2^P| = 80, |O_3^P| = 20, |O_4^P| = 16;$$
$$|O_1^L| = 160, |O_2^L| = 240, |O_3^L| = 30, |O_4^L| = 160, |O_5^L| = 120, |O_6^L| = 96.$$

The corresponding point-by-line orbit matrix $A = (a_{ij})_{4\times 6}$, where a_{ij} is the number of the points from the i-th point orbit incident with any line from the j-th line orbit is the following

$$A = \begin{pmatrix} 3 & 1 & 4 & 1 & 2 & 0 \\ 3 & 4 & 0 & 2 & 2 & 5 \\ 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}.$$

Set $w = (w_1, w_2, w_3, w_4)$. We look for solutions of the equation $wA \equiv 3\boldsymbol{j} \pmod 5$, where $\boldsymbol{j}$ is the all-one vector, subject to the conditions $w_i \le 3$ for all $i = 1, 2, 3, 4$. The set of all solutions is given by

$$\{w = (w_1, w_2, w_3, w_4) \mid w_i\{0, \ldots 4\}, \\ w_2 \equiv 1 - w_1 \pmod 5, w_3 \equiv 4 - 2w_1 \pmod 5, w_4 = 3\}.$$

There exist two solutions that satisfy $w_i \le 3$: $w = (3, 3, 3, 3)$ and $w = (1, 0, 2, 3)$. The first one yields the trivial $(3 \mod 5)$-arc formed by three copies of the whole space. The second one gives the desired arc of size 128.

It should be noted that the weight vectors $(0, 3, 2, 4)$, $(1, 2, 0, 4)$, $(2, 1, 3, 4)$, and $(3, 0, 1, 4)$ yield strong $(4 \mod 5)$-arcs of cardinalities 344, 264, 284, and 204, respectively, that are not lifted.

4 Strong $\left(\frac{q+1}{2} \mod q\right)$-arcs from quadrics and the arcs of size 143 and 168

For an arbitrary odd prime power q and an integer $r \geq 2$, let $\mathcal{Q}$ be a quadric of $\mathrm{PG}(r,q)$ and let F be the quadratic form defining $\mathcal{Q}$. This means that a point $P(x_0,\ldots,x_r)$ of $\mathrm{PG}(r,q^2)$ belongs to $\mathcal{Q}$ whenever $F(x_0,\ldots,x_r)=0$. The points of $\mathrm{PG}(r,q)$ outside $\mathcal{Q}$ are partitioned into two point classes, say $\mathcal{P}_1$ and $\mathcal{P}_2$. Indeed, if $P(x_0,\ldots,x_r)$ is a point of $\mathrm{PG}(r,q)\setminus\mathcal{Q}$, then P belongs to $\mathcal{P}_1$ or $\mathcal{P}_2$, according as $F(x_0,\ldots,x_r)$ is a non-square or a square in $\mathbb{F}_q$. Now we define the arcs $\mathcal{K}_1$ and $\mathcal{K}_2$ in the following way:

- $\mathcal{K}_1$: for a point P of $\mathrm{PG}(r,q)$ set

$$\mathcal{K}_1(P)=\begin{cases}\frac{q+1}{2} & \text{if } P\in\mathcal{Q},\\ 1 & \text{if } P\in\mathcal{P}_1,\\ 0 & \text{if } P\in\mathcal{P}_2.\end{cases} \tag{2}$$

- $\mathcal{K}_2$: for a point P of $\mathrm{PG}(r,q)$ set

$$\mathcal{K}_2(P)=\begin{cases}\frac{q+1}{2} & \text{if } P\in\mathcal{Q},\\ 0 & \text{if } P\in\mathcal{P}_1,\\ 1 & \text{if } P\in\mathcal{P}_2.\end{cases} \tag{3}$$

The following result is well-known.

Proposition 6. *Let $f(x)=ax^2+bx+c$, where $a,b,c,\in\mathbb{F}_q$, $a\neq 0$, q odd. If $\mathbb{F}_q=\{\alpha_0,\alpha_1,\ldots,\alpha_{q-1}\}$. Denote by S be the list of the following elements from $\mathbb{F}_q$:*

$$a, f(\alpha_0), f(\alpha_1),\ldots,f(\alpha_{q-1}).$$

Then

(a) if $f(x)$ has two distinct roots in $\mathbb{F}_q$ the list S contains two zeros, $(q-1)/2$ squares and $(q-1)/2$ non-squares;

(b) if $f(x)$ has one double root in $\mathbb{F}_q$ then S contains a zero and q squares, or a zero and q non-squares;

(c) if $f(x)$ is irreducible over $\mathbb{F}_q$ then S contains $(q+1)/2$ squares and $(q+1)/2$ non-squares.

