Differential forms on log canonical spaces in positive characteristic

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Abstract

Given a logarithmic 1-form on the snc locus of a log canonical surface pair (X, D) over a perfect field of characteristic $p \geqslant 7$, we show that it extends with at worst logarithmic poles to any resolution of singularities. We also prove the analogous statement for regular differential forms, under an additional tameness hypothesis. In addition, residue and restriction sequences for tamely dlt pairs are established. We give a number of examples showing that our results are sharp in the surface case, and that they fail in higher dimensions. On the other hand, our techniques yield a new proof of the characteristic zero Logarithmic Extension Theorem in any dimension.

Contents

1. Introduction

Differential forms play an essential role in the study of algebraic varieties. Given an algebraic variety X over a field k and a resolution of singularities $\pi\colon Y \to X$, it is natural to ask whether any p-form on the regular locus X_{reg} extends to a regular p-form on Y. There is also a version of this question which concerns pairs and allows certain logarithmic poles. In order to fix our terminology once and for all, we introduce the following language (for notation, see Section 2).

DEFINITION (Extension properties for differential forms). Let (X, D) be a pair (that is, X is normal and D is a Weil $\mathbb{Q}$-divisor with coefficients in $[0, 1] \cap \mathbb{Q}$) defined over a field k, and $1 \leqslant q \leqslant \dim X$ an integer.

[Correction added on 06 September 2021, after first online publication: The paper originally contained an abstract written in German. This was replaced with an abstract written in English.]

- We say that (X, D) *satisfies the regular extension theorem for q-forms* if for any proper birational map $\pi\colon Y \to X$ from a normal variety Y, the natural inclusion

$$\pi_*\Omega^{[q]}_{Y/k} \hookrightarrow \Omega^{[q]}_{X/k}$$

is an isomorphism. Equivalently, the sheaf $\pi_*\Omega^{[q]}_{Y/k}$ is reflexive. It is sufficient to check this for a resolution of singularities $Y \to X$ (if available): cf. [**10**, Lemma 2.13] and note that the proof given there is independent of the base field.

- We say that (X, D) *satisfies the logarithmic extension theorem for q-forms* if for any map π as above, with D_Y the strict transform of D and $E \subset Y$ the reduced divisorial part of the exceptional set $\mathrm{Exc}(\pi)$, the natural inclusion

$$\pi_*\Omega^{[q]}_{Y/k}(\log \lfloor D_Y \rfloor + E) \hookrightarrow \Omega^{[q]}_{X/k}(\log \lfloor D \rfloor)$$

is an isomorphism. Equivalently, the sheaf $\pi_*\Omega^{[q]}_{Y/k}(\log \lfloor D_Y \rfloor + E)$ is reflexive. Again, it is sufficient to check this for a log resolution $Y \to X$ of (X, D).

- We say that (X, D) *satisfies the regular extension theorem* if it satisfies the regular extension theorem for q-forms, for all values of q. Ditto for the logarithmic variant.

Over the complex numbers, the problem of when the extension theorems hold has a long history. It has been studied by several people using different methods — the following list is not exhaustive: [**4, 7, 10, 11, 22, 23**]. The paper mentioned last, [**11**], can in many ways be seen as the culmination[†] of this line of research. It proved the following.

(1.1.1) Any complex klt (=Kawamata log terminal) pair (X, D) satisfies the regular extension theorem [**11**, Theorem 1.4].

(1.1.2) Any complex log canonical pair (X, D) satisfies the logarithmic extension theorem [**11**, Theorem 1.5].

Given the importance of these results, it is not free of interest to ask whether similar results also hold in positive characteristic. Curiously enough, no research in this direction has been conducted so far. We have identified two main reasons for this.

- It has been known to experts for some time that (1.1.1) fails in a strong sense in positive characteristic. In fact, over any field of non-zero characteristic, there exists a strongly F-regular (in particular, klt) surface X violating the regular extension theorem (Example 10.2).

- The proof of (1.1.2) relies on rather subtle Hodge-theoretic vanishing theorems for Du Bois spaces. These are either false or not known in positive characteristic, inextricably linking the proof to the complex numbers. The same can be said of the techniques in [**16**].

The purpose of this paper is to overcome these obstacles, at least for surfaces (but see Theorem 1.6 for higher dimensions). Concerning the first issue, our approach is pretty straightforward: as (1.1.1) fails, we instead concentrate on (1.1.2) (cf. however Theorem 1.3, which explores the failure of (1.1.1) more thoroughly). To deal with the second problem, we develop a completely novel and much more hands-on approach to extension. Our first main result is as follows.

THEOREM 1.2 (Logarithmic extension for surfaces). *Let (X, D) be a log canonical surface pair over a perfect field k of characteristic $p \geqslant 7$. Then (X, D) satisfies the logarithmic extension theorem.*

[†]Very recently, it has been generalized further in [**16**], using perverse sheaves.

Our second main result explains when the logarithmic extension theorem does imply the regular extension theorem.

THEOREM 1.3 (Regular extension for surfaces). *Let $(0 \in X, D)$ be a surface singularity over a field k of characteristic $p > 0$. Assume that for some (not necessarily log) resolution $\pi \colon Y \to X$, with exceptional curves $E_1, \ldots, E_\ell$, the determinant of the intersection matrix $(E_i \cdot E_j)$ is not divisible by p. Then if $(0 \in X, D)$ satisfies the logarithmic extension theorem for 1-forms, it also satisfies the regular extension theorem for 1-forms.*

We would like to emphasize the advantages of our approach over the existing techniques. First of all, we feel that our proof offers a new level of both transparency and tangibility, as it does not explicitly use any Hodge theory (it does, however, rely on the minimal model program). Second, this very same feature also makes it, to a large extent, insensitive to the characteristics of the ground field. In fact, aside from some effortless changes our approach also yields a new proof of the characteristic zero extension theorem [**11**, Theorem 1.5] — the details are worked out in Section 9. Third and maybe most importantly, we obtain a lucid explanation of why the logarithmic extension theorem fails in low characteristics, even for surface rational double points (RDPs).

Further results in this paper

Apart from the above extension results, we establish residue and restriction sequences for reflexive differential forms on dlt pairs in positive characteristic, and symmetric powers thereof. This is analogous to known results in characteristic zero [**8, 11**]. However, it is important to note that actually a slightly stronger notion is required, called *tamely dlt* in this paper. A dlt pair (X, D) is tamely dlt if D is reduced and the Cartier index of $K_X + D$ is not divisible by p (Definition 4.2).

The precise statement is as follows. Even though we only use it as a technical tool in the proof of our main result, we believe that it is of independent interest.

THEOREM 1.4 (Residue sequence). *Let (X, D) be a tamely dlt surface pair (in particular, D is reduced), and let $P \subset D$ be an irreducible component. Set $P^c \coloneqq \mathrm{Diff}_P(D - P)$, so that $(K_X + D)|_P = K_P + P^c$. Then there is a short exact sequence*

$$0 \longrightarrow \Omega_X^{[1]}(\log D - P) \longrightarrow \Omega_X^{[1]}(\log D) \xrightarrow{\mathrm{res}_P} \mathscr{O}_P \longrightarrow 0 \qquad (1.4.1)$$

which on the simple normal crossings (snc) locus of (X, D) agrees with the usual residue sequence. Its restriction to P induces a short exact sequence[†]

$$0 \longrightarrow \Omega_P^1(\log\lfloor P^c \rfloor) \longrightarrow \Omega_X^{[1]}(\log D)\big|_P^{\vee\vee} \xrightarrow{\mathrm{res}_P^P} \mathscr{O}_P \longrightarrow 0. \qquad (1.4.2)$$

More generally, for every $m \in \mathbb{N}$ there is a surjective map

$$\mathrm{res}_P^m \colon \mathrm{Sym}^{[m]} \Omega_X^{[1]}(\log D) \longrightarrow\!\!\!\!\to \mathscr{O}_P$$

which generically coincides with the mth symmetric power of the residue map.

THEOREM 1.5 (Restriction sequence). *Notation as above. Then there is a short exact sequence[‡]*

$$0 \longrightarrow \Omega_X^{[1]}(\log D)(-P)^{\vee\vee} \longrightarrow \Omega_X^{[1]}(\log D - P) \xrightarrow{\mathrm{restr}_P} \Omega_P^1(\log\lfloor P^c \rfloor) \longrightarrow 0 \qquad (1.5.1)$$

[†]Here, of course, in the middle term we are taking the double dual on P and not on X (the latter would be zero).

[‡]By definition, $\Omega_X^{[1]}(\log D)(-P)^{\vee\vee}$ means the double dual of $\Omega_X^{[1]}(\log D) \otimes \mathscr{O}_X(-P)$. Taking the reflexive hull is necessary because $P \subset X$ is in general not a Cartier divisor.

which on the snc locus of (X, D) agrees with the usual restriction sequence. More generally, for every $m \in \mathbb{N}$ there is a surjective map

$$\mathrm{restr}_P^m \colon \mathrm{Sym}^{[m]} \Omega_X^{[1]}(\log D - P) \longrightarrow \mathscr{O}_P(mK_P + \lfloor mP^c \rfloor)$$

which generically coincides with the mth symmetric power of the restriction map.

Sharpness of results

In Section 10, we have gathered a number of examples to show that our results are sharp. First of all, Theorem 1.2 does fail in characteristic less than seven, even if k is algebraically closed, $D = 0$ and X is an RDP. More precisely, we show by explicit calculation that the singularity given by the equation $z^2 + x^3 + y^5 = 0$ violates the logarithmic extension theorem over *any* field of characteristic $p \leqslant 5$. In the terminology of Artin's classification of RDPs [**2**], this is the E_8^0 singularity. This failure also occurs for some singularities of types D_n $(p = 2)$ and E_6, E_7 $(p = 2, 3)$. We have omitted those calculations, as they are very similar in spirit to the E_8 case.

Turning to Theorem 1.3, its statement is sharp too, as shown by the example of contracting a smooth rational curve with self-intersection $-p$ in any characteristic $p > 0$. In this case, the logarithmic extension theorem holds for 1-forms, but the regular extension theorem does not. Again, this can be seen via explicit computation.

The latter example can also be elaborated upon to show that Theorem 1.5 fails for dlt pairs that are not tamely dlt. If one tries to run the proof of Theorem 1.2 on, say, a D_n singularity in characteristic two, the lack of a suitable restriction map is exactly where the argument breaks down: already the first contraction performed by the minimal model program (MMP) produces a pair that is not tamely dlt. This should be seen as the deeper reason for the failure of Theorem 1.2 in low characteristics.

Higher dimensions

For the majority of readers, a most pressing question will be to what extent Theorem 1.2 carries over to higher dimensions. As we will see in Section 9, in characteristic zero the higher dimensional logarithmic extension theorem is intimately linked to the fact that on a projective snc pair (X, D), a line bundle $\mathscr{L} \subset \Omega_X^p(\log D)$ cannot be big unless $p = \dim X$. This is the content of the Bogomolov–Sommese vanishing theorem [**5**, Corollary 6.9], while the weaker statement that $\mathscr{L}$ cannot be ample is a special case of (Kodaira–Akizuki–)Nakano vanishing [**1**, Theorem 1″]. Both results fail badly in positive characteristic and in fact there are counterexamples strong enough to show that Theorem 1.2 itself does not hold. The precise statement is as follows and the details of the construction can be found in Section 11.

THEOREM 1.6 (Failure of the higher dimensional logarithmic extension theorem). *Fix an algebraically closed field k of characteristic $p > 0$.*

(1.6.1) *In any dimension $n \geqslant p - 1$, there exists a log canonical pair $(X, \emptyset)$ over k that violates the logarithmic extension theorem for $(n - 2)$-forms.*

(1.6.2) *If $n \geqslant 2p - 1$, there exists a* canonical *pair $(X, \emptyset)$ for which the logarithmic extension theorem fails as above.*

(1.6.3) *If $n \geqslant 3p - 1$, there even exists a* terminal *pair $(X, \emptyset)$ as above.*

Furthermore, the above examples admit log resolutions.

As Theorem 1.2 already fails for surfaces if the characteristic is low, Theorem 1.6 becomes interesting only for $p \geqslant 7$. In this sense, the lowest dimensional example it provides is a six-dimensional singularity in characteristic seven. The following conjecture hence remains open.

One sacrilegious conjecture. Over a perfect field of characteristic $p \geqslant 7$, the logarithmic extension theorem holds for log canonical pairs of dimension at most $p - 2$.

Of course, we do not believe in the sacrilegious conjecture. Rather, our inability to disprove it is caused by a lack of techniques to produce meaningful counterexamples.

Relation to F-singularities

The examples in Theorem 1.6 are (un-)fortunately not F-pure. On the other hand, using classification results [**12**, Theorem 1.1] one can show that all normal F-regular surface singularities over a perfect field satisfy the logarithmic extension theorem. The same is probably true for F-pure surfaces, but the case distinctions get much more tedious. These observations have led us to the following intriguing question:

Question 1.7. Is there a version of the logarithmic extension theorem for strongly F-regular/F-pure singularities that does not exclude low characteristics and works in any dimension?

The following line of attack appears to be quite promising. By [**25**, Theorem 3.3], the affine cone over a smooth projective variety X is F-pure if and only if X is (globally) F-split. Hence one would need to investigate whether F-split varieties satisfy Nakano vanishing. Since at least Kodaira vanishing obviously holds for these, chances may not be that bad. This would immediately provide a positive answer to Question 1.7 for cones. On the other hand, if Nakano vanishing failed, we would obtain an F-pure counterexample to the logarithmic extension theorem.

2. *Notation and conventions*

Base field

Throughout this paper, we work over a field k, which except for Section 9 will be assumed to be of positive characteristic $p > 0$. Further assumptions (perfect, algebraically closed, etc.) will be expressly stated whenever necessary.

Pairs and divisors

A *pair* (X, D) consists of a normal variety X and a Weil $\mathbb{Q}$-divisor $D = \sum a_i D_i$ with coefficients $0 \leqslant a_i \leqslant 1$. The pair is called *reduced* if D is reduced. The *round-down* of D is denoted by $\lfloor D \rfloor := \sum \lfloor a_i \rfloor D_i$, and similarly for the *round-up* $\lceil D \rceil$. The *fractional part* $\{D\}$ is, by definition, $D - \lfloor D \rfloor$. For a uniform definition of the singularities of the MMP (klt, plt, dlt, lc, etc.), we refer to [**19**, Definition 2.8].

