Existence and stability of stellardynamic models

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Achim Schulze

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Contents

Abstract

In der vorliegenden Arbeit wird die Existenz und Stabilität von stationären Lösungen des Vlasov–Poisson–Systems untersucht. Dieses System wird u.a. in der Stellardynamik zur Beschreibung von Galaxien verwendet. Dabei werden Kollisionen vernachlässigt und die Interaktion zwischen den Teilchen, d.h der Sterne wird nur durch das von ihnen erzeugte Gravitationspotential bestimmt. Das System besteht aus folgenden Gleichungen:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0,$$

$$\Delta U = 4\pi\rho, \ \lim_{|x|\to\infty} U(t,x) = 0,$$

$$\rho(t,x) = \int f(t,x,v)dv.$$

Dabei bezeichnet $f : \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}$, $f = f(t,x,v) \geq 0$ für $t \in \mathbb{R}$, $x,v \in \mathbb{R}^3$ die Phasenraumdichte der Teilchen, $U = U(t,x)$ das Gravitationspotential, und $\rho = \rho(t,x)$ die räumliche Dichte.

Im ersten Teil der Arbeit werden Existenz und Stabilität stationärer Lösungen der Form

$$f_0(x,v) = (E_0 - E)_+^k (L - L_0)_+^l$$

unter dem Einfluss einer im Ursprung fixierten Punktmasse mittels einer Variationsmethode bewiesen. Dabei ist

$$E = E(x,v) = \frac{1}{2}|v|^2 + U(x) - \frac{M_c}{|x|},$$

$$L = |x \times v|^2,$$

und k,l mit $0 < k \leq l$, sowie E_0 und L_0 sind Konstanten. M_c ist die Masse der Punktmasse. Diese Lösungen können als einfache Modelle für Galaxien verwendet werden, die in ihrem Zentrum ein massives schwarzes Loch besitzen, das dann annähernd als fixe Punktmasse angesehen werden kann. Für das Vlasov–Poisson System unter der Wirkung eines solchen externen Potentials wird auch ein globaler Existenzsatz für geeignete Anfangsdaten bewiesen.

Im zweiten Teil wird die Existenz axialsymmetrischer Lösungen behandelt. Es werden stationäre Zustände konstruiert, die in einem langsam rotierenden System durch Deformation einer gegebenen sphärisch symmetrischen Lösung entstehen. Dazu wird ein Satz über implizite Funktionen auf einen modifizierten Lösungsoperator der Poisson–Gleichung angewendet. Im rotierenden System haben diese Zustände die Form

$$f_0(x,v) = \varphi\left(\frac{1}{2}v^2 + U(x) - \frac{1}{2}\omega^2(x_1^2 + x_2^2)\right),$$

wobei ω die Rotationsgeschwindigkeit bezeichnet und $\varphi \in C^1(\mathbb{R})$ noch gewissen Zusatzannahmen unterliegt. Der Ausdruck

$$E_J := \frac{1}{2}|v|^2 + U(x) - \frac{1}{2}\omega^2(x_1^2 + x_2^2)$$

wird auch als Jakobis Integral bezeichnet. Die hier präsentierten Ergebnisse sind in Übereinstimmung mit den numerischen Beobachtungen des Astrophysikers P.O. Vandervoort.

Preface

In the present work, we analyse the time evolution of large particle ensembles, e.g. stars, under the influence of a self-consistent force, which will be the gravitational force here. If relativistic effects are neglected and if the distances between the particles are large enough, an appropiate way to describe such a system are the equations of the N-body-problem:

$$m_j \ddot{X}_j = -\gamma \sum_{k \neq j}^{N} m_j m_k \frac{X_j - X_k}{|X_j - X_k|^3}, \quad j = 1, \ldots, N, \tag{0.1}$$

where $X_1(s), X_2(s), \ldots, X_N(s)$ are threedimensional time-dependent vectors denoting the position of the N point masses and $m_1, m_2, \ldots m_N$ are their masses. The gravitational constant is denoted by γ and we prescribe $X_j(0), \dot{X}_j(0),\ j = 1, \ldots, N$ with $X_j(0) \neq X_i(0)$ for $i \neq j$. Many important developments in astrophysics originate from the N-body-problem such as the analysis of motions of the solar system and with the growing computing power of modern processors, the problem is numerically tractable even for large N.

However, for very large N, such as in galaxies or star clusters, where $N > 10^{10}$, cf. Figure 1, one is in a very unpromising situation concerning analytical results for this system and it is indeed not reasonable to observe each particle if the focus is for example on stability or the distribution of the particles. In fact, an "averaged" view seems to be more adequate and for many questions it is sufficient to know only how the particles are distributed in phase space $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$. For that purpose, one introduces a function $f : \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}$, which is the phase space density $f = f(t,x,v) \geq 0$, where $t \in \mathbb{R}$ denotes time and $x, v \in \mathbb{R}^3$ denote position and velocity. Its value is - up to a multiplicative constant - the number of particles per phase space volume at the time t. More precisely, if we take a phase space volume $\Pi \subset \mathbb{R}^6$, the number of particles $N(\Pi, t)$ at time t in the volume Π is

$$N(\Pi, t) = \int_{\Pi} f(t,x,v)\, dx dv.$$

We have to ascertain the equations which determine f and we start by analysing the motion of a particle X in a fixed force field, induced by a given

Figure 1: The spiral galaxy M74, http://hubblesite.org

potential $U=U(t,x)$. The behaviour of $X=X(s)$ is again governed by Newton's Law,

$$\begin{cases} \dot{X} = V \\ \dot{V} = -\nabla_x U(s,X), \end{cases} \tag{0.2}$$

with some initial condition $X(0),V(0)$. If collissions are neglected, it is natural to require that f is constant along such particle paths, if U in (0.2) is the self-consistent potential which is collectively created by all particles. More precisely, we require

$$f(s,X(s),V(s))=\text{const.},$$

for all (X,V) solving (0.2) and differentiating with respect to s implies

$$\partial_t f(s,Z(s))+V(s)\cdot\nabla_x f(s,Z(s))-\nabla_x U(s,X(s))\cdot\nabla_v f(s,Z(s))=0, \tag{0.3}$$

where we abbreviated $(X(s),V(s))=:Z(s)$. If we take for given (t,x,v) a suitable $Z=Z(s)$ with $Z(t)=(X(t),V(t))=(x,v)$, we arrive at the so-called Vlasov equation,

$$\partial_t f+v\cdot\nabla_x f-\nabla_x U\cdot\nabla_v f=0, \tag{0.4}$$

and we easily verify that $f=f(t,x,v)$ solves (0.4), iff f is constant along every solution of (0.2). This statement is just the fact that (0.2) is the characteristic system of the first order PDE (0.4). To reflect the requirement, that the potential U is created by the particles, we have to couple Eq. (0.4) with Poisson's equation,

$$\Delta_x U(t,x)=4\pi\rho(t,x), \quad \lim_{|x|\to\infty} U(t,x)=0, \tag{0.5}$$

where we define

$$\rho(t,x) := \int_{\mathbb{R}^3} f(t,x,v)\,dv$$

as the spatial density. Equation (0.5) just says that

$$U(t,x) = -\left[\frac{1}{|.|} *_x \rho\right](t,x) = -\int_{\mathbb{R}^3} \frac{\rho(t,y)}{|x-y|}\,dy$$

and we also have

$$\nabla_x U(t,x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\rho(t,y)\,dy,$$

which is the averaged form of the right-hand side of equation (0.1), and we assumed $\rho(t,.) \in C_c(\mathbb{R}^3)$.

Equations (0.4) and (0.5) form a closed system, the so-called Vlasov–Poisson system. At the beginning of the last century, this system was used by Sir J. Jeans to model stellar clusters and galaxies [14] and to investigate their stability properties. In this context, it is also used by today's astrophysicists, cf. [4].

Meanwhile, the existence theory for this system is well-understood: Local existence and uniqueness was established by R. Kurth in 1952, cf. [15] and global existence for spherically symmetric solutions was proved by J. Batt in 1977, cf. [3]. Afterwards, global solutions with small initial data were obtained by C. Bardos and P. Degond in 1985, cf. [2]. The final breakthrough was achieved in 1989, when global existence for general classical solutions was independently from each other proved by Pfaffelmoser, cf. [21] and P. L. Lions/B. Perthame, cf. [19]. They proved that for every inital datum $\mathring{f} \in C_c^1(\mathbb{R}^6)$, the corresponding solution to the Vlasov–Poisson system exists for all time.

We are interested in existence and stability of stationary soltutions to (0.4)–(0.5). For the construction of stationary solutions, we recall that f is a solution of the Vlasov equation, iff it is constant along characteristics, i.e., the solutions of the charateristic system (0.2). Thus natural candidates for stationary solutions are functions of conserved quantities of the characteristic system. One immediate expression for a conserved quantity is the particle energy

$$E(x,v) := \frac{1}{2}|v|^2 + U(x), \tag{0.6}$$

so that the ansatz

$$f_0(x,v) := \varphi(E) \tag{0.7}$$

automatically satisfies the Vlasov equation for $\varphi \in C^1$. Of course, we still have to solve Poisson's equation. Plugging (0.7) into (0.5) yields

$$\Delta U(x) = 4\pi \int \varphi\big(\frac{1}{2}|v|^2 + U(x)\big)\,dv, \tag{0.8}$$

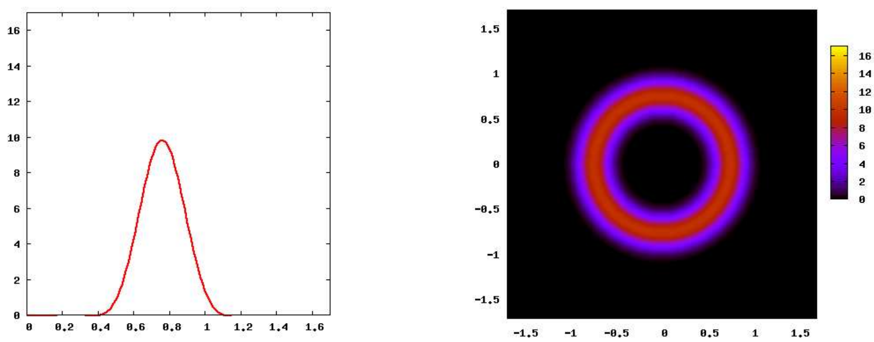

Figure 2: The spatial density $\rho = \rho(x) = \rho(|x|)$ of a shell

which is a semilinear elliptic equation for U. If φ only depends on the energy, one can show that the potential U then has to be spherically symmetric, which is a special case of a more general result of Gidas, Ni and Nirenberg, cf. [6]. Here spherical symmetry means that $U(Ax) = U(x)\ \ \forall A \in O(3)\ \ \forall x \in \mathbb{R}^3$, where $O(3)$ is the group of orthogonal 3×3 matrices. Equation (0.8) then reduces to the following ordinary differential equation

$$\frac{1}{r^2}\big(r^2 U'(r)\big)' = 4\pi \int \varphi\big(\frac{1}{2}|v|^2 + U(r)\big)\, dv,$$

which is much easier to solve.
If we a-priori require some symmetry on U, say spherical symmetry, we can also use other invariants than the energy, for example components of the angular momentum $x \times v$, or

$$L := |x \times v|^2,$$

the modulus of the angular momentum squared.

In Chapter 1, we are interested in stationary solutions of the form

$$f_0(x,v) = \varphi(E, L), \tag{0.9}$$

and one can show that for suitable φ, the support of the corresponding spatial densities is bounded away from the origin and also contained in some large ball. The spatial profile of such a shell is shown in Figure 2. These shells can be used as a simple model for a galaxy surrounding a black hole in the following way: The black hole is modeled by a fixed point mass in the origin which acts like an external force on all particles. The Vlasov–Poisson system then has to be modified such that the additional force term due to the point

mass is taken into account.
We will then prove existence and stability of shells of form (0.9) for the modified system with a variational approach based on energy methods. This approach, once established, yields existence and stability of a stationary solution from one source: The existence will be a consequence of the Euler-Lagrange equations of the considered energy functional and the stability follows from the convergence properties of minimizing sequences.

For the stability issue, we only can consider perturbations to the system which "stay away" in a suitable sense from the point mass singularity and we also have to prove an existence theorem to know that solutions exist in the neighbourhood of such a stationary shell. All this will be described in detail in the next Chapter. The variational techniques mentioned above were established by Y. Guo and G. Rein and meanwhile there is a quite extensive literature on this topic, we want to list [7, 8, 9, 10, 11, 24, 25, 26] here.

Chapter 2 evolved from the attempt to construct triaxial solutions to the Vlasov–Poisson system. A triaxial solution is a stationary solution with no axial symmetry, the support of its spatial density is an ellipsoid with three principal axes of different length. The central idea is to construct such a solution by deforming a given spherically symmetric solution (f_0,U_0) of form (0.7). This is tried by considering the Vlasov–Poisson system in a rotating frame for small rotational velocity $\omega>0$ around the x_3-axis. There, one conserved quantity of the characteristic system is the so-called Jacobi's integral

$$E_J := \frac{1}{2}|v|^2 + U(x) - \frac{1}{2}\omega^2\left(x_1^2+x_2^2\right),$$

and the ansatz $f_0(x,v)=\varphi(E_J)$ leads to

$$\Delta U(x) = 4\pi \int \varphi\left(\frac{1}{2}|v|^2 + U(x) - \frac{1}{2}\omega^2\left(x_1^2+x_2^2\right)\right) dv, \qquad (0.10)$$

which corresponds to Eq. (0.8) in this situation. For $\omega=0$, we have the original system with solution (f_0,U_0) and for small ω, we construct solutions of (0.10) which arise from deformations of (f_0,U_0) by applying an implicit function theorem on a modified solution operator of the Poisson equation. However, up to now we have no method to exclude symmetry with respect to the x_3-axis, though our class of allowed deformations could match a truely triaxial system. Our result is the existence of axially symmetric solutions of the above form to the Vlasov–Poisson system. The approach for the deformation of the spherically symmetric steady state described above is based on [23], where axially symmetric solutions to the nonrotating Vlasov–Poisson system were constructed. There, the deformed solution a-priori has to be axially symmetric, since the ansatz used for the steady state depends on $P=x_1v_2-x_2v_1$, the third component of the angular momentum, which is only a conserved quantity along characteristics, if U is axially symmetric. So far, our approach seems to allow less symmetry, since we have no symmetry assumptions on U.

Eq. (0.10) also has been studied numerically by P.O. Vandervoort, cf. [30], for $\varphi(E_J)=(E_0-E_J)_+^k$ for some constant E_0 and he observed that triaxial systems only can occur for ω large enough. For small ω, all calculated solutions are axially symmetric. Our result is in accordance with his one, since we achieve existence only for small ω.

Notation

Most of the notation used here is self-explaining, but we still want to fix some of it. For a set $M \subset \mathbb{R}^n$, χ_M denotes its indicator function, i.e.,

$$\chi_M(x) = 1 \text{ if } x \in M, \quad \chi_M(x) = 0 \text{ if } x \notin M.$$

If M is measurable, we denote its Lebesgue measure by $|M|$. For $1 \leq p \leq \infty$, we will write $\|\cdot\|_p$ for the norm on the Lebesgue space $L^p(\mathbb{R}^n)$.
For $x, y \in \mathbb{R}^n$, we denote by

$$x \cdot y := \sum_{i=1}^{n} x_i y_i, \quad |x| := \sqrt{x \cdot x}$$

the Euclidian scalar product and norm. For $\eta \in \mathbb{R}$, the positive part η_+ is defined by

$$\eta_+ := \max\{\eta, 0\}.$$

We will write

$$C^k(\mathbb{R}^n), \quad C_c^k(\mathbb{R}^n)$$

for the space of k times continuously differentiable functions on $\mathbb{R}^n$, the subscript "c" indicates compactly supported functions.
For a function $g \in C_c^1(\mathbb{R}^n)$ we define the norm $\|\cdot\|_{1,\infty}$ by

$$\|g\|_{1,\infty} := \|g\|_\infty + \|\nabla g\|_\infty$$

and the norm $\|\cdot\|_{2,\infty}$ is defined analoguously for functions which are twice differentiable. We call a function $g : \mathbb{R}^3 \to \mathbb{R}$ spherically symmetric, if

$$g(Ax) = g(x) \quad \forall \; A \in O(3),$$

where $O(3)$ denotes the group of orthogonal 3×3-matrices. Furthermore, a function $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, with $f = f(x, v)$, $x, v \in \mathbb{R}^3$ is called spherically symmetric, if

$$f(Ax, Av) = f(x, v) \quad \forall \; A \in O(3).$$

In Chapter 1, two weighed L^p-spaces play an important role. For given $L_0 > 0$, they are defined by

$$L^{k,l}(\mathbb{R}^6) := \Big\{ f : \mathbb{R}^6 \to \mathbb{R} \text{ measurable, spherically symmetric and } \iint |f|^{1+\frac{1}{k}} (L - L_0)_+^{-l/k} \, dx dv < \infty \Big\}$$

equipped with the norm

$$\|f\|_{k,l} := \left(\iint |f|^{1+\frac{1}{k}} (L - L_0)_+^{-l/k} \, dx dv \right)^{\frac{k}{k+1}},$$

where we define $L := |x \times v|^2$ for $x, v \in \mathbb{R}^3$, and

$$L^{n,l}(\mathbb{R}^3) := \Big\{ \rho : \mathbb{R}^3 \to \mathbb{R} \text{ measurable, spherically symmetric and } \int |\rho|^{1+\frac{1}{n}} |x|^{-2l/n} dx < \infty \Big\}$$

with norm

$$\|\rho\|_{n,l} := \left(\int |\rho|^{1+\frac{1}{n}} |x|^{-2l/n} dx \right)^{\frac{n}{n+1}}.$$

Chapter 1

Existence and stability of static shells with a fixed central point mass

1.1 Introduction

For the sake of clarity, we state the Vlasov–Poisson system again:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \tag{1.1}$$

$$\Delta U = 4\pi\rho, \ \lim_{|x|\to\infty} U(t,x) = 0, \tag{1.2}$$

$$\rho(t,x) = \int f(t,x,v)dv, \tag{1.3}$$

where $f = f(t,x,v) \geq 0$ is the phase-space density of the particles, $U = U(t,x)$ is the gravitational potential of the ensemble, and $\rho = \rho(t,x)$ is its spatial density. We want to investigate this system under the influence of a fixed point mass. If we assume that a point mass M_c is fixed in the origin and acts like an external force on the particles, the Vlasov equation reads

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x U - \nabla_x \frac{M_c}{|x|}) \cdot \nabla_v f = 0. \tag{1.4}$$

If we write $U_{\text{eff}} := U - M_c/|x|$, the Poisson equation becomes

$$\Delta U_{\text{eff}} = 4\pi\left(\rho + M_c\delta\right), \tag{1.5}$$

where δ denotes the δ-distribution. This chapter is dedicated to the existence and stability of steady states of the system (1.2)–(1.4).

As already mentioned in the Preface, for the construction of stationary solutions, one has to find conserved quantities of the characteristic system

$$\begin{cases} \dot{X} = V \\ \dot{V} = -\nabla_x \big(U(X) - \frac{M_c}{|X|} \big) \end{cases}, \tag{1.6}$$

where $(X,V)=(X,V)(s):=(X,V)(s,t,x,v)$ with $(X,V)(t,t,x,v)=(x,v)$ for an initial value $(x,v)\in\mathbb{R}^6$ and $s,t\in\mathbb{R}$. Again, one immediate expression for such a quantity is the particle energy

$$E=\frac{1}{2}|v|^2+U(x)-\frac{M_c}{|x|}$$

and if we make some additional symmetry assumptions on the potential U, we can find other conserved terms such as the angular momentum.

Here, we are interested in stationary solutions of the form

$$f_0(x,v)=(E_0-E)_+^k(L-L_0)_+^l, \tag{1.7}$$

where $0<k\le l$ and $E_0<0, L_0>0$ are constants. E is the particle energy as above and

$$L=|x\times v|^2=|x|^2|v|^2-(x\cdot v)^2 \tag{1.8}$$

denotes the modulus of angular momentum squared which is conserved along characteristics, if U is spherically symmetric.

If we want to construct the stationary solution (f_0,U_0) explicitely from the ansatz (1.7), we still have to solve the Poisson equation (1.2) to get a self-consistent potential U_0. The existence of stationary solutions with parameter range $k>-1,\ l>-1,\ k+l+1/2\ge 0,\ k<3l+7/2$ was established in [22] for $M_c=0$.

Without the exterior potential, the existence and stability of stationary solutions of the form (1.7) was done in [29], where the parameter range $l>-1$ and $0<k<l+3/2$ was covered. The details of this paper will be described in Section 1.7.

As mentioned above, for our ansatz (1.7) we require that the corresponding potential U is spherically symmetric and therefore the stationary solutions (1.7) also have to be spherically symmetric, i.e.,

$$f(x,v)=f(Ax,Av) \quad \forall A\in O(3), \tag{1.9}$$

where $O(3)$ is the group of orthogonal 3×3 matrices. For $L_0>0$ the support of the induced spatial density $\rho(x)=\rho(|x|)$ is contained in some interval $[R_1,R_2]$ with $R_1>0$ and the steady state describes a shell. This can be seen as follows.

If we introduce the new coordinates $r := |x|, w := x \cdot v/r$ and L as in (1.8), we can calculate the spatial density of f_0 as

$$\begin{aligned}\rho_{f_0}(x) &= \int_{\mathbb{R}^3} f_0(x,v)\,dv \\ &= \frac{\pi}{r^2}\int_{\mathbb{R}}\int_0^\infty \left(E_0 - \frac{1}{2}(w^2 + \frac{L}{r^2}) - U_0(r) + \frac{M_c}{r}\right)_+^k (L - L_0)_+^l \,dw dL \quad (1.10)\\ &= C(k,l)\,r^{2l}\left(E_0 - U_0(r) + \frac{M_c}{r} - \frac{L_0}{r^2}\right)_+^{k+l+3/2}, \quad (1.11)\end{aligned}$$

where

$$C(k,l) = 2^{l+3/2}\pi \int_0^1 \frac{s^l}{\sqrt{1-s}}\,ds \int_0^1 s^{l+1/2}(1-s)^k\,ds.$$

For small r the expression in the bracket of (1.11) becomes negative and this implies $\operatorname{supp}\rho_f \subset [R_1, \infty[$, for some $R_1 > 0$. On the other hand, because of $U_0'(r) = \int_0^r s^2\rho(s)\,ds/r^2 > 0$ for $r > 0$, the function $-U_0(r) + M_c/r$ is decreasing and with $E_0 < 0$ we conclude that $\rho_{f_0}(r) = 0$ for large r.

These shells together with the exterior potential induced by a point mass can be used as a simple model for a galaxy which encloses a black hole in the center.

The ansatz (1.7) also leads to steady states and shells of the Vlasov-Einstein system, the general relativistic counterpart of the newtonian Vlasov-Poisson system, and they provide an access to study stability and critical phenomena numerically, cf. [1].

We examine the shells in the newtonian framework and to investigate their stability, we will firstly prove a global existence result for the system (1.2)–(1.4) for initial data, which vanishes in a neighbourhood of the singularity $r = 0$. The corresponding solution then exists for all time, and will always vanish, if x is in a ball around the singularity, which is determined by the initial datum. We mention that, without the exterior potential, the existence problem for the Vlasov–Poisson system is well understood, see for example [19, 21, 28] for global existence of classical solutions. However, in our situation the exterior potential becomes unbounded in $r = 0$ and we have to ensure that the particles stay away from the singularity.

To show existence and stability of the shells, we use a similar approach as in [9], where existence and stability of the above steady states was shown in the case $L_0 = 0$ without the exterior potential. The main idea is to use an Energy-Casimir functional as a Lyapunov function with the help of variational methods. We briefly sketch the basic concept:

The Vlasov-Poisson system is conservative, i.e., the total energy

$$\begin{aligned}\mathcal{H}(f) &:= E_{\text{kin}}(f) + E_{\text{pot}}(f)\\ &:= \frac{1}{2}\int |v|^2 f(x,v)\,dv\,dx - \frac{1}{8\pi}\int \left(|\nabla U_f(x)|^2 + \frac{8\pi M_c}{|x|}\rho_f(x)\right)dx \qquad (1.12)\end{aligned}$$

of a state f is conserved along solutions and hence is a natural candidate for a Lyapunov function in a stability analysis; U_f denotes the potential induced by f, note also the interaction term $\int \rho_f\, M_c/|x|\,dx$ induced by the fixed central point mass. However, the energy does not have critical points, but for any reasonable function Φ the so-called *Casimir functional*

$$\mathcal{C}(f) := \iint \Phi(f(x,v))\,dv\,dx$$

is conserved as well. Now one tries to minimize the energy-Casimir functional

$$\mathcal{H}_C := \mathcal{H} + \mathcal{C}$$

in the class of allowed perturbations $\mathcal{F}_M$, which consists of positive $L^1(\mathbb{R}^6)$-functions with prescribed mass M, i.e. $\iint f = M$ and with finite kinetic energy and a finite Casimir functional to ensure that $\mathcal{H}_C$ is well-defined.

The aim is to prove that a minimizer f_0 is a stationary solution of (1.2)–(1.4) and to deduce its stability. One of the difficulties is to show that the weak limit of a minimizing sequence in $\mathcal{H}_C$, indeed is a minimizer. For this purpose, we will need that every function in the class of perturbations $\mathcal{F}_M$ vanishes on the set $0 \leq L < L_0$.

