Lyapunov's Second Method for Nonautonomous Differential Equations

Lars Grüne, Peter E. Kloeden*, Stefan Siegmund†, Fabian R. Wirth‡

Abstract

Converse Lyapunov theorems are presented for nonautonomous systems modelled as skew product flows. These characterize various types of stability of invariant sets and pullback, forward and uniform attractors in such nonautonomous systems.

Keywords and phrases: Lyapunov function, Lyapunov's second method, nonautonomous dynamical system, nonautonomous differential equation, stability, nonautonomous attractor.

2000 Mathematics Subject Classification. 37B25 37B55 93D30

1 Introduction

Asymptotic stability is one of the corner stones of the qualitative theory of dynamical systems and is of fundamental importance in many applications of the theory in virtually all fields where dynamical effects play a role. In the analysis of stability properties of invariant objects it is very often useful to employ what is now called Lyapunov's second method [4] (see [2] for a random version). This method relies on the observation that asymptotic stability is intimately linked to the existence of a Lyapunov function, that is, a proper, nonnegative function, vanishing only on an invariant set and decreasing along those trajectories of the system not evolving in the invariant set. Lyapunov proved that the existence of a Lyapunov function guarantees asymptotic stability and for linear time-invariant systems also showed the converse statement that asymptotic stability implies the existence of a Lyapunov function. Converse theorems usually are the harder part of the theory and the first general results for nonlinear systems were obtained by Massera [22, 23] and Kurzweil [18, 19]. Converse theorems are interesting because they show the universality of Lyapunov's second method. If an invariant object is asymptotically stable then there exists a Lyapunov function. Thus there is always the possibility that we may actually find it, though this may be hard.

A typical direct and converse result is the following found in Bhatia and Szegö [4, Theorem V.2.12].

Theorem 1. Let φ be a topological dynamical system on a locally compact space X, and let A be a nonvoid compact set which is invariant under φ.

Then A is asymptotically stable if and only if there exists a Lyapunov function for A, i.e. a function $V : X \to \mathbb{R}^+$ such that

(i) V is continuous,

(ii) V is uniformly unbounded, i.e. for all $C > 0$ there exists a compact set $K \subset X$ such that $V(x) \geq C$ for all $x \notin K$,

*Peter Kloeden was partially supported by the Ministerio de Educación y Ciencia in Spain under the Programa de Movilidad del Profesorado universitario español y extranjero, grant SAB2004-0146

†Stefan Siegmund was supported by the Emmy Noether program of the DFG.

‡Fabian Wirth acknowledges support by the Science Foundation Ireland grants 04-IN3-I460 and 00/PI.1/C067.

(iii) V is positive-definite, i.e. $V(x) = 0$ if $x \in A$, and $V(x) > 0$ if $x \notin A$,

(iv) V is strictly decreasing along orbits of φ, i.e. $V(\varphi(t,x)) < V(x)$ for $x \notin A$ and $t > 0$.

Furthermore, V can be chosen to satisfy

$$V(\varphi(t,x)) = e^{-t}V(x) \qquad \text{for all} \quad x \in X, t \in \mathbb{R}.$$

Despite the considerable time and effort that has been spent on developing stability theory, important progress has still been made with respect to the theory of Lyapunov functions in recent times. Probably the most far-reaching extension is Conley's work on global Lyapunov functions with respect to Morse sets of a dynamical system that allows a precise characterization of the system's global behavior [9, 7, 8]. Secondly, for systems in $\mathbb{R}^d$ several results have been obtained that show the existence of smooth Lyapunov functions under minimal assumptions on the regularity of the differential equation [29, 24, 21]. Constructive methods to find Lyapunov functions numerically for arbitrary systems (methods that are usually feasible in low dimensions) have been presented in [32, 6, 13].

Finally, the question of whether the rate of attraction can be recovered by an appropriate choice of a Lyapunov function has been shown for different system classes in [11, 30]. The latter fact has been known for linear time invariant systems since the work of Massera, [22, 23]. One approach to describe attraction rates for general nonlinear systems relies on so-called comparison functions. This approach goes back at least to Hahn [14] and has been popularized again in the past decade by influential works as [26, 15].

In this paper we study the problem of converse Lyapunov theorems for nonautonomous dynamical systems. The asymptotically stable objects are given naturally by pullback, forward and uniform attractors. We prove converse Lyapunov theorems for these attractor types. The focus of the paper lies on obtaining Lyapunov functions that recover certain attraction rates given in terms of comparison functions, that is functions of class $\mathcal{K}$ and class $\mathcal{KL}$. To this end we show how the different notions of stability and attractivity that play a role within the nonautonomous framework can be characterized in terms of attraction rates given by comparison functions. We then show that the existence of an attraction rate in terms of comparison functions is equivalent to the existence of a Lyapunov function guaranteeing this attraction rate. We note that for hyperbolic skew product flows some Lyapunov theory is available in [5].

The paper is organized as follows: In the following Section 2 we introduce the formalism of skew product flows and nonautonomous sets. Invariant objects will be found in this class. In Section 3 we define several notions of stability and attractivity of invariant nonautonomous sets. In particular, the notions of pullback, forward and uniform attractors are defined. For the proofs to come it turns out to be vital that the notions of attractor is defined with respect to the attraction of arbitrary compact sets. We comment on this and show that this implies stability properties as well. In Section 4 Lyapunov functions are defined and it is shown that if the base space of the skew product flow is compact, then only maximal invariant sets can possess Lyapunov functions. In Section 5 we first show that a skew product flow satisfies a decay condition expressed in terms of comparison functions if and only if there exists a Lyapunov function characterizing this decaying behavior. The next step is obtained in Section 6, where it is shown how the different notions of stability and attractivity may be equivalently expressed in terms of nonautonomous comparison functions. The section starts with a case study to highlight the various phenomena that can occur within this theory. The main result of the section is obtained in Subsection 6.2. The final result in Subsection 6.3 then provides Lyapunov and converse Lyapunov theorems for the various stability notions of interest for skew product flows.

Notation: The open ball in $\mathbb{R}^d$ of radius ε centered at x is denoted by $B_\varepsilon(x)$ and its closure is denoted $\overline{B}_\varepsilon(x)$. For $x \in \mathbb{R}^d$ and a closed nonempty set A we define the distance of x to A by

$$\|x\|_A := \min\{\, \|x-y\| \mid y \in A\}\,.$$

For non-empty closed sets A and B the Hausdorff semi-metric $d(A|B)$ is defined by

$$d(A|B) := \sup\{\, \|x\|_B \mid x \in A\}\,.$$

So $d(A|B)$ measures how far A is from B ($d(A|B) = 0$ only implies that $A \subseteq B$), while

$$d_H(A, B) := d(A|B) + d(B|A)$$

denotes the Hausdorff metric.

2 Skew Product Flows and Nonautonomous Sets

The concept of *skew product flows* arose from topological dynamics during the 1960s as a description of dynamical systems with "nonautonomy", i.e. showing an explicit dependence on the actual time rather than just on the elapsed time as in autonomous systems. Since then, skew product flows have extensively been studied, [5, 3, 12, 17, 28]. They are tailor-made to nonautonomous systems such as nonautonomous differential equations

$$\dot{x} = f(t, x)\,.$$

We do not obtain a dynamical system directly from solving the respective differential equation. Instead, the solution gives rise to a so-called *cocycle* over a dynamical system which models the nonautonomy of the equation.

Here is a formal definition, where for the sake of not overburdening the presentation we restrict ourselves to the case of a state space $\mathbb{R}^d$ and continuous two-sided time $\mathbb{R}$.

Definition 2 (Skew Product Flow (SPF))**.** A *skew product flow*, shortly denoted by φ, consists of two ingredients:

(i) A model of the nonautonomy, namely a continuous dynamical system $\theta : \mathbb{R} \times P \to P$, where P is a complete metric space.

(ii) A model of the system perturbed or forced by nonautonomy, namely a *cocycle* φ *over* θ, i.e. a continuous mapping $\varphi : \mathbb{R} \times P \times \mathbb{R}^d \to \mathbb{R}^d$, $(t, p, x) \mapsto \varphi(t, p, x)$, such that the family $\varphi(t, p, \cdot) = \varphi(t, p) : \mathbb{R}^d \to \mathbb{R}^d$ of self-mappings of $\mathbb{R}^d$ satisfies the *cocycle property*

$$\varphi(0, p) = \mathrm{id}_X\,, \quad \varphi(t + s, p) = \varphi(t, \theta(s)p) \circ \varphi(s, p)\,, \tag{1}$$

for all $t, s \in \mathbb{R}$ and $p \in P$. ■

The pair of mappings

$$(\theta, \varphi) : \mathbb{R} \times P \times \mathbb{R}^d \to P \times \mathbb{R}^d\,, \quad (t, p, x) \mapsto (\theta(t, p), \varphi(t, p, x))\,,$$

is called the corresponding *skew product*. If $P = \{p\}$ consists of a single point, then the cocycle φ is a dynamical system on $\mathbb{R}^d$. We often use the less clumsy notation θ_t instead of $\theta(t)$. The well-known trick of making a nonautonomous differential equation

$$\dot{x} = f(t, x) \tag{2}$$

autonomous by introducing a new variable for the time suggests to investigate a corresponding skew product flow with base $P := \mathbb{R}$ and driving system $(t, s) \mapsto \theta_t s := t + s$. However, as P does not depend on f, we should not expect a specific kind of nonautonomy (e.g. periodicity in t) to be captured by this base dynamics. Moreover, P is not compact which may cause additional difficulties. For a fairly general class of right hand sides f the Bebutov flow $(t, p) \mapsto \theta_t p := p(\cdot + t, \cdot)$ on the *hull* $P := H(f) = \mathrm{cl}\{f(\cdot + t, \cdot) : t \in \mathbb{R}\}$ of f can serve as a model for the nonautonomy (Sell [25]). Here the closure is taken with respect to an adequate topology. The evaluation mapping

$$\bar{f} : P \times \mathbb{R}^d \to \mathbb{R}^d, \qquad (p, x) \mapsto p(0, x)$$

satisfies $\bar{f}(\theta_t p, x) = p(t, x)$ and, since $f \in H(f)$ and therefore $\bar{f}(\theta_t f, x) = f(t, x)$, it is a natural "extension" of f to $P \times \mathbb{R}^d$. As a slight abuse of notation we will sometimes omit the bar. Instead of looking at the single equation (2) we consider the associated family of equations

$$\dot{x} = \bar{f}(\theta_t p, x), \qquad p \in P = H(f). \tag{3}$$

By using standard results about linearly bounded equations as in Amann [1] and Arzela-Ascoli's theorem the following may be shown, [3].

Theorem 3 (SPF from Nonautonomous Differential Equation)**.** Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function, and consider the nonautonomous differential equation (2). If $(t, x) \mapsto f(t, x)$ is locally Lipschitz in x and

$$\|f(t, x)\| \leq \alpha(t)\|x\| + \beta(t)\,,$$

where $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$ are locally integrable, then the hull $P := H(f)$ is a metric space (where the closure is taken in $C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ with the compact-open topology), the Bebutov flow $(t, p) \mapsto \theta_t p = p(\cdot + t, \cdot)$ is continuous, and (2) uniquely generates an SPF φ over θ through the solution

$$\varphi(t, p, x) = x + \int_0^t \bar{f}(\theta_s p, \varphi(s, p, x))\, ds \tag{4}$$

of the associated family of equations (3). Moreover, $H(f)$ is compact if and only if $(t, x) \mapsto f(t, x)$ is bounded and uniformly continuous on every set of the form $\mathbb{R} \times K$ where $K \subset \mathbb{R}^d$ is compact.

We now turn to the concepts we need in order to be able to define attractors for nonautonomous systems. In general, there is no reason to assume that these should be autonomous objects themselves. The following notion of sets depending on the parameter p is standard.

Definition 4 (Nonautonomous Set)**.** A function $M : p \mapsto M(p)$ taking values in the non-empty closed/compact/bounded subsets of $\mathbb{R}^d$ is called a *nonautonomous closed/compact/bounded set.* ■

For convenience we will often suppress the p argument of M. The set $M(p)$ is called the p fibre of the nonautonomous set M. In general the term p fibre of an expression will be used in discussing the expression for the specific parameter value p.

Definition 5 (Invariance of Nonautonomous Set)**.** A nonautonomous set M is called *forward invariant* under the SPF φ if $\varphi(t, p, M(p)) \subset M(\theta_t p)$ for all $t \geq 0$. It is called *invariant* if $\varphi(t, p, M(p)) = M(\theta_t p)$ for all $t \in \mathbb{R}$. ■

3 Asymptotic Stability of Nonautonomous Sets

Asymptotic stability is usually defined through the properties of stability and attractivity. For nonautonomous attractors, there are various ways to define stability, as well as attraction. We present some of the standard definitions here.