The regular and singular loci of a variety X are denoted X_{reg} and X_{sg}, respectively. We say that a closed subset $Z \subset X$ is *small* if $\mathrm{codim}_X(Z) \geqslant 2$, and that an open subset $U \subset X$ is *big* if $X \setminus U$ is small.

A Weil divisor D on a normal variety X is said to be $\mathbb{Z}_{(p)}$-*Cartier* if it has a multiple not divisible by p which is Cartier. Equivalently, D is in the image of the natural map

$$\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \longrightarrow \mathrm{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

Since $\mathbb{Z}_{(0)} = \mathbb{Q}$, in characteristic zero we recover the usual notion of being $\mathbb{Q}$-Cartier. More generally, the *Cartier index* of D is the smallest integer $m > 0$ with mD Cartier (or $+\infty$ if no such m exists).

Reflexive sheaves

Let X be a normal variety and $\mathscr{E}$ a coherent sheaf on X. The $\mathscr{O}_X$-double dual (or reflexive hull) of $\mathscr{E}$ is denoted by $\mathscr{E}^{\vee\vee}$. The sheaf $\mathscr{E}$ is called *reflexive* if the canonical map $\mathscr{E} \to \mathscr{E}^{\vee\vee}$ is an isomorphism. A *Weil divisorial sheaf* is a reflexive sheaf of rank one. A coherent subsheaf $\mathscr{A} \subset \mathscr{E}$ of a reflexive sheaf is said to be *saturated* if the quotient $\mathscr{E}/\mathscr{A}$ is torsion-free. We use square brackets $^{[-]}$ as an abbreviation for taking the double dual, for example, $\mathscr{E}^{[k]} = (\mathscr{E}^{\otimes k})^{\vee\vee}$ and $f^{[*]}\mathscr{E} = (f^*\mathscr{E})^{\vee\vee}$ for a map $f \colon Y \to X$ with Y normal.

Let $D \subset X$ be a reduced divisor. Then we denote by

$$\mathscr{E}(*D) \coloneqq \varinjlim (\mathscr{E} \otimes \mathscr{O}_X(mD))^{\vee\vee}$$

the quasi-coherent sheaf of sections of $\mathscr{E}$ with arbitrarily high-order poles along D. If $i \colon U \hookrightarrow X$ is the inclusion of the snc locus of (X, D), the sheaf of *reflexive differential q-forms* is defined to be $\Omega^{[q]}_{X/k}(\log D) \coloneqq i_*\Omega^q_{U/k}(\log D|_U)$. The base field k will usually be dropped from notation.

Following are some useful properties of reflexive sheaves which will be used implicitly or explicitly. For proofs, we refer to [8, Section 3].

LEMMA 2.1. *Let $\mathscr{E}$ be a reflexive sheaf on the normal variety X and $\mathscr{A}, \mathscr{B} \subset \mathscr{E}$ coherent subsheaves, with $\mathscr{A}$ saturated.*

(2.1.1) *The sheaf $\mathscr{A}$ is reflexive.*

(2.1.2) *Let s be a rational section of $\mathscr{A}$ which is regular as a section of $\mathscr{E}$. Then s is also regular as a section of $\mathscr{A}$.*

(2.1.3) *Suppose that for some dense open subset $U \subset X$, the subsheaves $\mathscr{A}|_U$ and $\mathscr{B}|_U$ of $\mathscr{E}|_U$ are equal. Then it follows that $\mathscr{B} \subset \mathscr{A}$.*

3. Factorizing resolutions

It is well known that in characteristic zero, the MMP can be used to obtain log crepant partial resolutions for log canonical pairs (called 'minimal dlt models', 'dlt blowups' or 'dlt modifications'); see, for example, [20, Theorem 3.1]. Here we would like to point out that the same argument also works for surfaces over arbitrary fields. The reason is that the MMP for log canonical surfaces is very well developed [24]. In fact, our proof is even simpler than the one in [20] because we do not have to perturb the dlt pair of interest into a linearly equivalent klt pair.

Unlike [20], we are not only interested in the end product of the MMP (in the notation below, the map f), but also in the intermediate steps. Note that since we are on a surface, we can use Mumford's pullback to get the same result also for *numerically* log canonical pairs [21, Notation 4.1]. This will be important later.

THEOREM 3.1. *Let (X, D) be a numerically log canonical surface pair and $\pi \colon Y \to X$ a log resolution, with exceptional divisor E. Then π can be factored into a sequence of maps as follows:*

$$Y = Y_0 \xrightarrow{\varphi_0} Y_1 \xrightarrow{\varphi_1} \cdots\cdots\cdots \xrightarrow{\varphi_{r-2}} Y_{r-1} \xrightarrow{\varphi_{r-1}} Y_r = Z$$

with π and f mapping to X.

such that, setting $\widetilde{D}_0 \coloneqq \pi_*^{-1}D + E$ and $\widetilde{D}_{i+1} \coloneqq (\varphi_i)_*\widetilde{D}_i$, the following properties hold:

(3.1.1) *for any $0 \leqslant i \leqslant r$, the pair $(Y_i, \widetilde{D}_i)$ is dlt and Y_i is $\mathbb{Q}$-factorial;*

(3.1.2) *for any $0 \leqslant i \leqslant r - 1$, the exceptional locus of φ_i is irreducible;*

(3.1.3) *the map f is* (numerically) *log crepant, that is, $K_Z + \widetilde{D}_r = f^*(K_X + D)$.*

Proof. Let $F_1, \ldots, F_n$ be all the irreducible components of E, and consider the ramification formula $K_Y + \pi_*^{-1}D = \pi^*(K_X + D) + \sum_{i=1}^n a_i F_i$, where $\pi^*(-)$ denotes Mumford's pullback. We then have

$$K_Y + \widetilde{D}_0 = \pi^*(K_X + D) + \sum_{i=1}^n (a_i + 1)F_i. \tag{3.1.4}$$

We may run the MMP on the dlt pair $(Y, \widetilde{D}_0)$ and obtain a minimal model $\varphi \colon Y \to Z$ over X [**24**, Theorem 1.1]. This provides the maps in the statement to be proven. Also, (3.1.1) and (3.1.2) are clear by construction. It remains to show (3.1.3). To this end, push forward (3.1.4) to Z:

$$K_Z + \widetilde{D}_r = f^*(K_X + D) + \underbrace{\varphi_*\left(\sum_{i=1}^n (a_i + 1)F_i\right)}_{f\text{-nef by construction}}. \tag{3.1.5}$$

The negativity lemma [**21**, Lemma 3.40] implies that the underbraced term in the above formula is zero. Hence (3.1.5) simplifies to (3.1.3). □

4. *Adjunction and the different on dlt surface pairs*

The different is a correction term that makes the adjunction formula work in the presence of singularities. For a general treatment of the different, including the case of positive characteristic; see [**19**, Chapter 4]. On a surface, things are somewhat simpler, as explained in [**19**, Definition 2.34].

PROPOSITION/DEFINITION 4.1 (Different on surfaces). Let X be a normal $\mathbb{Q}$-factorial surface and $B \subset X$ a reduced irreducible curve with normalization $\overline{B} \to B$. Let B' be a $\mathbb{Q}$-divisor that has no common components with B. Then there is a canonically defined $\mathbb{Q}$-divisor $\mathrm{Diff}_{\overline{B}}(B')$ on $\overline{B}$, called the *different*, such that

$$(K_X + B + B')\big|_{\overline{B}} \sim_{\mathbb{Q}} K_{\overline{B}} + \mathrm{Diff}_{\overline{B}}(B').$$

We will mostly be interested in the case where (X, B) is dlt, in which case B is regular by Proposition 4.4. Hence $\overline{B} = B$ and we may write

$$\mathrm{Diff}_B(B') = \sum_{x \in B} \delta_x \cdot [x],$$

where $\delta_x \neq 0$ only for points x that are singular on X or contained in $\mathrm{supp}\, B'$. We need to compute the coefficients δ_x in relation to the singularities of $(X, B + B')$. In positive characteristic this is only possible under the following additional tameness hypothesis:

DEFINITION 4.2 (Tamely and fiercely dlt pairs). A pair (X, D) over a field of characteristic p is called *tamely dlt* if the following hold:

(4.2.1) (X, D) is reduced and dlt,

(4.2.2) $K_X + D$ is $\mathbb{Z}_{(p)}$-Cartier (see Section 2).

If Condition (4.2.1) is satisfied but (4.2.2) is not, the pair is said to be *fiercely dlt*.

In the case $p = 0$, we recover the usual notion of a reduced dlt pair. The main result concerning the different is then as follows. The reader may like to compare this to [**19**, Theorem 3.36], where a similar formula is proven under slightly different assumptions.

THEOREM 4.3 (Computation of the different). *Let (X, D) be a tamely dlt surface pair, and let $P \subset D$ be an irreducible component. Write*

$$\text{Diff}_P(D - P) = \sum_{x \in P} \delta_x \cdot [x]$$

as above. Then, referring to the dichotomy in Proposition 4.4:

(4.3.1) *if locally at x, (4.4.1) holds, then $\delta_x = 1$;*
(4.3.2) *if locally at x, (4.4.2) holds, then $\delta_x = 1 - \frac{1}{m}$, where m is the Cartier index of $K_X + D$ at x.*

4.A. *The local structure of dlt surfaces*

Locally, dlt surface pairs are in some sense quite simple (even if they are fierce).

PROPOSITION 4.4 (Dichotomy for dlt surfaces). *Let (X, D) be a reduced dlt surface pair, and let $x \in \text{supp } D$ be any point. Then either one of the following holds.*

(4.4.1) *The pair (X, D) is snc at x, and x is contained in exactly two components of D.*
(4.4.2) *The divisor D is regular at x and the pair (X, D) is plt at x.*

In particular, every irreducible component of D is regular.

Proof. Assume that we are not in case (4.4.1). Then either (X, D) is snc at x, but D has only one component at x. In this case, (4.4.2) clearly holds. Or the pair (X, D) is not snc at x, in which case it is plt at x by definition. Regularity of D at x then follows from [**19**, 3.35]. □

In the following corollary, the crucial point is the separability of the maps γ_α. Note that the U_α cover only $\text{supp } D$ and not all of X.

COROLLARY 4.5 (Dlt surfaces as quotients). *Let (X, D) be a tamely dlt surface pair. Then there exist finitely many Zariski-open subsets $\{U_\alpha\}_{\alpha \in I}$ of X that cover $\text{supp } D$ and admit maps*

$$\gamma_\alpha \colon V_\alpha \to U_\alpha \quad \text{finite quasi-étale separable cyclic Galois}$$

such that the pairs $(V_\alpha, \gamma_\alpha^* D)$ are snc for all indices $\alpha \in I$.

Proof. Let $x \in \text{supp } D$ be any point, and apply Proposition 4.4. If we are in case (4.4.1), we may take $\gamma_\alpha = \text{id}$ and there is nothing to show. In case (4.4.2), let γ_α be a local index one cover with respect to $K_X + D$. Then γ_α by construction has all the properties claimed, except separability. But separability is also clear because of our assumption that $K_X + D$ is $\mathbb{Z}_{(p)}$-Cartier. It remains to see that $(V_\alpha, \gamma_\alpha^* D)$ is snc. To this end, note that this pair is again plt [**19**, Corollary 2.43]. Furthermore, as $K_{V_\alpha} + \gamma_\alpha^* D$ is Cartier, the discrepancies are actually integral and hence non-negative. The pair $(V_\alpha, \gamma_\alpha^* D)$ is therefore canonical. Let $y \in V_\alpha$ be the unique point in $\gamma_\alpha^{-1}(x)$. Then $y \in \text{supp } \gamma_\alpha^* D$. The claim now follows from [**19**, Theorem 2.29]. □

4.B. *Proof of Theorem* 4.3

Case (4.3.1) is clear, hence we concentrate on Case (4.3.2). We follow the local computational approach as illustrated in [**19**, Example 4.3]. Let $\gamma\colon V \to U$ be a map as in Corollary 4.5, where $x \in U$, and put

$$D_V = \gamma^* D, \quad \text{a regular curve,}$$

$$\gamma_D = \gamma\big|_{D_V},$$

$$\sigma = \text{a local generator for } m\big(K_U + D\big|_U\big),$$

$$\sigma_V = \gamma^* \sigma,$$

$$\omega = \text{a local generator for } K_V + D_V.$$

Then $\sigma_V = \omega^{[m]}$ up to a unit, that is, for a suitable choice of ω. It follows that

$$\operatorname{res}(\sigma_V) = \operatorname{res}(\omega^{[m]}) = \operatorname{res}(\omega)^m = \omega'^m, \tag{4.5.1}$$

where ω' is a local generator for ω_{D_V}. On the other hand, as γ is quasi-étale and the residue map (in the snc case) commutes with étale pullback, we have

$$\operatorname{res}(\sigma_V) = \gamma_D^* \operatorname{res}(\sigma). \tag{4.5.2}$$

Let $t \in \mathscr{O}_{D,x}$ be a local parameter of D at x, and let $u \in \mathscr{O}_{D_V,y}$ be a local parameter of D_V at the unique point y lying over x such that $\omega' = \mathrm{d}u$. Then $\gamma_D^*(t) = \varepsilon u^m$ for some unit $\varepsilon \in \mathscr{O}_{D_V,y}^\times$. Hence, writing $\operatorname{res}(\sigma) = t^k (\mathrm{d}t)^m$ up to a unit, with k to be determined, combining (4.5.1) with (4.5.2) gives

$$\varepsilon^k u^{km} (\varepsilon m u^{m-1} \mathrm{d}u + u^m \mathrm{d}\varepsilon)^m = \left(\varepsilon^{k+m} m^m u^{m(k+m-1)} + \cdots \right) \cdot (\mathrm{d}u)^m$$

$$= (\mathrm{d}u)^m,$$

where the dots stand for terms involving higher powers of u. By the tameness assumption, $m \neq 0$ in the ground field and we obtain $m(k + m - 1) = 0$. So $k = 1 - m$ and $\delta_x = -k/m = 1 - 1/m$, as claimed.