Theorem 7. *Let the $\mathcal{K}_1$ and $\mathcal{K}_2$ be the arcs defined in (2) and (3), respectively. Then $\mathcal{K}_i$ is a $\left(\frac{q+1}{2} \mod q\right)$ arc of $\mathrm{PG}(r,q)$, $i=1,2$. Moreover, if $\mathcal{Q}$ is non-degenerate, then both arcs are not lifted.*

Proof. Let ℓ be a line of $\mathrm{PG}(r,q)$, then $\mathcal{Q}\cap\ell$ is a quadric of ℓ. Then, from Proposition 6, it follows that

$$\mathcal{K}_i(\ell)=\begin{cases}2\cdot\frac{q+1}{2}+\frac{q-1}{2} & \text{if } |\ell\cap\mathcal{Q}|=2,\\ \frac{q+1}{2}+q & \text{if } |\ell\cap\mathcal{Q}|=1 \text{ and } |\ell\cap\mathcal{P}_i|=q,\\ \frac{q+1}{2} & \text{if } |\ell\cap\mathcal{Q}|=1 \text{ and } |\ell\cap\mathcal{P}_i|=0,\\ \frac{q+1}{2} & \text{if } |\ell\cap\mathcal{Q}|=0.\end{cases}$$

Therefore $\mathcal{K}_i$ is a $\left(\frac{q+1}{2} \mod q\right)$ arc of $\mathrm{PG}(r,q)$, $i=1,2$. If $\mathcal{Q}$ is non-degenerate, then through every point of $\mathrm{PG}(r,q)$ there exists a line r that is secant to $\mathcal{Q}$. By construction, the line r has two $\frac{q+1}{2}$-points, $\frac{q-1}{2}$ 1-points and $\frac{q-1}{2}$ 0-points. Hence $\mathcal{K}_i$ is not lifted. □

Corollary 8. *If r is odd, then*

$$|\mathcal{K}_i| = \begin{cases} \frac{q+1}{2}\cdot\frac{(q^{\frac{r+1}{2}}+1)(q^{\frac{r-1}{2}}-1)}{q-1}+\frac{q^r+q^{\frac{r-1}{2}}}{2} & \text{if } \mathcal{Q} \text{ is elliptic,}\\ \frac{q+1}{2}\cdot\frac{(q^{\frac{r-1}{2}}+1)(q^{\frac{r+1}{2}}-1)}{q-1}+\frac{q^r-q^{\frac{r-1}{2}}}{2} & \text{if } \mathcal{Q} \text{ is hyperbolic.}\end{cases}$$

If r is even, then

$$|\mathcal{K}_1| = \frac{q+1}{2}\cdot\frac{(q^r-1)}{q-1}+\frac{q^r-q^{\frac{r}{2}}}{2},$$
$$|\mathcal{K}_2| = \frac{q+1}{2}\cdot\frac{(q^r-1)}{q-1}+\frac{q^r+q^{\frac{r}{2}}}{2}.$$

Remark 9. *In the case when the quadric $\mathcal{Q}$ is degenerate, then it is not difficult to see that the arc $\mathcal{K}_i$, $i=1,2$, is lifted. Let $\mathcal{Q}$ be a non-degenerate quadric of* $\mathrm{PG}(r,q)$*, then $\mathcal{K}_1$ and $\mathcal{K}_2$ are projectively equivalent if r is odd, but they are not in the case when r is even. On the other hand, if r is odd, there are two distinct classes of non-degenerate quadrics, namely the hyperbolic quadric and the elliptic quadric. Therefore in all cases Theorem 7 gives rise to two distinct examples of non lifted $\left(\frac{q+1}{2} \mod q\right)$ arcs of* $\mathrm{PG}(r,q)$.

4.1 The arcs of size 143 and 168

In [2], the following two strong non-lifted $(3 \mod 5)$-arcs in $\mathrm{PG}(3,5)$ were constructed by a computer search. The respective spectra are:

$$|\mathcal{F}_1| = 143, \quad a_{18}(\mathcal{F}_1)=26, a_{23}(\mathcal{F}_1)=0, a_{28}(\mathcal{F}_1)_{28}=65, a_{33}(\mathcal{F}_1)=65;$$

$$\lambda_0(\mathcal{F}_1)=65, \lambda_1(\mathcal{F}_1)=65, \lambda_2(\mathcal{F}_1)=0, \lambda_3(\mathcal{F}_1)=26,$$

and

$$|\mathcal{F}_2| = 168, \quad a_{28}(\mathcal{F}_2)=60, a_{33}(\mathcal{F}_2)=60, a_{43}(\mathcal{F}_2)36);$$

$$\lambda_0(\mathcal{F}_2)=60, \lambda_1(\mathcal{F}_2)=60, \lambda_2(\mathcal{F}_2)=0, \lambda_3(\mathcal{F}_2)=36.$$

In addition, $|\operatorname{Aut}(\mathcal{F}_1)|=62400$, and $|\operatorname{Aut}(\mathcal{F}_2)|=57600$.