The following notion of stability is taken from [20, Definition 2.3].

Definition 6 (Pullback Stability)**.** Let φ be an SPF and A be a nonautonomous compact set invariant under φ. Then A is called *(pullback) stable* under φ if for any $\varepsilon > 0$ there exists a function $p \mapsto \delta_\varepsilon(p) > 0$ such that for any $x \in \mathbb{R}^d, p \in P$ the relation $d(x, A(\theta_{-t} p)) \leq \delta_\varepsilon(p)$ implies that $d(\varphi(t, \theta_{-t} p, x), A(p)) \leq \varepsilon$ for any $t \geq 0$.

If, in addition, δ_ε may be chosen so that for each $p \in P$ we have $\delta_\varepsilon(p) \to \infty$ as $\varepsilon \to \infty$, then A is called *globally (pullback) stable.* ■

The next definition is inspired by [2, Definition 4.1]. We recall that a compact set $C \subset \mathbb{R}^d$ is called a neighborhood of $A \subset \mathbb{R}^d$ if $A \subset \operatorname{int} C$. Similarly, a nonautonomous compact set C is a neighborhood of A, if $C(p)$ is a neighborhood of $A(p)$ for all $p \in P$.

Definition 7 (Forward Stability)**.** Let φ be an SPF and A be a nonautonomous compact set invariant under φ. Then A is called *(forward) stable* under φ if for any $\varepsilon > 0$ there exists a nonautonomous compact set C which is a neighborhood of A such that

(i) $d_H(C(p), A(p)) \leq \varepsilon$ for each $p \in P$, i.e. C is ε-close to A,

(ii) $\varphi(t, p, C(p)) \subset C(\theta_t p)$ for all $t \geq 0, p \in P$, i.e. C is forward invariant. ■

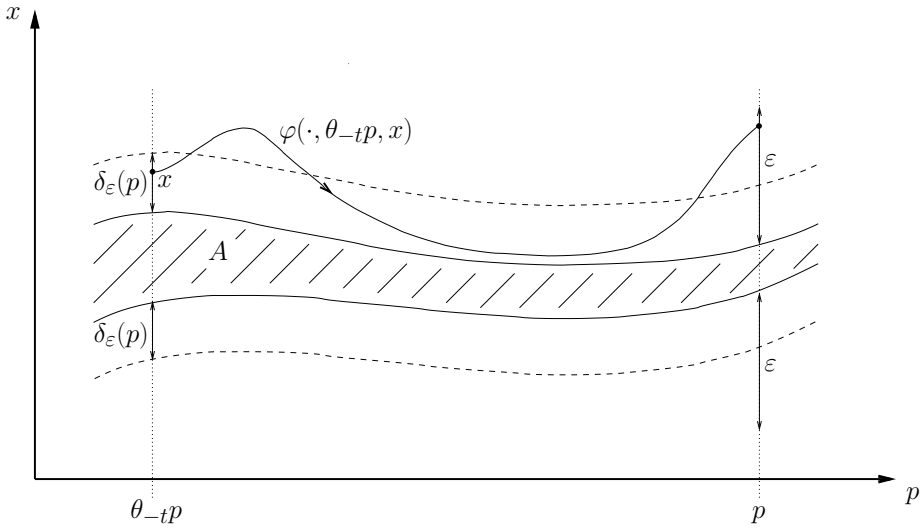

Figure 1: A is pullback stable, i.e. if x (in the $\theta_{-t}p$-fibre) is $\delta_\varepsilon(p)$-close to A then $\varphi(t, \theta_{-t}p, x)$ (in the p-fibre) is ε-close to A.

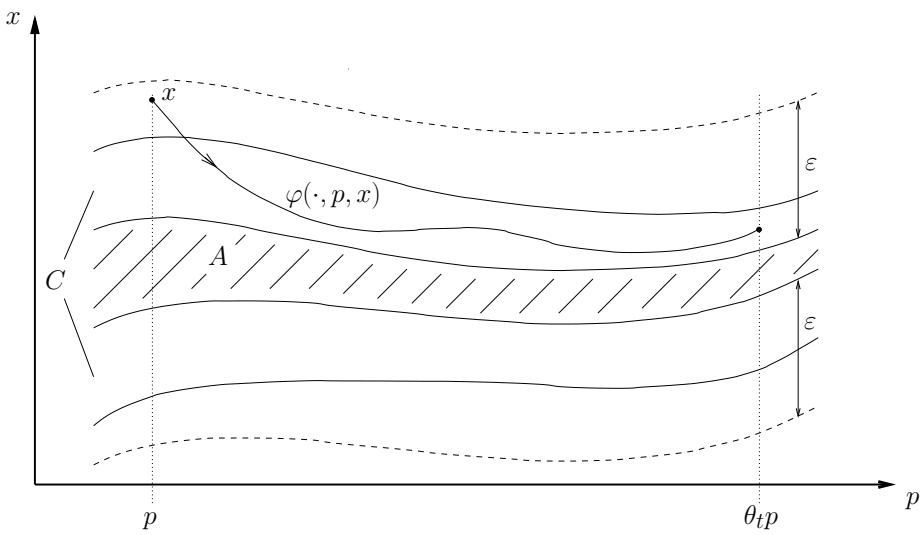

Figure 2: A is forward stable, i.e. there exists an ε-close forward invariant neighborhood C of A.

We note the following property implied by Definition 6 for further reference.

Lemma 8. Let φ be an SPF and A be a nonautonomous compact set which is invariant under φ and pullback stable. Then there exists a bounded forward invariant, nonautonomous set C, such that for every $p \in P$ there exists $\eta(p) > 0$ with $B_{\eta(p)}(A(\theta_{-t}p)) \subset C(\theta_{-t}p)$ for all $t \geq 0$.

Proof. It is sufficient to prove the result along a fixed orbit Γ of θ. Fix $\varepsilon > 0, p \in \Gamma$. By definition, there exists a constant $\delta = \delta(p) > 0$ such that

$$\varphi(t, \theta_{-t}p, B_\delta(A(\theta_{-t}p))) \subset B_\varepsilon(A(p)), \quad \text{for all } t \geq 0. \tag{5}$$

If p is a fixed point of θ, i.e. $\Gamma = \{p\}$, the result is standard in the theory of nonautonomous differential equations. If $t \mapsto \theta_t p$ is periodic with period $T > 0$, define

$$C(p) := \bigcup_{k=0}^{\infty} \varphi(kT, p, B_\delta(A(p))) = \bigcup_{k=0}^{\infty} \varphi(kT, \theta_{-kT}p, B_\delta(A(\theta_{-kT}p))).$$

Then $\varphi(T, p, C(p)) \subset C(p)$, $B_\delta(A(p)) \subset C(p)$ and by (5) $C(p) \subset B_\varepsilon(A(p))$, because every point in $C(p)$ is contained in a set of the form $\varphi(kT, \theta_{-kT}p, B_\delta(A(\theta_{-kT}p)))$. For $t \in [0, T)$ we now define

$$C(\theta_t p) := \varphi(t, p, C(p)),$$

and it follows that C has all the desired properties on Γ. Note that we may define η on Γ uniformly as

$$\eta := \frac{1}{2} \sup\{\gamma > 0 \mid B_\gamma(A(q)) \subset C(q) \text{ for all } q \in \Gamma\}.$$

Assume now that $t \mapsto \theta_t p$ is not periodic. Then we define for $t \geq 0$

$$C(\theta_{-t}p) := \bigcup_{\tau \geq 0} \varphi(\tau, \theta_{-t-\tau}, B_\delta(A(\theta_{-t-\tau}p))).$$

Then for $0 \leq s \leq t$ we have $\varphi(s, \theta_{-t}p, C(\theta_{-t}p)) \subset C(\theta_{-t+s}p)$, $B_\delta(A(\theta_{-t}p)) \subset C(\theta_{-t}p)$ and using (5) we have $C(p) \subset B_\varepsilon(A(p))$. By continuity of φ this implies that all of the sets $C(\theta_{-t}p), t \geq 0$ are bounded. Finally, we define for $t > 0$ the set $C(\theta_t p) := \varphi(t, p, C(p))$. This obviously defines a bounded, forward invariant nonautonomous set C on Γ. For $t \geq 0$ we may set $\eta(\theta_{-t}p) := \delta$. To choose $\eta(\theta_t p)$ for $t \geq 0$ note that for $t \geq 0$ we have

$$0 < \mu := \sup\{\gamma > 0 \mid \forall s \in [0, t] : B_\gamma(A(\theta_s p)) \subset C(\theta_s p)\} < \infty$$

by the continuity of φ, $A(p) \subset \operatorname{int} C(p)$ and as t is finite. We may thus set $\eta(\theta_t p) := \min\{\delta, \mu\}$. This choice satisfies the assertion by construction. □

We now define our notion of attraction, which is based on attraction of compact sets.

Definition 9 (Attractor)**.** Let φ be an SPF and A a nonautonomous compact set which is invariant under φ.

(i) A is called a *pullback attractor* of φ if for every $p \in P$ and every compact set $D \subset \mathbb{R}^d$

$$\lim_{t \to \infty} d_H(\varphi(t, \theta_{-t}p, D), A(p)) = 0$$

(ii) A is called a *forward attractor* of φ if for every $p \in P$ and every compact set $D \subset \mathbb{R}^d$

$$\lim_{t \to \infty} d_H(\varphi(t, p, D), A(\theta_t p)) = 0$$

(iii) A is called a *uniform attractor* of φ if for every compact set $D \subset \mathbb{R}^d$

$$\lim_{t \to \infty} d_H(\varphi(t, p, D), A(\theta_t p)) = 0 \qquad \text{uniformly in } p \in P.$$

■

For introduction and application of pullback attractors, see e.g. [12, 17].

Remark 10. (i) It is easy to see that a uniform attractor is a pullback and a forward attractor. The converse is false in general. An example to this effect can be given as follows. Let $P = [0, 1]$ and assume that all points in P are fixed points under θ, i.e. $\theta_t p \equiv p$. We are thus not really dealing with a nonautonomous system, but with a parameterized family of autonomous systems. Consider the differential equation in $\mathbb{R}$ given by

$$\dot{x} = -px - \exp\left(-\frac{1}{x^2}\right)\left(px + x(x+1)^2(x-1)^2\right) .$$

It is easy to see that the global attractor of the system is given by $A(p) = \{0\}$, $p \in (0, 1]$ and $A(0) = [-1, 1]$. For $p > 0$, the set $A(p) = \{0\}$ is exponentially attacting with rate of attraction $-p$. Hence, the attractor A is not uniform, but of course a pullback and a forward attractor, since there is essentially no dynamics on P.

(ii) Note that in Definition 9 we require attraction of *all compact sets* as opposed to attraction of points only. This issue has been discussed for stochastic systems in [10], from which we cite the following illuminating example. The time-invariant system $\dot{x} = x - x^3$ has the invariant set $A := \{-1, 0, 1\}$, which is a set that attracts all points, but which is not an attractor in the sense of Definition 9. The fixed point $x^* = 0$ is unstable and so the set A is neither pullback nor forward stable. Indeed we will show that our definition of attractivity has some implications on stability as well. Proposition 11 shows that if the attractor is always contained in a given compact set, then pullback attraction implies pullback stability. Without any further assumptions forward attraction always implies forward stability. Example 23 on the other hand shows that it is possible for pullback attractors not to be forward stable and by Example 24 forward attractors need not be pullback stable.

(iii) For autonomous and periodic systems (i.e., $\theta_T p = p$ for some $T > 0$) the definitions of pullback, forward and uniform attractor coincide.

(iv) For some problems it is useful to consider unbounded attracting sets, e.g. for problems in reference tracking. We are forced to assume compactness of the attractor for technical reasons in some of the later proofs. Also we note that our definition of attraction relates this property to the "'universe"' of bounded sets. It appears reasonable that attractors should also belong to this set. When studying attraction properties of an invariant set A we will therefore often assume the existence of a compact set $K \subset \mathbb{R}^d$ with the property

$$\bigcup_{p \in P} A(p) \subset K . \tag{6}$$

Otherwise, we would have to consider examples of the following kind:

$$\dot{x} = t(x - t) + 1 ,$$

where the base space is $P = \mathbb{R}$. Clearly, the diagonal $\{x = t\}$ is an invariant set for the corresponding SPF. It is easy to see that it is a pullback attractor (in fact, the equation is obtained under the transformation $\tilde{x} := x + t$ from Example 23.)

Note that in the above examples all trajectories below the diagonal are not important for attractivity. It appears strange that on "'half"' of the state space the system can be altered arbitrarily without any impact on the *global* attractivity properties of the invariant set.