5. *Residues and restriction on dlt surfaces*

In this section, we prove Theorems 1.4 and 1.5.

5.A. *Proof of Theorem* 1.4

The proof is divided into four steps.

Step 1: Symmetric residue maps.   First, we will construct the maps res_P^m. So fix a natural number m and consider the mth symmetric power of the residue map on the snc locus of (X, D). Pushing it forward to all of X yields a map

$$\operatorname{Sym}^{[m]} \Omega_X^{[1]}(\log D) \longrightarrow \mathscr{O}_P(*\lceil P^c \rceil) \tag{5.1.1}$$

to the sheaf of rational functions on P with arbitrarily high-order poles along $\operatorname{supp}\lceil P^c \rceil$. We need to show that (5.1.1) factorizes via $\operatorname{Sym}^{[m]} \Omega_X^{[1]}(\log D) \to \mathscr{O}_P$, for this will be the desired map res_P^m. So let σ be an arbitrary local section of $\operatorname{Sym}^{[m]} \Omega_X^{[1]}(\log D)$, defined on an open set $U \subset X$. Let $\widetilde{\sigma}$ be its image under (5.1.1), and (after possibly shrinking U) pick a map $\gamma\colon V \to U$ as in Corollary 4.5.

We will employ the following regularity criterion: $\widetilde{\sigma} \in \Gamma(U, \mathscr{O}_P)$ if and only if $\gamma^*(\widetilde{\sigma}) \in \Gamma(V, \mathscr{O}_{P_V})$, where $P_V := \gamma^* P$ and $D_V := \gamma^* D$. This criterion holds because P is regular, in particular normal. (Recall that if $A \subset B$ is a finite extension of normal domains and $Q(A)$ is the fraction field of A, then $B \cap Q(A) = A$. In our situation, A is a local ring of P and B is a suitable local ring of P_V.)

By Corollary 4.5, the pair (V, D_V) is snc, hence $\Omega^1_V(\log D_V)$ is locally free and we obtain a residue map

$$\mathrm{res}^m_{P_V} \colon \ \mathrm{Sym}^{[m]} \Omega^{[1]}_V(\log D_V) = \mathrm{Sym}^m \Omega^1_V(\log D_V) \longrightarrow \mathrm{Sym}^m \mathscr{O}_{P_V} = \mathscr{O}_{P_V}.$$

Furthermore, note that $\gamma \colon (V, D_V) \to (U, D|_U)$ is a 'morphism of logarithmic pairs' in the sense of [10, Definition 2.4] (this simply means that $\gamma^{-1}(D|_U) = D_V$ set-theoretically). Therefore, by [10, Remark 2.10] we can pullback σ to a regular section of $\mathrm{Sym}^m \Omega^1_V(\log D_V)$, at least off the preimage of the non-snc locus $(U, D|_U)_{\mathrm{sg}}$. But γ is finite (in particular equidimensional), so this preimage still has codimension two in V. Hence $\gamma^*\sigma$ is regular on all of V. In other words,

$$\gamma^*\sigma \in \Gamma\Big(V, \ \mathrm{Sym}^{[m]} \Omega^{[1]}_V(\log D_V)\Big).$$

Recall that the standard residue map commutes with étale pullback, and that γ is étale over the general point of P. So the two functions

$$(\gamma|_{P_V})^*(\widetilde{\sigma}) \in \Gamma\big(P_V, \mathscr{O}_{P_V}(* \operatorname{supp} \gamma|^*_{P_V}\lceil P^c\rceil)\big)$$

and

$$\mathrm{res}^m_{P_V}(\gamma^*\sigma) \in \Gamma(P_V, \mathscr{O}_{P_V})$$

agree on an open subset of P_V, hence everywhere. This shows that $\widetilde{\sigma}$ is a regular function on P_V, as desired.

Step 2: Surjectivity. It remains to show surjectivity of the maps res^m_P. This is a local question, so we may restrict ourselves to an open set $U \subset X$ admitting a map $\gamma \colon V \to U$ as in Corollary 4.5. Let $G = \mathrm{Gal}(\gamma)$ be the Galois group of γ. Start with the map

$$\mathrm{res}^m_{P_V} \colon \ \mathrm{Sym}^m \Omega^1_V(\log D_V) \longrightarrow \mathscr{O}_{P_V}$$

as before and note that we can also construct res^m_P by applying the functor $\gamma_*(-)^G$ to $\mathrm{res}^m_{P_V}$. This means that we consider $U = {}^V\!/_G$ with the trivial G-action and look at the invariant sections of the relevant push-forward sheaves (which are G-sheaves in a natural way). For more details, cf. [11, Appendix A].

The claim now follows from the surjectivity of $\mathrm{res}^m_{P_V}$ (which is due to the fact that the pair (V, D_V) is snc) and the exactness of the functor $\gamma_*(-)^G$. This exactness holds because the order of G is prime to p by the 'tamely dlt' assumption, and therefore we have the usual Reynolds operator argument at our disposal; cf. the characteristic zero version of this argument [11, Lemma A.3].

Step 3: Residue sequence on X. Next we prove the existence of sequence (1.4.1). The map res_P is of course nothing but the special case $m = 1$ of the maps just constructed. By what we already know, we thus only need to show that its kernel is isomorphic to $\Omega^{[1]}_X(\log D - P)$. But that kernel is a reflexive sheaf by [14, Corollary 1.5]. Furthermore, it is isomorphic to $\Omega^{[1]}_X(\log D - P)$ on $(X, D)_{\mathrm{snc}}$, by the usual residue sequence for snc pairs. The isomorphism then extends to all of X by reflexivity.

Step 4: Residue sequence on P. Finally we turn to sequence (1.4.2). Clearly, the reflexive restriction of res_P to P is a surjective map $\mathrm{res}^P_P \colon \Omega^{[1]}_X(\log D)|^{\vee\vee}_P \longrightarrow \mathscr{O}_P$, and it remains to show

$$0 \qquad\qquad\qquad 0$$

$$\downarrow \qquad\qquad\qquad\quad \downarrow$$

$$0 \longrightarrow \Omega_X^{[1]}(\log D)(-P)^{\vee\vee} \longrightarrow \Omega_X^{[1]}(\log D - P) \xrightarrow{\mathrm{restr}_P} \Omega_P^1(\log\lfloor P^c\rfloor) \longrightarrow 0$$

$$\big\| \qquad\qquad\qquad\quad \downarrow \qquad\qquad\qquad\quad \downarrow$$

$$0 \longrightarrow \Omega_X^{[1]}(\log D)(-P)^{\vee\vee} \longrightarrow \Omega_X^{[1]}(\log D) \longrightarrow \Omega_X^{[1]}(\log D)\big|_P^{\vee\vee} \longrightarrow 0$$

$$\qquad\qquad\qquad \downarrow {\scriptstyle \mathrm{res}_P} \qquad\qquad\qquad \downarrow {\scriptstyle \mathrm{res}_P^P}$$

$$\mathscr{O}_P =\!=\!=\!=\!=\!=\!= \mathscr{O}_P$$

$$\downarrow \qquad\qquad\qquad\quad \downarrow$$

$$0 \qquad\qquad\qquad 0$$
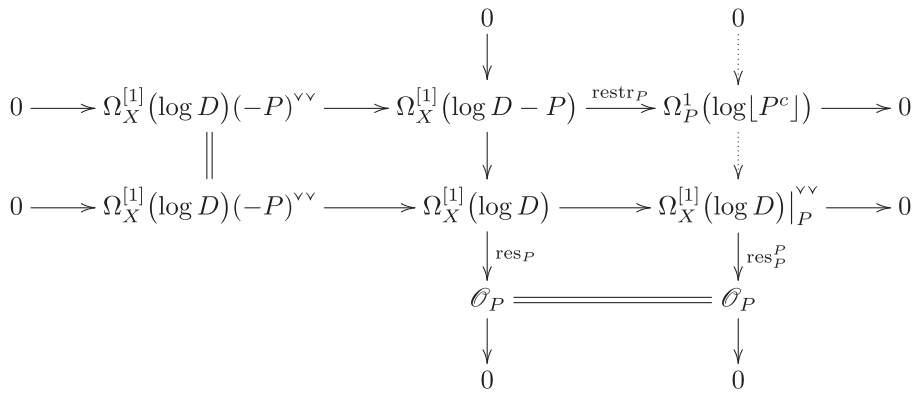
FIGURE 1. *Diagram used in the proof of* (1.4.2).

that its kernel is isomorphic to $\Omega_P^1(\log\lfloor P^c\rfloor)$. To this end, first note that there is a short exact sequence

$$0 \longrightarrow \Omega_X^{[1]}(\log D)(-P)^{\vee\vee} \longrightarrow \Omega_X^{[1]}(\log D) \longrightarrow \Omega_X^{[1]}(\log D)\big|_P^{\vee\vee} \longrightarrow 0. \qquad (5.1.2)$$

In fact, the second map is surjective because on a regular curve, taking the double dual really just amounts to dividing out the torsion. And by the same argument as in the previous step, the kernel is reflexive and thus isomorphic to $\Omega_X^{[1]}(\log D)(-P)^{\vee\vee}$.

Consider now the commutative diagram with exact rows and columns depicted in Figure 1 on the facing page. The first row is the restriction sequence $(1.5.1)^\dagger$, while the second row is (5.1.2). The middle column is (1.4.1), the residue sequence on X. The snake lemma then shows that the dotted arrow $\Omega_P^1(\log\lfloor P^c\rfloor) \dashrightarrow \Omega_X^{[1]}(\log D)|_P^{\vee\vee}$ exists, is injective and that its image is exactly the kernel of res_P^P. The column on the right-hand side is therefore likewise exact, and it is precisely sequence (1.4.2).

5.B. *Proof of Theorem* 1.5

The proof of Theorem 1.5 is analogous to the proof of Theorem 1.4, hence we will only provide an outline, with most details omitted. To begin with, if (X, D) is snc then sequence (1.5.1) reads

$$0 \longrightarrow \Omega_X^1(\log D)(-P) \longrightarrow \Omega_X^1(\log D - P) \xrightarrow{\mathrm{restr}_P} \Omega_P^1(\log P^c) \longrightarrow 0$$

and this exists by [5, 2.3(c)]. In particular, we already have restr_P and its mth symmetric power on the snc locus $(X, D)_{\mathrm{snc}}$. Pushing forward this symmetric power to all of X, we obtain a map

$$\mathrm{Sym}^{[m]}\,\Omega_X^{[1]}(\log D - P) \longrightarrow \mathscr{O}_P(mK_P)(*\lceil P^c\rceil), \qquad (5.1.3)$$

and we have to show that it factors via a map

$$\mathrm{Sym}^{[m]}\,\Omega_X^{[1]}(\log D - P) \longrightarrow \mathscr{O}_P(mK_P + \lfloor mP^c\rfloor),$$

for this will be the desired map restr_P^m. To this end, we have the following criterion:

CLAIM 5.2. *Notation as in the previous proof. A local section $\widetilde{\sigma}$ of $\mathscr{O}_P(mK_P)(*\lceil P^c\rceil)$ is contained in $\mathscr{O}_P(mK_P + \lfloor mP^c\rfloor)$ if and only if $\gamma^*(\widetilde{\sigma})$ is a regular section of $\mathscr{O}_{P_V}(mK_{P_V} + mP_V^c)$, where $P_V^c := \mathrm{Diff}_{P_V}(D_V - P_V) = (D_V - P_V)|_{P_V}$.*

---

†Needless to say, the proof of Theorem 1.5 does not rely on Theorem 1.4 — see Section 5.B.

Proof of Claim 5.2. The proof of this criterion is done by a local computation similar to the one in the proof of Theorem 4.3, whose notation we adopt. If locally at $x \in \operatorname{supp}\lceil P^c \rceil$, Case (4.4.1) holds, the claim is clear. Therefore we focus on Case (4.4.2). Note that in this case D_V is smooth, which implies $P_V = D_V$ and thus $P_V^c = 0$.

Write $\widetilde{\sigma} = t^k \, (\mathrm{d}t)^m$ with $k \in \mathbb{Z}$ (locally and up to units), and let ℓ be the Cartier index of $K_X + D$ at x. The coefficient of $\lfloor mP^c \rfloor$ at x is $\lfloor m(1 - \frac{1}{\ell}) \rfloor$, therefore $\widetilde{\sigma}$ is contained in $\mathscr{O}_P(mK_P + \lfloor mP^c \rfloor)$ if and only if

$$k \geqslant -\lfloor m(1 - \tfrac{1}{\ell}) \rfloor. \tag{5.2.1}$$

On the other hand, $\gamma^*(t) = \varepsilon u^\ell$ and hence

$$\gamma^*(\widetilde{\sigma}) = \left(\varepsilon^{k+m} \ell^m u^{k\ell + m(\ell-1)} + \cdots \right) \cdot (\mathrm{d}u)^m,$$

where the dots stand for terms involving higher powers of u. We see that $\gamma^*(\widetilde{\sigma})$ is regular if and only if $k\ell + m(\ell - 1) \geqslant 0$. This is equivalent to (5.2.1), proving the claim. $\square$

Once the maps restr_P^m are constructed, their surjectivity follows from the right-exactness of $\gamma_*(-)^G$, as before. Finally, to obtain sequence (1.5.1) we set $\operatorname{restr}_P := \operatorname{restr}_P^1$. On the snc locus $(X, D)_{\mathrm{snc}}$, the kernel agrees with $\Omega_X^{[1]}(\log D)(-P)^{\vee\vee}$ by the snc case mentioned in the beginning of the proof. Since both sheaves are reflexive, they agree everywhere.

6. Lifting forms along a non-positive map

The following theorem, while technical in nature, is at the heart of the paper. The 'non-positivity' in the title refers to property (6.1.2).

