Affine vector space partitions

John Bamberg[1], Yuval Filmus[2], Ferdinand Ihringer[3], and Sascha Kurz[4]

[1]The University of Western Australia, Australia, john.bamberg@uwa.edu.au
[2]The Henry and Marilyn Taub Faculty of Computer Science and Faculty of Mathematics, Technion — Israel Institute of Technology, Israel, yuvalfi@cs.technion.ac.il
[3]Ghent University, Belgium, Ferdinand.Ihringer@gmail.com
[4]University of Bayreuth, Germany, sascha.kurz@uni-bayreuth.de

November 2022

Abstract

An affine vector space partition of $\mathrm{AG}(n,q)$ is a set of proper affine subspaces that partitions the set of points. Here we determine minimum sizes and enumerate equivalence classes of affine vector space partitions for small parameters. We also give parametric constructions for arbitrary field sizes.

Keywords— finite geometry, vector space partitions, spreads, Klein quadric, Fano plane, hitting formulas

1 Introduction

A *vector space partition* $\mathcal{P}$ of the projective space $\mathrm{PG}(n-1,q)$ is a set of subspaces in $\mathrm{PG}(n-1,q)$ which partitions the set of points. For a survey on known results we refer to [Hed12]. We say that a vector space partition $\mathcal{P}$ has type $(n-1)^{m_{n-1}} \dots 2^{m_2} 1^{m_1}$ if precisely m_i of its elements have dimension i, where $1 \le i \le n$. The classification of the possible types of a vector space partition, given the parameters n and q, is an important and difficult problem. Based on [Hed86], the classification for the binary case $q = 2$ was completed for $n \le 7$ in [EZSS$^+$09]. Under the assumption $m_1 = 0$ the case $(q,n) = (2,8)$ has been treated in [EZHS$^+$10]. It seems quite natural to define a *vector space partition* $\mathcal{A}$ of the affine space $\mathrm{AG}(n,q)$ as a set of subspaces in $\mathrm{AG}(n,q)$ that partitions the set of points. However, it turns out that those partitions exist for all types which satisfy a very natural numerical condition. If we impose the additional condition of *tightness*, that is that the projective closures of the elements of $\mathcal{A}$ have an empty intersection, then the classification problem becomes interesting and challenging. This condition is natural in the context of hitting formulas as introduced in [Iwa89], that is for logical formulas in full disjunctive normal form (DNF) such that each truth assignment to the underlying variables satisfies precisely one term. For a more recent treatment and applications we refer to [PS22]. Here we consider the geometrical and the combinatorial point of view.

Variants of vector space partitions of $\mathrm{PG}(n-1,q)$ have been studied in the literature. In [EZSS$^+$11] the authors study (multi-)sets of subspaces covering each point exactly λ times. The problem of covering each k-space exactly once is considered in [HHKK19]. A more general partition problem for groups is studied in [Hed86]. However, we are not aware of any publication directly treating the introduced affine vector space partitions.

The paper is organized as follows. In Section 2 we formally introduce affine vector space partitions, state the preliminaries, and develop the first necessary existence conditions. Here we are guided by the published necessary conditions for vector space partitions. We also argue why tightness (see above) and *irreducibility*, that is there exists no proper subset $\mathcal{A}' \subsetneq \mathcal{A}$ such that the union of all elements of $\mathcal{A}'$ is a subspace of $\operatorname{AG}(n,q)$, are necessary to obtain an interesting existence question. In Section 3 we classify affine vector space partitions for arbitrary field sizes but small dimensions. Section 4 is concerned with the binary case. We completely determine the possible dimension distributions of tight irreducible affine vector space partitions of $\operatorname{PG}(n-1,2)$ for all $n \le 7$. In a few cases we give theoretical or computational classifications of the corresponding equivalence classes of tight irreducible vector space partitions. A very nice example consists of eight solids in $\operatorname{PG}(6,2)$ whose parts at infinity live on the Klein quadric $Q^+(5,2)$. A generalization to arbitrary finite fields of characteristic 2 is given in Subsection 5.2. Parametric constructions of tight irreducible affine vector space partitions using spreads or hitting formulas complete Section 5. In Section 6 we determine the smallest possible size of an irreducible tight affine vector space partition of $\operatorname{PG}(7,2)$ and give a parametric upper bound for $\operatorname{PG}(n-1,2)$ of size roughly $\frac{3n}{2}$, which is significantly smaller than the conjectured smallest size of an irreducible hitting formula mentioning all variables. We close with a conclusion and a list of open problems in Section 7. To keep the paper self-contained we present some additional material in an appendix. Section A contains details on integer linear programming formulations that we have utilized to obtain some computational results. Section B contains a few technical results that might be left to the reader or collected from the literature. Lists of hitting formulas that can be used to construct tight irreducible affine vector space partitions of the minimum possible size are given in Section C.

2 Preliminaries and necessary conditions

Definition 1. *An* affine vector space partition $\mathcal{A}$ *of* $\operatorname{AG}(n,q)$ *is a set* $\{A_1,\dots,A_r\}$ *of subspaces of* $\operatorname{AG}(n,q)$ *such that* $0 \le \dim(A_i) \le n-1$ *for all* $1 \le i \le r$ *and every point (element of* $\mathbb{F}_q^n\backslash\mathbf{0}$*) is contained in exactly one element* A_i*. The integer* r *is called the* size *of the affine vector space partition.*

We write $\#\mathcal{A}$ for the size of $\mathcal{A}$. For each affine subspace $A \in \operatorname{AG}(n,q)$ we write $\overline{A}$ for its projective closure. With this $\overline{\mathcal{A}} := \left\{\overline{A} \,:\, A \in \mathcal{A}\right\}$ is the natural embedding of an affine vector space partition of $\operatorname{AG}(n,q)$ in $\operatorname{PG}(n,q)$. Denoting the hyperplane at infinity by H_∞, we can directly define an affine vector space partition in $\operatorname{PG}(n,q)$:

Definition 2. *An* affine vector space partition $\mathcal{U}$ *of* $\operatorname{PG}(n-1,q)$ *is a set* $\{U_1,\dots,U_r\}$ *of subspaces of* $\operatorname{PG}(n-1,q)$ *such that* $1 \le \dim(U_i) \le n-1$ *for all* $1 \le i \le r$ *and there exists a hyperplane* H_∞ *such that every point (1-dimensional subspace) outside of* H_∞ *is contained in exactly one element* U_i *and* $U_i \not\le H_\infty$ *for all* $1 \le i \le r$*. The integer* r *is called the* size *of the affine vector space partition and also denoted by* $\#\mathcal{U}$*.*

Here we use the algebraic dimension for subspaces in $\operatorname{PG}(n-1,q)$, i.e., if $\dim(U) = u$, then $\#U = \begin{bmatrix} u \\ 1 \end{bmatrix}_q := \frac{q^u-1}{q-1}$ and we also speak of u-spaces. Using the geometric language, we call 1-, 2-, 3-, 4-, and $n-1$-spaces points, lines, planes, solids, and hyperplanes, respectively. For each $1 \le i \le r$ the set $U_i\backslash H_\infty$ is an affine space containing $q^{\dim(U_i)-1}$-points.

In the following we will mostly speak of an affine vector space partition, abbreviated as avsp, and will consider its embedding in $\operatorname{PG}(n-1,q)$. The *type* of an avsp $\mathcal{U} = \{U_1,\dots,U_r\}$ is given by $(n-1)^{m_{n-1}}\dots 2^{m_2}1^{m_1}$, where $m_i = \#\left\{U_j \,:\, 1 \le j \le r, \dim(U_j) = i\right\}$. Counting points outside of H_∞ gives

$$\sum_{i=1}^{n-1} m_i \cdot q^{i-1} = q^{n-1}. \qquad (1)$$

The analog of Equation (1) for vector space partitions of $\operatorname{PG}(n-1,q)$ is called the *packing condition.* While the packing condition for vector space partitions of $\operatorname{PG}(n-1,q)$ is just a necessary but not a sufficient condition for the existence with a given type, for avsps Equation (1) is both necessary and sufficient.

Lemma 3. *For each type* $(n-1)^{m_{n-1}}\dots 2^{m_2}1^{m_1}$ *that satisfies the packing condition (1) there exists an avsp* $\mathcal{U}$ *of* $\operatorname{PG}(n-1,q)$ *attaining that type.*

Proof. Consider a subspace K of H_∞ with $\dim(K) = n-2$. By $H_1, \ldots, H_q$ we denote the q hyperplanes containing K that are not equal to H_∞. Clearly, we have $0 \le m_{n-1} \le q$ and we can choose $H_1, \ldots, H_{m_{n-1}}$ as the first elements of $\mathcal{U}$. The remaining elements are constructed recursively. For each index $m_{n-1}+1 \le j \le q$ we consider an avsp of type $(n-2)^{m_{n-2}^{(j)}} \ldots 2^{m_2^{(j)}} 1^{m_1^{(j)}}$ where the $m_i^{(j)} \in \mathbb{N}_0$ are chosen such that the packing condition is satisfied for H_j and

$$\sum_{j=m_{n-1}+1}^{q} m_i^{(j)} = m_i \qquad (2)$$

for all $1 \le i \le n-2$. Such a decomposition can be easily constructed, see e.g. Algorithm 1 in Section B. □

Definition 4. *We call an avsp* $\mathcal{U} = \{U_1, \ldots, U_r\}$ reducible *if there exists a subspace* U *and a subset* $S \subsetneq \{1, \ldots, r\}$ *such that* $\dim(U) < n$, $\#S > 1$ *and* $\{U_i : i \in S\}$ *is an avsp of* S. *Otherwise* $\mathcal{U}$ *is called* irreducible.

Lemma 5. *The smallest size of an irreducible avsp* $\mathcal{U}$ *of* $\mathrm{PG}(n-1,q)$ *is given by* $\#\mathcal{U} = q$.

Proof. Let $\mathcal{U}$ be an avsp of $\mathrm{PG}(n-1,q)$. Since there are q^{n-1} points to cover and each subspace covers at most q^{n-2} points, we have $\#\mathcal{U} \ge q$. Now consider a hyperplane K of H_∞. By $H_1, \ldots, H_q$ we denote the q hyperplanes containing K and not being equal to H_∞. With this, $\{H_1, \ldots, H_q\}$ is an irreducible avsp of $\mathrm{PG}(n-1,q)$. □

For a vector space partition $\mathcal{P}$ of $\mathrm{PG}(n-1,q)$ we have $\dim(A) + \dim(B) \le n$ for each pair $\{A, B\}$ of different elements of $\mathcal{P}$, which is also called *dimension condition*. Using this it can be easily shown that $\#\mathcal{P} \ge \frac{q^n-1}{q^{n/2}-1} = q^{n/2}+1$ if n is even and $\#\mathcal{P} \ge q^{(n+1)/2}+1$ if n is odd. Both bounds can be attained by spreads, i.e., vector space partitions of type $(n/2)^{q^{n/2}+1}$, and lifted MRD codes of maximum possible rank distance, i.e., vector space partitions of type $((n+1)/2)^1((n-1)/2)^{q^{(n+1)/2}}$, respectively. In [NS11] the authors determine the minimum size $\sigma_q(n,t)$ of a vector space partition of $\mathrm{PG}(n,q)$ whose largest subspace has dimension t.

Lemma 6. *Let* $\mathcal{U}$ *be an irreducible avsp of* $\mathrm{PG}(n-1,q)$ *and* $U_1, \ldots, U_q \in \mathcal{U}$ *be* q *different elements with* $\dim(U_1) = \cdots = \dim(U_q)$ *and* $\dim(\langle U_1, \ldots, U_q \rangle) = \dim(U_1) + 1$. *Then we have* $\dim(U_1) = \cdots = \dim(U_q) = n-1$.

Proof. Let $U := \langle U_1, \ldots, U_q \rangle$ and $u := \dim(U_1)$. Since $U \backslash H_\infty$ contains q^u points and $U_i \backslash H_\infty$ contains q^{u-1} points for each $1 \le i \le q$, the set $\mathcal{U} \backslash \{U_1, \ldots, U_q\} \cup \{U\}$ is an avsp unless $\dim(U) = u+1 = n$. □

Corollary 7. *Let* $\mathcal{U}$ *be an irreducible avsp of* $\mathrm{PG}(n-1,2)$ *and* $U_1, U_2 \in \mathcal{U}$ *be two different elements with* $\dim(U_1) = \dim(U_2) = \dim(U_1 \cap U_2) + 1$. *Then, we have* $\dim(U_1) = \dim(U_2) = n-1$.

As an analog of the dimension condition for vector space partitions in $\mathrm{PG}(n-1,q)$ we have:

Lemma 8. *Let* $\mathcal{U}$ *be an avsp in* $\mathcal{U}$. *For each* $U, U' \in \mathcal{U}$ *we have*

$$\dim(U \cap U') = \dim(U \cap U' \cap H_\infty) \ge \dim(U) + \dim(U') - n. \qquad (3)$$

Proof. Since $U \backslash H_\infty$, $U' \backslash H_\infty$ are disjoint and $U, U' \not\le H_\infty$ we have $\dim(U \cap U') = \dim(U \cap U' \cap H_\infty)$. The inequality follows from $\dim(U_1 \cup U_2) + \dim(\langle U_1, U_2 \rangle) = \dim(U_1) + \dim(U_2)$ and $\dim(\langle U_1, U_2 \rangle) \le n$. □

Due to the following general construction for (irreducible) avsps we introduce a further condition.

Lemma 9. *Let* $\mathcal{U} = \{U_1, \ldots, U_r\}$ *be an avsp of* $\mathrm{PG}(n-1,q) =: V$ *and* P *be a point outside of* V *(embedded in* $\mathrm{PG}(n'-1,q)$ *for some larger value of* n'*). Then,* $\mathcal{U}' := \{\langle U_1, P \rangle, \ldots, \langle U_r, P \rangle\}$ *is an avsp of* $\langle V, P \rangle \cong \mathrm{PG}((n+1)-1,q)$. *Reducability inherits, i.e.* $\mathcal{U}'$ *is irreducible iff* $\mathcal{U}$ *is irreducible.*

Definition 10. *Let* $\mathcal{U} = \{U_1, \ldots, U_r\}$ *be an avsp of* $\mathrm{PG}(n-1,q)$. *We call* $\mathcal{U}$ tight *iff the intersection of all* U_i *does not contain a point, i.e. the intersection is trivial.*

We remark that an avsp $\mathcal{A}$ of $\mathrm{AG}(n, 2)$ is tight iff for any $x \in \mathbb{F}_2^n$, there exists an $A \in \mathcal{A}$ such that A is not invariant under addition of x, that is $A + x \neq A$.

