Suboptimality estimates for receding horizon controllers

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***Abstract*— Receding horizon control is a well established approach for control of systems with constraints and nonlinearities. Optimization over an infinite time-horizon, which is often computationally intractable, is therin replaced by a sequence of finite horizon problems. This paper provides a method to quantify the performance degradation that comes with this approximation. Results are provided for problems both with and without terminal costs and constraints. Stability proofs follow as special cases.**

***Keywords*— Receding horizon control, model predictive control, dynamic programming**

I. Introduction

Receding horizon control (RHC), often also termed model predictive control (MPC), is by now a well established method for the optimal control of linear and nonlinear systems [1], [3], [9]. On way of interpreting this method in a discrete time setting is the following: In order to approximate the solution to a (computationally intractable) infinite horizon optimal control problem, a sequence of — often suitably constrained — finite horizon optimal control problems is solved. Then in each time step the first element of the resulting optimal control sequence is used as a feedback control value for the current state.

This interpretation immediately leads to the question about the suboptimality of the resulting RHC feedback: how good is the resulting RHC controller with respect to the original infinite horizon cost functional? Despite the fact that this seems to be a very natural question, it is hardly ever addressed in the RHC literature. In fact, the only paper we are aware of which deals with this question is [4], where, however, only a negative result about the monotonicity of the infinite horizon performance of RHC controllers is shown.

A property which is related to our question is the inverse optimality of RHC controllers: it is well known that under suitable conditions RHC controllers are infinite horizon inversely optimal, i.e., they are optimal for an infinite horizon optimal control problem with a suitably adjusted running cost, see, e.g., [2], [9], [8]. However, this property does not yield immediate estimates for the suboptimality with respect to the original running cost. Furthermore, inverse optimality statements usually make strong assumptions on the terminal cost of the finite horizon problems and are not applicable to problems with arbitrary or without terminal cost and constraints. It is the goal of the present paper to close this gap.

It should be noted that also stability results for receding horizon schemes without terminal costs and constraints are quite recent [5], [6] and that our results imply stability via the infinite horizon suboptimality and thus also contribute to the stability of RHC schemes.

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II. Setup

We consider a nonlinear discrete time system given by

$$x(n+1) = f(x(n), u(n)), \quad x(0) = x_0 \tag{2.1}$$

with $x(n) \in X$ and $u(n) \in U$ for $n \in \mathbb{N}_0$. Here we denote the space of control sequences $u : \mathbb{N}_0 \to U$ by $\mathcal{U}$ and the solution trajectory for some $u \in \mathcal{U}$ by $x_u(n)$.

Our goal is to find a feedback control law $u = \mu(x)$ minimizing the infinite horizon cost

$$J_\infty(x_0, u) = \sum_{n=0}^{\infty} l(x_u(n), u(n)), \tag{2.2}$$

with running cost $l : X \times U \to \mathbb{R}_0^+$. We denote the optimal value function for this problem by

$$V_\infty(x_0) = \inf_{u \in \mathcal{U}} J_\infty(x_0, u).$$

If this optimal value function is known, it is easy to prove using Bellman's optimality principle that the optimal feedback law μ is given by

$$\mu(x) := \operatorname*{argmin}_{u \in U} \{V_\infty(f(x,u)) + l(x,u)\}.$$

Remark 2.1: We assume throughout this paper that in all relevant expressions the minimum with respect to $u \in U$ is attained. Although it is possible to give modified statements using approximate minimizers, we decided to make this assumption in order to simplify and streamline the presentation. □

