Set oriented optimal control using past information[1]

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Abstract: We propose a set oriented approach to the global optimal control of nonlinear systems with quantized state measurement in which information about the past measurements is taken into account. We investigate the theoretical properties of the approach and illustrate the performance by a numerical example.

1   Introduction

In this paper we consider the problem of optimally controlling a discrete time nonlinear system to a desired target state by means of a state feedback law. However, we assume that instead of the precise state of the system only a rather rough quantized information about the region containing the current state is available, as, for instance, in a multi tank system where for each tank only certain fill intervals like, e.g., $[0\%, 25\%]$, $[25\%, 50\%]$ etc. are distinguishable.

It was already observed in [2,3] that the set oriented approach to global optimal control problems for perturbed systems developed in [2] is suitable to handle this situation when the uncertainty about the exact state of the system is modelled as a perturbation. We summarize this approach in Section 2 of this paper. Proceeding this way significantly improves the results compared to not taking into account the uncertainty at all, cf. [3], however, due to its worst case dynamic game approach it is still rather conservative.

Motivated by conceptually similar methods in the discrete event system literature, see, e.g., [5] and the references therein, in this paper we propose to extend the method by including past information: We determine the feedback value at time k not only depending on the state region at the current time k but also on the regions visited in the past, i.e., at times $k - m + 1, \ldots, k - 1$. We formalize this idea in Section 3, present our main theorem about the relation between the optimization with and without considering past information in Section 4 and illustrate the efficiency of our approach with a numerical example in Section 5.

2   Problem formulation

We consider the discrete-time nonlinear control system

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots \quad , \tag{2.1}$$

where $f : \mathcal{X} \times U \longrightarrow \mathcal{X}$ is continuous, $x_k \in \mathcal{X}$ is the state of the system, $u_k \in U$ is the control input, chosen from compact sets $\mathcal{X} \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$. The set of all control sequences $\mathbf{u} = (u_k)_{k \in \mathbb{N}}$ is denoted by $U^{\mathbb{N}}$ and for each initial value x_0 and control sequence $\mathbf{u}$ we denote the corresponding trajectory by $x_k(x_0, \mathbf{u})$.

The control problem we consider is as follows: Given a target set $X^* \subset \mathcal{X}$, steer the system into X^* while minimizing the functional

$$J(x_0, \mathbf{u}) = \sum_{k=0}^{N(x_0, \mathbf{u})} g(x_k(x_0, \mathbf{u}), u_k) \tag{2.2}$$

over $\mathbf{u}$, where $N(x_0, \mathbf{u})$ denotes the minimal $k \geq 0$ such that $x_k(x_0, \mathbf{u}) \in X^*$ holds. Here $g : \mathcal{X} \times U \to \mathbb{R}$ is a continuous running cost satisfying $\min_{u \in U} g(x, u) > 0$ for all $x \notin X^*$.

Our goal now is to find a feedback law which approximately solves this problem, assuming, however, that the system's state is not exactly determinable. Instead, we assume that we can only identify regions of the state space where the current state is located. These regions are described by a partition $\mathcal{Q}$ consisting of finitely many connected and disjoint subsets $Q \subset \mathcal{X}$ with the properties

$$\bigcup_{Q \in \mathcal{Q}} Q = \mathcal{X} \quad \text{and} \quad Q \cap \widetilde{Q} = \emptyset \text{ for all } Q, \widetilde{Q} \in \mathcal{Q} \text{ with } Q \neq \widetilde{Q} \tag{2.3}$$

In contrast to, e.g., [1,2] we do not consider the sets $Q \in \mathcal{Q}$ as a discretization which we are able to change according to our demands but rather as a feature of the system which we cannot influence. One example — and our main

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motivation — for this setting are systems with coarse quantization of the measurement, i.e., systems in which the sensors do only yield information about the region $Q \in \mathcal{Q}$ the system's state is contained in.