Lemma 11. *For each integer $n \geq 2$ there exists a tight avsp of $\mathrm{PG}(n-1, q)$ with size $(q-1) \cdot (n-1) + 1$.*

Proof. Apply the following recursive construction. Start with an $(n-2)$-dimensional subspace K of H_∞ and consider the q hyperplanes $H_1, \ldots, H_q$ containing K but not being equal to H_∞. Choose $q-1$ out of these and continue the iteration with the remaining hyperplane until it becomes 2-dimensional, i.e. a line. In the final step replace the affine line by q points, so that the resulting avsp is trivially tight. □

A classical result in computer science, attributed to Tarsi, states that a minimally unsatisfiable CNF with m clauses mentions at most $m-1$ variables, see e.g. [DDKB98, Theorem 8]. The proof can be slightly modified to show that for $n \geq 2$ each tight avsp of $\mathrm{PG}(n-1, 2)$ has size at least n. We will prove the conjecture that Lemma 11 is tight in a subsequent paper. The determination of the minimum size of an irreducible tight avsp is quite a challenge and we will present our preliminary results in Sections 3 and 4.

Note that tightness and irreducibility can be checked efficiently. In particular, for irreducibility it suffices to calculate the affine closure for all pairs of subspaces in the avsp. We will show efficiency formally and provide detailed algorithms in future work on hitting formulas.

Lemma 12. *Let U, K, and H_∞ be subspaces in $\mathrm{PG}(n-1, q)$ with $K \leq H_\infty$, $\dim(K) = n-2$, $\dim(H_\infty) = n-1$, and $\dim(U \cap H_\infty) = \dim(U) - 1$, i.e. $U \not\leq H_\infty$. By $H_1, \ldots, H_q$ we denote the q hyperplanes containing K but not being equal to H_∞. Then the following statements are equivalent:*

(1) $U \cap H_\infty \leq K$;
(2) there exists an index $1 \leq i \leq q$ with $U \leq H_i$;
(3) there exists an index $1 \leq i \leq q$ with $U \leq H_i$ and $U \cap H_j = U \cap H_\infty = U \cap K$ for all $1 \leq j \leq q$ with $j \neq i$;
(4) $\dim(U \cap K) = \dim(U) - 1$.

Lemma 13. *Let U, K, and H_∞ be subspaces in $\mathrm{PG}(n-1, q)$ with $K \leq H_\infty$, $\dim(K) = n-2$, $\dim(H_\infty) = n-1$, and $\dim(U \cap H_\infty) = \dim(U) - 1$, i.e. $U \not\leq H_\infty$. By $H_1, \ldots, H_q$ we denote the q hyperplanes containing K but not being equal to H_∞. Then the following statements are equivalent:*

(1) $U \cap H_\infty \not\leq K$;
(2) $\dim(U \cap H_i) = \dim(U)$ *for all* $1 \leq i \leq q$;
(3) there are q $(\dim(U)-1)$-spaces in U containing $U \cap K$ and not being contained in H_∞;
(4) $\dim(U \cap K) = \dim(U) - 2$.

Assume that $\mathcal{P}$ is a vector space partition of $\mathrm{PG}(n-1, q)$ with type $k_1^{m_1} \ldots k_l^{m_l}$, where $k_1 > \cdots > k_l$ and $k_i > 0$ for all $1 \leq i \leq l$. The so-called *tail* $\mathcal{T}$ of $\mathcal{P}$ is the set of all k_l-spaces in $\mathcal{P}$, i.e., the set of all elements with the smallest occurring dimension. In [Hed09] several conditions on $\#\mathcal{T}$ have been obtained. In our situation we can also consider the *tail* $\mathcal{T} := \{U \in \mathcal{U} : \dim(U) = k_l\}$ of an avsp of $\mathrm{PG}(n-1, q)$ with type $k_1^{m_1} \ldots k_l^{m_l}$, where $k_1 > \cdots > k_l$ and $k_i > 0$ for all $1 \leq i \leq l$. The packing condition (1) directly implies that $q^{k_{l-1}-k_l}$ divides $\#\mathcal{T} = m_l$ if $l \geq 2$ and that q divides $\#\mathcal{T} = m_l$ if $l = 1$. In [Kur18] the results on the tail of a vector space partition of $\mathrm{PG}(n-1, q)$ were refined using the notion of Δ-divisible sets of k-spaces.

Definition 14. *A (multi-)set $\mathcal{S}$ of k-spaces in $\mathrm{PG}(n-1, q)$ is called Δ-divisible iff $\#\mathcal{S} \equiv \#(H \cap \mathcal{S}) \pmod{\Delta}$ for every hyperplane H, where $H \cap \mathcal{S}$ denotes the (multi-)set of elements of $\mathcal{S}$ that are contained in H.*

Lemma 15. *Let $\mathcal{U}$ be an avsp of $\mathrm{PG}(n-1, q)$ of type $k_1^{m_1} \ldots k_l^{m_l}$, where $k_1 > \cdots > k_l > 1$ and $k_i > 0$ for all $1 \leq i \leq l$. Let $\mathcal{T} := \{U \in \mathcal{U} : \dim(U) = k_l\}$ be the tail of $\mathcal{U}$ and $\mathcal{T}' := \{T \cap H_\infty : T \in \mathcal{T}\}$. If $l \geq 2$, then $\mathcal{T}'$ is $q^{k_{l-1}-k_l}$-divisible and $\#\mathcal{T} = \#\mathcal{T}' \equiv 0 \pmod{q^{k_{l-1}-k_l}}$. If $l = 1$, then $\mathcal{T}'$ is q-divisible and $\#\mathcal{T} = \#\mathcal{T}' \equiv 0 \pmod{q}$.*

Proof. Clearly we have $\#\mathcal{T} = \#\mathcal{T}'$. From the packing condition (1) we directly conclude $\#\mathcal{T} \equiv 0 \pmod{q^{k_{l-1}-k_l}}$ if $l \ge 2$ and $\#\mathcal{T} \equiv 0 \pmod{q}$ if $l = 1$. Let K be an arbitrary hyperplane of H_∞ and $H_1, \dots, H_q$ be the q hyperplanes of $\operatorname{PG}(n-1,q)$ not being equal to H_∞. Call the points outside of H_∞ that are contained in some element of $\mathcal{U}$ with dimension strictly larger than k_l covered and all others outside of H_∞ uncovered. Since each k-space covers either q^{k-1} or q^{k-2} points of $H_i \backslash H_\infty$, the number of uncovered points in $H_i \backslash H_\infty$ is divisible by $q^{k_{l-1}-2}$ if $l \ge 2$ and by q^{k_l-1} if $l = 1$, where $1 \le i \le q$ is arbitrary. Let a be the number of k_l-spaces in $\mathcal{U}$ that are completely contained in H_i, so that the number of uncovered points in H_i equals

$$x := a \cdot q^{k_l-1} + (\#\mathcal{T} - a) \cdot q^{k_l-2}.$$

If $l \ge 2$ we have $x \equiv 0 \pmod{q^{k_{l-1}-2}}$ and $\#\mathcal{T} \equiv 0 \pmod{q^{k_{l-1}-k_l}}$, so that $(q-1)a \equiv 0 \pmod{q^{k_{l-1}-k_l}}$ and $a \equiv 0 \pmod{q^{k_{l-1}-k_l}}$. If $l = 1$ we have $x \equiv 0 \pmod{q^{k_l-1}}$ and $\#\mathcal{T} \equiv 0 \pmod{q}$, so that $(q-1)a \equiv 0 \pmod{q}$ and $a \equiv 0 \pmod{q}$. □

Δ-divisible (multi-)sets $\mathcal{S}$ of k-spaces in $\operatorname{PG}(n-1,q)$ have been studied in [Kur18]. If we replace each k-space by its $\frac{q^k-1}{q-1}$ points we obtain a Δq^{k-1}-divisible multiset of $\#\mathcal{S} \cdot \frac{q^k-1}{q-1}$ points in $\operatorname{PG}(n-1,q)$. The possible cardinalities, given the divisibility constant and the field size, have been completely characterized in [KK20, Theorem 1]. Here we will use only a few results on the possible structure of the tail (or more precisely of $\mathcal{T}'$) which allow more direct proofs.

Lemma 16. *Let $\mathcal{U}$ be an avsp of $\operatorname{PG}(n-1,q)$ with tail $\mathcal{T}$. If $\#\mathcal{T} = q$, then either $\mathcal{U}$ is reducible or we have $\mathcal{U} = \mathcal{T}$ and $n = 2$.*

Proof. Denote the dimension of the elements of $\mathcal{T}$ by k. Lemma 15 yields that $\mathcal{T}' := \{T \backslash H_\infty : T \in \mathcal{T}\}$ is a q-divisible multiset of $(k-1)$-spaces. So, each hyperplane of H_∞ contains either all q or zero elements from $\mathcal{T}'$, so that $\mathcal{T}'$ is a q-fold $(k-1)$-space. With this, the stated results follows from Lemma 6. □

Corollary 17. *Let $\mathcal{U}$ be an irreducible avsp of $\operatorname{PG}(n-1,q)$ of type $k_1^{m_1} \dots k_l^{m_l}$, where $k_1 > \dots > k_l$ and $k_i > 0$ for all $1 \le i \le l$. If $m_l = q$, then we have $l = 1$ and $k_1 = n-1$.*

2.1 The structure of the tail for small parameters

If $\#\mathcal{T}$ is small, then we can also characterize the tail. To this end, let $\mathcal{S}$ denote a set of k-spaces in $\operatorname{PG}(n-1,q)$. The corresponding *spectrum* $(a_i)_{i \in \mathbb{N}_0}$ is given by the numbers a_i of hyperplanes that contain exactly i elements from $\mathcal{S}$, so that

$$\sum_{i=0}^{\#\mathcal{S}} a_i = \frac{q^n-1}{q-1}. \tag{4}$$

The condition that $\mathcal{S}$ is spanning, i.e. $\langle S : S \in \mathcal{S} \rangle = \operatorname{PG}(n-1,q)$, is equivalent to $a_{\#\mathcal{S}} = 0$. Double-counting the k-spaces gives

$$\sum_{i=0}^{\#\mathcal{S}} i a_i = \#\mathcal{S} \cdot \frac{q^{n-k}-1}{q-1}. \tag{5}$$

Lemma 18. *Let $\mathcal{S}$ be a 2-divisible set of four k-spaces in $\operatorname{PG}(n-1,2)$. Then there exists a $(k-1)$-space B, a plane E, and a line $L \le E$ with $\dim(\langle E, B \rangle) = k+2$, such that $\mathcal{S} = \{\langle P, B \rangle : P \in E \backslash L\}$.*

Proof. Assume that P is a point that is contained in at least one but not all elements from $\mathcal{S}$. Let x denote the number of elements of $\mathcal{S}$ that contain P. Since all hyperplanes contain an even number of elements from $\mathcal{S}$ we have $x \ne 3$. Assume $x = 2$ for a moment and let $S, S' \in \mathcal{S}$ be the two elements not containing P. There are 2^{n-k-1} hyperplanes that contain S but do not contain P, so that all of those hyperplanes contain S and S'. The intersection of these hyperplanes has dimension at most k and contains S as well as S', so that $S = S'$, which is a contradiction. Thus, each point P in $\operatorname{PG}(n-1,2)$ is contained in 0, 1 or 4 elements of $\mathcal{S}$.

By $(a_i)_{i\in\mathbb{N}_0}$ we denote the spectrum of $\mathcal{S}$. W.l.o.g. we assume that $\mathcal{S}$ is spanning, i.e., we have $a_4 = 0$. From the equations (4) and (5) we conclude

$$a_0 = 2^n - 2^{n-k+1} + 1 \quad \text{and} \quad a_2 = 2^{n-k+1} - 2.$$

If there is no point P that is contained in all four elements of $\mathcal{S}$, then the elements of $\mathcal{S}$ are pairwise disjoint and double-counting pairs yields

$$\binom{2}{2} a_2 = \binom{4}{2} \cdot \left(2^{n-2k} - 1\right), \tag{6}$$

so that

$$2^{n-k+1} - 2 = 6 \cdot \left(2^{n-2k} - 1\right) \quad \Leftrightarrow \quad 2^{n-k} - 3 \cdot 2^{n-2k} + 2 = 0,$$

which has the unique solution $n = 3$, $k = 1$.

So, by recursively quotienting out points P that are contained in all elements of $\mathcal{S}$ we conclude the existence of a $(k-1)$-space B that is contained in all four elements of $\mathcal{S}$. Quotienting out B yields a spanning 2-divisible set of points in $\mathrm{PG}(2,2)$ with $a_0 = 1$ and $a_2 = 6$. Choosing E as the ambient space and L as the empty hyperplane yields the stated characterization since in $\mathrm{PG}(2,2)$ there are exactly four points outside a hyperplane. □

If $k = 1$, i.e., the k-spaces are points, the equations (4)-(6) are also known as "standard equations" or the first three MacWilliams equations for the corresponding linear code.

We remark that Lemma 18 is based on the fact that each 2-divisible set of 4 points is an affine plane. For $q > 2$ there there further possibilities for q-divisible sets of q^2 points over $\mathbb{F}_q$, see [DBDMS19, KM21] on the so-called cylinder conjecture.

3 Classification of avsps in $\mathrm{PG}(n-1, q)$ for small parameters

By definition, there is no avsp in $\mathrm{PG}(1-1, q)$. In $\mathrm{PG}(2-1, q)$ there is a unique avsp. It has type 1^q and is irreducible and tight.