(iv) We note that some care has to be taken, when performing basic operations on the objects we have defined. Clearly, any good stability concept is invariant under changes of variables. In a time-dependent setting it is natural to allow for time-dependent transformation, but without further conditions, these may destroy stability. This is shown by an example in Section 6.1.

Proposition 11. Let φ be an SPF and A be a nonautonomous compact set invariant under φ.

(i) If A is a pullback attractor and $\bigcup_{p \in P} A(p)$ is bounded, then A is pullback stable.

(ii) If A is a forward attractor, then it is forward stable.

Proof. (i) Assume that A is a pullback attractor that is not pullback stable. Then there exist an $\varepsilon > 0$ and a $p \in P$ such that for all $n \geq 1$ there exist $x_n \in \mathbb{R}^d$ and $t_n \geq 0$ with

$$d(x_n, A(\theta_{-t_n}p)) \leq \frac{1}{n}\,, \quad \text{and } d(\varphi(t_n, \theta_{-t_n}p, x_n), A(p)) \geq \varepsilon\,. \tag{7}$$

By boundedness of $\bigcup_{p\in P} A(p)$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded and we may assume that $\lim_{n\to\infty} x_n =: x^* \in \mathbb{R}^d$ exists. Furthermore, we may choose $\eta > 0$ such that $\|x_n\| \leq \eta$ holds for all $n \in \mathbb{N}$. Then by assumption there exists a $T > 0$ such that for all $t \geq T$ we have

$$d_H(\varphi(t, \theta_{-t}p, \overline{B_\eta}(0)), A(p)) \leq \frac{\varepsilon}{2}\,.$$

This implies for all $n \geq 1$ that $t_n \leq T$ and so $t^* := \lim_{n\to\infty} t_n$ may be assumed to exist. Now the invariance of $A(\theta_t p)$ and the continuity of $\varphi(t, \theta_t p, x)$ in t implies that $A(\theta_t p)$ is continuous in t (although $A(p)$ might not be continuous in p). Thus from the first inequality in (7) we obtain $x^* \in A(\theta_{-t^*}p)$ while the second inequality in (7) implies $\varphi(t^*, \theta_{-t^*}p, x^*) \notin A(p)$. This contradicts the invariance of the attractor.

(ii) Fix $\varepsilon > 0$ and $p \in P$. We assume that A is a forward attractor. To prove forward stability it is sufficient to prove the existence of the sets $C(p)$ satisfying (i) and (ii) of Definition 7 for a single orbit $\{\theta_t p \mid t \in \mathbb{R}\}$, as the requirements for the overall compact set C only relate to particular orbits. So pick $p \in P$ and define

$$\tilde{C}(p) := \overline{B}_\varepsilon(A(p))\,.$$

By forward attraction there exists a $T \geq 0$ such that for all $t \geq T$ we have

$$d_H(\varphi(t, p, \tilde{C}(p)), A(\theta_t p)) \leq \frac{\varepsilon}{2}\,.$$

Define

$$C(p) := \tilde{C}(p) \setminus \{x \in \mathbb{R}^d \mid \exists s \in [0, T] : d(\varphi(s, p, x), A(\theta_s p)) > \varepsilon\}\,,$$

and $C(\theta_t p) := \varphi(t, p, C(p))$ for $t \geq 0$ (or for $t \in [0, T_p)$ if $\theta_t p$ is periodic with period T_p).

Then it is easy to see that $C(p)$ is compact and $A(p) \subset C(p)$. Indeed, $A(p) \subset \operatorname{int} C(p)$ because otherwise we easily obtain a contradiction to the continuity of φ. It follows that for all $t \geq 0$ we have $A(\theta_t p) \subset \operatorname{int} C(\theta_t p)$ and conditions (i) and (ii) of Definition 7 are satisfied. If $\theta_t p$ is not periodic, it remains to extend the construction to negative t. This can be done inductively. Assume we have defined $C(\theta_t p)$ for all $t \in [-n, \infty)$. Then we set

$$\tilde{C}(\theta_{-(n+1)}p) := \varphi(-1, \theta_{-n}p, C(\theta_{-n}p))\,,$$

and

$$C(\theta_{-(n+1)}p) := \tilde{C}(\theta_{-(n+1)}p) \setminus \{x \in \mathbb{R}^d \mid \exists s \in [-(n+1), -n] : d(\varphi(s, p, x), A(\theta_s p)) > \varepsilon\}\,.$$

We now set $C(\theta_{-(n+1)+s}p) := \varphi(s, \theta_{-(n+1)}p, C(\theta_{-(n+1)}p))$ for $s \in (0, 1)$, so that $C(\theta_t p)$ is now defined on $[-(n+1), \infty)$. By the same arguments as before, it follows that the sets $C(\theta_t p)$ satisfy all necessary conditions on the interval $[-(n+1), \infty)$. This shows the assertion. □

Definition 12 (Asymptotic Stability)**.** Let φ be an SPF and A a nonautonomous compact set which is invariant under φ. Then A is called *asymptotically stable* if it is stable and an attractor. ■

We note that the above definition is a bit loose, as we have to distinguish between the six notions of asymptotic stability that can be obtained by combining the two notions of stability with the three notions of attractivity. We will use the appropriate wording to distinguish between these notions where necessary.

4 Lyapunov Functions for Skew Product Flows

We now introduce Lyapunov functions with respect to stability and global attractivity of a compact invariant set. It is also shown that for compact P a Lyapunov function determines a maximal invariant set.

Definition 13 (Lyapunov function)**.** Let φ be an SPF in $\mathbb{R}^d$ and A be a nonautonomous compact set which is invariant under φ. A family of functions $\{V_p : \mathbb{R}^d \to \mathbb{R}^d\}_{p\in P}$ is called a *Lyapunov function* for A (with respect to φ) if it has the following properties:

(i) V is *uniformly unbounded*, i.e. $\lim_{\|x\|\to\infty} V_p(x) = \infty$ for all $p \in P$;

(ii) V is *positive-definite*, i.e. $V_p(x) = 0$ for $x \in A(p)$, and $V_p(x) > 0$ for $x \notin A(p)$;

(iii) V is *strictly decreasing along orbits* of φ, i.e.

$$V_{\theta_t p}(\varphi(t, p, x)) < V_p(x) \qquad \text{for all} \quad t > 0 \text{ and } x \notin A(p).$$

Remark 14. We note that in the previous definition item (i) can be weakened without any harm to the requirement that V be proper, i.e., for all $p \in P$ preimages of compact sets under $V_p(\cdot)$ should be compact if they are contained in the range of $V_p(\cdot)$. Both approaches to the definition of Lyapunov functions can be found in the literature.

First we show that Lyapunov functions ensure the uniqueness of invariant nonautonomous compact sets in the following sense.

Proposition 15. Let φ be an SPF in $\mathbb{R}^d$ and A be a nonautonomous compact set which is invariant under φ. Suppose there exists a Lyapunov function for A. If P is compact, any other invariant nonautonomous compact set A' satisfies $A'(p) \subset A(p)$ for each $p \in P$.

Proof. Assume the assertion is false, so that there are an invariant nonautonomous compact set A', $p \in P$ and $x \in A'(p) \setminus A(p)$. By assumption this implies that $V(p, x) > 0$. By compactness of A' and P and by the unboundedness of V there is a constant C such that $V(q, y) < C$ for all $q \in P, y \in A'(q)$. Now backwards in time $V(\theta_{-t}p, \varphi(-t, p, x))$ is monotonically increasing and bounded by C due to the invariance of A'. Thus $\eta := \lim_{t\to\infty} V(\theta_{-t}p, \varphi(-t, p, x))$ exists, and the α-limit set

$$\alpha(p, x) := \{(q, y) \in P \times \mathbb{R}^d \mid \exists t_k \to \infty : (\theta_{-t_k}p, \varphi(-t_k, p, x)) \to (q, y)\}$$

is contained in the compact set $V^{-1}(\eta)$. (Note that for this compactness argument we need that P is compact.) Now the set $\alpha(p, x)$ is nonempty and invariant under φ. This implies that V is constant along trajectories evolving in $\alpha(p, x)$ in contradiction to the decrease property of Lyapunov functions. □

The following example shows that the assertion of Proposition 15 is false if the assumption of compactness of P is omitted.

Example 16. Suppose $P = X = \mathbb{R}^1$ and let θ be the shift on P. To define the cocycle mapping we introduce the auxiliary function

$$h(t) := \begin{cases} 2 - e^t & \text{if } t \le 0 \\ e^{-t} & \text{if } t \ge 0 \end{cases}.$$

The cocycle is then given through the family of single valued complete orbits

$$x_\gamma : \mathbb{R}^1 \to \mathbb{R}^1 \quad \text{with} \quad x_\gamma(t) = \gamma h(t)\,.$$

Then for each fixed $\gamma \ge 0$ the family of sets

$$A_\gamma(t) = [-x_\gamma(t), x_\gamma(t)]$$

is forward attractive and pullback as well as forward stable. A Lyapunov function for A_γ is given by

$$V_\gamma(t_0, x) := d(x, A_\gamma(t_0))\,.$$

Indeed, if $x > x_\gamma(t_0)$ then there is a $\gamma' > \gamma$ such that $x = \gamma' h(t_0)$ and for $t \geq 0$ we have by the monotone decrease of h that

$$\begin{aligned} V_\gamma(t+t_0, \varphi(t, t_0, x)) &= V_\gamma(t+t_0, \gamma' h(t+t_0)) = d(\gamma' h(t+t_0), A_\gamma(t+t_0)) \\ &= (\gamma' - \gamma) h(t+t_0) < (\gamma' - \gamma) h(t_0) = V_\gamma(t_0, x)\,. \end{aligned} \tag{8}$$

The case $x < -x_\gamma(t_0)$ follows using symmetry. However, the sets A_γ increase as we increase γ, so that the statement of Proposition 15 does not hold in this example.

5 Rate preserving Lyapunov functions

In this section we introduce a finer notion of Lyapunov functions that have the property of characterizing the rate of decay of solutions. To this end we need the following function classes: a continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called of *class* $\mathcal{K}$, if $\gamma(0) = 0$ and γ is strictly increasing. If in addition, it is a homeomorphism of $\mathbb{R}_+$, then it is called of *class* $\mathcal{K}_\infty$. A continuous function $\beta : \mathbb{R}_+^2 \to \mathbb{R}_+$ is called of *class* $\mathcal{KL}$, if it is of class $\mathcal{K}$ in the first argument and decreases monotonically to 0 in the second argument, [14, 15].

5.1 The autonomous case

In order to motivate our approach we first sketch some known results for the autonomous case. We consider an SPF with a singleton base space $P = \{p\}$. Suppose for this SPF we are given a global attractor A, i.e., a compact invariant set with the property

$$d_H(\varphi(t, D), A) \to 0$$

for any compact set $D \subset \mathbb{R}^d$. This is equivalent to the existence of an *attraction rate* $\beta \in \mathcal{KL}$, such that

$$\|\varphi(t, x)\|_A \leq \beta(\|x\|_A, t)$$

holds for all $x \in \mathbb{R}^d$ and all $t \geq 0$, see [11, Remark B.1.5] or [21]. By Sontag's $\mathcal{KL}$–Lemma [27], for any $\mathcal{KL}$ function β there are functions $\rho, \sigma \in \mathcal{K}_\infty$ such that

$$\beta(r, t) \leq \rho(\sigma(r) e^{-t})\,.$$

It is of interest to obtain Lyapunov functions that reflect the growth rates modelled by the functions ρ and σ. Such Lyapunov functions are called *rate preserving.* This is always possible by setting

$$V(x) := \sup_{t \geq 0} \rho^{-1}(\|\varphi(t, x)\|_A) e^t. \tag{9}$$

It is straightforward to verify that this function satisfies

$$\rho^{-1}(\|x\|_A) \leq V(x) \leq \sigma(\|x\|_A) \tag{10}$$

and

$$V(\varphi(t, x)) \leq e^{-t} V(x)\,.$$

Thus V is a Lyapunov function which exactly represents the functions ρ and σ, in the sense that if (9),(10) hold, then $\|\varphi(t, x)\|_A \leq \rho(\sigma(\|x\|_A) e^{-t})$.

This construction, which generalizes a definition from Yoshizawa [31], in general yields a discontinuous Lyapunov function. A slight modification of this construction along with appropriate smoothing techniques result in continuous and even smooth V, even for perturbed dynamical systems [11, Section 3.5], however, at the cost of only approximately representing ρ and σ.