We assume that our target set X^* is a union of such regions, i.e., $X^* = \bigcup_{Q \in \mathcal{Q}^*} Q_i$ for some set $\mathcal{Q}^* \subset \mathcal{Q}$. For simplicity of notation we will frequently identify subsets $\{Q_1, \dots, Q_n\} \subset \mathcal{Q}$ with their corresponding subsets $\bigcup_{i=1,\dots,n} Q_i \subset \mathcal{X}$. The next definition is a tool for formalizing the feedback concept we intend to use.

Definition 2.1. *The **correlation function** $\rho : \mathcal{X} \to \mathcal{Q}$ is defined as $\rho(x) = Q$ for $x \in Q$.*

The correlation function returns the region $Q \in \mathcal{Q}$ in which the state x is located. Now the desired feedback law for solving our control problem is given by a map $\mu : \mathcal{Q} \to U$ which assigns a control value to each state $x \in \mathcal{X}$ via $x \mapsto \mu(\rho(x))$.

In the remainder of this section we describe the solution for the problem presented in [3] using, however, a different notation which is more suitable for our extension of the method in the following section. To this end we define the set $2^{\mathcal{X}}$ of all subsets of $\mathcal{X}$ and the set of sequences $(2^{\mathcal{X}})^{\mathbb{N}} := \{\mathbf{X} = (X_0, X_1, \dots) \,|\, X \subset \mathcal{X} \text{ for all } i \in \mathbb{N}\}$ and use the following concept of choice functions.

Definition 2.2. *A **choice function** $\beta : (2^{\mathcal{X}})^{\mathbb{N}} \times U^{\mathbb{N}} \longrightarrow \mathcal{X}^{\mathbb{N}}$ is a function of the form*

$$\beta(\mathbf{X}, \mathbf{u}) = (\hat{\beta}_0(X_0, u_0), \hat{\beta}_1(X_1, u_1), \dots),$$

with component functions $\hat{\beta}_i : 2^{\mathcal{X}} \times U \longrightarrow \mathcal{X}$ satisfying $\hat{\beta}_i(X, u) \in X$ for all $X \subseteq \mathcal{X}$. The set of all choice functions β is denoted by $\mathcal{B}$ and the set of all component functions $\hat{\beta}$ by $\widehat{\mathcal{B}}$.

The components $\hat{\beta}$ of the choice functions β model the uncertainty of the state x given that we only know $x \in X$ for regions $X \subset \mathcal{X}$ by choosing one $x \in X$ depending on the control u. The choice functions β then extend this concept to a sequence of regions and controls.

Using this concept we now define a perturbed set valued control system by

$$X_{k+1} = F(X_k, u_k, \hat{\beta}_k(X_k, u_k)), \quad k = 0, 1, \dots \quad , \tag{2.4}$$

with $F : 2^X \times U \times \widehat{\mathcal{B}} \to \mathcal{Q}$, $F(X, u, \hat{\beta}(X, u)) := \rho(f(\hat{\beta}(X, u), u))$. In what follows we will omit the arguments of $\hat{\beta}$ in order to simplify the notation.

The map F describes all possible transitions of f from a subset $X \subset \mathcal{X}$ of the state space to regions $Q \in \mathcal{Q}$, parametrized by $\hat{\beta}_k$ which is interpreted as a perturbation. More precisely, for each $u \in U$ we have the identity

$$\bigcup_{\hat{\beta} \in \widehat{\mathcal{B}}} F(X, u, \hat{\beta}) = \{Q \in \mathcal{Q} \,|\, f(x, u) \in Q \text{ for some } x \in X\}.$$

A trajectory $X_k = X_k(Q, \mathbf{u}, \beta)$, $k \in \mathbb{N}$ of (2.4) is now a sequence of regions defined by

$$X_0 = Q, \quad X_{k+1} = F(X_k, u_k, \hat{\beta}_k)$$

and depends on the start set $Q \in \mathcal{Q}$, the control sequence $\mathbf{u} \in U^{\mathbb{N}}$ and the choice function $\beta \in \mathcal{B}$.