We are only able to show stability against spherically symmetric perturbations, because our approach requires an L-dependence in the Casimir functional, more precisely, we define

$$\mathcal{C}(f) := \int_{\mathbb{R}^3} \Phi\left((L-L_0)_+^{-l} f(x,v)\right)(L-L_0)_+^{l}\,dvdx, \qquad (1.13)$$

with $0 < k \leq l$ as in (1.7), Φ convex, satisfying certain growth conditions, and this will be a conserved quantity for spherically symmetric f only. To simplify our presentation, we focus on the case

$$\Phi(f) = f^{1+1/k}$$

which will lead to stationary solutions of the form (1.7). The Casimir functional then reads

$$\mathcal{C}(f) := \int_{\mathbb{R}^3} f^{1+1/k}(x,v)(L-L_0)_+^{-l/k}\,dvdx. \qquad (1.14)$$

At one point we need a scaling argument, which gets complicated in the case of a translation in L in the Casimir-functional. Here we exploit the spherical symmetry and use coordinates adapted to it: If $f=f(x,v)$ is spherically symmetric, we have

$$f(x,v)=\tilde{f}(r,w,L),$$

with $r=|x|, w=\frac{x\cdot v}{r}$ and L as in (1.10), see Section 1.4. Altogether, we want to minimize the energy-Casimir functional

$$\mathcal{H}_C(f)=E_{\text{kin}}(f)+E_{\text{pot}}(f)+\mathcal{C}(f),$$

with $E_{\text{kin}}, E_{\text{pot}}$ from (1.12) and $\mathcal{C}(f)$ as in (1.14) over the set

$$\mathcal{F}_M := \Bigg\{ f\in L^1(\mathbb{R}^6) \ | \ f\geq 0, \ f \text{ is spherically symmetric}, \ \iint f = M, \\ E_{\text{kin}}(f)+\mathcal{C}(f)<\infty, \ f(x,v)=0 \text{ a.e. for } 0\leq L<L_0 \Bigg\}. \tag{1.15}$$

See (1.9) for the definition of spherical symmetry.

This chapter is organized as follows: In the next section, we prove a global existence result for the system (1.2)–(1.4). Afterwards, we examine the variational problem and we show that $\mathcal{H}_C$ is bounded from below in Section 1.3. Then we prove a scaling property and that the the infimum of $\mathcal{H}_C$ is negative in Section 1.4. In Sections 1.5 and 1.6 we show the existence of a minimizer and analyse its properties; it is a stationary solution, and it is nonlinearly stable against sperically symmetric perturbations. Finally, in Section 1.7, we treat the case $M_c=0$.

1.2 Global existence

In order to prepare the stability analysis, we want to prove a global existence result for classical solutions to the system (1.2)–(1.4), so that we know that solutions in a neighbourhood of the examined steady states exist. We want to prove the following theorem.

Theorem 1.1. *Consider the system (1.2)–(1.4). Let $\mathring{f}\in C_c^1$ be a spherically symmetric initial datum with $\mathring{f}(x,v)=0$ for $L:=|x\times v|^2\leq L_0$, where $L_0>0$ is given. Then the corresponding solution (f,U) exists for all time and there exists $R_{min}>0$, such that $f(t,x,v)=0$ for $|x|<R_{min}, \ t\geq 0$, where the number R_{min} only depends on M_c, L_0 and $\mathring{f}$.*

Remark. Without the exterior potential, the global existence result was proved by J. Batt, cf. [3] and this was also the first global existence result for the Vlasov–Poisson system in space dimension three. In our proof given below, the main idea for proving the boundedness of the velocities, is due to E. Horst, cf. [13].

Proof. We fix an initial datum $\mathring{f} \in C_c^1(\mathbb{R}^6)$ with $\mathring{f} \geq 0$ and we fix $\mathring{R}, \mathring{P}$ with

$$\mathring{f}(x,v) = 0 \quad \text{for } |x| \geq \mathring{R} \text{ or } |v| \geq \mathring{P}.$$

This implies $\mathring{f}(x,v) = 0$ for $|x| < \sqrt{L_0}/\mathring{P}$, since $L = \sin^2(\alpha)|v|^2|x|^2 > L_0$ on the support of $\mathring{f}$, where α denotes the angle between x and v.
In the following, we will denote first partial derivatives with respect to x with ∇_x and we will write ∂_x^2 for the second partial derivatives. We now consider the following iteration process to construct the classical solution. The 0th iterate is defined by

$$f_0(t,z) := \mathring{f}(z), \quad t \geq 0,\, z \in \mathbb{R}^6.$$

If the nth iterate f_n is already defined, we define

$$\rho_n := \rho_{f_n} := \int_{\mathbb{R}^3} f_n \, dv, \quad U_n := U_{\rho_n} := -\rho_n * \frac{1}{|\cdot|}, \quad U_{n,\text{eff}} := U_{\rho_n} - \frac{M_c}{|\cdot|}$$

on $[0,\infty[\times\mathbb{R}^3$, and for $L = |x \times v|^2 > L_0$ we denote by

$$Z_n(s,t,z) := (X_n, V_n)(s,t,x,v) \tag{1.16}$$

the solution of the characteristic system

$$\dot{X} = V, \quad \dot{V} = -\nabla_x U_{n,\text{eff}}(s,X) \tag{1.17}$$

with $Z_n(t,t,z) = z$, where we want to examine characteristics which start on the support of $\mathring{f}$. We claim that $|X_n(s,0,z)|$ is bounded from below by a positive constant for all $s \geq 0, n \in \mathbb{N}$, so that the right-hand side of the charcteristic system is well-defined for all time. Together with (1.16)–(1.17) this leads to the definition

$$f_{n+1}(t,x,v) := \begin{cases} \mathring{f}(Z_n(0,t,z)) & \text{for } z = (x,v):\ |x \times v|^2 > L_0 \\ 0 & \text{else.} \end{cases}$$

for the $(n+1)$st iterate. Note that, due to sperical symmetry, $L = |X \times V|^2$ is a conserved quantity of (1.17) and that $\|f_n(t)\|_1 = \|\rho_n(t)\|_1 = \|\mathring{f}\|_1$ since the characteristic flow is measure preserving. We introduce some notations:

$$P_0(t) := \mathring{P},$$

$$P_n(t) := \sup\left\{ |V_{k-1}(s,0,z)| \ \middle|\ z \in \operatorname{supp} \mathring{f},\ 0 \leq s \leq t,\ 1 \leq k \leq n \right\}, \quad n \in \mathbb{N},$$

$$R_{\min}^0(t) := \sqrt{L_0}/\mathring{P},$$

$$\begin{aligned} R_{\min}^n(t) &:= \inf\{|x|,\ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (x,v) \in \operatorname{supp} f_n(s),\ 0 \leq s \leq t\} \\ &= \inf\left\{ |X_{n-1}(s,0,z)| \ \middle|\ z \in \operatorname{supp} \mathring{f},\ 0 \leq s \leq t \right\}, \quad n \in \mathbb{N}. \end{aligned}$$

Next we show

$$P_n(t) \le C_0, \;\; R^n_{\min}(t) \ge C_1, \quad t \in \mathbb{R}^+, \quad n \in \mathbb{N},$$

where $C_0 > 0$ and $C_1 > 0$ only depend on M_c, $\|\mathring{f}\|_1$, $\|\mathring{f}\|_\infty$ and L_0. We abbreviate $X_n(s) := X_n(s,0,z)$ for some $z \in \operatorname{supp} \mathring{f}$ fixed. Now fix $t > 0$; we then have $X_n \in C^2([0,t])$ with

$$\ddot{X}_n(s) = -(m_{\rho_n}(s,|X_n(s)|) + M_c) \cdot \frac{X_n(s)}{|X_n(s)|^3}, \tag{1.18}$$

where we used the spherical symmetry and defined

$$m_{\rho_n}(s,r) := 4\pi \int_0^r \tau^2 \rho_n(s,\tau)\, d\tau.$$

To get suitable bounds for the right-hand side of equation (1.18), we firstly use [27], Lemma P1:

$$\begin{aligned} \frac{m_{\rho_n}(s,|X_n(s)|)}{|X_n(s)|^2} &= |\nabla_x U_n(s,X(s))| \\ &\le 3(2\pi)^{2/3} \|\rho_n(t)\|_1^{1/3} \|\rho_n(t)\|_\infty^{2/3} \\ &\le 4 \cdot 3^{1/3} \pi^{4/3} \|\mathring{f}\|_1^{1/3} \|\mathring{f}\|_\infty^{2/3} P_n^2(t) \\ &=: \kappa P_n^2(t). \end{aligned}$$

Furthermore, since $L_n(s) := |X_n(s) \times V_n(s)|^2$ is constant in s,

$$|X_n(s)|^2 \ge \frac{L_0}{|V_n(s)|^2 |\sin^2(\angle(X_n(s),V_n(s)))|} \ge \frac{L_0}{P^2_{n+1}(s)},$$

which implies

$$\left| \frac{X_n(s)}{|X_n(s)|^3} \right| \le \frac{1}{L_0} P^2_{n+1}(t), \quad 0 \le s \le t.$$

We also have

$$|\ddot{X}_n(s)| \le \frac{\|\mathring{f}\|_1 + M_c}{|X_n(s)|^2}.$$

Altogether,

$$|\ddot{X}_n(s)| \le C^* \min\left\{ \frac{1}{|X_n(s)|^2}, P^2_{n+1}(t) \right\}, \quad 0 \le s \le t,$$

where

$$C^* = C^*(\|\mathring{f}\|_1, \|\mathring{f}\|_\infty, M_c, L_0) = \max\left(\|\mathring{f}\|_1 + M_c, \frac{M_c}{L_0} + \kappa \right).$$

Now define $\xi_n(s) := (X_n(s))_i$ for $i \in \{1,2,3\}$ and $0 \le s \le t$. Then

$$|\ddot{\xi}_n(s)| \le g(\xi(s)), \quad 0 \le s \le t,$$

where

$$g(r) := C^* \min\left\{\frac{1}{r^2}, P_{n+1}^2(t)\right\}, \quad r \in \mathbb{R}.$$

If $\dot{\xi}_n(s) \neq 0$ on $]0,t[$, we have

$$\begin{aligned}\left|\dot{\xi}_n(t) - \dot{\xi}_n(0)\right|^2 &\le \left|\dot{\xi}_n(t) - \dot{\xi}_n(0)\right| \left|\dot{\xi}_n(t) + \dot{\xi}_n(0)\right| \\ &= \left|\dot{\xi}_n(t)^2 - \dot{\xi}_n(0)^2\right| = 2\left|\int_0^t \dot{\xi}_n(s)\ddot{\xi}_n(s)\,ds\right| \\ &\le 2\int_0^t \left|\dot{\xi}_n(s)\right| g\big(\xi_n(s)\big)\,ds = 2\int_{\xi([0,t])} g(r)\,dr \\ &\le 2\int_{\mathbb{R}} g(r)\,dr = 8C^* P_{n+1}(t),\end{aligned}$$

and hence

$$\left|\dot{\xi}_n(t) - \dot{\xi}_n(0)\right| \le 2\sqrt{2C^*}P_{n+1}^{1/2}(t).$$

If $\dot{\xi}_n(s) = 0$ for some $s \in]0,t[$, we define

$$s_- := \inf\{s \in]0,t[\mid \dot{\xi}_n(s) = 0\}, \qquad s_+ := \sup\{s \in]0,t[\mid \dot{\xi}_n(s) = 0\}$$

and the calculation made above implies

$$\begin{aligned}\left|\dot{\xi}_n(t) - \dot{\xi}_n(0)\right| &\le \left|\dot{\xi}_n(t) - \dot{\xi}_n(s_+)\right| + \left|\dot{\xi}_n(s_-) - \dot{\xi}_n(0)\right| \\ &\le 4\sqrt{2C^*}P_{n+1}^{1/2}(t).\end{aligned}$$

Since $\dot{\xi}_n = (\dot{X}_n)_i = (V_n)_i$, we conclude that

$$P_{n+1}(t) \le \mathring{P} + 4\sqrt{6C^*}P_{n+1}^{1/2}(t) \qquad t \ge 0, \quad n \in \mathbb{N}.$$

and therefore

$$P_n(t) \le C_0, \quad n \in \mathbb{N},$$

where C_0 only depends on $\|\mathring{f}\|_1, \|\mathring{f}\|_\infty, L_0, M_c$ and we also have

$$R_{\min}^n(t) \ge \frac{\sqrt{L_0}}{C_0}, \quad n \in \mathbb{N}.$$

Now we can continue with the iterates and prove their convergence. We have

$$f_n \in C^1([0,\infty[\times\mathbb{R}^6), \quad \|f_n(t)\|_\infty = \|\mathring{f}\|_\infty, \quad \|f_n(t)\|_1 = \|\mathring{f}\|_1, \quad t \ge 0,$$

$$f_n(t,x,v)=0 \quad \text{for } |v|\geq P_n(t) \text{ or } |x|\geq \mathring{R}+\int_0^t P_n(s)\,ds, \text{ or } L\leq L_0,$$

and

$$\begin{aligned}
&\rho_n\in C^1([0,\infty[\times\mathbb{R}^3),\\
&\|\rho_n(t)\|_1=\|\mathring{f}\|_1, \quad \|\rho_n(t)\|_\infty\leq\frac{4\pi}{3}\|\mathring{f}\|_\infty P_n^3(t), \quad t\geq 0,\\
&\rho_n(t,x)=0 \quad \text{for } |x|\geq \mathring{R}+\int_0^t P_n(s)\,ds
\end{aligned}$$

We define

$$\|\nabla_x U_{n,\text{eff}}(t)\|_{\min,\infty}:=\sup\left\{|\nabla_x U_{n,\text{eff}}(t,x)| \mid \frac{\sqrt{L_0}}{C_0}\leq|x|<\infty\right\}$$

and $\|\partial_x^2 U_{n,\text{eff}}(t)\|_{\min,\infty}$ is defined analoguously.

Now choose $T_0>0$. We want to prove that there exists a constant $C>0$, which only depends on $T_0,\mathring{f},L_0$ and M_c, such that

$$\|\nabla_x\rho_n(t)\|_\infty+\|\partial_x^2 U_{n,\text{eff}}(t)\|_{\min,\infty}\leq C, \quad t\in[0,T_0],\ n\in\mathbb{N}.$$

In the following, $C>0$ may change from line to line, but there is no dependence on $t\in[0,T_0]$ or $n\in\mathbb{N}$. We have

$$|\nabla_x\rho_{n+1}(t,x)|\leq\int_{|v|\leq P_n(t)}\left|\nabla_x\left[\mathring{f}\,(Z_n(0,t,x,v))\right]\right|dv\leq C\|\nabla_x Z_n(0,t,\cdot)\|_\infty^*,$$

where

$$\|\nabla_x Z_n(0,t,\cdot)\|_\infty^*:=\sup\left\{\left|\nabla_x Z_n(0,t,z)\right| \ \mid\ z: Z(0,t,z)\in\operatorname{supp}\mathring{f}\right\}.$$

Next, fix $x,v\in\mathbb{R}^3$, $t\in[0,T_0]$ and write $Z_n(s)=(X_n,V_n)(s):=(X_n,V_n)(s,t,x,v)$, where we require that $Z_n(0)\in\operatorname{supp}\mathring{f}$. Differentiating the characteristic system with respect to x, we get

$$|\nabla_x\dot{X}_n(s)|\leq|\nabla_x V_n(s)|, \quad |\nabla_x\dot{V}_n(s)|\leq\|\partial_x^2 U_{n,\text{eff}}(s)\|_{\min,\infty}|\nabla_x X_n(s)|.$$

By integrating and noticing $\nabla_x X_n(t)=E,\nabla_x V_n(t)=0$, we have

$$\begin{aligned}
&\left|\nabla_x X_n(s)\right|+\left|\nabla_x V_n(s)\right|\\
&\quad\leq 1+\int_s^t\left(1+\|\partial_x^2 U_{n,\text{eff}}(\tau)\|_{\min,\infty}\right)\left(\left|\nabla_x X_n(\tau)\right|+\left|\nabla_x V_n(\tau)\right|\right)d\tau.
\end{aligned}$$

Gronwall's lemma now implies

$$\left|\nabla_x X_n(s)\right| + \left|\nabla_x V_n(s)\right| \leq \exp \int_0^t \left(1 + \|\partial_x^2 U_{n,\mathrm{eff}}(\tau)\|_{\min,\infty}\right) d\tau,$$

and thus

$$\|\nabla_x \rho_{n+1}(t)\|_\infty \leq C \exp \int_0^t \|\partial_x^2 U_{n,\mathrm{eff}}(\tau)\|_{\min,\infty}\, d\tau.$$

A well known estimate for the Poisson equation then implies, cf.[27], Lemma P1,

$$\|\partial_x^2 U_{n,\mathrm{eff}}(t)\|_{\min,\infty} \leq C\left(1 + \int_0^t \|\partial_x^2 U_{n,\mathrm{eff}}(\tau)\|_{\min,\infty}\, d\tau\right).$$

By induction,

$$\|\partial_x^2 U_{n,\mathrm{eff}}(t)\|_{\min,\infty} \leq C e^{Ct}$$

and thus $\|\partial_x^2 U_{n,\mathrm{eff}}(t)\|_{\min,\infty} \leq C$. Now we show that the sequence (f_n) converges to some function f, uniformly on $[0,T_0] \times \mathbb{R}^3 \times \mathbb{R}^3$. For $n \in \mathbb{N}$ and $z \in \mathbb{R}^6$,

$$|f_{n+1}(t,z) - f_n(t,z)| \leq C|Z_n(0,t,z) - Z_{n-1}(0,t,z)|.$$

For $0 \leq s \leq t$, we have

$$\begin{aligned}
|X_n(s) - X_{n-1}(s)| &\leq \int_s^t |V_n(\tau) - V_{n-1}(\tau)| d\tau, \\
|V_n(s) - V_{n-1}(s)| &\leq \int_s^t \Big[|\nabla_x U_n(\tau, X_n(\tau)) - \nabla_x U_{n-1}(\tau, X_n(\tau))| \\
&\qquad + |\nabla_x U_{n-1,\mathrm{eff}}(\tau, X_n(\tau)) - \nabla_x U_{n-1,\mathrm{eff}}(\tau, X_{n-1}(\tau))| \Big] d\tau \\
&\leq \int_s^t \Big[\big\|\nabla_x U_n(\tau) - \nabla_x U_{n-1}(\tau)\big\|_\infty \\
&\qquad + 2\|\partial_x^2 U_{n-1,\mathrm{eff}}(\tau)\|_{\min,\infty} |X_n(\tau) - X_{n-1}(\tau)| \Big] d\tau,
\end{aligned}$$

where we used the mean value theorem and the factor $2|X_n(\tau) - X_{n-1}(\tau)|$ in the last line is an upper bound for the length of a curve which connects $X_n(\tau)$ with $X_{n-1}(\tau)$ $(s \leq \tau \leq t)$ and avoids the critical area $B_{\sqrt{L_0}/C_0}$ – note again that we have the inequality $R^n_{\min}(t) \geq \sqrt{L_0}/C_0$.

Recalling $\|\partial_x^2 U_{n,\mathrm{eff}}(t)\|_{\min,\infty} \leq C$, adding these estimates and applying Gronwall's lemma, we obtain

$$\begin{aligned}|Z_n(s)-Z_{n-1}(s)| &\leq C\int_s^t \left\|\nabla_x U_n(\tau)-\nabla_x U_{n-1}(\tau)\right\|_\infty d\tau \\ &\leq C\int_0^t \left\|\rho_n(\tau)-\rho_{n-1}(\tau)\right\|_\infty^{2/3}\left\|\rho_n(\tau)-\rho_{n-1}(\tau)\right\|_1^{1/3} d\tau \\ &\leq C\int_0^t \left\|\rho_n(\tau)-\rho_{n-1}(\tau)\right\|_\infty d\tau \\ &\leq C\int_0^t \left\|f_n(\tau)-f_{n-1}(\tau)\right\|_\infty,\end{aligned}$$

where the second inequality follows by splitting the expression

$$|\nabla_x U(x)| \leq \int \frac{\rho(x)}{|x-y|^2}\,dy \leq \int_{|x-y|<R} \frac{\rho(x)}{|x-y|^2}\,dy + \int_{|x-y|\geq R} \frac{\rho(x)}{|x-y|^2}\,dy,$$

and then using Hölder's inequality and an optimization in $R>0$, cf. [27], Lemma P1.

Also note that the support of both $\rho_n(t)$ and $f_n(t)$ is bounded, uniformly in n and $t\in[0,T_0]$. Altogether, we have

$$\left\|f_{n+1}(t)-f_n(t)\right\|_\infty \leq C_*\int_0^t \left\|f_n(\tau)-f_{n-1}(\tau)\right\|_\infty d\tau,$$

and by induction,

$$\left\|f_n(t)-f_{n-1}(t)\right\|_\infty \leq C\frac{C_*^n t^n}{n!} \leq C\frac{C^n}{n!}, \quad n\in\mathbb{N}_0,\ 0\leq t\leq T_0.$$

This implies that the sequence (f_n) is uniformly Cauchy and converges uniformly on $[0,T_0]\times\mathbb{R}^6$ to some function $f\in C([0,T_0]\times\mathbb{R}^6)$, which has the following property:

$$f(t,x,v)=0 \quad \text{for} \quad |v|\geq C_0 \quad \text{or} \quad |x|\geq \mathring{R}+C_0 t.$$

Furthermore,

$$\rho_n \to \rho := \rho_f, \quad U_n \to U := U_f, \quad (n\to\infty),$$

uniformly on $[0,T_0]\times\mathbb{R}^3$. Since $T_0>0$ was arbitrary, the proof is complete once we show that the limit function f has the regularity to be a solution to the Vlasov–Poisson system. With [27], Lemma P1, we have

$$\|\nabla_x U_n(t)-\nabla_x U_m(t)\|_\infty \leq C\|\rho_n(t)-\rho_m(t)\|_\infty^{2/3}\|\rho_n(t)-\rho_m(t)\|_1^{1/3}$$

and

$$\|\partial_x^2 U_n(t) - \partial_x^2 U_m(t)\|_\infty \le C\left[\left(1+\ln\frac{R}{d}\right)\|\rho_n(t)-\rho_m(t)\|_\infty \right.$$
$$\left. + d\|\nabla_x\rho_n(t) - \nabla_x\rho_m(t)\|_\infty + R^{-3}\|\rho_n(t)-\rho_m(t)\|_1\right]$$

for any $0<d\le R$. This implies that the sequences $(\nabla_x U_n)$ and $(\partial_x^2 U_n)$ are also uniform Cauchy sequences on $[0,T_0]\times\mathbb{R}^3$. Indeed, since all ρ_n have compact support, uniformly in n, we can estimate

$$\|\rho_n(t)-\rho_m(t)\|_1 \le C\|\rho_n(t)-\rho_m(t)\|_\infty \le C\|f_n(t)-f_m(t)\|_\infty$$

which converges to zero. For the term with the derivatives of ρ_n, we only know that

$$\|\nabla_x\rho_n(t) - \nabla_x\rho_m(t)\|_\infty \le C$$

with a not necessarily small constant C, but here we can choose $d>0$ in front of this term as small as we want. Hence we have

$$\nabla_x U, \partial_x^2 U \in C([0,T_0]\times\mathbb{R}^3).$$

Now we have for the characterstic flow Z, induced by the limiting field $-\nabla_x U$,

$$Z = \lim_{n\to\infty} Z_n \in C^1([0,T_0]\times[0,T_0]\times\mathbb{R}^6),$$

and finally,

$$f(t,z) = \lim_{n\to\infty} \mathring{f}\big(Z_n(0,t,z)\big) = \mathring{f}\big(Z(0,t,z)\big),$$

so that $f\in C^1\big([0,T_0]\times\mathbb{R}^6\big)$ is a classical solution. □

1.3 A lower bound on $\mathcal{H}_C$

We recall that we want to minimize

$$\mathcal{H}_C(f) = E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f) + \mathcal{C}(f),$$

with $E_{\mathrm{kin}}, E_{\mathrm{pot}}$ from (1.12) and $\mathcal{C}(f)$ as in (1.14) over the set

$$\mathcal{F}_M := \left\{ f\in L^1(\mathbb{R}^6) \;\mid\; f\ge 0,\; f \text{ is spherically symmetric}, \iint f = M, \right.$$
$$\left. E_{\mathrm{kin}}(f) + \mathcal{C}(f) < \infty,\; f(x,v)=0 \text{ a.e. for } 0\le L<L_0 \right\}.$$

Firstly, we want to establish a lower bound on $\mathcal{H}_C$ and we will need several estimates for ρ_f and U_f induced by an element $f \in \mathcal{F}_M$. We will show that $E_{\text{pot}}(f)$ makes sense, that is,

$$\nabla U_f \in L^2(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3} \frac{M_c}{|x|} \rho_f(x)\, dx < \infty.$$

Lemma 1.2. *Let $n := k + l + \frac{3}{2}$. Then there exists $C > 0$, such that*

$$\int \rho_f^{1+\frac{1}{n}}(x)|x|^{-2l/n} dx \le C(\mathcal{C}(f) + E_{\text{kin}}(f)), \quad f \in \mathcal{F}_M.$$

Proof. For any $R > 0$, we have

$$\begin{aligned}
\rho_f(x) &= \int f(x,v)\, dv \\
&= \int_{|v|\le R} f(x,v)\, dv + \int_{|v|\ge R} f(x,v)\, dv \\
&\le \int_{|v|\le R} (L - L_0)_+^{\frac{l}{k+1}} f(x,v)(L-L_0)_+^{-\frac{l}{k+1}} dv + \frac{1}{R^2}\int |v|^2 f(x,v)\, dv \\
&\le C\left(\int_{|v|\le R} (L-L_0)_+^l\, dv\right)^{\frac{1}{k+1}} \left(\int f^{1+\frac{1}{k}}(x,v)(L-L_0)_+^{-l/k}\, dv\right)^{\frac{k}{k+1}} \\
&\quad + \frac{1}{R^2}\int |v|^2 f(x,v)\, dv \\
&\le C|x|^{\frac{2l}{k+1}} R^{\frac{2l+3}{k+1}} \left(\int f^{1+\frac{1}{k}}(x,v)(L-L_0)_+^{-l/k}\, dv\right)^{\frac{k}{k+1}} \\
&\quad + \frac{1}{R^2}\int |v|^2 f(x,v)\, dv.
\end{aligned}$$

Optimization in R yields

$$R := \left[\left(2\int |v|^2 f(x,v)\, dv\right)|x|^{\frac{-2l}{k+1}} \left(\int f^{1+\frac{1}{k}}(x,v)(L-L_0)_+^{-l/k}\, dv\right)^{\frac{-k}{k+1}}\right]^{\frac{k+1}{2l+2k+5}},$$

and thus

$$\begin{aligned}
\rho_f(x) &\le C|x|^{\frac{2l}{k+l+5/2}} \left(\int f^{1+\frac{1}{k}}(x,v)(L-L_0)_+^{-l/k}\, dv\right)^{\frac{k}{l+k+5/2}} (E_{\text{kin}}(f))^{\frac{l+3/2}{l+k+5/2}} \\
&\le C|x|^{\frac{2l}{k+l+5/2}} \left(E_{\text{kin}}(f) + \int f^{1+\frac{1}{k}}(x,v)(L-L_0)_+^{-l/k}\, dv\right)^{\frac{n}{n+1}}.
\end{aligned}$$

Taking both sides of the inequality to the power $1+\frac{1}{n}$, dividing by $r^{\frac{2l}{n}}$ and integrating with respect to x proves the assertion. □

From Lemma 1.2 we see that a function f lying in $\mathcal{F}_M$ and its induced density ρ_f automatically are elements of certain Banach spaces which we now define:

$$L^{k,l}(\mathbb{R}^6) := \Big\{ f : \mathbb{R}^6 \to \mathbb{R} \text{ measurable, spherically symmetric and} \\ \iint |f|^{1+\frac{1}{k}} (L-L_0)_+^{-l/k}\, dxdv < \infty \Big\}$$

equipped with the norm

$$\|f\|_{k,l} := \left(\iint |f|^{1+\frac{1}{k}} (L-L_0)_+^{-l/k}\, dxdv \right)^{\frac{k}{k+1}}$$

and

$$L^{n,l}(\mathbb{R}^3) := \Big\{ \rho : \mathbb{R}^3 \to \mathbb{R} \text{ measurable, spherically symmetric and} \\ \int |\rho|^{1+\frac{1}{n}} |x|^{-2l/n} dx < \infty \Big\}$$

with norm

$$\|\rho\|_{n,l} := \left(\int |\rho|^{1+\frac{1}{n}} |x|^{-2l/n} dx \right)^{\frac{n}{n+1}}.$$

Both spaces are reflexive Banach spaces. More precisely, f and ρ_f are contained in the subsets $L_+^{k,l}(\mathbb{R}^6)$ and $L_+^{n,l}(\mathbb{R}^3)$, respectively, which consist of the a.e.-nonnegative functions of these spaces.