Lemma 19. *Let $\mathcal{U}$ be an avsp of $\mathrm{PG}(n-1,q)$, where $n \ge 3$. If there exist pairwise different hyperplanes $U_1, \dots, U_l \in \mathcal{U}$, then there exists an $(n-2)$-space $K \le H_\infty$ such that $K \le U_i$ for all $1 \le i \le l$.*

Proof. The statement is trivial for $l \le 1$, so that we assume $l \ge 2$. Due to the dimensions we have $\dim(U_i \cap U_j) = n-2$ for all $1 \le i < j \le l$. Since the sets of points $U_i \backslash H_\infty$ and $U_j \backslash H_\infty$ are disjoint we have $U_i \cap U_j \le H_\infty$ and $U_i \cap U_j = U_i \cap H_\infty = U_j \cap H_\infty$. So, we set $K = U_1 \cap H_\infty$. □

Proposition 20. *Let $\mathcal{U}$ be an irreducible avsp of $\mathrm{PG}(n-1,q)$, where $n \ge 3$. If $\mathcal{U}$ is of type $(n-1)^{m_{n-1}} \dots 2^{m_2} 1^{m_1}$, then we have $m_{n-1} \le q-2$ or $m_{n-1} = q$. In the latter case $\mathcal{U}$ is not tight.*

Proof. We assume $m_{n-1} = q-1 \ge 1$ and let $K \le H_\infty$ as in Lemma 19. With this, let $H \ne H_\infty$ be the unique hyperplane with $K \le H$ that is not contained as an element in $\mathcal{U}$ and $\mathcal{U}'$ arise from $\mathcal{U}$ by removing the $q-1$ $(n-1)$-dimensional elements. Thus, $\mathcal{U}'$ is an avsp of H, i.e., $\mathcal{U}$ is reducible.

If $m_{n-1} = q$, then the $(n-2)$-space $K \le H_\infty$ (as in Lemma 19) is contained in all elements of $\mathcal{U}$, i.e., $\mathcal{U}$ is not tight. □

Corollary 21. *Let $\mathcal{U}$ be an irreducible tight avsp of $\mathrm{PG}(n-1,2)$ of type $(n-1)^{m_{n-1}} \dots 1^{m_1}$, where $n \ge 3$. Then we have $m_{n-1} = 0$.*

Let $\mathcal{U} = \{U_1, \dots, U_r\}$ be an avsp of $\mathrm{PG}(n-1,q)$, $I \subseteq \{1, \dots, r\}$, and V be a proper subspace with $V \not\le H_\infty$. If $\#I \ge 2$ and $\{U_i : i \in I\}$ is an avsp of V, then we say that the spaces U_i with $i \in I$ can be *joined* to V. Note that this is exactly the situation when $\mathcal{U}$ is reducible. In $\mathrm{PG}(n-1,2)$ any two points outside of H_∞ can be joined to a line, so that:

Lemma 22. *Let $\mathcal{U}$ be an irreducible tight avsp of* $\operatorname{PG}(n-1,2)$ *of type* $(n-1)^{m_{n-1}}\dots 1^{m_1}$*, where* $n\ge 3$*. Then,we have* $m_1=0$*.*

Theorem 23. *Let $\mathcal{U}$ be an avsp of* $\operatorname{PG}(3-1,q)$ *with type* $2^{m_2}1^{m_1}$*. Then, we have* $0\le m_2\le q$*,* $m_1=q\cdot(q-m_2)$*, all lines in $\mathcal{U}$ contain a common point* $P\le H_\infty$*, and the* 1*-dimensional elements can be grouped into pairwise disjoint sets of size q that can be joint to a line each.*

Proof. The existence of P follows from Lemma 19 and the parameterization of m_2,m_1 follows from the packing condition (1). If $m_2=0$ then choose an arbitrary point $P\le H_\infty$. By $L_1,\dots,L_q$ we denote the q lines containing P that are not equal to H_∞. For each line L_i that is not an element of $\mathcal{U}$ there exist q points in $\mathcal{U}$ that can be joined to L_i. (Note that $L_i\cap L_j=P$ for all $1\le i<j\le q$.) □

We remark that all possibilities for $0\le m_2\le q$ can indeed by attained. In general there exist several non-isomorphic examples.

Corollary 24.

(1) Let $\mathcal{U}$ be an irreducible avsp of $\operatorname{PG}(3-1,q)$*. Then $\mathcal{U}$ is of type* 2^q *and non-tight.*

(2) Let $\mathcal{U}$ be an avsp of $\operatorname{PG}(3-1,q)$ *with type* $2^{m_2}1^{m_1}$*. Then, $\mathcal{U}$ is tight iff* $m_1\ge 1$*. In that case $\mathcal{U}$ is reducible.*

(3) No irreducible tight avsp of $\operatorname{PG}(3-1,q)$ *exists.*

Let $\mathcal{U}$ be an avsp of $\operatorname{PG}(n-1,q)$, where $n\ge 3$, and $K\le H_\infty$ be an arbitrary $(n-2)$-space. We say that $\mathcal{U}^{(1)},\dots,\mathcal{U}^{(q)}$ is a *K-decomposition* of $\mathcal{U}$ if the q hyperplanes containing K and not being equal to H_∞ can be labeled as $H_1,\dots,H_q$ such that

$$\mathcal{U}^{(i)}=\{U\cap H_i\,:\,U\in\mathcal{U},U\cap H_i\not\le H_\infty\}\tag{7}$$

for all $1\le i\le q$. Note that $\mathcal{U}^{(i)}$ is an avsp of H_i for each $1\le i\le q$ (including the case $\mathcal{U}^{(i)}=\{H_i\}$). Moreover, any labeling of the q hyperplanes H_i induces a K-decomposition. Observe that for a fixed $(n-2)$-space $K\le H_\infty$ each pair of K-decompositions arises just by relabeling, so that we also speak of *the* K-decomposition of $\mathcal{U}$ since the actual labeling will not matter in our context.

Proposition 25. *Let $\mathcal{U}$ be an avsp of* $\operatorname{PG}(n-1,q)$*, where* $n\ge 3$*, with type* $(n-1)^{m_{n-1}}\dots 2^{m_2}1^{m_1}$*. If* $1\le m_{n-1}\le q$*, then there exists an* $(n-2)$*-space* $K\le H_\infty$ *such that the K-decomposition* $\mathcal{U}^{(1)},\dots,\mathcal{U}^{(q)}$ *partitions $\mathcal{U}$, i.e.,*

$$\bigcup_{1\le i\le q}\mathcal{U}^{(i)}=\mathcal{U}.$$

Moreover, if $m_{n-1}\le q-1$*, then $\mathcal{U}$ is reducible. (More precisely, for each index* $1\le i\le q$ *with* $\#\mathcal{U}^{(i)}>1$ *the elements in* $\mathcal{U}^{(i)}$ *can be joined to* H_i*.)*

Proof. Choose some arbitrary $U\in\mathcal{U}$ with $\dim(U)=n-1$, set $K:=U\cap H_\infty$, and let $\mathcal{U}^{(1)},\dots,\mathcal{U}^{(q)}$ be the K-decomposition of $\mathcal{U}$ and $H_1,\dots,H_q$ be the corresponding hyperplanes. Due to Lemma 19 each $U\in\mathcal{U}$ with $\dim(U)=n-1$ results in the same $(n-2)$-space K and the same K-decomposition $\mathcal{U}^{(1)},\dots,\mathcal{U}^{(q)}$ (up to relabeling). Especially we have that for each $U'\in\mathcal{U}$ with $\dim(U')=n-1$ there exists an index $1\le i\le q$ with $\mathcal{U}^{(i)}=\{U'\}$. W.l.o.g. we assume $\#\mathcal{U}^{(1)}=1$.

Now consider an element $U\in\mathcal{U}$ with $\dim(U)<n-1$. From Lemma 13 we conclude $U\cap H_\infty\le K$ since otherwise $\#\mathcal{U}^{(1)}>1$ (more precisely, U would split into q $(\dim(U)-1)$-spaces where one of these would be contained in $\mathcal{U}^{(1)}$ that also contains an entire hyperplane, which contradicts the packing condition (1)), which would contradict our assumption. Thus, for each $U\in\mathcal{U}$ there exists exactly one index $1\le i\le q$ with $U\in\mathcal{U}^{(i)}$ and for each index $1\le j\le q$ either $U\in H_j$ or $U\cap H_j\le K\le H_\infty$. □

Corollary 26. *Let $\mathcal{U}$ be an avsp of* $\operatorname{PG}(n-1,q)$ *with type* $(n-1)^{m_{n-1}}\dots 2^{m_2}1^{m_1}$*, where* $n\ge 3$*. If $\mathcal{U}$ is irreducible, then we have* $m_{n-1}=0$*.*

Affine vector space partitions of $\mathrm{PG}(4-1,q)$ that contain at least one hyperplane as an element can be characterized easily.

Proposition 27. *Let $\mathcal{U}$ be an avsp of $\mathrm{PG}(4-1,q)$ of type $3^{m_3}2^{m_2}1^{m_1}$ with $m_3 \ge 1$. Then, we have $1 \le m_3 \le q$, $0 \le m_2 q \cdot (q-m_3)$, and $m_1 = q^3 - q^2 m_3 - q m_2$. Moreover, there exists an $(n-2)$-space $K \le H_\infty$ such that the K-decomposition $\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(q)}$ partitions $\mathcal{U}$, so that $\mathcal{U}$ especially is reducible if $m_{n-1} \le q-1$.*

Proof. The equation $m_1 = q^3 - q^2 m_3 - q m_2$ directly follows from the packing condition (1) and the ranges $0 \le m_2 \le q \cdot (q-m_3)$, $0 \le m_3 \le q$ follow from the non-negativity of m_1, m_2, m_3. Note that we have $m_3 \ge 1$ by assumption. The remaining part follows from Proposition 25. □

4 Classification of tight irreducible avsps in $\mathrm{PG}(\mathbf{n-1,2})$ for small dimensions n

The cases $n \le 3$ have already been treated in Section 3, so that we assume $n \ge 4$ in the following. Our aim is to classify all possible types $(n-1)^{m_{n-1}} \ldots 1^{m_1}$ such that a tight irreducible avsp $\mathcal{U}$ exists in $\mathrm{PG}(n-1,2)$. We have $m_{n-1} = 0$ and $m_1 = 0$ due to Corollary 21 and Lemma 22. From Lemma 17 we conclude $m_l \ne 2$ for the smallest index $1 \le l \le n-1$ with $m_l > 0$. The possible vectors $(m_{n-2}, \ldots, m_2) \in \mathbb{N}_0^{n-3}$ are quite restricted by the packing condition (1). For $n = 4$ the only remaining possibility is type 2^4. From Lemma 6 we conclude that the four lines are pairwise disjoint, i.e., they form a partial line spread of cardinality 4. It is well known that each partial line spread of cardinality q^2 in $\mathrm{PG}(3,q)$ can be extended to a line spread, which has size q^2+1.[1] For $q = 2$ there is only the Desarguesian line spread and since it has a transitive automorphism group, there is only one equivalence class. The numbers of line spreads in $\mathrm{PG}(3,q)$ are 1, 2, 3, 21, 1347 for $q = 2, 3, 4, 5, 7$.

In the three subsequent subsections we will consider tight irreducible avsps in $\mathrm{PG}(n-1,2)$ for $n \in \{5,6,7\}$. The possible types are completely determined in all cases, where realizations are computed using an integer linear programming (ILP) formulation, see Section A in the appendix for the details. If the sizes of the avsps are not too large we were able to also compute all equivalence classes of avsps using a slight modification of an algorithm from [Lin04], see also [KÖ06, Algorithm 4.5]. A `GAP` implementation , based on the `GAP` package "FinInG" [BBC+18] for computations in finite incidence geometry, can be obtained from the authors upon request. In the theoretical parts we will also use classification for 2-divisible sets points that can e.g. be found in [HHK+17] or [Kur21]. For the convenience of the reader we will also give a few selected proofs in Section B in the appendix.

4.1 Tight irreducible avsps in $\mathrm{PG}(\mathbf{4,2})$

We may use Lemma 15 and Lemma 18 to conclude that each avsp of $\mathrm{PG}(n-1,2)$ of type $(n-2)^4$ is non-tight if $n \ge 5$. However, we can further tighten the statement to:

Proposition 28. *Let $\mathcal{U} = \{U_1, \ldots, U_r\}$ be an avsp of $\mathrm{PG}(n-1,2)$, where $n \ge 4$, $r \ge 4$, and $\dim(U_i) = n-2$ for all $1 \le i \le 3$. Then, the elements $\{U_4, \ldots, U_r\}$ can be joined to an $(n-2)$-space B (including the case $r = 4$ and $U_4 = B$) and there exists an $(n-4)$-space C that is contained in all elements of $\{U_1, U_2, U_3, B\}$.*

Proof. First we assume that two elements of $\{U_1, U_2, U_3\}$ can be joined to an $(n-1)$-space H. Without loss of generality, we assume that U_1 and U_2 can be joined to H. Let $K := H \cap H_\infty$, so that $\dim(K) = n-2$. By H' we denote the unique hyperplane containing K that is not equal to H or H_∞. Observe that $\{U_3, \ldots, U_r\}$ is an avsp of H' and K is "the hyperplane at infinity" of H'. Next we set $K' := K \cap U_3$, so that $\dim(K') = n-3$. Let B denote the unique $(n-2)$-space in H' that contains K' and is not equal to U_3 or K. With this,

[1] One argumentation is based on the fact that each q^k-divisible (multi-) set of $\frac{q^{k+1}-1}{q-1}$ points forms a $(k+1)$-space for each positive integer k, see e.g. [HKK18].

$\{U_4, \ldots, U_r\}$ is an avsp of B (including the case $r = 4$, $U_4 = B$). Note that the $(n-3)$-space K' is contained in all elements of $\{H, U_3, B\}$. Since $\{U_1, U_2\}$ forms an avsp of H and $\dim(U_1) = \dim_(U_2) = n-2$, there exists an $(n-4)$-space C that is contained in all elements of $\{U_1, U_2, U_3, B\}$.

