Asset Pricing with Loss Aversion*

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Abstract

Using standard preferences for asset pricing has not been very successful in matching asset price characteristics such as the risk-free interest rate, equity premium and the Sharpe ratio to time series data. Behavioral finance has recently proposed more realistic preferences such as those with loss aversion. Research is starting to explore the implications of behaviorally founded preferences for asset price characteristics. Encouraged by some studies of Benartzki and Thaler (1995) and Barberis et al. (2001) we study asset pricing with loss aversion in a production economy. Here, we employ a stochastic growth model and use a stochastic version of a dynamic programming method with adaptive grid scheme to compute the above mentioned asset price characteristics of a model with loss aversion in preferences. As our results show using loss aversion we get considerably better results than one obtains from pure consumption-based asset pricing models including the habit formation variant.

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1 Introduction

Consumption based asset pricing models with time separable preferences, such as power utility, have been shown to encounter serious difficulties in matching financial market characteristics such as the risk-free interest rate, the equity premium and the Sharpe ratio with time series data. In those models, even if the coefficient of relative risk aversion in the power utility function, is significantly raised, neither the risk free rate nor the mean equity premium and Sharpe ratio fit the observed data. The former is usually too high and the latter two are much too low in the model when compared to the data.

One important concern has been that asset pricing models have often used models with exogenous dividend streams.[1] The difficulties of matching stylized financial statistics may have come from the fact that consumption was not endogenized. There is a tradition in asset pricing that is based on the stochastic growth model which endogenizes consumption, see Brock and Mirman (1972) and Brock (1979, 1982). The Brock approach extends the asset pricing strategy beyond endowment economies to economies that have endogenous state variables including capital stocks that are used in production. Authors, building on this tradition,[2] have argued that how consumption is endogenized is crucial. In stochastic growth models the randomness occurs to the production function of firms and consumption and dividends are derived endogenously. Yet, models with production have turned out to be even less successful. Given a production shock, consumption can be smoothed through savings and, thus, asset market features are even harder to match.[3]

Recent developments in asset pricing have focused attention on extensions of intertemporal models, conjecturing that the difficulties in matching real and financial time series characteristics may be related to the simple structure of the basic model. In order to better match the asset price characteristics of the model to the data, economic research has explored numerous extensions of the baseline stochastic growth model.[4] An enormous effort has been invested into models with time non-separable preferences[5], such as habit formation

[1]Those models originate in Lucas (1978) and Breeden (1979) for example.

[2]See Rouwenhorst (1995, Akdeniz and Dechert (1997), Jerman (1998), Boldrin, Christiano and Fisher (2001), Lettau and Uhlig (1999) and Hansen and Sargent (2002), the latter in a linear-quadratic economy. The Brock model has also been used to evaluate the effect of corporate income tax on asset prices, see McGrattan and Prescott (2001).

[3]For a recent account of the gap between such models and facts, see Boldrin, Christiano and Fisher (2001), Cochrane (2001, ch. 21), Lettau and Gong and Semmler (2001).

[4]For detailed studies of those extensions see, for example, Campbell and Cochrane (2000), Jerman (1998), Boldrin, Christiano and Fisher (2001) and Cochrane (2001, ch. 21).

[5]For an extensive exploration of the role of preferences for asset pricing, see Backus et al.

models, which allow for adjacent complementarity in consumption. This type of habit specification gives rise to time non-separable preferences and time varying risk aversion. The risk aversion falls with rising surplus consumption and the reverse holds for falling surplus consumption. A high volatility in the surplus consumption will lead to a high volatility in the growth of marginal utility and thus to a high volatility in the stochastic discount factor.

Such habit persistence was introduced in asset pricing models by Constantinides (1990) in order to account for high equity premia. Asset pricing models along this line have been further explored by Campbell and Cochrane (2000), Jerman (1998) and Boldrin et al. (2001). As the literature has demonstrated [6] one needs not only habit formation but also adjustment costs of investment in order to reduce the elasticity of the supply of capital to generate higher equity premium and Sharpe ratio. Yet, a habit formation models can only slightly improve the equity premium and Sharpe ratio. It also does not generate enough co-variance of consumption growth with asset returns so as to match the data.[7]

Current research has focused on prospect theory which moves away from consumption based asset pricing models. The new strategy is to look for the impact of wealth fluctuation on the households' welfare. Here then decision on a portfolio is impacted by both preferences over a consumption stream as well as by changes in financial wealth. In the preferences there will be thus an extra term representing the change of wealth. Furthermore, as prospect theory has taught us, an investor may be much more sensitive to losses than to gains. This is known as loss aversion. Loss aversion, in particular, seems to hold if there have been prior losses already. By extending the asset pricing model in this direction one does not need to raise the co-variance of consumption growth and asset returns, a feature not found in the data.[8]

Not only low variance of consumption growth, but a higher mean and volatility of asset returns, might be achieved by a time varying risk aversion arising from the fluctuation of asset value. The idea is that after an asset price boom the agents may become less risk averse because their gains may dominate any fear of losses. On the other hand, after an asset price fall the agents become more cautious and more risk averse. This way, the variation of risk aversion would allow the asset returns to be more volatile than the underlying pay-offs, the dividend payments, a property that Shiller (1991) has studied extensively. Generous dividend payments and an asset price boom makes the investor less risk averse and drives the asset price still higher.

(2004).

[6]See Jerman (1998), and Boldrin et al. (2001)

[7]For details, see Grüne and Semmler (2006).

[8]See Semmler (2003, ch. 9)

The reverse can be predicted to happen if large losses occur. This may give rise to some waves of optimism and pessimism and associated asset price movements.

Habit formation models attempt to increase the equity premium and Sharpe ratio by constructing a time varying risk aversion arising from the change of consumption. Loss aversion models do not rely on surplus consumption, as in the habit formation model, but rather on the fluctuation in asset value affecting the stochastic discount factor. This is likely to produce a substantial equity premium and Sharpe ratio, high volatility of returns, yet it allows for a low co-variance of the growth rate of consumption and asset returns.

The basic idea of loss aversion as developed in the context of prospect theory goes back to Kahneman and Tversky (1979) and Tversky and Kahneman (1992). It was further developed for applications in asset pricing by Benartzi and Thaler (1995), although their work is set in the context of a single period portfolio decision model. Barberis et al. (2001) have extended it to an intertemporal model of an endowment economy.[9] Yet, without the asymmetry in gains and losses, whereby prior losses will play an important role, the risk aversion will be constant over time and the theory cannot contribute to the explanation of the equity premium.[10]

The most interesting feature of the loss aversion model, the feedback effect of asset value – and changes of wealth – on preferences on the one hand, and the choice of consumption path on asset value, on the other hand, creates a complicated stochastic dynamic optimization problem that we propose to be solved by a dynamic programming algorithm as presented in Grüne and Semmler (2004).