We now need some notations which clarify what $E_{\text{pot}}(f)$ and ∇U_f means for $f \in \mathcal{F}_M$. For spherically symmetric $\rho \in C_c^1(\mathbb{R}^3)$ Poisson's equation becomes

$$\frac{1}{r^2}\left(r^2 U'(r)\right)' = 4\pi\rho(r),$$

where $r := |x|$ and we have $U \in C^2(\mathbb{R}^3)$ and $U'(r) = 4\pi \int_0^r s^2 \rho(s)\, ds / r^2$; in particular, $\nabla U(x) = U'(r)\frac{x}{r}$. This motivates the following definitions. For $f \in \mathcal{F}_M$, i.e., $\rho := \rho_f \in L^{n,l}(\mathbb{R}^3)$, we define

$$m_\rho(r) := \int_{|x|\le r} \rho(x)\, dx = 4\pi \int_0^r s^2 \rho(s)\, ds. \tag{1.19}$$

$$U_\rho'(r) := \frac{m_\rho(r)}{r^2} \tag{1.20}$$

$$\nabla U_\rho(x) := \frac{m_\rho(r)}{r^2}\frac{x}{r} \tag{1.21}$$

$$U_\rho(r) := -\int_r^\infty U_\rho'(s)\, ds \tag{1.22}$$

and we will sometimes write U_f or ∇U_f instead of $U_{\rho_f} = U_\rho$ or $\nabla U_{\rho_f} = \nabla U_\rho$, if ρ_f is induced by f. The definition (1.21) implies

$$\int_{\mathbb{R}^3} |\nabla U_{\rho_f}(x)|^2\,dx = 4\pi \int_0^\infty \frac{m_\rho^2(r)}{r^2}\,dr.$$

Now we can state the next lemma.

Lemma 1.3. *(a) Define the function $\zeta \in C(\mathbb{R}^+)$ by*

$$\zeta(R) = \begin{cases} R^{q_1} \text{ for } 0 \le R \le 1 \\ R^{q_2} \text{ for } 1 < R < \infty \end{cases},$$

where $q_1 := l - k + 1/2 > 0$ and $q_2 := 4l + 5 - n > 0$. Then there exists a constant $C > 0$ such that for $\rho \in L^{n,l}(\mathbb{R}^3)$ with $\int \rho(x)dx = M$ we have

$$\begin{aligned} -E_{\text{pot}}(\rho) &:= \frac{1}{8\pi}\int |\nabla U_\rho|^2 dx + \int \frac{M_c}{|x|}\rho(x)dx \\ &\le \frac{1}{2}\int_0^R \left(\frac{m_\rho^2(r)}{r^2} + 8\pi M_c r\rho(r)\right)dr + \frac{1}{2R}\left(M^2 + 2MM_c\right) \\ &\le C\zeta(R)(1 + \|\rho\|_{n,l}^{1+\frac{1}{n}}) + \frac{1}{2R}\left(M^2 + 2MM_c\right), \qquad R > 0 \end{aligned}$$

where U_ρ denotes the potential induced by ρ.

(b) For every $R > 0$ the mapping

$$T : L^{n,l}(\mathbb{R}^3) \ni \rho \mapsto \frac{m_\rho}{r}|_{[0,R]} \in L^2([0,R])$$

is compact.

(c) For $\rho_1, \rho_2 \in L^{n,l}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ we have

$$\int \nabla U_{\rho_1} \cdot \nabla U_{\rho_2}\,dx = -4\pi \int U_{\rho_1}\rho_2\,dx.$$

Proof. Obviously, we have $m_\rho(r) \le M$, and this shows the first estimate of (a). Now for $\rho \in L^{n,k}(\mathbb{R}^3)$, we have

$$\begin{aligned} \int_0^R r\rho(r)dr &= \int_0^R r^{\frac{l-k-1/2}{n+1}} r^{\frac{2k+3}{n+1}} \rho(r)\,dr \\ &\le \left(\int_0^R r^{l-k-1/2}\,dr\right)^{\frac{1}{n+1}} \left(\int_0^R r^{\frac{2k+3}{n}} \rho^{1+1/n}(r)\,dr\right)^{\frac{n}{n+1}} \\ &\le CR^{l-k+1/2}\|\rho\|_{n,l} \\ &\le CR^{l-k+1/2}(1 + \|\rho\|_{n,l}^{1+\frac{1}{n}}) \end{aligned}$$

where we used Hölder's inequality in the second line. Furthermore, again by Hölder's inequality,

$$|m_\rho(r)| \le C r^{(2l+3)/(n+1)} \|\rho\|_{n,l}, \quad r \ge 0, \tag{1.23}$$

and thus

$$\begin{aligned}\int_0^R \frac{m_\rho^2(r)}{r^2} dr &\le C\|\rho\|_{n,l}^2 R^{(4l+5-n)/(n+1)} \\ &\le C R^{(4l+5-n)/(n+1)} (1+\|\rho\|_{n,l}^{1+\frac{1}{n}}),\end{aligned} \tag{1.24}$$

which implies the estimate in (a). As to (b), by (1.24) we already know that the operator T is bounded. To show the compactness of T, we use the Fréchet-Kolmogorov criterion, cf. [31], Theorem X.1. We take a bounded set $K \in L^{n,l}$ and to show the precompactness of TK, we redefine $T\rho := \frac{m_\rho}{r}\chi_{[0,R]} \in L^2(\mathbb{R})$. The crucial part is to show that

$$\|(T\rho)_h - T\rho\|_2 \to 0, \quad h \to 0,$$

uniformly in $\rho \in K$, where $(T\rho)_h := (T\rho)(\cdot + h)$. For $h > 0$, we have

$$\begin{aligned}&\left\| \frac{m_\rho(r+h)}{r+h}\chi_{[0,R]}(r+h) - \frac{m_\rho(r)}{r}\chi_{[0,R]}(r)\right\|_2^2 \\ &\quad \le \int_0^h \frac{m_\rho^2(r+h)}{r^2} dr + \int_0^h \frac{m_\rho^2(r)}{r^2} dr + \int_{R-h}^R \frac{m_\rho^2(r)}{r^2} dr \\ &\qquad + \int_h^{R-h} m_\rho^2(r) \left| \frac{1}{r+h} - \frac{1}{r}\right|^2 dr \\ &\qquad + \int_h^{R-h} \frac{1}{(r+h)^2} \left| m_\rho(r+h) - m_\rho(r)\right|^2 dr\end{aligned}$$

For the first four terms, one can use the estimate (1.23). Indeed, for example,

$$\begin{aligned}\int_{R-h}^R \frac{m_\rho^2(r)}{r^2} dr &\le C \int_{R-h}^R \|\rho\|_{n,l}^2 r^{(4l+4-2n)/(n+1)} dr \\ &= C\|\rho\|_{n,l}^2 \left(R^{(4l+5-n)/(n+1)} - (R-h)^{(4l+5-n)/(n+1)}\right)\end{aligned}$$

and

$$\int_h^{R-h} m_\rho^2(r) \left| \frac{1}{r+h} - \frac{1}{r}\right|^2 dr = \int_h^{R-h} \frac{m_\rho^2(r)}{r^2} \left(\frac{h}{r+h}\right)^2 dr,$$

which converges to zero by Lebesgue's theorem. We have

$$|m_\rho(r+h) - m_\rho(r)| \le C\|\rho\|_{n,l} \left((r+h)^{(2l+3)/(n+1)} - r^{(2l+3)/(n+1)}\right),$$

and again by Lebesgue's theorem, also the last term converges to zero. Each term coneverges uniformly in $\rho \in K$ and the case $h<0$ is completely analoguous.
As to (c), we firstly show the assertion for $\rho_1,\rho_2 \in C^\infty \cap L^{n,l} \cap L^1$. An integration by parts gives

$$\begin{aligned}\int \nabla U_{\rho_1} \cdot \nabla U_{\rho_2}\, dx &= 4\pi \int_{\mathbb{R}^+} U'_{\rho_1}(r) m_{\rho_2}(r)\, dr \\ &= 4\pi U_{\rho_1}(r) m_{\rho_2}(r)\Big|_{r=0}^{r=\infty} - (4\pi)^2 \int_{\mathbb{R}^+} U_{\rho_1}(r) r^2 \rho_2(r)\, dr \\ &= -4\pi \int U_{\rho_1}\rho_2\, dx,\end{aligned}$$

where the boundary term at infinity vanishes since $|U_{\rho_1}(r)| \le \|\rho_1\|_1/r$ and $m_{\rho_2}(r) \le \|\rho_2\|_1$ and the boundary term at zero vanishes since $m_{\rho_2}(r) = O(r^2),\ r\to 0$. Now we consider approximating sequences $(\rho_1^j),(\rho_2^j) \subset L^{n,l} \cap C^\infty \cap L^1$ such that for $i=1,2$

$$\rho_i^j \to \rho_i \quad \text{in } L^{n,l} \quad (j\to\infty),$$

and $\|\rho_i^j\|_1 \le \|\rho_i\|_1$. Using the estimates of (a), we conclude that the above identity still holds for $\rho_i \in L^{n,l}\cap L^1$ and the proof is complete. □

Lemma 1.4. *There exists a constant $C>0$, such that*

$$\mathcal{H}_C(f) \ge \frac{1}{2}(E_{\mathrm{kin}}(f) + \mathcal{C}(f)) - C, \quad f\in \mathcal{F}_M$$

in particular,

$$h_M := \inf\{\mathcal{H}_C(f) \,|\, f\in\mathcal{F}_M\} > -\infty. \tag{1.25}$$

Proof. Using the previous two lemmas we have

$$\begin{aligned}\mathcal{H}_C(f) &\ge E_{\mathrm{kin}}(f) + \mathcal{C}(f) - C\zeta(R)(1+\|\rho_f\|_{n,l}^{1+\frac{1}{n}}) - \frac{M^2+2MM_c}{2R} \\ &\ge (E_{\mathrm{kin}}(f)+\mathcal{C}(f))(1-C\zeta(R)) - C\zeta(R) - \frac{M^2+2MM_c}{2R},\end{aligned}$$

where $C>0$ is some constant which does not depend on $R>0$. The assertion follows by a suitable choice of R. □

1.4 A scaling lemma

In this section we show that h_M is negative. We also examine the behaviour of $\mathcal{H}_C(f)$, if f is rescaled.

Lemma 1.5. *Define h_M as in (1.25). Then for $M>0$ we have $-\infty<h_M<0$.*

Proof. As already mentioned in the introduction, we will use coordinates adapted to spherical symmetry. If $f(x,v)=f(Ax,Av)\ \forall\ A\in O(3)$, we have

$$f(x,v)=\underline{f}(r,w,L),$$

where $r:=|x|$, $w:=\frac{x\cdot v}{r}$, $L:=|x\times v|^2$ and we will write again f instead of $\underline{f}$.

It is easy to check that, in the new coordinates, the energies and the Casimir functional read

$$\begin{aligned}
E_{\mathrm{kin}}(f)&=2\pi^2\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}(w^2+\frac{L}{r^2})f(r,w,L)\,dLdwdr,\\
E_{\mathrm{pot}}(f)&=-\frac{1}{2}\int_{\mathbb{R}^+}\frac{m_f^2(r)}{r^2}dr-4\pi M_c\int_{\mathbb{R}^+}r\rho_f(r)dr,\\
\mathcal{C}(f)&=4\pi^2\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}f^{1+1/k}(r,w,L)(L-L_0)_+^{-l/k}\,dLdwdr,
\end{aligned}$$

with $\mathbb{R}^+:=[0,\infty[$ and $m_f=m_{\rho_f}$ as in (1.19).

Given any function $f\in\mathcal{F}_M$, we define a rescaled and translated function

$$\bar{f}(r,w,L)=af\left(br,cw,b^2c^2L-(b^2c^2-1)L_0\right), \tag{1.26}$$

where $a,b,c>0$.
Then $\bar{f}(r,w,L)=0$ a.e. if $L<L_0$,

$$\iiint\bar{f}(r,w,L)\,dLdwdr=a(bc)^{-3}\iiint f(r,w,L)\,dLdwdr$$

and if $f\in\mathcal{F}_M$, we have $\bar{f}\in\mathcal{F}_{\bar{M}}$ with $\bar{M}=a(bc)^{-3}M$. Furthermore,

$$E_{\mathrm{kin}}(\bar{f})=2\pi^2ab^{-3}c^{-5}\iiint\left(w^2+\frac{L+(b^2c^2-1)L_0}{r^2}\right)f\,(r,w,L)\,drdwdL, \tag{1.27}$$

$$\mathcal{C}(\bar{f})=a^{1+\frac{1}{k}}b^{-3+\frac{2l}{k}}c^{-3+\frac{2l}{k}}\mathcal{C}(f), \tag{1.28}$$

$$\begin{aligned}
E_{\mathrm{pot}}(\bar{f})&=-\frac{1}{2}\int_{\mathbb{R}^+}a^2b^{-6}c^{-6}\frac{m_f^2(br)}{r^2}dr-4\pi ab^{-2}c^{-3}\int_{\mathbb{R}^+}M_cr\rho_f(r)dr\\
&=-\frac{1}{2}a^2b^{-5}c^{-6}\int_{\mathbb{R}^+}\frac{m_f^2(r)}{r^2}dr-4\pi M_cab^{-2}c^{-3}\int_{\mathbb{R}^+}r\rho_f(r)dr.
\end{aligned} \tag{1.29}$$

To prove the lemma, we consider the case $bc<1$. Here we have

$$E_{\mathrm{kin}}(\bar{f})\le ab^{-3}c^{-5}E_{\mathrm{kin}}(f). \tag{1.30}$$

Now we fix some $f\in\mathcal{F}_1$ with compact support and let

$$a=M(bc)^3.$$

Consequently,

$$\begin{aligned}\mathcal{H}_C(\bar{f}) &\le a^{1+\frac{1}{k}} b^{-3+\frac{2l}{k}} c^{-3+\frac{2l}{k}} \mathcal{C}(f) + ab^{-3}c^{-5} E_{\mathrm{kin}}(f) \\ &\quad - \frac{1}{2} a^2 b^{-5} c^{-6} \int_{\mathbb{R}^+} \frac{m_f^2(r)}{r^2} dr - 4\pi M_c a b^{-2} c^{-3} \int_{\mathbb{R}^+} r\rho_f(r) dr \\ &\le C_1 a^{\frac{1}{k}} (bc)^{\frac{2l}{k}} + C_2 c^{-2} - C_3 b,\end{aligned}$$

where $C_1, C_2, C_3 > 0$ depend on f and M. Since we want the last term to dominate as $b \to 0$, we let $c = b^{-\eta/2}$, so that $bc = b^{1-\frac{\eta}{2}}$ for some $\eta \in]1,2[$. For b small enough we have $bc < 1$ and

$$\mathcal{H}_C(\bar{f}) \le C_1 b^{(1-\frac{\eta}{2})(2l+3)/k} + C_2 b^{\eta} - C_3 b.$$

Now fix $\eta \in]1,2[$ such that $(1-\frac{\eta}{2})(2l+3)/k > 1$; such an η exists by the assumptions on k and l. For $b > 0$ sufficiently small, the sum of the last three terms will be negative and the assertion follows. □

In the next section, we will use the rescaling formulas (1.27)–(1.29) to show that a function f_0, constructed by the weak limit of a minimizing sequence actually is a minimizer with mass M.

1.5 Existence and properties of minimizers

Theorem 1.6. *Let $M > 0$, $L_0 > 0$ and let $(f_j) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}_C$. Then there is a minimizer f_0 and a subsequence (f_{j_k}) such that $\mathcal{H}_C(f_0) = h_M$ and $f_{j_k} \rightharpoonup f_0$ weakly in $L^{k,l}$. For the induced potentials we have $\nabla U_{j_k} \to \nabla U_0$ strongly in $L^2(\mathbb{R}^3)$.*

Proof. By Lemma 1.4, $E_{\mathrm{kin}}(f_j) + \mathcal{C}(f_j)$ is bounded and thus (f_j) is bounded in $L^{k,l}$. Now there exists a weakly convergent subsequence, denoted by (f_j) again:

$$f_j \rightharpoonup f_0 \quad \text{weakly in } L^{k,l}.$$

Clearly, $f_0 \ge 0$ a.e. and $f_0(x,v) = 0$ a.e. for $0 \le L < L_0$. By weak convergence,

$$E_{\mathrm{kin}}(f_0) \le \limsup_{j\to\infty} E_{\mathrm{kin}}(f_j) < \infty. \tag{1.31}$$

By Lemma 1.2, $(\rho_j) = (\rho_{f_j})$ is bounded in $L^{n,l}(\mathbb{R}^3)$. After choosing another subsequence, we conclude that

$$\rho_j \rightharpoonup \rho_0 \quad \text{weakly in } L^{n,l}, \tag{1.32}$$

where we have the identity

$$\rho_0 = \rho_{f_0} := \int f_0(x,v)\, dv.$$

Indeed, assume we would have $\rho_{f_0} > \rho_0$ a.e. on the measurable set $A := A_{R_1,R_2} := \{x \in \mathbb{R}^3 \,|\, R_1 < |x| < R_2 \text{ with } 0 < R_1 < R_2 < \infty\}$, note that both ρ_0 and ρ_{f_0} are spherically symmetric. Then for $R > 0$, by weak convergence we have

$$\begin{aligned} 0 < \gamma &:= \int_A \big(\rho_{f_0}(x) - \rho_0(x)\big)\, dx \\ &= \lim_{j\to\infty} \int_A \int_{|v|<R} f_j(x,v)\, dv dx + \int_A \int_{|v|>R} f_0(x,v)\, dv dx - \\ &\quad - \lim_{j\to\infty} \int_A \rho_j(x)\, dx, \end{aligned}$$

where we used the fact that $\chi_A \in \big(L^{n,l}\big)^*$ and $\chi_{A\times B_R} \in \big(L^{k,l}\big)^*$. Now $E_{\mathrm{kin}}(f_j)$ is bounded and this implies

$$\int_A \int_{|v|>R} f_0(x,v)\, dv dx \le \frac{2}{R^2} E_{\mathrm{kin}}(f_0) \le \frac{2}{R^2} \limsup_{j\to\infty} E_{\mathrm{kin}}(f_j) \le \frac{C}{R^2}.$$

We conclude

$$|\gamma| \le \frac{C}{R^2} + \lim_{j\to\infty} \int_A \int_{|v|>R} f_j(x,v)\, dv dx \le \frac{2C}{R^2},$$

which is a contradiction.

Next, from (1.32) together with Lemma 1.3 (a) (b), the strong convergence

$$\nabla U_j \to \nabla U_0 \quad \text{strongly in } L^2(\mathbb{R}^3), \tag{1.33}$$

follows, and we have

$$E_{\mathrm{pot}}(f_j) \to E_{\mathrm{pot}}(f_0).$$

Indeed, from Lemma 1.3 we have

$$\begin{aligned} \frac{1}{4\pi} \int |\nabla U_j - \nabla U_0|^2\, dx &= \int_0^\infty \frac{m^2_{\rho_{f_j} - \rho_{f_0}}}{r^2}\, dr \\ &\le \int_0^R \frac{m^2_{\rho_{f_j} - \rho_{f_0}}}{r^2}\, dr + \frac{M^2}{R} =: I + II. \end{aligned}$$

Now let $\epsilon > 0$ be given. Choose $R > 0$ large enough so that $II < \epsilon/2$. For j sufficiently large, the first term also will be smaller than $\epsilon/2$ because of the compactness of T, defined in Lemma 1.3 (b): The weak convergence $\rho_{f_j} \rightharpoonup \rho_{f_0}$ implies the strong convergence $m_{\rho_{f_j} - \rho_{f_0}}/r \to 0$ in $L^2([0,R])$.

Furthermore, we can estimate the interaction term as

$$\left| \int_{\mathbb{R}^3} \frac{1}{|x|} (\rho_j(x) - \rho_0(x))\, dx \right| \le \left| \int_{B_R} \frac{1}{|x|} (\rho_j(x) - \rho_0(x))\, dx \right| + \frac{2M}{R}.$$

Here the first term tends to zero, because of the weak convergence (1.32) together with the fact that $\langle \frac{1}{|x|}, \cdot \rangle_{L^2(B_R)} \in \left(L^{n,l}(\mathbb{R}^3)\right)^*$ which we have shown in the proof of Lemma 1.3(a). The same argument as above then proves

$$\left| \int_{\mathbb{R}^3} \frac{1}{|x|} (\rho_j(x) - \rho_0(x))\, dx \right| \to 0.$$

Next, we show that f_0 actually is a minimizer, in particular $E_{\text{kin}}(f_0) + \mathcal{C}(f_0) < \infty$. By weak covergence, we have

$$\mathcal{C}(f_0) = \|f_0\|_{k,l}^{(k+1)/k} \leq \liminf_{j\to\infty} \|f_j\|_{k,l}^{(k+1)/k} < \infty.$$

Together with (1.31) and (1.33) this implies

$$E_{\text{kin}}(f_0) + \mathcal{C}(f_0) \leq \lim_{j\to\infty} (E_{\text{kin}}(f_j) + \mathcal{C}(f_j)) < \infty,$$

note that the $\lim_{j\to\infty}$ in the above inequality exists. Finally,

$$\mathcal{H}_C(f_0) = \mathcal{C}(f_0) + E_{\text{kin}}(f_0) + E_{\text{pot}}(f_0) \leq \lim_{j\to\infty} (\mathcal{C}(f_j) + E_{\text{kin}}(f_j) + E_{\text{pot}}(f_j)) = h_M.$$

It remains to show that $\|f_0\|_1 = M$. By weak convergence, we have $\|f_0\|_1 \leq M$ and we already know that $\|f_0\|_1 > 0$, since $h_M < 0$. Now assume that $M_0 := \|f_0\|_1 < M$. We consider the rescaled function $\bar{f}_0$ defined in (1.26) in section 4 and recall formulas (1.27)–(1.29). Now define

$$a := 1, \quad c := \left(\frac{M_0}{M}\right)^{-1/3}, \quad b := c^{-2}.$$

This implies $(bc)^{-3} = M/M_0$ and thus $\|\bar{f}_0\|_1 = M$. We have

$$\begin{aligned} h_M &\leq \mathcal{H}_C(\bar{f}_0) \\ &\leq c E_{\text{kin}}(f_0) + c^{3-2l/k}\mathcal{C}(f_0) - \frac{1}{2}c^4 \int_{\mathbb{R}^+} \frac{m_f^2(r)}{r^2} dr - c4\pi M_c \int_0^\infty r\rho_f(r) dr, \end{aligned}$$

where we used (1.30), note that $bc = c^{-1} < 1$. Since $c > 1$ and $0 < k \leq l$ we conclude

$$h_M \leq \mathcal{H}_C(\bar{f}_0) \leq c\mathcal{H}_C(f_0) = \left(\frac{M}{M_0}\right)^{1/3} h_M, \tag{1.34}$$

which is a contradiction. □

Theorem 1.7. *Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}_C$. Then there exists $E_0 < 0$ such that*

$$f_0(x,v) = \frac{k}{k+1}(E_0 - E)_+^k (L - L_0)_+^l \tag{1.35}$$

where

$$E := \frac{1}{2}v^2 + U_0(x) - \frac{M_c}{|x|} \tag{1.36}$$

and U_0 is the potential induced by f_0. Moreover, f_0 is a steady state of the Vlasov-Poisson system (1.2)–(1.4).