THEOREM 6.1 (Lifting forms). *Let $g \colon Y \to X$ be a proper birational map of normal surfaces over a field k, with $E = \operatorname{Exc}(g)$ the reduced exceptional divisor. Furthermore, let D be a reduced divisor on X, and set $D_Y := g_*^{-1} D + E$. Assume the following:*

(6.1.1) *the pair (Y, D_Y) is tamely dlt, and*
(6.1.2) *the anticanonical divisor $-(K_Y + D_Y)$ is g-nef.*

Then the natural map

$$g_* \Omega_{Y/k}^{[1]}(\log D_Y) \hookrightarrow \Omega_{X/k}^{[1]}(\log D)$$

is an isomorphism.

Step 0: Setup of notation and outline of proof strategy

Let

$$\sigma \in \mathrm{H}^0\!\left(X, \Omega_X^{[1]}(\log D) \right) \setminus \{0\}$$

be a non-zero reflexive logarithmic 1-form, and let $g^*\sigma$ be its pullback to Y, considered as a rational section of the sheaf $\Omega_Y^{[1]}(\log D_Y)$. We want to show that $g^*\sigma$ is in fact a regular section of that sheaf. To this end, first pick an effective g-exceptional divisor G such that

$$g^*\sigma \in \mathrm{H}^0\!\left(Y, \Omega_Y^{[1]}(\log D_Y)(G)^{\vee\vee} \right). \tag{6.1.3}$$

For example, G may be taken to be the pole divisor of the rational section $g^*\sigma$. We will show that whenever G is non-zero, there is a curve $P \subset \operatorname{supp} G$ such that (6.1.3) continues to hold with G replaced by $G - P$. Iterating this argument finitely often, we arrive at $G = 0$, hence $g^*\sigma \in \mathrm{H}^0(Y, \Omega_Y^{[1]}(\log D_Y))$ as desired.

Step 1: Residue sequence

Assume that (6.1.3) holds for some $G \neq 0$. Then $G^2 < 0$ by the negativity lemma (applied on some resolution of Y) and consequently, $G \cdot P < 0$ for some exceptional curve $P \subset \operatorname{supp} G \subset E$. Twisting by $\mathscr{O}_Y(-G)$ and taking the reflexive hull, (6.1.3) induces a map $i \colon \mathscr{O}_Y(-G) \to \Omega_Y^{[1]}(\log D_Y)$. As $g^*\sigma \neq 0$, this map is non-zero and hence injective. On the tamely dlt pair (Y, D_Y), we have the residue sequence (1.4.1)

$$
\begin{array}{ccccccccc}
& & & & \mathscr{O}_Y(-G) & & & & \\
& & & {\scriptstyle j} \nearrow\!\!\!- & \Big\downarrow {\scriptstyle i} & \searrow & & & \\
0 & \longrightarrow & \Omega_Y^{[1]}\big(\log D_Y - P\big) & \longrightarrow & \Omega_Y^{[1]}\big(\log D_Y\big) & \xrightarrow{\ \mathrm{res}_P\ } & \mathscr{O}_P & \longrightarrow & 0.
\end{array}
$$

CLAIM 6.2. *The composition $\mathrm{res}_P \circ i$ is zero, and hence i factors via a map j as indicated by the dashed arrow in the above diagram.*

Proof of Claim 6.2. Let $m \geqslant 1$ be sufficiently divisible so that mG is Cartier (recall that Y is $\mathbb{Q}$-factorial). The mth reflexive symmetric power of i, composed with the map res_P^m from Theorem 1.4, yields a map

$$
\mathscr{O}_Y(-mG) \xrightarrow{\ \mathrm{Sym}^{[m]} i\ } \mathrm{Sym}^{[m]} \Omega_Y^{[1]}(\log D_Y) \xrightarrow{\ \mathrm{res}_P^m\ } \mathscr{O}_P \tag{6.2.1}
$$

which is nothing but the mth reflexive symmetric power of $\mathrm{res}_P \circ i$. Hence in order to show that $\mathrm{res}_P \circ i$ vanishes, it is sufficient to prove the vanishing of (6.2.1). As the target of the latter map is supported on P, it is zero if and only if its restriction to P is zero. But that restriction is a map $\mathscr{O}_P(-mG) \to \mathscr{O}_P$, or in other words, an element of $\mathrm{H}^0(P, \mathscr{O}_P(mG))$. As $G \cdot P < 0$ and mG is Cartier, the latter space is zero. $\qquad\square$

Step 2: Restriction sequence

We essentially repeat Step 1, but with the residue sequence replaced by the restriction sequence (1.5.1):

$$
\begin{array}{ccccccccc}
& & & & \mathscr{O}_Y(-G) & & & & \\
& & & {\scriptstyle \iota} \nearrow\!\!\!- & \Big\downarrow {\scriptstyle j} & \searrow & & & \\
0 & \longrightarrow & \Omega_Y^{[1]}\big(\log D_Y\big)(-P)^{\vee\vee} & \longrightarrow & \Omega_Y^{[1]}\big(\log D_Y - P\big) & \xrightarrow{\ \mathrm{restr}_P\ } & \Omega_P^1\big(\log \lfloor P^c \rfloor\big) & \longrightarrow & 0.
\end{array}
$$

CLAIM 6.3. *The composition $\mathrm{restr}_P \circ j$ is zero, and hence j factors via a map ι as indicated by the dashed arrow in the above diagram.*

Proof of Claim 6.3. Let m be as in the proof of Claim 6.2, so that mG is Cartier. The mth reflexive symmetric power of j, composed with the map restr_P^m from Theorem 1.5, is the mth reflexive symmetric power of $\mathrm{restr}_P \circ j$:

$$
\mathscr{O}_Y(-mG) \xrightarrow{\ \mathrm{Sym}^{[m]} j\ } \mathrm{Sym}^{[m]} \Omega_Y^{[1]}(\log D_Y - P) \xrightarrow{\ \mathrm{restr}_P^m\ } \mathscr{O}_P(mK_P + \lfloor mP^c \rfloor). \tag{6.3.1}
$$

As in Claim 6.2, it suffices to show that the restriction of (6.3.1) to P vanishes. This is a map $\mathscr{O}_P(-mG) \to \mathscr{O}_P(mK_P + \lfloor mP^c \rfloor)$, or in other words, an element of $\mathrm{H}^0(P, \mathscr{O}_P(mK_P + \lfloor mP^c \rfloor + mG))$. As

$$
\deg\big(mK_P + \lfloor mP^c \rfloor + mG\big|_P\big) \leqslant \deg\big(m(K_P + P^c) + mG\big|_P\big) \qquad \text{round-down}
$$

$$
\leqslant \deg\big(m(K_Y + D_Y + G)\big|_P\big) \qquad \text{by adjunction}
$$

$$\leqslant \deg\left(mG\big|_P\right) \qquad\qquad \text{by (6.1.2)}$$

$$= mG \cdot P \qquad\qquad\qquad mG \text{ is Cartier}$$

$$< 0, \qquad\qquad\qquad\qquad\quad \tilde{}$$

the latter space is zero. This ends the argument. □

The proof of Theorem 6.1 is now easily finished: the existence of the map ι is equivalent to giving a global section of the sheaf $\Omega_Y^{[1]}(\log D_Y)(G - P)^{\vee\vee}$, which of course is exactly the form $g^*\sigma$ we started with. This shows that (6.1.3) holds with $G - P$ in place of G, as desired.

7. Proof of Theorem 1.2

The aim of this section is to prove our first main result: any log canonical surface pair (X, D) over a perfect field of characteristic at least seven satisfies the logarithmic extension theorem. The following notion will play a key role.

DEFINITION 7.1 (Tame resolutions). Let (X, D) be a reduced log canonical surface pair over a field k. A *tame resolution* of (X, D) is a log resolution $\pi\colon Y \to X$ together with a factorization of π as in Theorem 3.1 such that

(7.1.1) for any $0 \leqslant i \leqslant r - 1$, the pair $(Y_i, \widetilde{D}_i)$ is tamely dlt, and
(7.1.2) if f is not an isomorphism (this can happen only if (X, D) is not plt), then also $(Z, \widetilde{D}_r)$ is required to be tamely dlt.

7.A. Auxiliary results

First we show that when dealing with log canonical surface pairs, there is no loss of generality in assuming them to be reduced. We also prove that having a tame resolution implies the logarithmic extension theorem and that the logarithmic extension theorem is invariant under separable base change. The latter property is used for reducing to the case of an algebraically closed ground field, where the classification of surface singularities becomes simpler.

PROPOSITION 7.2 (Rounding down). *Let (X, D) be a log canonical surface pair. Then also $(X, \lfloor D \rfloor)$ is log canonical.*

PROPOSITION 7.3 (Tameness is sufficient). *Let (X, D) be a reduced log canonical surface pair admitting a tame resolution. Then (X, D) satisfies the logarithmic extension theorem for 1-forms.*

PROPOSITION 7.4 (Base change). *Let (X, D) be a pair defined over a field k, and consider a separable field extension k'/k. Set $X' := X \times_k k'$ and $D' := D \times_k k'$.*

(7.4.1) If (X', D') satisfies the regular extension theorem for q-forms, for some value of q, then so does (X, D).
(7.4.2) If (X, D) admits a log resolution, the converse of (7.4.1) also holds.

Ditto for the logarithmic extension theorem.

Proof of Proposition 7.2. If (X, D) is numerically log canonical, then so is $(X, \lfloor D \rfloor)$. Thus it suffices to show that $K_X + \lfloor D \rfloor$ is $\mathbb{Q}$-Cartier. The question is local, so we may concentrate attention on a point $x \in \operatorname{supp}\{D\}$, the fractional part of D. At such a point, the pair $(X, \lfloor D \rfloor)$ is

even numerically dlt and then $(X, \emptyset)$ is numerically klt. Applying Theorem 3.1 to the latter pair, we get that $f \colon Z \to X$ is an isomorphism, as there are no exceptional divisors of discrepancy -1, and hence X is even $\mathbb{Q}$-factorial because Z is.

An alternative (yet closely related) argument goes by noting that the characteristic zero proof of [21, Proposition 4.11] still works if we replace the use of the basepoint-free theorem [21, Theorem 3.3] by [24, Theorem 4.2]. □

Proof of Proposition 7.3. Let $\pi \colon Y \to X$ be a tame resolution of (X, D), where we keep notation from Theorem 3.1. It suffices to extend 1-forms along each step of the given factorization separately. That is, we will prove the following two statements:

CLAIM 7.5. *The sheaf $f_* \Omega_Z^{[1]}(\log \widetilde{D}_r)$ is reflexive.*

CLAIM 7.6. *For any $0 \leqslant i \leqslant r - 1$, the sheaf $(\varphi_i)_* \Omega_{Y_i}^{[1]}(\log \widetilde{D}_i)$ is reflexive.*

Proof of Claim 7.5. If (X, D) is plt, then f is an isomorphism and there is nothing to prove. Otherwise, we would like to apply Theorem 6.1. The tameness condition (6.1.1) is satisfied by (7.1.2). It remains to check (6.1.2), that is, $-(K_Z + \widetilde{D}_r)$ is f-nef. To this end, let $P \subset Z$ be any f-exceptional curve and note that

$$\left(K_Z + \widetilde{D}_r \right) \cdot P = f^*(K_X + D) \cdot P \qquad \text{by (3.1.3)}$$

$$= 0 \qquad \text{as } f_* P = 0.$$

So $K_Z + \widetilde{D}_r$ is even f-numerically trivial. Claim 7.5 is proved. □

Proof of Claim 7.6. Again, we will apply Theorem 6.1 and only Condition (6.1.2), the φ_i-nefness of $-(K_{Y_i} + \widetilde{D}_i)$, needs to be checked. Let $P \subset Y_i$ be the unique φ_i-exceptional curve. Since $(Y_{i+1}, \widetilde{D}_{i+1})$ is dlt (in particular, log canonical) and $\widetilde{D}_i = (\varphi_i^{-1})_*(\widetilde{D}_{i+1}) + P$, we have

$$K_{Y_i} + \widetilde{D}_i = \varphi_i^*\left(K_{Y_{i+1}} + \widetilde{D}_{i+1} \right) + \lambda P, \qquad (7.6.1)$$

where $\lambda = a(P, Y_{i+1}, \widetilde{D}_{i+1}) + 1 \geqslant 0$. On the other hand, $P^2 < 0$ by the negativity lemma. Hence

$$(K_{Y_i} + \widetilde{D}_i) \cdot P = \left(\varphi_i^*\left(K_{Y_{i+1}} + \widetilde{D}_{i+1} \right) + \lambda P \right) \cdot P \qquad \text{by (7.6.1)}$$

$$= \lambda \cdot P^2 \qquad \text{as } (\varphi_i)_* P = 0$$

$$\leqslant 0. \qquad \sim$$

Claim 7.6 now follows from Theorem 6.1. □

By Claims 7.5 and 7.6, also the sheaf

$$\pi_* \Omega_Y^1\left(\log \pi_*^{-1} D + E\right) = (f \circ \varphi_{r-1} \circ \cdots \circ \varphi_0)_* \Omega_{Y_0}^1\left(\log \widetilde{D}_0\right)$$

$$= (f \circ \varphi_{r-1} \circ \cdots \circ \varphi_1)_* \Omega_{Y_1}^{[1]}\left(\log \widetilde{D}_1\right)$$

$$= \cdots$$
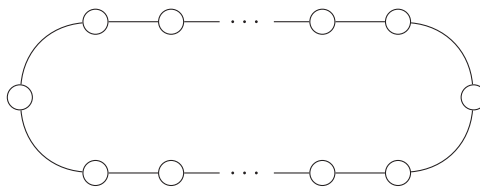
FIGURE 2. *Cusp* [**19**, (3.39.2)].

$$= f_* \Omega_{Y_r}^{[1]} \left(\log \widetilde{D}_r \right)$$

$$= \Omega_X^{[1]} (\log D)$$

is reflexive. The proof of Proposition 7.3 is thus finished. □

Proof of Proposition 7.4. For any object (variety, map, sheaf, etc.) over k, we denote the base change to k' by $(-)'$. Concerning (7.4.1), let $\pi \colon Y \to X$ be proper birational, with Y normal. Then there is a commutative diagram

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\pi' \downarrow & & \downarrow \pi \\
X' & \longrightarrow & X
\end{array}
$$

Since k'/k is a separable field extension, the horizontal maps are étale and faithfully flat. In particular, X' and Y' are still normal, and after possibly replacing them by suitable connected components, π' is proper birational. By assumption, $\pi'_* \Omega_{Y'/k'}^{[q]}$ is reflexive. But

$$
\begin{aligned}
\pi'_* \Omega_{Y'/k'}^{[q]} &= \pi'_* \left(\left(\Omega_{Y'/k'}^q \right)^{\vee\vee} \right) && \text{by definition} \\
&= \pi'_* \left[\left(\left(\Omega_{Y/k}^q \right)' \right)^{\vee\vee} \right] && \text{[13, Chapter II, Proposition 8.10]} \\
&= \pi'_* \left(\left(\Omega_{Y/k}^{[q]} \right)' \right) \\
&= \left(\pi_* \Omega_{Y/k}^{[q]} \right)' && \text{[13, Chapter III, Proposition 9.3],}
\end{aligned}
$$

hence the claim follows.