5.2 The nonautonomous case

In our following constructions we assume that the base flow θ does not have periodic or stationary solutions, i.e., that

$$\theta_{t_1} p \neq \theta_{t_2} p \text{ for all } t_1 \neq t_2 \text{ and all } p \in P.$$

If this is not the case then — denoting the original parameter space by $\widetilde{P}$ and the original skew product flow by $(\tilde{\theta}, \tilde{\varphi})$ — we can augment our parameter space by setting

$$P := \widetilde{P} \times \mathbb{R}, \;\; \theta_t(\tilde{p}, s) := (\tilde{\theta}_t \tilde{p}, s + t), \;\; \varphi(t, (\tilde{p}, s), x) = \tilde{\varphi}(t, \tilde{p}, x). \tag{11}$$

In Remark 32 we show how to interpret our results in case of periodic base flows.

A natural idea of generalizing the concept of attraction rates to the nonautonomous setting is to allow β to depend on p. That is, we are interested in "nonautonomous" $\mathcal{KL}$ functions β_p such that we have

$$\|\varphi(t, \theta_{-t} p, x)\|_{A(p)} \leq \beta_p(\|x\|_{A(\theta_{-t} p)}, t). \tag{12}$$

In order to capture both local and global stability effects we use the following definition.

Definition 17. We say that (12) is satisfied *locally* if there exists an open and forward invariant nonautonomous set $C(p) \supset A(p)$, $p \in P$, such that (12) holds for all $t \geq 0$, $p \in P$ and $x \in \mathbb{R}^d$ with $x \in C(\theta_{-t} p)$.
We say that (12) is satisfied *globally* if $C(p) = \mathbb{R}^d$ for all $p \in P$.

In fact, the Lyapunov function construction for pullback attractors of Kloeden [16] yields a global estimate of the form (12) with

$$\beta_p(r, t) = a_p^{-1}(e^{-t} r)\,.$$

As in the autonomous case a suitable class of attraction rates β_p has to be identified for which a similar construction as sketched in Section 5.1 is possible. The main conceptional question is, which structure of β_p is (i) general enough to represent a wide range of different attraction speeds while (ii) still allowing to be "encoded" into a Lyapunov function. To this end the following class of function turns out to be suitable.

Definition 18. A family of functions $\beta_p : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $p \in P$, is called a *nonautonomous $\mathcal{KL}$–function*, if there exists families of $\mathcal{K}_\infty$ functions ρ_p, σ_p, $p \in P$, such that the inequality

$$\beta_p(r, t) \leq \rho_p(\sigma_{\theta_{-t} p}(r) e^{-t}) \tag{13}$$

holds for all $r, t \geq 0$ and all $p \in P$.

In the following we restrict our attention to nonautonomous $\mathcal{KL}$–functions which are given in the form

$$\beta_p(r, t) = \rho_p(\sigma_{\theta_{-t} p}(r) e^{-t}) \tag{14}$$

for suitable families of $\mathcal{K}_\infty$ functions ρ_p, σ_p, $p \in P$.

Note that in this definition ρ and σ depend on different parameters p and $\theta_{-t} p$, respectively. This is natural if we combine (12) and (14), because the argument r of σ measures the distance in the fibre $\theta_{-t} p$ while the value of ρ gives an estimate for the distance in the fibre p.

The following theorem shows that one can indeed encode the information about ρ_p and σ_p from (14) in suitable Lyapunov functions.

Theorem 19. Let β_p be a nonautonomous $\mathcal{KL}$ function satisfying (14) for functions $\rho_p, \sigma_p \in \mathcal{K}_\infty$. An SPF φ satisfies (12) locally on an open, forward invariant, nonautonomous set C for β_p if and only if there exists a family of functions $V_p : C(p) \to \mathbb{R}$ with the properties

$$\rho_p^{-1}(\|x\|_{A(p)}) \leq V_p(x) \leq \sigma_p(\|x\|_{A(p)}) \tag{15}$$

for all $x \in C(p)$ and

$$V_{\theta_t p}(\varphi(t,p,x)) \le e^{-t} V_p(x)\,, \quad \text{for all } x \in C(p). \tag{16}$$

If these equivalent conditions hold then the functions V_p may be chosen to be equal to one of the alternative formulas

$$V_p(x) := \sup_{t\ge 0} \rho^{-1}_{\theta_t p}(\|\varphi(t,p,x)\|_{A(\theta_t p)})e^t, \tag{17}$$

or

$$V_p(x) := \inf_{t\ge 0:\varphi(-t,p,x)\in C(\theta_{-t}p)} \sigma_{\theta_{-t}p}(\|\varphi(-t,p,x)\|_{A(\theta_{-t}p)})e^{-t}. \tag{18}$$

Proof. The existence of V_p with (15) and (16) immediately implies (12), (14).

Conversely, we show that if (12), (14) holds, then both formulas (17) and (18) yield a function satisfying (15) and (16). We start with (17).

The lower inequality in (15) is immediate setting $t = 0$ in (17). For the upper inequality, from (12) and (14) for $t \ge 0$ and $x \in C(\theta_{-t}p)$ we obtain

$$\|\varphi(t,\theta_{-t}p,x)\|_{A(p)} \le \beta_p(\|x\|_{A(\theta_{-t}p)}, t)$$

which, using the transformation $p \to \theta_{-t}p$, implies

$$\|\varphi(t,p,x)\|_{A(\theta_t p)} \le \beta_{\theta_t p}(\|x\|_{A(p)}, t) = \rho_{\theta_t p}(\sigma_p(\|x\|_{A(p)})e^{-t}).$$

for $t \ge 0$ with $x \in C(p)$. This yields

$$\rho^{-1}_{\theta_t p}(\|\varphi(t,p,x)\|_{A(\theta_t p)})e^t \le \sigma_p(\|x\|_{A(p)})$$

for all $t \ge 0$ with $x \in C(p)$, hence (17) satisfies the upper inequality in (15).

Finally, we pick $x \in C(p)$ and $\tau \ge 0$. Due to the forward invariance of the $C(p)$ the value $V_{\theta_\tau p}(\varphi(\tau,p,x))$ is defined and we can estimate

$$\begin{aligned}
V_{\theta_\tau p}(\varphi(\tau,p,x)) &= \sup_{t\ge 0} \rho^{-1}_{\theta_{t+\tau}p}(\|\varphi(t,\theta_\tau p,\varphi(\tau,p,x))\|_{A(\theta_{t+\tau}p)})e^t \\
&= \sup_{t\ge \tau} \rho^{-1}_{\theta_t p}(\|\varphi(t,p,x)\|_{A(\theta_t p)})e^{t-\tau} \\
&= e^{-\tau} \sup_{t\ge \tau} \rho^{-1}_{\theta_t p}(\|\varphi(t,p,x)\|_{A(\theta_t p)})e^t \\
&\le e^{-\tau} \sup_{t\ge 0} \rho^{-1}_{\theta_t p}(\|\varphi(t,p,x)\|_{A(\theta_t p)})e^t \\
&= e^{-\tau} V_p(x)
\end{aligned}$$

holds, we also obtain (16).

In order to show that the formula (18) also yields a suitable Lyapunov function we proceed similarly. Here the upper inequality follows from (18) for $t = 0$. For the lower inequality, from (12) and (14) with $y = \varphi(t,\theta_{-t}p,x) \in C(p)$ we obtain

$$\|y\|_{A(p)} \le \beta_p(\|\varphi(-t,p,y)\|_{A(\theta_{-t}p)}, t)$$

implying

$$\rho_p^{-1}(\|y\|_{A(p)}) \le \sigma_{\theta_{-t}p}(\|\varphi(-t,p,y)\|_{A(\theta_{-t}p)})e^t$$

for all $t \ge 0$ with $\varphi(t,\theta_{-t}p,x) \in C(p)$, hence V_p satisfies the lower inequality in (15).

In order to show (16) for any $\tau > 0$ with $x \in C(p)$ we obtain

$$\begin{aligned}
V_{\theta_\tau p}(\varphi(\tau,p,x)) &= \inf_{t\geq 0:\varphi(-t,\theta_\tau p,\varphi(\tau,p,x))\in C(\theta_{-t}\theta_\tau p)} \sigma_{\theta_{-t}\theta_\tau p}(\|\varphi(-t,\theta_\tau p,\varphi(\tau,p,x))\|_{A(\theta_{-t}\theta_\tau p)})e^{-t} \\
&= \inf_{t\geq 0:\varphi(-t+\tau,p,x)\in C(\theta_{-t+\tau}p)} \sigma_{\theta_{-t+\tau}p}(\|\varphi(-t+\tau,p,x))\|_{A(\theta_{-t+\tau}p)})e^{-t} \\
&\leq \inf_{t\geq \tau:\varphi(-t+\tau,p,x)\in C(\theta_{-t+\tau}p)} \sigma_{\theta_{-t+\tau}p}(\|\varphi(-t+\tau,p,x))\|_{A(\theta_{-t+\tau}p)})e^{-t} \\
&= e^{-\tau} \inf_{t-\tau\geq 0:\varphi(-t+\tau,p,x)\in C(\theta_{-t+\tau}p)} \sigma_{\theta_{-t+\tau}p}(\|\varphi(-t+\tau,p,x)\|_{A(\theta_{-t+\tau}p)})e^{-t+\tau} \\
&= e^{-\tau} \inf_{t\geq 0:\varphi(-t,p,x)\in C(\theta_{-t}p,t)} \sigma_{\theta_{-t}p}(\|\varphi(-t,p,x)\|_{A(\theta_{-t}p)})e^{-t} \\
&= e^{-\tau}V_p(x).
\end{aligned}$$

This proves (16). □

Remark 20. Note that (17) and (18) do not coincide in general. The difference between these two constructions is that in the first formula only the function ρ_p enters the construction explicitly, while in the second only the function σ_p is used.

Note that the Lyapunov function obtained from either (17) or (18) may be discontinuous. The following theorem gives a modified construction which yields a Lyapunov function which is continuous in t and Lipschitz in x.

Theorem 21. Let β_p be a nonautonomous $\mathcal{KL}$ function satisfying (14) for functions $\rho_p, \sigma_p \in \mathcal{K}_\infty$ and consider an SPF φ. Assume that for each $p \in P$ the map

$$(t,x) \mapsto \|\varphi(t,p,x)\|_{A(\theta_t p)}$$

is continuous and Lipschitz in x with uniform Lipschitz constant $L_\varphi(p,R,T)$ for all $t \in [-T,T]$, all $x \in \mathbb{R}^d$ with $\|x\|_{A(p)} \leq R$ and all $R,T > 0$. Assume furthermore that the maps

$$(t,r) \mapsto \rho^{-1}_{\theta_t p}(r) \quad \text{or} \quad (t,r) \mapsto \sigma_{\theta_t p}(r)$$

are continuous and Lipschitz in r with uniform Lipschitz constant $L(p,R,T)$ for all $t \in [-T,T]$, all $r \in [0,R]$ and all $R,T > 0$.

Then φ satisfies (12) locally on an open, forward invariant, nonautonomous set C for β_p if and only if for each $\varepsilon \in (0,1)$ there exists a family of functions $V_p^\varepsilon : C(p) \to \mathbb{R}$ such that for each $p \in P$ the map

$$(t,x) \mapsto V^\varepsilon_{\theta_t p}(x)$$

is continuous and Lipschitz in x, and which satisfies the properties

$$\rho_p^{-1}(\|x\|_{A(p)}) \leq V_p^\varepsilon(x) \leq \sigma_p(\|x\|_{A(p)}) \tag{19}$$

for all $x \in C(p)$ and

$$V^\varepsilon_{\theta_t p}(\varphi(t,p,x)) \leq e^{-(1-\varepsilon)t}V_p^\varepsilon(x)\,, \quad \text{for all } x \in C(p). \tag{20}$$

If these equivalent conditions hold then the functions V_p^ε may be chosen to be equal to one of the alternative formulas

$$V_p^\varepsilon(x) := \sup_{t\geq 0} \rho^{-1}_{\theta_t p}(\|\varphi(t,p,x)\|_{A(\theta_t p)})e^{(1-\varepsilon)t}, \tag{21}$$

or

$$V_p^\varepsilon(x) := \inf_{t\geq 0:\varphi(-t,p,x)\in C(\theta_{-t}p)} \sigma_{\theta_{-t}p}(\|\varphi(-t,p,x)\|_{A(\theta_{-t}p)})e^{-(1-\varepsilon)t}. \tag{22}$$

Proof. The existence of V_p^ε satisfying (19) and (20) for each $\varepsilon > 0$ immediately implies (12), (14).

Conversely, the proof of (19) and (20) for V_p^ε defined by the formulas (21) or (22) is completely analogous to the proof of Theorem 19.

It thus remains to show the asserted continuity property. We will do this for formula (21); similar arguments work for (22).