Proof. Let f_0 be a minimizer. We choose a suitable representative for f_0 and define for $\epsilon > 0$ the set

$$K_\epsilon := \left\{ (x,v) | \epsilon < f_0(x,v) \le \frac{1}{\epsilon},\ L_0 + \epsilon \le L \le L_0 + \frac{1}{\epsilon} \right\}.$$

Since $f_0 \in L^{k,l}$ we have $0 < |K_\epsilon| < \infty$ for ϵ sufficiently small. Now let $g \in L^\infty(\mathbb{R}^6)$ be spherically symmetric with supp $g \subset K_\epsilon$, and

$$h := g - \frac{1}{|K_\epsilon|}\left(\iint g\, dv dx\right) \cdot \chi_{K_\epsilon}.$$

Then for $\tau \in \mathbb{R}$ small enough we have $f_0 + \tau h \ge 0$ and $f_0 + \tau h \in \mathcal{F}_M$, indeed, $E_{\text{kin}}(f_0 + \tau h) < \infty$ and

$$\mathcal{C}(f_0 + \tau h) = \mathcal{C}(f_0) + \tau \iint \Phi'(f_0)(L - L_0)_+^{-\frac{l}{k}} h + o(\tau) < \infty,$$

where we recall that $\Phi(f) = f^{1+1/k}$. Now we have

$$\begin{aligned} 0 \le \mathcal{H}_C(f_0 + \tau h) - \mathcal{H}_C(f_0) = & \\ = \tau \iint \left(\Phi'(f_0)(L - L_0)_+^{-l/k} + \frac{1}{2}v^2 + U_0(x) - \frac{M_c}{|x|} \right) h\, dv dx + o(\tau) & \\ = \tau \iint \left(\Phi'(f_0)(L - L_0)_+^{-l/k} + E \right) h\, dv dx + o(\tau), & \end{aligned}$$

where we used Lemma 1.3 (c) to calculate the potential energy term. Since $-h$ is also an admissible function, this implies

$$\iint \left(\Phi'(f_0)(L - L_0)_+^{-l/k} + E \right) h\, dv dx = 0.$$

Inserting the definition of h we get

$$\iint \left[\left(\Phi'((L - L_0)_+^{-l} f_0) + E \right) - \frac{1}{|K_\epsilon|} \iint_{K_\epsilon} \left(\Phi'((L - L_0)_+^{-l} f_0) + E \right) \right] g\, dv dx = 0.$$

Consequently,

$$\Phi'((L - L_0)_+^{-l} f_0) + E = E_\epsilon \quad \text{a.e. on } K_\epsilon,$$

where

$$E_\epsilon := \frac{1}{|K_\epsilon|} \iint_{K_\epsilon} \left(\Phi'((L-L_0)_+^{-l} f_0) + E \right) dv dx.$$

Thus for ϵ small, E_ϵ will be a constant which we denote by E_0 and we conclude

$$\Phi'((L-L_0)_+^{-l} f_0) + E = E_0 \quad \text{a.e. on } \{(x,v) | f_0(x,v) > 0\}. \tag{1.37}$$

Suppose now, there would exist a measurable set $A \subset \{(x,v) | f_0(x,v) = 0, L_0 \leq L\}$ with

$$E < E_0 \quad \text{a.e. on } A$$

and $0 < |A| < \infty$. We can also assume that A is spherically symmetric, i.e. χ_A is spherically symmteric. Next, define

$$h := \chi_A - \frac{1}{|K_\epsilon|} \left(\iint \chi_A dv dx \right) \cdot \chi_{K_\epsilon}$$

with K_ϵ as above and small $\epsilon > 0$. Then for $\tau > 0$ sufficiently small we have $f_0 + \tau h \in \mathcal{F}_M$ and again

$$0 \leq \mathcal{H}_C(f_0 + \tau h) - \mathcal{H}_C(f_0) = \tau \iint \left(\Phi'((L-L_0)_+^{-l} f_0) + E \right) h dv dx + o(\tau).$$

Plugging the definition of h into the above equation, we have

$$\begin{aligned} 0 &\leq \iint \left(\Phi'((L-L_0)_+^{-l} f_0) + E \right) \chi_A - E_0 \iint \chi_A \\ &= \iint_A (E - E_0) < 0, \end{aligned}$$

a contradiction and thus $E \geq E_0$ a.e. on $\{(x,v) | f_0(x,v) = 0, L_0 \leq L\}$. Together with (1.37) this implies that f_0 is of the form given in the theorem.

Since f_0 is a function of the microscopic energy E defined by (1.36) and L, it is constant along solutions of the characteristic system

$$\begin{cases} \dot{X} = V \\ \dot{V} = -\nabla_x U_0(X) - \frac{M_c}{|X|^3} X \end{cases}$$

and thus f_0 is a solution of the Vlasov equation, provided the potential U_0 is sufficiently smooth. But one can indeed show that $U_0 \in C^2(\mathbb{R}^3)$. This can be seen as follows. We firstly recall the formula for ρ_{f_0}, if f_0 is of form (1.35),

$$\rho_0(r) := \rho_{f_0}(r) = C(k,l) r^{2l} \left(E_0 - U_0(r) + \frac{M_c}{r} - \frac{L_0}{r^2} \right)_+^{k+l+3/2} \tag{1.38}$$

and we claim that $U_0 \in L^\infty(\mathbb{R}^+)$ and thus the above equation implies $\rho_{f_0} \in L^1 \cap L^\infty$. Indeed, for any $R > r$,

$$\begin{aligned} -U_0(r) &= \int_r^R \frac{m_{\rho_0}(s)}{s^2} ds + \int_R^\infty \frac{m_{\rho_0}(s)}{s^2} ds \\ &\leq C \int_r^R s^{(-2k-2)/(n+1)} \|\rho_0\|_{n,l} ds + \frac{M}{R} \\ &= C\|\rho_0\|_{n,l} \big(R^{(-k+l+1/2)/(n+1)} - r^{(-k+l+1/2)/(n+1)}\big) + \frac{M}{R}, \end{aligned}$$

and because of $0 < k < l + 1/2$, the claim follows. Now $\rho_{f_0} \in L^1 \cap L^\infty$ implies $U_0 \in C^1$ and because of (1.38) also $\rho_0 \in C^1$. Together with $U_0'(r) = \frac{1}{r^2}\int_0^r s^2 \rho_0(s)\, ds$, the asserted regularity of U_0 is proved.

By construction, we have

$$\Delta U_0 = 4\pi\rho_0,$$

so that (f_0, ρ_0, U_0) is indeed a solution of the Vlasov-Poisson system. It remains to show that $E_0 < 0$. Recall the formula for ρ_0 from (1.38) and the fact that $\|f_0\|_1 = M$. If $E_0 \geq 0$, we would have

$$\|f_0\|_1 = \|\rho_0\|_1 \geq C(k,l) \int_{R_0}^\infty r^{2l+2} \left(\frac{M_c}{2r}\right)^{k+l+3/2} dr = C \int_{R_0}^\infty r^{l-k+1/2} dr = \infty,$$

where we have chosen $R_0 > 0$ sufficiently large so that $L_0/r^2 < M_c/2r,\ r > R_0$. Consequently, we conclude $E_0 < 0$. □

1.6 Dynamical stability

We investigate the nonlinear stability of f_0. For $f \in \mathcal{F}_M$,

$$\mathcal{H}_C(f) - \mathcal{H}_C(f_0) = d(f, f_0) - \frac{1}{8\pi} \int |\nabla U_f - \nabla U_{f_0}|^2 dx, \tag{1.39}$$

where

$$d(f, f_0) := \iint \left[\left(f^{1+1/k} - f_0^{1+1/k}\right)(L - L_0)_+^{-l/k} + (E - E_0)(f - f_0)\right] dv dx,$$

where E is defined as in (1.36). We have $d(f, f_0) \geq 0,\ f \in \mathcal{F}_M$ with $d(f, f_0) = 0$, iff $f = f_0$. Indeed,

$$d(f, f_0) \geq \iint \left[\Phi'((L - L_0)_+^{-l} f_0) + (E - E_0)\right](f - f_0)\, dv dx \geq 0,$$

which is due to the convexity of Φ, and on the support of f_0 the bracket vanishes. This fact allows us to use $d(., f_0)$ to measure the distance to the stationary solution f_0.

Theorem 1.8. *Assume that the minimizer f_0 is unique in $\mathcal{F}_M$. Then for all $\epsilon > 0$ there is $\delta > 0$ such that for any solution $f(t)$ of the Vlasov-Poisson system with $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$,*

$$d(f(0), f_0) + \frac{1}{8\pi} \int |\nabla U_{f(0)} - \nabla U_{f_0}|^2 dx < \delta$$

implies

$$d(f(t), f_0) + \frac{1}{8\pi} \int |\nabla U_{f(t)} - \nabla U_{f_0}|^2 dx < \epsilon, \quad t \geq 0.$$

Proof. We observe that $\mathcal{H}_C$ is conserved along any solution $f(t)$ of the Vlasov-Poisson system with $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$. This follows from conservation of energy and the fact that both $f(t)$ and L are conserved along characteristics. Assume the theorem were false. Then there exists $\epsilon_0 > 0$, $t_j > 0$, and $f_j(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$ such that

$$d(f_j(0), f_0) + \frac{1}{8\pi} \int |\nabla U_{f_j(0)} - \nabla U_{f_0}|^2 dx \leq \frac{1}{j}$$

and

$$d(f_j(t_j), f_0) + \frac{1}{8\pi} \int |\nabla U_{f_j(t_j)} - \nabla U_{f_0}|^2 dx \geq \epsilon_0.$$

From (1.39), we have

$$\lim_{j \to \infty} \mathcal{H}_C(f_j(0)) = h_M,$$

and because $\mathcal{H}_C(f_j(t))$ is conserved,

$$\lim_{j \to \infty} \mathcal{H}_C(f_j(t_j)) = \lim_{j \to \infty} \mathcal{H}_C(f_j(0)) = h_M.$$

Thus $(f_j(t_j)) \subset \mathcal{F}_M$ is a minimizing sequence of $\mathcal{H}_C$ and with Theorem 1.6 we have

$$\frac{1}{8\pi} \int |\nabla U_{f_j(t_j)} - \nabla U_{f_0}|^2 dx \to 0,$$

which implies

$$d(f_j(t_j), f_0) \to 0$$

by (1.39), a contradiction. □

Corollary 1.9. *If in Theorem 1.8 the assumption $\|f(0)\|_{k,l} = \|f_0\|_{k,l}$ is added, then for any $\epsilon > 0$ the parameter $\delta > 0$ can be chosen such that the stability estimate*

$$\|f(t) - f_0\|_{k,l} < \epsilon, \quad t \geq 0$$

holds.

Proof. We repeat the proof of Theorem 1.8 except that in the contradiction assumption have

$$\|f_j(t_j)-f_0\|_{k,l}+d(f_j(t_j),f_0)+\frac{1}{8\pi}\int|\nabla U_{f_j(t_j)}-\nabla U_{f_0}|^2dx\geq\epsilon_0.$$

From the minimizing sequence $f_j(t_j)$ we can now extract a subsequence which converges weakly in $L^{k,l}$ to f_0. But due to our additional restriction we have

$$\|f_j(t_j)\|_{k,l}=\|f_0\|_{k,l},\quad j\in\mathbb{N}.$$

Now the lower semicontinuity of the norm and the uniform convexity of $L^{k,l}(\mathbb{R}^6)$ imply $f_j(t_j)\to f_0$ strongly in $L^{k,l}$. Together with the rest of the proof of Theorem 1.8, the assertion follows. □

Remarks.

(a) The technical assumption $f=0$ a.e. for $0<L<L_0$ in the class of perturbations $\mathcal{F}_M$, see (1.15), is needed for the scaling argument in Lemma 1.5 and it would be desirable to improve it to $f=0$ a.e. for $0<L<\gamma L_0$ for some $0<\gamma<1$.

(b) For $M_c=0$ one can show existence and stability for steady states of form (1.7) for the parameter range $l>-1$, $0<k<l+3/2$, see the next Section 1.7. For $M_c>0$, we had to restrict the parameter range to $0<k\leq l$ in order to guarantee that the scaling argument (1.34) works.

(c) The uniqueness of the minimizer f_0 subject to the fixed mass constraint can be shown by a scaling argument in the case $L_0=0$ and $M_c=0$. For $L_0>0$, at least numerically the minimizer seems to be unique, but the scaling argument fails because of the translation in L. We mention that, for Theorem 1.8, it would suffice if the minimizers of $\mathcal{H}_C$ were isolated.

(d) We only obtain stability against spherically symmetric perturbations, because the quantity L is conserved by the characteristic flow only for spherically symmetric solutions. Stability against asymmetric perturbations is an open problem and more delicate mathematical tools have to be invented to address this question.

1.7 The case $M_c=0$

In this Section, we want to prove existence and stability of stationary solutions of form 1.7, but now with $M_c=0$, i.e., we consider the unmodified Vlasov–Poisson system. Of course, for the parameter range $0<k\leq l$ we can follow Sections 1.3–1.6.

We want to extend the covered parameter range to $0<k<l+3/2$ and $l>-1$. As already mentioned, in this case the scaling argument (1.34) fails, but with the point mass vanishing, we can analyze how h_M behaves, if M varies. We then will show that the mass remains concentrated along a minimizing sequence for the variational problem introduced in Section 1.1, which will easily imply that f_0 has mass M and has compact support.

All lemmas of Section 1.3 hold true and we can use the proofs given there also for the extended parameter range, except for Lemma 1.3, where the case $4l+5-n<0$ can appear, if $n>1$. But here we can use the estimate

$$\int_0^R \frac{m_\rho^2(r)}{r^2} dr \le M^{1-\frac{1}{n}} \int_0^R \frac{m_\rho^{1+\frac{1}{n}}(r)}{r^2} dr \le C \|\rho\|_{n,l}^{1+\frac{1}{n}} R^{(2l+3-n)/n},$$

where $2l+3-n = l-k+3/2 > 0$ and the assertions still hold if we make suitable changes for the parameters q_1, q_2.
We now want to prove an extended version of Lemma 1.5.

Lemma 1.10. *Define $h_M := \inf_{f\in\mathcal{F}_M} \mathcal{H}_C(f)$ with $M_c = 0$ in the definition of $\mathcal{H}_C$.*

(a) Let $M>0$. Then $-\infty < h_M < 0$.

(b) There exists $\alpha > 0$ such that for $0 < M_1 \le M_2$

$$h_{M_1} \ge \left(\frac{M_1}{M_2}\right)^{1+\alpha} h_{M_2}.$$

Proof. We again will use coordinates adapted to spherical symmetry. Recall formula (1.26), where we defined for $f \in \mathcal{F}_M$ the rescaled and translated function

$$\bar{f}(r,w,L) = af\left(br, cw, b^2c^2L - (b^2c^2-1)L_0\right), \tag{1.40}$$

where $a,b,c>0$.
Then $\bar{f}(r,w,L) = 0$ a.e. if $L<L_0$ and if $f\in\mathcal{F}_M$, we have $\bar{f}\in\mathcal{F}_{\bar{M}}$ with $\bar{M} = a(bc)^{-3}M$. Furthermore, we recall the scaling behaviour of the terms of $\mathcal{H}_C$, cf. (1.27) – (1.29), now with $M_c = 0$:

$$E_{\text{kin}}(\bar{f}) = 2\pi^2 ab^{-3}c^{-5} \iiint \left(w^2 + \frac{L+(b^2c^2-1)L_0}{r^2}\right) f(r,w,L)\, dr dw dL, \tag{1.41}$$

$$\mathcal{C}(\bar{f}) = a^{1+\frac{1}{k}} b^{-3+\frac{2l}{k}} c^{-3+\frac{2l}{k}} \mathcal{C}(f),$$

$$E_{\text{pot}}(\bar{f}) = -\frac{1}{2}\int_{\mathbb{R}^+} a^2 b^{-6} c^{-6} \frac{m_f^2(br)}{r^2} dr = a^2 b^{-5} c^{-6} E_{\text{pot}}(f).$$

The mapping $\mathcal{F}_M \to \mathcal{F}_{\bar{M}}$, $f \mapsto \bar{f}$ is bijective and its inverse is

$$f(r,w,L) \mapsto a^{-1} f\left(b^{-1}r, c^{-1}w, \frac{L+(b^2c^2-1)L_0}{b^2c^2}\right).$$

For the proof of (a), we can follow the same lines as in the proof of Lemma 1.5.

As to (b), we distinguish the cases $l \geq 0$ and $-1 < l < 0$. If $l \geq 0$, we take $\bar{f}$ defined by (1.40), but now with $a = c = 1$ and $b > 1$. From (1.41), we have

$$E_{\text{kin}}(\bar{f}) \leq b^{-1} E_{\text{kin}}(f)$$

and

$$E_{\text{kin}}(\bar{f}) \geq b^{-3} E_{\text{kin}}(f).$$

We take $b = (M_1/M_2)^{-1/3} > 1$. This implies

$$\begin{aligned}\mathcal{H}_C(\bar{f}) &\geq b^{-3} E_{\text{kin}}(f) + b^{-5} E_{\text{pot}}(f) + b^{-3+\frac{2l}{k}} \mathcal{C}(f)\\ &\geq b^{-5} E_{\text{kin}}(f) + b^{-5} E_{\text{pot}}(f) + b^{-5} \mathcal{C}(f)\\ &= b^{-5} \mathcal{H}_C(f).\end{aligned}$$

This implies $\alpha = \frac{2}{3}$, notice that $b^{\omega} < 1$ for $\omega < 0$.

If $-1 < l < 0$, we first consider the case $0 < k \leq 1/2$ and we assume $bc \geq 1$ in (1.40). Together with (1.41),

$$E_{\text{kin}}(\bar{f}) \leq ab^{-1}c^{-3} E_{\text{kin}}(f),$$

and

$$E_{\text{kin}}(\bar{f}) \geq ab^{-3}c^{-5} E_{\text{kin}}(f).$$

Now we choose $f \in \mathcal{F}_{M_2}$ and $\bar{f} \in \mathcal{F}_{M_1}$, so that

$$ab^{-3}c^{-3} = \frac{M_1}{M_2} =: m \leq 1. \tag{1.42}$$

Then

$$\begin{aligned}\mathcal{H}_C(\bar{f}) &\geq ab^{-3}c^{-5} E_{\text{kin}}(f) + a^2 b^{-5} c^{-6} E_{\text{pot}}(f) + a^{1+\frac{1}{k}} (bc)^{-3+\frac{2l}{k}} \mathcal{C}(f)\\ &= mc^{-2} E_{\text{kin}}(f) + m^2 b E_{\text{pot}}(f) + m a^{\frac{1}{k}} (bc)^{\frac{2l}{k}} \mathcal{C}(f).\end{aligned}$$

We require

$$m a^{\frac{1}{k}} (bc)^{\frac{2l}{k}} = m^2 b = mc^{-2}.$$

Hence we choose

$$c = m^{\frac{2l+2}{2k-3-2l}}, \quad b = m^{\frac{2l+2k+1}{3+2l-2k}}, \quad a = (bc)^3 m$$

and this implies $bc = m^{(2l+2)/(3+2l-2k)-1} = m^{(2k-1)/(3+2l-2k)} \geq 1$. Then

$$\mathcal{H}_C(\bar{f}) \geq m^{1+\alpha}(E_{\rm kin}(f) + E_{\rm pot}(f) + \mathcal{C}(f)) = m^{1+\alpha}\mathcal{H}_C(f),$$

where $\alpha := (4l+4)/(2l+3-2k) > 0$.

For $-1 < l < 0$ and $k > \frac{1}{2}$ we assume $bc < 1$ in (1.40) and it is easy to check that

$$E_{\rm kin}(\bar{f}) \geq ab^{-1}c^{-3}E_{\rm kin}(f).$$

We take f, $\bar{f}$ and m as in (1.42) above. Finally,

$$\begin{aligned}\mathcal{H}_C(\bar{f}) &\geq ab^{-1}c^{-3}E_{\rm kin}(f) + a^2b^{-5}c^{-6}E_{\rm pot}(f) + a^{1+\frac{1}{k}}(bc)^{-3+\frac{2l}{k}}\mathcal{C}(f)\\ &= mb^2E_{\rm kin}(f) + m^2bE_{\rm pot}(f) + ma^{\frac{1}{k}}(bc)^{\frac{2l}{k}}\mathcal{C}(f).\end{aligned}$$

Defining

$$b := m, \quad c := m^{(2k-4-2l)/(3+2l)},$$

so that $bc = m^{(2k-1)/(3+2l)} < 1$, the assertion follows with $\alpha = 2$. □

The scaling estimate above can be used to show that, along a minimizing sequence, the mass has to remain concentrated.

Lemma 1.11. *Let $M > 0$ and $M_c = 0$. Then there exists a radius $R_M > 0$ such that if $(f_j) \subset \mathcal{F}_M$ is a minimizing sequence of $\mathcal{H}_C$,*

$$\lim_{j\to\infty} \int_{|x|>R} \int f_j \, dv dx = 0, \quad R > R_M.$$

Proof. We define the ball $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$. Let $\chi_{B_R\times\mathbb{R}^3}$ be the characterisitc function of $B_R \times \mathbb{R}^3$. For $f \in \mathcal{F}_M$ we split

$$f_1 := \chi_{B_R\times\mathbb{R}^3}f, \quad f_2 = f - f_1$$

and define $m_i(r) := m_{f_i}(r)$, $i = 1,2$, with $m_{f_i}(r) := 4\pi\int_0^r s^2\rho_{f_i}(s)\,ds$. We abbreviate $\lambda = M - m_f(R)$. Then

$$\begin{aligned}\mathcal{H}_C(f) &= \mathcal{H}_C(f_1) + \mathcal{H}_C(f_2) - \int_0^\infty \frac{m_1(r)m_2(r)}{r^2}dr\\ &\geq h_{M-\lambda} + h_\lambda - \int_0^\infty \frac{m_1(r)m_2(r)}{r^2}dr,\end{aligned}$$

since $f_1 \in \mathcal{F}_{M-\lambda}$ and $f_2 \in \mathcal{F}_\lambda$. Next,

$$\int_0^\infty \frac{m_1(r)m_2(r)}{r^2}dr \leq \lambda(M-\lambda)\int_R^\infty \frac{1}{r^2}dr = \frac{\lambda(M-\lambda)}{R}.$$

Using Lemma 1.10 (b), we find that

$$\mathcal{H}_C(f) \geq \left[\left(1-\frac{\lambda}{M}\right)^{1+\alpha} + \left(\frac{\lambda}{M}\right)^{1+\alpha}\right] h_M - \frac{\lambda(M-\lambda)}{R}.$$

Since the function q defined by

$$q(x) := x^{\alpha+1} + (1-x)^{\alpha+1} + C_\alpha x(1-x)$$

is convex in $[0,1]$ for suitable $C_\alpha > 0$, we have the inequality

$$(1-x)^{1+\alpha} + x^{1+\alpha} - 1 \leq -C_\alpha(1-x)x \quad 0 \leq x \leq 1.$$

Choosing $x = \frac{\lambda}{M}$ and noticing that by Lemma 1.10 (a) $h_M < 0$, we have

$$\begin{aligned}
\mathcal{H}_C(f) - h_M &\geq \left[\left(1-\frac{\lambda}{M}\right)^{1+\alpha} + \left(\frac{\lambda}{M}\right)^{1+\alpha} - 1\right] h_M - \frac{\lambda(M-\lambda)}{R} \\
&\geq -C_\alpha h_M \left(1-\frac{\lambda}{M}\right)\frac{\lambda}{M} - \frac{\lambda(M-\lambda)}{R} \\
&= \left(-\frac{C_\alpha h_M}{M^2} - \frac{1}{R}\right)(M-\lambda)\lambda \\
&= \left(\frac{1}{R_M} - \frac{1}{R}\right) m_f(R)(M - m_f(R)), \qquad (1.43)
\end{aligned}$$

where

$$R_M := -\frac{M^2}{C_\alpha h_M} > 0.$$

Now let $(f_j) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}_C$, and assume the assertion of the lemma is wrong. Then there exist some $R > R_M, \lambda > 0$, and a subsequence called (f_j) again, such that

$$\lim_{j\to\infty} \int_{|x|>R} \int f_j \, dv dx = \lambda.$$

For every $j \in \mathbb{N}$ we can choose $R_j > R$ such that

$$\lambda_j := \int_{|x|>R_j} \int f_j dv dx = \frac{1}{2} \int_{|x|>R} \int f_j \, dv dx.$$

Then

$$\lim_{j\to\infty} \int_{|x|>R_j} \int f_j \, dv dx = \lim_{j\to\infty} \lambda_j = \frac{\lambda}{2} > 0.$$

Applying (1.43), we get

$$\begin{aligned}\mathcal{H}_C(f) - h_M &\geq \left(\frac{1}{R_M} - \frac{1}{R_j}\right)(M - \lambda_j)\lambda_j \\ &> \left(\frac{1}{R_M} - \frac{1}{R}\right)(M - \lambda_j)\lambda_j \\ &\to \left(\frac{1}{R_M} - \frac{1}{R}\right)(M - \frac{\lambda}{2})\frac{\lambda}{2} \ > 0, \quad j \to \infty,\end{aligned}$$

since $0 < \frac{\lambda}{2} < M$. This contradicts (f_j) being a minimizing sequence. □

Now the theorems of Section 1.5 still hold, where we now can show the mass property of f_0 in Theorem 1.6 as follows: By Lemma 1.11 we have

$$\begin{aligned}M &= \lim_{j\to\infty} \int_{|x|<R_1} \int_{|v|\leq R_2} f_j \, dv dx + \lim_{j\to\infty} \int_{|x|<R_1} \int_{|v|\geq R_2} f_j \, dv dx \\ &\leq \lim_{j\to\infty} \int_{|x|<R_1} \int_{|v|\leq R_2} f_j \, dv dx + \frac{C}{R_2^2},\end{aligned}$$

where $R_1 > R_M$ and $R_2 > 0$ are arbitrary. This implies

$$\int_{|x|<R} \int f_0 \, dv dx = M$$

for every $R_1 > R_M$. This proves $\iint f_0 = M$ and also that supp $\rho_{f_0} \subset [0, R]$ for some $R > 0$. This support property also shows that the Lagrange multiplier E_0 in Theorem 1.7 has to be negative. We then can calculate ρ_{f_0} as

$$\rho_{f_0}(r) = C(k,l)\, r^{2l} \left(E_0 - U_0(r) - \frac{L_0}{r^2}\right)_+^{k+l+3/2}.$$

Together with the fact that $-U(r) \leq M/r, \ r > 0$ this implies that supp$\rho_{f_0} \subset [R_1, R]$, for some $0 < R_1 < R$ and with a similar argument as in the proof Theorem 1.7 we can show the boundedness and the regularity properties of U_0. For the extended parameter range, the stability assertions in Section 1.6 hold true without changes in the proofs.