For (7.4.2), keep notation but assume additionally that π is a resolution of singularities. By the above argument, the sheaf $\pi'_* \Omega_{Y'/k'}^{[q]}$ is reflexive. Because $Y' \to Y$ is étale, π' is in fact a resolution and it follows that X' satisfies the regular extension theorem for q-forms.

The proof in the logarithmic case is similar, and therefore omitted. □

7.B. *Proof of Theorem* 1.2

By Proposition 7.2, we may assume that (X, D) is reduced. Furthermore, since our ground field k is assumed to be perfect, its algebraic closure $\bar{k}$ is separable over k and hence by Proposition 7.4, we may assume that $k = \bar{k}$. The singularities of reduced log canonical surface pairs over an algebraically closed ground field have been classified in [**19**, Corollaries 3.31, 3.39, 3.40]. According to this classification, there are seven cases to be considered. Their dual graphs are depicted in Figures 2–7 (the first case is not shown since it has only one exceptional curve).
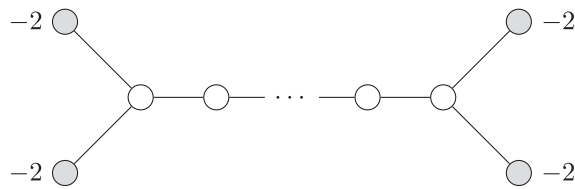
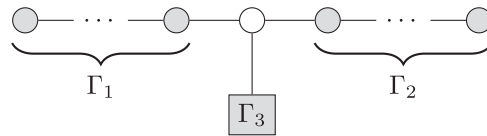FIGURE 3. $\mathbb{Z}/2$-quotient of cusp or simple elliptic [**19**, (3.39.3)].

FIGURE 4. *Other quotient of simple elliptic* [**19**, (3.39.4)]. Γ_3 *is likewise a chain and* $(\det \Gamma_i)$ *is either* $(3,3,3)$, $(2,4,4)$ *or* $(2,3,6)$.
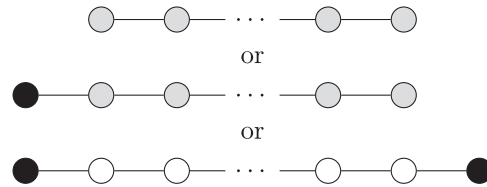

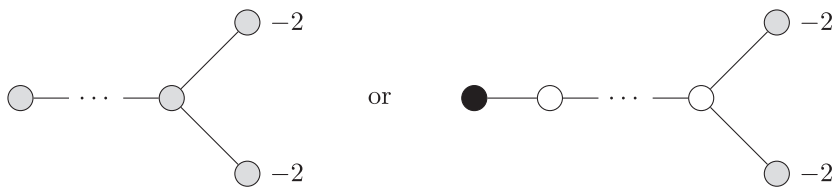FIGURE 5. *The three possibilities for a cyclic quotient* [**19**, (3.40.1)].

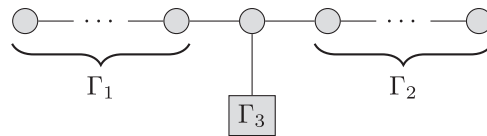FIGURE 6. *The two possibilities for a dihedral quotient* [**19**, (3.40.2)].

FIGURE 7. *Other quotient of a smooth surface* [**19**, (3.40.3)]. $(\det \Gamma_i)$ *is either* $(2,3,3)$, $(2,3,4)$ *or* $(2,3,5)$.

Here we use the following color and labeling pattern. The extra information thus contained in the figures is easily verified.

NOTATION 7.7. A *plain circle* denotes an exceptional curve with discrepancy equal to -1. A node shaded in *gray* denotes an exceptional curve with discrepancy > -1. All exceptional curves are smooth rational. The components of $\pi_*^{-1}D$ are shown in *black*. A *negative number* attached to a vertex denotes the self-intersection of the corresponding curve. A *leaf* is a curve intersecting at most one other curve, while a *fork* intersects at least three other curves.

Since Theorem 1.2 is local, we may shrink X and assume that (X, D) has only one singular point. We use notation from Theorem 3.1, applied to the minimal resolution $\pi\colon Y \to X$ of (X, D). In particular, E is the exceptional locus of π and r is the number of contractions performed by the MMP before the minimal dlt model is reached. The classification is then as follows. (The names are actually valid only in characteristic zero. Here they are only meant for easier reference and should not be taken literally.)

(7.8.1) (Simple elliptic, [**19**, (3.39.1)]) Here $D = 0$ and E consists of a single smooth elliptic curve, which has discrepancy -1. So $r = 0$ and the tameness condition on π is automatically satisfied. In this case, Theorem 1.2 thus follows directly from Proposition 7.3.

(7.8.2) (Cusp, Figure 2) Again, there are no curves of discrepancy > -1, so $r = 0$ and we conclude as before.

(7.8.3) ($\mathbb{Z}/2$-quotient of cusp or simple elliptic, Figure 3) Here $r = 4$ and each step φ_i contracts a curve $C_i \subset U_i \subset Y_i$, where U_i is a smooth open subset of Y_i and $C_i^2 = -2$. By [**19**, Theorem 3.32], the resulting singularity is étale locally[†] isomorphic to the vertex of $\operatorname{Spec} k[u^2, uv, v^2]$. Since $\operatorname{char} k \geqslant 7 > 2$, this singularity is actually a $\mathbb{Z}/2$-quotient of a smooth surface. Then by the usual norm argument, $2 \cdot \Delta_i$ is Cartier for every integral Weil divisor Δ_i on Y_i, where $0 \leqslant i \leqslant 4$. Applying this to $K_{Y_i} + \widetilde{D}_i$, we see that π is a tame resolution and we conclude by Proposition 7.3.

(7.8.4) (Other quotient of simple elliptic, Figure 4) The three chains of rational curves Γ_i can obviously be treated independently of each other, hence we will concentrate on, say, Γ_1. The first curve contracted has to be the leaf, since otherwise there would be two components of $\widetilde{D}_1$ intersecting at a singular point of Y_1, contradicting Proposition 4.4. Repeating this argument, we see that the curves in Γ_1 are contracted in sequence, starting from the leaf and proceeding toward the fork. In particular, at each step there is only one singular point and it is obtained by contracting a subchain of Γ_1. But $\det \Gamma_1 \leqslant 6$ and by the recurrence relation in [**19**, (3.33.1)], the same also holds for all its subchains. By [**19**, Theorem 3.32] and the assumption $\operatorname{char} k \geqslant 7$, the singular point of each Y_i is a quotient by a finite group of order at most 6. As in the previous case (7.8.3), this implies that π is tame and hence Theorem 1.2 holds also in this case.

(7.8.5) (Cyclic quotient, Figure 5) There are three subcases, according to whether the boundary D has zero, one or two components. In the first two cases, it actually not true that (X, D) has a tame resolution, since the chain E can be arbitrarily long and hence infinitely many (in fact, all) primes would have to be excluded. So we cannot apply Proposition 7.3. But note that for every exceptional curve $P \subset Y = Y_0$, we have $\deg(K_P + P^c) \leqslant 0$, where $P^c := \operatorname{Diff}_P(\widetilde{D}_0 - P)$. (The degree is -1 for the leaves and 0 for the other curves, since there is no fork.) Also the pair $(Y_0, \widetilde{D}_0)$ is clearly tamely dlt, since it is even snc. Hence in these cases, Theorem 1.2 is a direct consequence of Theorem 6.1 applied to π. In the third subcase, we may follow the same argument or else note that $r = 0$, so π is tame — it boils down to the same thing.

(7.8.6) (Dihedral quotient, Figure 6) We have two subcases: either $D = 0$ or $D \neq 0$. If $D = 0$, again there may not be a tame resolution. Instead, the MMP needs to be chosen in such a way that first the two (-2)-curves intersecting the fork are contracted. The resulting pairs $(Y_i, \widetilde{D}_i)$, $i = 1, 2$, are tamely dlt by the same reasoning as in

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[†]As stated, [**19**, Theorem 3.32] gives the result only up to completion (which would also be sufficient), but the proof shows that there is a map $\varphi_i(U_i) \to \operatorname{Spec} k[u^2, uv, v^2]$ which is étale at the point $\varphi_i(C_i)$.

case (7.8.3). If $P \subset Y_2$ is the image of the fork, both singular points of Y_2 appear in $P^c := \mathrm{Diff}_P(\widetilde{D}_2 - P)$ with coefficient either[†] zero or $\frac{1}{2}$, by Theorem 4.3. Hence

$$\deg(K_P + P^c) \leqslant -2 + 1 + \frac{1}{2} + \frac{1}{2} = 0.$$

If $P \subset Y_2$ is any other exceptional curve, the above inequality also holds, as in case (7.8.5). We can therefore apply Theorem 6.1 to the map $Y_2 \to X$ to conclude. If $D \neq 0$, then $r = 2$ and only the two (-2)-curves are contracted. The resolution π is then tame by exactly the same argument as in case (7.8.3).

(7.8.7) (Other quotient of a smooth surface, Figure 7) The argument is similar to case (7.8.4). First the chains Γ_i are contracted, starting from the leaves and progressing toward the fork. As $\det \Gamma_i \leqslant 5 < 7 \leqslant \mathrm{char}\, k$, this implies that $(Y_i, \widetilde{D}_i)$ is tamely dlt for $0 \leqslant i \leqslant r - 1$. Furthermore, X is log terminal and $D = 0$, so (X, D) is plt and case (7.1.2) of the definition of tameness applies. So π is tame and Proposition 7.3 gives the result.

Since we have now worked our way through all the cases, the proof of Theorem 1.2 is finished.

8. Proof of Theorem 1.3

This section contains the proof of our second main result, Theorem 1.3. The argument proceeds in three steps.

8.A. Passing to a log resolution

First of all, by blowing up Y further we may turn $\mathrm{Exc}(\pi) + D_Y$ into an snc divisor. We need to show that this does not change $\det(E_i \cdot E_j)$ up to sign. Indeed, after renumbering we may assume that we are blowing up a point $p \in Y$ which is contained exactly in the exceptional curves $E_1, \dots, E_k$, where $k \leqslant \ell$. Let r_i be the multiplicity of E_i at p. Set $A = (a_{ij}) = (-E_i \cdot E_j)$, the negative of the intersection matrix on Y and $\widetilde{A}$ the analogous matrix after blowing up p, with the new exceptional curve put first. Also, let $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$ with one additional zero row and column. Then

$$\widetilde{A} = A_0 + \begin{pmatrix} 1 & -r_1 & -r_2 & \cdots & -r_k & 0 \\ -r_1 & r_1^2 & r_1 r_2 & \cdots & r_1 r_k & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -r_k & r_1 r_k & r_2 r_k & \cdots & r_k^2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

where there are $\ell - k$ zero rows and columns, respectively. By adding $r_i \cdot$ (first column) to the $(i+1)$st column for all $1 \leqslant i \leqslant k$, the matrix $\widetilde{A}$ is transformed into

$$\widetilde{A}' = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -r_1 & a_{11} & \cdots & a_{1\ell} \\ \vdots & \vdots & & \vdots \\ -r_k & \vdots & & \vdots \\ 0 & a_{\ell 1} & \ldots & a_{\ell\ell} \end{pmatrix},$$

[†]As $K_{Y_2} + \widetilde{D}_2$ is not Cartier, the coefficient is actually $1/2$, but we only need an upper bound on the different.

while keeping the determinant unchanged. Expanding $\det \widetilde{A}'$ by the first row, we see that $\det \widetilde{A}' = \det A$. Hence we may make the following

ADDITIONAL ASSUMPTION 8.1. *The map $\pi\colon Y \to X$ is a log resolution of (X, D).*

8.B. *Dropping the non-exceptional divisor*

Pick an irreducible component $P \subset \lfloor D \rfloor$, and let P_Y be its strict transform on Y. Then consider the short exact sequence given by the residue map [**5**, 2.3(b)]

$$0 \longrightarrow \Omega^1_Y(\log \lfloor D_Y \rfloor + E - P_Y) \longrightarrow \Omega^1_Y(\log \lfloor D_Y \rfloor + E) \longrightarrow \mathscr{O}_{P_Y} \longrightarrow 0.$$

Pushing it forward via π yields

$$0 \longrightarrow \pi_* \Omega^1_Y(\log \lfloor D_Y \rfloor + E - P_Y) \longrightarrow \pi_* \Omega^1_Y(\log \lfloor D_Y \rfloor + E) \longrightarrow \mathscr{Q} \longrightarrow 0,$$

where $\mathscr{Q} \subset \pi_* \mathscr{O}_{P_Y}$. In particular, $\mathscr{Q}$ is supported on P and it is torsion-free as an $\mathscr{O}_P$-module. Hence $\mathscr{Q}$ has only one associated prime, which is of height 1. It then follows from [**14**, Corollary 1.5] that the sheaf $\pi_* \Omega^1_Y(\log \lfloor D_Y \rfloor + E - P_Y)$ is likewise reflexive. Repeating this argument for all components $P \subset \lfloor D \rfloor$, we arrive at the conclusion that

$$\pi_* \Omega^1_Y(\log E)$$

is reflexive, and hence isomorphic to $\Omega^{[1]}_X$. In other words, $(X, \emptyset)$ satisfies the logarithmic extension theorem.