The next object defines the set of regions from which the system (2.4) can be steered to the target set X^* regardless of the choice of β.

Definition 2.3. *The **domain of controllability** of X^* is defined as*

$$S = \{Q \in \mathcal{Q} \,|\, \text{for each } \beta \in \mathcal{B} \text{ there exists } \mathbf{u} \in U^{\mathbb{N}} \text{ and } k \in \mathbb{N} \text{ with } X_k(Q, \mathbf{u}, \beta) \subset X^*\}.$$

*and the **first hitting time** is defined as $N(Q, \mathbf{u}, \beta) = \inf\{k \in \mathbb{N} \,|\, X_k(Q, \mathbf{u}, \beta) \in S\}$.*

Note that for fixed Q we can interpret β as a map from $U^{\mathbb{N}}$ to $\mathcal{X}^{\mathbb{N}}$. In the language of dynamic game theory this map defines a nonanticipating strategy, cf. [2].

Using the running cost g we now define a running cost for the set valued perturbed control system (2.4)

$$g_1 : \mathcal{Q} \times U \longrightarrow \mathbb{R}_0^+, \quad g_1(Q, u) := \sup_{x \in Q} g(x, u).$$

By this definition we assume the worst case, i.e., the highest cost, over all the uncertain states in Q. Using g_1 we now define the functional

$$J_1(Q, \mathbf{u}, \beta) := \sum_{k=0}^{N(Q, \mathbf{u}, \beta)} g_1(X_k(Q, \mathbf{u}, \beta), u_k) \ \in \mathbb{R}_0^+ \cup \{+\infty\}$$

and the optimal value function $V_1(Q) = \sup_{\beta \in \mathcal{B}} \inf_{\mathbf{u} \in U^{\mathbb{N}}} J_1(Q, \mathbf{u}, \beta)$. By standard arguments one sees that V_1 fulfills the optimality principle

$$V_1(Q) = \inf_{u \in U} \left\{ g_1(Q, u) + \sup_{\hat{\beta} \in \widehat{\mathcal{B}}} V_1(F(Q, u, \hat{\beta})) \right\} \tag{2.5}$$

for all $Q \not\subset X^*$ and $V(Q) = 0$ for all $Q \subset X^*$. Since $\mathcal{Q}$ consists of finitely many sets, from this it is easy to see by induction that, if $Q \notin S$ then $V_1(Q) = \infty$ and if $Q \in S$ then $V_1(Q) < \infty$.

We will now investigate the behavior of V_1 along an optimal trajectory for the original system (2.1). To this end, observe that the optimal feedback law $\mu : \mathcal{Q} \to U$ is given by

$$\mu(Q) = \operatorname{argmin}_{u \in U} \left\{ g_1(Q, u) + \sup_{\hat{\beta} \in \widehat{\mathcal{B}}} V(F(Q, u, \hat{\beta})) \right\}$$

Using this μ we get the following theorem.

Theorem 2.4. *For all $x \in S$ the inequality*

$$g(x, \mu(\rho(x))) + V_1(\rho(f(x, \mu(\rho(x))))) \leq V_1(\rho(x)) \tag{2.6}$$

holds.

Proof. Using the optimality principle (2.5) and the definition of μ, $\hat{\beta}$ and g_1 we get

$$
\begin{aligned}
V_1(\rho(x)) &= \inf_{u \in U} \left\{ g_1(\rho(x), u) + \sup_{\hat{\beta} \in \widehat{\mathcal{B}}} V_1(F(\rho(x), u, \hat{\beta})) \right\} \\
&= g_1(\rho(x), \mu(\rho(x))) + \sup_{\hat{\beta} \in \widehat{\mathcal{B}}} V_1(F(\rho(x), \mu(\rho(x)), \hat{\beta})) \geq g(x, \mu(\rho(x))) + V_1(\rho(f(x, \mu(\rho(x)))))
\end{aligned}
$$