Chapter 2

Existence of axially symmetric solutions to the Vlasov-Poisson system depending on Jacobi's integral

2.1 Introduction

In this chapter, we consider the Vlasov–Poisson system in the following form

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \tag{2.1}$$

$$\Delta U = 4\pi\rho, \tag{2.2}$$

$$\rho(t,x) = \int f(t,x,v)dv. \tag{2.3}$$

We are looking for stationary solutions of (2.1)-(2.3). As already mentioned in the Preface, if f_0 only depends on the particle energy, i.e., if is of the form

$$f_0(x,v) = \Phi(E) = \Phi(\frac{1}{2}v^2 + U(x)), \tag{2.4}$$

this leads to

$$\Delta U = 4\pi h_\Phi(U) = 4\pi \int \Phi(\frac{1}{2}v^2 + U(x))dv, \tag{2.5}$$

and this equation only possesses spherically symmetric solutions, a fact which follows from a more general result of Gidas, Ni and Nirenberg, cf. [6].

If one is interested in stationary solutions with less symmetry, one can try to add more invariants to (2.4), so that the right-hand side of (2.5) explicitly depends on x. These invariants ideally should require no symmetry assumptions on U.

One possibility is to consider a rotating system. If the ensemble is rotating about a given axis, say the x_3-axis, we can change to the rotating frame and change coordinates as follows:

$$\zeta := R_t x, \qquad \eta := R_t v - \Omega \times (R_t x),$$

where

$$R_t := \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Omega := \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

and the (rotational) velocity $\omega > 0$ is given. The Vlasov-Poisson system then takes the form

$$\partial_t f + \eta \cdot \nabla_\zeta f - (\nabla_\zeta U + \Omega \times (\Omega \times \zeta) + 2(\Omega \times \eta)) \cdot \nabla_\eta f = 0, \tag{2.6}$$

$$\Delta_\zeta U(t,\zeta) = 4\pi \rho(t,\zeta), \tag{2.7}$$

$$\rho(t,\zeta) = \int f(t,\zeta,\eta) d\eta \tag{2.8}$$

and the characteristic system of the Vlasov equation (2.6) reads

$$\begin{cases} \dot{\zeta} = \eta \\ \dot{\eta} = -\partial_\zeta U(t,\zeta) - 2\Omega \times \eta - \Omega \times (\Omega \times \zeta) \end{cases},$$

which has the following expression as a conserved quantity, if U is time-independent:

$$E_J := \frac{1}{2}\eta^2 + U(\zeta) - \frac{1}{2}|\Omega \times \zeta|^2,$$

where E_J is also called Jacobi's integral. A natural ansatz for the construction of stationary solutions of (2.6)-(2.8) is now

$$f(\zeta,\eta) = \varphi(E_J) = \varphi(\frac{1}{2}|\eta|^2 + U(\zeta) - \frac{1}{2}\omega^2 r^2) \tag{2.9}$$

for a suitable function $\varphi : \mathbb{R} \to \mathbb{R}^+$, where $r := r(x) = \sqrt{\zeta_1^2 + \zeta_2^2}$. In the original coordinates x, v one easily verfies that this ansatz leads to

$$g(x,v) := f(\zeta,\eta) = \varphi(\frac{1}{2}v^2 + U(R_t x) - \omega P),$$

where we define P as the third component of the angular momentum, that is $P := x_1 v_2 - x_2 v_1$, which is a conserved quantity of the characteristic system of the Vlasov equation (2.1), if U is axially symmetric with respect to the x_3-axis. Obviously, the function $f = f(\zeta,\eta)$ then automatically satisfies (2.6) and one has to solve the Poisson equation, where we relabel ζ and η to x and v,

$$\Delta U = \int \varphi(\frac{1}{2}v^2 + U(x) - \frac{1}{2}\omega^2 r^2)\, dv =: \tilde{h}(\omega, r(x), U(x)). \tag{2.10}$$

So if we construct an axially symmetric U solving (2.10), the corresponding functions (g, U), with g defined as above also will be a stationary solution of (2.1)-(2.3). Clearly, our ansatz for f satisfies (2.6) without any symmetry assumptions on U and this gives hope for the construction of stationary solutions with less symmetry, for example triaxial systems.
Equation (2.10) has been studied, among others, by Vandervoort, cf. [30]. He observed numerically, that if φ is of the form

$$\varphi(E_J) = (E_0 - E_J)_+^{\beta - 3/2}, \tag{2.11}$$

then for $0.5 < \beta \leq 0.808$ there are triaxial solutions to (2.10) for sufficiently large ω. For small ω or $\beta > 0.808$, all numerically constructed solutions are axially symmetric. Consequently, (2.10) seems to be of particular interest for the construction of ellipsoidal systems, but to our knowledge no self-consistent ellipsoidal systems to (2.1)-(2.3) or (2.6)-(2.8) have been constructed analytically yet.
We will prove that there exist axially symmetric solutions to (2.10) for small ω under suitable assumptions on φ, where we treat the case $\beta > 5/2$ in (2.11). For this purpose, we require, that for $\omega = 0$, we have a nontrivial, spherically symmetric solution (f_0, U_0) of (2.10). Note, that in this case the righthand-side of (2.10) only depends on U_0. For $\omega \neq 0$, we want to apply an implicit function theorem to get solutions, which arise by deforming U_0, where certain symmetries are conserved. The central idea, which makes this approach work is to look for a solution U^ω as a deformation of U_0, i.e., $U^\omega = U_0(g(x))$ for some diffeomorphism g on $\mathbb{R}^3$, and to formulate the problem in terms of finding zeros of a suitable operator T over the space of such deformations instead of the space of the potentials. Whereas the original problem (2.10) had to be solved in $\mathbb{R}^3$, we will only need to know the deformation on a compact neighbourhood of the support of the original solution (f_0, ρ_0, U_0), and this provides useful compactness properties. Furthermore, finite radius and finite mass of the constructed solutions then are just consequences of the corresponding properties of (f_0, ρ_0, U_0).
Although the allowed perturbations for the potential U_0 only have mirror symmetry which would match a triaxial system, we have up to now no method to exclude axial symmetry with respect to the x_3-axis for the perturbations constructed by the implicit function theorem.
The approach described above has been used by Lichtenstein for proving the existence of slowly rotating Newtonian stars, as described by selfgravitating fluid balls, cf. [16, 17]. A translation of Lichtenstein's approach into modern

mathematical language is due to Heilig, cf. [12].
The investigations made there were applied to the Vlasov-Poisson system in [23], where stationary solutions to (2.1)-(2.3) of the form $f(x,v)=\varphi(E)\psi(\omega P)$ were constructed. There, the potential U a-priori was axially symmetric, so that the expression $P=x_1v_2-x_2v_1$ is a conserved quantity with respect to the characteristic system. The procedure described there is the basis of our approach.
This chapter is organized as follows: In the next section we rewrite the problem in terms of finding zeros of the operator T, we then state the main result and prove it using an implicit function theorem. For this, we need certain properties of T which can be proved as in [23], except some minor technical modifications. For the convenience of the reader, the corresponding proofs are given in Sections 2.3 and 2.4.

2.2 The main result

The mappings, which leave our solutions invariant, are in the set

$$\begin{aligned} S:=\{&\tau_{110}:(x_1,x_2,x_3)\mapsto(x_1,x_2,-x_3),\ \tau_{101}:(x_1,x_2,x_3)\mapsto(x_1,-x_2,x_3),\\ &\tau_{011}:(x_1,x_2,x_3)\mapsto(-x_1,x_2,x_3)\}. \end{aligned}$$

Now let $B_R:=\{x\in\mathbb{R}^3\,|\,|x|\leq R\}$ and define

$$C_S(B_R):=\{f\in C(B_R)|\ f(Ax)=f(x),\ A\in S,\ x\in B_R\}. \tag{2.12}$$

Then we have

$$\nabla f(0)=0, \quad \text{if} \quad f\in C^1(B_R)\cap C_S(B_R).$$

For $\varphi:\mathbb{R}\to[0,\infty[$ we require

(φ1) $\varphi\in C^1(\mathbb{R})$ and there is $E_0\in\mathbb{R}$ with $\varphi(E_J)=0$ for $E_J\geq E_0$ and $\varphi(E_J)>0$ for $E_J<E_0$.

(φ2) φ is strictly decreasing in $]-\infty,E_0[$.

(φ3) The ansatz $f_0(x,v)=\varphi(E_J)$ with $\omega=0$ produces a nontrivial, spherically symmetric solution (f_0,ρ_0,U_0) of (2.1)-(2.3) with $\rho_0\in C^1_c(\mathbb{R}^3)$, supp $\rho_0=B_1$ and $U_0\in C^2(\mathbb{R}^3)$ with $\lim_{|x|\to\infty}U_0(x)=0$.

Examples for a functions satisfying (φ1)–(φ3) are the so-called polytropes

$$\varphi(E_J):=(E_0-E_J)^k_+$$

for $k>1$ and suitable $E_0<0$. Now we can state the main theorem.

Theorem 2.1. *Let $r := \sqrt{x_1^2 + x_2^2}$. There exists $\omega_0 > 0$, such that for all $\omega \in]-\omega_0, \omega_0[$ there exits a nontrivial solution $(f^\omega, \rho^\omega, U^\omega)$ of (2.6)-(2.8) with*

(i) $f^\omega(x,v) = \begin{cases} \varphi(\frac{1}{2}v^2 + U^\omega(x) - \frac{1}{2}\omega^2 r^2) & \text{for} \quad |x| < 4 \\ 0 & \text{else} \end{cases}$

(ii) $(f^0, \rho^0, U^0) = (f_0, \rho_0, U_0)$ *and for* $|\omega| < \omega_0$, $(f^\omega, \rho^\omega, U^\omega)$ *has the following symmetry properties: For all* $A \in S$ *we have*

$$f^\omega(Ax, Av) = f^\omega(x,v), \quad \rho^\omega(Ax) = \rho(x), \quad U^\omega(Ax) = U^\omega(x)$$

and $(f^\omega, \rho^\omega, U^\omega)$ *is not spherically symmetric for* $\omega \neq 0$.

(iii) $\rho^\omega \in C_c^1(\mathbb{R}^3)$ *and* $U^\omega \in C_b^2(\mathbb{R}^3)$, *where* $\rho^\omega(x) = \int f^\omega(x,v)\,dv$.

(iv) The mappings $]-\omega_0, \omega_0[\ni \omega \mapsto \rho^\omega$ *and* $]-\omega_0, \omega_0[\ni \omega \mapsto U^\omega$ *are continuous with respect to the norms* $\|\cdot\|_{1,\infty}$ *or* $\|\cdot\|_{2,\infty}$, *respectively.*

Remark. If we add rotations about the x_3- axis to the set S, the proof of Theorem 2.1 still holds – we can essentially follow the proof given here, and this shows that the constructed solutions in Theorem 2.1 have to be axially symmetric a-posteriori. This follows by the uniqueness of the mapping given in the implicit function theorem, cf. Theorem B.1.

For the proof of Theorem 2.1, we need some lemmata.

Lemma 2.2. *The spherically symmetric solution* (f_0, ρ_0, U_0) *has the following properties.*

(a) The potential U_0 *is given by*

$$U_0(x) = -\int \frac{\rho_0(y)}{|x-y|}dy = -\frac{4\pi}{|x|}\int_0^{|x|} s^2 \rho_0(s)ds - 4\pi \int_{|x|}^\infty s\rho_0(s)ds, \;\; x \in \mathbb{R}^3.$$

(b) ρ_0 *is decreasing with* $\rho_0(0) > 0, U_0''(0) > 0$ *and for every* $R > 0$ *there exists* $C > 0$, *such that* $U_0'(r) \geq Cr, \quad r \in [0,R]$, *and* $U_0(1) = E_0$.

(c) ρ_0' *is Hölder continuous and* $U_0' \in C^2(\dot{\mathbb{R}}^3)$, *where* $\dot{\mathbb{R}}^3 := \mathbb{R}^3 \backslash \{0\}$.

Proof. The formula

$$U_0'(r) = \frac{4\pi \int_0^r s^2 \rho_0(s)\,ds}{r^2}$$

easily follows from the Poisson equation with spherical symmetry and since we require $\lim_{|x|\to\infty} U_0(x) = 0$, the representation for U_0 holds by uniqueness. As to (b), for $\omega = 0$ we have $f_0(x,v) = f_0(E) = f_0(\frac{1}{2}v^2 + U_0(x))$ and this implies

$$\rho_0(x) = \int_{\mathbb{R}^3} f_0(x,v)\,dv = h_0(U_0(x)) := 4\pi\sqrt{2}\int_{U_0(x)}^{E_0} \varphi(E)\sqrt{E - U_0(x)}\,dE, \quad (2.13)$$

where the function h is continuously differentiable and with $(\varphi 1)$, $(\varphi 2)$ we have $h'(s)<0$ for $s<E_0$. Consequently, ρ_0 is decreasing because U_0 is increasing and since the steady state (f_0,U_0) is assumed to be nontrivial, we must have $\rho_0(0)>0$. Thus actually $U_0'(r)>0$, $r>0$, and since $U_0''(0)=(4\pi/3)\rho_0(0)>0$ this implies the estimate on U_0' from below. The assertion that $U_0(1)=E_0$ follows from (2.13) and the assumption $\operatorname{supp}\rho_0=B_1$. The regularity of U_0' follows from the formula for U_0' above and the fact that $\rho_0\in C_c^1$, which we deduce again from (2.13). Finally, the Hölder continuity of ρ_0' will be part of the next Lemma. □

Lemma 2.3. *Let $E_1:=U_0(2)-E_0$ and define f by*

$$f(x,v)=\begin{cases}\varphi(\frac{1}{2}v^2+U(x)-\frac{1}{2}\omega^2r^2) & \textit{for} \quad U(x)<E_0+E_1\\ 0 & \textit{else}\end{cases},$$

where φ satisfies $(\varphi 1)$, $(\varphi 2)$ and $U\in C_b^2(\mathbb{R}^3)$ with $U(x)>E_0+E_1$ for $|x|>4$. Then the following holds:

$$\begin{aligned}\rho_f(x) &:= \int_{\mathbb{R}^3} f(x,v)\,dv\\ &= \tilde h(\omega,r(x),U(x))\\ &= \begin{cases} h(U(x)-\frac{1}{2}\omega^2r^2) & \textit{for} \quad U(x)<E_0+E_1\\ 0 & \textit{else}\end{cases}\end{aligned} \tag{2.14}$$

with

$$h(s)=4\pi\sqrt{2}\int_s^{E_0}\sqrt{E-s}\,\varphi(E)\,dE.$$

Furthermore, $\tilde h\in C^1(\mathbb{R}\times[0,\infty[\times\mathbb{R})$ and for every bounded set $B\subset\mathbb{R}\times[0,\infty[\times\mathbb{R}$ there are constants $C>0$ and $\mu\in]0,1[$ such that for $(\omega,r,u),(\omega',r,u')\in B$ we have

$$\begin{aligned}&|\partial_r\tilde h(\omega,r,u)|\le Cr,\\ &|\tilde h(\omega,r,u)-\tilde h(\omega',r,u')|\le C(|\omega-\omega'|r+|u-u'|),\\ &|\partial_u\tilde h(\omega,r,u)-\partial_u\tilde h(\omega',r,u')|\le C(|\omega-\omega'|+|u-u'|^\mu).\end{aligned}$$

In addition, for $\omega=0$, the function $\tilde h(0,\cdot,\cdot)$ does not depend on $r(x)$ and we can write $h_0:=\tilde h(0,0,u)$.

Proof. Introducing polar coordinates, we have for $U(x) < E_0 + E_1$

$$\begin{aligned}\rho(x) &= \int \varphi\left(\frac{1}{2}v^2 + U(x) - \frac{1}{2}\omega^2 r^2\right) dv \\ &= 4\pi \int_0^\infty t^2 \varphi\left(\frac{1}{2}t^2 + U(x) - \frac{1}{2}\omega^2 r^2\right) dt \\ &= 4\pi\sqrt{2} \int_{U(x)-\frac{1}{2}\omega^2 r^2}^{E_0} \left(E - U(x) + \frac{1}{2}\omega^2 r^2\right)^{1/2} \varphi(E)\, dE,\end{aligned}$$

and (2.14) follows.
We have $h \in C^1(\mathbb{R})$ with

$$h'(s) = -4\pi\sqrt{2} \int_s^{E_0} \frac{1}{2\sqrt{E-s}} \varphi(E) dE$$

for $s < E_0$ and $h'(s) = 0$ for $s \geq E_0$ and the first two estimates follow. Next,

$$\begin{aligned}h''(s) &= -4\pi\sqrt{2} \frac{d}{ds} \int_0^{E_0 - s} \frac{1}{2\sqrt{E}} \varphi(E+s) dE \\ &= -4\pi\sqrt{2} \int_0^{E_0 - s} \frac{1}{2\sqrt{E}} \varphi'(E+s) dE \\ &= -4\pi\sqrt{2} \int_s^{E_0} \frac{1}{2\sqrt{E-s}} \varphi'(E) dE\end{aligned}$$

yields local Lipschitz continuity of $\partial_u \tilde{h}$ with respect to ω and u and the proof is complete. ☐

We want to find solutions of the equation

$$\Delta U = 4\pi \tilde{h}(\omega, r(x), U) \tag{2.15}$$

and the main idea is to rewrite problem (2.15) in terms of finding zeros of an operator T, which does not act directly on the space of potentials, but on deformations of the given spherically symmetric potential U_0. We define Banach spaces, which will serve as domain and range of T

$$\begin{aligned}X := \{ f \in C_S(B_4) | f(0) = 0, f \in C^1(\dot{B}_4), \exists C > 0 : |\nabla f(x)| \leq C, x \in \dot{B}_4, \\ \forall x \in \partial B_1 : \lim_{t \to 0, t > 0} \nabla f(tx) =: \nabla f(0x) \text{ exists, uniformly in } x \in \partial B_1 \},\end{aligned}$$

where $\partial B_1 := \{x \in \mathbb{R}^3 \mid |x| = 1\}$ and $\dot{B}_4 := B_4 \backslash \{0\}$. We equip X with the norm

$$\|f\|_X := \sup_{x \in \dot{B}_4} |\nabla f(x)|, \quad f \in X$$

and

$$Y := \{f \in C_S(B_4) | f(0) = 0, f \in C^1(B_4), \exists C > 0 : |\nabla f(x)| \leq C|x|, x \in B_4,$$
$$\forall x \in \partial B_1 : \lim_{t \to 0, t > 0} \frac{\nabla f(tx)}{t} =: \frac{\nabla f(0x)}{0} \text{ exists, uniformly in } x \in \partial B_1\}$$

with norm

$$\|f\|_Y := \sup_{x \in \dot{B}_4} \frac{|\nabla f(x)|}{|x|}, \quad f \in Y.$$

To state more precisely, how to use functions in X to deform the potential U_0, we need the next lemma.

Lemma 2.4. *For $\zeta \in X$ let*

$$g_\zeta : B_4 \to \mathbb{R}^3, \quad g_\zeta(x) := x + \zeta(x) \frac{x}{|x|}, \quad x \in \dot{B}_4, \quad g_\zeta(0) = 0$$

Then there exists $r > 0$, such that for all $\zeta \in \Omega$, where

$$\Omega := \{\zeta \in X | \|\zeta\|_X < r\}$$

we have:

(a) $g_\zeta : B_4 \to B_{4,\zeta} := g_\zeta(B_4)$ is a homeomorphism, $g_\zeta : \dot{B}_4 \to \dot{B}_{4,\zeta}$ is a C^1-diffeomorphism, with

$$|Dg_\zeta(x) - id| < \frac{1}{2}, \quad x \in \dot{B}_4$$

and for every $x \in \partial B_1$ the mapping

$$g_\zeta : \overline{0, 4x} \ni y \mapsto g_\zeta(y) \in \overline{0, |g_\zeta(4x)| x}$$

is one-to-one, onto and preserves the natural ordering of points in $\overline{0, 4x}$, where we defined $\overline{x_1, x_2} := \{x_1 + \lambda(x_2 - x_1) \,|\, \lambda \in [0, 1]\}$ for $x_1, x_2 \in \mathbb{R}^3$.

(b) $\frac{1}{2}|x| \leq |g_\zeta(x)| \leq \frac{3}{2}|x|, x \in B_4$, and $g_\zeta(B_2) \subset \mathring{B}_3, B_3 \subset g_\zeta(B_4) \subset B_5$

(c) $g_\zeta(Ax) = Ag_\zeta(x), x \in B_4$ and $g_\zeta^{-1}(Ax) = Ag_\zeta^{-1}(x), x \in B_{4,\zeta}, A \in S$

(d) $|Dg_\zeta^{-1}(x) - id| < \frac{1}{2}, x \in \dot{B}_{4,\zeta}$ and there exists a constant $C > 0$, such that for all $\zeta, \zeta' \in \Omega$:

$$\frac{1}{|x|}|g_\zeta(x) - g_{\zeta'}(x)| + |Dg_\zeta(x) - Dg_{\zeta'}(x)| \leq C\|\zeta - \zeta'\|_X, \quad x \in \dot{B}_4,$$

and

$$|g_\zeta^{-1}(x) - g_{\zeta'}^{-1}(x)| \leq C\|\zeta - \zeta'\|_X |x|, \quad x \in B_3$$

Proof. In $\dot{B}_4$, we have for $i,j=1,2,3$:

$$\partial_{x_i} g_{\zeta,j}(x) = \delta_{ij} + \partial_{x_i}\zeta(x)\frac{x_j}{|x|} + \frac{\zeta(x)}{|x|}\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right) \tag{2.16}$$

and therefore

$$|Dg_\zeta(x) - \mathrm{id}| < 3\|\zeta\|_X.$$

With the inverse function theorem the first two assertions in (a) follow. For $x \in \partial B_1$,

$$g_\zeta(tx) = tx + \zeta(tx)x = x(t + \zeta(tx))$$

and

$$\frac{d}{dt}(t + \zeta(tx)) = 1 + \nabla\zeta(tx)\cdot x > 0 \quad \text{for } \|\zeta\|_X \text{ small}$$

and the proof of (a) is complete.
We have $|\zeta(x)| \le \|\zeta\|_X |x|$ for $x \in B_4$ and this implies (b) for $r > 0$ sufficiently small. Assertion (c) is easily verified, too. If we choose r even smaller we also have the first claim of (d), because

$$Dg_\zeta^{-1}(x) = (Dg_\zeta)^{-1}(g_\zeta^{-1}(x)).$$

The estimate for $g_\zeta - g_{\zeta'}$ follows from the definition of g_ζ and the estimate for $Dg_\zeta - Dg_{\zeta'}$ follows from (2.16).
For $x \in \dot{B}_3$, we have with (b): $x \in g_\zeta(B_4) \cap g_{\zeta'}(B_4)$. Consequently, there exists $y \in \dot{B}_4$ mit $x = g_{\zeta'}(y)$. Now we have

$$\begin{aligned}|g_\zeta^{-1}(x) - g_{\zeta'}^{-1}(x)| &= |g_\zeta^{-1}(g_{\zeta'}(y)) - y| \\ &= |g_\zeta^{-1}(g_{\zeta'}(y)) - g_\zeta^{-1}(g_\zeta(y))| \\ &\le 2|g_\zeta(y) - g_{\zeta'}(y)| \le 2\|\zeta - \zeta'\|_X |y| \\ &\le 4\|\zeta - \zeta'\|_X |x|,\end{aligned}$$

where we used the mean value theorem, the estimate for Dg_ζ^{-1} and $\overline{g_\zeta(y), g_{\zeta'}(y)} \subset g_\zeta(\dot{B}_4)$. □

We want to find solutions of (2.15) with the following structure

$$U(x) = U_\zeta(x) := U_0(g_\zeta^{-1}(x)), \quad x \in B_{4,\zeta},$$

with a suitable $\zeta \in \Omega$. Obviously, we need U on the whole space $\mathbb{R}^3$, but this is only a technical problem. We use the fundamental solution of the Poisson equation to integrate (2.15) and we then have to solve

$$U_0(x) + \int_{B_{4,\zeta}} \frac{\tilde{h}(\omega, r(y), U_0(g_\zeta^{-1}(y)))}{|g_\zeta(x) - y|} dy = 0, \quad x \in B_4. \tag{2.17}$$

This equation essentially contains the operator we are looking for, but we have to modify things a little and also we want to get rid of the dependence on ζ in the integration domain.

