Combining subspace codes

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Abstract

In the context of constant–dimension subspace codes, an important problem is to determine the largest possible size $A_q(n, d; k)$ of codes whose codewords are k-subspaces of $\mathbb{F}_q^n$ with minimum subspace distance d. Here in order to obtain improved constructions, we investigate several approaches to combine subspace codes. This allow us to present improvements on the lower bounds for constant–dimension subspace codes for many parameters, including $A_q(10, 4; 5)$, $A_q(12, 4; 4)$, $A_q(12, 6, 6)$ and $A_q(16, 4; 4)$.

Keywords: constant–dimension subspace code; finite projective geometry; network coding.

1 Introduction

Let V be an n-dimensional vector space over the finite field $\mathbb{F}_q$, q any prime power. The set $S(V)$ of all subspaces of V, or subspaces of the projective space $\mathrm{PG}(V) = \mathrm{PG}(n-1, q)$, forms a metric space with respect to the *subspace distance* defined by $d(U, U') = \dim(U + U') - \dim(U \cap U')$. In the context of subspace codes, an important problem is to determine the largest possible size $A_q(n, d)$ of codes in the space $(S(V), d)$ with a given minimum distance, and to classify the corresponding optimal codes. The interest in these codes is a consequence of the fact that codes in the projective space have been proposed for error control in random linear network coding, see [30]. In this application the codewords are mostly assumed to be contained in a Grassmannian over a finite field, i.e., they all have the same vector space dimension k. These codes are referred to as *constant–dimension* codes (CDCs for short) and their maximum cardinality is denoted by $A_q(n, d; k)$.

Here we will consider several approaches to combine subspace codes in order to improve lower bounds for $A_q(n, d; k)$. The currently best known lower and upper bounds for $A_q(n, d; k)$ can be found at the online tables `http://subspacecodes.uni-bayreuth.de` and the associated survey

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[22]. For the parameters $2 \le q \le 9$, $4 \le n \le 9$, $2 \le k \le \frac{n}{2}$, $2 \le \frac{d}{2} \le k$ covered there, we obtain more than 200 improved constructions.

The remaining part of this paper is structured as follows. In Section 2 we introduce the necessary preliminaries and in Section 3 we briefly review the known constructions and bounds for $A_q(n,d;k)$. Our main results, i.e., improved constructions for CDCs are presented in Section 4 and Section 5.

More precisely, in Section 4 we consider constructions of CDCs based on rank metric codes. The results therein provide not only a generalization of several recent results [4, 17, 18, 19, 35, 43], but they also offer a more general point of view with respect to techniques that have been previously investigated in the literature, as for instance the so called linkage construction [16, 39]. In particular, by using rank metric codes in different variants, we are able to obtain CDCs that either give improved lower bounds for many parameters, including $A_2(12,4;4)$, $A_q(12,6;6)$, $A_q(4k,2k;2k)$, $k \ge 4$ even, $A_q(10,4;5)$, or whose size matches the best known lower bounds. Note that these bounds have been previously established with different approaches as the so-called Echelon-Ferrers construction [12, 39].

In Section 5 we investigate a construction method introduced for specific parameters by the first, third and fourth author in [5] and further developed by the second author in [34]. Here this approach is generalized and applied to a wide range of parameters. In particular, it enables us to obtain improved lower bounds for many parameters including $A_q(12,4;4)$, $q \ge 3$, $A_q(16,4;4)$. Finally we discuss this new approach in a more general framework. By way of examples the cases $A_q(3k,4;k)$, $k \ge 5$, and $A_q(6k,2k;2k)$, $k \ge 4$ even, are considered.

2 Preliminaries

Let V denote an n-dimensional vector space over $\mathbb{F}_q$. Since $V \simeq \mathbb{F}_q^n$ induces an isometry $(S(V),d) \simeq (S(\mathbb{F}_q^n),d)$, the particular choice of the ambient vector space V does not matter here, so that we will always write $\mathbb{F}_q^n$ in the following. An $(n,\Lambda,d;k)_q$ *constant–dimension code (CDC)* is a set $\mathcal{C}$ of k-dimensional subspaces of $\mathbb{F}_q^n$ with $\#\mathcal{C} = \Lambda$ and minimum subspace distance $d(\mathcal{C}) = \min\{d(U,U') \,:\, U,U' \in \mathcal{C}, U \ne U'\} = d$. In the terminology of projective geometry, an $(n,\Lambda,2\delta;k)_q$ constant–dimension code, $\delta > 1$, is a set $\mathcal{C}$ of $(k-1)$-dimensional projective subspaces of $\mathrm{PG}(n-1,q)$ such that $\#\mathcal{C} = \Lambda$ and every $(k-\delta)$-dimensional projective subspace of $\mathrm{PG}(n-1,q)$ is contained in at most one member of $\mathcal{C}$ or, equivalently, any two distinct codewords of $\mathcal{C}$ intersect in at most a $(k-\delta-1)$-dimensional projective space. The maximum size of an $(n,\star,d;k)_q$-CDC is denoted by $A_q(n,d;k)$. The number of k-dimensional subspaces of $\mathbb{F}_q^n$ is given by $\left[{n \atop k} \right]_q = \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}$.

If U is a k-dimensional subspace of $\mathbb{F}_q^n$ we write $U \le \mathbb{F}_q^n$ and call U a *k-subspace*. The row space $R(M)$ of any full-rank matrix $M \in \mathbb{F}_q^{k \times n}$ gives rise to such a k-subspace U. Here M is called a *generator matrix* of U. For the other direction we denote by $\tau(U)$ the unique full-rank matrix in $\mathbb{F}_q^{k \times n}$ that is in *reduced row echelon form* (rre). By $p(U) \in \mathbb{F}_2^n$ we denote the binary vector whose 1-entries coincide with the pivot columns of $\tau(U)$. Its *Hamming weight* $w_{\mathrm{h}}(p(U))$, i.e., the number of non-zero entries, equals the dimension k of U. Slightly abusing notation, we also write

$\tau(M) = \tau(R(M))$ and $p(M) = p(R(M))$ for a matrix $M \in \mathbb{F}_q^{k\times n}$. For $M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \in \mathbb{F}_2^{2\times 4}$ we have $\tau(M) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ and $p(M) = (1,1,0,0)$. The subspace distance $d(U,U')$ between two subspaces U and U' of $\mathbb{F}_q^n$ can be expressed via the ranks of their generator matrices:

$$\begin{aligned} d(U,U') &= \dim(U+U') - \dim(U\cap U') = 2\dim(U+U') - \dim(U) - dim(U') \\ &= 2\operatorname{rk}\begin{pmatrix} \tau(U) \\ \tau(U') \end{pmatrix} - \operatorname{rk}(\tau(U)) - \operatorname{rk}(\tau(U')). \end{aligned} \quad (2.1)$$

Note that this equation remains true if we replace τ by any other normal form of k-subspaces in $\mathbb{F}_q^n$ as full-rank $(k\times n)$-matrices over $\mathbb{F}_q$. If U and U' have the same dimension, say k, then their subspace distance has to be even and is at most $2k$. In the case $d = 2k$ a CDC is also known as a *partial k-spread* and we say that the codewords are pairwise *disjoint*, i.e., they intersect trivially. We speak of a *k-spread* if the cardinality $\left[{n \atop 1}\right]_q / \left[{k \atop 1}\right]_q$ is attained, which is possible if and only if k divides n, see e.g. [1, 37].