From our continuity assumptions it follows that the map

$$(t,x) \mapsto w(t,w) := \rho_{\theta_t p}^{-1}(\|\varphi(t,p,x)\|_{A(\theta_t p)})e^{(1-\varepsilon)t}$$

from (21) is continuous and Lipschitz in x with uniform Lipschitz constant $L_w(p,T,R)$ for $t \in [-T,T]$ and $\|x\|_{A(p)} \leq R$. From (14) it follows that for each $p \in P$, $R > 0$ and $\varepsilon \in (0,1)$ the supremum over $w(t,x)$ is a maximum which is attained for

$$t \in [0,T] \quad \text{for} \quad T = T(p,R,\varepsilon) = -\frac{\ln\left(\rho_p^{-1}(R)/\sigma_p(R)\right)}{\varepsilon}$$

for all $x \in C(p)$ with $\|x\|_{A(p)} \leq R$. Thus for $x, y \in C(p)$ with $\|x\|_{A(p)}, \|y\|_{A(p)} \leq R$ we obtain

$$\begin{aligned} |V_p^\varepsilon(x) - V_p^\varepsilon(y)| &= \left| \max_{t\in[0,T]} w(t,x) - \max_{t\in[0,T]} w(t,y) \right| \\ &\leq \max_{t\in[0,T]} |w(t,x) - w(t,y)| \;\leq\; L_w(p,T,R)\|x-y\| \end{aligned}$$

which shows the Lipschitz continuity of V_p^ε in x. Continuity of $(t,x) \mapsto V_{\theta_t p}^\varepsilon(x)$ follows similarly from the continuity of $w(t,x)$ in (t,x). □

Remark 22. (i) Note that the continuity property comes at the cost of a slower decay of V_p^ε, because while (15) remains true for V_p^ε, (16) changes to

$$V_{\theta_t p}^\varepsilon(\varphi(t,p,x)) \leq e^{-(1-\varepsilon)t} V_p^\varepsilon(x) \tag{23}$$

for all $x \in C(p)$, i.e, the decay is slightly slower.

(ii) The continuity assumptions on ρ_p^{-1} and σ_p are rather mild, cf. Remark 28, below.

6 Necessary and sufficient conditions

In this section we prove that the stability and attraction properties for nonautonomous systems are equivalent to the existence of attraction rates which (i) satisfy (12) and (ii) have a suitable limiting behavior. In order to motivate our approach we first illustrate possible limiting behaviors in a case study with several simple examples in Section 6.1. Afterwards, in Section 6.2 we provide the general statements.

6.1 A case study

With our choice of β_p in (14) neither the limiting behavior of $\beta_p(r,t)$ as $t \to \infty$ nor the limiting behavior of $\beta_{\theta_{-t}p}(r,t)$ as $t \to \infty$ is determined. What may seem as a disadvantage is in fact an advantage, because for this reason the estimate (12) can be interpreted as a very flexible device which can characterize several types of long time behavior.

Before we turn to a rigorous classification of the different possible behaviors, we illustrate this fact by explicitly computing rates β_b of the type (14) for a number of simple 1d examples. They fit into the SPF setting by defining

$$P := \mathbb{R}, \qquad \theta_t t_0 = t + t_0.$$

Example 23. Consider the equation

$$\dot{x} = tx =: f(t,x)$$

and the nonautonomous set $A = A(t_0) = \{0\}$ (since in this example the parameter set is time we use the notation t_0 for the fibre of interest). The set A is

(i) a pullback attractor, but not a forward attractor,

(ii) pullback stable, but not forward stable.

A Lyapunov function for this equation is given by $V_{t_0}(x) = |x|e^{-\frac{1}{2}t_0^2 - t_0}$ which is checked using

$$\frac{\partial}{\partial x}V_t(x)f(t,x) + \frac{\partial}{\partial t}V_t(x) = \frac{\partial}{\partial x}V_t(x)tx + \frac{\partial}{\partial t}V_t(x) = -V_t(x), \tag{24}$$

in fact this PDE was used to compute V_t. It is easily seen that (15) is satisfied, if we choose

$$\sigma_{t_0}(r) = re^{-\frac{1}{2}t_0^2 - t_0}\,, \quad \text{and} \quad \rho_{t_0}(r) = re^{\frac{1}{2}t_0^2 + t_0}$$

which leads to

$$\beta_{t_0}(r,t) = re^{\frac{1}{2}t(-t+2t_0)}.$$

Hence we obtain

$$\lim_{t\to\infty}\beta_{t_0}(r,t) = 0 \quad \text{and} \quad \beta_{\theta_t t_0}(r,t) \to \infty \text{ for } t \to \infty\,.$$

The first convergence reflects the pullback attraction while the divergence reflects the non–forward convergence and the instability.

We also use this example to show that state transformations depending on the base space can lead to a change of the notion of stability. Consider the transformation

$$\Psi(t_0,x) = e^{-\frac{1}{2}t_0^2}x\,,$$

then the transformed trajectory $\Psi(t+t_0,\varphi(t,t_0,x))$ satisfies the differential equation

$$\frac{d}{dt}\Psi(t+t_0,\varphi(t,t_0,x)) = \frac{d}{dt}\left(e^{-\frac{1}{2}(t+t_0)^2}\varphi(t,t_0,x)\right)$$
$$= -(t+t_0)e^{-\frac{1}{2}(t+t_0)^2}\varphi(t,t_0,x) + e^{\frac{1}{2}(t+t_0)^2}(t+t_0)\varphi(t,t_0,x) = 0\,.$$

And the differential equation $\dot{x} = 0$ clearly does not have attractive sets.

Example 24. Consider the equation

$$\dot{x} = -tx$$

and the nonautonomous set $A = A(t_0) = \{0\}$. The set A is

(i) not a pullback attractor, but a forward attractor,

(ii) not pullback stable, but forward stable.

Here a Lyapunov function is given by $V_{t_0}(x) = |x|e^{\frac{1}{2}t_0^2 - t_0}$, which again can be checked and was obtained using (24). It follows that (15) is satisfied, if we choose

$$\sigma_{t_0}(r) = re^{\frac{1}{2}t_0^2 - t_0}\,, \quad \text{and} \quad \rho_{t_0}(r) = re^{-\frac{1}{2}t_0^2 + t_0}$$

which leads to

$$\beta_{t_0}(r,t) = re^{\frac{1}{2}t(t-2t_0)}.$$

Hence we obtain

$$\begin{array}{rcll} \beta_{t_0}(r,t) & \to & \infty & \text{for} \quad t\to\infty \text{ and } t_0 \text{ fixed} \\ \beta_{\theta_t t_0}(r,t) & \to & 0 & \text{for} \quad t\to\infty \\ \beta_{\theta_t t_0}(r,t) & \le & re^{\max\{0,-t_0\}^2} & \text{for} \quad t\ge 0 \end{array}$$

In this example the divergence reflects the non–pullback attraction while the convergence shows the forward convergence and the boundedness indicates stability.

Example 25. Consider the equation

$$\dot{x} = \begin{cases} tx, & t < 0 \\ 0, & t \geq 0 \end{cases}$$

and the nonautonomous set $A = A(t_0) = \{0\}$. In this case the set A is

(i) a pullback attractor, but not a forward attractor,

(ii) pullback and forward stable.

A Lyapunov function is obtained by appropriately modifying the V from Example 23 for $t \geq 0$ which leads to

$$V_{t_0}(x) = \begin{cases} |x|e^{-\frac{1}{2}t_0^2 - t_0}, & t_0 < 0 \\ |x|e^{-t_0}, & t_0 \geq 0 \end{cases}$$

which again can be checked by (24). In this example we obtain

$$\sigma_{t_0}(r) = \begin{cases} re^{-\frac{1}{2}t_0^2 - t_0}, & t_0 < 0 \\ re^{-t_0}, & t_0 \geq 0 \end{cases}, \quad \text{and} \quad \rho_{t_0}(r) = \begin{cases} re^{\frac{1}{2}t_0^2 + t_0}, & t_0 < 0 \\ re^{t_0}, & t_0 \geq 0 \end{cases}$$

which leads to

$$\beta_{t_0}(r,t) = \begin{cases} re^{\frac{1}{2}t(-t+2t_0)}, & t_0 < 0 \\ r, & t_0 \geq 0 \text{ and } t_0 - t \geq 0 \\ \beta_0(r, t - t_0), & \text{else} \end{cases}.$$

Thus

$$\begin{array}{rcll} \beta_{t_0}(r,t) & \to & 0 & \text{for} \quad t \to \infty \text{ and } t_0 \text{ fixed} \\ \beta_{\theta_t t_0}(r,t) & \not\to & 0 & \text{for} \quad t \to \infty \\ \beta_{\theta_t t_0}(r,t) & \leq & \beta_0(r, \max\{t_0, 0\}) & \text{for} \quad t \geq 0 \end{array}.$$

The first convergence again reflects the pullback attraction while the non convergence to 0 indicates the non–forward convergence. However, the boundedness of β indicates the stability of A.

Example 26. Consider the equation

$$\dot{x} = \begin{cases} -tx, & t < 0 \\ 0, & t \geq 0 \end{cases}$$

and the nonautonomous set $A = A(t_0) = \{0\}$. In this case the set A is

(i) neither a pullback attractor nor a forward attractor,

(ii) not pullback stable, but forward stable.

We obtain a Lyapunov function by appropriately modifying the V from Example 24 for $t \geq 0$ which leads to

$$V_{t_0}(x) = \begin{cases} |x|e^{\frac{1}{2}t_0^2 - t_0}, & t_0 < 0 \\ |x|e^{-t_0}, & t_0 \geq 0 \end{cases}$$

which again can be checked using (24). This yields

$$\sigma_{t_0}(r) = \begin{cases} re^{\frac{1}{2}t_0^2 - t_0}, & t_0 < 0 \\ re^{-t_0}, & t_0 \geq 0 \end{cases} \quad \text{and} \quad \rho_{t_0}(r) = \begin{cases} re^{-\frac{1}{2}t_0^2 + t_0}, & t_0 < 0 \\ re^{t_0}, & t_0 \geq 0 \end{cases}$$

which leads to

$$\beta_{t_0}(r,t) = \begin{cases} re^{\frac{1}{2}t(t-2t_0)}, & t_0 < 0 \\ r, & t_0 \geq 0 \text{ and } t_0 - t \geq 0 \\ \beta_0(r, t - t_0), & \text{else} \end{cases}.$$

Thus

$$\begin{aligned}
\beta_{t_0}(r,t) &\to \infty && \text{for} \quad t \to \infty \text{ and } t_0 \text{ fixed}\,,\\
\beta_{\theta_t t_0}(r,t) &\not\to 0 && \text{for} \quad t \to \infty\,,\\
\beta_{\theta_t t_0}(r,t) &\le \beta_0(r, \max\{t_0, 0\}) && \text{for} \quad t \ge 0\,.
\end{aligned}$$

Neither of the limits is 0 which shows that neither pullback nor forward attraction holds. However, the boundedness of β indicates that A is stable.

6.2 Necessary and sufficient $\mathcal{KL}$ conditions

The following proposition gives necessary and sufficient conditions for our different types of stability and attraction in terms of nonautonomous $\mathcal{KL}$ functions.

Proposition 27 (Necessary and Sufficient $\mathcal{KL}$ Conditions for Stability and Attraction)**.** Let φ be an SPF in $\mathbb{R}^d$ and A be a nonautonomous compact set which is invariant under φ. Then

(i) A is pullback stable if and only if there exists a nonautonomous $\mathcal{KL}$ function β_p satisfying (12) locally with

$$\lim_{r\to 0} \sup_{t\ge 0} \beta_p(r,t) = 0 \quad \forall p \in P$$

on a nonautonomous set C such that for each $p \in P$ there exists $\eta(p) > 0$ with $B_{\eta(p)}(A(\theta_{-t}p)) \subset C(\theta_{-t}p)$ for all $t \ge 0$. A is globally pullback stable, if and only if in addition

$$\sup_{t\ge 0} \beta_p(r,t) < \infty$$

holds for each $r \ge 0$ and (12) is satisfied globally.

(ii) A is forward stable if and only if there exists a nonautonomous $\mathcal{KL}$ function β_p satisfying (12) locally with

$$\lim_{r\to 0} \sup_{t\ge 0} \beta_{\theta_t p}(r,t) = 0.$$

(iii) A is pullback attracting if and only if there exists a nonautonomous $\mathcal{KL}$ function β_p satisfying (12) globally such that for each $r > 0$

$$\lim_{t\to\infty} \beta_p(r,t) = 0, \quad \forall p \in P.$$

(iv) A is forward attracting and forward stable if and only if there exists a nonautonomous $\mathcal{KL}$ function β_p satisfying (12) such that for each $r > 0$, $p \in P$,

$$\lim_{t\to\infty} \beta_{\theta_t p}(r,t) = 0, \quad \forall p \in P.$$

(v) A is uniformly attracting and pullback stable with δ_ε independent of p, if and only if there exists an autonomous $\mathcal{KL}$ function β such that (12) is satisfied with $\beta_p \equiv \beta$.