Since the accuracy of the solution method is an intricate issue for models with more complicated decision structure, confidence in the accuracy of the solution method when solving such models is essential. In Grüne and Semmler (2004, 2007) a stochastic dynamic programming method with flexible grid size is used to solve such models. In that method an efficient and reliable local error estimation is undertaken and used as a basis for a local refinement of the grid in order to deal with regions of steep slopes or other non-smooth properties of the value function (such as non-differentiability). This procedure allows for a global dynamic analysis of deterministic as well as stochastic intertemporal decision problems. [11] This dynamic program-

[9]Further important literature along this line is Thaler et al. (1997), Barberis and Huang (2003), Barberis et al. (2004a,b), Barberis and Thaler (2003).

[10]As has been pointed out by a referee the model without asymmetry can still generate a high Sharpe ratio.

[11]For deterministic versions of this paper, see Grüne (1997), Santos and Vigo–Aguiar

ming method is used in this paper. A model of loss aversion, as proposed by Benartzi and Thaler (1995) and Barberis et al. (2001), is reformulated for a production economy and numerically solved.

The paper is organized as follows. Section 2 presents the model. Section 3 illustrates the expected results in a simpler setting. Section 4 introduces the stochastic dynamic programming algorithm. Section 5 studies our model of loss aversion and reports numerical results. Section 6 evaluates the results in the context of other recent asset pricing models with production. Section 7 concludes the paper. The appendix presents the details of the algorithm used in this paper.

2 The Asset Pricing Model with Loss Aversion

In order to formalize the new idea on asset pricing we may follow Barberis et al. (2001) and specify the following preference

$$E\left[\sum_{t=0}^{\infty}\left(\rho^t \frac{C_t^{1-\gamma}}{1-\gamma} + b_t\rho^{t+1}\nu(X_{t+1}, S_t, z_t)\right)\right] \tag{1}$$

The first term in equ. (1) represents, as usual, the utility over consumption, using power utility, ρ is the discount factor and γ, the parameter of relative risk aversion. For $\gamma = 1$ we replace $\frac{C_t^{1-\gamma}}{1-\gamma}$ by the log–utility $\ln C_t$. The second term captures the effect of the change of the value of risky asset on the agent's welfare. More precisely it represents the value of risky assets. Hereby X_{t+1} is the change of wealth relative to a benchmark and S_t, the value of the agent's risky assets. Finally, we want to note that z_t is a variable, measuring the agent's gains or losses prior to period t as fraction of S_t. The variables S_t and z_t express the way of how the agent has experienced gains or losses in the past affecting his or her willingness to take risk.

In particular it is presumed that

$$X_{t+1} = S_t R_{t+1} - S_t R_{f,t} \tag{2}$$

which means that the gain or loss $S_t R_t$, with R_t the risky return, $R_{f,t}$ the risk free return, is measured relative to a return $S_t R_{f,t}$ from a risk-free asset which serves as benchmark. The difference $R_t - R_{f,t}$ can be positive, zero or negative and the variable z_t can be greater, equal or smaller than one, with

(1998), and Grüne and Semmler (2004).

$$\nu(X_{t+1}, S_t, z_t) = \begin{cases} X_{t+1}, & R_{t+1} \geq z_t R_{f,t} \text{ and } z_t \leq 1 \\ S_t(z_t R_{f,t} - R_{f,t}) \\ +\lambda S_t(R_{t+1} - z_t R_{f,t}), & R_{t+1} < z_t R_{f,t} \text{ and } z_t \leq 1 \\ X_{t+1} & R_{t+1} \geq R_{f,t} \text{ and } z_t > 1 \\ \hat{\lambda}(z_t) X_{t+1} & R_{t+1} < R_{f,t} \text{ and } z_t \leq 1 \end{cases} \tag{3}$$

with $\lambda \geq 1$ and

$$\hat{\lambda}(z_t) = \lambda + k(z_t - 1) \tag{4}$$

expressing the fact that a loss is more severe than a gain with $k > 0$, and

$$z_{t+1} = \eta z_t \frac{\overline{R}}{R_{t+1}} + (1 - \eta) \tag{5}$$

with $\eta \in [0, 1]$ and $\overline{R}$ a fixed parameter which is chosen to be the long time average of the asset return. Moreover, it is presumed that

$$b_t = b_0 \tilde{C}_t^{-\gamma} \tag{6}$$

with b_0, a scaling factor, and $\tilde{C}_t$ some aggregate consumption which will be specified below, so that the price-dividend ratio and the risky asset premium remain stationary. Hereby b_0 is an important parameter indicating the relevance that financial wealth has in utility gains or losses relative to consumption. In case $b_0 = 0$, we recover the consumption based asset pricing model with power utility.

Barberis et al. (2001) employ such a model of loss aversion and asset pricing to two stochastic variants of an endowment economy without production. In their model variant there is only one stochastic pay-off for the asset holder, a stochastic dividend, whereby dividend pay-offs are always equal to consumption. In their second model variant dividends and consumption follow different stochastic processes.

From the agent's Euler equation for the optimality of the equilibrium, Barberis et al. (2001) obtain a characterization of the risk-free rate, $R_{f,t}$, given by

$$1 = R_{f,t} \rho E_t \left[(\tilde{C}_{t+1}/\tilde{C}_t)^{-\gamma} \right] \tag{7}$$

Let us define

$$m_{f,t+1} = \rho(\tilde{C}_{t+1}/\tilde{C}_t)^{-\gamma}.$$

The risk free rate then is

$$R_{f,t} = 1/E_t[m_{f,t+1}]$$

which coincides with the stochastic discount factor for the consumption based model, see Cochrane (2001, sect. 1.2). The stochastic discount factor for risky asset in the context of the loss aversion model is

$$1 = \rho E_t \left[R_{t+1} (\tilde{C}_{t+1}/\tilde{C}_t)^{-\gamma} \right] + b_0 \rho E_t \left[\hat{\nu}(R_{t+1}, z_t) \right] \tag{8}$$

with

$$\hat{\nu}(R_{t+1}, z_t) = \begin{cases} R_{t+1} - R_{f,t}, & R_{t+1} \geq z_t R_{f,t} \text{ and } z_t \leq 1 \\ (z_t - 1) R_{f,t} + \lambda (R_{t+1} - z_t R_{f,t}), & R_{t+1} < z_t R_{f,t} \text{ and } z_t \leq 1 \\ R_{t+1} - R_{f,t}, & R_{t+1} \geq R_{f,t} \text{ and } z_t > 1 \\ \hat{\lambda}(z_t)(R_{t+1} - R_{f,t}), & R_{t+1} < R_{f,t} \text{ and } z_t > 1 \end{cases} \tag{9}$$

As compared to (7), equ. (8) has two terms. The first term is obtained from consumption based asset pricing and is also found in (7). The second term expresses the fact that if the agent consumes less today and invests in risky assets the agent is exposed to the risk of greater losses, a risk that is represented by the state variable z_t.