If $p(U) = p(U')$, then Equation (2.1) simplifies to $d(U,U') = 2\operatorname{rk}(\tau(U) - \tau(U'))$. More generally, for two matrices $M, M' \in \mathbb{F}_q^{m\times n}$ we define the *rank distance* via $d_{\mathrm{r}}(M,M') = \operatorname{rk}(M - M')$, so that $\left(\mathbb{F}_q^{m\times n}, d_{\mathrm{r}}\right)$ is a metric space. A subset $\mathcal{M} \subseteq \mathbb{F}_q^{m\times n}$ is called a *rank metric code*. More precisely, we speak of an $(m\times n, d_r)_q$-rank metric code, where d_r is the minimum rank distance $d_{\mathrm{r}}(\mathcal{M}) = \min\{d_{\mathrm{r}}(M,M') \;:\; M, M' \in \mathcal{M}, M \neq M'\}$. A rank metric code is called *linear* if it is a subspace of $\mathbb{F}_q^{m\times n}$ over $\mathbb{F}_q$ and *additive* if it is closed under addition. The maximum size of an $(m\times n, d_r)_q$-rank metric code is given by $m(q,m,n,d_{\mathrm{r}}) := q^{\max\{m,n\}\cdot(\min\{m,n\}-d_{\mathrm{r}}+1)}$. A rank metric code $\mathcal{M} \subseteq \mathbb{F}_q^{m\times n}$ attaining this bound is said to be a *maximum rank distance (MRD) code* with parameters $(m\times n, d_{\mathrm{r}})_q$ or $(m\times n, d_r)_q$*–MRD code*, see e.g. the recent survey [38]. Linear MRD codes exist for all parameters. Moreover, for $d_{\mathrm{r}} < d_{\mathrm{r}}'$ we can assume the existence of a linear $(m\times n, d_{\mathrm{r}})_q$–MRD code that contains an $(m\times n, d_{\mathrm{r}}')_q$–MRD code as a subcode. The rank distribution of an additive $(m\times n, d_{\mathrm{r}})_q$–MRD code is completely determined by its parameters, i.e., the number of codewords of rank r is given by

$$a(q,m,n,d_{\mathrm{r}},r) = \left[{\min\{n,m\} \atop r}\right]_q \sum_{s=0}^{r-d_r} (-1)^s q^{\binom{s}{2}} \cdot \left[{r \atop s}\right]_q \cdot \left(q^{\max\{n,m\}\cdot(r-d_r-s+1)} - 1\right) \quad (2.2)$$

for all $d_r \le r \le \min\{n,m\}$, see e.g. [10, Theorem 5.6] or [38, Theorem 5]. Clearly, there is a unique codeword of rank strictly smaller than d_{r} – the zero matrix.

The *Hamming distance* $d_{\mathrm{h}}(u,u') = \#\{i \mid u_i \neq u_i'\}$, for two vectors $u, u' \in \mathbb{F}_2^n$, can be used to lower bound the subspace distance between two subspaces U and U' (not necessarily of the same dimension) of $\mathbb{F}_q^n$:

Lemma 2.1. *[12, Lemma 2] For $U, U' \le \mathbb{F}_q^n$, we have $d(U,U') \ge d_h(p(U), p(U'))$.*

3 Known constructions and bounds for constant–dimension codes

First we note that for bounds on $A_q(n,d;k)$ it suffices to consider the cases $k \le \frac{n}{2}$. Given a non-degenerate bilinear form, we denote by $U^\perp$ the orthogonal subspace of a subspace U, which then has dimension $n-\dim(U)$. With this, we have $d(U,W)=d(U^\perp,W^\perp)$, so that $A_q(n,d;k)=A_q(n,d;n-k)$ and we assume $k\le\frac{n}{2}$ in the following. Since each $(k-\frac{d}{2}+1)$-subspace is contained in at most one codeword, we have $A_q(n,d;k)\le\left[{n \atop k-\frac{d}{2}+1}\right]_q/\left[{k \atop k-\frac{d}{2}+1}\right]_q$, see [41, Theorem 5.2]. For fixed parameters d and k this bound is asymptotically optimal, see [14]. For $d<2k$ the recursive Johnson bound $A_q(n,d;k)\le\left\lfloor\left[{v \atop 1}\right]_q\cdot A_q(n-1,d;k-1)/\left[{k \atop 1}\right]_q\right\rfloor$, see [42], improves upon that. Besides the tightening of this bound, based on divisible codes, see [29, Theorem 5], the only known improvements are $A_2(6,4;3)=77<81$ [26] and $A_2(8,6;4)=257<289$ [21]. For partial spreads all known upper bounds can be derived from the non-existence of projective divisible codes of a certain length and divisibility, see [27]. This includes the exact determination of $A_q(n,2k;k)$ for large k, see [36], as well as several explicit analytical lower bounds, see [32]. For other known, but weaker, upper bounds for CDCs we refer e.g. to the survey in [23].

With respect to the best known constructions, or lower bounds for $A_q(n,d;k)$, the situation is not that overseeable. Here we only mention few general approaches, that give the record codes in the majority of the parameter cases covered in [22], and refer e.g. to the recent survey [28]. Based on Lemma 2.1, in [12] the *Echelon-Ferrers construction* was introduced, see e.g. [39] for refinements. Here different subcodes with diverse pivot vectors are combined according to Lemma 2.1. Considering only the pivot vector $(1,\dots,1,0,\dots 0)$ this contains the so-called lifted MRD (LMRD) codes from [40]. Here, the codewords are of the form $R(I_k|M)$, where I_k denotes the $k\times k$ unit matrix and M is a matrix from an MRD code. This construction yields $A_q(n,d;k)\ge m(q,k,n-k,\frac{d}{2})$. A bit more general, we can consider codewords of the form $R(\tau(U)|M)$, where M is an element of an $(k\times(n-m),\frac{d}{2})_q$-MRD code and U is an element of an $(m,d;k)_q$-CDC. Since this *lifting step* created an $(n-m)$-subspace that is disjoint to all codewords, more codewords can be added. This approach is called the *linkage construction* [16], see also [39], and yields $A_q(n,d;k)\ge A_q(m,d;k)\cdot m(q,k,n-m,\frac{d}{2})+A_q(n-m,d;k)$. We will generalize this approach in Lemma 4.1. Finally, for constructions obtained by using geometrical techniques we refer the interested reader to [5, 6, 7, 8, 9].

4 Constructions based on rank metric codes

In this subsection we aim at constructive lower bounds for $A_q(n,d;k)$ using rank metric codes in different variants. As mentioned before, we assume $2k\le n$.

Lemma 4.1. *For a subspace distance d, let $\bar{n}=(n_1,\dots,n_l)\in\mathbb{N}^l$, where $l\ge 2$, be such that $\sum_{i=1}^l n_i=n$ and $n_i\ge k$ for all $1\le i\le l$. Let $\mathcal{C}_i$ be an $(n_i,\star,d;k)_q$-CDC and $\mathcal{M}_i$ be a $(k\times n_i,\frac{d}{2})_q$-rank metric code for $1\le i\le l$. Then $\mathcal{C}=\bigcup_{i=1}^l\mathcal{C}^i$, where*

$$\mathcal{C}^i=\Big\{R(M_1|\dots|M_{i-1}|\tau(U_i)|M_{i+1}|\dots|M_l)\quad:\quad U_i\in\mathcal{C}_i, M_j\in\mathcal{M}_j,\ \forall 1\le j\le l, i\ne j,$$
$$\textit{and}\ \operatorname{rk}(M_j)\le k-\tfrac{d}{2},\ \forall 1\le j<i\Big\},$$

is an $(n,\star,d;k)_q$*-CDC of cardinality*

$$\#\mathcal{C}=\sum_{i=1}^{l}\left(\prod_{j=1}^{i-1}\#\left\{M\in\mathcal{M}_j\,:\,\operatorname{rk}(M)\le k-\tfrac{d}{2}\right\}\right)\cdot\#\mathcal{C}_i\cdot\left(\prod_{j=i+1}^{l}\#\mathcal{M}_j\right).$$

Proof. Since $\operatorname{rk}(\tau(U_i))=k$ for all $U_i\in\mathcal{C}_i$ the elements of $\mathcal{C}^i$ are k-subspaces of $\mathbb{F}_q^n$ for all $1\le i\le l$; so the elements of $\mathcal{C}$ are k-subspaces of $\mathbb{F}_q^n$.

For the distance analysis let $U\in\mathcal{C}^i$, $U'\in\mathcal{C}^{i'}$ for some indices $1\le i\le i'\le l$. By construction there exist $U_i\in\mathcal{C}_i$ and $M_j\in\mathcal{M}_j$ for $1\le j\le l$, $j\ne i$, with

$$U=R(M_1|\dots|M_{i-1}|\tau(U_i)|M_{i+1}|\dots|M_l)$$

and $\operatorname{rk}(M_j)\le k-\frac{d}{2}$ for all $1\le j<i$. Similarly, there exist $U'_{i'}\in\mathcal{C}_{i'}$ and $M'_j\in\mathcal{M}_j$ for $1\le j\le l$, $j\ne i'$, with $U'=R(M'_1|\dots|M'_{i'-1}|\tau(U'_{i'})|M'_{i'+1}|\dots|M'_l)$ and $\operatorname{rk}(M'_j)\le k-\frac{d}{2}$ for all $1\le j<i'$.