8.C. *Dropping the exceptional divisor*

Set $E = E_1 + \cdots + E_\ell$, and consider the residue sequence [**5**, 2.3(a)]

$$0 \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_Y(\log E) \longrightarrow \bigoplus_{i=1}^{\ell} \mathscr{O}_{E_i} \longrightarrow 0.$$

We need to show that

$$\mathrm{H}^0(Y, \Omega^1_Y) \longrightarrow \mathrm{H}^0(Y, \Omega^1_Y(\log E))$$

is an isomorphism. It suffices to show that the connecting homomorphism

$$\delta\colon \bigoplus_{i=1}^{\ell} \mathrm{H}^0(E_i, \mathscr{O}_{E_i}) \longrightarrow \mathrm{H}^1(Y, \Omega^1_Y)$$

is injective. To this end, consider the restriction map

$$r\colon \mathrm{H}^1(Y, \Omega^1_Y) \longrightarrow \bigoplus_{i=1}^{\ell} \mathrm{H}^1(E_i, \Omega^1_{E_i}).$$

We will show that the composition

$$r \circ \delta\colon \bigoplus_{i=1}^{\ell} \mathrm{H}^0(E_i, \mathscr{O}_{E_i}) \longrightarrow \bigoplus_{i=1}^{\ell} \mathrm{H}^1(E_i, \Omega^1_{E_i}) \tag{8.1.1}$$

is an isomorphism. In fact, on the left-hand side choose the basis consisting of the constant functions $\mathbf{1}_{E_i}$, and on each summand of the right-hand side, choose the basis canonically determined by the trace map. It is easy to see[†] that with respect to these bases, (8.1.1) is

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[†]For more details, the reader is advised to consult the proof of [**9**, Proposition 3.2], which is independent of the characteristic.

given by the intersection matrix $A := (E_i \cdot E_j)$. By the negativity lemma [21, Lemma 3.40], A is negative definite (in particular, invertible) when considered as an integer matrix. Here, of course, we have to regard A as defined over our ground field k instead. However, by our assumption, the characteristic p of k does not divide $\det A$. Hence the matrix A remains invertible when reduced modulo p. In other words, $r \circ \delta$ is an isomorphism, and then δ is injective. It follows that the sheaf $\pi_* \Omega_Y^1$ is reflexive.

9. *The characteristic zero extension theorem revisited*

The purpose of this section is to explain how the ideas in this paper yield a new proof of [11, Theorem 1.5], repeated below as Theorem A. Even though we are ultimately only interested in that statement, in order to give a self-contained argument we have to set up an inductive procedure involving Theorem B. The latter statement has already been proven in much greater generality in [8, Theorem 1.2], but we must not use that result in our proof in order to avoid a circular dependence on [11].

For a very brief run-down of $\mathcal{C}$-pairs and $\mathcal{C}$-differentials, we refer to Section 9.A. In the whole section, all varieties are assumed to be defined over the complex numbers.

THEOREM A. *Let (X, D) be a complex log canonical pair. Then (X, D) satisfies the logarithmic extension theorem.*

THEOREM B (Bogomolov–Sommese vanishing). *Let (X, D) be a complex-projective dlt $\mathcal{C}$-pair and $\mathscr{A} \subset \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^r(\log D) = \Omega_X^{[r]}(\log\lfloor D \rfloor)$ a rank one reflexive subsheaf. Then the $\mathcal{C}$-Kodaira dimension $\kappa_{\mathcal{C}}(\mathscr{A}) \leqslant r$.*

The induction runs as follows, where the start of induction (dimension one) is trivial. Here, of course, 'Theorem A_n' means 'Theorem A for X of dimension at most n', and ditto for Theorem B_n.

- Theorem A_n implies Theorem B_n, and
- Theorem B_n implies Theorem A_{n+1}.

While the proofs of both directions do draw on some of the more elementary arguments in [11], we stress that the technical core of that paper is not used. Hence it still seems fair to say that our proof is 'new'.

9.A. *Background on $\mathcal{C}$-pairs*

For a more thorough treatment, see [8, Section 5] and the references therein. A $\mathcal{C}$-*pair* is a pair (X, D) in the usual sense, where $D = \sum_i (1 - \frac{1}{n_i}) D_i$ with $n_i \in \mathbb{N} \cup \{\infty\}$. One then defines *adapted morphisms* $\gamma \colon Y \to X$, essentially by requiring that the ramification order over D_i is equal to n_i. The sheaves of $\mathcal{C}$-*differentials* are subsheaves

$$\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_X^r(\log D) \subset \left(\mathrm{Sym}^{[m]} \Omega_X^r\right)(*\lceil D \rceil)$$

defined by the condition that the pullback of a local section under an adapted morphism has at worst logarithmic poles along $\mathrm{supp}\, \gamma^* \lfloor D \rfloor$. We have

$$\mathrm{Sym}^{[m]} \Omega_X^r(\log\lfloor D \rfloor) \subset \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_X^r(\log D)$$

and for $m = 1$, this is an equality. Let $\mathscr{A} \subset \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^r(\log D)$ be a Weil divisorial subsheaf. The $\mathcal{C}$-Kodaira dimension $\kappa_{\mathcal{C}}(\mathscr{A})$ is defined to be the maximum of the dimensions of $\overline{\varphi_m(X)}$,

where φ_m is the rational map given by the global sections of $\mathrm{Sym}_{\mathcal{C}}^{[m]}\mathscr{A}$. Here $\mathrm{Sym}_{\mathcal{C}}^{[m]}\mathscr{A}$ is the saturation of $\mathscr{A}^{[m]}$ inside $\mathrm{Sym}_{\mathcal{C}}^{[m]}\Omega_X^r(\log D)$.

9.B. *Residue and restriction sequences*

We need to have the results of Section 5 at our disposal in this setting. These are, to a large extent, already contained in [**8**, Section 6; **11**, Section 11]. Hence we only give a sketch of the proof.

THEOREM 9.1 (Residue sequence). *Let (X, D) be a dlt $\mathcal{C}$-pair and $P \subset \lfloor D \rfloor$ an irreducible component. Setting $P^c := \mathrm{Diff}_P(D - P)$, the pair (P, P^c) is again a dlt $\mathcal{C}$-pair, and the following holds: For any integer $r \geqslant 1$, there is a sequence*

$$0 \longrightarrow \Omega_X^{[r]}(\log\lfloor D \rfloor - P) \longrightarrow \Omega_X^{[r]}(\log\lfloor D \rfloor) \xrightarrow{\mathrm{res}_P} \Omega_P^{[r-1]}(\log\lfloor P^c \rfloor) \longrightarrow 0 \qquad (9.1.1)$$

which is exact on X off a codimension three subset and on $(X, D)_{\mathrm{snc}}$ agrees with the usual residue sequence. Its restriction to P induces a sequence

$$0 \longrightarrow \Omega_P^{[r]}(\log\lfloor P^c \rfloor) \longrightarrow \Omega_X^{[r]}(\log\lfloor D \rfloor)\big|_P^{\vee\vee} \xrightarrow{\mathrm{res}_P^P} \Omega_P^{[r-1]}(\log\lfloor P^c \rfloor) \longrightarrow 0 \qquad (9.1.2)$$

which is exact on P off a codimension two subset. More generally, for every $m \in \mathbb{N}$ there is a map

$$\mathrm{res}_P^m \colon \mathrm{Sym}_{\mathcal{C}}^{[m]}\Omega_X^r(\log D) \longrightarrow \mathrm{Sym}_{\mathcal{C}}^{[m]}\Omega_P^{r-1}(\log P^c),$$

surjective off a codimension three subset of X, which generically coincides with the mth symmetric power of the residue map.

THEOREM 9.2 (Restriction sequence). *Notation as above. Then there is a sequence*

$$0 \longrightarrow \Omega_X^{[r]}(\log\lfloor D \rfloor)(-P)^{\vee\vee} \longrightarrow \Omega_X^{[r]}(\log\lfloor D \rfloor - P) \xrightarrow{\mathrm{restr}_P} \Omega_P^{[r]}(\log\lfloor P^c \rfloor) \longrightarrow 0, \qquad (9.2.1)$$

exact off a codimension three subset, which on $(X, D)_{\mathrm{snc}}$ agrees with the usual restriction sequence. More generally, for every $m \in \mathbb{N}$ there is a map

$$\mathrm{restr}_P^m \colon \mathrm{Sym}_{\mathcal{C}}^{[m]}\Omega_X^r(\log D - P) \longrightarrow \mathrm{Sym}_{\mathcal{C}}^{[m]}\Omega_P^r(\log P^c)$$

which is surjective in codimension two and generically coincides with the mth symmetric power of the restriction map.

Proof sketch of Theorems 9.1 and 9.2. Sequences (9.1.1) and (9.1.2) along with the respective properties are in [**11**, Theorem 11.7]. The existence of res_P^m is shown in [**8**, Theorem 6.9(i)]. Likewise, the maps restr_P^m are constructed in [**8**, Theorem 6.9(ii)]. What is missing is the following:

- the existence of sequence (9.2.1), and
- the surjectivity properties of res_P^m and restr_P^m.

For the first item, the argument is similar to Step 3 in the proof of Theorem 1.4, that is, we reduce to the snc case by a reflexivity argument. For the second item (which is in fact not used in this paper), we resort to Step 2 in the above proof, but instead of Corollary 4.5 we use [**8**, Proposition 6.12]. $\qquad\square$

9.C. *Lifting along a non-positive map*

The analog of Theorem 6.1 is as follows.

THEOREM 9.3 (Lifting forms). *Assume Theorem* B_n. *Let* $g \colon Y \to X$ *be a proper birational map of normal varieties of dimension at most* $n + 1$, *with* $E = \mathrm{Exc}(g)$ *the reduced divisorial part of the exceptional locus. Furthermore, let* D *be an effective divisor on* X, *and set* $D_Y := g_*^{-1}D + E$. *Assume both of the following:*

(9.3.1) *The pair* (Y, D_Y) *is dlt and* $\mathbb{Q}$-*factorial.*
(9.3.2) *For any irreducible component* $P \subset E$, *setting* $P^c := \mathrm{Diff}_P(D_Y - P)$, *we have that* $K_P + P^c$ *is not* $g|_P$-*big.*

Then for any integer $r \geqslant 1$, *the natural map*

$$g_* \Omega_Y^{[r]}(\log \lfloor D_Y \rfloor) \hookrightarrow \Omega_X^{[r]}(\log \lfloor D \rfloor)$$

is an isomorphism.

Recall that if f is any map, a divisor on the source of f is called f-big if its restriction to a *general* fiber of f is big.

The proof relies crucially on the following negativity lemma, which should be compared to the usual one [**3**, Lemma 3.6.2(1)]. Indeed our version is somewhat stronger, as it does not merely make a numerical statement, but actually produces sections of a suitable line bundle.

PROPOSITION 9.4 (Big negativity lemma, cf. [**8**, Proposition 4.1]). *Let* $\pi \colon Y \to X$ *be a proper birational map between normal quasi-projective varieties. Then for any non-zero effective* π-*exceptional* $\mathbb{Q}$-*Cartier divisor* E, *there is an irreducible component* $P \subset E$ *such that* $-E|_P$ *is* $\pi|_P$-*big.*

Proof of Theorem 9.3. We first contend that we may replace D by $\lfloor D \rfloor$ and thus assume that D is reduced. To this end, note that $\lfloor D_Y \rfloor = g_*^{-1}\lfloor D \rfloor + E$, so the conclusion we are aiming at only depends on $\lfloor D \rfloor$. Also, as Y is $\mathbb{Q}$-factorial, the pair $(Y, \lfloor D_Y \rfloor)$ remains dlt. Finally, for any component $P \subset E$ we have

$$\mathrm{Diff}_P(\lfloor D_Y \rfloor - P) = \mathrm{Diff}_P(0) + (\lfloor D_Y \rfloor - P)\big|_P \qquad [19, (4.2.10)]$$

$$= \mathrm{Diff}_P(0) + (D_Y - P)\big|_P - \{D_Y\}\big|_P$$

$$= P^c - \{D_Y\}\big|_P, \qquad\qquad [19, (4.2.10)]$$

where $\{D_Y\} := D_Y - \lfloor D_Y \rfloor \geqslant 0$ is the fractional part of D_Y. So Condition (9.3.2) is likewise preserved.

Now let $\sigma \in \mathrm{H}^0(X, \Omega_X^{[r]}(\log D)) \setminus \{0\}$ be an arbitrary non-zero logarithmic r-form, and pick an effective g-exceptional divisor G such that

$$g^* \sigma \in \mathrm{H}^0\Big(Y, \Omega_Y^{[r]}(\log D_Y)(G)^{\vee\vee}\Big). \qquad\qquad (9.4.1)$$

Equivalently, there is an injective map $i \colon \mathscr{O}_Y(-G) \to \Omega_Y^{[r]}(\log D_Y)$. We may assume that $G \neq 0$, in which case by Proposition 9.4 there is a component $P \subset G$ such that $-G|_P$ is $g|_P$-big. Set $P^c := \mathrm{Diff}_P(D_Y - P)$. Let $F \subset P$ be a general fiber of $g|_P$, and set $F^c := P^c|_F$. Then F is normal and (F, F^c) is again a dlt $\mathcal{C}$-pair. Also, define $\mathscr{A} \subset \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_Y^r(\log D_Y)$ to be the image of i. Now consider the residue sequence (9.1.1) along P:

$$0 \longrightarrow \Omega_Y^{[r]}\big(\log D_Y - P\big) \longrightarrow \Omega_Y^{[r]}\big(\log D_Y\big) \xrightarrow{\ \mathrm{res}_P\ } \Omega_P^{[r-1]}\big(\log \lfloor P^c \rfloor\big) \longrightarrow 0.$$

CLAIM 9.5. *We have $\mathrm{res}_P(\mathscr{A}) = 0$, and hence $\mathscr{A}$ is contained in $\Omega_Y^{[r]}(\log D_Y - P)$ as indicated by the dashed arrow in the above diagram.*

In the following, note that the restriction of a reflexive sheaf on P to the general fiber F remains reflexive and hence in this case the double dual may be omitted.