In all these cases the nonautonomous $\mathcal{KL}$–functions with the stated properties can be chosen such that equality holds in (14).

Proof. **Sufficiency:** We first show that the existence of the nonautonomous $\mathcal{KL}$ functions with the stated properties is sufficient for the respective stability properties.

(i) Let $\varepsilon > 0$. Then for each $p \in P$ there exists a $\delta_\varepsilon(p) > 0$ such that

$$\beta_p(r,t) \le \varepsilon \qquad \text{for all } t \ge 0 \text{ and } r \le \delta_\varepsilon(p)\,.$$

Without loss of generality we can choose $\delta_\varepsilon(p) \le \eta(p)$. Then, using the decay inequality (12) we get that

$$\|x\|_{A(\theta_{-t}p)} \le \delta_\varepsilon(p) \qquad \text{implies} \qquad \|\varphi(t,\theta_{-t}p,x)\|_{A(p)} \le \varepsilon \qquad \text{for all } t \ge 0,$$

proving that A is pullback stable. If the additional requirement holds then for each $r > 0$, $p \in P$ we find $b_r \in \mathbb{R}$ with

$$\beta_p(r,t) \le b_r(p) \qquad \text{for all } t \ge 0$$

which implies that for $\varepsilon \ge b_r$ we can choose $\delta_\varepsilon(p) = r$. Thus, $\delta_\varepsilon(p) \to \infty$ as $\varepsilon \to \infty$.

(ii) Let $\varepsilon > 0$. Then for each $p \in P$ there exists a $\delta = \delta(p) > 0$ such that

$$\beta_{\theta_t p}(r,t) \le \varepsilon \qquad \text{for } t \ge 0 \text{ and } r \le \delta(p).$$

As in (i) without loss of generality we can choose $\delta_\varepsilon(p) \le \eta(p)$. With inequality (12) we get that

$$\|x\|_{A(p)} \le \delta(p) \qquad \text{implies} \qquad \|\varphi(t,p,x)\|_{A(\theta_t p)} \le \varepsilon \qquad \text{for } t \ge 0. \tag{25}$$

We define the nonautonomous set

$$C(p) := \bigcup_{t\ge 0} \varphi(t,\theta_{-t}p, B_{\delta(\theta_{-t}p)}(A))$$

and show that it is contained in the ε-neighborhood of A and is forward invariant under φ.

Let $x \in C(p)$, then there exists a $t_0 \ge 0$ with $x \in \varphi(t_0,\theta_{-t_0}p, B_{\delta(\theta_{-t_0}p)}(A))$, i.e. $x = \varphi(t_0,\theta_{-t_0}p,y)$ for a y with $\|y\|_{A(\theta_{-t_0}p)} \le \delta(\theta_{-t_0}p)$. Using (25) with $x = y$, $t = t_0$ and $\theta_{-t_0}p$ instead of p we get $\|x\|_{A(p)} \le \varepsilon$.

To show that C is forward invariant we use the cocycle property to see that

$$\begin{aligned}
\varphi(t,p)C(p) &= \bigcup_{s\ge 0} \varphi(t,p,\varphi(s,\theta_{-s}p, B_{\delta(\theta_{-s}p)}(A))) \\
&= \bigcup_{s\ge 0} \varphi(t+s,\theta_{-s}p, B_{\delta(\theta_{-s}p)}(A)) \\
&= \bigcup_{s\ge 0} \varphi(t+s,\theta_{-(t+s)}\theta_t p, B_{\delta(\theta_{-(t+s)}\theta_t p)}(A)) \\
&= \bigcup_{s\ge t} \varphi(s,\theta_{-s}\theta_t p, B_{\delta(\theta_{-s}\theta_t p)}(A)) \quad \subset \quad C(\theta_t p)
\end{aligned}$$

(iii) Since $A(p) \subset K$, $p \in P$ for a compact set $K \subset \mathbb{R}^d$

$$\sup_{t\ge 0} \|x\|_{A(\theta_{-t}p)} =: r < \infty$$

for each fixed $x \in \mathbb{R}^d$ and $p \in P$. Inequality (12) yields

$$\|\varphi(t,\theta_{-t}p,x)\|_{A(p)} \le \beta_p(\|x\|_{A(\theta_{-t}p)},t) \le \beta_p(r,t) \to 0 \qquad \text{as } t \to \infty,$$

proving that A is a pullback attractor.

(iv) Using inequality (12) we get

$$\|\varphi(t,p,x)\|_{A(\theta_t p)} \le \beta_{\theta_t p}(\|x\|_{A(p)},t) \to 0 \qquad \text{as } t \to \infty,$$

proving that A is a forward attractor.

(v) As in (iii) we get

$$\|\varphi(t,p,x)\|_{A(\theta_t p)} \le \beta_{\theta_t p}(\|x\|_{A(p)},t) = \beta(\|x\|_{A(p)},t) \overset{t\to\infty}{\to} 0 \qquad \text{uniformly in } p \in P,$$

proving that A is a uniform attractor. Furthermore we get

$$\|\varphi(t,p,x)\|_{A(\theta_t p)} \le \beta_{\theta_t p}(\|x\|_{A(p)},0) = \beta(\|x\|_{A(p)},0)$$

proving that for the $\mathcal{K}_\infty$–function $\alpha(r) = \beta(r,0)$ we can chose $\delta_e ps(p) = \alpha^{-1}(\varepsilon)$ which is obviously independent of p.

Necessity: Now we show that the existence of the nonautonomous $\mathcal{KL}$ functions with the stated properties is also necessary. Here, in the proofs of (i)–(iv) for each orbit Γ of θ we pick a parameter value $p_0(\Gamma) \in P$ and for each value $p \in P$ on the orbit Γ we define the time $t_0(p) \in \mathbb{R}$ by $\theta_{t_0(p)} p_0(\Gamma) = p$. Note that this time satisfies

$$t_0(\theta_t p) = t_0(p) + t. \tag{26}$$

(i) By Lemma 8 there exists a forward invariant $C(p)$ with the asserted properties such that in the global case $C(p) = \mathbb{R}^d$ holds for each $p \in P$ while in the local case each $C(p)$ is bounded. Forward invariance of $C(p)$ implies

$$\bigcup_{t\ge 0} \varphi(t, \theta_{-t}p, C(\theta_{-t}p)) = C(p). \tag{27}$$

We now define functions $\tilde{\alpha}_p$ by

$$\tilde{\alpha}_p(r) := d_H\left(\bigcup_{t\ge 0} \varphi(t,\theta_{-t}p, C(\theta_{-t}p) \cap B_r(A(\theta_{-t}p))),\ A(p)\right).$$

From the stability property we obtain that $r \le \delta_\varepsilon(p)$ implies $\tilde{\alpha}_p(r) \le \varepsilon$ which in particular implies $\tilde{\alpha}_p(r) \to 0$ as $r \to 0$. In the global case this also ensures finiteness of $\tilde{\alpha}_p$ while in the non–global case (27) and the boundedness of the $C(p)$ does so. Thus we can find class $\mathcal{K}_\infty$ functions α_p with $\tilde{\alpha}_p \le \alpha_p$.

Now we define

$$\rho_p(r) := \alpha_p(e^{t_0(p)} r) \quad \text{and} \quad \sigma_p(r) := e^{-t_0(p)} r.$$

From the construction it immediately follows that ρ_p and σ_p are of class $\mathcal{K}_\infty$. This definition of σ_p implies the inequality

$$\sigma_{\theta_{-t}p}(r)e^{-t} = e^{-t_0(\theta_{-t}p)} r e^{-t} = e^{-t_0(p)+t} e^{-t} r = e^{-t_0(p)} r.$$

For $x \in C(\theta_{-t}p)$ this yields

$$\|\varphi(t,\theta_{-t}p,x)\|_{A(p)} \le \alpha_p(\|x\|_{A(\theta_{-t}p)}) = \rho_p(e^{-t_0(p)}\|x\|_{A(\theta_{-t}p)}) = \rho_p(\sigma_{\theta_{-t}p}(\|x\|_{A(\theta_{-t}p)})e^{-t}).$$

Thus, $\beta_p(t,r) = \rho_p(\sigma_{\theta_{-t}p}(r)e^{-t})$ from (14) satisfies (12) on $C(p)$ and

$$\sup_{t\ge 0} \beta_p(r,t) = \sup_{t\ge 0} \rho_p(\sigma_{\theta_{-t}p}(r)e^{-t}) = \sup_{t\ge 0} \rho_p(e^{-t_0(p)} r) = \alpha_p(r)$$

which shows the desired properties since $\alpha_p(r) \to 0$ for $r \to 0$ and $\alpha_p(r) < \infty$ for all $r \ge 0$.

(ii) We fix an arbitrary $\varepsilon_0 > 0$ and use the nonautonomous set $C(p)$ from the stability property for $\varepsilon = \varepsilon_0$. Now we define

$$\tilde{\alpha}_p(r) := d_H\left(\bigcup_{t\ge 0} \varphi(t,p, C(p) \cap B_r(A(p))),\ A(\theta_t p)\right).$$

From the choice of $C(p)$ we obtain that $\tilde{\alpha}_p$ is bounded and from the stability assumption we have that $\tilde{\alpha}_p(r) \to 0$ as $r \to 0$. Thus we can find a $\mathcal{K}_\infty$ function α_p with $\tilde{\alpha}_p \le \alpha_p$.

Now we proceed similar to (i), above: we define

$$\rho_p(r) := e^{t_0(p)} r \quad \text{and} \quad \sigma_p(r) := e^{-t_0(p)} \alpha_p(r)$$

which yields $\mathcal{K}_\infty$ functions satisfying

$$\|\varphi(t,p,x)\|_{A(\theta_t p)} \le \alpha_p(\|x\|_{A(p)}) = e^{t_0(p)} \sigma_p(\|x\|_{A(p)}) = e^{t_0(\theta_t p)} \sigma_p(\|x\|_{A(p)}) e^{-t} = \rho_{\theta_t p}(\sigma_p(\|x\|_{A(p)}) e^{-t}).$$

Thus, $\beta_p(t,r) = \rho_p(\sigma_{\theta_{-t}p}(r) e^{-t})$ satisfies (12) on $C(p)$ and

$$\sup_{t\ge 0} \beta_{\theta_t p}(r,t) = \sup_{t\ge 0} \rho_{\theta_t p}(\sigma_p(r) e^{-t}) = \sup_{t\ge 0} \rho_{\theta_t p}(e^{-t_0(\theta_t p)} \alpha_p(r)) = \alpha_p(r)$$

which shows the desired properties since $\alpha_p(r) \to 0$ for $r \to 0$.

(iii) We define the function

$$\tilde{\zeta}(r,t) := \sup_{\tau \ge t} d_H\left(\varphi(\tau, \theta_{-\tau} p_0, B_r(A(p_0))),\ A(p_0)\right)$$

for $r \ge 0$ and $t \in \mathbb{R}$. This function is continuous, monotone increasing in r, monotone decreasing in t and satisfies $\tilde{\zeta}(0,t) = 0$ for each $t \ge 0$ and $\lim_{t\to\infty} \tilde{\zeta}(r,t) = 0$ for each $r \ge 0$. Hence we obtain that

$$\zeta(r,t) := \tilde{\zeta}(r,t) + r e^{-t}$$

is of class $\mathcal{KL}$ and — by construction of $\tilde{\zeta}$ — satisfies

$$\|\varphi(t, \theta_{-t} p_0, x)\|_{A(p_0)} \le \zeta(\|x\|_{A(\theta_{-t} p_0)}, t)$$

for all $x \in \mathbb{R}^d$ and all $t \in \mathbb{R}$.

Now we define the functions

$$\rho_p(r) := d_H\left(\varphi(t_0(p), p_0, B_{e^{t_0(p)} r}(A(p_0))),\ A(p)\right) \text{ and } \sigma_p(r) = \zeta(r, t_0(p)) e^{-t_0(p)}.$$

These functions are of class $\mathcal{K}_\infty$, furthermore ρ_p satisfies

$$\|\varphi(t_0(p), p_0, x)\|_{A(p)} \le \rho_p(\|x\|_{A(p_0)} e^{-t_0(p)})$$

for all $p \in P$, $x \in \mathbb{R}^d$, and σ_p satisfies

$$\sigma_{\theta_t p_0}(r) e^t = \zeta(r,t)$$

for all $t \in \mathbb{R}$, $r \ge 0$.