If $i<i'$ we set $\overline{U}=R(\tau(U_i)|M_{i'})$ and $\overline{U}'=R(M'_i|\tau(U'_{i'}))$, which are both k-subspaces of $V\simeq\mathbb{F}_q^{n_i+n_{i'}}$ and satisfy $d(U,U')\ge d(\overline{U},\overline{U}')\ge d$. The later inequality follows from Lemma 2.1 and $d_{\mathrm{h}}(p(\overline{U}),p(\overline{U}'))\ge d$, which is true since $p(\overline{U})$ has its k ones in the first n_i components while $p(\overline{U}')$ has at least $\frac{d}{2}$ of its k ones in the last $n_{i'}$ components.

If $i=i'$ and $U_i\ne U'_i$ we have $d(U,U')\ge d(U_i,U'_i)\ge d$ since $U_i,U'_i\in\mathcal{C}_i$ and $d(\mathcal{C}_i)\ge d$. Now let $i=i'$, $U_i=U'_i$, and $1\le j\le l$ be an index with $M_j\ne M'_j$ and $j\ne i$. For $\overline{U}=R(\tau(U_i)|M_j)$, $\overline{U}'=R(\tau(U_i),M'_j)$, we have $d(U,U')\ge d(\overline{U},\overline{U}')$ and $d(\overline{U},\overline{U}')\ge d$, since $M_j\ne M'_j\in\mathcal{M}_j$ and $d_{\mathrm{r}}(\mathcal{M}_j)\ge\frac{d}{2}$. □

We remark that Lemma 4.1 generalizes [43, Theorem 3], [4, Theorem 3.1], [4, Theorem 4.1], [17, Theorem 1], and [18, Theorem 3].[1] Rank-metric codes of constant rank with a lower bound on the minimum rank-distance have been studied in [15] and generalized in [19, 35]. Here we restrict ourselves on subcodes contained in additive MRD codes.

Corollary 4.2. *For a subspace distance* d, $\bar{n}=(n_1,\dots,n_l)\in\mathbb{N}^l$, $l\ge 2$, *be such that* $\sum_{i=1}^{l}n_i=n$ *and* $n_i\ge k$ *for all* $1\le i\le l$*. Then, we have*

$$A_q(n,d;k)\ge\sum_{i=1}^{l}\left(\prod_{j=1}^{i-1}\left(1+\sum_{r=\frac{d}{2}}^{k-\frac{d}{2}}a(q,k,n_j,\tfrac{d}{2},r)\right)\right)\cdot A_q(n_i,d;k)\cdot\left(\prod_{j=i+1}^{l}m(q,k,n_j,\tfrac{d}{2})\right).$$

Note that $\bar{n}$ also specifies l. While we have no restriction on d in principle, $d>k$ forces $1+\sum_{r=\frac{d}{2}}^{k-\frac{d}{2}}a(q,k,n_j,\frac{d}{2},r)=1$. If we apply Corollary 4.2 with $\bar{n}=(4,4,4)$ and use $A_2(4,4;4)=1$,

[1] We remark that in Lemma 4.1 we can increase the dimension of the ambient space of the CDC $\mathcal{C}_i$ if we further restrict the possible ranks of the elements in $\mathcal{M}_j$ for $1\le j<i$. For details for the special case $l=2$ see [19, Theorem 24], which e.g. allows to also treat the improved linkage construction from [23] in that framework. Also the subsequent results in Lemma 4.3 and Lemma 4.4 can be adjusted to the end of that modification. However, as we are not aware of any specific parameters, where this approach leads to a strict improvement over a known code, we refrain from discussing the details.

we obtain $A_2(12,4;4) \geq 19\,208\,388$. With $\bar{n} = (8,4)$, $A_2(8,4;4) \geq 4801$ [3], and $A_2(4,4;4) = 1$, we obtain

$$A_2(12,4;4) \geq 19\,673\,822. \tag{4.1}$$

Aside from very recent preprints, the previously best known lower bound was $A_2(12,4;4) \geq 19\,664\,917$, obtained from the improved linkage construction [23]. Moreover, the constant–dimension codes from Lemma 4.1 have some special structure that allows to add more codewords.

Lemma 4.3. *With the same notation used in Lemma 4.1, set* $\sigma_i = \sum_{j=1}^{i} n_j$, $1 \leq i \leq l$ *and* $\sigma_0 = 0$. *Let* E_i *denote the* $(n-n_i)$*-subspace of* $\mathbb{F}_q^n$ *consisting of all vectors in* $\mathbb{F}_q^n$ *that have zeroes for the coordinates between* $\sigma_{i-1}+1$ *and* σ_i *for all* $1 \leq i \leq l$. *Then, the elements of* $\mathcal{C}^i$ *are disjoint from* E_i *for all* $1 \leq i \leq l$.

Proof. Let $U \in \mathcal{C}^i$ be arbitrary. By construction there exist $U_i \in \mathcal{C}_i$ and $M_j \in \mathcal{M}_j$ for $1 \leq j \leq l$, $j \neq i$ with $U = R(M)$, where $M = (M_1|\ldots|M_{i-1}|\tau(U_i)|M_{i+1}|\ldots|M_l)$, and $\operatorname{rk}(M_j) \leq k - \frac{d}{2}$ for all $1 \leq j < i$. Note that $E_i = R(N)$ and $\tau(E_i) = N$, where $N \in \mathbb{F}_q^{(n-n_i)\times n}$ is obtained from the unit matrix I_n by deleting the rows in position between $\sigma_{i-1}+1$ and σ_i. Consider a non-trivial linear combination of the k rows of M. The entries in the coordinates between $\sigma_{i-1}+1$ and σ_i are obtained by the same non-trivial linear combination applied to $\tau(U_i)$. Since $\operatorname{rk}(\tau(U_i)) = k$ the statement follows. □

In our next construction we want to combine several CDCs in the same ambient space. In order to express that every codeword from a CDC $\mathcal{C}$ has a subspace distance of at least d to any codeword from another CDC $\mathcal{C}'$ we write $d(\mathcal{C},\mathcal{C}') \geq d$.

Lemma 4.4. *Let* $\mathcal{C}$ *be a subspace code as in Lemma 4.1 with corresponding* $\bar{n} \in \mathbb{N}^l$, $\bar{a} = (a_1,\ldots,a_l) \in \mathbb{N}^l$ *and* $\bar{b} = (b_1,\ldots,b_l) \in \mathbb{N}^l$ *with* $\sum_{i=1}^{l} a_i = k$, $\sum_{i=1}^{l} b_i = k - \frac{d}{2}$, *and* $\frac{d}{2} \leq a_i, b_i < a_i \leq n_i$, *for all* $1 \leq i \leq l$. *For an integer* s, *let* $\mathcal{D}_i^j$ *be an* $(n_i,\star,d;a_i)_q$*-CDC, for all* $1 \leq i \leq l$ *and all* $1 \leq j \leq s$, *such that* $d(\mathcal{D}_i^{j_1},\mathcal{D}_i^{j_2}) \geq 2a_i - 2b_i$, *for all* $1 \leq i \leq l$ *and all* $1 \leq j_1 < j_2 \leq s$. *Then, there exists an* $(n,\star,d;k)_q$*-CDC, say* $\mathcal{D}$, *with cardinality*

$$\#\mathcal{D} = \sum_{j=1}^{s} \prod_{i=1}^{l} \#\mathcal{D}_i^j,$$

such that $\mathcal{C} \cap \mathcal{D} = \emptyset$ *and* $\mathcal{C} \cup \mathcal{D}$ *is also an* $(n,\star,d;k)_q$*-CDC.*

Proof. Let $\sigma_i = \sum_{h=1}^{i} n_h$ for all $1 \leq i \leq l$ and $\sigma_0 = 0$. For each $1 \leq i \leq l$ and $1 \leq j \leq s$ let $\overline{\mathcal{D}}_i^j$ be an embedding of $\mathcal{D}_i^j$ in $\mathbb{F}_q^n$ such that the vectors contained in an element of $\overline{\mathcal{D}}_i^j$ have non-zero entries only in the coordinates between $\sigma_{i-1}+1$ and σ_i. With this we set

$$\mathcal{D} = \bigcup_{j=1}^{s} \left\{ U_1 \times U_2 \times \cdots \times U_l \,:\, U_h \in \overline{\mathcal{D}}_h^j,\ \forall 1 \leq h \leq l \right\},$$

where $U_1 \times U_2 \times \cdots \times U_l$ denotes the smallest subspace that contains $U_1,\ldots,U_l$, i.e., the span of these subspaces. Since $\sum_{i=1}^{l} a_i = k$ and U_{h_1}, U_{h_2}, $1 \leq h_1, h_2 \leq l$, $h_1 \neq h_2$, are disjoint, then the elements of $\mathcal{D}$ are k-subspaces of $\mathbb{F}_q^n$.