which shows the assertion. $\qquad\square$

The result has two immediate consequences for the trajectory $x_k(x_0, \mu)$ of (2.1) corresponding to μ defined by

$$x_{k+1} = f(x_k, \mu(\rho(x_k))).$$

On the one hand, (2.6) shows that the value $V_1(\rho(x_k(x_0, \mu)))$ in decreasing in k until X^* is reached, impying that $x_k(x_0, \mu)$ eventually reaches X^* provided $x_0 \in S$ (or equivalently $V_1(\rho(x_0)) < \infty$) holds. On the other hand, we obtain the inequality

$$J(x_0, \mu) = \sum_{k=0}^{N(x_0, \mu)} g(x_k(x_0, \mu), \mu(\rho(x_k(x_0, \mu)))) \leq V_1(\rho(x_0)),$$

i.e., the function V_1 is a guaranteed upper bound for the "performance" of μ with respect to the original functional (2.2).

3 Including past information

The approach described in the previous section is conservative because by maximizing over β we implicitly assume the worst case in each step, i.e., that for each k among all the possible states in X_k the actual state x_k is the one which produces the largest cost. Of course, this is not necessarily the case. The approach we propose in order to reduce the conservatism relies on the idea that at time k we consider m state regions, namely the current and $m-1$ past regions, i.e., $X_{k-m+1}, \ldots, X_k$. This way we can collect more information and thus obtain a less conservative result. In other words, we are now looking at an approximately optimal feedback map of the form $\mu(X_{k-m+1}, \ldots, X_k)$.

In order to keep the exposition simple, in this paper we restrict ourselves to $m = 2$. All arguments can, however, be extended to the more general setting $m \geq 2$ (note that the setting in the last section corresponds to $m = 1$).

In order to formalize our idea, we define $\mathcal{Q}_2 := (\mathcal{Q} \cup \{\delta\}) \times \mathcal{Q}$ and introduce a new set valued state $Z_k = (Z_k^1, Z_k^2)^T \in \mathcal{Q}_2$ which represents $(X_{k-1}, X_k)^T$. For Z we define the set valued control system as

$$Z_{k+1} = F_2(Z_k, u_k, \hat{\beta}_k) := \begin{pmatrix} Z_k^2 \\ F(X(Z_k), u_k, \hat{\beta}_k) \end{pmatrix} \tag{3.1}$$

with F from (2.4) and

$$X(Z) := \begin{cases} Z^2, & \text{if } Z^1 = \delta \\ f(Z^1, U) \cap Z^2, & \text{else.} \end{cases} \tag{3.2}$$

The symbol δ is introduced to represent the "unknown" state, which appears when the system is started at time $k = 0$ with initial region $X_0 \in \mathcal{Q}$, because at $k = 0$ a past region does not exist. Therefore, at time k a trajectory starts with the vector $Z_0 = (\delta, X_0)^T$.

By including the extra information in the defininition of F_2 the uncertainty of the system is reduced. Instead of using $F(X_k, u, \hat\beta)$ as in the previous section now we use $F(X(Z_k), u, \hat\beta)$, where $X(Z_k)$ is the subset of the current region X_k which can be reached from the past region $Z_k^1 = X_{k-1}$, i.e., we exclude states which the system cannot reach.

Clearly, not all the pairs $Z = (Q_1, Q_2)^T \in \mathcal{Q}_2$ are actually attained by the systems dynamics. In fact, only those pairs with $X(Z) \neq \emptyset$ can appear on the left hand side of (3.1) which is why we define the active state regions $\mathcal{Q}_2^a := \{Z \in \mathcal{Q}_2 \mid X(Z) \neq \emptyset\}$. We denote the trajectories of (3.1) by $Z_k(Z_0, \mathbf{u}, \beta)$ and adapt the definitions from the previous section to our new setting.