Since infinite horizon optimal control problems are often computationally infeasible, we use a receding horizon approach in order to compute a controller by considering the finite horizon problem given by

$$J_N(x_0, u) = \sum_{n=0}^{N-1} l(x_u(n), u(n)) \tag{2.3}$$

for $N \in \mathbb{N}_0$ (using $\sum_{n=0}^{-1} = 0$) with optimal value function

$$V_N(x_0) = \inf_{u \in \mathcal{U}} J_N(x_0, u). \tag{2.4}$$

A variant of this approach often considered in the literature is obtained when we add a terminal cost $F : X \to \mathbb{R}_0^+$ to the problem. In this case, (2.3) is changed to

$$J_N(x_0, u) = \sum_{n=0}^{N-1} l(x_u(n), u(n)) + F(x_u(N)). \quad (2.5)$$

Another common extension is the introduction of a terminal constraint set $X_f \subset X$ for the finite horizon optimization, which amounts to replacing (2.4) by

$$V_N(x_0) = \inf_{u \in \mathcal{U}: x_u(N) \in X_f} J_N(x_0, u). \quad (2.6)$$

Here we assume that the set X_f is forward invariant, i.e., for each $x \in X_f$ there exists $u \in U$ with $f(x, u) \in X_f$. When imposing such a terminal constraint, the domain of points on which V_N is defined is restricted to the feasible set X_N, which is the set of initial values which can be controlled to X_f in at most N steps, i.e.,

$$X_N := \{x_0 \in X \,|\, \text{there exists } u \in \mathcal{U} \text{ with } x_u(N) \in X_f\}.$$

Note that (2.3) is a special case of (2.5), with $F \equiv 0$, and that (2.4) is a special case of (2.6) with $X_f = X$. Here we have stated (2.3) and (2.4) explicitly because it is the simplest version of receding horizon control and a major part of our results apply particularly to this case.

Based on this finite horizon optimal value function we define a feedback law μ_N by picking the first element of the optimal control sequence for this problem. Since Bellman's optimality principle for the functions V_N reads

$$V_N(x) = \min_u \{V_{N-1}(f(x, u)) + l(x, u)\}, \quad (2.7)$$

this amounts to defining

$$\mu_N(x) := \operatorname*{argmin}_u \{V_{N-1}(f(x, u)) + l(x, u)\}. \quad (2.8)$$

Note that the feedback law μ_N is not the optimal controller for the problem (2.4). However, the optimal trajectory for this problem can be expressed via the controllers $\mu_1, \ldots, \mu_N$ in the following inductive fashion

$$\begin{array}{l} x(0) = x_0 \\ x(n+1) = f(x(n), \mu_{N-n}(x(n))) \\ \text{for } N = 0, \ldots, N-1. \end{array} \quad (2.9)$$

The goal of the present paper is to give estimates about the suboptimality of the feedback μ_N for the infinite horizon problem. More precisely, if x_{μ_N} denotes the solution of the closed loop system

$$x_{\mu_N}(n+1) = f(x_{\mu_N}(n), \mu_N(x_{\mu_N}(n))), \quad x_{\mu_N}(0) = x_0$$

and we define the infinite horizon cost corresponding to μ_N by

$$V_\infty^{\mu_N}(x_0) := \sum_{n=0}^{\infty} l(x_{\mu_N}(n), \mu_N(x_{\mu_N}(n)))$$

then we are interested in upper bounds for this infinite horizon value, either in terms of the finite horizon optimal value function V_N or in terms of the infinite horizon optimal value function V_∞. In particular, the latter will give us estimates about the "degree of suboptimality" of the controller μ_N.

A result closely related to our problem is the inverse optimality of receding horizon schemes, see [9, Section 3.5], [8] or [2]. This result states that the controller μ_N is an infinite horizon optimal controller for the cost

$$\begin{aligned} \bar{l}(x, u) \;:=\; & l(x, u) + V_{N-1}(f(x, \mu_N(x))) \\ & - V_N(f(x, \mu_N(x))) \end{aligned}$$

and that V_N is the corresponding infinite horizon optimal value function. The importance of this result lies in the fact that it establishes infinite horizon optimality for the resulting controller. However, its disadvantage is that it only does so for the new running cost $\bar{l}$. Thus, this result does not tell us much about the performance of μ_N with respect to the *original* cost l, which is what we are interested in.