Next we want to show that $d(W,U) \geq d$, for all $W \in \mathcal{C}$ and for all $U \in \mathcal{D}$. With the same notation used in Lemma 4.1, there exists an index $1 \leq i_0 \leq l$ with $W \in \mathcal{C}^{i_0}$, so that Lemma 4.3 gives that $\dim(W \cap E_{i_0}) = 0$. Since $\dim(U \cap E_{i_0}) = \sum_{i=1,\, i \neq i_0}^{l} a_i \geq (l-1)\frac{d}{2} \geq \frac{d}{2}$, we have that $\dim(W \cap U) \leq k - \frac{d}{2}$ and $d(W,U) \geq d$.

Now let $U, U' \in \mathcal{D}$, $U \neq U'$, with $U = U_1 \times \cdots \times U_l$ and $U' = U'_1 \times \cdots \times U'_l$ for $U_i \in \overline{\mathcal{D}}_i^j$ and $U'_i \in \overline{\mathcal{D}}_i^{j'}$, where $1 \leq i \leq l$ and $1 \leq j, j' \leq s$. If $j = j'$ then there exists an index $1 \leq i^* \leq l$ with $U_{i^*} \neq U'_{i^*}$. Since $U_{i^*}, U'_{i^*} \in \overline{\mathcal{D}}_{i^*}^j$, we have $d(U_{i^*}, U'_{i^*}) \geq d$, so that $\dim(U_{i^*} \cap U'_{i^*}) \leq a_i - \frac{d}{2}$, which implies $\dim(U \cap U') \leq k - \frac{d}{2}$ and $d(U,U') \geq d$. It remains to consider the case $j \neq j'$, where we have $\dim(U_i \cap U'_i) \leq b_i$ for all $1 \leq i \leq l$. Thus, $\dim(U \cap U') \leq \sum_{i=1}^{l} b_i = k - \frac{d}{2}$ and $d(U,U') \geq d$.

The formula for the cardinality of $\mathcal{D}$ is obvious from the construction. □

Note that $a_i \geq \frac{d}{2}$ and $\sum_{i=1}^{l} a_i = k$ imply $k \geq \frac{ld}{2} \geq d$, i.e., the construction of Lemma 4.4 works for small subspace distances d only.

In what follows we apply Lemma 4.4 in order to achieve a lower bound for $A_q(8,4;4)$ and $A_q(12,4;4)$. In the former case we obtain the best known lower bound if $q > 2$ [8, 13], whereas in the latter case we get an improvement for any q. We need the following definition. A *k-parallelism* of $\mathbb{F}_q^{kt}$ is a set of $\left[\begin{smallmatrix} kt \\ k \end{smallmatrix}\right]_q \cdot \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} kt \\ 1 \end{smallmatrix}\right]_q$ pairwise disjoint k-spreads of $\mathbb{F}_q^{kt}$. For each q a 2-parallelism of $\mathbb{F}_q^4$ exists, see [2, 11].

Let $\bar{n} = (4,4)$, $\bar{a} = (2,2)$ and $\bar{b} = (1,1)$. Lemma 4.1 gives a CDC with $q^{12} + (q^2-1)(q^2+1)^2(q^2+q+1)+1$ codewords. To apply Lemma 4.4 we can choose $\mathcal{D}_i^j$, $i = 1,2$, as a 2-spread of $\mathbb{F}_q^4$ such that $s := q^2+q+1$ and $\{\mathcal{D}_i^j \,:\, 1 \leq j \leq s\}$ is a 2-parallelism of $\mathbb{F}_q^4$, $i = 1,2$. Here the CDC $\mathcal{D}_i^j$ matches the upper bound $A_q(4,4;2) = q^2+1$. It follows that

$$A_q(8,4;4) \geq q^{12} + q^2(q^2+1)^2(q^2+q+1) + 1. \tag{4.2}$$

In order to apply Lemma 4.4 for $A_q(12,4;4)$, we can choose $\bar{n} = (8,4)$, $\bar{a} = (2,2)$, and $\bar{b} = (1,1)$. Similarly to the previous case, we can choose $\mathcal{D}_2^j$ as a 2-spread of $\mathbb{F}_q^4$ such that $s := q^2+q+1$ and $\{\mathcal{D}_2^j \,:\, 1 \leq j \leq s\}$ is a 2-parallelism of $\mathbb{F}_q^4$. We can define $\mathcal{D}_1^j$ as a 2-spread of $\mathbb{F}_q^8$ such that $\{\mathcal{D}_1^j \,:\, 1 \leq j \leq s\}$ is a collection of s pairwise disjoint 2-spreads of $\mathbb{F}_q^8$. In order to do so let $\mathcal{S}$ be a 4-spread of $\mathbb{F}_q^8$, i.e., $\#\mathcal{S} = q^4+1$. For each 4-subspace $S \in \mathcal{S}$ we replace S with a 2-parallelism of S. This results in $\#\mathcal{D} = (q^2+q+1)(q^2+1)^2(q^4+1)$. Taking into account the lower bound for $A_q(8,4;4)$ obtained above, the previous discussion together with Lemma 4.1 and Corollary 4.2, gives rise to a $(12,\star,4;4)_q$-CDC $\mathcal{C}$ with cardinality

$$\begin{aligned}\#\mathcal{C} &= q^{12}\left(q^{12}+q^2(q^2+1)^2(q^2+q+1)+1\right) + (q+1)(q^2+1)^2(q-1)(q^2+q+1)(q^4+1)+1 \\ &= q^{24}+q^{20}+q^{19}+3q^{18}+2q^{17}+3q^{16}+q^{15}+q^{14}+2q^{12}+q^{11}+2q^{10}+q^9+q^8-q^4-q^3-2q^2-q.\end{aligned}$$

For $q \geq 3$ the previously best known lower bound was $A_q(12,4;4) \geq q^{24}+q^{20}+q^{19}+3q^{18}+2q^{17}+3q^{16}+q^{15}+q^{14}+q^{12}+q^{10}+2q^8+2q^6+2q^4+q^2$, see [33, Proposition 4.6]. Something more can be said in the case when $q = 2$, indeed by combining the previous argument together with (4.1) we obtain

$$A_2(12,4;4) \geq 19\,676\,797,$$

which strictly improves upon the corresponding results in [4, 17, 18, 19, 43].

We will further improve the lower bound for $A_q(12,4;4)$, $q \geq 3$, in Section 5.

We remark that for each $1 \leq i \leq l$, the CDC $\bigcup_{j=1}^{s} \mathcal{D}_i^j$ is an $(n_i, \star, 2a_i - 2b_i; a_i)_q$-code. Partitioning it into subcodes with subspace distance $d > 2a_i - 2b_i$ is a hard problem in general and was e.g. considered in the context of the *coset construction* for CDCs, see [24]. If we restrict ourselves to LMRD codes, then one can determine an analytical lower bound.