From these inequalities we obtain

$$\begin{aligned}
\|\varphi(t, \theta_{-t} p, x)\|_{A(p)} &= \|\varphi(t_0(p), p_0, \varphi(t - t_0(p), \theta_{-t+t_0(p)} p_0, x))\|_{A(p)} \\
&\le \rho_p(\|\varphi(t - t_0(p), \theta_{-t+t_0(p)} p_0, x)\|_{A(p_0)} e^{-t_0(p)}) \\
&\le \rho_p(\sigma_{\theta_{-t+t_0(p)} p_0}(\|x\|_{A(\theta_{-t+t_0 p} p_0)}) e^{-t+t_0(p)} e^{-t_0(p)}) \\
&= \rho_p(\sigma_{\theta_{-t} p}(\|x\|_{A(\theta_{-t} p)} e^{-t})) \ =: \ \beta_p(\|x\|_{A(\theta_{-t} p)}, t)
\end{aligned}$$

which shows (12), (14). Since from the same inequalities we obtain

$$\beta_p(r,t) = \rho_p(\sigma_{\theta_{-t} p}(r) e^{-t}) = \rho_p(\sigma_{\theta_{-t+t_0(p)} p_0}(r) e^{-t+t_0(p)} e^{-t_0(p)}) = \rho_p(\zeta(r, t_0(p) + t) e^{-t_0(p)}) \to 0$$

as $t \to \infty$, the function β_p satisfies the asserted limit property.

(iv) We define the function

$$\tilde{\zeta}(r,t) := \sup_{\tau \ge t} d_H\left(\varphi(\tau, p_0, B_r(A(p_0))),\ A(\theta_t p_0)\right)$$

for $r \geq 0$ and $t \in \mathbb{R}$. This function is continuous, monotone increasing in r, monotone decreasing in t and satisfies $\tilde{\zeta}(0,t) = 0$ for each $t \geq 0$ and $\lim_{t\to\infty} \tilde{\zeta}(r,t) = 0$ for each $r \geq 0$. Hence we obtain that

$$\zeta(r,t) := \tilde{\zeta}(r,t) + re^{-t}$$

is of class $\mathcal{KL}$ and — by construction of $\tilde{\zeta}$ — satisfies

$$\|\varphi(t,p_0,x)\|_{A(\theta_t p_0)} \leq \zeta(\|x\|_{A(p_0)}, t)$$

for all $x \in \mathbb{R}^d$ and all $t \in \mathbb{R}$.

Now we define the functions

$$\rho_p(s) := \zeta(t_0(p), e^{t_0(p)}s) \text{ and } \sigma_p(r) := d_H\left(\varphi(-t_0(p), p, B_r(A(p))),\ A(p_0)\right) e^{-t_0(p)}.$$

These functions are of class $\mathcal{K}_\infty$, furthermore ρ_p satisfies

$$\rho_{\theta_{t_0(p)}p_0}(re^{-t_0(p)}) = \zeta(r, t_0(p))$$

for all $p \in P$, $r \geq 0$, and for $y = \varphi(-t_0(p), p, x)$ the function σ_p satisfies

$$\|y\|_{A(p_0)} \leq \sigma_p(\|x\|_{A(p)} e^{t_0(p)})$$

for all $p \in P$, $x \in \mathbb{R}^d$.

From these inequalities we obtain

$$\begin{aligned}
\|\varphi(t,p,x)\|_{A(\theta_t p)} &= \|\varphi(t+t_0(p), p_0, \underbrace{\varphi(-t_0(p),p,x)}_{=y})\|_{A(\theta_t p)} \\
&\leq \zeta(\|y\|_{A(p_0)}, t+t_0(p)) \\
&= \rho_{\theta_{t_0(p)+t}p_0}(\|y\|_{A(p_0)} e^{-t_0(p)-t}) \\
&= \rho_{\theta_t p}(\|y\|_{A(p_0)} e^{-t_0(p)} e^{-t}) \\
&\leq \rho_{\theta_t p}(\sigma_p(\|x\|_{A(p)}) e^{-t}) \ =: \ \beta_{\theta_t p}(\|x\|_{A(p)}, t)
\end{aligned}$$

which shows (12), (14). Since by the same computation we obtain

$$\beta_{\theta_t p}(r,t) = \rho_{\theta_t p}(\sigma_p(r)e^{-t}) = \rho_{\theta_{t_0(p)+t}p_0}(\sigma_p(r)e^{t_0}(p)e^{-t_0(p)-t}) = \zeta(\sigma_p(r)e^{t_0(p)}, t_0(p)+t) \to 0$$

as $t \to \infty$, the function β_p satisfies the asserted limit property.

(v) From the uniformity of the attraction we obtain that for all $\varepsilon > 0$ and all $R \geq 0$ there exists $T > 0$ such that for all $p \in P$ the inequality $\|x\|_{A(p)} \leq R$ implies $\|\Phi(s,p,x)\|_{A(\theta_t p)} \leq \varepsilon$ for all $t \geq T$.

The stability assumption yields that for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all $p \in P$ the inequality $\|x\|_{A(\theta_{-t}p)} \leq \delta_\varepsilon$ implies $\|\Phi(t, \theta_{-t}p, x)\|_{A(p)} \leq \varepsilon$ for all $t \geq 0$. By substituting $\theta_{-t}p \to p$ this yields the implication

$$\|x\|_{A(p)} \leq \delta \quad \Rightarrow \quad \|\Phi(t,p,x)\|_{A(\theta_t p)} \leq \varepsilon.$$

Thus, [11, Remark B.1.5] or [21] imply the existence of $\beta \in \mathcal{KL}$ with

$$\|\Phi(t,p,x)\|_{A(\theta_t p)} \leq \beta(\|x\|_{A(p)}, t)$$

for all $t \geq 0$, $p \in P$ and $x \in \mathbb{R}^d$. By Sontag's $\mathcal{KL}$–Lemma [27] each $\mathcal{KL}$–function is also a nonautonomous $\mathcal{KL}$–function with σ and ρ independent of p, and the assertion follows. □

Remark 28. Note that the functions ρ_p and σ_p constructed in (i)–(iv) satisfy the continuity assumptions in Theorem 21 if the respective functions α_p and ζ used in the construction satisfy this property. In (i) and (ii) we can use the regularization techniques from [11, Appendix B] in order to obtain this property while in (iii) and (iv) this property is inherited from the continuity assumption on $(t,x) \mapsto \|\varphi(t,p,x)\|_{A(\theta_t p)}$ in Theorem 21.

In (v), again the regularization techniques from [11, Appendix B] can be applied in order to obtain Lipschitz continuity of ρ^{-1} and σ.

6.3 Necessary and sufficient Lyapunov function conditions

The following main theorem of our paper combines Theorem 19 and Proposition 27.

Theorem 29. For a nonautonomous system and a nonautonomous compact and invariant set A the following properties hold.

(i) A is pullback stable if and only if there exists a local Lyapunov function satisfying (15) and (16) with
$$\lim_{r\to 0}\sup_{t\geq 0}\sigma_{\theta_{-t}p}(r)e^{-t}=0.$$
on a nonautonomous set $C(p)$ such that for each $p\in P$ there exists $\eta(p)>0$ with $B_{\eta(p)}(A(\theta_{-t}p))\subset C(\theta_{-t}p)$ for all $t\geq 0$. A is globally pullback stable if and only if, in addition, the Lyapunov function is global and
$$\sup_{t\geq 0}\sigma_{\theta_{-t}p}(r)e^{-t}<\infty$$
holds for each $r\geq 0$.

(ii) A is forward stable if and only if there exists a local Lyapunov function satisfying (15) and (16) with
$$\lim_{r\to 0}\sup_{t\geq 0}\rho_{\theta_t p}(re^{-t})=0.$$

(iii) A is a pullback attractor if and only if there exists a global Lyapunov function satisfying (15) and (16) with
$$\lim_{t\to\infty}\sigma_{\theta_{-t}p}(r)e^{-t}=0$$
for each $r\geq 0$.

(iv) A is a forward attractor if and only if there exists a global Lyapunov function satisfying (15) and (16) with
$$\lim_{t\to\infty}\rho_{\theta_t p}(re^{-t})=0.$$
for each $r\geq 0$.

(iv) A is a uniform attractor and pullback stable with δ_ε independent of p if and only if there exists a local Lyapunov function satisfying (15) and (16) with σ_p and ρ_p which are independent of p.

Proof. The existence of the Lyapunov functions with the stated bounds follows from Proposition 27 followed by applying Theorem 19, using the fact that the nonautonomous $\mathcal{KL}$ functions in Proposition 27 are of the form $\beta_p(r,t)=\rho_p(\sigma_{\theta_{-t}p}(r)e^{-t})$.

The converse implications follow from applying Theorem 19 followed by Proposition 27, observing that in case (v) the independence of δ_ε of p is immediate from the independence of the bounds on V of p. □

Remark 30. Expressed in terms of the Lyapunov function V_p, the conditions from Theorem 29 imply

(i) $\lim_{r\to 0}\sup_{\|x\|_{A(\theta_t p)}\leq r}\sup_{t\leq 0}V_{\theta_t p}(x)e^t=0$ (pullback stability)
($e^tV_{\theta_t p}$ does not blow up locally for $t\to-\infty$)

$\sup_{\|x\|_{A(\theta_t p)}\leq r}\sup_{t\leq 0}V_{\theta_t p}(x)e^t<\infty$ for each $r>0$ (global pullback stability)
($e^tV_{\theta_t p}$ does not blow up globally for $t\to-\infty$)

(ii) $\sup_{\|x\|_{A(\theta_t p)}\leq r}\inf_{t\geq 0}V_{\theta_t p}(x)e^t>0$ for each $r>0$ (forward stability)
($e^tV_{\theta_t p}$ does not vanish for $t\to\infty$)

(iii) $\lim_{t\to-\infty} V_{\theta_t p}(x)e^t = 0$ for each $r > 0$ (pullback attractor)
($e^t V_{\theta_t p}$ vanishes for $t \to -\infty$)

(iv) $\sup_{\|x\|_{A(\theta_t p)}\le r} \lim_{t\to\infty} V_{\theta_t p}(x)e^t = \infty$ for each $r > 0$ (forward attractor)
($e^t V_{\theta_t p}$ blows up for $t \to \infty$)

If the bounds in (15) are tight (i.e., when σ_p and ρ_p are the smallest possible bounds in (15), which is always the case when the Lyapunov functions are generated by Theorem 19), then the conditions in Theorem 29 are, in turn, implied by these Lyapunov function conditions.

Example 31. We illustrate Theorem 29 and Remark 30 by the examples from Section 6.1 plotting the respective phase portraits functions $(t, x) \mapsto e^t V_t(x)$ in Figures 3.

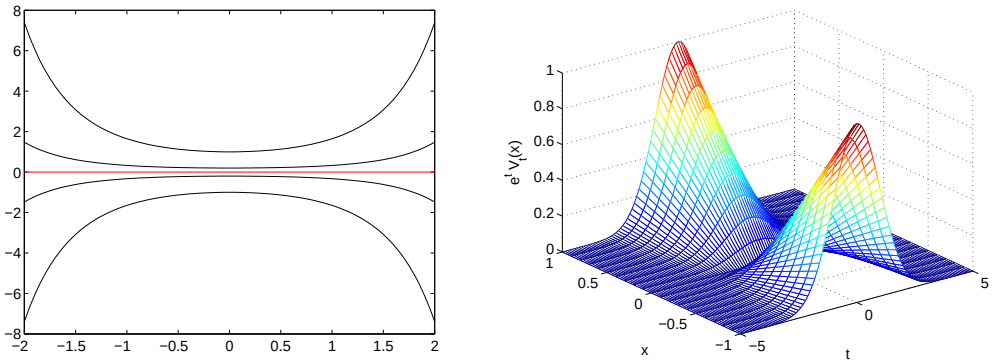

Figure 3: Phase portrait (left) and $(t, x) \mapsto e^t V_t(x)$ (right) for Example 23

Figure 3 shows that the Lyapunov function from example 23 vanishes both for $t \to +\infty$ and for $t \to -\infty$. This implies that $A = \{0\}$ is a pullback attractor and pullback stable but no forward attractor and not forward stable.

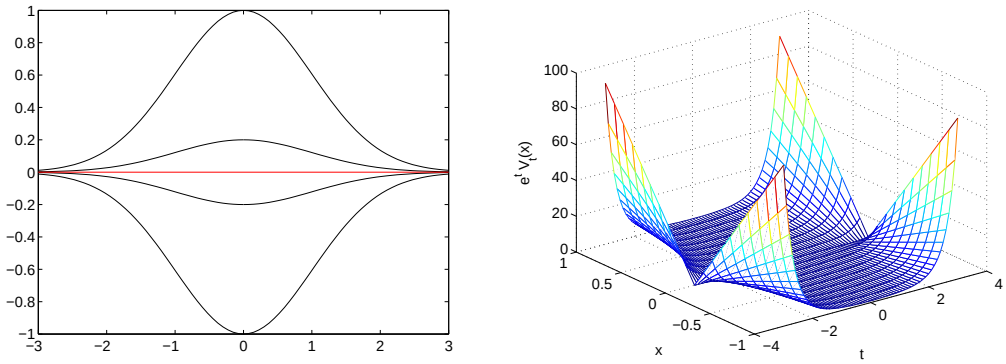

Figure 4: Phase portrait (left) and $(t, x) \mapsto e^t V_t(x)$ (right) for Example 24

Figure 4 shows that the Lyapunov function from example 24 blows up both for $t \to +\infty$ and for $t \to -\infty$. This implies that $A = \{0\}$ is no pullback attractor and not pullback stable but it is a forward attractor and it is forward stable.