Proof of Theorem 2.1. For $\zeta \in \Omega$ and $\omega \in \mathbb{R}$, we define

$$\begin{aligned} T(\omega,\zeta)(x) := U_0(x) + \int_{B_3} \frac{\tilde{h}(\omega, r(y), U_0(g_\zeta^{-1}(y)))}{|g_\zeta(x) - y|} dy \\ - U_0(0) - \int_{B_3} \frac{\tilde{h}(\omega, r(y), U_0(g_\zeta^{-1}(y)))}{|y|} dy, \qquad x \in B_4. \end{aligned} \tag{2.18}$$

Suppose we already know that this defines a continuous operator

$$T :]-\tilde{\omega}, \tilde{\omega}[\times \Omega \to Y$$

for some $\tilde{\omega} > 0$ and T is continuously Frechet-differentiable with respect to ζ, where

$$\partial_\zeta T(0,0) : X \to Y$$

is an isomorphism – we will verify this in Section 2.3 and 2.4. The definition of Y requires $T(\omega,\zeta)(0) = 0$ and therefore we substracted the constant in (2.18). With assumption (φ3), we know $T(0,0) = 0$, because $g_0 =$ id and $\operatorname{supp} \rho_0 = \operatorname{supp} h_0 \circ U_0 = B_1 \subset B_3$. The implicit function theorem, cf. [5], Theorem 15.1, also stated in the Appendix as Theorem B.1, cf. Section 2.5, now guarantees the existence of $\omega_1 \in]0, \tilde{\omega}[$ and the existence of a continuous mapping

$$]-\omega_1, \omega_1[\ni \omega \mapsto \zeta^\omega \in \Omega$$

such that

$$T(\omega, \zeta^\omega) = 0, \qquad \omega \in]-\omega_1, \omega_1[$$

and $\zeta^0 = 0$. We also will require that $\omega^2 r^2 < E_1$ in B_4, where E_1 is defined in Lemma 2.3 and therefore define

$$\omega_0 := \min\left\{ \omega_1, \frac{\sqrt{|E_1|}}{4} \right\}. \tag{2.19}$$

Now let $\zeta = \zeta^\omega$, where we choose a fixed $\omega \in]-\omega_0, \omega_0[$ and define

$$\rho_\zeta(x) := \tilde{h}(\omega, r(x), U_0(g_\zeta^{-1}(x))), \qquad x \in B_3. \tag{2.20}$$

Then we have $\rho_\zeta \in C_S(B_3) \cap C^1(\dot{B}_3)$. By Lemma 2.3, $\rho_\zeta > 0$ at most, if $U_0(g_\zeta^{-1}(x)) < E_0 + E_1$, which is equivalent to $|g_\zeta^{-1}(x)| < 2$ by Lemma 2.2. Consequently,

$$\operatorname{supp} \rho_\zeta = g_\zeta(B_2) \subset \mathring{B}_3.$$

We extend ρ_ζ by 0 to all of $\mathbb{R}^3$ and we achieve

$$\rho_\zeta \in C_c(\mathbb{R}^3), \quad \mathrm{supp}\rho_\zeta \subset \mathring{B}_3.$$

We want equation (2.20) to hold everywhere, but we have not defined g_ζ globally.
We can rewrite $T(\omega,\zeta)=0$ as

$$U_0(x) = -\int_{B_3} \frac{\rho_\zeta(y)}{|g_\zeta(x)-y|} dy + C, \quad x \in B_4,$$

or

$$U_0(g_\zeta^{-1}(x)) = -\int_{B_3} \frac{\rho_\zeta(y)}{|x-y|} dy + C, \quad x \in B_{4,\zeta},$$

where

$$C := U_0(0) + \int_{B_3} \frac{\rho_\zeta(y)}{|y|} dy.$$

Now define

$$U_\zeta(x) := -\int_{\mathbb{R}^3} \frac{\rho_\zeta(y)}{|x-y|} dy + C.$$

Then we have $U_\zeta \in C^1(\mathbb{R}^3)$ with

$$U_\zeta(x) = U_0(g_\zeta^{-1}(x)), \quad x \in B_3 \subset B_{4,\zeta} \tag{2.21}$$

and thus $\rho_\zeta \in C_0^1(\mathbb{R}^3)$ and $U_\zeta \in C_b^2(\mathbb{R}^3)$ with $\Delta U_\zeta = 4\pi\rho_\zeta$ in $\mathbb{R}^3$.
Furthermore,

$$\Delta U_\zeta = 4\pi \tilde{h}(\omega, r(x), U_\zeta(x)), \qquad x \in B_3 \subset B_{4,\zeta}. \tag{2.22}$$

The last equation holds even in $\mathbb{R}^3$. We have to show

$$\rho_\zeta(x) = \tilde{h}(\omega, r(x), U_\zeta(x)), \quad x \in \mathbb{R}^3,$$

that is, $U_\zeta(x) > E_0 + E_1$ for $x \in \mathbb{R}^3 \backslash g_\zeta(B_2)$. We know

$$\Delta U_\zeta(x) = 0, \qquad x \in \mathbb{R}^3 \backslash g_\zeta(B_2),$$

$\lim_{|x|\to\infty} U_\zeta(x) = C$ and

$$\begin{aligned} U_\zeta(x) &= E_0 + E_1, \qquad && x \in \partial g_\zeta(B_2), \\ U_\zeta(x) &> E_0 + E_1, \qquad && x \in B_3 \backslash g_\zeta(B_2). \end{aligned}$$

Here we used (2.21) and the monotonicity of $U_0(|x|)$ with $U_0(2) = E_0 + E_1$. If $C \le E_0 + E_1$, we have a contradiction to the maximum principle. Therefore, $C > E_0 + E_1$ and again by the maximum principle: $U_\zeta > E_0 + E_1$ on $\mathbb{R}^3 \backslash g_\zeta(B_2)$

and consequently, (2.22) holds in $\mathbb{R}^3$.

Now define $\rho^\omega := \rho_\zeta, \quad U^\omega := U_\zeta$ and

$$f^\omega(x,v) := \begin{cases} \varphi(\frac{1}{2}v^2 + U^\omega(x) - \frac{1}{2}\omega^2 r^2), & \text{for } U^\omega(x) < E_0 + E_1 \\ 0 & \text{else} \end{cases}$$
$$= \begin{cases} \varphi(\frac{1}{2}v^2 + U^\omega(x) - \frac{1}{2}\omega^2 r^2), & \text{for } |x| < 4 \\ 0 & \text{else.} \end{cases} \tag{2.23}$$

Now f^ω defined by (2.23) solves the Vlasov equation (2.6) because it is constant along characteristics. More precisely, we have $U_\zeta(x) - \frac{1}{2}\omega^2 r^2 > E_0$ in a neighbourhood of ∂B_4, if we choose ω_0 sufficiently small as in (2.19). If we then fix (x,v) with $E_J(x,v) < E_0$ and consider a characteristic (X,V) going through (x,v) we conclude that if $x \in B_4$, we have $X \in B_4$ for all time. On the other hand, if $x \notin B_4$, we have $X \notin B_4$ for all time.
Altogether, assertions (i)-(iii) of the theorem follow, except the non-spherical symmetry in the case $\omega \neq 0$. Choose $x \in \mathbb{R}^3$ with $\rho^\omega(x) > 0, x_1 := a \neq 0, x_2 = x_3 = 0$. Then there exists some $\eta \in \mathbb{R}^3$, such that

$$\frac{1}{2}\eta^2 + U^\omega(x) - \frac{1}{2}\omega^2 a^2 < E_0.$$

Now if (f^ω, U^ω) were spherically symmetric, there would exist a rotation A around the x_2-axis such that $(Ax)_1 = (Ax)_2 = 0$ and $f^\omega(Ax, Av) = f^\omega(x,v)$. But the monotonicity of φ implies

$$f^\omega(x,v) = \varphi(\frac{1}{2}v^2 + U^\omega(x) - \frac{1}{2}\omega^2 a^2) = \varphi(E_J(x,v))$$
$$\neq \varphi(E_J(Ax,Av)) = \varphi(\frac{1}{2}v^2 + U^\omega(x)) = f^\omega(Ax,Av),$$

which contradicts our assumption of spherical symmetry. With a similar argument, one can also show that the constructed solutions cannot be axially symmetric with respect to any axis in $\mathbb{R}^3$ except for the x_3-axis. Though our deformations only have mirror symmetry with respect to every coordinate plane, which would match a triaxial system, we would still have to prove that the constructed ζ^ω are not axially symmetric with repect to the x_3-axis to construct triaxial solutions.

The asserted continuity properties (iv) can be proved as follows: For $x \in B_3$ we have

$$|U^\omega(x) - U^{\omega'}(x)| \leq \|U_0'\|_\infty |g_{\zeta_\omega}^{-1}(x) - g_{\zeta_{\omega'}}^{-1}(x)| \leq C\|\zeta_\omega - \zeta_{\omega'}\|_X.$$

By the implicit function theorem, ζ^ω continuously depends on ω with respect to the $\|\cdot\|_X$-norm and we have $\rho^\omega(x)=\tilde{h}(\omega,r(x),U^\omega(x))$.
Lemma 2.3 implies that ρ^ω is continuous in ω with respect to $\|\cdot\|_\infty$ and

$$U^\omega(x)=-\int_{B_3}\frac{\rho^\omega(y)}{|x-y|}dy+U_0(0)+\int_{B_3}\frac{\rho^\omega(y)}{|y|}dy,\quad x\in\mathbb{R}^3$$

implies the continuity of U^ω in ω with respect to $\|\cdot\|_{1,\infty}$. Differentiating the above expression for ρ^ω yields the continuity of ρ^ω with respect to $\|\cdot\|_{1,\infty}$ and therefore also the continuity of U^ω in the norm $\|\cdot\|_{2,\infty}$. $\square$

2.3 Fréchet-Differentiability of T

Theorem 2.5. *Let $\omega_2 := \sqrt{|E_1|}/4$, where E_1 is defined in Lemma 2.3. The continuous mapping $T:]-\omega_2, \omega_2[\times \Omega \to Y$, defined by (2.18), is continuously Fréchet-differentiable with respect to ζ and the Fréchet-derivative is given by*

$$
\begin{aligned}
&[\partial_\zeta T(\omega,\zeta)\Lambda](x) = \\
&= -\int_{B_3} \left(\frac{1}{|g_\zeta(x)-y|} - \frac{1}{|y|} \right) \partial_u \tilde{h}(\omega, r(y), U_\zeta(y)) \nabla U_\zeta(y) \cdot \frac{g_\zeta^{-1}(y)}{|g_\zeta^{-1}(y)|} \Lambda(g_\zeta^{-1}(y)) dy \\
&\quad - \int_{B_3} \frac{g_\zeta(x)-y}{|g_\zeta(x)-y|^3} \tilde{h}(\omega, r(y), U_\zeta(y)) dy \cdot \frac{x}{|x|} \Lambda(x), \quad x \in B_4, \qquad (2.24)
\end{aligned}
$$

where $\omega \in]-\omega_2, \omega_2[$, $\zeta \in \Omega$, $\Lambda \in X$, und $U_\zeta(y) := U_0(g_\zeta^{-1}(y))$, $y \in B_3$.

For the proof we need some preliminary results.

Lemma 2.6. *Let $\zeta \in \Omega$. Then we have:*

(a) $|\zeta(x) - \zeta(x')| \le \|\zeta\|_X |x - x'|$, $x, x' \in B_4$.

(b) For $x \in \partial B_1$, the mapping $[0,3] \ni t \mapsto \zeta(tx)$ is continuously differentiable and $\lim_{t\to 0, t>0} (\zeta(tx)/t) =: \nabla\zeta(0x) \cdot x$ exists uniformly in $x \in \partial B_1$.

(c) The mapping $[0,4] \times \partial B_1 \ni (t,x) \mapsto \nabla\zeta(tx)$ is uniformly continous.

(d) For $x \in \partial B_1$, the following limits

$$\lim_{t\to 0, t>0} \frac{g_\zeta(tx)}{t} =: \frac{g_\zeta(0x)}{0} \quad \text{and} \quad \lim_{t\to 0, t>0} Dg_\zeta(tx) =: Dg_\zeta(0x)$$

exist uniformly in $x \in \partial B_1$.

Proof. The assertion in (a) follows easily by distinguishing the cases $0 \in \overline{x, x'}$ and $0 \notin \overline{x, x'}$. (b) follows with

$$\frac{d}{dt}\zeta(tx) = \nabla\zeta(tx) \cdot x \to \nabla\zeta(0x) \cdot x, \quad t \to 0, t > 0$$

and the definition of X. With $\nabla\zeta \in C(\dot{B}_4)$ and $\nabla\zeta(tx) \to \nabla\zeta(0x)$, uniformly in $x \in \partial B_1$, (c) is obvious. The last assertion (d) follows with the definitions of g_ζ, X and equation (2.16). ☐

Next we establish some estimates for the spatial density induced by a deformation of the potential U_0:

Lemma 2.7. *For* $\omega \in]-\omega_2, \omega_2[$ *and* $\zeta \in \Omega$ *let*

$$\rho_{\omega,\zeta}(x) := \tilde{h}(\omega, r(x), U_0(g_\zeta^{-1}(x))), \quad x \in B_3.$$

Then the following holds:

(a) $\rho_{\omega,\zeta} \in C_S(B_3) \cap C^1(B_3)$ *with* $\operatorname{supp} \rho_{\omega,\zeta} \subset \mathring{B}_3$*, and there exists a constant* $C > 0$ *such that for all* $\omega \in]-\omega_2, \omega_2[$ *and* $\zeta \in \Omega$,

$$|\nabla \rho_{\omega,\zeta}(x)| \leq C|x|, \quad x \in B_3.$$

(b) There exists a constant $C > 0$ *such that for all* $\omega, \omega' \in]-\omega_2, \omega_2[$ *and* $\zeta, \zeta' \in \Omega$,

$$|\rho_{\omega,\zeta}(x) - \rho_{\omega',\zeta'}(x)| \leq C(|\omega - \omega'| + \|\zeta - \zeta'\|_X)|x|, \quad x \in B_3.$$

Proof. Lemmas 2.3 and 2.4 imply that $\rho = \rho_{\omega,\zeta} \in C_S(B_3) \cap C^1(\dot{B}_3)$. For $x \in \dot{B}_3$ we have

$$\nabla \rho_{\omega,\zeta}(x) = \partial_r \tilde{h}(\omega, r(x), U_0(g_\zeta^{-1}(x))) \nabla r(x) + \partial_u \tilde{h}(\omega, r(x), U_0(g_\zeta^{-1}(x))) \nabla U_0(g_\zeta^{-1}(x)) \cdot D g_\zeta^{-1}(x),$$

and Lemma 2.3, the fact that $U_0 \in C^2(\mathbb{R}^3)$ with $\nabla U_0(0) = 0$, and Lemma 2.4 imply the estimate

$$|\nabla \rho(x)| \leq C|x| + C|\nabla U_0(g_\zeta^{-1}(x))| \leq C|x| + C|g_\zeta^{-1}(x)| \leq C|x|, \quad x \in \dot{B}_2;$$

note that the range of U_0 is bounded. Since $x \notin g_\zeta(B_2)$ implies $U_0(g_\zeta^{-1}(x)) > E_0 + E_1$ and thus $\rho(x) = 0$, the assertion on the support of ρ follows by Lemma 2.4(b). The inequality in (b) is immediate from Lemmas 2.3 and 2.4(d). □

Lemma 2.8. *Let* $\sigma \in C_S(B_3)$ *be such that*

$$c_\sigma := \sup_{x \in \dot{B}_3} \frac{|\sigma(x)|}{|x|} < \infty$$

and define

$$V_\sigma := -\int_{B_3} \frac{\sigma(y)}{|x - y|} dy, \quad x \in \mathbb{R}^3$$

Then $V_\sigma \in C^1(\mathbb{R}^3)$*, and there exists* $C > 0$ *such that for all* σ *as above the following estimates hold:*

(a) $|\nabla V_\sigma(x)| \leq C c_\sigma |x|, \quad x \in \mathbb{R}^3$,

(b) $|\nabla V_\sigma(g_\zeta(x)) - \nabla V_\sigma(g_{\zeta'}(x))| \leq C c_\sigma \|\zeta - \zeta'\|_X^{1/2} |x|, \; x \in B_4, \; \zeta, \zeta' \in \Omega.$

Proof. For $\sigma \in C_S(B_3)$ we have $\nabla V_\sigma(0)=0$ and thus

$$|\nabla V_\sigma(x)| \le \left| \int_{B_3} \left(\frac{x-y}{|x-y|^3} + \frac{y}{|y|^3} \right) \sigma(y)\, dy \right|, \quad x \in \mathbb{R}^3.$$

Let $x \neq 0$ and $r := 2|x|$. Then we obtain the estimate

$$\begin{aligned} |\nabla V_\sigma(x)| \le & c_\sigma \int_{B_3 \setminus B_r} \left| \frac{x-y}{|x-y|^3} + \frac{y}{|y|^3} \right| |y|\, dy \\ & + c_\sigma \int_{B_3 \cap B_r} \left(\frac{1}{|x-y|^2} + \frac{1}{|y|^2} \right) |y|\, dy =: I_1 + I_2 \end{aligned}$$

For almost every $y \in B_3$ there exists $\tau \in [0,1]$ such that

$$\left| \frac{x-y}{|x-y|^3} + \frac{y}{|y|^3} \right| \le |x| \frac{C}{|\tau x - y|^3}$$

and since for $|y| \ge r$,

$$|\tau x - y| \ge |y| - |x| = |y| - \frac{r}{2} \ge \frac{|y|}{2},$$

we can estimate the first term as

$$I_1 \le C c_\sigma |x| \int_{B_3} \frac{1}{|y|^2}\, dy = C c_\sigma |x|.$$

For the second term we have

$$I_2 \le 3 c_\sigma \left(\int_{B_r} \frac{1}{|x-y|^2}\, dy + \int_{B_r} \frac{1}{|y|^2}\, dy \right) \le 6 c_\sigma \int_{B_r} \frac{1}{|y|^2}\, dy = C c_\sigma |x|,$$

and (a) follows. As to (b), we have

$$|\nabla V_\sigma(g_\zeta(x)) - \nabla V_\sigma(g_{\zeta'}(x))| \le C c_\sigma \int_{B_3} \left| \frac{g_\zeta(x)-y}{|g_\zeta(x)-y|^3} - \frac{g_{\zeta'}(x)-y}{|g_{\zeta'}(x)-y|^3} \right| \; |y|\, dy.$$

Let $x \in \dot{B}_3$ and $\delta := \|\zeta - \zeta'\|_X < 1$, $r_1 := 2\delta |x|$, and $r_2 := 4|x| > r_1$; recall that we choose that radius of the set Ω less than $1/3$. We split the integral above into three parts, I_1, I_2, I_3, according to the decomposition

$$B_3 = (B_3 \setminus B_{r_2}) \cup ((B_3 \cap B_{r_2}) \setminus B_{r_1}(g_\zeta(x))) \cup (B_3 \cap B_{r_1}(g_\zeta(x))).$$

As to I_1, we find for almost every $y \in B_3$ a τ between $\zeta(x)$ and $\zeta'(x)$ such that

$$\left| \frac{g_\zeta(x)-y}{|g_\zeta(x)-y|^3} - \frac{g_{\zeta'}(x)-y}{|g_{\zeta'}(x)-y|^3} \right| \le \frac{C}{|x + \tau \frac{x}{|x|} - y|^3} \; |\zeta(x) - \zeta'(x)|;$$

note that both $g_\zeta(x)$ and $g_{\zeta'}(x)$ lie on the line $\mathbb{R}x$. Since

$$|\zeta(x)-\zeta'(x)| \leq \|\zeta-\zeta'\|_X |x| = \delta|x|,$$

we have for $|y| \geq r_2$,

$$\begin{aligned} |x+\tau\frac{x}{|x|}-y| &= \left|y-g_\zeta(x)+(\zeta(x)-\tau)\frac{x}{|x|}\right| \geq |y|-|g_\zeta(x)|-|\zeta(x)-\zeta'(x)| \\ &\geq |y|-\frac{3}{2}|x|-\delta|x| \geq |y|-\frac{5}{2}|x| = |y|-\frac{5}{8}r_2 \geq \frac{3}{8}|y|. \end{aligned}$$

This implies the estimate

$$I_1 \leq Cc_\sigma \|\zeta-\zeta'\|_X |x|.$$

To estimate I_2, we have for $y \notin B_{r_1}(g_\zeta(x))$

$$\begin{aligned} |x+\tau\frac{x}{|x|}-y| &= \left|y-g_\zeta(x)+(\zeta(x)-\tau)\frac{x}{|x|}\right| \geq |y|-|g_\zeta(x)|-|\zeta(x)-\zeta'(x)| \\ &\geq |y-g_\zeta(x)|-\delta|x| \geq \frac{1}{2}|y-g_\zeta(x)|, \end{aligned}$$

and for $y \in B_{r_2}$

$$|y-g_\zeta(x)| \leq |y|+\frac{3}{2}|x| \leq r_2+\frac{3}{8}r_2 \leq 2r_2.$$

Altogether,

$$\begin{aligned} I_2 &\leq Cc_\sigma\delta|x| \int_{B_{2r_2}(g_\zeta(x))\setminus B_{r_1}(g_\zeta(x))} \frac{1}{|g_\zeta(x)-y|^3}\,dy = Cc_\sigma\delta|x|4\pi\ln\frac{2r_2}{r_1} \\ &= Cc_\sigma\delta|x|4\pi\ln\frac{4}{\delta} = Cc_\sigma\|\zeta-\zeta'\|_X^{1/2}|x|. \end{aligned}$$

As to the third term we have

$$\begin{aligned} I_3 &\leq 2c_\sigma\left(\int_{B_{r_1}(g_\zeta(x))} \frac{dy}{|g_\zeta(x)-y|^2}\,dy + \int_{B_{r_1}(g_\zeta(x))} \frac{dy}{|g_{\zeta'}(x)-y|^2}\,dy\right) \\ &\leq 4c_\sigma \int_{B_{r_1}(g_\zeta(x))} \frac{dy}{|g_\zeta(x)-y|^2}\,dy = Cc_\sigma r_1 = Cc_\sigma\|\zeta-\zeta'\|_X|x|, \end{aligned}$$

and the proof is complete. □

Lemma 2.9. *For $\omega \in]-\omega_2,\omega_2[$ and $\zeta \in \Omega$ we have $T(\omega,\zeta) \in Y$, and the mapping $T:]-\omega_2,\omega_2[\times\Omega \to Y$ is continuous.*

Proof. Let

$$V_{\omega,\zeta}(x) := -\int_{B_3} \frac{\rho_{\omega,\zeta}(y)}{|x-y|}\,dy, \quad x\in\mathbb{R}^3,\ (\omega,\zeta)\in]-\omega_2,\omega_2[\times\Omega$$

The assertions in Lemma 2.7(a) imply that $V_{\omega,\zeta}\in C^2(\mathbb{R}^3)$ with $\nabla V_{\omega,\zeta}(0)=0$. Since

$$T(\omega,\zeta)(x) = U_0(x) - V_{\omega,\zeta}(g_\zeta(x)) - U_0(0) + V_{\omega,\zeta}(0), \quad x\in B_4,$$

cf. equation (2.18), we have $T(\omega,\zeta)(0)=0$ and $T(\omega,\zeta)\in C^1(\dot{B}_4)\cap C_S(B_4)$. While we show that $T(\omega,\zeta)\in Y$ for $(\omega,\zeta)\in]-\omega_2,\omega_2[\times\Omega$, the arguments ω and ζ remain fixed, and we write $V := V_{\omega,\zeta}$. We have

$$\nabla T(\omega,\zeta)(x) = \nabla U_0(x) - \nabla V_{\omega,\zeta}(g_\zeta(x))Dg_\zeta(x), \quad x\in\dot{B}_4,$$

which implies

$$|\nabla T(\omega,\zeta)(x)| = \|D^2U_0\|_\infty|x| + 2\|D^2V\|_\infty|g_\zeta(x)| \le C|x|,$$

with some constant C, which depends on U_0 and V but not on x. In particular, this shows that $T(\omega,\zeta)\in C^1(B_3)$. Now we fix $x\in\partial B_1$. Since any point on the line segment $\overline{0,g_\zeta(tx)}$ can be written as $g_\zeta(\tau x)$ with $\tau\in[0,t]$, we have

$$\begin{aligned}
\frac{\partial_{x_i}T(\omega,\zeta)(tx)}{t} &= \frac{\partial_{x_i}U_0(tx)}{t} - \frac{1}{t}\nabla V(g_\zeta(tx))\cdot\partial_{x_i}g_\zeta(tx)\\
&= \frac{\partial_{x_i}U_0(tx)}{t} - \frac{1}{t}\left(D^2V(g_\zeta(\tau x))g_\zeta(tx)\right)\cdot\partial_{x_i}g_\zeta(tx)\\
&\to \nabla\partial_{x_i}U_0(0)\cdot x - \left(D^2V(0)\frac{g_\zeta(0x)}{0}\right)\cdot\partial_{x_i}g_\zeta(0x),
\end{aligned}$$

as $t\to 0^+$, uniformly in $x\in\partial B_1$, by Lemma 2.6(d). This verifies $T(\omega,\zeta)\in Y$. To show that T is continuous, we fix $(\omega',\zeta')\in]-\omega_2,\omega_2[\times\Omega$. Constants denoted by C may depend on (ω',ζ') but not on $(\omega,\zeta)\in]-\omega_2,\omega_2[\times\Omega$ or $x\in B_4$. We have

$$\begin{aligned}
\|T(\omega,\zeta)-T(\omega',\zeta')\|_Y &= \sup_{x\in\dot{B}_4}\frac{1}{|x|}|\nabla V_{\omega,\zeta}(g_\zeta(x))Dg_\zeta(x) - \nabla V_{\omega',\zeta'}(g_{\zeta'}(x))Dg_{\zeta'}(x)|\\
&\le \sup_{x\in\dot{B}_4}\frac{1}{|x|}(I_1+I_2+I_3),
\end{aligned}$$

where for $x\in\dot{B}_4$,

$$\begin{aligned}
I_1 &:= |Dg_\zeta(x)|\ \ |\nabla V_{\omega,\zeta}(g_\zeta(x)) - \nabla V_{\omega',\zeta'}(g_\zeta(x))|,\\
I_2 &:= |Dg_\zeta(x)|\ \ |\nabla V_{\omega,\zeta}(g_\zeta(x)) - \nabla V_{\omega',\zeta'}(g_{\zeta'}(x))|,\\
I_3 &:= |Dg_\zeta(x) - Dg_{\zeta'}(x)|\ \ |\nabla V_{\omega',\zeta'}(g_{\zeta'}(x))|.
\end{aligned}$$

Using Lemmas 2.7(b) and 2.8(a) with $\sigma := \rho_{\omega,\zeta} - \rho_{\omega',\zeta'}$, we find

$$|\nabla V_{\omega,\zeta}(g_\zeta(x)) - \nabla V_{\omega',\zeta'}(g_\zeta(x))| \leq C\,(|\omega-\omega'| + \|\zeta-\zeta'\|_X)\,|g_\zeta(x)|,$$

and by Lemma 2.4,

$$I_1 \leq C\,(|\omega-\omega'| + \|\zeta-\zeta'\|_X)\,|x|, \quad x \in B_4.$$

Since $V_{\omega',\zeta'} \in C^2(\mathbb{R}^3)$ with $\nabla V_{\omega',\zeta'}(0) = 0$, we have by Lemma 2.4(d),

$$I_2 \leq C|g_\zeta(x) - g_{\zeta'}(x)|| \leq C\|\zeta-\zeta'\|_X|x|, \quad x \in B_4,$$

and

$$I_3 \leq \|D^2 V_{\omega',\zeta'}\|_\infty |g_{\zeta'}(x)|\ \ \|\zeta-\zeta'\|_X \leq C\|\zeta-\zeta'\|_X|x|, \quad x \in B_4,$$

and the continuity of T follows. □

To deal with the differentiability of T, we have to investigate the integrand in the formula for $\partial_\zeta T$ in Theorem 2.5.