Corollary 4.5. *In Lemma 4.4 one can achieve*

$$\#\mathcal{D} \geq \min\{\alpha_i \,:\, 1 \leq i \leq l\} \cdot \prod_{i=1}^{l} m\big(q, a_i, n_i - a_i, \tfrac{d}{2}\big)\,,$$

where $\alpha_i = m(q, a_i, n_i - a_i, a_i - b_i) \,/ m\big(q, a_i, n_i - a_i, \frac{d}{2}\big)$.

Proof. For $1 \leq i \leq l$ let $\mathcal{M}_i$ be a linear $(a_i \times (n_i - a_i), a_i - b_i)_q$–MRD code that contains a linear $(a_i \times (n_i - a_i), \frac{d}{2})_q$–MRD code $\overline{\mathcal{M}}_i$ as a subcode. For each $M \in \mathcal{M}_i$ we can consider $M + \overline{\mathcal{M}}_i = \big\{M + \overline{M} \,:\, \overline{M} \in \overline{\mathcal{M}}_i\big\}$. Note that $M + \overline{\mathcal{M}}_i$ is an $(a_i \times (n_i - a_i), \frac{d}{2})_q$–MRD code. Moreover, we have $M + \overline{\mathcal{M}}_i = M' + \overline{\mathcal{M}}_i$ if and only if $M - M' \in \overline{\mathcal{M}}_i$ and $\big(M + \overline{\mathcal{M}}_i\big) \cap \big(M' + \overline{\mathcal{M}}_i\big) = \emptyset$ otherwise. In other words, we consider the α_i cosets of $\overline{\mathcal{M}}_i$ in $\mathcal{M}_i$. With this, $\mathcal{M}_i$ can be partitioned into α_i maximum rank distance codes with parameters $(a_i \times (n_i - a_i), \frac{d}{2})_q$; hence each of them has cardinality $m\big(q, a_i, n_i - a_i, \frac{d}{2}\big)$. By lifting with an $(a_i \times a_i)$-unit matrix we obtain the CDCs $\mathcal{D}_i^j$ with the required properties of Lemma 4.4 for $s = \min\{\alpha_i \,:\, 1 \leq i \leq l\}$. □

In order to obtain a good lower bound for $A_q(n,d;k)$ we can combine Corollary 4.2 with Lemma 4.4 and Corollary 4.5. As an example we consider $A_q(12,6;6)$. For $\bar{n} = (6,6)$ Corollary 4.2 gives a $(12, \star, 6; 6)_q$-CDC $\mathcal{C}$ with cardinality

$$\begin{aligned}\#\mathcal{C} &= q^{24} + (q^2+1)(q^5-1)(q^5+q^4+q^3+q^2+q+1)(q^3+1) \\ &= q^{24}+q^{15}+q^{14}+2q^{13}+3q^{12}+3q^{11}+3q^{10}+2q^9+q^8-q^7-2q^6-3q^5-3q^4-3q^3-2q^2-q-1.\end{aligned}$$

Actually this matches the best known lower bound for $A_q(12,6;6)$ for all field sizes q, see [43, Theorem 3]. By using Lemma 4.4 via Corollary 4.5 with $\bar{a} = (3,3)$ and $\bar{b} = (1,2)$ we get $\#\mathcal{D}_i^j = q^3$, $1 \leq j \leq q^3$, $i = 1,2$, and hence $q^3 \cdot q^3 \cdot q^3 = q^9$ additional codewords. In some cases, the CDC so obtained can be enlarged. Indeed, let E_1 or E_2 denote the 6-subspace of $\mathbb{F}_q^{12}$ consisting of all vectors in $\mathbb{F}_q^{12}$ that have zeroes in the first or in the last six coordinates, respectively. Let $\overline{\mathcal{D}}_i^j$ be an embedding of $\mathcal{D}_i^j$ in $\mathbb{F}_q^{12}$ such that the vectors contained in an element of $\overline{\mathcal{D}}_i^j$ are in E_i, $i = 1,2$. Note that, by construction, there exists a special 3–subspace of E_i, say ξ_i, such that every member of $\overline{\mathcal{D}}_i^j$ is disjoint from ξ_i, $i = 1,2$. Let $\mathcal{F}_1$ be the set consisting of the 6-subspaces of $\mathbb{F}_q^{12}$ spanned by ξ_1 and a member of $\overline{\mathcal{D}}_2^1$. Similarly, let $\mathcal{F}_2$ be the set consisting of the 6-subspaces of $\mathbb{F}_q^{12}$ spanned by ξ_2 and a member of $\overline{\mathcal{D}}_1^1$. Then it can be easily checked that the CDC constructed above can be enlarged by adding the $2q^3$ codewords of $\mathcal{F}_1 \cup \mathcal{F}_2$. This leads to

$$A_q(12,6;6) \geq q^{24}+q^{15}+q^{14}+2q^{13}+3q^{12}+3q^{11}+3q^{10}+3q^9+q^8-q^7-2q^6-3q^5-3q^4-q^3-2q^2-q-1.$$

Let $k \geq 4$ be a positive even integer and consider a CDC with parameters $(4k, 2k; 2k)_q$ obtained from Corollary 4.2 with $\bar{n} = (2k, 2k)$, Corollary 4.5 with $\bar{a} = (k, k)$, $\bar{b} = \left(\frac{k}{2}, \frac{k}{2}\right)$. A similar argument to that used in the previous paragraph shows that the CDC so obtained can be enlarged by adding a further $2q^k$ codewords. Then

$$\begin{aligned} A_q(4k, 2k; 2k) &\geq m(q, 2k, 2k, k) + a(q, 2k, 2k, k, k) + m(q, k, k, k) \cdot m(q, k, k, \tfrac{k}{2}) + 2q^k + 1 \\ &= q^{2k(k+1)} + a(q, 2k, 2k, k, k) + q^{k(k/2+2)} + 2q^k + 1. \end{aligned} \tag{4.3}$$

4.1 On constant–dimension codes with $d > k$

The drawback of the construction of Lemma 4.4 is that it is applicable for $d \leq k$ only. This is due to the "product-type" constructions where the elements of two (or more) codes are combined in all different ways. If we use a one-to-one correspondence for the combinations we can construct CDCs for $d > k$:

Lemma 4.6. *Let $\bar{n} = (n_1, n_2) \in \mathbb{N}^2$ with $n_1 + n_2 = n$, $\bar{a} = (a_1, a_2) \in \mathbb{N}^2$ with $a_1 + a_2 = k$, $a_1 \leq k - \frac{d}{2}$, and $\bar{b} = (b_1, b_2) \in \mathbb{N}^2$ with $b_1 + b_2 = k - \frac{d}{2}$. Let $\mathcal{C}_0$ be an $(n_1, \star, d; k)_q$-CDC, $\mathcal{C}_1$ be an $(n_1, \star, 2a_1 - 2b_1; a_1)_q$-CDC, and $\mathcal{C}_2$ be an $(n_2, \star, 2a_2 - 2b_2; a_2)_q$-CDC. Then there exists an $(n, \star, d; k)_q$-CDC with cardinality $\#C_0 \cdot m(q, k, n_2, \frac{d}{2}) + \min\{\#\mathcal{C}_1, \#\mathcal{C}_2\}$.*

Proof. W.l.o.g. we assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same cardinality. Let $\overline{\mathcal{C}}_1$ be an embedding of $\mathcal{C}_1$ in $\mathbb{F}_q^n$ such that the last n_2 entries of the vectors contained in the codewords are always zero. Similarly, let $\overline{\mathcal{C}}_2$ be an embedding of $\mathcal{C}_2$ in $\mathbb{F}_q^n$ such that the first n_1 entries of the vectors contained in the codewords are always zero. We choose an arbitrary numbering $U_1^1, \ldots, U_1^s$ and $U_2^1, \ldots, U_2^s$ of the elements of $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$, respectively, where $s = \#\mathcal{C}_1 = \#\mathcal{C}_2$. Let $\mathcal{M}_0$ be a $(k \times n_2, \frac{d}{2})_q$–MRD code. With this we set

$$\mathcal{C} = \{R(\tau(U)|M) \,:\, U \in \mathcal{C}_0, M \in \mathcal{M}_0\} \cup \left\{U_1^i \times U_2^i \,:\, 1 \leq i \leq s\right\}.$$

Obviously the elements of $\mathcal{C}$ are k-subspaces of $\mathbb{F}_q^n$ and we have $\#\mathcal{C} = \#C_0 \cdot m(q, k, n_2, \frac{d}{2}) + \min\{\#\mathcal{C}_1, \#\mathcal{C}_2\}$.

