# LOTSIZE OPTIMIZATION LEADING TO A p-MEDIAN PROBLEM WITH CARDINALITIES

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ABSTRACT. We consider the problem of approximating the branch and size dependent demand of a fashion discounter with many branches by a distributing process being based on the branch delivery restricted to integral multiples of lots from a small set of available lot-types. We propose a formalized model which arises from a practical cooperation with an industry partner. Besides an integer linear programming formulation and a primal heuristic for this problem we also consider a more abstract version which we relate to several other classical optimization problems like the p-median problem, the facility location problem or the matching problem.

## 1. INTRODUCTION

Usually, fashion discounters can only achieve small profit margins. Their economic success depends mostly in the ability to meet the customers' demands for individual products. More specifically: offer exactly what you can sell to your customers. This task has two aspects: offer what the customers would like to wear (attractive products) and offer the right volumes in the right places and the right sizes (demand consistent branch and size distribution).

In this paper we deal with the second aspect only: meet the branch and size specific demand for products as closely as possible. Our industry partner is a fashion discounter with more than 1 000 branches most of whose products are never replenished, except for the very few "never-out-of-stock"-products (NOS products): because of lead times of around three months, apparel replenishments would be too late anyway. In most cases the supplied items per product and apparel size lie in the range between 1 and 6. Clearly there are some difficulties to determine a good estimate for the branch and size dependent demand, but besides a few practical comments on this problem we will blind out this aspect of the problem completely.

The problem we deal with in this article comes from another direction. Our business partner is a discounter who has a lot of pressure to reduce its costs. So he is forced to have a lean distribution logistics that works efficiently. Due to this reason he, on the one hand, never replenishes and, on the other hand, tries to reduce the distribution complexity. To achieve this goal the supply of the branches is based on the delivery of lots, i.e., pre-packed assortments of single products in various sizes. Every branch can only be supplied with an integral multiple of one lot-type from a rather small number of available lot-types. So he has to face an approximation problem: which (integral) multiples of which (integral) lot-types should be supplied to a branch in order to meet a (fractional) mean demand as closely as possible?

We call this specific demand approximation problem the *lot-type design problem (LDP)*.

1.1. **Related Work.** The model we suggest for the LDP is closely related to the extensively studied p-median- and the facility location problem. These problems appear in various applications as some kind of clustering problems. Loads of heuristics have been applied onto them. Nevertheless the first constant-factor approximation algorithm, based on LP rounding, was given not until 1999 by Charikar,

Guha, Tardos, and Shmoys [5]. We will give some more detailed treatment or literature of approximation algorithms and heuristics for the p-median- and the facility location problem in Subsection 4.1.

1.2. **Our contribution.** In cooperation with our business partner, we identified the lot-type design problem as a pressing real-world task. We present an integer linear program (ILP) formulation of the LDE that looks abstractly like a p-median problem with an additional cardinality constraint. We call this problem the *cardinality constrained p-median problem (Card-p-MP)*. To the best of our knowledge, the Card-p-MP has not been studied in the literature so far.

Although the ILP model can be solved by standard software on a state-of-the-art PC in reasonable time, the computation times are prohibitive for the use in the field, where interactive decision support on a laptop is a must for negotiations with the supplier. Therefore, we present a very fast primal any-time heuristics, that yields good solutions almost instantly and searches for improvements as long as it is kept running. We demonstrate on real data that the optimality gaps of our heuristics are mostly way below 1 %. At the moment these heuristics are in test mode.

1.3. **Outline of the paper.** In Section 2 we will briefly describe the real world problem, which we will formalize and model in Section 3. In Section 4 we will present its abstract version, the *cardinality constrained p-median problem (Card-p-MP)*. Besides a formalized description we relate it to several other well known optimization problems like the matching problem, the facility location problem, or the p-median problem. In Section 5 we present a primal heuristic for the Card-p-MP, which we apply onto our real world problem. We give some numerical data on the optimality gap of our heuristic before we draw a conclusion in Section 6.

## 2. THE REAL WORLD PROBLEM

Our industry partner is a fashion discounter with over 1 000 branches. Products can not be replenished, and the number of sold items per product and branch is rather small. There are no historic sales data for a specific product available, since every product is sold only for one sales period. The challenge for our industry partner is to determine a suitable total amount of items of a specific product which should be bought from the supplier. For this part the knowledge and experience of the buyers employed by a fashion discounter is used. We seriously doubt that a software package based on historic sales data can do better.

But there is another task being more accessible for computer aided forecasting methods. Once the total amount of sellable items of a specific product is determined, one has to decide how to distribute this total amount to a set of branches B in certain apparel sizes with in general different demands. There are some standard techniques how to estimate branch- and size-dependent demand from historic sales data of related products, being, e.g., in the same commodity group. We will address the problem of demand forecasting very briefly in Subsection 3.1. But let us assume for simplicity that we know the exact (fractional) branch and size dependent mean demands for a given new product or have at least good estimates.

Due to cost reasons, our industry partner organizes his distribution process for the branches using a central warehouse. To reduce the number of necessary handholds in the distributing process he utilizes the concept of lots, by which we understand a collection of some items of one product. One could have in mind different sizes or different colors at this point. To reduce the complexity of the distribution process also the number of used lot-types, e.g., different collections of items, is limited to a rather small number.

One could imagine that the branch- and size-dependent demand for a specific product may vary broadly over the large set of branches. This is at least the case for the branches of our industry partner. The only flexibility to satisfy the demand in each single branch is to choose a suitable lot-type from the small sets of available lot-types and to choose a suitable multiplier, i.e., how many lots of a chosen lot-type a specific branch should get. One should keep in mind that we are talking about small multipliers here, i.e., small branches will receive only one lot, medium sized branches will receive two lots, and very big branches will receive three lots of a lot-type with, say, six items.

The cost reductions by using this lot-based distribution system are paid with a lack of possibility to approximate the branch and size-dependent demand. So one question is, how many different lot-types one should allow in order to be able to approximate the branch- and size-dependent demand of the branches up to an acceptable deviation on the one hand and to avoid a complex and cost intensive distribution process in the central warehouse on the other hand. But also for a fixed number of allowed lot-types the question of the best possible approximation of the demand by using a lot-based supply of the branches arises. In other words we are searching for an optimal assignment of branches to lot-types together with corresponding multipliers so that the deviation between the theoretical estimated demand and the planned supply with lots is minimal. This is the main question we will focus on in this paper.