In order to derive a stochastic discount factor from (8), observe that the definition of $\hat{\nu}$ in (9) implies that (8) can be rewritten as

$$1 = E_t \left[\rho \left(\left(\tilde{C}_{t+1}/\tilde{C}_t \right)^{-\gamma} + b_0 \alpha_1 \right) R_{t+1} - \rho b_0 \alpha_2 R_{f,t} \right]$$

with α_1 and α_2 given by

$$\begin{array}{llll} \alpha_1 = 1, & \alpha_2 = 1 & \text{for } R_{t+1} \geq z_t R_{f,t} & \text{and } z_t \leq 1 \\ \alpha_1 = \lambda, & \alpha_2 = (\lambda - 1) z_t + 1 & \text{for } R_{t+1} < z_t R_{f,t} & \text{and } z_t \leq 1 \\ \alpha_1 = 1, & \alpha_2 = 1 & \text{for } R_{t+1} \geq R_{f,t} & \text{and } z_t > 1 \\ \alpha_1 = \hat{\lambda}(z_t), & \alpha_2 = \hat{\lambda}(z_t) & \text{for } R_{t+1} < R_{f,t} & \text{and } z_t > 1 \end{array} \tag{10}$$

Since this equation is affinely linear in $R_{f,t}$, we can further rewrite it as

$$1 + \rho b_0 E_t[\alpha_2] R_{f,t} = E_t \left[\left(\rho \left(\tilde{C}_{t+1}/\tilde{C}_t \right)^{-\gamma} + b_0 \alpha_1 \right) R_{t+1} \right] \tag{11}$$

Now, using the equation

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} \tag{12}$$

for the risky return with P_t denoting the asset price and D_t the dividend, which we chose equal to $\tilde{C}_t$ in our model and plugging this equation into (11) we obtain

$$P_t = E_t \left[\underbrace{\frac{\rho \left(\tilde{C}_{t+1}/\tilde{C}_t\right)^{-\gamma} + \rho b_0 \alpha_1}{1 + \rho b_0 E_t[\alpha_2] R_{f,t}}}_{=m_{t+1}} (\tilde{C}_{t+1} + P_{t+1}) \right] \tag{13}$$

Note, again, that for $b_0 = 0$ this equation coincides with the stochastic discount factor for the consumption based model, see Cochrane (2001, sect. 1.2) Note, however, that for $b_0 \neq 0$, in contrast to the consumption based case the stochastic discount factor depends on R_{t+1},[12] which in turn depends on P_{t+1}. Thus, the right hand side of (10) becomes nonlinear and even discontinuous in P_{t+1}.

In order to generate the consumption $\tilde{C}_t$, we use the basic growth model from Brock and Mirman (1972). This amounts to choosing $\tilde{C}_t$ to be the optimal control of the problem[13]

$$\max_{\tilde{C}_t} E \left(\sum_{t=0}^{\infty} \rho^t \frac{\tilde{C}_t^{1-\gamma}}{1-\gamma} \right) \tag{14}$$

subject to the dynamics

$$k_{t+1} = y_t A k_t^{\alpha} - \tilde{C}_t \tag{15}$$

$$\ln y_{t+1} = \sigma \ln y_t + \varepsilon_t \tag{16}$$

with k the capital stock, g the stochastic shock to output, and ε_t being i.i.d. random variables. Here γ is the same as in (1) and as there we replace the utility function by log–utility $\ln \tilde{C}_t$ for $\gamma = 1$. In this case, i.e. for log–utility, the optimal consumption policy is known and is given by

$$\tilde{C}(k_t, y_t) = (1 - \alpha\rho) A y_t k_t^{\alpha}. \tag{17}$$

For $\gamma \neq 1$ we compute $\tilde{C}_t$ numerically.

For this model we want to compute a number of financial measures: the risk free interest rate $R_{f,t}$, the equity return R_{t+1}, the stochastic discount

[12]See equ. (11) where it is visible that R_{t+1} relative to the risk free rate impacts the stochastic discount factor in equ. (12).

[13]Note that we start with the subsequent utility functional in order to generate a consumption stream to be used as part of the stochastic discount factor in equ. (13).

factor m_{t+1} as well as $m_{f,t+1}$, all of which are specified above. In addition we will compute the Sharpe Ratio given by[14]

$$SR = \left| \frac{E(R_{t+1}) - E(R_{f,t})}{\sigma(R_{t+1})} \right|. \tag{18}$$

Next, in a simplified set up we want to illustrate the link between loss aversion and the stochastic discount factor.

3 Intuitions from the Loss Aversion Model

The essential point in loss aversion theory is that it de-links asset returns from consumption growth and gains and losses in asset value have a significant impact on the stochastic discount factor (SDF). This comes to the forefront if one compares the SDF from the loss aversion theory to the SDF based consumption based asset pricing. As one can observe from equ. (13) the SDF, derived from loss aversion, encompasses the SDF from consumption based asset pricing. If we set the parameter in equ. (13) to $b_0 = 0$, the latter case is recovered from the former. Some intuitions on asset pricing, implied by the SDF of equ. (10) can be best spelled out if we view equ. (10) simply as model that allows for gains and losses in consecutive periods. Using some benchmark parameters – which we will also employ in our intertemporal infinite horizon version of sect. 5 – we can define the asset price characteristics that one might expect from the loss aversion model. As parameters we presume $\rho = 0.99, \lambda = 7.5, k = 3, R_{f,t} = 1.03$ and for $b_0 = \{0, 0.3, 1, 3\}$ We study the four cases as stated in (10); Case 1: no current gain, no prior loss; Case 2: current loss, prior gain; Case 3: current gain, prior loss; Case 4: current loss, prior loss.

We take R_{t+1} as a constant for each case but vary it across the cases. In studying the four cases we use the above parameters in equs. (4), (5) and equ. (10).

[14]See Cochrane (2001)

	$b_0 = 0$	$b_0 = 0.3$	$b_0 = 1$	$b_0 = 3$
Case 1 (no current gains, no prior loss)	0.99	0.98557	0.98034	0.97558
Case 2 (current loss, prior gain)	0.99	0.97726	0.97383	0.97248
Case 3 (current gain prior loss	0.99	0.98552	0.98034	0.97558
Case 4 (current loss, prior loss)	0.99	0.97652	0.97300	0.97164

Table 1: Stochastic discount factor for varying b_0

In table 1 we have employed the expected value of the SDF for the consumption based asset pricing model (fixed at 0.99 and obtained by setting $b_0 = 0$). As one can observe from table 1, for all $b_0 > 0$ the SDF is lower than for $b_0 = 0$, the benchmark case. Moreover, as one would expect the SDF decreases with increasing b_0. The greater b_0 is, the more current and past gains and losses exert their influence on the SDF.

Moreover, as current and past losses are allowed to enter the SDF, one can observe that the SDF falls. One can see this by moving down from case 1 to case 4, except that the SDF is the same for case 1 and case 3. The latter result comes from the fact that, for reason of simplicity, we have assumed that case 1 represents a break-even case. In our computation this is presumed to give the same outcome as case 3 (current gain, prior loss).