For the distance analysis let $W, W' \in \mathcal{C}$ be arbitrary. If W and W' are both of the form $R(\tau(U)|M)$, then $d(W, W') \geq d$ follows as in the proof of Lemma 4.1 (or as in the literature on lifting and the linkage construction). If only W is of the first type, then $p(W)$ contains its k ones in the first n_1 coordinates, while $p(W')$ contain only a_1 of its k ones in the first coordinates. So using $a_1 \leq k - \frac{d}{2}$ Lemma 2.1 gives $d(W, W') \geq |k - a_1| + |a_2| = 2(k - a_1) \geq d$. If $W = U_1^i \times U_2^i$ and $W' = U_1^{i'} \times U_2^{i'}$ for $1 \leq i < i' \leq s$, then $\dim(W \cap W') = \dim(U_1^i \cap U_1^{i'}) + \dim(U_2^i \cap U_2^{i'}) \leq b_1 + b_2 = k - \frac{d}{2}$, so that $d(W, W') \geq d$. □

Corollary 4.7. *Let $\bar{n} = (n_1, n_2) \in \mathbb{N}^2$ with $n_1 + n_2 = n$, $\bar{a} = (a_1, a_2) \in \mathbb{N}^2$ with $a_1 + a_2 = k$, $a_1 \leq k - \frac{d}{2}$, and $\bar{b} = (b_1, b_2) \in \mathbb{N}^2$ with $b_1 + b_2 = k - \frac{d}{2}$. Then*

$$A_q(n, d; k) \geq A_q(n_1, d; k) \cdot m(q, k, n_1, \tfrac{d}{2}) + \min\{A_q(n_1, 2a_1 - 2b_1; a_1), A_q(n_2, 2a_2 - 2b_2; a_2)\}.$$

As an example we consider a lower bound for $A_q(12,8;6)$ and apply Corollary 4.7 with $\bar{n}=(6,6)$, $\bar{a}=(2,4)$, and $\bar{b}=(0,2)$. Since $A_q(6,8;6)=1$ and $A_q(6,4;2)=A_q(6,4;4)=q^4+q^2+1$ we have $A_q(12,8;6)\ge q^{18}+q^4+q^2+1$. The same lower bound can also be obtained using the optimal code within the class of Echelon-Ferrers constructions, see [12, 22]. Within the $(12,\star,8;6)_q$-CDCs that contain an LMRD the construction is indeed optimal. For $A_2(16,12;8)$ we can apply Corollary 4.7 with $\bar{n}=(8,8)$, $\bar{a}=(2,6)$, and $\bar{b}=(0,2)$.

Another possible approach is to start with the same lifting $\{R(\tau(U)|M)\,:\,U\in\mathcal{C}_0, M\in\mathcal{M}_0\}$ where $\bar{n}=(n_1,n_2)$. As observed in [33] we can add all codewords from an $(n,\star,d;k)_q$ code $\mathcal{C}'$, without decreasing the minimum subspace distance, if all elements of $\mathcal{C}'$ intersect an arbitrary but fixed n_2-subspace S in dimension at least $\frac{d}{2}$ (as it is the case in Lemma 4.6). We can start with an $(n_2,\star,2d-2k;\frac{d}{2})_q$-CDC and then step by step enlarge the dimension of the codewords without creating intersections of codewords with a dimension strictly larger than $k-\frac{d}{2}$. For the special case $d=2k-2$ it was shown in [33, Theorem 4.2] that $\#\mathcal{C}=A_q(n_2,2d-2k;\frac{d}{2})$ can indeed be attained. Whether this is possible for $d\le 2k-4$ is an interesting open problem. As a stimulation for further research in this direction we pose an explicit open problem:

Open Problem 4.8. *Do there exist* $\left[{5 \atop 3}\right]_q=\left[{5 \atop 2}\right]_q=(q^4+q^3+q^2+q+1)(q^2+1)$ 5*-subspaces of* $\mathbb{F}_q^{10}$ *pairwise intersecting in dimension at most* 2 *such that all elements intersect a special fixed* 5*-subspace in dimension* 3*?*

If true this would improve the best know lower bound for $A_q(10,6;5)$ for $q\ge 3$ and indeed match the upper bound within the class of such codes containing an LMRD subcode, see [13]. For $q=2$ such a code was found by computer search, see [20].

5 Duplicating CDCs in several subspaces of a large-dimensional CDC

In [5] the authors combined several $(6,\star,4;3)_q$-CDCs to show $A_q(9,4;3)\ge q^{12}+2q^8+2q^7+q^6+q^5+q^4+1$, which improves the previously best known lower bound $A_q(9,4;3)\ge q^{12}+2q^8+2q^7+q^6+1$ obtained from the improved linkage construction, see [23]. In [34] the bound was further improved to $A_q(9,4;3)\ge q^{12}+2q^8+2q^7+q^6+2q^5+2q^4-2q^2-2q+1$. Here we want to generalize the approach of [5, 34] and apply it to a much wider range of parameters.

Definition 5.1. *An* (n,d,k)-sequence *of CDCs is a list* $(\mathcal{D}_0,\dots,\mathcal{D}_r)$ *of* $(n,\star,d;k)_q$*-CDCs such that for each index* $0\le i\le r$ *there exists a codeword* $U\in\mathcal{D}_i$ *and a disjoint* $(n-k)$*-subspace* S *such that* $\dim(U'\cap S)\le i$ *for all* $U'\in\mathcal{D}_i$*, where* $r=k-\frac{d}{2}$.

We remark that an LMRD code gives an example for $\mathcal{D}_0$ and for $\mathcal{D}_i$, with $i\ge 1$, we can take $\mathcal{D}_0$. Another possibility is to start with an arbitrary $(n,\star,d;k)_q$-CDC, pick the special subspace S, and remove all codewords whose dimension of the intersection with S is too large.

Definition 5.2. *A list* $(\mathcal{C}_0,\dots,\mathcal{C}_r)$ *is called a* distance-partition *of an* $(n,\star,d;k)_q$*-CDC* $\mathcal{C}$*, where* $r=k-\frac{d}{2}$*, if* $\mathcal{C}_0,\dots,\mathcal{C}_r$ *is a partition of* $\mathcal{C}$ *and* $\bigcup_{j=0}^{i}\mathcal{C}_j$ *is an* $(n,\star,2k-2i;k)_q$*-CDC for all* $0\le i\le r$*.*

A trivial distance-partition of an $(n, \star, d; k)_q$-CDC $\mathcal{C}$ is given by $(\emptyset, \ldots, \emptyset, \mathcal{C})$. A subcode $\mathcal{C}' \subseteq \mathcal{C}$ with maximal subspace distance $d = 2k$ is called a *partial-spread subcode*. Given such a partial-spread subcode $\mathcal{C}'$, if $d < 2k$, then $(\mathcal{C}', \emptyset, \ldots, \emptyset, \mathcal{C}\backslash\mathcal{C}')$ is a distance-partition of $\mathcal{C}$.

Lemma 5.3. *Let $(\mathcal{C}_0, \ldots, \mathcal{C}_r)$ be a distance-partition of a $(k+t, \star, d; k)_q$-CDC $\mathcal{C}$ and $(\mathcal{D}_0, \ldots, \mathcal{D}_r)$ be a $(k+s, d, k)$-sequence, where $r = k - \frac{d}{2}$. If $\mathcal{A}$ is an $(s, \star, d; k)_q$-CDC, then there exists a $(k+s+t, \star, d; k)_q$-CDC $\mathcal{C}'$ with cardinality*

$$\#\mathcal{C}' = \#\mathcal{A} + \sum_{i=0}^{r} \#\mathcal{C}_i \cdot \#\mathcal{D}_{r-i}.$$

Proof. In order to build up $\mathcal{C}'$ step by step we embed $\mathcal{C}$ in a $(k+t)$-subspace of $\mathbb{F}_q^{k+s+t}$ and let S be an s-subspace of $\mathbb{F}_q^{k+s+t}$ disjoint from it. For each codeword $U \in \mathcal{C}$ let $0 \le i \le r$, be the index such that $U \in \mathcal{C}_i$. With this, we embed an isomorphic copy $\mathcal{C}_U$ of $\mathcal{D}_{r-i}$ in the $(k+s)$-subspace $\langle U, S\rangle$ such that U is a codeword, S the special subspace, and add all those codewords to $\mathcal{C}'$. Let $\overline{\mathcal{A}}$ denote the CDC obtained by embedding the codewords of $\mathcal{A}$ in S. As a last step add the codewords of $\overline{\mathcal{A}}$ to $\mathcal{C}'$. Such a procedure gives rise to a $(k+s+t, \star, ?; k)_q$-CDC $\mathcal{C}'$ with the stated cardinality. It remains to check the minimum subspace distance.