The target set now becomes $Z^* = \{Z \in \mathcal{Q}_2 \mid Z^2 \subseteq X^*\}$ and the definition of the domain of controllability S and the first hitting time N changes accordingly. For the cost function

$$g_2 : \mathcal{Q}_2 \times U \to \mathbb{R}_0^+, \quad g_2(Z, u) = \sup_{x \in X(Z)} g(x, u),$$

we define the functional

$$J_2(Z_0, \mathbf{u}, \beta) = \sum_{k=0}^{N(Z_0, \mathbf{u}, \beta)} g_2(Z_k(Z_0, \mathbf{u}, \beta), u_k) \in \mathbb{R}_0^+ \cup \{+\infty\}$$

and the optimal value function $V_2(Z) = \sup_{\beta \in \mathcal{B}} \inf_{\mathbf{u} \in U^{\mathbb{N}}} J_2(Z_0, \mathbf{u}, \beta)$. V_2 again fulfills the optimality principle

$$V_2(Z) = \inf_{u \in U} \left\{ g_2(Z, u) + \sup_{\hat\beta \in \widehat{\mathcal{B}}} V_2(F_2(Z, u, \hat\beta)) \right\}. \tag{3.3}$$

and the optimal feedback $\mu_2(Z)$ is given by the argmin of this expression. The following theorem is the counterpart of Theorem 2.4.

Theorem 3.1. *For all $x \in X$ and all $Z \subset S$ with $x \in X(Z)$ the inequality*

$$g(x, \mu_2(Z)) + V_2((\rho(x), \rho(f(x, \mu_2(Z)))^T) \leq V_2(Z)$$

holds. In particular, the inequality holds for $Z = (\delta, \rho(x))^T$.

Proof. Completely analogous to Theorem 2.4. $\qquad\square$

By formally setting $\rho(x_{-1}) = \delta$ we define the closed loop trajectory $x_k(x_0, \mu_2)$ of (2.1) by

$$x_{k+1} = f(x_0, \mu_2((\rho(x_{k-1}), \rho(x_k))^T)), \quad k = 0, 1, 2, \dots$$

Then from Theorem 3.1 by a straightforward induction we obtain

$$J(x_0, \mu_2) = \sum_{k=0}^{N(x_0, \mu_2)} g(x_k(x_0, \mu_2), \mu_2((\rho(x_{k-1}(x_0, \mu_2)), \rho(x_k(x_0, \mu_2)))^T)) \leq V_2((\delta, \rho(x_0))^T),$$

4 Comparison of the two approaches

In the preceding sections we have seen that the optimal value functions V_1 and V_2 yield upper bounds for the performance of the feedback laws μ and μ_2. Our main theorem now shows that the upper bound V_2 for the formulation including the past region is at least as good as the upper bound V_1.

Theorem 4.1. *The optimal value functions V_1 and V_2 satisfy*

$$V_2(Z) \leq V_1(Q) \quad \text{for all } Z \in \mathcal{Q}_2^a, Q \in \mathcal{Q} \text{ with } Z^2 = Q.$$

Proof. We prove the theorem by induction over the elements $Q_1, Q_2, \ldots, Q_l \in \mathcal{Q}$ which we number according to their values in the optimal value function V_1, i.e., $V_1(Q_i) \leq V_1(Q_j)$ for all $1 \leq i < j \leq l$. We will frequently use the obvious inclusion $X(Z) \subseteq Z^2$ for $X(Z)$ from (3.2) and all $Z = (Z^1, Z^2)^T \in \mathcal{Q}_2$.

Induction start $n = 1$: Since $V_1(Q) = 0$ holds if any only if $Q \subseteq X^*$ we obtain $Q_1 \subseteq X^*$. Since $Z \subseteq Z^*$ for all $Z \in \mathcal{Q}_2$ with $Z^2 = Q_1 \subseteq X^*$ we obtain $V_2(Z) = 0 = V_1(Q_1)$ and thus the assertion for Q_1.