## 3. MATHEMATICAL MODELING OF THE PROBLEM

In this section we will prescind the real world problem from the previous section and will develop an formulation as a well defined optimization problem. Crucial and very basic objects for our considerations are the set of branches $\mathcal{B}$, the set of sizes $\mathcal{S}$ (in a more general context one could also think of a set of variants of a product, like, e.g., different colors), and the set of products $\mathcal{P}$.

In practice, we may want to sell a given product $p \in \mathcal{P}$ only in some branches $\mathcal{B}_p \subseteq \mathcal{B}$ and only in some sizes $\mathcal{S}_p \subseteq \mathcal{S}$ (clearly there are different types of sizes for, e.g., skirts or socks). To model the demand of a given branch $b \in \mathcal{B}_p$ for a given product $p \in \mathcal{P}$ we use the symbol $\eta_{b,p}$, by which we understand a mapping $\varphi_{b,p}$ from the set of sizes $\mathcal{S}_p$ into a suitable mathematical object. This object may be a random variable or simply a real number representing the mean demand. In this paper we choose the latter possibility. For the sake of a brief notation we regard $\eta_{b,p}$ as a vector $\left( \varphi_{b,p}\left(s_{i_1}\right) \quad \varphi_{b,p}\left(s_{i_2}\right) \quad \ldots \quad \varphi_{b,p}\left(s_{i_r}\right) \right) \in \mathbb{R}^r$, where we assume that $\mathcal{S} = \{s_1, \ldots, s_t\}$ and $\mathcal{S}_p = \{s_{i_1}, \ldots, s_{i_r}\}$ with $i_j < i_{j+1}$ for all $j \in \{1, \ldots, r-1\}$.

3.1. **Estimation of the branch- and size-dependent demand.** For the purpose of this paper, we may assume that the demands $\eta_{b,p}$ are given, but, since this is a very critical part in practice, we would like to mention some methods how to obtain these numbers. Marketing research might be a possible source. Another possibility to estimate the demand for a product is to utilize historic sales information. We may assume that for each product $p$ which was formerly sold by our retailer, each branch $b \in \mathcal{B}$, each size $s \in \mathcal{S}$ and each day of sales $d$ we know the number $\tau_{b,p}(d, s)$ of items which where sold in branch $b$ of product $p$ in size $s$ during the first $d$ days of sales. Additionally we assume, that we have a set $\mathcal{U} \subseteq \mathcal{P}$ of formerly sold products which are in some sense similar (one might think of the set of jeans if our new product is also a jeans) to the new product $\tilde{p}$. By $\mathcal{U}_{b,s}$ we denote the subset of products in $\mathcal{U}$, which were traded by a positive amount in size $s$ in branch $b$ and by $\chi_{b,s}(p)$ we denote a characteristic function which equals $1$ if product $p$ is distributed in size $s$ to branch $b$, and equals $0$ otherwise. For a given day of sales $d$ the value

$$\tilde{\eta}_{b,\tilde{p},d}(s) := \frac{c}{|\mathcal{U}_{b,s}|} \sum_{u \in \mathcal{U}_{b,s}} \frac{\tau_{b,u}(d, s) \cdot \sum\limits_{b' \in \mathcal{B}_{\tilde{p}}} \sum\limits_{s' \in \mathcal{S}_{\tilde{p}}} \chi_{b',s'}(u)}{\sum\limits_{b' \in \mathcal{B}_{\tilde{p}}} \sum\limits_{s' \in \mathcal{S}_{\tilde{p}}} \tau_{b',u}(d, s')} \tag{1}$$

might be a useable estimate for the demand $\eta_{b,\tilde{p}}(s)$, after choosing a suitable scaling factor $c \in \mathbb{R}$ so that the total estimate demand

$$\sum_{b \in \mathcal{B}_{\tilde{p}}} \sum_{s \in \mathcal{S}_{\tilde{p}}} \tilde{\eta}_{b,\tilde{p},d}(s)$$

over all branches and sizes equals the total requirements. We would like to remark that for small days of sale $d$ the quality of the estimate $\tilde{\eta}_{b,\tilde{p},d}(s)$ suffers from the fact that the stochastic noise of the consumer behavior is to dominating and for large $d$ the quality of the estimate suffers from the fact of stockout-substitution.

There are parametric approaches to this problem in the literature (like Poisson-type sales processes). In the data that was available to us, we could not verify the main assumptions of such models, though (not even close).

In our real world data set we have observed the fact that the sales period of a product (say, the time by which 80 % of the supply is sold) varies a lot depending on the product. This effect is due to the attractiveness of a given product (one might think of two T-shirts which only differ in there color, where one color hits the vogue and the other color does not). To compensate this effect we have chosen the day of sales $d$ in dependence of the product $u \in \mathcal{U}_{b,s}$. More precisely, we have chosen $d_u$ so that in the first $d_u$ days of sales a certain percentage of all items of product $u$ where sold out over all branches and sizes.

Another possibility to estimate the demand is to perform the estimation for the branch-dependent demand aggregated over all sizes and the size-dependent demand for a given branch separately.

More sophisticated methods of demand estimation from historic sales based on small data sets are, e.g., described in [19, 20]. Also research results from forecasting NOS (never-out-of-stock) items, see, e.g., [1, 17, 24] for some surveys, may be utilized. Also quite a lot of software-packages for demand forecasting a available, see [31] for an overview.

3.2. **Supply of the branches by lots.** To reduce handling costs in logistic and stockkeeping our business partner orders his products from its external suppliers in so called lots. These are assortments of several items of one product in different sizes which form an entity. One could have a set of T-shirts in different sizes in mind which are wrapped round by a plastic foil. The usage of lots has the great advantage of reducing the number of picks during the distribution process in a high-wage country like Germany, where our partner operates.

Let us assume that the set of sizes for a given product $p$ is given by $\mathcal{S}_p = \{s_{i_1}, \ldots, s_{i_r}\}$ with $i_j < i_{j+1}$ for all $j \in \{1, \ldots, r-1\}$. By a lot-type $l$ we understand a mapping $\varphi : \mathcal{S}_p \to \mathbb{N}$, which can also be denoted by a vector $\begin{pmatrix} \varphi(s_{i_1}) & \varphi(s_{i_2}) & \ldots & \varphi(s_{i_r}) \end{pmatrix}$ of non-negative integers.