Lemma 2.10. *For $\omega \in]-\omega_0,\omega_0[$, $\zeta \in \Omega$, and $\Lambda \in X$ define*

$$\sigma_{\omega,\zeta,\Lambda}(x) := \partial_u \tilde{h}(\omega, r(x), U_\zeta(x)) \nabla U_\zeta(x) \cdot \frac{g_\zeta^{-1}(x)}{|g_\zeta^{-1}(x)|} \Lambda(g_\zeta^{-1}(x)), \quad x \in B_3,$$

where we recall that $U_\zeta(x) = U_0(g_\zeta^{-1}(x))$, $x \in B_3$. Then $\sigma_{\omega,\zeta,\Lambda} \in C_S(B_3)$, and there exists $C > 0$ such that for every $\omega \in]-\omega_2,\omega_2[$, $\zeta \in \Omega$ and $\Lambda \in X$,

$$|\sigma_{\omega,\zeta,\Lambda}(x)| \leq C\|\Lambda\|_X|x|, \quad x \in B_3.$$

Furthermore, if we fix $(\omega',\zeta') \in]-\omega_2,\omega_2[\times\Omega$, there exists for each $\epsilon > 0$ a $\delta > 0$ such that for all $(\omega,\zeta) \in]-\omega_2,\omega_2[\times\Omega$ with $|\omega-\omega'| + \|\zeta-\zeta'\|_X < \delta$ and $\Lambda \in X$,

$$|\sigma_{\omega,\zeta,\Lambda}(x) - \sigma_{\omega',\zeta',\Lambda}(x)| \leq \epsilon\|\Lambda\|_X|x|, \quad x \in B_3.$$

Proof. The range of U_0 and therefore also of U_ζ is bounded. Thus the first factor in $\sigma_{\omega,\zeta,\Lambda}$ is bounded, uniformly in ω and ζ, and the same is obviously true for the second and third factor. Together with

$$|\Lambda(g_\zeta^{-1}(x))| \leq \|\Lambda\|_X|g_\zeta^{-1}(x)| \leq 2\|\Lambda\|_X|x|, \quad x \in B_3, \tag{2.25}$$

the estimate for $\sigma_{\omega,\zeta,\Lambda}$ follows. The continuity of $\sigma_{\omega,\zeta,\Lambda}$ on $\dot{B}_3$ is clear, and at $x = 0$ it follows from the estimate above. The symmetry follows from the corresponding properties of U_0, g_ζ and Λ In the following, C denotes a constant which may depend on U_0 and (ω',ζ'), but not on ω,ζ,Λ or x. We find that

$$\begin{aligned}
&|\sigma_{\omega,\zeta,\Lambda}(x)-\sigma_{\omega',\zeta',\Lambda}(x)|\\
&\quad\le C\left|\partial_u\tilde h(\omega,r(x),U_\zeta(x))-\partial_u\tilde h(\omega',r(x),U_{\zeta'}(x))\right|\,|\Lambda(g_\zeta^{-1}(x))|\\
&\qquad+C|\nabla U_\zeta(x)-\nabla U_{\zeta'}(x)|\,|\Lambda(g_\zeta^{-1}(x))|\\
&\qquad+C\left|\frac{g_\zeta^{-1}(x)}{|g_\zeta^{-1}(x)|}-\frac{g_{\zeta'}^{-1}(x)}{|g_{\zeta'}^{-1}(x)|}\right|\,|\Lambda(g_\zeta^{-1}(x))|\\
&\qquad+C|\Lambda(g_\zeta^{-1}(x))-\Lambda(g_{\zeta'}^{-1}(x))|\\
&\quad=:I_1+I_2+I_3+I_4,\quad x\in\dot B_3.
\end{aligned}$$

Now (2.25) together with Lemma 2.3 and Lemma 2.4 imply that

$$\begin{aligned}
I_1&\le C\left(|\omega-\omega'|+|U_0(g_\zeta^{-1}(x))-U_0(g_{\zeta'}^{-1}(x))|^\mu\right)\|\Lambda\|_X|x|\\
&\le C\left(|\omega-\omega'|+|g_\zeta^{-1}(x)-g_{\zeta'}^{-1}(x)|^\mu\right)\|\Lambda\|_X|x|\\
&\le C\,(|\omega-\omega'|+\|\zeta-\zeta'\|_X^\mu)\,\|\Lambda\|_X|x|,\quad x\in B_3.
\end{aligned}$$

For I_3 and I_4, we have

$$\begin{aligned}
I_3&\le C\left(\frac{1}{|g_\zeta^{-1}(x)|}+\frac{1}{|g_{\zeta'}^{-1}(x)|}\right)|g_\zeta^{-1}(x)-g_{\zeta'}^{-1}(x)|\,\|\Lambda\|_X|x|\\
&\le C\|\zeta-\zeta'\|_X\|\Lambda\|_X|x|,\quad x\in\dot B_3,
\end{aligned}$$

and

$$I_4\le C\|\Lambda\|_X|g_\zeta^{-1}(x)-g_{\zeta'}^{-1}(x)|\le C\|\zeta-\zeta'\|_X\|\Lambda\|_X|x|,\quad x\in\dot B_3.$$

The estimate of term I_2 is more difficult, here we need the limit condition in the definition of the space X. Firstly, we have

$$\begin{aligned}
I_2&\le C\left|\nabla U_0(g_\zeta^{-1}(x))-\nabla U_0(g_{\zeta'}^{-1}(x))\right|\,\|\Lambda\|_X|x|\\
&\quad+C\left|Dg_\zeta^{-1}(x)-Dg_{\zeta'}^{-1}(x)\right|\,\|\Lambda\|_X|x|\\
&\le C\|\zeta-\zeta'\|_X\|\Lambda\|_X|x|+C\left|Dg_\zeta^{-1}(x)-Dg_{\zeta'}^{-1}(x)\right|\,\|\Lambda\|_X|x|,
\end{aligned}$$

and with $z:=g_\zeta^{-1}(x)$ and $z':=g_{\zeta'}^{-1}(x)$ we can estimate

$$\begin{aligned}
\left|Dg_\zeta^{-1}(x)-Dg_{\zeta'}^{-1}(x)\right|&=\left|(Dg_\zeta)^{-1}(z)-(Dg_{\zeta'})^{-1}(z')\right|\\
&\le C\,|Dg_\zeta(z)-Dg_{\zeta'}(z')|\\
&\le C\,|Dg_\zeta(z)-Dg_{\zeta'}(z)|+C\,|Dg_{\zeta'}(z)-Dg_{\zeta'}(z')|\\
&\le C\|\zeta-\zeta'\|_X+C\,|Dg_{\zeta'}(z)-Dg_{\zeta'}(z')|,
\end{aligned}$$

where the first inequality relies on the identity $A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}$ for two quadratic and invertible matrices A,B. We now have to deal with the term $|Dg_{\zeta'}(z)-Dg_{\zeta'}(z')|$. From equation (2.16) we get

$$\begin{aligned}|Dg_{\zeta'}(z)-Dg_{\zeta'}(z')| &\le C|\nabla\zeta'(z)-\nabla\zeta'(z')| \\ &\qquad + C|\nabla\zeta'(z')|\left|\frac{z}{|z|}-\frac{z'}{|z'|}\right|+\frac{C}{|z|}|\zeta'(z)-\zeta'(z')| \\ &\qquad C|\zeta'(z')|\left(\left|\frac{1}{|z|}-\frac{1}{|z'|}\right|+\max_{i,j=1,2,3}\left|\frac{z_iz_j}{|z|^3}-\frac{z'_iz'_j}{|z'|^3}\right|\right) \\ &=: J_1+J_2+J_3+J_4\end{aligned}$$

With $\bar{x}:=x/|x|$, there exist $s,s'>0$ such that $z=g_\zeta^{-1}(x)=s\bar{x}$ and $z'=g_{\zeta'}^{-1}(x)=s'\bar{x}$ so that $s=|z|$, $s'=|z'|$, and

$$|s-s'|=\big||z|-|z'|\big|\le|g_\zeta^{-1}(x)-g_{\zeta'}^{-1}(x)|\le C\|\zeta-\zeta'\|_X,\quad x\in\dot{B}_3.$$

Now given $\epsilon>0$ we can choose $\delta>0$ according to Lemma 2.6(c) such that $\|\zeta-\zeta'\|_X<\delta$ implies

$$J_1=C|\nabla\zeta'(s\bar{x})-\nabla\zeta'(s'\bar{x})|<\epsilon,\quad x\in\dot{B}_3.$$

With Lemma 2.4 and Lemma 2.6 we obtain

$$\begin{aligned}J_2 &\le C\left(\frac{1}{|z|}+\frac{1}{|z'|}\right)|z-z'|\le\frac{C}{|x|}\|\zeta-\zeta'\|_X|x|=C\|\zeta-\zeta'\|_X, \\ J_3 &\le\frac{C}{|z|}|z-z'|\le C\|\zeta-\zeta'\|_X,\end{aligned}$$

and

$$J_4\le C|z'|\left(\frac{1}{|z|^2}+\frac{1}{|z'|^2}\right)|z-z'|\le C\|\zeta-\zeta'\|_X,$$

so that finally

$$I_2\le C\|\zeta-\zeta'\|_X\|\Lambda\|_X|x|+C\epsilon\|\Lambda\|_X|x|,\quad x\in\dot{B}_3,$$

provided $\|\zeta-\zeta'\|_X<\delta$, and the proof is complete. □

The next step in the proof of Theorem 2.5 is to show that the right-hand side of the formula for $\partial_\zeta T(\omega,\zeta)$ is indeed the Gateaux derivative. For fixed $(\omega,\zeta)\in]-\omega_2,\omega_2[\times\Omega$, we will denote by $L\Lambda$ the right-hand side of the definition of $\partial_\zeta T(\omega,\zeta)\Lambda$, $\Lambda\in X$, cf. equation (2.24).

Lemma 2.11. *$L\in\mathfrak{L}(X,Y)$ is a bounded, linear operator, and for all $\Lambda\in X$,*

$$\lim_{t\to 0}\frac{T(\omega,\zeta+t\Lambda)-T(\omega,\zeta)}{t}=L\Lambda$$

with respect to $\|\cdot\|_Y$.

Proof. It is convenient to introduce the Banach space

$$\overline{Y}:=\{f\in C_S(B_4)\big|f(0)=0,\;\; f\in C^1(B_4),\exists C>0:\;\; |\nabla f(x)|\le C|x|,\;\; x\in B_4\},$$

which we equip with the norm $\|\cdot\|_Y$. Clearly, Y is a closed subspace of $\overline{Y}$. Since we already know that T maps X into Y it is then sufficient to show that $L\in\mathfrak{L}(X,\overline{Y})$ and that the asserted convergence holds. To see the former define

$$V_\Lambda(x):=-\int_{B_3}\frac{1}{|x-y|}\sigma_{\omega,\zeta,\Lambda}(y)\,dy,\quad x\in\mathbb{R}^3,$$

and

$$W(x):=-\int_{B_3}\frac{1}{|x-y|}\rho_{\omega,\zeta}(y)\,dy,\quad x\in\mathbb{R}^3,$$

where $\rho_{\omega,\zeta}$ is defined as in Lemma 2.7. Then $V_\Lambda\in C^1(\mathbb{R}^3),\;\; W\in C^2(\mathbb{R}^3)$, and we can write

$$(L\Lambda)(x)=V_\Lambda(g_\zeta(x))-V_\Lambda(0)-\nabla W(g_\zeta(x))\cdot\frac{x}{|x|}\Lambda(x),\quad x\in B_4.$$

This implies that for $\Lambda\in X$, we have $L\Lambda\in C^1(\dot{B}_4),\;\; (L\Lambda)(0)=0$, and

$$\begin{aligned}(\nabla L\Lambda)(x)=&\nabla V_\Lambda(g_\zeta(x))Dg_\zeta(x)-D^2W(g_\zeta(x))Dg_\zeta(x)\frac{x}{|x|}\Lambda(x)\\&-\nabla W(g_\zeta(x))D\left(\frac{x}{|x|}\right)\Lambda(x)-\nabla W(g_\zeta(x))\cdot\frac{x}{|x|}\nabla\Lambda(x),\quad x\in\dot{B}_4.\end{aligned}$$

Using Lemma 2.10 and Lemma 2.8(a), we get the estimate

$$\begin{aligned}|(\nabla L\Lambda)(x)|\le& C\|\Lambda\|_X|g_\zeta(x)|+C|\Lambda(x)|+C\|D^2W\|_\infty|g_\zeta(x)|\left(\frac{|\Lambda(x)|}{|x|}+|\nabla\Lambda(x)|\right)\\ \le& C\|\Lambda\|_X|x|,\quad x\in\dot{B}_4.\end{aligned}$$

In particular, this implies that $L\Lambda$ is differentiable also at $x=0$, and

$$\|L\Lambda\|_Y\le C\|\Lambda\|_X,\quad \Lambda\in X.$$

The symmetry of $L\Lambda$ follows easily from the corresponding properties of V_λ,W,ζ, and Λ. In order to show that $L\Lambda$ is indeed the Gateaux derivative of

T at (ω,ζ) in the direction of Λ we choose $t_0>0$ such that $\zeta+t\Lambda\in\Omega$ for $|t|<t_0$ and we will write

$$g_t(x)=g_{\zeta+t\Lambda}(x):=x+(\zeta(x)+t\Lambda(x))\,\frac{x}{|x|},\quad x\in B_4,\ t\in]-t_0,t_0[.$$

We now define for fixed $y\in\dot{B}_3$ the mapping

$$G(t,x):=g_t(x)-y,\quad t\in]-t_0,t_0[,\ x\in\dot{B}_4.$$

Now since $G(t,g_t^{-1}(y))=0,\ t\in]-t_0,t_0[$, the fact that $\partial_x G(t,x)=Dg_t(x)$ is invertible and the implicit function theorem imply that g_t^{-1} is continuously differentiable with respect to t. We can calculate its derivate by differentiating the identity $x=g_t(g_t^{-1}(x))$ with respect to t and obtain

$$\frac{d}{dt}g_t^{-1}=-(Dg_t)^{-1}(g_t^{-1}(x))\Lambda(g_t^{-1}(x))\frac{g_t^{-1}(x)}{|g_t^{-1}(x)}.$$

It will also be convenient to abbreviate

$$\rho_t(x):=\rho_{\omega,\zeta+t\Lambda}(x),\quad \sigma_t:=\sigma_{\omega,\zeta+t\Lambda,\Lambda}(x),\quad t\in]-t_0,t_0[,\quad x\in B_3,$$

and we define

$$F(t,x):=\int_{B_3}\left(\frac{1}{|x-y|}-\frac{1}{|y|}\right)\rho_t(y)\,dy,\quad x\in\mathbb{R}^3,\ t\in]-t_0,t_0[.$$

Then except for $\partial_t^2 F$ all derivatives of F up to second order exist and are continuous on $]-t_0,t_0[\times\mathbb{R}^3$, and

$$\begin{aligned}
\partial_t F(t,x)&=-\int_{B_3}\left(\frac{1}{|x-y|}-\frac{1}{|y|}\right)\sigma_t(y),dy,\\
\nabla F(t,x)&=-\int_{B_3}\frac{x-y}{|x-y|}\rho_t(y)\,dy.
\end{aligned}$$

These results follow from the fact that $\rho_t\in C_c^1(B_3)$ and

$$\begin{aligned}
\frac{d}{dt}\rho_t(y)&=\partial_u\tilde{h}(\omega,r(x),U_0(g_t^{-1}(y)))\nabla U_0(g_t^{-1}(y))\cdot\frac{d}{dt}g_t^{-1}(x)\\
&=-\partial_u\tilde{h}(\omega,r(x),U_0(g_t^{-1}(y)))\nabla U_0(g_t^{-1}(y))(Dg_t)^{-1}(g_t^{-1}(x))\\
&\qquad\times\Lambda(g_t^{-1}(x))\frac{g_t^{-1}(x)}{|g_t^{-1}(x)}\\
&=-\partial_u\tilde{h}(\omega,r(x),U_0(g_t^{-1}(y)))\nabla\left(U_0(g_t^{-1}(y))\right)\Lambda(g_t^{-1}(x))\frac{g_t^{-1}(x)}{|g_t^{-1}(x)}\\
&=-\sigma_t(y).
\end{aligned}$$

Now

$$\begin{aligned}\frac{T(\omega,\zeta+t\Lambda)(x)-T(\omega,\zeta)(x)}{t} &= \frac{F(t,g_t(x))-F(0,g_t(x))}{t}\\ &\quad+\frac{F(0,g_t(x))-F(0,g_0(x))}{t},\\ &\quad t\in]-t_0,t_0[,\ \ x\in B_4,\end{aligned}$$

and

$$(L\Lambda)(x)=\partial_t F(0,g_0(x))+\nabla F(0,g_0(x))\cdot\frac{x}{|x|}\Lambda(x),\quad x\in B_4;$$

one should note here that $g_0=g_{\zeta+0\Lambda}=g_\zeta$. To prove that L is the Gateaux differential of T at (ω,ζ), we have to show

$$\frac{F(t,g_t(x))-F(0,g_t(x))}{t}\to\partial_t F(0,g_0(x)) \tag{2.26}$$

and

$$\frac{F(0,g_t(x))-F(0,g_0(x))}{t}\to\nabla F(0,g_0(x))\cdot\frac{x}{|x|}\Lambda(x), \tag{2.27}$$

where the limits are understood with respect to $\|\cdot\|_Y$. As to Eq. (2.26), we observe that

$$\begin{aligned}&\left|\nabla\frac{F(t,g_t(x))-F(0,g_t(x))}{t}-\nabla\left(\partial_t F(0,g_0(x))\right)\right|\\ &=\left|\frac{\nabla F(t,g_t(x))-\nabla F(0,g_t(x))}{t}Dg_t(x)-\nabla\partial_t F(0,g_0(x))Dg_0(x)\right|\\ &\le\left|\frac{\nabla F(t,g_t(x))-\nabla F(0,g_t(x))}{t}-\nabla\partial_t F(0,g_t(x))\right||Dg_t(x)|\\ &\quad+|\nabla\partial_t F(0,g_t(x))-\nabla\partial_t F(0,g_0(x))|\ \ |Dg_t(x)|\\ &\quad+|\nabla\partial_t F(0,g_0(x))|\ \ |Dg_t(x)-Dg_0(x)|\\ &=:I_1+I_2+I_3.\end{aligned}$$

Let $\epsilon>0$. For every $z\in\mathbb{R}^3$ there exists τ between 0 and t such that

$$\begin{aligned}\left|\frac{\nabla F(t,z)-\nabla F(0,z)}{t}-\nabla\partial_t F(0,z)\right| &= |\nabla\partial_t F(\tau,z)-\nabla\partial_t F(0,z)|\\ &=\left|\nabla\int_{B_3}\frac{1}{|z-y|}(\sigma_\tau(y)-\sigma_0(y))\,dy\right|\end{aligned}$$

and using Lemma 2.8(a), the latter integral can be estimated by $C\epsilon\|\Lambda\|_X|z|$, provided

$$|\sigma_\tau(y)-\sigma_0(y)|\le\epsilon\|\Lambda\|_X|y|,\quad y\in B_3.$$

This is guaranteed by Lemma 2.10 and we have for $\delta > 0$ sufficiently small, $|t| < \delta$,

$$I_1 \leq C\epsilon |g_t(x)| \leq C\epsilon |x|, \quad x \in B_4.$$

Again by Lemma 2.10 and Lemma 2.8(b) we find the estimate

$$I_2 \leq C\|\Lambda\|_X \|\zeta + t\Lambda - \zeta\|_X^{1/2} |x| \leq C|t|^{1/2}|x|, \quad x \in B_4,$$

and by Lemmas 2.10, 2.8(a) and 2.4(d) we conclude that

$$I_3 \leq C\|\Lambda\|_X |g_0(x)| \, \|\zeta + t\Lambda - \zeta\|_X \leq C|t| \, |x|, \quad x \in B_4.$$

This proves convergence in Eq. (2.26). As to Eq. (2.27), we have for every $x \in B_4$,

$$\frac{F(0, g_t(x)) - F(0, g_0(x))}{t} = \frac{d}{dt} F(0, g_t(x))_{|t=\tau} = \nabla F(0, g_\tau(x)) \cdot \frac{x}{|x|} \Lambda(x),$$

where τ lies between 0 and t. Therefore,

$$\begin{aligned}
&\left| \nabla \frac{F(0, g_t(x)) - F(0, g_0(x))}{t} - \nabla \left(\nabla F(0, g_0(x)) \cdot \frac{x}{|x|} \Lambda(x) \right) \right| \\
&\quad = \left| \nabla \left[(\nabla F(0, g_\tau(x)) - \nabla F(0, g_0(x))) \cdot \frac{x}{|x|} \Lambda(x) \right] \right| \\
&\quad \leq |D^2 F(0, g_\tau(x)) - D^2 F(0, g_0(x))| C|x| \\
&\qquad + |D^2 F(0, g_0(x))| \, |D g_\tau(x) - D g_0(x)| C|x| \\
&\qquad + |\nabla F(0, g_\tau(x)) - \nabla F(0, g_0(x))| \left| D \left(\Lambda(x) \frac{x}{|x|} \right) \right|, \quad x \in B_4.
\end{aligned}$$

Since $D^2 F(0, \cdot)$ is uniformly continuous on B_5, which contains $g_\tau(x)$ for $x \in B_4$ and $\tau \in]-t_0, t_0[$, cf. Lemma 2.4(b), and

$$|g_\tau(x) - g_0(x)| \leq \|\Lambda\|_X |\tau| \, |x| \leq C|t|, \quad x \in B_4,$$

we obtain the asserted convergence with respect to the norm $\| \cdot \|_Y$. □

Since a continuous Gateaux derivative is a Fréchet derivative, we need to verify the next Lemma to complete the proof of Theorem 2.5

Lemma 2.12. *The mapping* $]-\omega_2, \omega_2[\times \Omega \ni (\omega, \zeta) \to \partial_\zeta T(\omega, \zeta) \in \mathfrak{L}(X, Y)$ *is continuous.*

Proof. We fix $(\omega',\zeta')\in]-\omega_2,\omega_2[\times\Omega$ and we take $(\omega,\zeta)\in]-\omega_2,\omega_2[\times\Omega$ and $\Lambda\in X$ with $\|\Lambda\|_X=1$. Since

$$\begin{aligned}
&[\partial_\zeta T(\omega,\zeta)\Lambda](x)-[\partial_\zeta T(\omega',\zeta')\Lambda](x)\\
&\quad=-\int_{B_3}\left[\left(\frac{1}{|g_\zeta(x)-y|}-\frac{1}{|y|}\right)\sigma_{\omega,\zeta,\Lambda}(y)-\left(\frac{1}{|g_{\zeta'}(x)-y|}-\frac{1}{|y|}\right)\sigma_{\omega',\zeta',\Lambda}(y)\right]dy\\
&\quad\quad-\int_{B_3}\left[\frac{g_\zeta(x)-y}{|g_\zeta(x)-y|^3}\rho_{\omega,\zeta}(y)-\frac{g_{\zeta'}(x)-y}{|g_{\zeta'}(x)-y|^3}\rho_{\omega',\zeta'}(y)\right]dy\cdot\frac{x}{|x|}\Lambda(x),\quad x\in B_4.
\end{aligned}$$

we have

$$\left|\nabla\left([\partial_\zeta T(\omega,\zeta)\Lambda](x)-[\partial_\zeta T(\omega',\zeta')\Lambda](x)\right)\right|\le\sum_{j=1}^{6}I_j,$$

where

$$\begin{aligned}
I_1&:=\left|\nabla\int_{B_3}\left(\frac{1}{|g_\zeta(x)-y|}-\frac{1}{|y|}\right)(\sigma_{\omega,\zeta,\Lambda}(y)-\sigma_{\omega',\zeta',\Lambda}(y))\,dy\right|,\\
I_2&:=\left|\nabla\int_{B_3}\left(\frac{1}{|g_\zeta(x)-y|}-\frac{1}{|g_{\zeta'}(x)-y|}\right)\sigma_{\omega',\zeta',\Lambda}(y)\,dy\right|,\\
I_3&:=\left|D\int_{B_3}\frac{g_\zeta(x)-y}{|g_\zeta(x)-y|^3}(\rho_{\omega,\zeta}(y)-\rho_{\omega',\zeta'}(y))\,dy\right||\Lambda(x)|,\\
I_4&:=\left|D\int_{B_3}\left(\frac{g_\zeta(x)-y}{|g_\zeta(x)-y|^3}-\frac{g_{\zeta'}(x)-y}{|g_{\zeta'}(x)-y|^3}\right)\rho_{\omega',\zeta'}(y)\,dy\right||\Lambda(x)|,\\
I_5&:=\left|\int_{B_3}\frac{g_\zeta(x)-y}{|g_\zeta(x)-y|^3}(\rho_{\omega,\zeta}(y)-\rho_{\omega',\zeta'}(y))\,dy\right|\;\left|D\left(\Lambda(x)\frac{x}{|x|}\right)\right|,\\
I_6&:=\left|\int_{B_3}\left(\frac{g_\zeta(x)-y}{|g_\zeta(x)-y|^3}-\frac{g_{\zeta'}(x)-y}{|g_{\zeta'}(x)-y|^3}\right)\rho_{\omega',\zeta'}(y)\,dy\right|\;\left|D\left(\Lambda(x)\frac{x}{|x|}\right)\right|.
\end{aligned}$$

Given $\epsilon>0$, we can choose $\delta>0$ so that the second estimate in Lemma 2.10 holds for $|\omega-\omega'|+\|\zeta-\zeta'|_X<\delta$. Then Lemma 2.8(a) implies, with $z=g_\zeta(x)$, the estimate

$$\begin{aligned}
I_1&\le\left|\nabla\int_{B_3}\frac{1}{|z-y|}(\sigma_{\omega,\zeta,\Lambda}(y)-\sigma_{\omega',\zeta',\Lambda}(y))\,dyDg_\zeta(x)\right|\\
&\le C\epsilon|z|\le C\epsilon|x|,\quad x\in B_4.
\end{aligned}$$