Otherwise, we assume that no two elements of $\{U_1, U_2, U_3\}$ can be joined to an $(n-1)$-space, so that $\dim(U_i \cap U_j) = n-4$ for all $1 \le i < j \le 3$. We set $E_i := U_i \cap H_\infty$, so that $\dim(E_i) = n-3$, for all $1 \le i \le 3$ and $\dim(E_i \cap E_j) = n-4$ for all $1 \le i < j \le 3$. Let $K := \langle E_1, E_2, E_3\rangle \le H_\infty$, so that $n-2 \le \dim(K) \le n-1$. If $\dim(K) = n-2$, then consider the K-decomposition $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}$ of $\mathcal{U}$ and let H_1, H_2 be the corresponding hyperplanes. Since $E_1, E_2, E_3 \le K$, we have that either $U_i \le H_1$ or $U_i \le H_2$ for all indices $1 \le i \le 3$. By the pigeonhole principle two of the three $(n-2)$-spaces in $\mathcal{U}$ have to be contained in the same hyperplane, so that they can be joined, which is a contradiction. Thus, we have $\dim(K) = n-1$, i.e., $K = H_\infty$. Since $\dim\left(\langle E_1, E_2\rangle\right) = n-2$, $\dim(E_3) = n-3$, and $\dim(K) = n-1$, we have

$$\dim(\langle E_1, E_2\rangle \cap E_3) = n-4.$$

Since $\dim(E_1 \cap E_3) = \dim(E_2 \cap E_3) = n-4$, we have $\dim(C) = n-4$ for $C := E_1 \cap E_2 \cap E_3$. Pick three linearly independent vectors v_1, v_2, v_3 such that $E_1 = \langle C, v_1\rangle$, $E_2 = \langle C, v_2\rangle$, $E_3 = \langle C, v_3\rangle$, and $H_\infty = \langle C, v_1, v_2, v_3\rangle$. Let P_1, P_2 be two different arbitrary points outside of H_∞ that or not covered by U_1, U_2, or U_3. For pairwise different $i, j, h \in \{1, 2, 3\}$ consider the $(n-2)$-space $K_{i,j,j} := \langle C, v_i, v_j + v_h\rangle$ and let $H_{i,j,j}$ be the hyperplane that contains $K_{i,j,h}$ and U_i. Since all points in $H_{i,j,h} \backslash H_\infty$ are covered by U_1, U_2, U_3 the points P_1, P_2 have to be contained in the other hyperplane containing $K_{i,j,h}$ not equal to $H_{i,j,h}$ and H_∞, so that $P_1 - P_2 \in \langle C, v_i, v_j + v_h\rangle$. Since

$$\langle C, v_1, v_2 + v_3\rangle \cap \langle C, v_2, v_1 + v_3\rangle \cap \langle C, v_3, v_1 + v_2\rangle = \langle C, v_1 + v_2 + v_3\rangle\,,$$

the 2^{n-3} points outside of H_∞ that are not covered by U_1, U_2, or U_3 have to form an affine subspace $B \ge C$. If $\#\mathcal{U} = r = 4$, then $B = U_4$. If $\#\mathcal{U} > 5$, then the elements in $\{U_4, \ldots, U_r\}$ form an avsp of B. □

Corollary 29. *Let $\mathcal{U}$ be an irreducible tight avsp of* $\mathrm{PG}(n-1, 2)$ *of type* $(n-2)^{m_{n-2}} \ldots 2^{m_2}$*, where* $n \ge 5$*. Then, we have* $m_{n-2} \le 2$*.*

Together with the conditions $m_{n-1} = m_1 = 0$ and the packing condition (1) we obtain:

Corollary 30. *Let $\mathcal{U}$ be an irreducible tight avsp of* $\mathrm{PG}(5-1, 2)$*. Then the type of $\mathcal{U}$ is given by* $3^2 2^4$*,* $3^1 2^6$*, or* 2^8*.*

All types can indeed be realized and corresponding numbers of equivalence classes are given by 3, 4, and 2, respectively. I.e., for $n = 4$ we have 9 non-isomorphic examples in total. Below are representatives:

E1, 2^8: $\langle 10000, 01000\rangle$, $\langle 10100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10010, 01100\rangle$, $\langle 11010, 00101\rangle$, $\langle 10001, 01010\rangle$, $\langle 10011, 00110\rangle$, $\langle 10111, 01110\rangle$.

E2, $2^6 3^1$: $\langle 10000, 01000\rangle$, $\langle 10100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10010, 01100\rangle$, $\langle 10001, 00100\rangle$, $\langle 11001, 00011\rangle$, $\langle 10011, 01000, 00100\rangle$.

E3, $2^6 3^1$: $\langle 10000, 01000\rangle$, $\langle 10100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10010, 01100\rangle$, $\langle 10001, 01010\rangle$, $\langle 10111, 01101\rangle$, $\langle 10011, 01010, 00110\rangle$.

E4, $2^6 3^1$: $\langle 10000, 01000\rangle$, $\langle 10100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10010, 01100\rangle$, $\langle 11001, 00011\rangle$, $\langle 10011, 00100\rangle$, $\langle 10001, 01010, 00100\rangle$.

E5, $2^6 3^1$: $\langle 10000, 01000\rangle$, $\langle 10100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10010, 01001\rangle$, $\langle 10001, 01111\rangle$, $\langle 10111, 01101\rangle$, $\langle 10011, 01010, 00110\rangle$.

E6, 2^8: $\langle 10000, 01000\rangle$, $\langle 10100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10010, 01101\rangle$, $\langle 10001, 01011\rangle$, $\langle 11001, 00111\rangle$, $\langle 10101, 01110\rangle$, $\langle 10011, 00100\rangle$.

E7, $2^4 3^2$: $\langle 10000, 01000\rangle$, $\langle 10100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10010, 01011\rangle$, $\langle 11010, 00100, 00001\rangle$, $\langle 10001, 00100, 00010\rangle$.

E8, $2^4 3^2$: $\langle 10000, 01000\rangle$, $\langle 10100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10010, 01011\rangle$, $\langle 10001, 01010, 00100\rangle$, $\langle 10011, 01001, 00100\rangle$.

E9, $2^4 3^2$: $\langle 10000, 01000\rangle$, $\langle 10100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10101, 01011\rangle$, $\langle 10010, 01000, 00001\rangle$, $\langle 10001, 01000, 00110\rangle$.

We remark that the hypothetical type $3^3 2^2$ is also excluded by Corollary 17.

Similarly as we have constructed $\mathcal{T}'$ from the tail $\mathcal{T}$ in Lemma 15, we can consider the set $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$ for an avsp $\mathcal{U}$ of $\mathrm{PG}(n-1, q)$. If $\mathcal{U}$ is an irreducible tight avsp of $\mathrm{PG}(n-1, q)$ of type $2^{m_2} 3^{m_3}$, where $m_2 = q^{n-1} - q m_3$, then $\mathcal{U}'$ is a configuration of m_2 points and m_3 lines in $H_\infty \cong \mathrm{PG}(n-2, q)$. The points are pairwise disjoint, so that Lemma 15 yields that they form a q-divisible set. Any two lines can meet in at most a point. If $n = 5$, then any two lines indeed intersect in a point. So, the maximum point multiplicity is at most $m_3 + 1$. In the following we will theoretically classify the possibilities for $\mathcal{U}'$ for tight irreducible avsps of $\operatorname{PG}(4,2)$ of type $2^{m_2}3^{m_3}$ up to symmetry. Corollary 29 gives $m_3 \in \{0,1,2\}$. First we will deduce two general necessary criteria for $\mathcal{U}'$.

Lemma 31. *Let $\mathcal{U}$ be an irreducible avsp of $\operatorname{PG}(n-1,q)$ not of type $(n-1)^q$ and $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$. Then $\mathcal{U}'$ is spanning, i.e., $\mathcal{U}'$ spans H_∞.*

Proof. Assume that K is a hyperplane of H_∞ that contains all elements of $\mathcal{U}'$. From Lemma 12 we can conclude that the K-decomposition $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(q)}$, with corresponding hyperplanes $H_1, \dots, H_q$, is a partition of $\mathcal{U}$, i.e., the elements of $\mathcal{U}^{(i)}$ can be joined to H_i for all $1 \le i \le q$. Since we have assumed that $\mathcal{U}$ is not of type $(n-1)^q$ we obtain a contradiction. □

Lemma 32. *Let $\mathcal{U}$ be an avsp of $\operatorname{PG}(n-1,q)$ of type $(n-1)^{m_{n-1}} \dots 2^{m_2}$ and $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$. For each hyperplane K of H_∞ let a_i^K denote the number of i-dimensional elements of $\mathcal{U}'$ that are contained in K and $b_i^K = m_{i+1} - a_i^K$ the number of i-dimensional elements of $\mathcal{U}'$ that are contained in K, where $1 \le i \le n-2$. Then there exist $c_{i,j}^K \in \mathbb{N}_0$ for all $1 \le j \le q$, $1 \le i \le n-2$ such that*

$$\sum_{j=1}^{q} c_{i,j}^K = a_i^K \quad \forall 1 \le i \le n-2, \tag{8}$$

$$\sum_{i=1}^{n-2} c_{i,j}^K \cdot q^i + b_i^K \cdot q^{i-1} = q^{n-2} \quad \forall 1 \le j \le q. \tag{9}$$

Proof. For an arbitrary but fix hyperplane K of H_∞ let $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(q)}$ be the K-decomposition of $\mathcal{U}$ with corresponding hyperplanes $H_1, \dots, H_q$. From Lemma 12 we conclude that for each element $U \in \mathcal{U}$ with $U \cap H_\infty \le K$ there exists an index $1 \le j \le q$ such that $U \le H_j$. The $c_{i,j}^K$ just count how many $(i+1)$-dimensional elements of that type are contained in H_j. Since the hyperplanes $H_1, \dots, H_q$ are pairwise disjoint, we obtain Equation (8). From Lemma 13 we conclude that for each element $U \in \mathcal{U}$ such that $U \cap H_\infty \not\le K$ we have $\#\left(U \cap H_j \backslash H_\infty\right) = q^{\dim(U)-2}$, so that the packing condition yields Equation (9). □

Let us consider the case $m_3 = 0$ first. Here the 2^3 points in $H_\infty \cong \operatorname{PG}(3,2)$ form a spanning 2-divisible set $\mathcal{P}$ of points. So, $\mathcal{P}$ is either an affine solid attained in Example E6 or given by the points of a plane and an intersecting line without the intersection point attained in Example E1, cf. Lemma 52. In the first case the geometrical object of the pairwise disjoint eight lines, that are disjoint from a special plane, in $\operatorname{PG}(4,2)$ is also known under the name of a lifted MRD code or a vector space partition of $\operatorname{PG}(4,2)$ of type $2^8 3^1$. The uniqueness up to symmetry is a well known result.

For $m_3 = 1$ the six points in $H_\infty \cong \operatorname{PG}(3,2)$ form a 2-divisible set $\mathcal{P}$ of points, so that $\mathcal{P}$ is given by two disjoint lines L_1, L_2, cf. Lemma 51. Let us denote the unique line in H_∞ by L. Up to symmetry, the lines L_1, L_2, and L can be arranged as follows:

- all three lines are pairwise disjoint, so that there are no multiple points, see Example E5;
- $L = L_1$, so that there are three double points forming a line, see Example E2;
- L intersects both L_1 and L_2 in a point, so that we have two double points, see Example E3;
- L intersects exactly one of the lines L_1 and L_2 in a point, so that we have a unique double points, see Example E4.

For $m_3 = 2$ let us denote the two lines by L_1, L_2, their intersection point by P, and their span by E, which is a plane. The four points in $\mathcal{P}$ form an affine plane, cf. Lemma 18. I.e., there exist a plane E' and a line $L' \le E'$ with $\mathcal{P} = E' \backslash L'$. From Lemma 31 we conclude $E \ne E'$, so that we set $L^* := E \cap E'$ with $\dim(L^*) = 2$. From Lemma 32 we conclude that each hyperplane H of H_∞ that contains either the line L_1 or the line L_2, but not both, has to intersect $\mathcal{P}$ in at least two points. (More technically, $a_2^K = 1$ implies $a_1^K \ge 2$.) With this we obtain the following list of possible arrangements of L_1, L_2, and $E' \backslash L'$:

- P is outside of E'. Since $L_i \not\leq \langle P, L'\rangle$ for $i = 1, 2$, we have that $|L_i \cap E'\backslash L'| = 1$ for $i = 1, 2$, i.e., we have three double points forming a basis, see Example E7;
- $P \in \mathcal{P} = E'\backslash L'$. Since $E \neq E'$, one of lines L_1, L_2 intersects E' exactly in the point P, so that the other line meets $\mathcal{P}$ in a second point, i.e., we have one triple and one double point, see Example E9;
- $P \in L'$. W.l.o.g. we assume $L_1 \cap E' = P$, so that $L_2 \leq E'$. Since the plane $\pi := \langle L_1, L'\rangle$ is disjoint to $\mathcal{P}$, we conclude $L_2 \in \pi$, so that $L_2 = L'$. So, P is the unique double point, see Example E8.

Thus we have completed the classification of possibilities for $\mathcal{U}'$ that may correspond to tight irreducible avsps $\mathcal{U}$ of $\mathrm{PG}(4, 2)$. We call the process of moving from $\mathcal{U}'$ to $\mathcal{U}$ the *extension problem*. An integer linear programming formulation is given in Section A in the appendix. Note that the extension problem comprises additional symmetry given by the pointwise stabilizer of H_∞ of order q^{n-1}.

Given a set $\mathcal{U}'$ satisfying all of the necessary conditions mentioned so far it is neither clear that an extension to a corresponding avsp $\mathcal{U}$ always exists nor that it is, in the case of existence, unique up to symmetry. Indeed, we will give counter examples later on. However, for the nine classified configurations $\mathcal{U}'$ in $\mathrm{PG}(3, 2)$ it turns out that there always is an up to symmetry unique extension.