Proof of Claim 9.5. Arguing by contradiction, let us assume that $\mathrm{res}_P(\mathscr{A}) \neq 0$ and denote its saturation by $\mathscr{B} \subset \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_P^{r-1}(\log P^c)$, a Weil divisorial sheaf. By [**8**, Proposition 7.3], there are a number $0 \leqslant q \leqslant r - 1$ and embeddings

$$\alpha_k \colon \mathscr{C}_k := (\mathrm{Sym}_{\mathcal{C}}^{[k]} \mathscr{B})\big|_F \hookrightarrow \mathrm{Sym}_{\mathcal{C}}^{[k]} \Omega_F^q(\log F^c)$$

for all k, satisfying the compatibility conditions that $\mathscr{C}_k$ and $\mathscr{C}_1^{[k]}$ generically agree as subsheaves of $\mathrm{Sym}_{\mathcal{C}}^{[k]} \Omega_F^q(\log F^c)$. We will show that $\mathscr{C} := \mathscr{C}_1$ is '$\mathcal{C}$-big' in the sense that $\kappa_{\mathcal{C}}(\mathscr{C}) = \dim F$. If $q < \dim F$, this contradicts Theorem B_n. If $q = \dim F$, it contradicts Assumption (9.3.2), which says that $K_F + F^c = (K_P + P^c)|_F$ is not big.

To this end, we claim that for any natural number m there is an inclusion

$$\left(\mathscr{O}_Y(-mG)\big|_P^{\vee\vee}\right)\big|_F \subset \left(\left(\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{A}\right)\big|_P^{\vee\vee}\right)\big|_F \subset \mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{C}. \tag{9.5.1}$$

The first inclusion holds because i does not vanish along P nor F (otherwise we would necessarily have $\mathrm{res}_P(\mathscr{A}) = 0$). For the second one, the map res_P^m from Theorem 9.1 gives an inclusion

$$(\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{A})\big|_P^{\vee\vee} \hookrightarrow \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_P^{r-1}(\log P^c)$$

which by (2.1.3) factors via the saturated subsheaf $\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{B}$. Restricting to F, we see that the middle term of (9.5.1) is contained in $\mathscr{C}_m$. But $\mathscr{C}_m$, $\mathscr{C}^{[m]}$ and $\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{C}$ all generically agree. Hence $\mathscr{C}_m \subset \mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{C}$ by another application of (2.1.3) and we obtain (9.5.1).

Now let m be sufficiently divisible so that mG is Cartier. In this case, on the left-hand side of (9.5.1) the double dual may be dropped and we simply get the big line bundle $\mathscr{O}_F(-mG|_F)$. As a consequence, also $\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{C}$ is big, establishing our claim that $\kappa_{\mathcal{C}}(\mathscr{C}) = \dim F$. $\qquad\square$

We next consider the restriction sequence (9.2.1):

$$0 \longrightarrow \Omega_Y^{[r]}(\log D_Y)(-P)^{\vee\vee} \longrightarrow \Omega_Y^{[r]}(\log D_Y - P) \xrightarrow{\mathrm{restr}_P} \Omega_P^{[r]}(\log \lfloor P^c \rfloor) \longrightarrow 0.$$

The same line of argument as in the proof of Claim 9.5 then shows that we have $\mathrm{restr}_P(\mathscr{A}) = 0$ and so $\mathscr{A}$ is contained in $\Omega_Y^{[r]}(\log D_Y)(-P)^{\vee\vee}$. The details are omitted for their similarity. The proof of Theorem 9.3 is now finished in exactly the same fashion as Theorem 6.1: we have shown that G can be replaced by $G - P$ in (9.4.1). Hence after finitely many repetitions we arrive at $G = 0$. $\qquad\square$

9.D. *Proof of '$\mathrm{A}_n \Rightarrow \mathrm{B}_n$'*

This implication is by far the easier of the two. By [**3**], the dlt pair (X, D) admits a $\mathbb{Q}$-factorialization. This is essentially a dlt modification in the special case of dlt pairs; cf. Section 3. For the proof, see [**20**, Theorem 3.1] and [**8**, Theorem 9.2].

As the Kodaira dimension is invariant under small morphisms, we may assume that $\mathscr{A}$ is $\mathbb{Q}$-Cartier. Under this assumption, the proof of [**11**, Theorem 7.2] shows how to deduce that

$\kappa(\mathscr{A}) \leqslant r$ from Theorem A_n and the standard Bogomolov–Sommese vanishing theorem for snc pairs [5, Corollary 6.9]. The stronger statement that $\kappa_\mathcal{C}(\mathscr{A}) \leqslant r$ can then be obtained from this by a branched covering trick as explained in [15, Section 7]. Compare also [11, Theorem 7.3] (which erroneously contains the extra assumption that $\dim X \leqslant 3$).

9.E. *Proof of '$B_n \Rightarrow A_{n+1}$'*

Let $\pi\colon Y \to X$ be a log resolution of (X, D). Then π can be factored as $f \circ \varphi_{r-1} \circ \cdots \circ \varphi_0$ just as in Theorem 3.1, whose notation we adopt here. The only difference is that some of the φ_i might be flips. Also the proof is essentially the same, except that instead of [24] we need to appeal to [3]. Also, because we cannot in general run the MMP on a dlt pair, we have to use the perturbation trick from the proof of [20, Theorem 3.1] to reduce the situation to the case of klt pairs.

We will lift forms along each step separately, just as in Proposition 7.3 but using Theorem 9.3 instead of Theorem 6.1. For this, we need to make sure Condition (9.3.2) is satisfied. As far as the map f is concerned, this is quite clear: since $K_Z + \widetilde{D}_r = f^*(K_X + D)$, the restriction of $K_Z + \widetilde{D}_r$ to any fiber of f is trivial and in particular not big. But for any component $P \subset E$, we have $(K_Z + \widetilde{D}_r)|_P = K_P + P^c$ by adjunction and so (9.3.2) is satisfied.

We now fix $0 \leqslant i \leqslant r - 1$ and turn to the map $\varphi_i\colon Y_i \to Y_{i+1}$. If φ_i is a flip, then it is an isomorphism in codimension one and hence extension of forms from Y_{i+1} to Y_i is automatic by reflexivity. We may therefore assume that φ_i is a divisorial contraction, with irreducible exceptional divisor P. By construction, $K_{Y_i} + \widetilde{D}_i$ is φ_i-anti-ample. Also, $(K_{Y_i} + \widetilde{D}_i)|_P = K_P + P^c$ by adjunction. Thus $K_P + P^c$ is $\varphi_i|_P$-anti-ample and in particular (9.3.2) is satisfied.

10.   *Sharpness of results*

In this section we discuss some examples that show to what extent our main results are optimal. First, we show that Theorem 1.2 fails for the RDP E_8^0 in characteristic $p \leqslant 5$, using notation from [2].

EXAMPLE 10.1 (No logarithmic extension theorem in low characteristics). Fix any field k of characteristic $p = 2, 3$ or 5. Then for the (non-F-pure) E_8^0 singularity

$$X = \left\{ f = z^2 + x^3 + y^5 = 0 \right\} \subset \mathbb{A}_k^3,$$

the logarithmic extension theorem does not hold. More precisely, note that Kähler differentials on X satisfy the relation

$$\mathrm{d}f = 3x^2\mathrm{d}x + 5y^4\mathrm{d}y + 2z\mathrm{d}z = 0$$

and hence we may consider the 1-form

$$\sigma := \begin{cases} y^{-4}\mathrm{d}x = x^{-2}\mathrm{d}y, & p = 2, \\ z^{-1}\mathrm{d}y = -y^{-4}\mathrm{d}z, & p = 3, \\ z^{-1}\mathrm{d}x = -x^{-2}\mathrm{d}z, & p = 5. \end{cases}$$

As any two coordinate functions on X vanish simultaneously only at the origin, $\sigma \in \mathrm{H}^0(X_{\mathrm{reg}}, \Omega_X^1) = \mathrm{H}^0(X, \Omega_X^{[1]})$ is a reflexive differential form on X. We blow up the origin of $\mathbb{A}_k^3$ (and points lying over it) four times in a row, yielding a map $\varphi\colon \widetilde{\mathbb{A}_k^3} \to \mathbb{A}_k^3$. In suitable coordinates on $\widetilde{\mathbb{A}_k^3}$, this map is given by

$$\varphi(u, v, w) = (u^2v^5, uv^3, u^2v^7w).$$

We compute

$$\varphi^*(f) = u^4 v^{14} w^2 + u^6 v^{15} + u^5 v^{15} = u^4 v^{14} \cdot \underbrace{\left(w^2 + uv(u+1) \right)}_{\substack{\text{equation of strict} \\ \text{transform } \widetilde{X} \text{ of } X}}.$$

We see that $\widetilde{X}$ can be parametrized rationally by the (u,w)-plane, namely by setting $v = -\frac{w^2}{u(u+1)}$. In this parametrization, for $p = 2$ the pullback of σ is given by

$$\varphi^*(\sigma) = (uv^3)^{-4}\, \mathrm{d}(u^2 v^5)$$

$$= u^{-2} v^{-8}\, \mathrm{d}v$$

$$= \frac{u^6(u+1)^8}{w^{16}}\, \mathrm{d}\left(\frac{w^2}{u(u+1)} \right)$$

$$= \cdots$$

$$= \frac{u^4(u+1)^6}{w^{14}}\, \mathrm{d}u.$$

A similar calculation for the other characteristics gives

$$\varphi^*(\sigma) = \begin{cases} \dfrac{u^2(u+1)^4}{w^9}\, \mathrm{d}u, & p = 3, \\[2ex] \dfrac{u(u+1)^2}{w^5}\, \mathrm{d}u, & p = 5. \end{cases}$$

This shows that the extension of σ to $\widetilde{X}$ has worse than logarithmic poles along the exceptional divisor $\{w = 0\}$.

Next, we show that in Theorem 1.3, the assumption on $\det(E_i \cdot E_j)$ not being divisible by p really is necessary.

EXAMPLE 10.2 (No regular extension theorem in spite of logarithmic extension theorem). Let k be a field of characteristic $p > 0$, and consider the pth Veronese subring R of $k[x,y]$, that is, $R = k[x^p, x^{p-1}y, \ldots, y^p]$. Then $X = \mathrm{Spec}\, R$ is a strongly F-regular surface, since R is a direct summand of the regular ring $k[x,y]$. In particular, X is klt. If $\pi\colon Y \to X$ is the minimal resolution, then $E = \mathrm{Exc}(\pi)$ consists of a single smooth rational curve of self-intersection $-p$. In particular, the assumptions of Theorem 1.3 are not satisfied. For later use, let us record the discrepancy $a = a(E, X)$ along E: by adjunction,

$$-2 = (K_Y + E) \cdot E = (\pi^* K_X + (a+1)E) \cdot E = -(a+1)p$$

and hence $a = -1 + \frac{2}{p}$.

One can see by direct calculation that X satisfies the logarithmic extension theorem, but not the regular extension theorem. More precisely, Y is covered by two open sets $U_0, U_1 \cong \mathbb{A}_k^2$, where U_i has coordinates x_i, y_i and the coordinate change is given by $(x_1, y_1) = (x_0^{-1}, x_0^p y_0)$. The exceptional curve E is given by the equation $y_i = 0$ in the chart U_i. Consider the form $\sigma \in \mathrm{H}^0(Y \setminus E, \Omega_Y^1)$ given by $y_0^{-1}\mathrm{d}y_0$ on U_0 and by $y_1^{-1}\mathrm{d}y_1$ on U_1. It obviously does not extend regularly over E, showing that the regular extension theorem fails for X. But σ has only a logarithmic pole along E, and it generates the quotient $\Omega_X^{[1]}\big/_{\pi_* \Omega_Y^1}$. Thus the logarithmic extension theorem does hold for X. (Of course, this last fact also follows from Theorem 1.2, at least if $p \geqslant 7$.)

An elaboration of the previous example shows that Theorem 1.5 fails for dlt pairs whose canonical divisor is not $\mathbb{Z}_{(p)}$-Cartier.

EXAMPLE 10.3 (No restriction sequence for fiercely dlt pairs). Using notation from Example 10.2, let $D \subset X$ be the π-image of the curve $D_Y \subset U_0 \subset Y$ given by the equation $\{x_0 = \text{const.}\}$ (where the constant is arbitrary). Then D is a smooth curve passing through the singular point $x \in X$, and it is isomorphic to its strict transform D_Y.

$$
\begin{array}{ccc}
D_Y & \hookrightarrow & Y \\
\wr\downarrow & & \downarrow \pi \\
D & \hookrightarrow & X
\end{array}
$$

We have $\pi^* D = D_Y + rE$, where $0 = \pi^* D \cdot E = 1 - rp$ and hence $r = 1/p$. It follows that the discrepancy of (X, D) along E is

$$a(E, X, D) = a(E, X) - r = -\frac{p-1}{p}.$$

This shows that the pair (X, D) is plt. On the other hand, $K_X + D$ cannot be $\mathbb{Z}_{(p)}$-Cartier, since otherwise p would not appear in the denominator of $a(E, X, D)$ (written in lowest terms). Hence (X, D) is fiercely dlt. In particular, we cannot apply Theorem 4.3 to compute $\text{Diff}_D(0)$. But [19, Theorem 3.36] tells us that we still have $D^c := \text{Diff}_D(0) = (1 - \frac{1}{p}) \cdot [x]$. So, if Theorem 1.5 is held, we would have a restriction map as in (1.5.1):

$$\text{restr}_D \colon \Omega_X^{[1]} \to \Omega_D^1(\log\lfloor D^c \rfloor) = \Omega_D^1.$$

Consider however the form σ from the previous example, viewed as a section of $\Omega_X^{[1]}$. As $D_Y \cong D$, we can compute $\text{restr}_D(\sigma)$ on Y. We have already seen that σ acquires a logarithmic pole along E. So $\sigma|_{D_Y}$ has a logarithmic (that is, simple) pole at the unique point in the intersection $D_Y \cap E$, which under π maps to x. Summing up, this means that we do have a restriction map

$$\text{restr}_D \colon \Omega_X^{[1]} \to \Omega_D^1(\log x),$$

but it does not factor via Ω_D^1. Looking at higher powers $\sigma^{[m]}$, we see that also the other maps restr_D^m from Theorem 1.5 do not exist in this example.