These arcs can be recovered from Theorem 7. Indeed, if $\mathcal{Q}$ is an elliptic quadric of $\mathrm{PG}(3,5)$, then $\mathcal{K}_1$ is a non lifted $(3 \mod 5)$ arc of $\mathrm{PG}(3,5)$ of size 143, whereas if $\mathcal{Q}$ is a hyperbolic quadric of $\mathrm{PG}(3,5)$, then $\mathcal{K}_1$ is a non lifted $(3 \mod 5)$ arc of $\mathrm{PG}(3,5)$ of size 168.

5 Further examples $(t \mod q)$-arcs

A set of type (m, n) in $\mathrm{PG}(r, q)$ is a set $\mathcal{S}$ of points such that every line of $\mathrm{PG}(r, q)$ contains either m or n points of $\mathcal{S}$, $m < n$, and both values occur. Assume $m > 0$. Then the only sets of type (m, n) that are known, exist in $\mathrm{PG}(2, q)$, q square, and are such that $n = m + \sqrt{q}$. In particular, sets of type $(1, 1 + \sqrt{q})$ either contains $q + \sqrt{q} + 1$ and are Baer subplanes or $q\sqrt{q} + 1$ points and are known as *unitals*. For more details on sets of type (m, n) in $\mathrm{PG}(2, q)$ see [8] and references therein. If $\mathcal{S}$ is an (m, n) set in $\mathrm{PG}(r, q)$, $r > 2$, then necessarily q is an odd square, $m = (\sqrt{q} - 1)^2/2$, $n = m + \sqrt{q}$ and $|\mathcal{S}| = \frac{1 + \frac{q^r - 1}{q - 1}(q - \sqrt{q}) \pm \sqrt{q}^r}{2}$, see [10]. However no such a set is known to exist if $r > 2$.

Theorem 10. *Let $\mathcal{S}$ be a set of type $(m, m + \sqrt{q})$ in $\mathrm{PG}(r, q)$, q square. Let $\mathcal{K}$ be the arc of $\mathrm{PG}(r, q)$ such that $\mathcal{K}(P) = \sqrt{q}$, if $P \in \mathcal{S}$ and $\mathcal{K}(P) = 0$, if $P \notin \mathcal{S}$. Then $\mathcal{K}$ is a $(\sqrt{q} \mod q)$-arc of $\mathrm{PG}(r, q)$.*

Proof. Let ℓ be a line of $\mathrm{PG}(r, q)$. If $|\ell \cap \mathcal{S}| = m$, then $\mathcal{K}(\ell) = m\sqrt{q}$, whereas if $|\ell \cap \mathcal{S}| = m + \sqrt{q}$, then $\mathcal{K}(\ell) = m\sqrt{q} + q$. □

In $\mathrm{PG}(r, q)$, q square, let $\mathcal{H}$ be a Hermitian variety of $\mathrm{PG}(r, q)$, i.e., the variety defined by a Hermitian form of $\mathrm{PG}(r, q)$. It is well-known that a line of $\mathrm{PG}(r, q)$ has 1, $\sqrt{q} + 1$ or $q + 1$ points in common with $\mathcal{H}$. Let $\mathcal{K}'$ be the arc of $\mathrm{PG}(r, q)$ such that $\mathcal{K}'(P) = \sqrt{q}$, if $P \in \mathcal{H}$ and $\mathcal{K}'(P) = 0$, if $P \notin \mathcal{H}$.

Theorem 11. *$\mathcal{K}'$ is a $(\sqrt{q} \mod q)$-arc of $\mathrm{PG}(r, q)$. Moreover, if $\mathcal{H}$ is non-degenerate, then $\mathcal{K}'$ is not lifted.*

Proof. Let ℓ be a line of $\mathrm{PG}(r, q)$. Then

$$\mathcal{K}'(\ell) = \begin{cases} \sqrt{q} & \text{if } |\ell \cap \mathcal{H}| = 1, \\ \sqrt{q} + q & \text{if } |\ell \cap \mathcal{H}| = \sqrt{q} + 1, \\ \sqrt{q}(1 + q) & \text{if } |\ell \cap \mathcal{H}| = q + 1. \end{cases}$$

If $\mathcal{H}$ is non-degenerate, then through every point of $\mathrm{PG}(r, q)$ there exists a line r such that $|\mathcal{H} \cap r| = \sqrt{q} + 1$. By construction, the line r has $\sqrt{q} + 1$ $\sqrt{q}$-points and $q - \sqrt{q}$ 0-points. Hence $\mathcal{K}'$ is not lifted. □

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