Note that in (undiscounted) infinite horizon optimal control one is in general interested in nonnegative running cost functions, in order to be able to conclude, e.g., stability of the closed loop system. Thus, in this context the inverse optimality result is only useful if $\bar{l}(x, u) \geq 0$, implying the condition $V_{N-1}(f(x, \mu_N(x))) - V_N(f(x, \mu_N(x))) \geq -l(x, u)$. We will use a similar condition in the sequel.

III. Preliminary results

The approach we take in this paper relies on recently developed results on relaxed dynamic programming [7], [10]. Before we formulate these results, we need to introduce some invariance concepts. These are necessary because we will formulate our suboptimality estimates for general subsets $\widetilde{X} \subseteq X$. Those subsets, however, must have certain invariance properties as defined next.

Definition 3.1: Let $\widetilde{X} \subseteq X$ be a subset of the state space.

(i) We call $\widetilde{X}$ optimally invariant *for V_N for some $N \in \mathbb{N}$, if for each $x \in \widetilde{X}$ there exists an optimal optimal trajectory $x(n)$ with $x(0) = x$ satisfying*

$$x(n) \in \widetilde{X} \text{ for all } n = 0, \ldots, N-1.$$

(ii) We call $\widetilde{X}$ invariant with respect to the feedback *μ_N for some $N \in \mathbb{N}$ if for each $x \in \widetilde{X}$ the property*

$$f(x, \mu_N(x)) \in \widetilde{X}$$

holds. □

Remark 3.2: Note that Definition 3.1 (i) and (ii) are indeed different concepts, because μ_N is not the optimal feedback law for the problem (2.4)*, cf.* (2.9)*. However,* (2.9) *immediately yields that (i) implies (ii), because the feedback μ_N defines the first element of the optimal control sequence for* (2.4)*. Conversely,* (2.9) *yields that if (ii) holds for $\mu_1, \ldots, \mu_N$, then (i) is implied for this N. Condition (ii) for $\mu_1, \ldots, \mu_N$, however, is in general stronger than (i).* □

Remark 3.3: We will frequently use the following consequence from Definition 3.1 (i) and (2.9)*: If we assume* *optimal invariance of $\widetilde{X}$ for V_{N-1} and inductively define the sets $\widetilde{X}_N := \widetilde{X}$, $\widetilde{X}_{N-1} := \widetilde{X}$ and*

$$\widetilde{X}_{k-1} := \{f(x,\mu_k(x))\,|\,x\in\widetilde{X}_k\} \quad \textit{for} \quad k = N-1,\dots,1,$$

then the optimal invariance implies the inclusion

$$\widetilde{X}_k \subseteq \widetilde{X}.$$

□

Note that the global case $\widetilde{X} = X$ is always included in our setting as a special case. In this global case, both invariance conditions of Definition 3.1 are automatically satisfied.

Now we turn to the mentioned relaxed dynamic programming results. Here we use slight variants of the results in [7], [10] which are more adapted to our receding horizon setting.

Proposition 3.4: Consider a set $\widetilde{X} \subseteq X$, a feedback law $\tilde\mu : \widetilde{X} \to U$ satisfying $f(x,\tilde\mu(x)) \in \widetilde{X}$ for all $x \in \widetilde{X}$ and a function $\widetilde{V} : \widetilde{X} \to \mathbb{R}_0^+$ satisfying the inequality

$$\widetilde{V}(x) \geq \widetilde{V}(f(x,\tilde\mu(x))) + \alpha l(x,\tilde\mu(x)) \qquad (3.1)$$

for some $\alpha \in [0,1]$ and all $x \in \widetilde{X}$. Then for all $x \in \widetilde{X}$ the estimate

$$\alpha V_\infty(x) \leq \alpha V_\infty^{\tilde\mu}(x) \leq \widetilde{V}(x)$$

holds.