Induction step $n = n + 1$: We use the induction hypothesis $V_2(Z) \leq V_1(Q_j)$ for all $j = 0, \ldots, n$ and all $Z \in \mathcal{Q}_2^a$ with $Z^2 = Q_j$ in order to show $V_2(Z) \leq V_1(Q_{n+1})$ for all $Z \in \mathcal{Q}_2^a$ with $Z^2 = Q_{n+1}$.
The optimality principle for V_1 yields

$$V_1(Q_{n+1}) = \inf_{u \in U} \left\{ g_1(Q_{n+1}, u) + \sup_{\hat{\beta} \in \widehat{\mathcal{B}}} V_1(F(Q_{n+1}, u, \hat{\beta})) \right\} = g_1(Q_{n+1}, \mu(Q_{n+1})) + \sup_{\hat{\beta} \in \widehat{\mathcal{B}}} V_1(F(Q_{n+1}, \mu(Q_{n+1}), \hat{\beta})).$$

By positivity of g_1 this implies $V_1(F(Q_{n+1}, \mu(Q_{n+1}), \hat{\beta})) < V_1(Q_{n+1})$ for all $\hat{\beta}$ and thus the numbering of the Q_i yields

$$F(Q_{n+1}, \mu(Q_{n+1}), \hat{\beta}) \in \{Q_1, \ldots, Q_n\}. \tag{4.1}$$

Now the optimality principle for V_2 yields

$$
\begin{aligned}
V_2(Z) &= \inf_{u \in U} \left\{ g_2(Z, u) + \sup_{\hat{\beta} \in \widehat{\mathcal{B}}} V_2(F_2(Z, u, \hat{\beta})) \right\} \\
&\leq g_2(Z, \mu(Q_{n+1})) + \sup_{\hat{\beta} \in \widehat{\mathcal{B}}} V_2(F_2(Z, \mu(Q_{n+1}), \hat{\beta})) = g_2(Z, \mu(Q_{n+1})) + V_2(Z_{\max}),
\end{aligned} \tag{4.2}
$$

where $Z_{\max} = (Q_{n+1}, Q_i)^T$ denotes the element from $\{F_2(Z, \mu(Q_{n+1}), \hat{\beta}) \,|\, \hat{\beta} \in \widehat{\mathcal{B}}\}$ realizing the supremum, which exists because F_2 can only assume finitely many values.
Now $X(Z) \subseteq Q_{n+1}$ implies $Q_i = F(Q_{n+1}, \mu(Q_{n+1}), \hat{\beta})$ for some suitable $\hat{\beta}$ and thus from (4.1) we can conclude $i \leq n$. Furthermore, from the optimality principle for V_1 we obtain

$$V_1(Q_{n+1}) = g_1(Q_{n+1}, \mu(Q_{n+1})) + \sup_{\hat{\beta} \in \widehat{\mathcal{B}}} V_1(F(Q_{n+1}, \mu(Q_{n+1}), \hat{\beta})) \geq g_1(Q_{n+1}, \mu(Q_{n+1})) + V_1(Q_i).$$

Using the induction assumption $V_1(Q_i) \geq V_2(Z_{\max})$ (which is applicable since $i \leq n$) and

$$g_2(Z, \mu(Q_{n+1})) = \sup_{x \in X(Z)} g(x, \mu(Q_{n+1})) \leq \sup_{x \in Q_{n+1}} g(x, \mu(Q_{n+1})) = g_1(Q_{n+1}, \mu(Q_{n+1}))$$

we can continue to estimate $V_1(Q_{n+1}) \geq g_1(Q_{n+1}, \mu(Q_{n+1})) + V_1(Q_i) \geq g_2(Z, \mu(Q_{n+1})) + V_2(Z_{\max})$ which together with (4.2) yields the assertion. $\square$

In practice, we expect V_2 to be considerably smaller than V_1, as the numerical example in the following section as well as further numerical examples in [6] confirm. Theorem 4.1, however, only yields $V_2 \leq V_1$ because system (3.1) may not yield any useful additional information compared to (2.4), which is theoretically possible but appears to be an exceptional case.