By $\mathcal{L}$ we denote the set of applicatory lot-types. One could imagine that a lot of a certain lot-type should not contain too many items in order to be manageable. In the other direction it should also not contain too few items in order to make use of the cost reduction potential of the lot idea. Since the set of applicatory lot-types may depend on a the characteristics of a certain product $p$ we specialize this definition to a set $\mathcal{L}_p$ of manageable lot-types. (One might imagine that a warehouseman can handle more T-shirts than, e.g., winter coats; another effect that can be modeled by a suitable set of lot-types is to enforce that each size in $\mathcal{S}_p$ is supplied to each branch in $\mathcal{B}_p$ by a positive amount due to juridical requirements for advertised products.)

To reduce the complexity and the error-proneness of the distribution process in a central warehouse, each branch $b \in \mathcal{B}_p$ is supplied only with lots of one lot-type $l_{b,p} \in \mathcal{L}_p$. We model the assignment of lot-types $l \in \mathcal{L}_p$ to branches $b \in \mathcal{B}_p$ as a function $\omega_p : \mathcal{B}_p \to \mathcal{L}_p$, $b \mapsto l_{b,p}$. Clearly, this assignment $\omega_p$ is a decision variable which can be used to optimize some target function. The only flexibility that we have to approximate the branch-, size- and product dependent demand $\eta_{b,p}$ by our delivery in lots is to supply an integral multiple of $m_{b,p}$ items of lot-type $\omega_p(b)$ to branch $b$. Again, we can denote this connection by a function $m_p : \mathcal{B}_p \to \mathbb{N}$, $b \mapsto m_{b,p}$. Due to practical reasons, also the total number $|\omega_p(\mathcal{B}_p)|$ of used lot-types for a given product is limited by a certain number $\kappa$.

3.3. **Deviation between supply and demand.** With the notation from the previous subsection, we can represent the replant supply for branch $b$ with product $p$ as a vector $m_p(b) \cdot \omega_p(b) \in \mathbb{N}^r$. To measure the deviation between the supply $m_p(b) \cdot \omega_p(b)$ and the demand $\eta_{b,p}$ we may utilize an arbitrary vector norm $\| \cdot \|$. Mentionable vector norms in our context are the sum of absolute values

$$\| \begin{pmatrix} v_1 & v_2 & \ldots & v_r \end{pmatrix} \|_1 := \sum_{i=1}^{r} |v_i|,$$

the maximum norm

$$\| \begin{pmatrix} \nu_1 & \nu_2 & \dots & \nu_r \end{pmatrix} \|_\infty := \max\{|\nu_i| : 1 \leqslant i \leqslant r\},$$

and the general p-norm

$$\| \begin{pmatrix} \nu_1 & \nu_2 & \dots & \nu_r \end{pmatrix} \|_p := \sqrt[p]{\sum_{i=1}^{r} |\nu_i|^p}$$

for real numbers $p > 0$, which is also called the Euclidean norm for $p = 2$. With this we can define the deviation

$$\sigma_{b,l,m} := \|\eta_{b,p} - m \cdot l\|_\star$$

between demand $\eta_{b,p}$ and supply $m \in \{1, \dots, M\} =: \mathcal{M} \subset \mathbb{N}$ times lot-type $l \in \mathcal{L}_p$ for each branch $b \in \mathcal{B}_p$ and an arbitrary norm $\| \cdot \|_\star$ for a given product $p \in \mathcal{P}$. It depends on practical considerations which norm to choose. The $\| \cdot \|_1$-norm is very insensitive in respect to outliers in contrast to the $\| \cdot \|_\infty$-norm which is absolutely sensitive with respect to outliers. A possible compromise may be the Euclidean norm $\| \cdot \|_2$, but for most considerations we choose the $\| \cdot \|_1$-norm because of its robustness. (We do not trust every single exact value in our demand forecasts that much.)

For given functions $m_p$ and $\omega_p$ we can consider the deviation vector

$$\Sigma_p := \begin{pmatrix} \sigma_{b_1, \omega_p(b_1), m_p(b_1)} & \sigma_{b_2, \omega_p(b_2), m_p(b_2)} & \cdots & \sigma_{b_q, \omega_p(b_q), m_p(b_q)} \end{pmatrix}$$

if the set of branches is written as $\mathcal{B}_p := \{b_1, \dots, b_q\}$. To measure the total deviation of supply and demand we can apply an arbitrary norm $\| \cdot \|_\star$, which may be different from the norm to measure the deviation of a branch, onto $\Sigma_p$. In this paper we restrict ourselves on the $\| \cdot \|_1$-norm, so that we have

$$\|\Sigma_p\|_1 = \sum_{b \in \mathcal{B}_p} \sigma_{b, \omega_p(b), m_p(b)}.$$

3.4. **The cardinality condition.** For a given assignment $\omega_p$ of lot-types to branches and corresponding multiplicities $m_p$ then quantity

$$I := \sum_{b \in \mathcal{B}_p} m_p(b) \cdot \|\omega_p(b)\|_1 \quad \in \mathbb{N}$$

gives the total number of replant distributed items of product $p$ over all sizes and branches. From a practical point of view we introduce the condition

$$\underline{I} \leqslant I \leqslant \overline{I}, \tag{2}$$

where $\underline{I}, \overline{I}$ are suitable integers. One might imagine that our retailer may buy a part of already produced products so that there is a natural upper bound $\overline{I}$ or that there are some minimum quantities. Another interpretation may be that the buying department of our retailer has a certain idea on the value of $I$ but is only able to give an interval $[\underline{I}, \overline{I}]$.

During our cooperation with our busines partner we have learned that in practice you do not get what you order. If you order exactly $I$ items of a given product you will obtain $I$ plus minus some certain percentage items in the end. (And their actually exists a certain percentage up to which a retailer accepts a deviation between the original order and the final delivery by its external suppliers as a fulfilled contract.)

Besides these and other practical reason to consider an interval $[\underline{I}, \overline{I}]$ for the total number of items of a given product, there are very strong reasons not to replace Inequalities (2) by an equation, as we will explain in the following. Let us consider the case where our warehouse (or our external suppliers in a low-cost-country) is only able to deal with a single lot-type per product. This is the case $\kappa = 1$. Let us further assume that there exists a rather small integer $k$ (e.g. $k = 20$) fulfilling $\|l\|_1 \leqslant k$ for all $l \in \mathcal{L}_p$. If $I$ contains a prime divisor being larger than $k$, then there exist no assignments multiplicities $m_p \in \mathbb{N}$ ($\omega_p$ is a constant function due to $\kappa = 1$) which lead to a feasible solution of our problem. These number-theoretic influences are somewhat ugly. In some cases the lead to the infeasibility of our problem or to

bad solutions with respect to the quality of the demand-supply approximation in comparison to a relaxed version of the problem, where the restrictions on I are weaker. One could have in mind the possibility of throwing one item into the garbage if this will have a large impact on the quality of the demand-supply approximation.