Proof: The proof is similar to that of [10, Proposition 3]: Consider $x_0 \in \widetilde{X}$, the trajectory $x(n) = x_{\tilde\mu}(n)$ generated by the closed loop system using $\tilde\mu$, and the control sequence generated by $u(n) = \tilde\mu(x(n))$. Then the trajectory lies in $\widetilde{X}$ for all $n \in \mathbb{N}_0$ and from (3.1) we obtain

$$\alpha l(x(n),u(n)) \leq \widetilde{V}(x(n)) - \widetilde{V}(x(n+1)).$$

Thus, summing over n yields

$$\alpha \sum_{n=0}^{N-1} l(x(k),u(k)) \leq \widetilde{V}(x(0)) - \widetilde{V}(x(N)) \leq \widetilde{V}(x(0)).$$

Thus, $\widetilde{V}$ is an upper bound on $V_\infty^{\tilde\mu}$ and we immediately obtain

$$\alpha V_\infty(x) \leq \alpha V_\infty^{\tilde\mu}(x) \leq \widetilde{V}(x).$$

□

Our idea which is carried out in the remainder of this paper is to apply Proposition 3.4 to $\widetilde{V} = V_N$. Hence, we need to establish conditions under which V_N satisfies (3.1). For this purpose, the following simple observation is useful.

Lemma 3.5: Consider $N \in \mathbb{N}$ and a set $\widetilde{X} \subseteq X$ which is invariant under the receding horizon feedback law μ_N. Assume that

$$\begin{aligned} V_N(f(x,\mu_N(x))) - V_{N-1}(f(x,\mu_N(x))) \\ \leq\ (1-\alpha) l(x,\mu_N(x)) \end{aligned} \qquad (3.2)$$

holds for some $\alpha \in [0,1]$ and all $x \in \widetilde{X}$. Then $\widetilde{V} = V_N$ satisfies (3.1) on $\widetilde{X}$ and, in particular,

$$\alpha V_\infty^{\mu_N}(x) \leq V_N(x)$$

holds for all $x \in \widetilde{X}$.

Proof: Combining (2.7) and (2.8) and inserting (3.2) yields

$$\begin{aligned} V_N(x) &= \min_u\{V_{N-1}(f(x,u)) + l(x,u)\} \\ &= V_{N-1}(f(x,\mu_N(x))) + l(x,\mu_N(x)) \\ &\geq V_N(f(x,\mu_N(x))) - (1-\alpha) l(x,\mu_N(x)) \\ &\qquad +\ l(x,\mu_N(x)) \\ &= V_N(f(x,\mu_N(x))) + \alpha l(x,\mu_N(x)), \end{aligned}$$

which shows (3.1). Now Proposition 3.4 yields the assertion. □

IV. Results without terminal cost

The first observation is that without terminal cost the inequality

$$V_M(x) \leq V_N(x) \leq V_\infty(x) \qquad (4.1)$$

always holds for all $M, N \in \mathbb{N}_0$ with $M \leq N$.

The next assumption is crucial for our analysis.

Assumption 4.1: For a given $N \in \mathbb{N}$ there exists $\widetilde{X} \subseteq X$, optimally invariant for V_N and V_{N-1}, and $\gamma > 0$ such that the inequality

$$V_k(f(x,\mu_k(x))) \leq \gamma l(x,\mu_k(x))$$

holds for all $k = 1,\dots,N$ and all $x \in \widetilde{X}_k$ using the sets $\widetilde{X}_k$ from Remark 3.3. □

Two simple observations concerning this assumption are given in the next Lemma.

Lemma 4.2: (i) If Assumption 4.1 holds, then the inequality $V_k(x) \leq (\gamma+1) l(x,\mu_k(x))$ holds for all $x \in \widetilde{X}_k$ and all $k = 1,\dots,N$.