Table 1 and the asset pricing equation (13) capture the varying loss aversion and its impact on the stochastic discount factor m_{t+1}. This can help us intuitively understand the impact of loss aversion on asset pricing characteristics. In pursuing this, we note, that in table 1 we assume that the discount factor will vary only due to gains and losses in asset value. The effects of stochastic shocks on the growth rates of marginal utility are left aside. Given this simplification we can state the following

- Asset prices and returns are likely to be more volatile than dividend payments when the discount factor varies due to gains and losses in the investor's asset value affecting the loss aversion.

- Investors' loss aversion depend on past investment performance and is thus state dependent. Switching from bull to bear positions in asset markets is thus endogenous. Loss aversion increases when investors experience current loss in addition to a prior loss.[15] When, subsequently, the investors increase the degree of loss aversion the discount factor falls and asset prices move down – and may stay down[16].

- As the loss aversion varies over time, following the experience of current and prior gains and losses, so the risk premium for risky assets expected to vary over time.

- Investors' loss aversion, as formulated in the cases (1) - (4) in equ. (10), predicts substantial equity premia of risky assets. This is also exhibited in table 1. It is observable that the discount factor is lowest for the highest $b_0 = 0$. Moreover, it decreases as one moves from case 1 to case 4.

- The price-dividend ratio is expected to be mean reverting. [17] It has been found to be a good predictor for stock price movements. Loss aversion theory captures this feature.[18] During a bull market the discount rate decreases, asset prices rise, the average returns tend to be lower and, thus, tend to revert to the mean.

- Loss aversion theory predicts a de-linking of consumption growth and asset returns. Theories of risk aversion using power utility and habit formation, can increase the equity premium by increasing the risk aversion parameter γ or introducing time varying risk aversion (as in habit formation models). Yet this creates the puzzle of the risk-free rates (which rises too) when γ rises.

We note that this section is included to help understand some asset price characteristics by referring to a rather simplified theory of loss aversion and its impact on the discount factor. An extended version of a loss aversion model could presume further asymmetric effects of gains and losses on the value function. In Tversky and Kahneman (1992) the value function is concave in gains and convex in losses. This is achieved by introducing an exponent for gains and losses. Because of the difficulties in deriving an asset

[15] Keynes (1936, ch. 17) extensively discusses this case, in the "General Theory", however, with respect to the risk-free rate and long bonds.

[16] For example, if case 3 becomes case 4.

[17] For classic studies on this point, see Campbell and Shiller (1988) and Fama and French (1988) who predict a positive autocorrelation for the short horizon but a negative one for the long horizon.

[18] For details see Barberis et al. (2001: 26)

pricing equation such as equ. (13) and the SDF, denoted there as m_{t+1}, we keep the exponent equal to one as Barberis et al (2001). These same difficulties arise from an endogenous degree of loss aversion whereby the loss aversion could be allowed to vary with the size of the loss.[19] In fact, it is expected that introducing an endogenous degree of loss aversion would even accentuate our result since the SDF would fluctuate even more. For example, switching from $b_0 = 0.3$ to $b_0 = 3$ accentuates the discount factor and thus the asset price movements and the risk premium. Yet, given the computational complexity of such an extended version, we will stay with a rather simple benchmark model to explore a loss aversion model with production.

4 Solving the Model through Stochastic Dynamic Programming

Next we briefly sketch the algorithm that is used to solve our proposed model of loss aversion with production, for details see the appendix. Our approach is a stochastic dynamic programming method using the risk free rate and the stochastic discount factors from the previous section. More precisely, using the state vector

$$x_t = (k_t, \ln y_t, z_t),$$

the equations (15), (16) and (5) for k_{t+1}, $\ln y_{t+1}$ and z_{t+1} define dynamics for x_t which we can write consisely as

$$x_{t+1} = \varphi(x_t, \tilde{C}_t, \varepsilon_t).$$

Now using Bellman's optimality principle the optimal value function V of the problem (14) is characterized by

$$V(x) = \max_{\tilde{C}} E_t \left[\frac{\tilde{C}^{1-\gamma}}{1-\gamma} + \rho V(\varphi(x, \tilde{C}, \varepsilon)) \right] \tag{19}$$

which can be used as the basis of our algorithm.

In contrast to other stochastic dynamic programming methods, here the dynamic programming principle (19) is not sufficient to solve the problem because the dynamics, φ depend not only on the state vector x_t, the control $\tilde{C}_t$ and the random variable ε_t, but also on the the risk free interest rate, $R_{f,t}$, and on the risky return R_{t+1}, i.e., the equations are *externally coupled.* Since $R_{f,t}$ and R_{t+1} are, in turn, obtained from stochastic factors and from

[19]See Frydman and Goldberg (2006).

the asset price function P, now the crucial observation is that using m_f and m, the values P_t and $R_{f,t}$ are again characterized by the equations

$$R_{f,t} = 1/E_t\left[m_{f,t+1}\right] \tag{20}$$

$$P_t = E_t\left[m_{t+1}(\tilde{C}_{t+1} + P_{t+1})\right]. \tag{21}$$

While (20) is explicit, equation (21) is implicit, more precisely it is again of dynamic programming type and thus needs to be solved iteratively in conjunction with (19). Due to the fact that the t-dependence in the equations (19), (20) and (21) is induced by the underlying optimal control problem (14), which admits an optimal control strategy in feedback form, i.e., $\tilde{C}_t = \tilde{C}^*(x_t)$, the functions V, P and R_f as well as all auxiliary functions can be expressed as functions in x instead of t. Our numerical approximation now solves the two dynamic programming equations (19) and (21) simultaneously in order to obtain the solutions V, P as functions of x. The algorithm, whose details can be found in the appendix, yields approximations[20] of V, C^*, R, and R_f.

Using these values and the dynamics (5), (15) and (16) it is easy to simulate approximately optimal solution trajectories x_t and evaluate the numerically computed functions along these trajectories in order to obtain simulated data for

$$C^*_t = C^*(x_t),\ R_{t+1} = R(x_t),\ \text{and}\ R_{f,t} = R_f(x_t).$$

For each set of parameters we have used a numerical simulation for $t = 0, \ldots, 20000$ in order to compute the expectations and standard deviations reported in our tables and from which we computed the Sharpe Ratio according to (18).

5 Presentation of the Results

Using the method described in the previous section we have chosen certain benchmark parameters to be evaluated. For the underlying Brock–Mirman production model the parameters were chosen as

$$A = 5,\ \ \alpha = 0.34,\ \ \rho = 0.99,$$

Note that in equ. (16), σ is a persistence parameter and our variable in (16) is a Gaussian distributed random variable with standard deviation σ_ε. The latter is chosen in such a way as to obtain a standard deviation in

[20]For our problem one can even find the exact C^*, cf. Grüne and Semmler (2007), which was used in our computations.

consumption that is consistent with the literature. Since the volatility of consumption (and its covariance with the asset return) is typically considered crucial for asset pricing we will explore a range of the standard deviations of consumption. We also explore the effect of variation in the persistence parameter, σ, in equ. (16).