In Equation (1) for the demand estimation we have used a certain number $\tilde{I}$ for the total number of items to scale the demands $\eta_{b,p}$ by a factor c. From a more general point of view it may also happen that the total demand

$$\sum_{b \in \mathcal{B}_p} \sum_{s \in \mathcal{S}_p} \eta_{b,p}(s)$$

is not contained in the interval $\left[\underline{I}, \overline{I}\right]$. In this case the $\|\cdot\|_1$-norm may not be very appropriate. In our estimation process, however, the demand forecasts in fact yield demand percentages rather than absolute numbers. The total volume is then used to calculate the absolute (fractional) mean demand values, so that in our work-flow the total demand is always in the target interval.

3.5. **The optimization problem.** Summarizing the ideas and using the notations from the previous subsections we can formulate our optimization problem in the following form. We want to determine an assignment function $\omega_p : \mathcal{B}_p \to \mathcal{L}_p$ and multiplicities $m_p : \mathcal{B}_p \to \mathcal{M} = \{1, \dots, M\} \subset \mathbb{N}$ such that the total deviation between supply and demand

$$\sum_{b \in \mathcal{B}_p} \sigma_{b, \omega_p(b), m_p(b)} \tag{3}$$

is minimized with respect to the conditions

$$|\omega_p(\mathcal{B}_p)| \leqslant \kappa \tag{4}$$

and

$$\underline{I} \leqslant \sum_{b \in \mathcal{B}_p} m_p(b) \cdot \|\omega_p(b)\|_1 \leqslant \overline{I}. \tag{5}$$

We use binary variables $x_{b,l,m}$, which are equal to 1 if and only if lot-type $l \in \mathcal{L}_p$ is delivered with multiplicity $m \in \mathcal{M}$ to Branch b, and binary variables $y_l$, which are 1 if and only if at least one branch in $\mathcal{B}_p$ is supplied with Lottype $l \in \mathcal{L}_p$. With this, we can easily model out problem as an integer linear program:

$$\min \quad \sum_{b \in \mathcal{B}_p} \sum_{l \in \mathcal{L}_p} \sum_{m \in \mathcal{M}} \sigma_{b,l,m} \cdot x_{b,l,m} \tag{6}$$

$$\text{s.t.} \quad \sum_{l \in \mathcal{L}_p} \sum_{m \in \mathcal{M}} x_{b,l,m} = 1 \qquad \forall b \in \mathcal{B}_p \tag{7}$$

$$\sum_{b \in \mathcal{B}_p} \sum_{l \in \mathcal{L}_p} \sum_{m \in \mathcal{M}} m \cdot \|l\|_1 \cdot x_{b,l,m} \leqslant \overline{I} \tag{8}$$

$$\sum_{b \in \lfloor_p} \sum_{l \in \mathcal{L}_p} \sum_{m \in \mathcal{M}} m \cdot \|l\|_1 \cdot x_{b,l,m} \geqslant \underline{I} \tag{9}$$

$$\sum_{m \in \mathcal{M}} x_{b,l,m} \leqslant y_l \qquad \forall b \in \mathcal{B}_p \, \forall l \in \mathcal{L}_p \tag{10}$$

$$\sum_{l \in \mathcal{L}_p} y_l \leqslant \kappa \tag{11}$$

$$x_{b,l,m} \in \{0, 1\} \qquad \forall b \in \mathcal{B}_p \, \forall l \in \mathcal{L}_p \, \forall m \in \mathcal{M} \tag{12}$$

$$y_l \in \{0, 1\} \qquad \forall l \in \mathcal{L}_p \tag{13}$$

The objective function (6) represents the sum (3), since irrelevant tuples $(b, l, m)$ may be downtrodened by $x_{b,l,m} = 0$. Condition (7) states that we assign for each Branch $b$ exactly one lot-type with a unique multiplicity. The cardinality condition (5) is modeled by Conditions (8) and (9) and the restriction (4) on the number of used lot-types is modeled by Condition (11). The connection between the $x_{b,l,m}$ and the $y_l$ is fixed in the usual Big-M condition (10). We would like to remark that the LP-relaxation of this ILP formulation is very strong above all in comparison to the more direct ILP formulation, where we assume the branch deviation between supply and demand is measured by the $\| \cdot \|_1$-norm:

$$\min \quad \sum_{b \in \mathcal{B}_p} \sum_{s \in \mathcal{S}_p} z_{b,s}$$

$$\text{s.t.} \quad \eta_{b,p}(s) - \alpha_{b,s} \leqslant z_{b,s} \qquad \forall b \in \mathcal{B}_p \; \forall s \in \mathcal{S}_p$$

$$\alpha_{b,s} - \eta_{b,p}(s) \leqslant z_{b,s} \qquad \forall b \in \mathcal{B}_p \; \forall s \in \mathcal{S}_p$$

$$\sum_{l \in \mathcal{L}_p} \sum_{m \in \mathcal{M}} x_{b,l,m} = 1 \qquad \forall b \in \mathcal{B}_p$$

$$\sum_{b \in \mathcal{B}_p} \sum_{l \in \mathcal{L}_p} \sum_{m \in \mathcal{M}} m \cdot \|l\|_1 \cdot x_{b,l,m} \leqslant \bar{I}$$

$$\sum_{b \in \mathcal{B}_p} \sum_{l \in \mathcal{L}_p} \sum_{m \in \mathcal{M}} m \cdot \|l\|_1 \cdot x_{b,l,m} \geqslant \underline{I}$$

$$\sum_{m \in \mathcal{M}} x_{b,l,m} \leqslant y_l \qquad \forall b \in \mathcal{B}_p \; \forall l \in \mathcal{L}_p$$

$$\sum_{l \in \mathcal{L}_p} y_l \leqslant \kappa$$

$$\sum_{l \in \mathcal{L}_p} \sum_{m \in \mathcal{M}} m \cdot l[s] \cdot x_{b,l,m} = \alpha_{b,s} \qquad \forall b \in \mathcal{B}_p \; \forall s \in \mathcal{S}_p$$

$$x_{b,l,m} \in \{0, 1\} \qquad \forall b \in \mathcal{B}_p \; \forall l \in \mathcal{L}_p \; \forall m \in \mathcal{M}$$

$$y_l \in \{0, 1\} \qquad \forall l \in \mathcal{L}_p$$

$$\alpha_{b,s} \in \mathbb{R}_0^+ \qquad \forall b \in \mathcal{B}_p \; \forall s \in \mathcal{S}_p,$$

where $l[s]$ is the entry in Vector $l$ corresponding to Size $s$.