EXAMPLE 10.4 (No logarithmic extension theorem for singularities reduced from characteristic zero). Finally, we would like to remark that if we start with a log canonical singularity in characteristic zero and then reduce it modulo some small prime p, the resulting singularity may not satisfy the logarithmic extension theorem even if it remains log canonical. Indeed, Example 10.1 furnishes a counterexample since $z^2 + x^3 + y^5 = 0$ defines an E_8 RDP also in characteristic zero.

11. Counterexamples in higher dimensions

In this final section, we will prove Theorem 1.6. As a starting point, in [17] Kollár has given a fairly explicit method for constructing counterexamples to Bogomolov–Sommese vanishing over fields of positive characteristic. We will recall Kollár's construction in Section 11.A, both for the benefit of the reader and in order to bring the result in the precise form we need. It turns out that in the examples, the line bundle in question is not just big, but even ample. Thus also Nakano vanishing is violated:

PROPOSITION 11.1 (Failure of Nakano vanishing). *Fix an algebraically closed field k of characteristic $p > 0$, and an integer $n \geqslant 2$.*

(11.1.1) If $n \geqslant 2p - 2$, then there exists an n-dimensional Fano variety Y/k with only isolated canonical hypersurface singularities such that

$$\mathrm{H}^0\Big(Y, \Omega_Y^{[n-1]} \otimes \omega_Y\Big) \neq 0.$$

(11.1.2) If $n \geqslant 3p - 2$, then there exists Y as above, but such that ω_Y^{-1} admits a square root L (that is, an ample line bundle with $L^2 \cong \omega_Y^{-1}$) and

$$\mathrm{H}^0\Big(Y, \Omega_Y^{[n-1]} \otimes L^{-1}\Big) \neq 0.$$

(11.1.3) If $n \geqslant p - 2$, then there exists an n-dimensional variety Y/k with only isolated canonical hypersurface singularities, satisfying $\omega_Y \cong \mathscr{O}_Y$ and

$$\mathrm{H}^0\Big(Y, \Omega_Y^{[n-1]} \otimes L^{-1}\Big) \neq 0$$

for some ample line bundle L on Y.

In all cases, Y actually has F-pure singularities. If $n \geqslant 3$, then Y is even terminal and strongly F-regular.

In Section 11.B, we will turn our attention to cones over projective varieties and study when the logarithmic extension theorem holds for such spaces. The conclusion is that cones over the examples from Proposition 11.1 are sufficient to prove Theorem 1.6, which is accomplished in Section 11.C.

11.A. *Kollár's construction*

Kollár's method is quite flexible in the sense that it does not rely on resolution of singularities and gives very good control on the canonical divisor of the resulting example. On the other hand, it only works in dimensions that satisfy a certain lower bound depending on the characteristic, and the spaces obtained are virtually never smooth. Also, the violation of Nakano vanishing is only guaranteed in degree $n - 1$, where n is the dimension.

Let X be an n-dimensional smooth projective variety over an algebraically closed field of characteristic $p > 0$ and L a line bundle on X. Assume that L^p is 'globally generated to second order' in the sense that the restriction map

$$\mathrm{H}^0(X, L^p) \longrightarrow L^p \otimes_{\mathscr{O}_X} \Big(\mathscr{O}_X \big/ \mathfrak{m}_x^3\Big)$$

is surjective for every (closed) point $x \in X$ with ideal sheaf $\mathfrak{m}_x \subset \mathscr{O}_X$. Choose a general section $s \in \mathrm{H}^0(X, L^p)$ and consider the cover

$$Y := X[\sqrt[p]{s}] \xrightarrow{\ \pi\ } X$$

as before. By [**17**, (14.2)] there is a short exact sequence

$$0 \longrightarrow \pi^* \operatorname{coker} \underbrace{\left[L^{-p} \xrightarrow{\ \mathrm{d}s\ } \Omega_X^1 \right]}_{\substack{\text{const. rank one} \\ \text{off small subset}}} \longrightarrow \Omega_Y^1 \longrightarrow \pi^* L^{-1} \longrightarrow 0. \tag{11.1.4}$$

Taking determinants, we see that $K_Y = \pi^*(K_X + (p-1)L)$. On the other hand, the $(n-1)$th reflexive wedge power of the first map in (11.1.4) shows that

$$\mathrm{H}^0\Big(Y, \Omega_Y^{[n-1]} \otimes \pi^*(K_X + pL)^{-1}\Big) \neq 0. \tag{11.1.5}$$

Thus we obtain interesting examples if $K_X + pL$ is ample, but $K_X + (p-1)L$ is not.

Proof of Proposition 11.1. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d = n - 2p + 3 \geqslant 1$, and take $L = \mathcal{O}_X(2)$. The global generation hypothesis on L^p is automatically satisfied, hence we may construct $\pi \colon Y \to X$ as above. Then Y is Fano since

$$K_X + (p-1)L = \mathcal{O}_X(-(n+2) + d + 2(p-1)) = \mathcal{O}_X(-1)$$

is anti-ample. On the other hand, $K_X + pL = \mathcal{O}_X(1) = (K_X + (p-1)L)^{-1}$ is ample and so by (11.1.5), the variety Y violates Nakano vanishing in the required form. By [**17**, (20.3), (22.1)], the singularities of Y are locally of the form

$$y^p = \underbrace{x_{n-1}x_n + f_2(x_1, \ldots, x_{n-2})}_{\text{non-degenerate quadric}} + \text{(higher order terms with respect to } x). \qquad (11.1.6)$$

Using this description, it can be checked that Y has only isolated canonical hypersurface singularities, which are terminal for $n \geqslant 3$. This proves (11.1.1).

The argument for (11.1.2) is very similar. We start with X a smooth hypersurface of degree $d = n - 3p + 3 \geqslant 1$ and $L = \mathcal{O}_X(3)$. Then

$$K_X + (p-1)L = \mathcal{O}_X(-(n+2) + d + 3(p-1)) = \mathcal{O}_X(-2)$$

and $K_X + pL = \mathcal{O}_X(1)$. Again we conclude by (11.1.5).

For (11.1.3), we tweak the numbers once more. Let X be of degree $d = n - p + 3 \geqslant 1$ and $L = \mathcal{O}_X(1)$. Then

$$K_X + (p-1)L = \mathcal{O}_X(-(n+2) + d + (p-1)) = \mathcal{O}_X,$$

so $\omega_Y \cong \mathcal{O}_Y$, and $K_X + pL = \mathcal{O}_X(1)$.

The claim about F-purity can likewise be checked using (11.1.6) and Fedder's criterion [**6**]. If $n \geqslant 3$, then even (11.1.6) multiplied by the non-unit x_1 is F-pure and so Y is strongly F-regular. Note also that a strongly F-regular Gorenstein singularity is automatically canonical, therefore this provides an alternative proof of Y being canonical. $\qquad \square$

REMARK 11.2. One might be tempted to try and construct lower dimensional examples by starting with a more interesting X than just a hypersurface in $\mathbb{P}^{n+1}$. This, however, is not possible because the Fano index of X is always $\leqslant \dim X + 1$ by [**18**, Chapter V, Theorem 1.6].

11.B. *Extension properties on cones*

Fix an integer $n \geqslant 2$, a smooth projective variety Y with $\dim Y = n - 1$, and an ample line bundle L on Y. Following [**19**, Chapter 3.1], let

$$X := \operatorname{Spec} \bigoplus_{m \geqslant 0} \mathrm{H}^0(Y, L^m)$$

be the affine cone over (Y, L). Blowing up the vertex gives a log resolution $\pi \colon \widetilde{X} \to X$, where $\widetilde{X}$ is the total space of the line bundle L^{-1} and the exceptional locus E is the zero section of L^{-1}. In particular, there is an affine map $r \colon \widetilde{X} \to Y$, which maps E isomorphically onto Y.

For any integer $q \geqslant 0$, we will say that Condition $(*)_q$ holds if

$$\mathrm{H}^0\big(Y, \Omega_Y^q \otimes L^{-m}\big) = 0 \qquad \text{for all } m \geqslant 1. \qquad (*)$$

Note that $(*)_q$ always holds in any of the following cases: $q = 0$, $q \geqslant n$, or if L is sufficiently ample. In characteristic zero, $(*)_q$ holds for any $q \neq n - 1$ by Nakano vanishing.

With this notation in place, we have the following result. It should be compared to the non-logarithmic, characteristic zero version in [**16**, Proposition B.2].

FIGURE 8. *Relative (log) differential sequences for the map* r.

PROPOSITION 11.3 (Logarithmic extension theorem on cones). *Notation as above. Then the following equivalences hold:*

(11.3.1) X *satisfies the logarithmic extension theorem for 1-forms* $\Leftrightarrow$ $(*)_1$ *holds.*

(11.3.2) X *satisfies the logarithmic extension theorem for n-forms* $\Leftrightarrow$ $(*)_{n-1}$ *holds.*

More generally, for arbitrary values of q *we have the following:*

(11.3.3) *If* $(*)_q$ *and* $(*)_{q-1}$ *hold, then* X *satisfies the logarithmic extension theorem for q-forms.*

(11.3.4) *Conversely, if* X *satisfies the logarithmic extension theorem for q-forms, then* $(*)_q$ *holds. If in addition* $n \geqslant 3$ *and* L *is sufficiently ample, then also* $(*)_{q-1}$ *holds.*

Proof. The sequence of relative differentials for r reads

$$0 \longrightarrow r^*\Omega_Y^1 \longrightarrow \Omega_{\widetilde{X}}^1 \longrightarrow r^*L \longrightarrow 0 \tag{11.3.5}$$

and its logarithmic version is

$$0 \longrightarrow r^*\Omega_Y^1 \longrightarrow \Omega_{\widetilde{X}}^1(\log E) \longrightarrow r^*L(E) \longrightarrow 0. \tag{11.3.6}$$

For (11.3.6), choose a system of local parameters $y_1, \ldots, y_{n-1}$ of Y and let t be a nowhere vanishing local section of L, considered as a fiberwise linear coordinate on $\widetilde{X}$. Then the middle term of (11.3.6) is locally freely generated by

$$r^*dy_1, \ldots, r^*dy_{n-1}, dt/t,$$

and the first map is the inclusion of the subsheaf generated by the r^*dy_i. Consequently, the quotient sheaf is invertible, locally generated by dt/t. Since $E = \{t = 0\}$, this shows that the quotient is isomorphic to $r^*L(E)$.

Now (11.3.5) and (11.3.6) sit inside the diagram shown in Figure 8. Also, for forms of higher degree, from (11.3.6) we get [**13**, Chapter II, Example 5.16]

$$0 \longrightarrow r^*\Omega_Y^q \longrightarrow \Omega_{\widetilde{X}}^q(\log E) \longrightarrow r^*\left(\Omega_Y^{q-1} \otimes L\right)(E) \longrightarrow 0. \tag{11.3.7}$$

Recalling that both r and its restriction $r' := r|_{\widetilde{X} \setminus E}$ are affine, with

$$r_*\mathcal{O}_{\widetilde{X}} = \bigoplus_{m \geqslant 0} L^m,$$

$$r_*\mathcal{O}_{\widetilde{X}}(E) = \bigoplus_{m \geqslant -1} L^m \quad \text{and}$$

$$r'_*\mathcal{O}_{\widetilde{X} \setminus E} = \bigoplus_{m \in \mathbb{Z}} L^m,$$

from (11.3.7) we obtain the following diagram with exact rows and injective vertical arrows:

$$
\begin{array}{ccccccc}
0 \to \bigoplus_{m \geq 0} \mathrm{H}^0(\Omega_Y^q \otimes L^m) & \to & \mathrm{H}^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^q(\log E)\big) & \to & \bigoplus_{m \geq 0} \mathrm{H}^0(\Omega_Y^{q-1} \otimes L^m) & \to & \bigoplus_{m \geq 0} \mathrm{H}^1(\Omega_Y^q \otimes L^m) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\
0 \to \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^0(\Omega_Y^q \otimes L^m) & \longrightarrow & \mathrm{H}^0\big(\widetilde{X} \setminus E, \Omega_{\widetilde{X}}^q\big) & \longrightarrow & \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^0(\Omega_Y^{q-1} \otimes L^m) & \to & \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^1(\Omega_Y^q \otimes L^m)
\end{array}
$$

It is clear that α is an isomorphism $\Leftrightarrow (*)_q$ holds, β is an isomorphism $\Leftrightarrow$ the logarithmic extension theorem for q-forms holds on X and γ is an isomorphism $\Leftrightarrow (*)_{q-1}$ holds. Furthermore, if $n \geqslant 3$ and L is sufficiently ample then δ is an isomorphism by Serre vanishing and Serre duality. All claims thus follow from straightforward diagram chases (cf. [**11**, Lemma B.2]). $\qquad\square$

11.C. *Proof of Theorem* 1.6

With all preliminaries in place, the construction of counterexamples to the logarithmic extension theorem becomes very easy. Take (Y, L) as in (11.1.3), and let X be the affine cone over (Y, L). Blowing up the vertex gives an exceptional divisor of discrepancy -1 because $\omega_Y \cong \mathscr{O}_Y$. The result is the total space of L^{-1}, which has canonical singularities just as Y. We conclude that X is log canonical. By (11.3.4), the logarithmic extension theorem for $(n-2)$-forms does not hold on X. This proves (1.6.1).

For (1.6.2), we use the Fano variety Y from (11.1.1) instead. In this case, X is the cone over (Y, ω_Y^{-1}). A calculation shows that the first discrepancy is zero. Hence, since Y has canonical singularities, so does X. The logarithmic extension theorem fails for the same reason as above.

For (1.6.3), we appeal to (11.1.2), that is, the cone X is taken with respect to a square root of ω_Y^{-1}. In this case the first discrepancy is equal to one. Since $\dim Y \geqslant 3p - 2 \geqslant 4$, we know that Y has only terminal singularities and then the same is true of X.

In each case, a log resolution of X can be obtained by first blowing up the vertex of the cone and then pulling back everything along a resolution of Y, which exists by [**17**, §21].

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