We also study the effect of the loss aversion parameter, b_0. With $b_0 = 0$, one can turn the loss aversion model, with the asset price fluctuations, affecting the stochastic discount factor, into the standard consumption based asset pricing model working with risk aversion only. We thus expect, as b_0 goes to zero, the risk premium and Sharpe ratio to fall. For the loss aversion model, for $b_0 > 0$, we also explore the role of the asymmetry in gains and losses on the risk premium and the Sharpe ratio.

Our standard set of parameters are

$$\gamma = 1, \;\; \lambda = 7.5 \;\; \eta = 0.9, \;\; b_0 = 3 \;\text{ and }\; k = 3.$$

In each of the tables below one of the parameters $\sigma, \sigma_\varepsilon, \lambda$ and b_0 are varied in order to study the variation of the financial characteristics with respect to the parameter. The parameters listed in the caption of the figure indicate the standard parameters.

In a first set of tests we explore the role of stochastic shocks. Barberis et al (2001) do not include production in their model. They explore two variants of a loss aversion model: Model I, where consumption is equal to dividend payments and another model, Model II, where dividend payments are correlated to the consumption stream but the former is more volatile than the latter. In Model I they have an annual $\sigma_D = \sigma_c = 3.79$ percent, in Model II they postulate a $\sigma_D = 12$ percent. In order to come close to their experiment in our model with production, where consumption is endogenous but consumption is equal to dividend payments, we explore the volatilities of dividend flows that are above and below the ones they assume in their Model II.

Using our first set of parameters, employed for the results in Table 2, we do not want a strong persistence in the stochastic shocks, represented by σ in equ. (16). So we choose $\sigma = 0.5$. We vary the standard deviation of the shocks σ_ε in such a way that we obtain various volatilities for the consumptions stream, one of them similar to the one presumed by Barberis et al (2001) in their Model II. The results are reported in table 2.

	$\sigma_\varepsilon = 0.072$	$\sigma_\varepsilon = 0.036$	$\sigma_\varepsilon = 0.0180$
$E(R_f)$	1.01212	1.01343	1.01290
$E(R)$	1.02948	1.02037	1.01602
$SD(R)$	0.1034	0.0441	0.0227
$E(R) - E(R_f)$ %	1.73603	0.6939	0.3117
$annual \;\; E(R) - E(R_f)$ %	6.9442	2.7756	1.2468
$E(c)$	4.3600	4.3427	4.3380
$SD(c)$	0.4558	0.2264	0.1130
$SD(c)$ %	10.4557	5.2137	2.6048
SR	0.1730	0.1575	0.1373
$annual \;\; SR$	0.346	0.3150	0.2746
$Cov(m, R)$	-0.00770	-0.0017	-0.00044

Table 2: Variation of shock σ_ε, with $\sigma = 0.5$, $\lambda = 7.5, b_0 = 3$, $k = 3$

In all of our tables 2-5, $E(R_f)$ denotes the expected risk free rate, $E(R)$, the expected equity return, $SD(R)$, the standard deviation of it, $E(R) - E(R_f)$ and $annual E(R) - E(R_f)$ are the equity premium quarterly and annually respectively, $E(c), SD(c), SD(c)\%$ the expected value of consumption, its absolute and relative standard deviation, SR, the Sharpe ratio, from the model and annually respectively, and $Cov(m, R)$, the covariance of the stochastic discount factor with the asset return.[21]

Our result in the third column of table 2, for the volatility of the consumption with $SD(c) = 5.2137\%$ is particularly interesting. It gives roughly an annual volatility of 10.4%, which is still below the value of Barberis et al (2001), but the annual equity premium of roughly 2.8 percent and the Sharpe ratio is 0.32. Both are reasonable values for a model with loss aversion in a production economy. Correspondingly, a higher volatility of consumption gives a higher equity premium and Sharpe ratio; lower volatility corresponds to lower values. Note that in our model with production, which is still kept simple, consumption is equal to dividend payments.

Table 2, however, shows, in all cases, the $Cov(m, R)$ is very low, indicating that the consumption correlation with the asset return does not contribute to the equity premium and Sharpe ratio. Note that here we have $b_0 = 3$; the effects of the variation of b_0 is explored in table 4, where we also can observe that for the standard consumption based asset pricing model, which in our model is achieved for $b_0 = 0$, the equity premium disappears and the Sharpe ratio becomes very small. Next we explore the effects of the variation of shocks for higher persistence parameter σ of equ. (16).

[21] For the computation of the annual SR and corresponding data of the US economy, see table 6, column 5.

	$\sigma_\varepsilon = 0.072$	$\sigma_\varepsilon = 0.036$	$\sigma_\varepsilon = 0.018$
$E(R_f)$	1.01375	1.01207	1.01229
$E(R)$	1.04069	1.02157	1.01629
$SD(R)$	0.1249	0.0502	0.02389
$E(R) - E(R_f)$	2.6939	0.9496	0.3996
$annual\ E(R) - E(R_f)$	10.7766	3.7984	1.5984
$E(c)$	4.4580	4.3660	4.3440
$SD(c)$	1.0804	0.5232	0.2594
$SD(c)\%$	24.2307	11.9833	5.9727
SR	0.2156	0.1893	0.1672
$annual\ SR$	0.4312	0.3786	0.3354
$Cov(m, R)$	-0.00893	-0.00187	-0.00045

Table 3: Variation of shock σ_ε, with $\sigma = 0.9$, $\lambda = 7.5, b_0 = 3$, $k = 3$

The results are reported in table 3. Here too, in the case of a larger persistence parameter for the stochastic shock, σ, in equ. (16), we can observe that for larger shocks to consumption, see $SD(c)\%$, the equity premium and Sharpe ratio will also, as in the previous case, correspondingly rise. Note, that the volatility of consumption in column 2 is very extreme as compared to the data used by Barberis et al. (2001). Here now the equity premium is much too high. Overall, in table 3 too we observe a very low covariance of the stochastic discount factor with asset returns.

Table 4 reports the results on the impact of the variation of b_0 on the financial characteristics of our loss aversion model with production.

	$b_0 = 3$	$b_0 = 1$	$b_0 = 0.1$	$b_0 = 0$
	$\sigma_\varepsilon = 0.036$	$\sigma_\varepsilon = 0.036$	$\sigma_\varepsilon = 0.036$	$\sigma_\varepsilon = 0.036$
$E(R_f)$	1.01207	1.01207	1.01290	1.01068
$E(R)$	1.02157	1.0203	1.01068	1.01082
$SD(R)$	0.0502	0.0486	0.04151	0.03857
$E(R) - E(R_f)$	0.9499	0.8231	0.3652	0.0144
$annual\ E(R) - E(R_f)$	3.7996	3.4124	1.4608	0.0576
$E(c)$	4.3664	4.3664	4.3664	4.3664
$SD(c)$	0.5234	0.5232	0.5232	0.5232
$SD(c)\%$	11.9833	11.9833	11.9833	11.9833
SR	0.1892	0.1693	0.0879	0.00374
$annual\ SR$	0.3784	0.3386	0.1758	0.00748
$Cov(m, R)$	-0.00187	-0.00181	-0.00156	-0.00145

Table 4: Variation of loss aversion b_0, with $\sigma = 0.9$, $\lambda = 7.5$, $k = 3$

As table 4 shows, equity premia and Sharpe ratios are higher with higher b_0. This is as one would expect from the intuition laid out in sect. 3 for the static case. Table 4 shows that if the stochastic discount factor responds to past gains and losses in asset value, the financial characteristics of the model are brought closer to the data. On the other hand, with b_0 converging to zero, the effect of the asset price fluctuations on the stochastic discount factor dissipates and, with the reemergence of the consumption based asset pricing model, the equity premium and Sharpe ratio disappear (see column 5 of table 4). In this exercise too, one can observe a very low covariance of the discount factor and the asset return.