We would like to remark that our strong ILP formulation of the problem of Subsection 3.5 can be used to solve all real world instances of our business partner in at most 30 minutes by using a standard ILP solver like CPLEX 11. Unfortunately, this is not fast enough for our real world application. The buyers of our retailer need a software tool which can produce a near optimal order recommendation in real time on a standard laptop. The buying staff travels to one of the external suppliers to negotiate several orderings. When they get to the details, the buyer inserts some key data like $\underline{I}$, $\bar{I}$, $\mathcal{B}_p$, $\mathcal{S}_p$, and $\mathcal{L}_p$ into his laptop and immediately wants a recommendation for an order in terms of multiples of lot-types. For this reason, we consider in Section 5 a fast heuristic, which has only a small gap compared to the optimal solution on a test set of real world data of our business partner.

## 4. The Cardinality Constrained p-Median Problem

In the previous section we have modeled our real world problem from Section 2. Now we want to abstract from this practical problem and formulate a more general optimization problem which we will relate to several well known optimization problems.

For the general Cardinality Constrained p-Median Problem let $p$ be an integer, $\mathcal{S}$ a set of chooseable items, $\mathcal{D}$ a set of demanders, a demand function $\delta : \mathcal{D} \to \mathbb{R}^+$, and $[\underline{I}, \bar{I}] \subseteq \mathbb{N}$ an interval. We are looking

for an assignment $\omega : \mathcal{D} \to \mathcal{S}$ with corresponding multipliers $\mathfrak{m} : \mathcal{D} \to \mathbb{N}$, such that the sum of distances

$$\sum_{d \in \mathcal{D}} \|\delta(d) - \mathfrak{m}(d) \cdot \omega(d)\|$$

is minimized under the conditions

$$|\omega(\mathcal{D})| \leqslant p$$

and

$$\underline{I} \quad \leqslant \quad \sum_{d \in \mathcal{D}} \mathfrak{m}(d) \cdot |\omega(d)| \quad \leqslant \quad \bar{I}.$$

Let us now bring this new optimization problem in line with known combinatorial optimizations problems. Since we have to choose an optimal subset of $\mathcal{S}$ to minimize a cost function subject to some constraints the cardinality constrained $p$-median problem belongs to the large class of generic selection problems.

Clearly, it is closely related to the $p$-median problem. The only characteristics of our problem that are not covered by the $p$-median problem are the multipliers $\mathfrak{m}$ and the cardinality condition. If we relax the cardinality condition we can easily transform our problem into a classical $p$-median problem. For every element $d \in \mathcal{D}$ and every element $s \in \mathcal{S}$ there exists an optimal multiplier $\mathfrak{m}_{d,s}$ such that $\|\delta(d) - \mathfrak{m}_{d,s} \cdot s\|$ is minimal.

If we do not bound $|\omega(\mathcal{D})|$ from above but assign costs for using elements of $\mathcal{S}$ instead, which means using another lot-type in our practical application, we end up with the facility location problem. Clearly we also have some kind of an assignment-problem, since the have to determine an assignment $\omega$ between the sets $\mathcal{D}$ and a subset of $\mathcal{S}$.

One can also look at our problem from a completely different angle. Actually we are given a set of $|\mathcal{B}|$ real-valued demand-vectors, which we want to approximate by a finite number of integer-valued vectors using integral multiples. There is a well established theory in number theory on so called Diophantine approximation [4, 21] or simultaneous approximation, which is somewhat related to our approximation problem. Here one is interested in simultaneously minimizing

$$\left\| \alpha_i - \frac{p_i}{q} \right\|$$

for linearly independent real numbers $\alpha_i$ by integers $p_i$ and $q$ [27, 22]. One might use some results from this theory to derive some bounds for our problem. One might also have a look at [9].

For a more exhaustive and detailed analysis of the taxonomy of the broad field of facility-location problems and their modeling we refer to [26].

4.1. **Approximation algorithms and heuristics for related problems.** Facility location problems and the $p$-median problem are well known and much research has been done. Since, moreover, these problems are closely related to our optimization problem, we would like to mention some literature and methods on approximation algorithms and heuristics for these problems.

Lin and Vitter [23] have developed a filtering and rounding technique which rounds fractional solutions of the standard LP for these problems to obtain good integer solution. For the metric case some some bounds for approximation quality are given. Based on this work some improvements were done in [28], were the authors give a polynomial-time 3.16-approximation algorithm for the metric facility location problem, and [6, 5], where the authors give a polynomial-time $\frac{20}{3}$-approximation algorithm for the metric $p$-median problem and a $9.8$-approximation algorithm for the $p$-facility location problem.

Besides Rounding techniques of LP-solutions also greedy techniques have been applied to the facility location problem and the $p$-median problems. Some results are given in [12, 15, 16]. Since these problems are so prominent in applications the whole broadness of heuristics are applied onto it. Examples are scatter search [10, 8], local search [2, 18], and neighborhood search [11, 14].

Good overviews for the broad topic of approximation algorithms and heuristics for the facility location and the p-median problem are given in [28, 29, 7, 25].

Besides results for the metric case there are also results for the non-metric case, see, e.g., [30].

Unfortunately, none of the theoretical guarantees seems to survive the introduction of the cardinality constraint in general.

## 5. A PRACTICAL HEURISTIC FOR THE CARDINALITY CONSTRAINED p-MEDIAN PROBLEM

As already mentioned in Section 3 solving our ILP formulation of our problem is too slow in practical applications. So there is a real need for a fast heuristic which yields good solutions, which is the top of this section.

In Section 4 we have analyzed our problem from different theoretical point of views. What happens if we relax some conditions or fix some decisions. A very important decision is: which lot-types should be used in the first place? Here one should have in mind that the cardinality $|\mathcal{L}_p|$ of the set of feasible lot-types is very large compared to the number $\kappa$ of lot-types which can be used for the delivery process of a specific product $p$.