4.2 Tight irreducible avsps in $\mathrm{PG}(5, 2)$

Lemma 33. *For $n \geq 6$ no tight irreducible avsp of type $(n-2)^2(n-3)^4$ in $\mathrm{PG}(n-1, 2)$ exists.*

Proof. Assume that such an avsp $\mathcal{U}$ exists and consider the intersections of the elements with the hyperplane H_∞ at infinity, i.e., $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$. By E_1, E_2 we denote the two $(n-3)$-spaces and by $L_1, \dots, L_4$ the four $(n-4)$-spaces. The intersection of E_1 and E_2 is an $(n-4)$-space L' and $\dim(E_i, L_j) \geq n-5$ for all $i = 1, 2$ and $j = 1, \dots, 4$. From Lemma 15 we conclude that $\mathcal{T}' = \{L_1, \dots, L_4\}$ is a 2-divisible set of four $(n-4)$-spaces, so that Lemma 18 implies the existence of a plane $E \leq H_\infty$, a line $L \leq E$, and an $(n-5)$-space $B \leq H_\infty$ with $B \cap E = \emptyset$ and

$$\{L_1, \dots, L_4\} = \{\langle Q, B\rangle : Q \in E\backslash L\}.$$

Since $\mathcal{U}$ is tight we have $B \cap L' = \emptyset$. However, $\dim(E_i, L_j) \geq n-5$ implies $\dim(L', L_j) \geq n-6$ for all $1 \leq j \leq 4$. So, we clearly have $n \leq 7$.

For $n = 7$ we conclude $E \leq E_1$, $E \leq E_2$, $\dim(B) = 2$, $\dim(E_1) = \dim(E_2) = 4$, $\dim(E) = 3$, and $\dim(L') = 3$, so that $L' = E$ and $\dim(E_i \cap B) \geq 1$ for $i = 1, 2$. Thus, we have $\langle \mathcal{U}'\rangle \leq \langle E, B\rangle$, i.e., $\mathcal{U}'$ is not spanning, which is a contradiction.

For $n = 6$ we have $\dim(B) = 1$, i.e., B is a point. Since $\mathcal{U}$ is tight we have $B \not\leq L'$. W.l.o.g. we assume $B \not\leq E_1$. Since E_1 intersects each of the lines L_j in at least a point, we have $E_1 = E$. Since $\mathcal{U}$ is irreducible E_2 is not contained in the solid $S := \langle E, B\rangle$. Since E_1 intersects each of the lines L_j in at least a point, we have that the line $L' \leq E$ intersects each of the lines L_j in at least a point. Since $B \not\leq L'$ this is impossible. □

Proposition 34. *Let $\mathcal{U}$ be a tight irreducible avsp of $\mathrm{PG}(5, 2)$, then $\mathcal{U}$ has one of the following types:*

- $4^2 3^i 2^{8-2i}$ *for $i \in \{0, 1, 2\}$;*
- $4^1 3^i 2^{12-2i}$ *for $i \in \{0, \dots, 6\}\backslash\{5\}$; and*
- $3^i 2^{16-2i}$ *for $i \in \{0, \dots, 8\}\backslash\{7\}$.*

All types are realizable.

Proof. Let the type of $\mathcal{U}$ be $5^{m_5} \dots 1^{m_1}$. From Corollary 21 and Lemma 22 we conclude $m_5 = 0$ and $m_1 = 0$, so that the packing condition (1) gives $4m_4 + 2m_3 + m_2 = 16$. Corollary 29 gives $m_4 \leq 2$ and Lemma 33 excludes $(m_4, m_3, m_2) = (2, 4, 0)$. Moreover, Corollary 17 implies $m_2 \neq 2$. All remaining possibilities $(m_4, m_3, m_2) \in \mathbb{N}_0^3$ are listed in the statement and for each type we found a realization using ILP computations. □

Corollary 35. *If $\mathcal{U}$ is a tight irreducible avsp of* $\mathrm{PG}(5,2)$ *of minimum possible size, then* $\#\mathcal{U} = 7$ *and* $\mathcal{U}$ *has type* $4^1 3^6$.

For small sizes we have enumerated the isomorphism types of tight irreducible avsps in $\mathrm{PG}(5,2)$, see Table 1. The last row concerns the parts $\mathcal{U}'$ at the hyperplane H_∞ at infinity w.r.t. the avsps $\mathcal{U}$ counted up to isomorphy in the second row. So, for e.g. types $4^1 3^4 2^4$ and $4^2 2^8$ there exist configurations $\mathcal{U}'$ that allow more than one extension up to symmetry.

type	$4^1 3^6$	$4^2 3^2 2^4$	3^8	$4^2 3^1 2^6$	$4^1 3^4 2^4$	$4^2 2^8$
#	6	38	32	55	827	83
in H_∞	6	38	32	55	811	50

Table 1: Number of isomorphism types of tight irreducible avsps in $\mathrm{PG}(5,2)$.

For the minimum possible size of a tight irreducible avsp in $\mathrm{PG}(5,2)$ we can write down all implications of the stated necessary conditions for the part $\mathcal{U}'$ at infinity. So, for type $4^1 3^6$ configuration $\mathcal{U}'$ consists of one plane E and six lines $\mathcal{L} = \{L_1, \dots, L_6\}$ satisfying the following conditions:

(1) the configuration is spanning, i.e., $\langle E, L_1, \dots, L_6 \rangle = \mathrm{PG}(4,2)$;
(2) the configuration is tight, i.e., there does not exist a point P that is contained in E and all lines in $\mathcal{L}$;
(3) the lines in $\mathcal{L}$ form a 2-divisible set of lines, i.e., each hyperplane contains an even number of lines;
(4) each line L_i intersects E in at least a point;
(5) hyperplanes that contain E also contain at least two lines.

Up to symmetry ten such configurations exist:

E1: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 01000\rangle$, $\langle 10000, 00100\rangle$, $\langle 01000, 00010\rangle$, $\langle 01000, 00110\rangle$, $\langle 10100, 00001\rangle$, $\langle 10100, 01101\rangle$
E2: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 01000\rangle$, $\langle 10000, 00010\rangle$, $\langle 10000, 00110\rangle$, $\langle 01100, 00010\rangle$, $\langle 01100, 00001\rangle$, $\langle 10011, 01100\rangle$
E3: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 01000\rangle$, $\langle 10000, 00010\rangle$, $\langle 10000, 00001\rangle$, $\langle 01000, 00011\rangle$, $\langle 10100, 01011\rangle$, $\langle 01011, 00111\rangle$
E4: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 01000\rangle$, $\langle 10000, 00010\rangle$, $\langle 01000, 00110\rangle$, $\langle 00100, 00010\rangle$, $\langle 11100, 00001\rangle$, $\langle 10111, 01011\rangle$
E5: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 00010\rangle$, $\langle 10000, 01010\rangle$, $\langle 00100, 00001\rangle$, $\langle 01100, 00011\rangle$, $\langle 11001, 00100\rangle$, $\langle 10111, 01100\rangle$
E6: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 00010\rangle$, $\langle 10000, 00001\rangle$, $\langle 10000, 01011\rangle$, $\langle 01000, 00010\rangle$, $\langle 01000, 00001\rangle$, $\langle 10011, 01000\rangle$
E7: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 00010\rangle$, $\langle 10000, 00001\rangle$, $\langle 10000, 01011\rangle$, $\langle 01000, 00010\rangle$, $\langle 01000, 00101\rangle$, $\langle 10111, 01000\rangle$
E8: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 00010\rangle$, $\langle 10000, 00001\rangle$, $\langle 01000, 00010\rangle$, $\langle 01000, 00001\rangle$, $\langle 10100, 01111\rangle$, $\langle 10111, 01100\rangle$
E9: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 00010\rangle$, $\langle 10000, 00001\rangle$, $\langle 01000, 00110\rangle$, $\langle 01000, 00101\rangle$, $\langle 10100, 01111\rangle$, $\langle 10111, 01100\rangle$
E10: $\langle 10000, 01000, 00100\rangle$, $\langle 10000, 00010\rangle$, $\langle 01000, 00010\rangle$, $\langle 00100, 00001\rangle$, $\langle 10100, 01011\rangle$, $\langle 11001, 00101\rangle$, $\langle 10011, 01100\rangle$

It turns out that E2, E4, E7, and E9 are not extendable to an avsp while the other six cases are. Moreover, the extension is unique up to symmetry in these cases.

4.3 Tight irreducible avsps in $\mathrm{PG}(6,2)$

Lemma 36. *In* $\mathrm{PG}(6,2)$ *no tight irreducible avsp of type* $5^2 4^2 3^4$ *or* $5^1 4^6$ *exists.*

Proof. All two possibilities are excluded using ILP computations, see Section A. They are also excluded using `GAP` computations. □

Proposition 37. *Let* $\mathcal{U}$ *be a tight irreducible avsp of* $\mathrm{PG}(6,2)$*, then* $\mathcal{U}$ *has one of the following types:*

- $5^2 4^i 3^j 2^{16-2j-4i}$ *for* $i \in \{0,1,2\}$ *and* $0 \le j \le 8-2i$*, where* $j+2i \ne 7$ *and* $(i,j) \ne (2,4)$*;*
- $5^1 4^i 3^j 2^{24-2j-4i}$ *for* $0 \le i \le 4$ *and* $0 \le j \le 12-2i$*, where* $j+2i \ne 11$*;*
- $4^i 3^j 2^{32-2j-4i}$ *for* $0 \le i \le 8$ *and* $0 \le j \le 16-2i$*, where* $j+2i \ne 15$ *and* $i \ne 7$*.*

All types are realizable.

Proof. Let the type of $\mathcal{U}$ be $6^{m_6} \ldots 1^{m_1}$. From Corollary 21 and Lemma 22 we conclude $m_6 = 0$ and $m_1 = 0$, so that the packing condition (1) gives $8m_5 + 4m_4 + 2m_3 + m_2 = 32$. Corollary 29 yields $m_5 \le 2$ and Lemma 33 excludes $(m_5, m_4, m_3, m_2) = (2, 4, 0, 0)$. Moreover, Corollary 17 implies $m_l \neq 2$ for the smallest index with $m_l > 0$. The two hypothetical types $5^2 4^2 3^4$ and $5^1 4^6$ are excluded in Lemma 36. All remaining possibilities $(m_5, m_4, m_3, m_2) \in \mathbb{N}_0^4$ are listed in the statement and for each type we found a realization using an ILP formulation, see Section A. □

Corollary 38. *If $\mathcal{U}$ is a tight irreducible avsp of* PG(6, 2) *of minimum possible size, then $\#\mathcal{U} = 8$ and $\mathcal{U}$ has type 4^8.*

Here we describe all four isomorphism types of homogeneous irreducible tight avsps $\mathcal{U}$ of PG(6, 2) of type 4^8. Geometrically each $\mathcal{U}$ is given by eight solids $S_1, \ldots, S_8$ in PG(6, 2) intersecting a hyperplane H_∞ in a plane (plus some extra conditions). Here we directly consider the part $\mathcal{U}'$ at infinity, i.e. the eight planes $\pi_1, \ldots, \pi_8 \in H_\infty \cong \mathrm{PG}(5, 2)$ given by $\pi_i = S_i \cap H_\infty$. The conditions for the pairwise intersections are

$$1 \le \dim(\pi_i \cap \pi_j) \le 2 \quad \forall 1 \le i < j \le 8. \tag{10}$$

Since the planes form a spanning 2-divisible set we have

$$\# \{1 \le i \le 8 \, : \, \pi_i \not\le H\} \in \{2, 4, 6, 8\} \tag{11}$$

for every hyperplane H of $H_\infty \cong \mathrm{PG}(5, 2)$.

Let e_i denote the ith unit vector, i.e., the vector with a 1 at the i-th position and zeros everywhere else. If the pairwise intersection of the planes π_i is a line in all cases then they span a solid, which contradicts the condition that not all eight planes can be contained in a hyperplane. W.l.o.g. we assume $\pi_1 = \langle e_1, e_2, e_3\rangle$ and $\pi_2 = \langle e_3, e_4, e_5\rangle$, i.e., the intersection point between π_1 and π_2 is $\langle e_3\rangle$. Since the intersection of all eight planes is empty we assume w.l.o.g. that π_3 does not contain $\pi_1 \cap \pi_2 = e_3$. Up to symmetry we have the following three cases for π_3:

(a) $\dim(\pi_1 \cap \pi_3) = 2$, $\dim(\pi_2 \cap \pi_3) = 1$: $\pi_3 = \langle e_1, e_2, e_4\rangle$;

(b) $\dim(\pi_1 \cap \pi_3) = \dim(\pi_2 \cap \pi_3) = 1$, $\dim(\langle \pi_1, \pi_2, \pi_3\rangle) = 5$: $\pi_3 = \langle e_1, e_4, e_2 + e_5\rangle$; and

(c) $\dim(\pi_1 \cap \pi_3) = \dim(\pi_2 \cap \pi_3) = 1$, $\dim(\langle \pi_1, \pi_2, \pi_3\rangle) = 6$: $\pi_3 = \langle e_1, e_4, e_6\rangle$.

Starting from the three possibilities for π_1, π_2, π_3 we build up a graph whose vertices consist of the planes that have intersection dimension 1 or 2 with π_i for $1 \le i \le 3$, c.f. Condition (10). Two vertices π and π' are connected by an edge if $1 \le \dim(\pi \cap \pi') \le 2$, c.f. Condition (10). For these graphs we determine all cliques of size five and check Condition (11) afterwards:

(a) 3,014,435,152 cliques → 432 cases;
(b) 2,198,293,872 cliques → 0 cases;
(c) 1,218,975,648 cliques → 320 cases.

The overall computation took just a few minutes. Note that the constructed 752 cases are just candidates for the extension problem to eight solids. Up to symmetry they decompose into just four non-isomorphic examples. It turns out that they can be distinguished by the maximum number γ_0 of incidences of a point and the eight planes, which has to lie between 2 and 5. In Table 2 we summarize incidence counts.

For $\gamma_0 = 2$ we consider an arbitrary plane π contained in the hyperbolic quadric $\mathcal{Q} = Q^+(5, 2)$, which form a single orbit under its collineation group $\mathrm{PGO}^+(6, 2) = C_2 \times \mathrm{PGL}(3, 2) = S_8$ of order 40,320. From the 35 points on $\mathcal{Q}$ the points in π have no incidences with the eight planes while all other 28 points on $\mathcal{Q}$ have exactly two incidences. This example is obtained in 16 cases. The symmetry group of the eight planes has order 1344 and type $C_2^3 : \mathrm{PGL}(3, 2)$.