	$\lambda = 3$	$\lambda = 5$	$\lambda = 10$	$\lambda = 20$
	$\sigma_\varepsilon = 0.036$	$\sigma_\varepsilon = 0.036$	$\sigma_\varepsilon = 0.036$	$\sigma_\varepsilon = 0.036$
$E(R_f)$	1.01343	1.01343	1.01343	1.01343
$E(R)$	1.01724	1.01916	1.02107	1.02320
$SD(R)$	0.0438	0.04433	0.04382	0.04296
$E(R) - E(R_f)$	0.3825	0.5737	0.7647	0.9776
$annual\ E(R) - E(R_f)$	1.5300	2.2948	3.0588	3.9104
$E(c)$	4.3427	4.3427	4.3427	4.3427
$SD(c)$	0.2264	0.2264	0.2264	0.2264
$SD(c)\%$	5.2137	5.2137	5.2137	5.2137
SR	0.0872	0.1294	0.1745	0.2275
$annual\ SR$	0.1744	0.2588	0.3490	0.4550
$Cov(m, R)$	-0.00170	-0.00171	-0.00169	-0.00166

Table 5: Variation of asymmetry of loss and gains λ, with $\sigma = 0.5, b_0 = 3, k = 3$

In table 5 we explore the effect of the degree of asymmetry of gains and losses for asset price characteristics. To be closer to the exercise proposed by Barberis et al (2001) we here go back to the persistence parameter $\sigma = 0.5$ of table 2, which avoids the impact of the strong persistence of shocks on consumption volatility. Here we also presume $\sigma_\varepsilon = 0.036$ so that we come close to what Barberis et al (2001) use as their standard case for the the volatility of divident payments.[22].

As one can observe in table 5, the equity premium and Sharpe ratio rise with the degree of asymmetry in the weight of losses for the value function as compared to gains. The larger increases in the equity premium and Sharpe ratio occur in the range of $3 < \lambda < 7.5$. Since we do not use a nonlinear loss aversion function as in Kahneman and Tversky (1979) or Tversky and Kahnemann (1992) which gives more curvature to the value function by using exponents for the loss part of the value function, [23] the use of a higher λ as our standard case may be justifiable.

Overall, we stress that the loss aversion model, since it considers the echo effects of past gains and losses in the stochastic discount factor brings the financial characteristics of the model closer to the data. Our loss aversion

[22]Note that they take annual values of $\sigma_D = 12\%$

[23]For a discussion on this point see di Georgi, Hens and Mayer (2006). We want to note that Barberis et al (2001:42) obtain higher values for the equity premium and Sharpe ratio for lower values of λ in their Model II, but in their model this comes from the fact that the k is chosen rather high, namely $k = 10$, we want to stay with our lower k to make the result comparable to the other tables.

model with production yields slightly lower equity premia and Sharpe ratios than those obtained by Barberis et al. (2001) in their Model II, a result to be discussed below. Note also, that in our loss aversion model with production we have, with the Brock-Mirman model, taken a rather parsimonious version of a production model. Other advances in asset pricing models such as models of habit formation and adjustment cost of investment have not been considered yet in our model.[24]. It maybe worthwhile interpreting our results in the context of other recent advances in modelling asset prices in the context of a production economy. This is undertaken next.

6 Interpretation of the Results

We focus here on recent advances in consumption based asset pricing studies that include production.[25] We, in particular, will restrict ourselves to a comparison with the results of habit formation models as proposed and studied in Boldrin et al. (2001), Jerman (1998) and Grüne and Semmler (2006), since those are models that include production.

Whereas Boldrin et al. (2001) use a model with log utility for internal habit, but endogenous labor supply in the household's preferences, Jerman studies the asset price implication of a production economy, also with internal habit formation, but, as in Grüne and Semmler (2006), labor effort is not a choice variable. In order to allow for some frictions in the model, all three papers, Boldrin et al. (2001), Jerman (1998) and Grüne and Semmler (2006), use adjustment costs of investment in a model with habit formation. As has been shown by Boldrin et al. (2001) habit formation is not sufficient to obtain proper characteristics of asset prices and returns. Adjustments costs of investment are needed.

Both, Boldrin et al. and Jerman claim that habit formation models with adjustment costs can match the financial characteristics of the data. Yet, both studies have chosen parameters that appear to be conducive to results which replicate better the financial characteristics such as risk free rate, equity premium and the Sharpe ratio. In comparison to their parameter choices, Grüne and Semmler (2006) have chosen parameters that have commonly been used for stochastic growth models[26] and that seem to describe the first and

[24] Since it is extremely difficult in the context of a loss aversion model to build in the latter features and to derive the stochastic discount factor we were forced to work with our simple version.

[25] For a recent comparison of the relative performance of models with different types of preferences, see Lettau and Uhlig (2002).

[26] See Santos and Vigo-Aguiar (1998).

second moments of the data well. Table 6 reports the parameters and the results.

Both, the study by Boldrin et al. (2001) and Jerman (1998), have chosen to set $\overline{\varphi} = 4.05$, in the adjustment costs of investment, a very high value which is at the very upper bound found in the data.[27] Since a high parameter $\overline{\varphi}$ strongly increases frictions in the fluctuation of the capital stock and makes the supply of capital very inelastic, Grüne and Semmler (2006) have rather worked with a $\overline{\varphi} = 0.8$ in order to avoid such strong volatility of returns. Moreover, both papers use a higher parameter for past consumption, b, than Grüne and Semmler (2006) have chosen. Those parameters increase the volatility of the stochastic discount factor, a crucial ingredient to raise the equity premium and the Sharpe ratio.

[27] See for example, Kim (2002) for a summary of the empirical results reported on $\overline{\varphi}$ in empirical studies.