### 5.1. **Heuristic selection of lot-types.**
For this selection problem of lot-types we utilize a scoring method. For every branch $b \in \mathcal{B}_p$ with demand $\eta_{b,p}$ there exists a lot-type $l \in \mathcal{L}_p$ and a multiplicity $m \in \mathbb{N}$ such that $\|\eta_{b,p} - m \cdot l\|$ is minimal in the set $\{\|\eta_{b,p} - m' \cdot l'\| : l' \in \mathcal{L}_p, m' \in \mathbb{N}\}$. So for every branch $b \in \mathcal{B}_p$ there exists a lot-type that fits best. More general, for a given $k \leqslant |\mathcal{L}_p|$ there exist lot-types $l_1, \ldots, l_k$ such that $l_i$ fits $i$-best if one uses the corresponding optimal multiplicity. Let us examine this situation from the point of view of the different lot-types. A given lot-type $l \in \mathcal{L}_p$ is the $i$-best fitting lot-type for a number $\rho_{l,i}$ of branches in $\mathcal{B}_p$. Writting these numbers $\rho_{l,i}$ as a vector $\rho_l \in \mathbb{N}^k$ we obtain score vectors for all lot-types $l \in \mathcal{L}_p$.

Now we want to use these score vectors $\rho_l$ to sort the lot-types of $\mathcal{L}_p$ in decreasing *approximation quality*. Using the lexicographic ordering $\preceq$ on vectors we can determine a bijective rank function $\lambda : \mathcal{L}_p \to \{1, \ldots, |\mathcal{L}_p|\}$. (We simply sort the score vectors according to $\preceq$ and for the case of equality we choose an arbitrary succession.) We extend $\lambda$ to subsets $\mathcal{L}' \subseteq \mathcal{L}_p$ by $\lambda(\mathcal{L}') = \sum_{l \in \mathcal{L}'} \lambda(l) \in \mathbb{N}$.

To fix the lot-types we simply loop over subsets $\mathcal{L}' \subseteq \mathcal{L}_p$ of cardinality $\kappa$ in decreasing order with respect to $\lambda(\mathcal{L}')$. In principle we consider all possible selections $\mathcal{L}'$ of $\kappa$ lot-types, but in practise we stop our computations after a adequate time period with the great advantage that we have checked the in some heuristic sense most promising selections $\mathcal{L}'$ first.

Now we have to go into detail how to efficiently determine the $p$ best fitting lot-types with corresponding optimal multiplicities for each branch $b \in \mathcal{B}_p$. We simply loop over all branches $b \in \mathcal{B}_p$ and determine the set of the $p$ best fitting lot-types separately. Here we also simply loop over all lot-types $l \in \mathcal{L}_p$ and determine the corresponding optimal multiplier $m$ by binary search (it is actually very easy to effectively determine lower and upper bounds for $m$ from $\eta_{b,p}$ and $l$) due to the convexity of norm functions. Using a heap data structure the sorting of the $p$ best fitting lot-types can be done in $O(|\mathcal{L}_p|)$ time if $k \log k \in O(|\mathcal{L}_p|)$, which is not a real restriction for practical problems. We further want to remark that we do not have to sort the score vectors completely since in practice we will not loop over all $\binom{|\mathcal{L}_p|}{\kappa}$ possible selections of lot-types. If one does not want to use a priori bounds (meaning that one excludes the lot-types with high rank $\lambda$) one could use a *lazy* or delayed computation of the sorting of $\lambda$ by utilizing again a heap data structure.

### 5.2. **Adjusting a delivery plan to the cardinality condition.**
If we determine assignments $\omega_p$ with corresponding multipliers $m_p$ with the heuristic being described in Subsection 5.1 in many cases we will not satisfy the cardinality condition (2) since it is totally unaccounted by our heuristic. Our strategy to satisfy the cardinality condition (2) is to adjust $m_p$ afterwards by decreasing or increasing the calculated multipliers unless condition (2) is fulfilled by pure chance.

Here we want to use a greed algorithm and have to distinguish two cases. If $I(\omega_p, m_p)$ is smaller then $\underline{I}$, then we increase some of the values of $m_p$, other wise we have $I(\omega_p, m_p) > \bar{I}$ and we decrease some of the values of $m_p$. Our procedure works iteratively and we assume that the current multipliers are given by $\widetilde{m}_p$. Our stopping criteria is given by $\underline{I} \leqslant I(\omega_p, \widetilde{m}_p) \leqslant \bar{I}$ or that there are no feasible operations left. We restrict our explanation of a step of the iteration to the case where we want to decrease the values of $\widetilde{m}_p$. For every branch $b \in \mathcal{B}_p$ the reduction of $\widetilde{m}_p(b)$ by one produces costs

$$\Delta_b^- = \sigma_{b, \omega_p(b), \widetilde{m}_p(b)-1} - \sigma_{b, \omega_p(b), \widetilde{m}_p(b)}$$

if the reduction of $\widetilde{m}_p(b)$ by one is allowed (a suitable condition is $\widetilde{m}_p \geqslant 1$ or $\widetilde{m}_p \geqslant 2$) and $\Delta_b^- = \infty$ if we do not have the possibility to reduce the multiplier $\widetilde{m}_p(b)$ by one. A suitable data structure for the $\Delta_b^-$ values is a heap, for which the update after an iteration can be done in $O(1)$ time. If we reach $I(\omega_p, \widetilde{m}_p) < \underline{I}$ at some point, we simply discard this particular selection $\omega_p$ and consider the next selection candidate.

Since this adjustment step can be performed very fast one might also take some kind of general swap techniques into account. Since for these techniques there exists an overboarding amount of papers in the literature we will not go into detail here, but we would like to remark that in those cases (see Subsection 5.3) where the optimality gap of our heuristic lies above 1 % swapping can improve the solutions of our heuristic by a large part.

5.3. **Optimality gap.** To substantiate the usefullness of our heuristic we have compared the quality of the solutions given by this heuristic after one second of computation time (on a standard laptop) with respect to the solution given by CPLEX 11.

Our business partner has provided us historic sales information for nine different commodity groups each ranging over a sales period of at least one and a half year. For each commodity group we have performed a test calculation for $\kappa \in \{1, 2, 3, 4, 5\}$ distributing some amount of items to almost all branches.