γ_0	2	3	4	5
point incidences	2^{28}	$1^{21}2^{7}3^{7}$	$1^{16}2^{12}4^{4}$	$1^{20}2^{6}3^{2}4^{2}5^{2}$
line incidences:	1^{56}	1^{56}	$1^{48}2^{4}$	$1^{46}2^{2}3^{2}$
solid incidences:	1^{56}	1^{56}	$1^{48}2^{4}$	$1^{40}2^{8}$
hyperplane incidences:	2^{28}	2^{28}	$2^{24}4^{2}$	$2^{23}4^{1}6^{1}$
case triples:	c^{168}	c^{168}	$a^{48}c^{96}$	$a^{72}c^{48}$

Table 2: Irreducible tight avsps of PG(6, 2) of type 4^8.

For $\gamma_0 = 3$ choose a projective base of PG(5, 2), i.e., put $f_i = e_i$ for $1 \le i \le 6$ and $f_7 = \sum_{i=1}^{6} e_i$. Consider a Fano plane on the set $\{1, 2, 3, 4, 5, 6, 7\}$:

$$\ell_1 = \{1,2,3\}, \quad \ell_2 = \{1,4,5\}, \quad \ell_3 = \{1,6,7\}, \quad \ell_4 = \{2,4,6\},$$
$$\ell_5 = \{3,4,7\}, \quad \ell_6 = \{2,5,7\}, \quad \ell_7 = \{3,5,6\}.$$

Choose seven planes $\pi_i := \langle f_j : j \in \ell_i \rangle$ for $1 \le i \le 7$ and an eight plane. $\pi_8 = K := \langle \sum_{j\in\ell_i} f_j : 1 \le j \le 7 \rangle$. Note that K itself is also a Fano plane (of course with a different embedding). The points with three incidences with the eight planes are the f_i for $1 \le i \le 7$ and the points with two incidences with the eight planes are the points of K. This example is obtained in 112 cases. The symmetry group of the eight planes has order 168 and type PGL(3, 2).

For $\gamma_0 = 4$ let $\{Q_1, Q_2, Q_3, Q_4, R_1, R_2\}$ be a basis of H_∞. With this, we construct the eight planes as

$$\langle Q_{i+j}, Q_{i+j+1}, R_i \rangle \text{ for } i \in \{1,2\}.j \in \{0,2\},$$
$$\langle Q_{i+j}, Q_{i+j+1}, R_i + A \rangle \text{ for } i \in \{1,2\}.j \in \{0,2\},$$

where $A = Q_1 + Q_2 + Q_3 + Q_4$ and $Q_5 = Q_1$. The points with four incidences with the eight planes are $Q_1, \ldots, Q_4$. The lines with two incidences with the eight planes are $\langle Q_i, Q_{i+1} \rangle$ for $1 \le i \le 4$ (again setting $Q_5 = Q_1$; so this is some kind of a cyclic construction). This example is obtained in 192 cases. The symmetry group of the eight planes has order 128 and type $D_8^2 : C_2$.

For $\gamma_0 = 5$ let $\{Q_1, Q_2, R_1, R_2, S_1, T_1\}$ be a basis of H_∞. With this, we set $S_2 := S_1 + Q_1 + Q_2$, $T_2 := T_1 + Q_1 + Q_2 + R_1 + R_2$ and construct the eight planes as

$$\langle Q_1, Q_2, R_i \rangle \text{ for } i \in \{1,2\},$$
$$\langle R_1, R_2, S_i \rangle \text{ for } i \in \{1,2\}, and$$
$$\langle Q_i, R_i, T_j \rangle \text{ for } i,j \in \{1,2\},$$

which also reflects the three orbits of the eight planes w.r.t. the action of their automorphism group. The points with five incidences with the eight planes are R_1 and R_2. The points with four incidences with the eight planes are Q_1 and Q_2. The points with three incidences with the eight planes are $R_1 + Q_1$ and $R_2 + Q_2$. The lines with three incidences with the eight planes are $\langle R_1, Q_1 \rangle$ and $\langle R_2, Q_2 \rangle$. The lines with two incidences with the eight planes are $\langle R_1, R_2 \rangle$ and $\langle Q_1, Q_2 \rangle$. This example is obtained in 432 cases. The symmetry group of the eight planes has order 1024.

5 Constructions of tight irreducible avsps

In this section we collect a few general constructions for tight irreducible avsps using different combinatorial objects. We use spreads, the Klein quadric, and hitting formulas in sections 5.1, 5.2, and 5.3, respectively.

5.1 Constructions from projective spreads

A k-spread in $\mathrm{PG}(n-1,q)$ is a disjoint set of k-spaces that partitions $\mathrm{PG}(n-1,q)$. It is well known that k-spreads exist iff k divides n.

Proposition 39. *For each positive even integer n there exists a tight irreducible avsp $\mathcal{U}$ of $\mathrm{PG}(n-1,q)$ of type $(n/2)^m$, where $m = q^{n/2}$.*

Proof. Let $k = n/2$ and $\mathcal{P}$ be a k-spread of $\mathrm{PG}(n-1,q)$, which has size q^k+1. Now choose an arbitrary element $K \in \mathcal{P}$ and an arbitrary hyperplane H containing K. With this we set $\mathcal{U} = \mathcal{P}\backslash\{K\}$ where we choose H as the hyperplane at infinity. By construction $\mathcal{U}$ is an avsp of $\mathrm{PG}(n-1,q)$. Since all elements are pairwise disjoint $\mathcal{U}$ is tight and since any two elements span $\mathrm{PG}(n-1,q)$ $\mathcal{U}$ is irreducible. □

We have seen that in $\mathrm{PG}(5,2)$ there exist tight irreducible avsps of types 3^8 and 2^{16}. Starting from a 2-spread of $\mathrm{PG}(5,q)$ we can clearly obtain a tight avsp $\mathcal{U}$ by removing all lines that are completely contained in an arbitrarily chosen hyperplane H. However, it may happen that $\mathcal{U}$ is reducible. This is indeed the case if we start with the Desarguesian line spread. In $\mathrm{PG}(5,2)$ there exist 131,044 non-isomorphic line spreads [MT09].

Conjecture 40. *For each integer $1 < k < n$ that divides n there exists a tight irreducible avsp $\mathcal{U}$ of $\mathrm{PG}(n-1,q)$ of type k^m, where $m = q^{n-k}$.*

If n is odd no $\lfloor (n-1)/2 \rfloor$-spread exists, but we can construct tight irreducible avsps from some special large partial spreads.

Proposition 41. *For each odd integer $n \geq 5$ there exists a tight irreducible avsp $\mathcal{U}$ of $\mathrm{PG}(n-1,q)$ of type $((n-1)/2)^m$, where $m = q^{(n+1)/2}$.*

Proof. Let $k = (n-1)/2$ and $\mathcal{P}$ be a vector space partition of $\mathrm{PG}(n-1,q)$ of type $(k+1)^2 k^m$, where $m = q^{k+1}$. Now choose an arbitrary hyperplane H containing the unique $(k+1)$-dimensional element K of $\mathcal{P}$. With this we set $\mathcal{U} = \mathcal{P}\backslash\{K\}$ where we choose H as the hyperplane at infinity. By construction $\mathcal{U}$ is an avsp of $\mathrm{PG}(n-1,q)$. Since all elements are pairwise disjoint $\mathcal{U}$ is tight. Any two elements of $\mathcal{U}$ span a hyperplane of $\mathrm{PG}(n-1,q)$. Since the elements of $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$ span H_∞, not all elements of $\mathcal{U}'$ can be contained in a hyperplane of H_∞ and $\mathcal{U}$ is irreducible. □

Vector space partitions of the used type can be obtained from lifted MRD codes, see e.g. [SSW19] for a survey on MRD codes. They also occur as extendible partial k-spreads, where $k = (n-1)/2$, of the second largest size q^{k+1} and are the main building block in the construction of partial k-spreads of size $q^{k+1}+1$ as described by Beutelspacher [Beu75]. For more details on the relations between these different geometrical objects we refer e.g. to [HKK19].

For each $n \geq 5$ there also exist a vector space partition $\mathcal{P}$ of $\mathrm{PG}(n-1,q)$ of type $(n-2)^1 2^m$, where $m = q^{n-2}$. Choosing a hyperplane that contains the unique $(n-2)$-space as the hyperplane at infinity we can obtain a tight avsp $\mathcal{U}$ of $\mathrm{PG}(n-1,q)$ of type $2^{q^{n-2}}$. The remaining question is whether we can choose $\mathcal{P}$ in such a way that $\mathcal{U}$ becomes irreducible.

5.2 Constructions from the Klein quadric

It seems very likely that the avsp of $\mathrm{PG}(6,2)$ of type 4^8 with maximum point multiplicity 2, see Subsection 4.2, can be generalized to arbitrary field sizes.

Theorem 42. *There exists a tight irreducible avsp of type 4^{q^3} in $\mathrm{AG}(6,q)$ for q even.*

Proof. We will use the following finite field model of AG$(6,q)$. Let $V = \mathbb{F}_{q^3} \times \mathbb{F}_{q^3} \times \mathbb{F}_q$ and let H_∞ be the hyperplane $X_3 = 0$. So we identify AG$(6,q)$ with the elements of V of the form (a,b,c), where $c \neq 0$. Consider the following quadratic form on H_∞:

$$Q(x,y,0) := Tr_{q^3/q}(xy).$$

Then Q defines the points of a hyperbolic quadric $\mathcal{Q}$. Next, let π be the plane $\{(0,y,0) : y \in \mathbb{F}_{q^3}^*\}$. Then π is totally singular with respect to Q. Let $S_0 := \{(x,0,1) : x \in \mathbb{F}_{q^3}\}$ and $S_1 := \{(y, y^{q^2} + y^q + 1, 1) : y \in \mathbb{F}_{q^3}\}$. Let α be a primitive element of $\mathbb{F}_{q^3}$, let σ be the map

$$\sigma : (x,y,z) \mapsto (\alpha^{-1}x, \alpha y, z),$$

and let $G := \langle \sigma \rangle$. We will show that $\mathcal{S} := \{S_0\} \cup S_1^G$ is a tight irreducible avsp of size q^3 in AG$(6,q)$.

First note that σ has order $q^3 - 1$. Let $(a,b,1)$ be a point P of AG$(6,q)$. We show that P lies in a unique element of $\mathcal{S}$. If $b = 0$, then P lies in S_0. The condition that P lies in $S_1^{\sigma^m}$ (where $1 \leq m \leq q^3 - 1$) can be restated as

$$a = \alpha^{-m} y, \qquad b = \alpha^m (y^{q^2} + y^q + 1).$$

for some $y \in \mathbb{F}_{q^3}$. We have

$$ab = y^{q^2+1} + y^{q+1} + y$$

and the polynomial $y^{q^2+1} + y^{q+1} + y$ is a permutation on $\mathbb{F}_{q^3}$, by [TZH14, Theorem 4]. Hence, y and, thus, m are determined by a and b. Therefore, $\mathcal{S}$ is an avsp.

Note that $\pi_1 := \overline{S_1} \cap H_\infty = \{(y, y^{q^2} + y^q, 0) : y \in \mathbb{F}_{q^3}\}$ and $\pi_0 := \overline{S_0} \cap H_\infty = \{(x,0,0) : x \in \mathbb{F}_{q^3}\}$. To compute the image of π_1 under σ^m, notice that

$$(\alpha^{-m}y, \alpha^m(y^{q^2} + y^q), 0) = (\alpha^{-m}y, \alpha^{(q^2+1)m}(\alpha^{-m}y)^{q^2} + \alpha^{(q+1)m}(\alpha^{-m}y)^q, 0) = (w, \zeta w^{q^2} + \zeta^q w^q, 0)$$

where $w = \alpha^{-m}y$ and $\zeta = \alpha^{(q^2+1)m}$. Therefore, upon application of G,

$$\mathcal{S}_\infty := \{\overline{S} \cap H_\infty : S \in \mathcal{S}\} = \{\pi_0\} \cup \pi_1^G = \{\pi_\zeta : \zeta \in \mathbb{F}_{q^3}\}$$

where $\pi_\zeta := \{(y, \zeta y^{q^2} + \zeta^q y^q, 0) : y \in \mathbb{F}_{q^3}^*\}$. Note that $|\mathcal{S}_\infty| = q^3$ and that $\mathcal{S}$ consists of totally singular planes of $\mathcal{Q}$ disjoint from π. As these are all totally singular planes of $\mathcal{Q}$ disjoint from π, these cover the points of $\mathcal{Q}$ uniformly and their common intersection is empty and, thus, $\mathcal{S}$ is tight. As these pairwise meet in a point, any two elements of $\mathcal{S}$ span PG$(6,q)$. This shows irreducibility. ☐

Let $\mathcal{P}$ be the set of planes in the Klein quadric $\mathcal{Q} = Q^+(5,q)$ that is disjoint to an arbitrary but fixed plane π in $\mathcal{Q}$. One can verify that $\mathcal{P}$ is a spanning q-divisible set of q^3 planes in PG$(5,q)$ such that the intersection of a pair of planes is a point, i.e., all known conditions for the part $\mathcal{U}'$ at infinity of a tight irreducible avsp of PG$(6,q)$ of type 4^{q^3} are satisfied. The remaining question is whether a solution of the extension problem for $\mathcal{P}$ exists.

Conjecture 43. *The extension problem for $\mathcal{P}$ admits a solution for all prime powers q.*

Theorem 42 shows the conjecture for q even. By computer we showed Conjecture 43 for $q = 3, 5$.

5.3 Constructions using hitting formulas

A *hitting formula* is a DNF such that each truth assignment to the underlying variables satisfies precisely one term [Iwa89]. For example:

$$(x \wedge y \wedge z) \vee (\bar{x} \wedge \bar{y} \wedge \bar{z}) \vee (\bar{x} \wedge y) \vee (\bar{y} \wedge z) \vee (\bar{z} \wedge x).$$

We say that a variable *appears* in the DNF if one of the two corresponding literals appears in one of the terms. The variables mentioned in the above DNF are x, y, z. We can represent hitting formulas over $x_1, \dots, x_n$ as collections of strings in $\{0, 1, *\}^n$, where 0 in the i'th position represents $\bar{x}_i$, 1 in the i'th position represents x_i, and $*$ in the i'th position represents the absence of x_i in the term. For example, the above hitting formula corresponds to the strings $111, 000, 01*, *01, 1*0$.