Boldrin et al.[a)]	Jerman[b)]	Grüne et al. [c)]	LAP [d)]	US Data [e)]
b= 0.73-0.9 $\overline{\varphi}$=4.15 $\sigma = 0.9$ ρ= 0.999 γ= 1	b= 0.83 $\overline{\varphi}$=4.05 $\sigma = 0.99$ ρ= 0.99 γ= 5	b= 0.5 $\overline{\varphi}$= 0.8 $\sigma = 0.9$ ρ= 0.95 γ= 1	$\sigma = 0.5$ $\rho = 0.99$ γ= 1	
$r_f = 1.2$	$r_f = 0.81$	$r_f = 24.0$	$r_f = 5.32$	$r_f = 0.8$
EP =6.63	EP =6.2	EP =1.32	EP =2.78	EP =6.18
SR= 0.36	SR= 0.33	SR=0.11	SR=0.32	SR=0.35

a) Boldrin et al.(2001) use a model with endogenous labor supply, log utility for habit formation and adjustment costs. Here, as in the following columns, r_f, EP and SR denote the riskfree rate (in percent), the equity premium and Sharpe ratio respective. Return data and SR are annualized.

b) Jerman (1998) uses a model with exogenous labor supply, habit formation with coefficient of RRA of 5, and adjustment costs, annualized data.

c) In Grüne and Semmler (2006) the labor supply is also exogenous. Note that here the risk-free rate, r_f, is high, because the subjective discount factor, $\rho = 0.95$, is low (which implies a high subjective discount rate). Return data and the SR are annualized.

d) LAP stands for our loss aversion model with production, detailed results of various parameter constellations are reported in sect. 5. We present here the results of the third column of table 2, which may be taken as representative for the loss aversion model with production, since it avoids extreme parameter values. Here we have $\sigma = 0.5$ which avoids a high persistence in the technology shock.

e) The following financial characteristics of the data are reported in Jerman (1998). The data range from 1954.1 to 1990.4. Note that the covariance between consumption growth and asset return with $Cov(\Delta C, R)$=0.0027 is low in the data, which translates also in a low $Cov(m, R)$.

Table 6: Models with Production

Jerman sets the relative risk aversion parameters $\gamma = 5$, which also increases the volatility of the discount factor and increases the equity premium when used for the pricing of assets. Jerman also presumes a much higher persistence parameter for the technology shocks, a $\sigma = 0.99$, from which one knows that it will make the stochastic discount factor more volatile too. All in all, both studies have chosen parameters which are known to bias the results toward the empirically found financial characteristics.[28] Grüne and Semmler (2006) have chosen a model variant with no endogenous labor supply. As Lettau and Uhlig (2000) show, this is the most favorable model for asset pricing in a production economy, since including labor supply as a choice variable, would even reduce the equity premium and the Sharpe ratio.

One is thus inclined to state that previous studies on habit formation models with production have not satisfactorily solved the dynamics of asset prices and the equity premium puzzle. At the heart of the consumption based asset pricing model, including the habit formation model, is the co-variance of consumption growth with asset return, which, in the context of that model, needs to be increased to get a higher equity premium and Sharpe ratio. Yet as the empirical data show,[29] this co-variance is very low.

As compared to models with habit formation, column 4 of table 6 reports some typical results of the loss aversion model with production. As argued previously, the model with loss aversion does not have to increase the covariance of consumption with asset returns. As shown above, the proposed loss aversion model with production includes echo effects from gains and losses in asset value, which appear in the preferences, producing a time varying loss aversion, a low risk free rate (with low volatility), a higher equity premium (with high volatility) and a reasonably high Sharpe ratio. All this holds for a $\gamma = 1$ and thus the risk aversion parameter does not have to be increased in order to increase the equity premium and Sharpe ratio.

As table 6, column 4, shows, for a discount factor of $\rho = 0.98$ one obtains a risk-free interest rate of approximately 5 percent which is still high as compared to the data. The equity premium, if we take the case as reported in table 6, column 4, as representing the loss aversion model, is still too low. But the tables 2 to 5 have shown results from different parameter constellations which are significantly higher.[30] We want to note that the

[28] We also want to remark that both, Jerman and Boldrin et al., do not provide any accuracy test for their procedure that they have chosen to solve the intertemporal decision problem. Boldrin et al. use the Lagrangian multiplier from the corresponding planner's problem to solve for asset prices with no accuracy test for the procedure. Jerman uses a log-linear approach to solve the model and an accuracy test of this procedure is also not provided in the paper. An accuracy test for the dynamic programming procedure that is used in Grüne and Semmler (2006) is provided in Grüne and Semmler (2007). We also want to note that there is a crucial constraint in habit formation models, namely that the surplus consumption has to remain non-negative when the optimal solution, C_t, is computed. As shown in Grüne and Semmler (2006) this constraint has to be treated properly in the numerical solution method.

[29] See table 6, column 5 and the note e)

[30] For computing the annual Sharpe ratio we have used here, and for the tables 2-5, a conversion formula developed by Lo (2002) with $SR(q) = \sqrt{q}SR$, with q the number of

habit formation model in Grüne and Semmler (2006), see table 6, column 3, is the same, in terms of its basic structures and parameters, as the here solved model with loss aversion. One can therefore be quite confident that the loss aversion model produces quantitatively important contributions to the equity premium and Sharpe ratio puzzles. Yet, in these types of models with production there is no friction in consumption and capital investment as one has in the habit formation models.

Overall, however, in our loss aversion model with production we obtain lower values of the equity premia and Sharpe ratios as in Barberis et al. (2001) with exogenous consumption. This comes from the fact that in models with production and endogenous consumption, consumption can be smoothed through intertemporal decisions, see Lettau and Uhlig (2002). Moreover, as mentioned above, in order to solve the model analytically for the stochastic discount factor, the loss aversion model with production presented here, does not include habit formation and adjustment costs of investment. Including those, will presumably improve the model further.

7 Conclusion

Extensive research has recently been devoted to the study of asset price characteristics, such as the risk-free interest rate, the equity premium and the Sharpe ratio, arising from the stochastic growth model of the Brock-Mirman type. The failure of the basic model to match the empirical characteristics of asset prices and returns has given rise to numerous attempts to extend the model by allowing for different preferences and technology shocks, adjustment costs of investment, the effect of leverage on asset prices and heterogenous households and firms.[31]

In this paper we have gone beyond the consumption based asset pricing model and have studied asset price characteristics when a consumption stream as well as the fluctuation of the agent's value of assets affect the utility of the agent. We have presumed, as recently proposed that agents become even more loss averse when they have prior experiences large with losses in asset value and are again hit by a decline in their asset value in the current period. This gives rise, as we have shown in sect. 2 of the paper,

periods. We have, even for our case reported in table 6, column 4 an annual Sharpe ratio of 0.32 which is in the vicinity of the actual annual Sharpe ratio as reported in table 6, column 5.

[31] A model with heterogenous firms in the context of a Brock type stochastic growth model can be found in Akdeniz and Dechert (1997) who are able to match, to some extent, the equity premium by building on idiosynchratic stochastic shocks to firms.

to a new form of a stochastic discount factor pricing the income stream in a production economy.

In the context of this model, the agents do not have to experience large losses in current consumption in order to induce them to change asset holdings. [32] In our model, as one finds in time series data, consumption growth is de-linked from asset prices booms and busts and the co-variance of consumption growth and asset returns can, as the empirical data show, allowed to be weak. Thus one might want to design empirical estimation strategies that accepts a de-linked relationship of consumption growth and asset returns.[33] The next step of research would be to integrate more frictions such as habit formation and adjustment costs of investment into the loss aversion model with production studied here.