**Commodity group 1**:
Cardinality interval: $[10630, 11749]$
number of sizes: $|\mathcal{S}_p| = 5$
number of branches: $|\mathcal{B}_p| = 1119$

|  | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 3$ | $\kappa = 4$ | $\kappa = 5$ |
|---|---|---|---|---|---|
| CPLEX | 4033.34 | 3304.10 | 3039.28 | 2951.62 | 2891.96 |
| heuristic | 4033.85 | 3373.95 | 3076.55 | 3011.49 | 2949.31 |
| gap | 0.013% | 2.114% | 1.226% | 2.028% | 1.983% |

TABLE 1. Optimality gap in the $\| \cdot \|_1$-norm for our heuristic on commodity group 1

**Commodity group 2**:
Cardinality interval: $[10000, 12000]$
number of sizes: $|\mathcal{S}_p| = 5$
number of branches: $|\mathcal{B}_p| = 1091$

|  | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 3$ | $\kappa = 4$ | $\kappa = 5$ |
|---|---|---|---|---|---|
| CPLEX | 2985.48 | 2670.04 | 2482.23 | 2362.75 | 2259.57 |
| heuristic | 3371.64 | 2671.72 | 2483.52 | 2362.90 | 2276.32 |
| gap | 12.934% | 0.063% | 0.052% | 0.006% | 0.741% |

TABLE 2. Optimality gap in the $\| \cdot \|_1$-norm for our heuristic on commodity group 2

**Commodity group 3**:
Cardinality interval: $[9785, 10815]$
number of sizes: $|\mathcal{S}_\mathrm{p}| = 5$
number of branches: $|\mathcal{B}_\mathrm{p}| = 1030$

|  | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 3$ | $\kappa = 4$ | $\kappa = 5$ |
|---|---|---|---|---|---|
| CPLEX | 3570.3282 | 3022.2655 | 2622.8209 | 2488.1009 | 2413.55 |
| heuristic | 3571.61 | 3023.91 | 2625.29 | 2492.07 | 2417.65 |
| gap | 0.036% | 0.054% | 0.094% | 0.160% | 0.170% |

TABLE 3. Optimality gap in the $\|\cdot\|_1$-norm for our heuristic on commodity group 3

**Commodity group 4**:
Cardinality interval: $[10573, 11686]$
number of sizes: $|\mathcal{S}_\mathrm{p}| = 5$
number of branches: $|\mathcal{B}_\mathrm{p}| = 1119$

|  | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 3$ | $\kappa = 4$ | $\kappa = 5$ |
|---|---|---|---|---|---|
| CPLEX | 4776.36 | 4364.63 | 4169.94 | 4023.60 | 3890.87 |
| heuristic | 5478.19 | 4365.47 | 4170.23 | 4024.55 | 3892.35 |
| gap | 14.694% | 0.019% | 0.007% | 0.024% | 0.038% |

TABLE 4. Optimality gap in the $\|\cdot\|_1$-norm for our heuristic on commodity group 4

**Commodity group 5**:
Cardinality interval: $[16744, 18506]$
number of sizes: $|\mathcal{S}_\mathrm{p}| = 5$
number of branches: $|\mathcal{B}_\mathrm{p}| = 1175$

|  | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 3$ | $\kappa = 4$ | $\kappa = 5$ |
|---|---|---|---|---|---|
| CPLEX | 4178.71 | 3418.37 | 3067.74 | 2874.70 | 2786.69 |
| heuristic | 4179.23 | 3418.87 | 3068.25 | 2875.21 | 2787.21 |
| gap | 0.013% | 0.015% | 0.017% | 0.018% | 0.019% |

TABLE 5. Optimality gap in the $\|\cdot\|_1$-norm for our heuristic on commodity group 5

**Commodity group 6**:
Cardinality interval: $[11000, 13000]$
number of sizes: $|\mathcal{S}_\mathrm{p}| = 4$
number of branches: $|\mathcal{B}_\mathrm{p}| = 1030$

|  | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 3$ | $\kappa = 4$ | $\kappa = 5$ |
|---|---|---|---|---|---|
| CPLEX | 2812,22 | 2311,45 | 2100,78 | 1987,46 | 1909,21 |
| heuristic | 2812,63 | 2311,87 | 2101,25 | 1987,93 | 1909,63 |
| gap | 0.015% | 0.018% | 0.022% | 0.024% | 0.022% |

TABLE 6. Optimality gap in the $\|\cdot\|_1$-norm for our heuristic on commodity group 6

**Commodity group 7**:
Cardinality interval: $[15646, 17293]$
number of sizes: $|\mathcal{S}_p| = 5$
number of branches: $|\mathcal{B}_p| = 1098$

|           | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 3$ | $\kappa = 4$ | $\kappa = 5$ |
|-----------|---------|---------|---------|---------|---------|
| CPLEX     | 4501.84 | 3917.96 | 3755.20 | 3660.32 | 3575.55 |
| heuristic | 4719.06 | 3918.46 | 3755.70 | 3660.84 | 3576.04 |
| gap       | 4.825%  | 0.013%  | 0.013%  | 0.014%  | 0.014%  |

TABLE 7.  Optimality gap in the $\|\cdot\|_1$-norm for our heuristic on commodity group 7

**Commodity group 8**:
Cardinality interval: $[11274, 12461]$
number of sizes: $|\mathcal{S}_p| = 5$
number of branches: $|\mathcal{B}_p| = 989$

|           | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 3$ | $\kappa = 4$ | $\kappa = 5$ |
|-----------|---------|---------|---------|---------|---------|
| CPLEX     | 3191.66 | 2771.89 | 2575.37 | 2424.31 | 2331.67 |
| heuristic | 3579.35 | 2772.33 | 2575.81 | 2424.75 | 2332.11 |
| gap       | 12.147% | 0.016%  | 0.017%  | 0.018%  | 0.019%  |

TABLE 8.  Optimality gap in the $\|\cdot\|_1$-norm for our heuristic on commodity group 8

**Commodity group 9**:
Cardinality interval: $[9211, 10181]$
number of sizes: $|\mathcal{S}_p| = 5$
number of branches: $|\mathcal{B}_p| = 808$

|           | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 3$ | $\kappa = 4$ | $\kappa = 5$ |
|-----------|---------|---------|---------|---------|---------|
| CPLEX     | 3616.71 | 3215.17 | 2981.02 | 2837.66 | 2732.29 |
| heuristic | 3617.09 | 3215.53 | 3009.01 | 2860.85 | 2758.39 |
| gap       | 0.010%  | 0.011%  | 0.939%  | 0.817%  | 0.955%  |

TABLE 9.  Optimality gap in the $\|\cdot\|_1$-norm for our heuristic on commodity group 9

Besides these nine test calculations we have done several calculations on our data sets with different parameters, we have, e.g., considered case with fewer sizes, fewer branches, smaller or larger cardinality intervals, larger $\kappa$, or other magnitudes for the cardinality interval. The results are from a qualitative point of view more or less the same, as for the presented test calculations.

## 6. CONCLUSION AND OUTLOOK

Starting from a real world optimization problem we have formalized a new general optimization problem, which we call cardinality p-facility location problem. It turns out that this problem is related to several other well known standard optimization problems. In Subsection 3.5 we have given an integer linear programming formulation which has a very strong LP-relaxation. Nevertheless this approach is quit fast (computing times below one hour), there was a practical need for fast heuristics to solve the

problem. We have presented one such heuristic which performs very well on real world data sets with respect to the optimality gap.