Figure 5 shows that the Lyapunov function from example 25 does neither blow up nor vanish for $t \to +\infty$ but it vanishes for $t \to -\infty$. This implies that $A = \{0\}$ is a pullback attractor and pullback and forward stable, but is is no forward attractor.

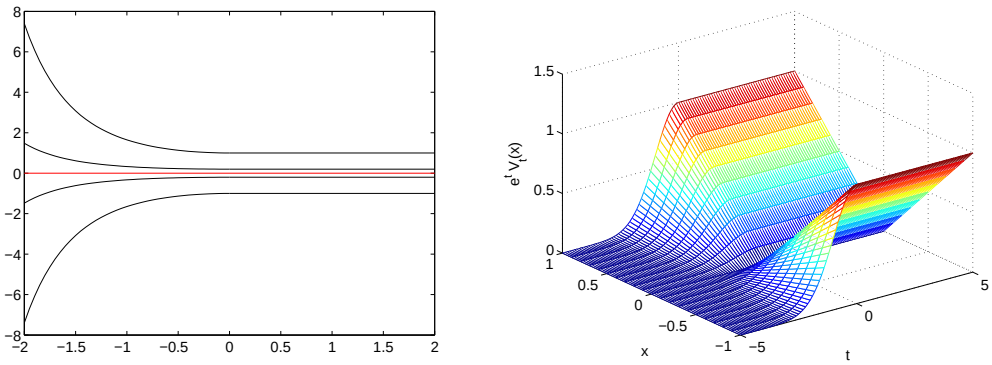

Figure 5: Phase portrait (left) and $(t, x) \mapsto e^t V_t(x)$ (right) for Example 25

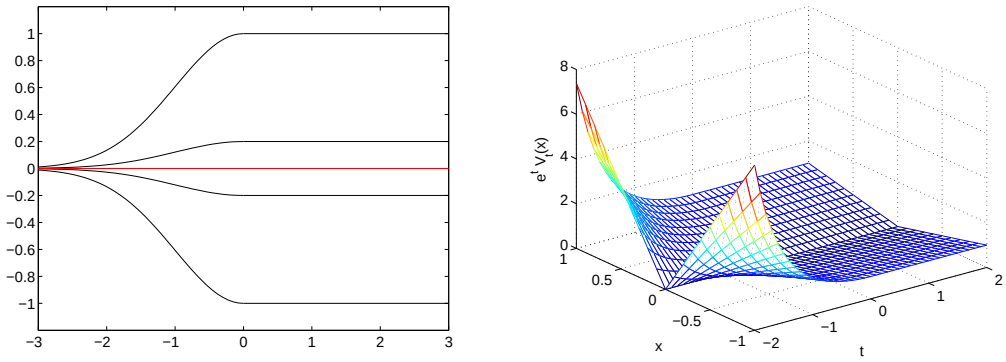

Figure 6: Phase portrait (left) and $(t, x) \mapsto e^t V_t(x)$ (right) for Example 26

Finally, Figure 6 shows that the Lyapunov function from example 26 does neither blow up nor vanish for $t \to +\infty$ and blows up for $t \to -\infty$. This implies that $A = \{0\}$ neither a pullback nor a forward attractor and that the system is forward stable but not pullback stable.

Remark 32. In our construction the non–periodicity of the base flow is crucial because we have constructed the comparison functions as well as the Lyapunov functions independently for each $\theta_t p$, $t \in \mathbb{R}$.

If, however, our original base flow $\tilde{\theta}_t \tilde{p}$ has periodic solutions with a certain period $T > 0$ which we removed by the augmentation (11), then from our Lyapunov function V_p we can construct a Lyapunov function $\widetilde{V}_{\tilde{p}}$ for the original base flow:

This construction is rather straightforward in case of a *pullback or forward attractor*: in this case, for an augmented periodic orbit of the base flow using continuity arguments and periodicity of the solution $\varphi(t, \theta_t p, x)$ in t one easily sees that both pullback and forward attractivity imply uniform attractivity and uniform stability. Thus, we are in the situation of Proposition 27(v) which yields that both ρ_p and σ_p can be chosen independent of p. Consequently, the Lyapunov function constructed in Theorem 19 satisfies

$$V_{\theta_T p}(x) = V_p(x),$$

i.e., it is T–periodic along periodic orbits. Thus, in order to obtain $\widetilde{V}_{\tilde{p}}$ on a periodic orbit $\tilde{\theta}_t \tilde{p}_0$ we can simply define

$$\widetilde{V}_{\tilde{\theta}_t \tilde{p}_0}(x) := V_{(\tilde{\theta}_t \tilde{p}_0, t)}(x),$$

which is well defined, since $V_{(\tilde{\theta}_t\tilde{p}_0,t)} = V_{(\tilde{\theta}_{t+T}\tilde{p}_0,t+T)}$. This defines a Lyapunov function for the original skew product flow in the sense of Definition 13, satisfies (15) with ρ and σ independent of $\tilde{p}$ and inherits the exponential decay property (16) from V_p.

In the case of mere *pullback or forward stability* one cannot expect to obtain a periodic *and* decaying Lyapunov function $\widetilde{V}_{\tilde{p}}$ for the original skew product flow, because the existence of such a function would immediately imply pullback and forward attraction. Thus, if one aims at constructing a periodic Lyapunov function $\widetilde{V}_{\tilde{p}}$, then this function can only expected to be bounded along solutions. Such a bounded function can be constructed as follows: if our nonautonomous set is pullback stable or forward stable then the functions constructed in the necessity part of the proof of Proposition 27 (i) and (ii), respectively, satisfy

$$\rho_{\theta_T p}(r) = \rho_p(re^{-T}) \quad \text{and} \quad \sigma_{\theta_T p}(r) = \sigma_p e^{-T}.$$

Consequently, the Lyapunov function constructed in Theorem 19 satisfies

$$V_{\theta_T p}(x) = V_p(x)e^{-T},$$

i.e., it is T–periodic up to the factor e^{-T}. Thus, if we fix a point $\tilde{p}_0$ on a periodic orbit of the original base flow $\tilde{\theta}$ and define

$$\widetilde{V}_{\tilde{\theta}_t\tilde{p}_0}(x) := e^t V_{(\tilde{\theta}_t\tilde{p}_0,t)}(x)$$

then we obtain a T–periodic function for the original skew product flow. This function satisfies Properties (i) and (ii) of Definition 13 and satisfies (15) with T–periodic bounds $\rho_{\tilde{p}}$ and $\sigma_{\tilde{p}}$. However, it only satisfies Properties (iii) of Definition 13 with "$\leq$" instead of "$<$" implying that it remains bounded but is not necessarily strictly decaying along solutions (in fact, it is strictly decaying if and only if the set A is attracting).

References

[1] H. Amann. *Ordinary differential equations*, volume 13 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1990.

[2] L. Arnold and B. Schmalfuss. Lyapunov's second method for random dynamical systems. *J. Differential Equations*, 177(1):235–265, 2001.

[3] A. Berger and S. Siegmund. On the gap between random dynamical systems and continuous skew products. *J. Dynam. Differential Equations*, 15(2-3):237–279, 2003. Special issue dedicated to Victor A. Pliss on the occasion of his 70th birthday.

[4] N. P. Bhatia and G. P. Szegő. *Stability theory of dynamical systems*. Classics in Mathematics. Springer-Verlag, Berlin, 2002.

[5] I. U. Bronstein and A. Y. Kopanskii. *Smooth Invariant Manifolds and Normal Forms*. World Scientific Publishing Co., River Edge, NJ, 1994.

[6] F. Camilli, L. Grüne, and F. Wirth. A generalization of Zubov's method to perturbed systems. *SIAM J. Control Optim.*, 40(2):496–515 (electronic), 2001.

[7] C. Conley. *Isolated invariant sets and the Morse index*, volume 38 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I., 1978.

[8] C. Conley. The gradient structure of a flow. I. *Ergodic Theory Dynam. Systems*, 8*(Charles Conley Memorial Issue):11–26, 9, 1988.

[9] C. Conley and R. Easton. Isolated invariant sets and isolating blocks. In *Advances in differential and integral equations (Conf. Qualitative Theory of Nonlinear Differential and Integral Equations, Univ. Wisconsin, Madison, Wis., 1968; in memoriam Rudolph E. Langer (1894–1968))*, pages 97–104. Studies in Appl. Math., No. 5. Soc. Indust. Appl. Math., Philadelphia, Pa., 1969.

[10] H. Crauel. Random point attractors versus random set attractors. *J. London Math. Soc. (2)*, 63(2):413–427, 2001.

[11] L. Grüne. *Asymptotic behavior of dynamical and control systems under perturbation and discretization*, volume 1783 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[12] L. Grüne and P. E. Kloeden. Discretization, inflation and perturbation of attractors. In *Ergodic theory, analysis, and efficient simulation of dynamical systems*, pages 399–416. Springer, Berlin, 2001.

[13] S. F. Hafstein. A constructive converse Lyapunov theorem on exponential stability. *Discrete Contin. Dyn. Syst.*, 10(3):657–678, 2004.

[14] W. Hahn. *Stability of motion*. Translated from the German manuscript by Arne P. Baartz. Die Grundlehren der mathematischen Wissenschaften, Band 138. Springer-Verlag New York, Inc., New York, 1967.

[15] H. K. Khalil. *Nonlinear systems*. Macmillan Publishing Company, New York, 1992.

[16] P. E. Kloeden. A Lyapunov function for pullback attractors of nonautonomous differential equations. In *Proceedings of the Conference on Nonlinear Differential Equations (Coral Gables, FL, 1999)*, volume 5 of *Electron. J. Differ. Equ. Conf.*, pages 91–102 (electronic), San Marcos, TX, 2000. Southwest Texas State Univ.

[17] P. E. Kloeden and S. Siegmund. Bifurcations and continuous transitions of attractors in autonomous and nonautonomous systems. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 15(3):743–762, 2005.

[18] J. Kurzweil. On the reversibility of the first theorem of Lyapunov concerning the stability of motion. *Czechoslovak Math. J.*, 5(80):382–398, 1955.

[19] J. Kurzweil and I. Vrkoč. On the converses of Ljapunov's theorem on stability and Persidskiĭ's theorem on uniform stability. *Amer. Math. Soc. Transl. (2)*, 29:271–288, 1963.

[20] J. A. Langa, J. C. Robinson, and A. Suárez. Stability, instability, and bifurcation phenomena in non-autonomous differential equations. *Nonlinearity*, 15(3):887–903, 2002.

[21] Y. Lin, E. D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM J. Control Optim.*, 34(1):124–160, 1996.

[22] J. L. Massera. On Liapounoff's conditions of stability. *Ann. of Math. (2)*, 50:705–721, 1949.

[23] J. L. Massera. Contributions to stability theory. *Ann. of Math. (2)*, 64:182–206, 1956.

[24] T. Nadzieja. Construction of a smooth Lyapunov function for an asymptotically stable set. *Czechoslovak Math. J.*, 40(115)(2):195–199, 1990.

[25] G. R. Sell. *Topological dynamics and ordinary differential equations*. Van Nostrand Reinhold Co., London, 1971.

[26] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.

[27] E. D. Sontag. Comments on integral variants of ISS. *Systems Control Lett.*, 34(1-2):93–100, 1998.

[28] S. Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2 of *Texts in Applied Mathematics*. Springer-Verlag, New York, NY, 2nd edition, 2003.

[29] F. W. Wilson, Jr. Smoothing derivatives of functions and applications. *Trans. Amer. Math. Soc.*, 139:413–428, 1969.

[30] F. Wirth. A converse Lyapunov theorem for linear parameter-varying and linear switching systems. *SIAM J. Control Optim.*, 44(1):210–239 (electronic), 2005.

[31] T. Yoshizawa. *Stability theory by Liapunov's second method.* Publications of the Mathematical Society of Japan, No. 9. The Mathematical Society of Japan, Tokyo, 1966.

[32] V. I. Zubov. *Methods of A. M. Lyapunov and their application.* Translation prepared under the auspices of the United States Atomic Energy Commission; edited by Leo F. Boron. P. Noordhoff Ltd, Groningen, 1964.