This notion describes subcubes of affine points. Taking the projective closure we end up with the list

$$\langle 1111\rangle, \langle 1000\rangle, \langle 1010, 0001\rangle, \langle 1001, 0100\rangle, \langle 1100, 0010\rangle$$

of subspaces of $\mathrm{PG}(3, 2)$ that form an avsp, which obviously is not irreducible. However, we can join the first two elements to $\langle 1000, 0111\rangle$ and obtain a tight irreducible avsp. While every string corresponds to an affine subspace, not every affine subspace corresponds to a string. It turns out that any two strings having its stars at the same positions can be joined to an affine subspace. For brevity, we speak of *compression*. Interestingly enough, several tight irreducible avsps of $\mathrm{PG}(n-1, 2)$ of the minimum possible size can be obtained by compression, see Section C in the appendix. More theoretical insights on the relations between hitting formulas and avsps will be treated in an upcoming article focusing on irreducible hitting formulas.

6 The minimum possible size of tight irreducible avsps

We have discussed the minimum possible size of a (tight) avsp of $\mathrm{PG}(n-1, q)$ in Section 2. Before we consider the minimum possible size $\sigma_q(n)$ of a tight irreducible avsp $\mathcal{U}$ of $\mathrm{PG}(n-1, q)$ we remark that Lemma 22 implies the upper bound $\#\mathcal{U} \le 2^{n-2}$ for $q = 2$. The constructions mentioned in Section 5 suggest that this upper bound can be attained. In Section 3 and Section 4 we have determined the exact values $\sigma_q(2) = q$, $\sigma_q(3) = \infty$, $\sigma_2(4) = 4$, $\sigma_2(5) = 6$, $\sigma_2(6) = 7$, and $\sigma_2(7) = 8$.

Lemma 44. *Let $\mathcal{U}'$ be a configuration of a 5-space K', four 4-spaces $S'_1, \dots, S'_4$, and four planes $E'_1, \dots, E'_4$ in $\mathrm{PG}(6, 2)$ such that there exists a 5-space A' and three 6-spaces B'_1, B'_2, B'_3 satisfying*

- $\dim(K' \cap A') = 4$, $S'_1, S_2 \le A'$, $\dim(S'_3 \cap A') = \dim(S'_4 \cap A') = 3$;
- $K' \not\le B'_1$, $S'_3, S'_4 \le B'_1$, $K', S'_3, S'_4 \not\le B'_2$, $K' \le B'_3$, $S'_3, S'_4 \not\le B'_3$.

Then, then extension problem of $\mathcal{U}'$ to a tight irreducible avsp $\mathcal{U}$ of $\mathrm{PG}(7, 2)$ has no solution.

Proof. Assume that $\mathcal{U} = \{K, S_1, \dots, S_4, E_1, \dots, E_4\}$ is an irreducible avsp of $\mathrm{PG}(7, 2)$ with $K \cap H_\infty = K'$, $S_i \cap H_\infty = S'_i$, $E_i \cap H_\infty = E'_i$ for all $1 \le i \le 4$. Let $A_1, \dots, A_4$ be the four 6-spaces that contain A' but are not contained in H_∞. From the known intersections with A' we conclude

- $\#\left(K \backslash H_\infty \cap A_i\right) \in \left\{0, 2^4\right\}$;
- $\#\left(S_j \backslash H_\infty \cap A_i\right) \in \left\{0, 2^4\right\}$ for $j = 1, 2$;
- $\#\left(S_j \backslash H_\infty \cap A_i\right) \in \left\{0, 2^3\right\}$ for $j = 3, 4$;
- $\#\left(E_j \backslash H_\infty \cap A_i\right) \in \left\{0, 2^3\right\}$ for $j = 1, \dots, 4$

for all $1 \le i \le 4$. From the intersections of the B'_i, where $1 \le i \le 3$, with S'_3 and S'_4 we conclude $\#\left(S_3 \backslash H_\infty \cap A_j\right) = \#\left(S_4 \backslash H_\infty \cap A_j\right)$ for all $1 \le j \le 4$. W.l.o.g. we assume $S_i \backslash H_\infty \cap A_j \ne \emptyset$ for all $i = 1, 2$ and $j = 1, 2$. From the intersection of K' with B'_1 we conclude that either $\#\left(K \backslash H_\infty \cap A_1\right) = 2^4$ or $\#\left(K \backslash H_\infty \cap A_2\right) = 2^4$. W.l.o.g. we assume $K \backslash H_\infty \cap A_4 = \emptyset$. With this the affine points of the elements of $\mathcal{U}$ are either completely contained in A_4 or disjoint to A_4. This contradicts our assumption that $\mathcal{U}$ is irreducible. □

Lemma 45. *In $\mathrm{PG}(6, 2)$ every configuration $\mathcal{U}'$ of type $5^2 4^2 3^4$, $5^2 4^1 3^6$, $5^1 4^6$, $5^1 4^4 3^4$, or 4^8 that does not contain a configuration as in Lemma 44 and satisfies the conditions of Lemma 31, Lemma 32, and the dimension condition, cf. Lemma 8, admits a point P that is contained in all elements of $\mathcal{U}'$.*

Proof. All cases have been excluded by ILP computations. □

Corollary 46. *In* $\mathrm{PG}(7,2)$ *no tight irreducible avsp of the following types exist:* $6^2 5^2 4^4$, $6^2 5^1 4^6$, $6^1 5^6$, $6^1 5^4 4^4$, 5^8.

Corollary 47. *The minimum size* $\sigma_2(8)$ *of an irreducible tight avsp of* $\mathrm{PG}(7,2)$ *is given by* 10.

For attaining examples we refer to Section C in the appendix.

Our next aim is a recursive construction which implies an asymptotic upper bound of roughly $\frac{3n}{2}$ for $\sigma_2(n)$.

Theorem 48. *Let* $\mathcal{U} = \{U_1, \dots, U_r\}$ *be an irreducible tight avsp of* $\mathrm{PG}(n-1,2)$ *with* $\dim(U_1) = n-2$ *and* $n \geq 3$*. Then, there exists an irreducible tight avsp* $\mathcal{U}'$ *of* $\mathrm{PG}(n+2-1,2)$ *of size* $\#\mathcal{U}+3 = r+3$ *that contains an element of dimension* n*.*

Proof. Let $V = \mathrm{PG}(n+2-1,2)$, H_∞ be the hyperplane at infinity, and $K \leq H_\infty$ be an arbitrary subspace with $\dim(K) = n$. With this, denote the two hyperplanes containing K and not being equal to H_∞ by H_1 and H_2. Choose an arbitrary point $P \leq K$ and a subspace $K' \leq K$ such that $\dim(K') = n-1$ and $\langle P, K' \rangle = K$. Now choose an irreducible tight avsp $\mathcal{U} = \{U_1, \dots, U_r\}$ of H_1/P such that $\dim(U_1) = n-2$. We set $A_i := \langle U_i, P \rangle$ for all $1 \leq i \leq r$. Choose an n-space B with $B \cap H_1 = A_1$ and $B \not\leq H_\infty$, so that $C_1 := B \cap H_2$ is an $(n-1)$-space in H_2 with $C_1 \not\leq H_\infty$ and $P \leq C_1$. In H_2 choose three further $(n-1)$-spaces C_2, C_3, C_4 such that $\dim(C_i \cap C_j) = \dim(C_1 \cap C_2 \cap C_3 \cap C_4) = n-3$ for all $1 \leq i < j \leq 4$, $C_1 \cap C_2 \cap C_3 \cap C_4 \leq K'$, and that $\{C_1, C_2, C_3, C_4\}$ forms an avsp of H_2. (This boils down to an avsp of $\mathrm{PG}(4-1,2)$ of type 2^4, which is a union of four disjoint lines.) Then,

$$\mathcal{U}' := \{A_2, \dots, A_r, B, C_2, C_3, C_4\}$$

is an irreducible tight avsp of V of size $\#\mathcal{U}+3 = r+3$. The size follows directly from the construction and $\dim(B) = n$. Since $B \cap H_1 \cap H_\infty = B \cap H_2 \cap H_\infty$ we have $B \cap C_2 \cap C_3 \cap C_4 = C_1 \cap C_2 \cap C_3 \cap C_4 \leq K'$ and $B \cap A_2 \cap \cdots \cap A_r = A_1 \cap \cdots \cap A_r = P$, so that $\mathcal{U}'$ is tight. Noting that $\mathcal{U}'' := \{A_1, \dots, A_r\}$ is an avsp of H_1, $\{C_1, \dots, C_4\}$ is an avsp of H_2, and $\{A_1, C_1\}$ is an avsp of B, we conclude that $\mathcal{U}'$ is indeed an avsp of V.

It remains to show that $\mathcal{U}'$ is irreducible. So, assume that there exists a proper subset $\tilde{\mathcal{U}} \subsetneq \mathcal{U}'$ that can be joined to an x-space X. If $\tilde{\mathcal{U}} \cap \{B, C_2, C_3, C_4\} = \emptyset$, then we have $\tilde{\mathcal{U}} \subseteq \mathcal{U}''$ contradicting the fact that $\mathcal{U}''$ is irreducible. So, especially we have $x \in \{n, n+1\}$. Noting that any two elements in $\{C_1, C_2, C_3, C_4\}$ span H_2, we conclude $\#\left(\tilde{\mathcal{U}} \cap \{B, C_2, C_3, C_4\}\right) = 1$.

(i) If $x = n$, then let $2 \leq i \leq 4$ be the unique index such $C_i \in \tilde{\mathcal{U}}$. Clearly, $B \notin \tilde{\mathcal{U}}$. Let $\tilde{C}$ be the the other $(n-1)$ space in X not contained in H_∞ and not equal to C_i with $\tilde{C} \cap H_\infty = C_i \cap \tilde{C}$, so that the elements of $\tilde{\mathcal{U}} \backslash \{C_i\}$ form a vector space partition of $\tilde{C}$. However, since $P \not\leq C_i$ and all elements in $\mathcal{U} \backslash \{B, C_1, C_2, C_3\}$ contain P, this is impossible.

(ii) If $x = n+1$ and $\#\left(\tilde{\mathcal{U}} \cap \{B, C_2, C_3, C_4\}\right) = 1$, then we have $\dim(X') = n$ for $X' := X \cap H_1$. If $B \in \tilde{\mathcal{U}}$, then $\tilde{\mathcal{U}} \backslash \{B\} \cup \{A_1\}$ can be joined to X' in H_1, which is a contradiction. If $C_i \in \tilde{\mathcal{U}}$, then $\tilde{\mathcal{U}} \backslash \{C_i\}$ can be joined to X' in H_1, which is also a contradiction.

Thus, $\mathcal{U}'$ is irreducible. □

Corollary 49. *For each* $n \geq 4$ *an irreducible tight avsp* $\mathcal{U}$ *of* $\mathrm{PG}(n-1,2)$ *of size* $\left\lfloor \frac{3n-3}{2} \right\rfloor$ *exists.*

Proof. For $n = 4$ there exists such an example with type 2^4 and for $n = 5$ there exists such an example with type $3^2 2^4$. Then, iteratively apply the construction from Theorem 48. □

We remark that the constructive upper bound for $q_2(n)$ is tight for $n \in \{4, 5, 6, 8\}$.

7 Conclusion

We have introduced the geometrical object of affine vector space partitions. To make their study interesting we need the additional conditions of tightness and irreducibility, which are natural in the context of hitting formulas. A very challenging problem is the determination of the minimum possible size of an irreducible tight avsp of $\mathrm{PG}(n-1,q)$. To this end we have obtained some preliminary results for arbitrary field sizes but small dimensions and for the binary case with medium sized dimensions. We also gave a parametric construction that matches the known exact values in many cases. That irreducible tight avsps are nice geometric objects can be e.g. seen at their sometimes large automorphism groups as well as the mentioned connection to the hyperbolic quadric $Q^+(5,q)$. While we have obtained a few insights, many questions remain open. So, we would like to close with a list of a few open problems:

1. Consider tight irreducible avsps of $\mathrm{PG}(4,q)$ of type $2^{m_2}3^{m_3}$. What is the largest possible value for m_3?
2. Determine a solution of the extension problem for the set $\mathcal{P}$ of q^3 planes in $\mathrm{PG}(5,q)$ obtained from the hyperbolic quadric $Q^+(5,q)$ for q odd, cf. Conjecture 43.
3. Determine further constructions for tight irreducible avsps of $\mathrm{PG}(n-1,q)$ with large automorphism groups.
4. Construct a tight irreducible avsp of $\mathrm{PG}(n-1,q)$ of type $2^{q^{n-2}}$ for all $n \geq 5$.
5. Is it possible that a tight irreducible avsp of $\mathrm{PG}(n-1,q)$ contains 1-dimensional elements if $n \geq 3$ and $q \geq 3$?
6. Determine further exact values of the minimum size $\sigma_q(n)$ of a tight irreducible avsp of $\mathrm{PG}(n-1,q)$.
7. Determine $\lim_{n\to\infty} \sigma_q(n)/n$.
8. Is $\sigma_q(n)$ strictly increasing in n?

Acknowledgements

We thank Esmeralda Năstase and Artur Riazanov for helpful discussions. Further thanks go to Tomáš Peitl and Stefan Szeider for sharing with us the results of the computer search reported in [PS22]. Ferdinand Ihringer and Sascha Kurz would like to thank the organizers of the Sixth Irsee Conference on Finite Geometries for their invitation. During that conference the idea of analyzing and introducing avsps slowly evolved, being triggered by some open problems for hitting formulas.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 802020-ERC-HARMONIC. Ferdinand Ihringer is supported by a postdoctoral fellowship of the Research Foundation – Flanders (FWO).

A Integer linear programming formulations

Let $\mathcal{U}'$ be an arbitrary set of subspaces of H_∞ in $\mathrm{PG}(n-1,q)$. For the question whether $\mathcal{U}'$ can be extended to an avsp $\mathcal{U}$ of $\mathrm{PG}(n-1,q)$ we utilize binary variables x_C for all subspaces C of $\mathrm{PG}(n-1,q)$ such that $C \not\leq H_\infty$ and $C \cap H_\infty \in \mathcal{U}'$ with the meaning $x_C = 1$ iff $C \in \mathcal{U}$. We denote the set of all of these subspaces by $\mathcal{C}$. For each point P in $\mathrm{PG}(n-1,q)\backslash H_\infty$ the equation

$$\sum_{c\in\mathcal{C}\,:\,P\leq C} x_C = 1 \tag{12}$$

and for each $U \in \mathcal{U}'$ the equation

$$\sum_{c\in\mathcal{C}\,:\,U\leq C} x_C = 1 \tag{13}$$

has to be satisfied. The 0/1 solutions of this equation system are in one-to-one correspondence to extensions of $\mathcal{U}'$ to avsps $\mathcal{U}$ in $\mathrm{PG}(n-1,q)$.