[32] We think this is an important feature of loss aversion models, since the financial loss of large institutional investors, such as pension funds, foundations and university endowments does not result directly in a consumption loss. The loss in wealth appears to be more painful for those institutions since they have to adjust downward their operations.

[33] For empirical results on prospect theory, using a Markov regime change model, see Zhang and Semmler (2005)

8 Appendix: details of the numerical algorithm

The aim of the numerical algorithm is to simultaneously solve the equations (19) and (21) using the fact that the time dependence of the values $R_{f,t}$ and P_t can be replaced by the dependence on x_t, such that we can compute these values as functions of the state x. Thus, we can compute V and P by solving the dynamic programming equations

$$\begin{aligned} V(x) &= \max_{\tilde{C}} E\left[\frac{\tilde{C}^{1-\gamma}}{1-\gamma} + \rho V(\varphi(x,\tilde{C},\varepsilon))\right] \\ P(x) &= E\left[m(x,\varphi(x,\tilde{C}^*(x),\varepsilon))(\tilde{C}^*(\varphi(x,\tilde{C}^*(x),\varepsilon)) + P(\varphi(x,\tilde{C}^*(x),\varepsilon)))\right] \end{aligned}$$

where $\tilde{C}^*(x)$ denotes the maximizing value for the right hand side of the V–equation. Instead of solving the P–equation directly we solve

$$\tilde{P}(x) = E\left[\tilde{C}^*(x) + m(x,\varphi(x,\tilde{C}^*(x),\varepsilon))\tilde{P}(\varphi(x,\tilde{C}^*(x),\varepsilon))\right]$$

which has the same structure as the V–equation and from which P is easily obtained via $P(x) = \tilde{P}(x) - \tilde{C}^*(x)$.

In order to approximate these functions numerically, we chose an appropriate domain $\Omega \subset \mathbb{R}^3$ for our state vector (in all our examples this was chosen as $\Omega = [0.2, 22] \times [-0.32, 0.32] \times [0.5, 2]$) and a three dimensional cuboidal grid Γ on Ω with nodes x^i. On this grid, we compute sequences of continuous and piecewise multilinear approximations $\widetilde{V}^{(j)} \approx V$, $\widetilde{P}^{(j)} \approx \tilde{P}$ and $\widetilde{C}^{(j)} \approx \tilde{C}^*$, $j = 0, 1, 2, \ldots$. Note that each function is uniquely determined by its values in the nodes x^i of the grid, hence we only need to store these values. We proceed iteratively by setting $\widetilde{V}^{(0)} = \widetilde{P}^{(0)} \equiv 0$ and computing

$$\begin{aligned} \widetilde{C}^{(j)}(x^i) &= \operatorname*{argmax}_{\tilde{C}} E\left[\frac{\tilde{C}^{1-\gamma}}{1-\gamma} + \rho\widetilde{V}^{(j)}(\varphi(x^i,\tilde{C},\varepsilon))\right] \\ \widetilde{V}^{(j+1)}(x^i) &= E\left[\frac{\widetilde{C}^{(j)}(x^i)^{1-\gamma}}{1-\gamma} + \rho\widetilde{V}^{(j)}(\varphi(x^i,\widetilde{C}^{(j)}(x^i),\varepsilon))\right] \\ \widetilde{P}^{(j+1)}(x^i) &= E\Big[C^{(j)}(x^i) + m^{(j)}(x^i,\varphi(x^i,\widetilde{C}^{(j)}(x^i),\varepsilon))\widetilde{P}^{(j)}(\varphi(x^i,\widetilde{C}^{(j)}(x^i),\varepsilon)\Big] \end{aligned}$$

for all nodes x^i of the grid Γ using the auxiliary functions

$$m^{(j)}(x_1,x_2) = \frac{\rho\left(\widetilde{C}^{(j)}(x_2)/\widetilde{C}^{(j)}(x_1)\right)^{-\gamma} + \rho b_0\alpha_1}{1+\rho b_0 E_t[\alpha_2]R_f^{(j)}(x_1,x_2)}$$

$$R_f^{(j)}(x_1,x_2) = E\left[(\widetilde{C}^{(j)}(x_2)/\widetilde{C}^{(j)}(x_1))^{-\gamma}\right]$$

whose formulas are derived from (13) and (20), (7), respectively. The value α_2 appearing in the equation for $m^{(j)}$ is computed according to (10) using the approximations

$$R_{f,t} \approx R_f^{(j)}(x^i, \varphi(x^i, \widetilde{C}^{(j)}(x^i), \varepsilon))$$

$$R_{t+1} \approx R^{(j)}(x^i, \varphi(x^i, \widetilde{C}(j)(x^i), \varepsilon))$$

for $R_f^{(j)}$ from above and

$$R^{(j)}(x_1,x_2) = \frac{\widetilde{P}^{(j)}(x_2)}{\widetilde{P}^{(j)}(x_1) - \widetilde{C}^{(j)}(x_1)},$$

which is derived from (12) observing that $\widetilde{P}^{(j)} \approx \tilde{P} = P + \tilde{C}^*$.

Note that this iteration resembles a Jacobi iteration for solving systems of linear equation. In order to speed up the iteration process we use the Gauss-Seidel type increasing coordinate algorithm described in Grüne (1997) and perform the iteration for $j = 0, 1, 2, \ldots$ until $\|\widetilde{V}^{(j+1)} - \widetilde{V}^{(j+1)}\|_\infty \leq \delta_V$ and $\|\widetilde{P}^{(j+1)} - \widetilde{P}^{(j+1)}\|_\infty \leq \delta_P$, where we chose $\delta_V = 10^{-5}$ and $\delta_P = 10^{-3}$. If the exact value C^* for the optimal feedback law of (14) is known analytically, which happens to be the case in the log-utility setting $\gamma = 1$, cf. e.g., Grüne and Semmler (2007), then the evaluation of the argmax can be replaced by $C^{(j)}(x^i) = C^*(x^i)$. At the end of the iteration we also store the auxiliary functions $R_f^{(j)}$ and $R^{(j)}$ for later evaluation along optimal trajectories.

Note that the value α_2 — which enters the equation for $\widetilde{P}^{(j+1)}$ via $m^{(j)}$, $R^{(j)}$ and (10) — depends nonlinearly on $\widetilde{P}^{(j)}$. Hence, the equation for $\widetilde{P}^{(j+1)}$ becomes nonlinear and it is not clear a priori whether this iteration will converge at all and if so, for which initial values. This issue certainly deserves further mathematical investigation which is, however, beyond the scope of this paper. Nevertheless, numerically we observed convergence for all considered parameter sets starting from $P^{(0)} \equiv 0$.

In order to make the grid node distribution efficient, we choose the grid adaptively using the a posteriori error estimation based grid generation technique described in Grüne and Semmler (2004). For each set of parameters we have performed 1–3 adaptation steps depending on the error estimates resulting in a grid with ≈ 5000–10000 cuboidal elements and an error of order 10^{-5} (measured accumulated along the optimal trajectories). This procedure results in computational times of ≈ 2–4 minutes per parameter set on a Pentium 4 Linux Computer with 3.06GHz.

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