Some more theoretic work on the cardinality p-facility location problem and its relationships to other classical optimization methods may lead to even stronger integer linear programming formulations or faster branch-and-bound frameworks enhanced with some graph theoretic algorithms.

We leave also the question of a good approximation algorithm for the cardinality p-facility location problem. Having the known approximation algorithms for the other strongly related classical optimization problems in mind, we are almost sure that it should be not too difficult to develop good approximation algorithms for our problem.

For the practical problem the uncertainties and difficulties concerning the demand estimation have to be faced. There are several ways to make solutions of optimization problems more robust. One possibility is to utilize robust optimization methods. Another possibility is to consider the branch- and size dependent demands as stochastic variables and to utilize integer linear stochastic programming techniques. See, e.g., [3] or more specifically [29]. These enhanced models, however, will challenge the solution methods a lot, since the resulting problems are of a much larger scale than the one presented in this paper. Nevertheless, this is exactly what we are looking at next.

## References

1. J.S. Armstrong (ed.), *Principles of forecasting: A handbook for researchers and practitioners*, Kluwer, 2001.
2. V. Arya, N. Garg, R. Khandekar, K. Munagala, A. Meyerson, and V. Pandit, *Local search heuristics for k-median and facility location problems*, 33rd Annual ACM Symposium on Theory of Computing (STOC), 2001, pp. 21–29.
3. J. R. Birge and Louveaux F., *Introduction to stochastic programming*, Springer Series in Operations Research and Financial Engineering, Springer, 1997.
4. J. W. S. Cassels, *An introduction to diophantine approximation*, Cambridge, 1965.
5. M. Charikar, S. Guha, E. Tardos, and D. Shmoys, *A constant-factor approximation algorithm for the k-median problem*, 31st Annual ACM Symposium on the Theory of Computing, 1999, pp. 1–10.
6. ———, *A constant-factor approximation algorithm for the k-median problem*, JCSS (to appear).
7. G. Cornuéjols, G. L. Nemhauser, and L. A. Wolsey, *The uncapacitated facility location problem*, Discrete Location Theory (P. Mirchandani and R. Francis, eds.), John Wiley and Sons, Inc., New York, 1990, pp. 119–171.
8. J. A. Díaza and E. Fernández, *Hybrid scatter search and path relinking for the capacitated p-median problem*, European Journal of Operational Research **169** (2006), no. 2, 570–585.
9. A. Frank and E. Tardos, *An application of simultaneous diophantine approximation in combinatorial optimization*, Combinatorica **7** (1987), no. 1, 49–65.
10. F. García, B. Melian, J.A. Moreno, and J.M. Moreno-Vega, *Scatter search for multiple objective p-facility location problems*, Workshop on Multiobjective Metaheuristics, 2002.
11. D. Ghosh, *Neighborhood search heuristics for the uncapacitated facility location problem*, European Journal of Operational Research **150** (2003), no. 1, 150–162.
12. S. Guha and S. Khuller, *Greedy strikes back: improved facility location algorithms*, ninth annual ACM-SIAM symposium on Discrete algorithms, 1998, pp. 649–657.
13. M. T. Hajiaghayi, M. Mahdian, and V. S. Mirrokni, *The facility location problem with general cost functions*, Networks **42** (2003), no. 1, 6 pages.
14. P. Hansen and N. Mladenović, *Variable neighborhood search for the p-median*, Location Science **5** (1997), no. 4, 207–226.
15. K. Jain, M. Mahdian, E. Markakis, A. Saberi, and V. V. Vazirani, *Greedy facility location algorithms analyzed using dual fitting with factor-revealing lp*, JACM **50** (2003), no. 6, 795–824.
16. K. Jain and V. V. Vazirani, *Primal-dual approximation algorithms for metric facility location and k-median problems*, 40th Annual Symposium on Foundations of Computer Science, 1999, p. 2.
17. A. Kok and M. Fisher, *Demand estimation and assortment optimization under substitution: Methodology and application*, Operations Research (to appear).
18. M. R. Korupolu, C. G. Plaxton, and R. Rajaraman, *Analysis of a local search heuristic for facility location problems*, Journal of Algorithms **37** (2000), no. 1, 146–188.
19. S. Kurz and J. Rambau, *Demand forecasting for companies with many branches, low sales numbers per product, and non-recurring orderings*, Proceedings of the Seventh International Conference on Intelligent Systems Design and Applications, 2007, pp. 196–201.
20. S. Kurz, J. Rambau, J. Schlüchtermann, and R. Wolf, *The top-dog index: A new measurement for the demand consistency of the size distribution in pre-pack orders for a fashion discounter with many small branches*, (in preparation).

21. S. Lang, *Introduction to diophantine approximations (new expanded edition)*, Springer-Verlag, 1995.
22. D. Leviatan and V. Temlyakov, *Simultaneous approximation by greedy algorithms*, Advances in Computational Mathematics **25** (2006), no. 1-3, 73–90.
23. J.-H. Lin and J. S. Vitter, *e-approximations with minimum packing constraint violation (extended abstract)*, twenty-fourth annual ACM symposium on Theory of computing, 1992, pp. 771–782.
24. S. Makridakis, S. Wheelwright, and R. Hyndman, *Forecasting:methods and applications*, Wiley, 2004.
25. G. L. Nemhauser and L. A. Wolsey, *Integer and combinatorial optimization*, John Wiley and Sons, Inc., New York, 1988.
26. C. S. ReVelle, H. A. Eiselt, and M. S. Daskin, *A bibliography for some fundamental problem categories in discrete location science*, European Journal of Operational Research **184** (2008), 817–848.
27. W. M. Schmidt, *Simultaneous approximation to algebraic numbers by rationals*, Acta Mathematica **125** (1970), no. 1, 189–201.
28. D. B. Shmoys, E. Tardos, and K. Aardal, *Approximation algorithms for facility location problems (extended abstract)*, twenty-ninth annual ACM symposium on Theory of computing, 1997, pp. 265–274.
29. C. Swamy, *Approximation algorithms for clustering problems*, Ph.D. thesis, Cornell University, 2004.
30. N. E. Young, *K-medians, facility location, and the chernoff-wald bound*, eleventh annual ACM-SIAM symposium on Discrete algorithms, 2000, pp. 86–95.
31. J. Yurkiewicz, *Software survey: Forecasting 2000*, OR/MS Today **27** (2000), no. 1.

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