For $W, W' \in \mathcal{C}'\backslash\overline{\mathcal{A}}$ there exist unique $U, U' \in \mathcal{C}$ such that $W \in \mathcal{C}_U$ and $W' \in \mathcal{C}_{U'}$. Moreover, there exist unique indices $0 \le i, i' \le r$ with $U \in \mathcal{C}_i$ and $U' \in \mathcal{C}_{i'}$. If $i = i'$ and $W \neq W'$, then $d(W, W') \ge d(\mathcal{D}_{r-i}) \ge d$. If $i \neq i'$, then w.l.o.g. we assume $i' < i$, so that $\dim(U \cap U') \le i$. By construction we have $\dim(W \cap W') \le \dim(U \cap U') + \dim(W \cap W' \cap S) \le i + \dim(W \cap S) \le r$, so that $d(W, W') \ge d$. If W, W' are both contained in $\overline{\mathcal{A}}$, then clearly $d(W, W') \ge d(\mathcal{A}) \ge d$. Finally, if $W \in \overline{\mathcal{A}}$ and $W' \in \mathcal{C}'\backslash\overline{\mathcal{A}}$, then $\dim(W' \cap S) \le k - \frac{d}{2}$ and hence $d(W, W') \ge d$. □

Let us briefly mention how Lemma 5.3 can be used in order to obtain the best known lower bound for $A_q(9, 4; 3)$ [5, 34] and $A_q(10, 4; 3)$,[34]. First let $(\mathcal{D}_0, \mathcal{D}_1)$ be a $(6, 4, 3)$-sequence. Here $\mathcal{D}_0$ is an LMRD code of cardinality q^6, and $\mathcal{D}_1$ is a $(6, \star, 4; 3)_q$-CDC with cardinality $q^6 + 2q^2 + 2q$, where we have removed one codeword from a pair of disjoint codewords, see [6, 26] for constructions of CDCs with cardinality $q^6 + 2q^2 + 2q + 1$.

As regards $A_q(9, 4; 3)$, we take as code $\mathcal{C}$ a $(6, \star, 4; 3)_q$-CDC with cardinality $q^6 + 2q^2 + 2q + 1$, see [6, 26]. In order to determine a distance-partition $(\mathcal{C}_0, \mathcal{C}_1)$ of $\mathcal{C}$, we need to find a large partial-spread subcode of $\mathcal{C}$. In [5, Theorem 3.12], it is shown that we can choose $\mathcal{C}_0$ of cardinality $q^3 - 1$ if we choose $\mathcal{C}$ from [6]. However, as shown in [34], the same can be done if we choose $\mathcal{C}$ from [26]. This can be made more precise in the language of linearized polynomials. For [26, Lemma 12, Example 4] the representation $\mathbb{F}_q^6 \cong \mathbb{F}_{q^3} \times \mathbb{F}_{q^3}$ is used and the planes removed from the lifted MRD code correspond to $ux^q - u^q x$ for $u \in \mathbb{F}_{q^3}$, so that the monomials ax for $a \in \mathbb{F}_{q^3}\backslash\{\mathbf{0}\}$ correspond to a partial-spread subcode of cardinality $q^3 - 1$. As subcode $\mathcal{A}$ we choose a single 3-space, so that we obtain

$$\begin{aligned} A_q(9, 4; 3) &\ge 1 + \#\mathcal{C}_0 \cdot \#\mathcal{D}_1 + \#\mathcal{C}_1 \cdot \#\mathcal{D}_0 \\ &= 1 + \left(q^3 - 1\right) \cdot \left(q^6 + 2q^2 + 2q\right) + \left(q^6 - q^3 + 2q^2 + 2q + 2\right) \cdot q^6 \\ &= q^{12} + 2q^8 + 2q^7 + q^6 + 2q^5 + 2q^4 - 2q^2 - 2q + 1. \end{aligned}$$

For $A_q(10,4;3)$ we let $\mathcal{C}$ be the $(7,\star,4;3)_q$-CDC of cardinality $q^8+q^5+q^4+q^2-q$ constructed in [25, Theorem 4]. Again we need to find a large partial-spread subcode $\mathcal{C}_0$ of $\mathcal{C}$. Here $\#\mathcal{C}_0 = q^4$ can be achieved, see [34]. Thus, we obtain

$$\begin{aligned} A_q(10,4;3) &\geq 1 + \#\mathcal{C}_0 \cdot \#\mathcal{D}_1 + \#\mathcal{C}_1 \cdot \#\mathcal{D}_0 \\ &= 1 + q^4 \cdot \left(q^6 + 2q^2 + 2q\right) + \left(q^8 + q^5 + q^2 - q\right) \cdot q^6 \\ &= q^{14} + q^{11} + q^{10} + q^8 - q^7 + 2q^6 + 2q^5 + 1. \end{aligned}$$

The determination of a large partial-spread subcode is mostly the hardest part in the analytic evaluation of the construction of Lemma 5.3. However, if $\mathcal{C}$ contains an $(n,\star,d;k)$-CDC that is an LMRD as a subcode, then it contains an $(n, m(q,k,n-k,k), 2k; k)$-CDC that is again an LMRD, i.e., a partial-spread subcode.

Theorem 5.4. $A_q(12,4;4) \geq q^{24}+q^{20}+q^{19}+3q^{18}+2q^{17}+3q^{16}+q^{15}+q^{14}+2q^{12}+q^{11}+3q^{10}+2q^9+4q^8+2q^7+4q^6+2q^5+3q^4+q^3+q^2+1$.

Proof. In order to apply Lemma 5.3, let $(\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2)$ be an $(8,4,4)$-sequence and let $\mathcal{C}$ be the $(8,\star,4;4)_q$-CDC with cardinality $q^{12} + q^2(q^2+1)^2(q^2+q+1)+1$ described in Section 4 and yielding (4.2). As $\mathcal{C}$ contains an LMRD subcode and a disjoint codeword we can choose a partial-spread subcode $\mathcal{C}_0$ of cardinality q^4+1. As distance-partition we use $(\mathcal{C}_0, \emptyset, \mathcal{C}\backslash\mathcal{C}_0)$. For $\mathcal{D}_0$ and $\mathcal{D}_1$ we choose an LMRD code of cardinality q^{12}. As $\mathcal{D}_2$ we choose the code obtained from $\mathcal{C}$ by removing one codeword from a pair of disjoint codewords. As regards $\mathcal{A}$, we choose a single 4-subspace. Thus, we obtain

$$\begin{aligned} A_q(12,4;4) &\geq \#\mathcal{A} + \#\mathcal{C}_0 \cdot \#\mathcal{D}_2 + \#\mathcal{C}_1 \cdot \#\mathcal{D}_1 + \#\mathcal{C}_2 \cdot \#\mathcal{D}_0 \\ &= 1 + \left(q^4+1\right) \cdot \left(q^{12} + q^2(q^2+1)^2(q^2+q+1)\right) \\ &\quad + \left(q^{12} + q^2(q^2+1)^2(q^2+q+1) - q^4\right) \cdot q^{12} \\ &= q^{24} + q^{20} + q^{19} + 3q^{18} + 2q^{17} + 3q^{16} + q^{15} + q^{14} + 2q^{12} + q^{11} \\ &\quad + 3q^{10} + 2q^9 + 4q^8 + 2q^7 + 4q^6 + 2q^5 + 3q^4 + q^3 + q^2 + 1. \end{aligned}$$

□

This construction improves upon the recent improvement of [33, Proposition 4.6] for $A_q(12,4;4)$. The approach of the previous theorem is rather general and universal since many of the largest known CDCs contain an LMRD as a subcode. As an example we consider the $(2k,\star,4;k)_q$-CDCs from [8].