Searching a tight irreducible avsp $\mathcal{U}$ in $\operatorname{PG}(n-1,q)$ directly can be achieved by a similar model. Now let $\mathcal{C}$ be the set of subspaces of $\operatorname{PG}(n-1,q)$ that are not incident with H_∞. Again we use binary variables x_C for all $C \in \mathcal{C}$ with the meaning $x_C = 1$ iff $C \in \mathcal{U}$. Partitioning the affine points is modeled by

$$\sum_{c\in\mathcal{C}\,:\,P\le C} x_C = 1 \tag{14}$$

for all points P not contained in H_∞. The condition that $\mathcal{U}$ is tight can be written as

$$\sum_{C\in\mathcal{C}\,:\,Q\le C} x_C + 1 \le \sum_{C\in\mathcal{C}} x_C \tag{15}$$

for all points $Q \le H_\infty$. In order to model the condition that $\mathcal{U}$ is irreducible we say that a subspace A *escapes* a subspace B if A has both points that are contained and points that are not contained in B. So, for each $B \in \mathcal{C}$ we require

$$x_B + \sum_{C\in\mathcal{C} \text{ such that } C \text{ escapes } B} x_C \ge 1. \tag{16}$$

Of course we can fix the type of $\mathcal{U}$ by additional equations. Using a target function we can minimize or maximize $\#\mathcal{U}$ as well as the number of i-dimensional elements. We have to mention that this ILP formulation comprises a lot of symmetry, so that it can be solved in reasonable time for small parameters n and q only. However, we can use the inherent symmetry to fix some of the x_C variables. I.e. the symmetry group acts transitively on the set of a-spaces that are not contained in H_∞. For pairs of an a-space A and a b-space B that both are not contained in H_∞, the different orbits under the action of the symmetry group are characterized by the invariant $\dim(A \cap B)$.

B Technical details

In order to keep the paper more readable, we have moved some technical details, that may also be left to the reader, to this section. The proof of Lemma 3 uses the numbers $m_i^{(j)}$ satisfying certain constraints. For completeness we state how those number can be computed in Algorithm 1.

In the three subsequent lemmas we characterize 2-divisible sets in $\operatorname{PG}(3,2)$ of cardinality $s \in \{3,6,8\}$.

Lemma 50. *Let $\mathcal{P}$ be a 2-divisible set of three points in $\operatorname{PG}(3,2)$ then $\mathcal{P}$ forms a line.*

Proof. Let $\mathcal{P} = \{P_1, P_2, P_3\}$ and $L := \langle P_1, P_2\rangle$. Since all hyperplanes containing L have to contain $\mathcal{P}$, we have $P_3 \in L$. □

Lemma 51. *Let $\mathcal{P}$ be a 2-divisible set of six points in $\operatorname{PG}(3,2)$ then $\mathcal{P}$ is the disjoint union of two lines.*

Proof. If H is a hyperplane containing all points of $\mathcal{P}$, then there is a unique point $P \le H$ with $P \notin \mathcal{P}$. Since every hyperplane H' that does not contain P intersects $\mathcal{P}$ in cardinality 3, so that this case cannot occur, i.e., $\mathcal{P}$ is spanning. From the standard equations we compute $a_0 = 0$, $a_2 = 9$, and $a_4 = 6$ for the spectrum. From the MacWilliams transform for the corresponding linear code we conclude the existence of a triple of points $\mathcal{P}'$ forming a line. Since $\mathcal{P}\backslash\mathcal{P}'$ is also 2-divisible the statement follows from Lemma 50. □

We remark that there exists a second 2-divisible set of six points – a projective base of dimension 5, which clearly cannot be embedded in $\operatorname{PG}(3,2)$.

Lemma 52. *Let $\mathcal{P}$ be a 2-divisible set of eight points in $\operatorname{PG}(3,2)$ then $\mathcal{P}$ is either an affine solid or given by the points of a plane and an intersecting line without the intersection point.*

Algorithm 1 Computing $m_i^{(j)}$

Input: $m_{n-1},\dots,m_1 \in \mathbb{N}_0$ with $\sum_{i=1}^{n-1} m_i \cdot q^{i-1} = q^{n-1}$

Output: $m_i^{(j)} \in \mathbb{N}_0$ with $\sum_{i=1}^{n-2} m_i^{(j)} \cdot q^{i-1} = q^{n-2}$ for all $n_{n-1}+1 \le j \le q$
and $\sum_{j=m_{n-1}+1}^{q} m_i^{(j)} = m_i$ for all $1 \le i \le n-2$

```
h ← n − 2
for m_{n−1} + 1 ≤ j ≤ q do
  r ← q^{n−2}
  while r > 0 do
    t ← min{r/q^{h−1}, m_h}
    m_h ← m_h − t
    r ← r − t · q^{h−1}
    if t = 0 then
      h ← h − 1
    end if
  end while
end for
return m_i^{(j)}
```

Proof. Assume that π is a hyperplane, which is a plane in our situation, containing six of the eight points and denote the unique uncovered point of π by P. Each hyperplane that is incident with P contains either two or six of the points in π. Thus, the remaining two points form a line L containing P. Clearly, there is a unique example up to symmetry. Otherwise each hyperplane contains either 0, 2, or 4 points, so that the standard equations yield that there is a unique empty hyperplane and all other hyperplanes contain exactly four points, i.e., the point set is given by an affine solid. □

We remark that both point sets can also be described as unions of two 2-divisible point sets, i.e., the union of two affine planes in the first case and the union of a line and a projective basis of size five in the second case.

C Tight irreducible affine vector space partitions of minimum size that can be obtained by compression

In Subsection 5.3 we have shown how avsps of $\mathrm{PG}(n-1,2)$ can be obtained from hitting formulas by compression. In [PS22] irreducible hitting formulas of minimum possible mentioning all variables where enumerated up to seven variables. Going over their list we obtain the following examples of tight irreducible avsps that can be obtained by compression and that have the minimum possible size $\sigma_2(n)$, see Section 6. The pairs of strings that can be compressed to an affine subspace are separated by horizontal lines.

Examples for $n = 5$:

00∗∗	00∗∗
1∗0∗	1∗0∗
010∗	01∗0
1∗10	1∗10
$\underline{\ast 111}$	$\underline{\ast 111}$
0110	0101
1011	1011

Examples for $n = 6$:

00∗∗∗	00∗∗∗	00∗∗∗
100∗∗	1∗00∗	100∗∗
∗100∗	1∗1∗0	01∗0∗
1∗1∗0	∗101∗	1∗1∗0
$\underline{\ast 1\ast 11}$	$\underline{\ast 11\ast 1}$	$\underline{\ast 1\ast 11}$
101∗1	0100∗	01∗10
$\underline{011\ast 0}$	$\underline{1001\ast}$	$\underline{11\ast 01}$
∗1101	011∗0	101∗1
∗1010	101∗1	110∗0

For $n = 7$ there is a unique example:

000∗∗∗
10∗0∗∗
∗1∗∗00
∗∗111∗
$\underline{\ast 10\ast\ast 1}$
∗0110∗
$\underline{\ast 1101\ast}$
∗11∗01
$\underline{\ast 10\ast 10}$
0010∗∗
1001∗∗

For $n = 8$, there are 26 irreducible hitting formulas of size 13 mentioning all $n-1 = 7$ variables. Curiously enough, compression was always successful. Moreover, we can also obtain tight irreducible avsps of $\mathrm{PG}(7,2)$ of minimum size $\sigma_2(8) = 10$ by compression starting from an irreducible hitting formulas with strictly more than 13 terms:

$***1**0$	$*10****$	$1*10*1*$
$***10*1$	$*000*0*$	$0*11**0$
$00*0*1*$	$100**1*$	$111**0*$
$0*00*0*$	$00*11**$	$\underline{001***1}$
$1***1*1$	$\underline{1*100**}$	$0010**0$
$\underline{*110**0}$	$1001*0*$	$\underline{0111**1}$
$*010*00$	$\underline{0000*1*}$	$110*1**$
$\underline{*100*10}$	$0*10**1$	$\underline{000*0**}$
$0**11*1$	$\underline{1*11**0}$	$01*0***$
$\underline{1**00*1}$	$0*1*0*0$	$\underline{10*1***}$
$1*00*00$	$\underline{1*1*1*1}$	$0*011**$
$\underline{0*10*01}$	$01111**$	$\underline{1*000**}$
$10*0*10$	$\underline{00010**}$	$1010*0*$
$01*0*11$	$**101*0$	$\underline{1111*1*}$
	$**110*1$	$*0001**$
		$*1010**$

References

[BBC+18] John Bamberg, Anton Betten, Philippe Cara, Jan De Beule, Michel Lavrauw, and Max Neunhöffer. *FinInG – Finite Incidence Geometry, Version 1.4.1*, 2018. http://www.fining.org.

[Beu75] Albrecht Beutelspacher. Partial spreads in finite projective spaces and partial designs. *Mathematische Zeitschrift*, 145(3):211–229, 1975.

[DBDMS19] Jan De Beule, Jeroen Demeyer, Sam Mattheus, and Péter Sziklai. On the cylinder conjecture. *Designs, Codes and Cryptography*, 87(4):879–893, 2019.

[DDKB98] Gennady Davydov, Inna Davydova, and Hans Kleine Büning. An efficient algorithm for the minimal unsatisfiability problem for a subclass of CNF. *Ann. Math. Artificial Intelligence*, 23(3-4):229–245, 1998.

[EZHS+10] Saad Ibrahim El-Zanati, Olof Heden, George Francis Seelinger, Papa Amar Sissokho, Lawrence Edward Spence, and Charles Vanden Eynden. Partitions of the 8-dimensional vector space over GF(2). *Journal of Combinatorial Designs*, 18(6):462–474, 2010.

[EZSS+09] Saad Ibrahim El-Zanati, George Francis Seelinger, Papa Amar Sissokho, Lawrence Edward Spence, and Charles Vanden Eynden. On partitions of finite vector spaces of low dimension over GF(2). *Discrete Mathematics*, 309(14):4727–4735, 2009.

[EZSS+11] Saad Ibrahim El-Zanati, George Francis Seelinger, Papa Amar Sissokho, Lawrence Edward Spence, and Charles Vanden Eynden. On λ-fold partitions of finite vector spaces and duality. *Discrete Mathematics*, 311(4):307–318, 2011.

[Hed86] Olof Heden. Partitions of finite abelian groups. *European Journal of Combinatorics*, 7(1):11–25, 1986.

[Hed09] Olof Heden. On the length of the tail of a vector space partition. *Discrete Mathematics*, 309(21):6169–6180, 2009.

[Hed12] Olof Heden. A survey of the different types of vector space partitions. *Discrete Mathematics, Algorithms and Applications*, 4(01):1250001, 2012.

[HHK$^+$17] Daniel Heinlein, Thomas Honold, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. Projective divisible binary codes. In *The Tenth International Workshop on Coding and Cryptography 2017 : WCC Proceedings*. IEEE Information Theory Society, Saint-Petersburg, September 2017. URL: `https://eref.uni-bayreuth.de/40887/`.

[HHKK19] Daniel Heinlein, Thomas Honold, Michael Kiermaier, and Sascha Kurz. Generalized vector space partitions. *Australasian Journal of Combinatorics*, 73(1):162–178, 2019.

[HKK18] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Partial spreads and vector space partitions. In *Network Coding and Subspace Designs*, pages 131–170. Springer, 2018.

[HKK19] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Classification of large partial plane spreads in $\mathrm{PG}(6, 2)$ and related combinatorial objects. *Journal of Geometry*, 110(1):1–31, 2019.

[Iwa89] Kazuo Iwama. CNF-satisfiability test by counting and polynomial average time. *SIAM J. Comput.*, 18(2):385–391, 1989.

[KK20] Michael Kiermaier and Sascha Kurz. On the lengths of divisible codes. *IEEE Transactions on Information Theory*, 66(7):4051–4060, 2020.

[KM21] Sascha Kurz and Sam Mattheus. A generalization of the cylinder conjecture for divisible codes. *IEEE Transactions on Information Theory*, 68(4):2281–2289, 2021.

[KÖ06] Petteri Kaski and Patric RJ Östergård. *Classification algorithms for codes and designs*, volume 15. Springer, 2006.

[Kur18] Sascha Kurz. Heden's bound on the tail of a vector space partition. *Discrete Mathematics*, 341(12):3447–3452, 2018.

[Kur21] Sascha Kurz. Divisible codes. *arXiv preprint 2112.11763*, 2021.

[Lin04] Steve Linton. Finding the smallest image of a set. In *Proceedings of the 2004 international symposium on Symbolic and algebraic computation*, pages 229–234, 2004.

[MT09] Zlatka T. Mateva and Svetlana T. Topalova. Line spreads of $\mathrm{PG}(5, 2)$. *Journal of Combinatorial Designs*, 17(1):90–102, 2009.

[NS11] Esmeralda L. Năstase and Papa Amar Sissokho. The minimum size of a finite subspace partition. *Linear Algebra and its Applications*, 435(6):1213–1221, 2011.

[PS22] Tomáš Peitl and Stefan Szeider. Are hitting formulas hard for resolution?, 2022. URL: `https://arxiv.org/abs/2206.15225`.

[SSW19] John Sheekey, Kai-Uwe Schmidt, and Arne Winterhof. MRD codes: constructions and connections. In *Combinatorics and finite fields: Difference sets, polynomials, pseudorandomness and applications*, volume 23, pages 255–286. de Gruyter, 2019.

[TZH14] Ziran Tu, Xiangyong Zeng, and Lei Hu. Several classes of complete permutation polynomials. *Finite Fields Appl.*, 25:182–193, 2014. `doi:10.1016/j.ffa.2013.09.007`.