5 Numerical Example

We illustrate our approach with the example of an inverted pendulum

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{\frac{g}{l}\sin(x_1) - \frac{1}{2}m_r x_2^2 \sin(2x_1) - \frac{m_r}{ml}\cos(x_1)u}{\frac{4}{3} - m_r \cos^2(x_1)}$$

The equations model the motion of a (planar) inverted pendulum on a cart which moves under an applied horizontal force u. The position x_1 of the pendulum is measured relative to the position of the cart as an offset angle from the vertical up position. The parameters are $M = 8$ (mass of the cart), $m = 2$ (mass of the pendulum), $m_r = m/(m + M)$ (mass ratio), $l = 0.5$ (distance of the pendulum mass from the pivot) and $g = 9.8$ (gravitational constant), see also [1–3]. The discrete time system (2.1) has been obtained by sampling the continuous time system with sampling period $T = 0.1$. The target region X^* was chosen as the neighborhood of the origin consisting of 4x4 regions.
The optimal value functions are computed with a graph theoretic approach, in which each state region Q is represented as a vertex and each transition as a hyperedge in a weighted directed hyphergraph. Then we can compute the optimal
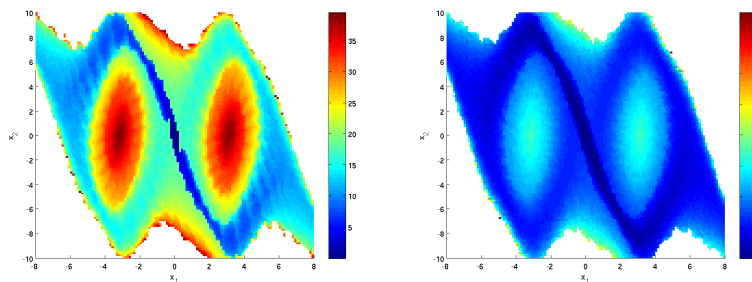
Figure 1: Value function V_1 (left) and value function V_2 (right) for 128x128 regions
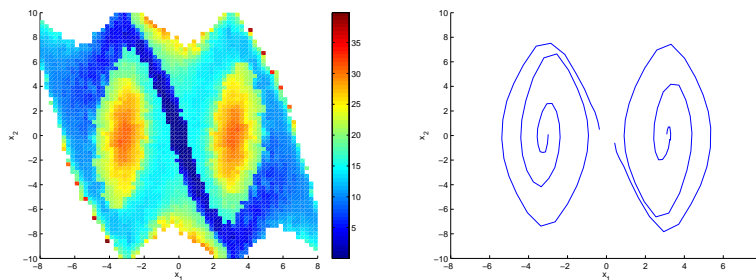
Figure 2: Value function V_2 (left) and trajectories with initial values $(3.1, 0.1)$ and $(-3, 0.1)$ (right) for 64x64 regions

value functions with a min-max version of Dijkstra's shortest path algorithm, see [2, 4] for details.

Figure 1 shows that the values for V_2 are considerably smaller than those for V_1.

On a partition with 64x64 regions, the approach via V_1 is no longer feasible because the domain of controllability S turns out to be empty. In contrast to this, for V_2 we still get get a useful solution and corresponding closed loop trajectory converging to X^* as shown in Figure 2.

The following table indicates that the computation time for V_2 is — at least in our current implementation — significantly larger than for V_1. However, this computation can be performed offline and since in return we get considerably better solutions with fewer regions this appears to be a reasonable price to pay.

number of regions	64^2	128^2	256^2	512^2
computation time for V_1 in seconds	17.7	70.6	286.1	1176.1
computation time for V_2 in seconds	265.8	1154.6	5344.7	

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