If we define

$$V(x):=\int_{B_3}\frac{1}{|x-y|}\sigma_{\omega',\zeta',\Lambda}(y)\,dy,\quad x\in\mathbb{R}^3,$$

we have $V\in C^1(\mathbb{R}^3)$ with $\nabla V(0)=0$, and

$$\begin{aligned} I_2 &= \left|\nabla V(g_\zeta(x))Dg_\zeta(x)-\nabla V(g_{\zeta'}(x))Dg_{\zeta'}(x)\right| \\ &\le C\left|\nabla V(g_\zeta(x))-\nabla V(g_{\zeta'}(x))\right|+|\nabla V(g_{\zeta'}(x))|\,|Dg_\zeta(x)-Dg_{\zeta'}(x)| \\ &\le C\|\zeta-\zeta'\|_X^{1/2}|x|+C|g_{\zeta'}(x)|\;\;\|\zeta-\zeta'\|_X\le C\|\zeta-\zeta'\|_X^{1/2}|x|,\quad x\in B_4, \end{aligned}$$

where we have used Lemma 2.10, Lemma 2.8 and Lemma 2.4(d). To estimate the remaining terms, we define

$$V_{\omega,\zeta}(x):=\int_{B_3}\frac{1}{|x-y|}\rho_{\omega,\zeta}(y)\,dy,\quad x\in\mathbb{R}^3.$$

Then $V_{\omega,\zeta}\in C^2(\mathbb{R}^3$ and

$$I_3\le\left|D^2V_{\omega,\zeta}(g_\zeta(x))-D^2V_{\omega',\zeta'}(g_\zeta(x))\right|\,|Dg_\zeta(x)|\,|\Lambda(x)|.$$

We now use [27], Lemma P1, where the second derivatives of a newtonian potential U_ρ, induced by $\rho\in C_c^1(\mathbb{R}^3)$ for any $0<d\le R$ are estimated as

$$\|D^2U_\rho\|_\infty\le C\left[R^{-3}\|\rho\|_1+d\|\nabla\rho\|_\infty+\left(1+\ln\frac{R}{d}\right)\|\rho\|_\infty\right].$$

If we choose $d=R$ and $R=(\|\rho\|_1/\|\rho\|_\infty)^{1/4}$, we have

$$\|D^2U_\rho\|_\infty\le C\left[\|\rho\|_1^{1/4}\|\nabla\rho\|_\infty^{3/4}+\|\rho\|_\infty\right],$$

and we can estimate I_3 as follows:

$$\begin{aligned} I_3 &\le C|x|\left[\|\rho_{\omega,\zeta}-\rho_{\omega',\zeta'}\|_1^{1/4}\|\nabla\rho_{\omega,\zeta}-\nabla\rho_{\omega',\zeta'}\|_\infty^{3/4}+\|\rho_{\omega,\zeta}-\rho_{\omega',\zeta'}\|_\infty\right] \\ &\le C|x|\left[\big(|\omega-\omega'|+\|\zeta-\zeta'\|_X\big)^{1/4}+\big(|\omega-\omega'|+\|\zeta-\zeta'\|_X\big)\right], \end{aligned}$$

where we used Lemma 2.7. As to I_4, we have

$$\begin{aligned} I_4 &= \left|D^2V_{\omega',\zeta'}(g_\zeta(x))Dg_\zeta(x)-D^2V_{\omega',\zeta'}(g_{\zeta'}(x))Dg_{\zeta'}(x)\right|\,|\Lambda(x)| \\ &\le C|x|\left|D^2V_{\omega',\zeta'}(g_\zeta(x))-D^2V_{\omega',\zeta'}(g_{\zeta'}(x))\right|+C|x|\;\;|Dg_\zeta(x)-Dg_{\zeta'}(x)| \\ &\le \epsilon|x|+C|x|\,\|\zeta-\zeta'\|_X,\quad x\in B_4, \end{aligned}$$

provided $\|\zeta-\zeta'\|_X$ is small enough, where we have used that $D^2V_{\omega',\zeta'}$ is uniformly continuous on $B_5\ni g_\zeta(x),g_{\zeta'}(x)$ and $|g_\zeta(x)-g_{\zeta'}(x)|\le C\|\zeta-\zeta'\|_X$. By Lemma 2.7(b) and Lemma 2.8(a) for $\sigma=\rho_{\omega,\zeta}-\rho_{\omega',\zeta'}$, we obtain

$$\begin{aligned} I_5 &\le C\left|\nabla\int_{B_3}\frac{1}{|g_\zeta(x)-y|}(\rho_{\omega,\zeta}(y)-\rho_{\omega',\zeta'}(y))\,dy\right| \\ &\le C|g_\zeta(x)|\sup_{x\in\dot{B}_3}\frac{|\rho_{\omega,\zeta}(y)-\rho_{\omega',\zeta'}(y)|}{|x|} \\ &\le C|x|\,(|\omega-\omega'|+\|\zeta-\zeta'\|_X),\quad x\in B_4, \end{aligned}$$

By Lemma 2.4 we finally have

$$\begin{aligned}I_6 &\le |\nabla V_{\omega',\zeta'}(g_\zeta(x)) - \nabla V_{\omega',\zeta'}(g_{\zeta'}(x))|\\ &\le C\|D^2 V_{\omega',\zeta'}\|_\infty |g_\zeta(x) - g_{\zeta'}(x)|\\ &\le C\|\zeta - \zeta'\|_X |x|, \ \ x \in B_4.\end{aligned}$$

and we have shown that for fixed (ω',ζ'), we have that for every $\epsilon > 0$ there exists $\delta = \delta(\omega',\zeta') > 0$ such that for all $(\omega,\zeta) \in]-\omega_2,\omega_2[\times\Omega$ with $|\omega-\omega'| + \|\zeta - \zeta'\| < \delta$ and all $\Lambda \in X$ with $\|\Lambda\|_X = 1$ we have

$$\|\partial_\zeta T(\omega,\zeta)\Lambda - \partial_\zeta T(\omega',\zeta')\Lambda\|_Y \le \epsilon.$$

The proof is complete. □

Putting Lemmas 2.9 - 2.12 together, the assertions of Theorem 2.5 follow.

2.4 $\partial_\zeta T(0,0)$ is an isomorphism

We want to prove the following result:

Proposition 2.13. *The mapping $\partial_\zeta T(0,0): X \to Y$ is a linear isomorphism.*

Let us abbreviate $L_0\Lambda := \partial_\zeta T(0,0)\Lambda$ for $\Lambda \in X$. We observe that $g_0 = id$ and therefore the function U_ζ in Theorem 2.5 coincides with the potential U_0 of the spherically symmetric steady state we started with, if $\zeta = 0$. We have

$$\begin{aligned}\rho_0'(|x|) &= \partial_u \tilde{h}(0, r(x), U_0(|x|))U_0'(|x|)\\ &= \partial_u \tilde{h}(0, r(x), U_0(|x|))\nabla U_0(x) \cdot \frac{x}{|x|}, \quad x \in \mathbb{R}^3.\end{aligned}$$

This implies

$$\begin{aligned}(L_0\Lambda)(x) &= -\int_{B_3}\left(\frac{1}{|x-y|} - \frac{1}{|y|}\right)\rho_0'(|y|)\Lambda(y)\,dy - \int_{B_3}\frac{x-y}{|x-y|^3}\rho_0(|y|)\,dy \cdot \frac{x}{|x|}\Lambda(y)\\ &= -U_0'(|x|)\Lambda(x) - \int_{B_3}\left(\frac{1}{|x-y|} - \frac{1}{|y|}\right)\rho_0'(|y|)\Lambda(y)\,dy, \quad x \in B_4, \Lambda \in X.\end{aligned}$$

Now let

$$(K\Lambda)(x) := -\frac{1}{U_0'(|x|)}\int_{B_3}\left(\frac{1}{|x-y|} - \frac{1}{|y|}\right)\rho_0'(|y|)\Lambda(y)\,dy, \quad x \in \dot{B}_4, \Lambda \in C_S(B_4).$$

Then we can write

$$(L_0\Lambda)(x) = -U_0'(|x|)[(id - K)\Lambda](x), \quad x \in B_4, \Lambda \in X. \tag{2.28}$$

In order to prove Proposition 2.13, we need

Lemma 2.14. *The linear operator $K : C_S(B_4) \to C_S(B_4)$ is compact, where $C_S(B_4)$ is equipped with the supremum norm $\|\cdot\|_\infty$.*

Proof. For $\Lambda \in C_S(B_4)$ let

$$V_\Lambda(x) := -\int_{B_3} \frac{1}{|x-y|} \rho_0'(|y|)\Lambda(y)\,dy, \quad x \in \mathbb{R}^3.$$

Then $V_\Lambda \in C^1(\mathbb{R}^3)$, $\nabla V_\Lambda(0) = 0$, and

$$(K\Lambda)(x) = \frac{1}{U_0'(|x|)}(V_\Lambda(x) - V_\Lambda(0)), \quad x \in \dot{B}_4.$$

Using Lemma 2.2(c), we obtain the estimate

$$|(K\Lambda)(x)| \le \frac{1}{C|x|}\|\nabla V_\Lambda\|_\infty |x| \le C\|\Lambda\|_\infty, \quad x \in \dot{B}_4,$$

where the constant C depends on ρ_0 and U_0, but not on Λ or x. Thus K maps bounded sets into bounded sets. We next show that $K\Lambda$ is Hölder continuous with exponent 1/2, uniformly on bounded sets in $C_S(B_4)$. Let $M > 0$ and assume $\|\Lambda\|_\infty \le M$. In the following, constants denoted by C depend on ρ_0, U_0 and M, but not on Λ. Obviously, $\rho_0'\Lambda \in L^\infty(\mathbb{R}^3)$ and we deduce from Lemma B.2 the existence of $C > 0$ with

$$|\nabla V_\Lambda(x) - \nabla V_\Lambda(x')| \le C\|\rho_0'\Lambda\|_\infty |x-x'|^{1/2}, \quad x, x' \in B_4$$

Since $\nabla V_\Lambda(0) = 0$, the latter implies

$$|\nabla V_\Lambda(x)| \le C|x|^{1/2}, \quad x \in B_4.$$

Now let $x, x' \in \dot{B}_4$ and $|x| \le |x'|$. Then

$$\begin{aligned} |(K\Lambda)(x) - (K\Lambda)(x')| \le & \left|\frac{1}{U_0'(|x|)} - \frac{1}{U_0'(|x'|)}\right| |V_\Lambda(x) - V_\Lambda(0)| \\ & + \frac{1}{U_0'(|x'|)}|V_\Lambda(x) - V_\Lambda(x')| =: I_1 + I_2 \end{aligned}$$

and we obtain for some $z \in B_4$ with $|z| \le |x'|$ the estimates

$$\begin{aligned} I_1 &\le \frac{|U_0'(|x|) - U_0'(|x'|)|}{|x|\,|x'|}|\nabla V_\Lambda(z)|\,|x| \le C|x-x'|^{1/2}\frac{(|x|+|x'|)^{1/2}}{|x'|}|z|^{1/2} \\ &\le C|x-x'|^{1/2}, \end{aligned}$$

and

$$I_2 \le \frac{C}{|x'|}|\nabla V_\Lambda(z)|\,|x-x'| \le \frac{C}{|x'|}|z|^{1/2}|x-x'| \le C|x-x'|^{1/2},$$

so that

$$|(K_\Lambda)(x)K_\Lambda)(x')| \le C|x-x'|^{1/2}, \quad x,x' \in \dot{B}_4$$

and

$$|(K_\Lambda)(x)| \le C|\nabla V_\Lambda(z)| \le C|x|^{1/2}, \quad x \in \dot{B}_4.$$

We have shown that K maps bounded sets of $C_S(B_4)$ into bounded and equicontinuous subsets of $C_S(B_4)$. Thus K is compact by the Arzela-Ascoli theorem and the proof is complete. □

Lemma 2.15. *$id - K : C_S(B_4) \to C_S(B_4)$ is one-to-one and onto.*

Proof. Since K is compact, it suffices to show that $id - K$ is one-to-one. Let $\Lambda \in C_S(B_4)$ with $\Lambda - K\Lambda = 0$. Now $\Lambda = 0$ can be shown by expanding Λ into spherical harmonics. For that purpose, let

$$\{\mathcal{S}_{n,j},\ n \in \mathbb{N},\ j = 1,\dots,2n+1\}$$

be the orthonormal set of spherical harmonics introduced in the Appendix, cf. Section 2.5, where for $n \in \mathbb{N}$, the functions $\mathcal{S}_{n,j} : \partial B_1 \to \mathbb{R},\ j = 1,\dots,2n+1$ are homogeneous polynomials of degree n. We define

$$\Lambda_{nj}(r) := \int_{\partial B_1} \mathcal{S}_{n,j}(\xi)\Lambda(r\xi)\,d\omega_\xi = \frac{1}{r^2}\int_{\partial B_r} \mathcal{S}_{n,j}(x/r)\Lambda(x)\,d\omega_x \tag{2.29}$$

and we use the expansion of the integral kernel $1/|x-y|$ into spherical harmonics, cf. Lemma B.3 and Lemma B.4: For $x,y \in \mathbb{R}^3$, $x = r\xi$ and $y = s\eta$ with $\xi,\eta \in \partial B_1$, $r,s \in \mathbb{R}^+$, $r \ne s$, we have

$$\frac{1}{|x-y|} = \max(r,s)^{-1} \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{4\pi}{2n+1}\left(\frac{\min(r,s)}{\max(r,s)}\right)^n \mathcal{S}_{n,j}(\xi)\mathcal{S}_{n,j}(\eta).$$

$K\Lambda - \Lambda = 0$ then implies

$$\begin{aligned}
\Lambda_{nj}(r) &= -\frac{1}{U_0'(r)}\int_{B_3}\int_{\partial B_1}\left(\frac{1}{|r\xi - y|} - \frac{1}{|y|}\right)\mathcal{S}_{n,j}(\xi)\,d\omega_\xi\,\rho_0'(|y|)\Lambda(y)\,dy \\
&= -\frac{4\pi}{2n+1}\frac{1}{U_0'(r)}\int_0^3 s^2\rho_0'(s)\frac{\min(r,s)^n}{\max(r,s)^{n+1}}\int_{\partial B_1}\mathcal{S}_{n,j}(\eta)\Lambda(s\eta)\,d\omega_\eta\,ds \\
&\quad + \frac{4\pi}{2n+1}\frac{1}{U_0'(r)}\int_0^3 s^2\rho_0'(s)\frac{0^n}{s^{n+1}}\int_{\partial B_1}\mathcal{S}_{n,j}(\eta)\Lambda(s\eta)\,d\omega_\eta\,ds \\
&= -\frac{4\pi}{2n+1}\frac{1}{U_0'(r)}\int_0^3 s^2\rho_0'(s)\left(\frac{\min(r,s)^n}{\max(r,s)^{n+1}} - \frac{0^n}{s^{n+1}}\right)\Lambda_{nj}(s)\,ds,
\end{aligned}$$

where we used that the functions $\mathcal{S}_{n,j}$ are orthonormal with repsect to $\langle .,.\rangle_{L^2(\partial B_1)}$. We find that

$$\Lambda_{01}(r)=-\frac{4\pi}{rU_0'(r)}\int_0^r \rho_0'(s)s(s-r)\Lambda_{01}(s)\,ds$$

and we obviously have $\lim_{r\to 0}\Lambda_{01}(r)=0$. Let $R\geq 0$ be maximal such that $\Lambda_{01}(r)$ vanishes on $[0,R]$. Then for $r\in[R,3]$,

$$|\Lambda_{01}(r)|\leq\frac{4\pi}{rU_0'(r)}\|\rho_0'\|_\infty\sup_{0\leq s\leq r}|\Lambda_{01}(s)|\int_R^r s(r-s)\,ds\leq C(r-R)\sup_{0\leq s\leq r}|\Lambda_{01}(s)|.$$

Thus for small $\epsilon>0$, we have $\Lambda_{01}(r)=0$ on the interval $[R,R+\epsilon]$ and we conclude that Λ_{01} vanishes on the whole interval $[0,3]$. Now up to linear combinations, the spherical harmonics for $n=1$ are given by x_1,x_2,x_3, and $\Lambda\in C_S$ implies

$$\int_{\partial B_1}\xi_1\Lambda(r\xi)\,d\omega_\xi=-\int_{\partial B_1}\xi_1\Lambda(r\xi)\,d\omega_\xi=0,$$

where we made the transformation $\xi\mapsto(-\xi_1,\xi_2,\xi_3)$. Analoguously,

$$\int_{\partial B_1}\xi_2\Lambda(r\xi)\,d\omega_\xi=\int_{\partial B_1}\xi_3\Lambda(r\xi)\,d\omega_\xi=0,$$

and we have $\Lambda_{11}=\Lambda_{12}=\Lambda_{13}\equiv 0$. Let $n\geq 2$. Then

$$\Lambda_{nj}(r)=-\frac{4\pi}{2n+1}\frac{1}{U_0'(r)}\left(\int_0^r s^2\rho_0'(s)\frac{s^n}{r^{n+1}}\Lambda_{nj}(s)\,ds+\int_r^3 s^2\rho_0'(s)\frac{r^n}{s^{n+1}}\Lambda_{nj}(s)\,ds\right),$$

and

$$\begin{aligned}|\Lambda_{nj}(r)|&\leq\frac{4\pi}{2n+1}\frac{1}{U_0'(r)}\|\Lambda_{nj}\|_\infty\left(\frac{1}{r^2}\int_0^r(-\rho_0')(s)\frac{s^{n-1}}{r^{n-1}}s^3\,ds+r\int_r^3(-\rho_0')(s)\frac{r^{n-1}}{s^{n-1}}\,ds\right)\\&\leq\frac{4\pi}{2n+1}\frac{1}{U_0'(r)}\|\Lambda_{nj}\|_\infty\left(\frac{1}{r^2}\int_0^r(-\rho_0')(s)s^3\,ds+r\int_r^3(-\rho_0')(s)\,ds\right)\\&=\frac{4\pi}{2n+1}\frac{1}{U_0'(r)}\|\Lambda_{nj}\|_\infty\left(\frac{1}{r^2}r^3(-\rho_0)(r)+\frac{3}{r^2}\int_0^r s^2\rho_0(s)\,ds+r\rho_0(r)\right)\\&=\frac{3}{2n+1}\|\Lambda_{nj}\|_\infty,\end{aligned}$$

where we integrated by parts in the third line and used the fact that $U_0'(r)=\frac{4\pi}{r^2}\int_0^r s^2\rho_0(s)\,ds$ in the last line, also recall from (2.13) that $-\rho_0'(r)\geq 0$.
Now $2n+1>3$ for $n\geq 2$ implies that $\Lambda_{nj}\equiv 0$ for $n\geq 2$ as well and the completeness of $\{\mathcal{S}_{n,j}\}$ induces $\Lambda\equiv 0$. We conclude that $id-K$ is one-to-one as claimed. □

It is now clear that $L_0 : X \to Y$ is one-to-one as well – this follows from Eq. (2.28) and the fact that $U_0'(r) > 0$ for $r > 0$. So once we have proved the next lemma, the proof of Proposition 2.13 will be complete.

Lemma 2.16. *$L_0 : X \to Y$ is onto.*

Proof. Let $g \in Y$ and define $q := g/U_0'$. We will show $q \in X$. We have $q \in C^1(\dot{B}_4) \cap C_S(B_4)$ and

$$|\nabla q| \le \frac{|\nabla g(x)|}{U_0'(|x|)} + |g(x)| \left| \frac{U_0''(|x|)}{U_0'(|x|)^2} \frac{x}{|x|} \right| \le C \left(\frac{|\nabla g(x)}{|x|} + \frac{|g(x)|}{|x|^2} \right) \le 2C \|g\|_Y.$$

By definition of Y and since $U_0 \in C^2([0,\infty[)$ with $U_0''(0) > 0$ we have that for every $x \in \partial B_1$,

$$\begin{aligned} \nabla q(tx) &= \frac{\nabla g(tx)}{t} \frac{t}{U_0'(t)} - \frac{g(tx)}{t^2} U_0''(t) \left(\frac{t}{U_0'(t)} \right)^2 x \\ &\to \frac{\nabla g(0x)}{0} \frac{1}{U_0''(0)} - \frac{g(0x)}{0^2} U_0''(0) \frac{1}{U_0''(0)^2} x \end{aligned}$$

as $t \to 0+$, uniformly in $x \in \partial B_1$.

Since $X \subset C_S(B_4)$, there exists by Lemma 2.15 an element $\Lambda \in C_S(B_4)$ such that

$$\Lambda - K\Lambda = -q = -\frac{g}{U_0'}.$$

This implies that $L_0 \Lambda = g$ and thus that L_0 is onto, provided $\Lambda \in X$. To see the latter we observe that $\Lambda = K\Lambda - q$ is Hölder continuous since $K\Lambda$ is Hölder continuous. If we now define V_Λ as above in the proof of Lemma 2.14 we also conclude that $V_\Lambda \in C^2(\mathbb{R}^3)$ and thus $K\Lambda \in C^1(\dot{B}_4)$. Denoting by H_{V_Λ} the Hessian of V_Λ we obtain for each $x \in \dot{B}_4$ a point $z \in \overline{0,x}$ such that

$$\begin{aligned} |\nabla(K\Lambda)(x)| &\le \left| \frac{U_0''(|x|)}{U_0'(|x|)^2} \right| |V_\Lambda(x) - V_\Lambda(0)| + \frac{1}{|U_0'(|x|)|} |\nabla V_\Lambda(x)| \\ &\le \frac{C}{|x|^2} |\langle H_{V_\Lambda}(z)x, x\rangle| + \frac{C}{|x|} |\nabla V_\Lambda(x)| \le C \|D^2 V_\Lambda\|_\infty \end{aligned}$$

Finally, for $x \in \partial B_1$, we have

$$\begin{aligned} \nabla(K\Lambda)(tx) &= -\frac{U_0''(t)}{U_0'(t)^2} x (V_\Lambda(tx) - V_\Lambda(0)) + \frac{1}{U_0'(t)} \nabla V_\Lambda(tx) \\ &= -U_0''(t) \left(\frac{t}{U_0'(t)} \right)^2 x \frac{1}{t^2} \frac{1}{2} \langle H_{V_\Lambda}(\tau x) tx, tx\rangle + \frac{t}{U_0'(t)} \frac{\nabla V_\Lambda(tx)}{t} \\ &\to -\frac{1}{2U_0''(0)} \langle H_{V_\Lambda}(0)x, x\rangle x + \frac{1}{U_0''(0)} D^2 V_\Lambda(0) x, \end{aligned}$$

as $t \to 0+$, uniformly in $x \in \partial B_1$. We have shown that $K\Lambda \in X$ and this implies $\Lambda = K\Lambda + q \in X$ and the proof is complete. ☐

2.5 Appendix

In this section, we firstly state the implicit function theorem which is used for the proof of Theorem 2.1. Then we give a regularity result for the Poisson equation and finally introduce spherical harmonics and state two important lemmas: an addition theorem and the expansion of the integral kernel $1/|x-y|$ in spherical harmonics.

Theorem B.1. *Let X,Y,Z be Banach spaces, $U\subset X$ and $V\subset Y$ neighbourhoods of $x_0\in X$ and $y_0\in Y$ respectively, $F:U\times V\to Z$ continuous and continously Fréchet-differentiable with respect to the second variable. Suppose also that $F(x_0,y_0)=0$ and $F_y^{-1}(x_0,y_0)\in\mathfrak{L}(Z,Y)$.*
Then there exist balls $\overline{B_r}(x_0)\subset U$, $\overline{B_\delta}(y_0)\subset V$ and exactly one continuous map $G:B_r(x_0)\to B_\delta(y_0)$ such that $Gx_0=y_0$ and $F(x,Gx)=0$ on $B_r(x_0)$.

Proof. [5], Theorem 15.1. □

Lemma B.2. *Let $n<p\le\infty$ and let $\rho(x)\in L^p(\mathbb{R}^n)$ with compact support. Define*

$$V_\rho(x):=-\int_{\mathbb{R}^n}\frac{1}{|x-y|}\rho(y)\,dy$$

Then for every $0<\alpha<1-n/p$ we have $V_\rho\in C^{1,\alpha}(\mathbb{R}^n)$ and

$$|\partial_i V_\rho(x)-\partial_i V_\rho(x')|\le C(n,\alpha,p)|x'-x|^\alpha\|f\|_p\mathcal{L}^n(\text{supp}\{\rho\})^{\frac{1-\alpha}{n}-\frac{1}{p}}$$

Proof. [18], Theorem 10.2. □

Some facts about spherical harmonics

In the following, we use the notation of [20] and we will always consider the case, where the space dimension q is equal to 3. For $n\in\mathbb{N}$, consider a homogeneous polynomial H_n of degree n, which satisfies

$$\Delta H_n(x)=0.$$

Then for $\xi\in\partial B_1:=\{x\in\mathbb{R}^3\,|\,|x|=1\}$,

$$S_n(\xi):=H_n(\xi)$$

is called a spherical harmonic of order n. For each n, there exist $2n+1$ linearly independent spherical harmonics, which we call $S_{n,j}$, $j=1,\dots 2n+1$, cf. [20], Lemma 4. We denote by $\{\mathcal{S}_{n,j},\ n=0,\dots,\infty,\ j=1,\dots,2n+1\}$ the orthonormal set of all spherical harmonics, where we orthonormalize with respect to $\langle .,.\rangle_{L^2(\partial B_1)}$. Then we have the following

Lemma B.3. *For a fixed* $n \in \mathbb{N}$ *and* $\xi, \eta \in \partial B_1$*, we have*

$$\sum_{j=1}^{2n+1} \mathcal{S}_{n,j}(\xi)\mathcal{S}_{n,j}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta),$$

where $P_n(t)$ *is the Legendre Polynomial of degree* n*.*

Lemma B.4. *Let* $x, y \in \mathbb{R}^3$ *with* $x = R\xi$*,* $y = r\eta$*, for suitable* $\xi, \eta \in \partial B_1$ *and* $r, R \in \mathbb{R}$*. Then we have for* $R > r$

$$\frac{1}{|x-y|} = R^{-1} \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n P_n(\xi \cdot \eta),$$

and for $R < r$

$$\frac{1}{|x-y|} = r^{-1} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^n P_n(\xi \cdot \eta),$$

where $P_n(t)$ *is the Legendre Polynomial of degree* n*.*

Proofs can be found in [20], Theorem 2 and Lemma 19.

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