Theorem 5.5. *For a positive integer $k \geq 5$, let $\mathcal{C}$ be the $(2k,\Lambda,4;k)_q$-CDC from [8, Theorem 3.8] or [8, Theorem 3.11], depending on whether k is even or odd. Then $A_q(3k,4;k) \geq 1 + \left(q^k+1\right)\cdot(\Lambda-1) + \left(\Lambda - q^k - 1\right)\cdot q^{k(k-1)}$.*

Proof. We apply Lemma 5.3 where $(\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2)$ is a $(2k,4,k)$-sequence and $\mathcal{C}$ is the $(2k,\Lambda,4;k)_q$-CDC from [8]. As $\mathcal{C}$ contains an LMRD subcode and a disjoint codeword we can choose a partial-spread subcode $\mathcal{C}_0$ of cardinality q^k+1. As distance-partition we use $(\mathcal{C}_0, \emptyset, \mathcal{C}\backslash\mathcal{C}_0)$. For $\mathcal{D}_0$ and $\mathcal{D}_1$

we choose an LMRD code of cardinality $q^{k(k-1)}$. As $\mathcal{D}_2$ we choose the code obtained from $\mathcal{C}$ by removing one codeword from a pair of disjoint codewords. As $\mathcal{A}$ we choose a single k-subspace. Thus, we obtain

$$\begin{aligned} A_q(3k,4;k) &\geq \#\mathcal{A} + \#\mathcal{C}_0 \cdot \#\mathcal{D}_2 + \#\mathcal{C}_1 \cdot \#\mathcal{D}_1 + \#\mathcal{C}_2 \cdot \#\mathcal{D}_0 \\ &= 1 + \left(q^k + 1\right) \cdot (\Lambda - 1) + \left(\Lambda - q^k - 1\right) \cdot q^{k(k-1)}. \end{aligned}$$

□

The construction described in Lemma 5.3 can be applied recursively, as we are going to see in the next lines for $A_q(16,4;4)$. Let $\mathcal{C}'$ be the $(12,\star,4;4)$-CDC code yielding the lower bound of $A_q(12,4;4)$ of Theorem 5.4. In order to find a partial-spread subcode of $\mathcal{C}'$, we remark that $\mathcal{C}_0$ is a partial-spread subcode of $\mathcal{C}$ of cardinality q^4+1. Thus, for each codeword of $\mathcal{C}_0$, via $\mathcal{D}_2$, we can select q^4 codewords in $\mathcal{C}'$ that are pairwise disjoint. By adding the elements of $\mathcal{A}$, we end up with a partial-spread subcode $\mathcal{C}_0'$ of $\mathcal{C}'$ of cardinality q^8+q^4+1. By choosing $\mathcal{A}$ and the $(8,4,4)$-sequence $(\mathcal{D}_0,\mathcal{D}_1,\mathcal{D}_2)$ as in the proof of Theorem 5.4, and by using the distance-partition $(\mathcal{C}_0', \emptyset, \mathcal{C}'\backslash\mathcal{C}_0')$, Lemma 5.3 gives

$$\begin{aligned} A_q(16,4;4) &\geq 1 + \left(q^8+q^4+1\right) \cdot \#\mathcal{D}_2 + (\#\mathcal{C}' - q^8 - q^4 - 1) \cdot q^{12} \\ &= q^{36} + q^{32} + q^{31} + 3q^{30} + 2q^{29} + 3q^{28} + q^{27} + q^{26} + 2q^{24} + q^{23} + 3q^{22} + 2q^{21} \\ &\quad +4q^{20} + 2q^{19} + 4q^{18} + 2q^{17} + 4q^{16} + 2q^{15} + 4q^{14} + 2q^{13} + 5q^{12} + 2q^{11} + 4q^{10} \\ &\quad +2q^9 + 4q^8 + 2q^7 + 4q^6 + 2q^5 + 3q^4 + q^3 + q^2 + 1, \end{aligned}$$

where $\#\mathcal{C}' = q^{24} + q^{20} + q^{19} + 3q^{18} + 2q^{17} + 3q^{16} + q^{15} + q^{14} + 2q^{12} + q^{11} + 3q^{10} + 2q^9 + 4q^8 + 2q^7 + 4q^6 + 2q^5 + 3q^4 + q^3 + q^2 + 1$ and $\#\mathcal{D}_2 = q^{12} + q^2(q^2+1)^2(q^2+q+1)$.

The next result considers the case of Lemma 5.3 when both $\mathcal{D}_r$ and $\mathcal{A}$ contain a partial-spread subcode.

Lemma 5.6. *Let $\mathcal{C}$ be a CDC obtained from the construction of Lemma 5.3 with a distance-partition $(\mathcal{C}_0,\dots,\mathcal{C}_r)$, a $(k+s,d,k)$-sequence $(\mathcal{D}_0,\dots,\mathcal{D}_r)$, and a CDC $\mathcal{A}$. If $\mathcal{D}_r$ contains a partial-spread subcode $\mathcal{P}$ and $\mathcal{A}$ contains a partial-spread subcode $\mathcal{P}'$, then $\mathcal{C}$ contains a partial-spread subcode of cardinality $\#\mathcal{C}_0 \cdot \#\mathcal{P} + \#\mathcal{P}'$.*

Of course we can also apply the construction of Lemma 5.3 on the CDCs constructed in Section 4. We do this exemplarily for the codes yielding improved lower bounds for $A_q(4k,2k;2k)$ to obtain a lower bound for $A_q(6k,2k;2k)$.

Theorem 5.7.

$$\begin{aligned} A_q(6k,2k;2k) &\geq 1 + \left(q^{2k}+1\right) \cdot \left(q^{2k(k+1)} + a(q,2k,2k,k,k) + q^{k(k/2+2)} + 2q^k\right) \\ &\quad + \left(q^{2k(k+1)} + a(q,2k,2k,k,k) + q^{k(k/2+2)} + 2q^k - q^{2k}\right) \cdot q^{2k(k+1)} \end{aligned}$$

for even $k \geq 4$.

Proof. Let $k \geq 4$ be a positive even integer. In order to apply Lemma 5.3 let $(\mathcal{D}_0, \ldots, \mathcal{D}_k)$ be a $(4k, 4k, 4k)$-sequence and let $\mathcal{C}$ be the $(4k, 2k; 2k)_q$-CDC described in Section 4 and yielding (4.3). Hence

$$\#\mathcal{C} = q^{2k(k+1)} + a(q, 2k, 2k, k, k) + q^{k(k/2+2)} + 2q^k + 1.$$

As $\mathcal{C}$ contains an LRMD subcode and a disjoint codeword we can choose a partial-spread subcode $\mathcal{C}_0$ of cardinality $q^{2k}+1$. As distance-partition we use $(\mathcal{C}_0, \emptyset, \ldots, \emptyset, \mathcal{C}\backslash\mathcal{C}_0)$. For $\mathcal{D}_0, \ldots, \mathcal{D}_{k-1}$ we choose an LMRD code of cardinality $q^{2k(k+1)}$. As $\mathcal{D}_k$ we choose the code $\mathcal{C}$ from above removing one codeword from a pair of disjoint codewords. As $\mathcal{A}$ we choose a single $2k$-subspace. Thus, we obtain

$$\begin{aligned} A_q(6k, 2k; 2k) &\geq \#\mathcal{A} + \#\mathcal{C}_0 \cdot \#\mathcal{D}_k + \#\mathcal{C}_k \cdot \#\mathcal{D}_0 \\ &= 1 + \left(q^{2k} + 1\right) \cdot \left(q^{2k(k+1)} + a(q, 2k, 2k, k, k) + q^{k(k/2+2)} + 2q^k\right) \\ &\quad + \left(q^{2k(k+1)} + a(q, 2k, 2k, k, k) + q^{k(k/2+2)} + 2q^k - q^{2k}\right) \cdot q^{2k(k+1)}. \end{aligned}$$

□

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