(ii) If $\widetilde{X}$ is optimally invariant for V_N and V_{N-1} and $V_N(f(x,u)) \leq \gamma l(x,u)$ holds for all $x \in \widetilde{X}$ and all $u \in U$, then Assumption 4.1 holds.

Proof: (i) Using the optimality principle, (4.1) and Assumption 4.1 we obtain

$$\begin{aligned} V_k(x) &= V_{k-1}(f(x,\mu_k(x))) + l(x,\mu_k(x)) \\ &\leq V_k(f(x,\mu_k(x)) + l(x,\mu_k(x)) \\ &\leq (1+\gamma) l(x,\mu_k(x)). \end{aligned}$$

(ii) Since $\widetilde{X}$ is optimally invariant for V_{N-1}, from Remark 3.3 we obtain $\widetilde{X}_k \subset \widetilde{X}$ and the assertion follows from (4.1). □

The next proposition is a slight modification of [10, Theorem 1].

Proposition 4.3: Let $N \in \mathbb{N}$ and assume that Assumption 4.1 holds for this N on a set $\widetilde{X} \subseteq X$. Then the inequality

$$(1-(1+\gamma^{-1})^{-N+1}) V_N(x) \leq V_{N-1}(x)$$

holds for $x \in \widetilde{X}$.

Proof: We prove the assertion showing

$$(1-(1+\gamma^{-1})^{-k+1})V_k(x) \le V_{k-1}(x)$$

for $x \in \widetilde{X}_{k-1}$ by induction over k. For $k=1$, the assertion is obvious because $(1-(1+\gamma^{-1})^{-k+1})=0$ and $V_0 \equiv 0$. The induction step $k \to k+1$ for $x \in \widetilde{X}_k$ is obtained from

$$\begin{aligned}
V_k(x) &= V_{k-1}(\underbrace{f(x,\mu_k(x))}_{\in \widetilde{X}_{k-1}}) + l(x,\mu_k(x)) \\
&\ge \underbrace{(1-(1+\gamma^{-1})^{-k+1})}_{=:\nu} V_k(f(x,\mu_k(x))) \\
&\quad + \ l(x,\mu_k(x)) \\
&\ge \left(\nu + \frac{1-\nu}{\gamma+1}\right) V_k(f(x,\mu_k(x))) \\
&\quad + \ \left(1-\gamma\frac{1-\nu}{\gamma+1}\right) l(x,\mu_k(x)) \\
&= \frac{\nu\gamma+1}{\gamma+1}\{V_k(f(x,\mu_k(x))) + l(x,\mu_k(x))\} \\
&\ge \frac{\nu\gamma+1}{\gamma+1}\min_{u\in U}\{V_k(f(x,u)) + l(x,u)\} \\
&= \frac{\nu\gamma+1}{\gamma+1} V_{k+1}(x),
\end{aligned}$$

where we have used the induction assumption in the first inequality and Assumption 4.1 together with (4.1) in the second inequality. This implies the assertion, because

$$\frac{\nu\gamma+1}{\gamma+1} = \frac{(1-(1+\gamma^{-1})^{-k+1})\gamma+1}{\gamma+1} = 1-(1+\gamma^{-1})^{-k}.$$

∎

Theorem 4.4: Consider $\gamma > 0$ and let $N \in \mathbb{N}$ be so large that $(\gamma+1)(1+\gamma^{-1})^{-N+1} < 1$ holds. Assume that Assumption 4.1 holds for this N on a set $\widetilde{X} \subseteq X$. Then

$$V_\infty^{\mu_N}(x) \le \frac{1}{1-(\gamma+1)(1+\gamma^{-1})^{-N+1}} V_\infty(x).$$

In particular, for sufficiently large $N \in \mathbb{N}$ the inequality

$$\frac{V_\infty^{\mu_N}(x) - V_\infty(x)}{V_\infty(x)} \le \frac{(\gamma+1)(1+\gamma^{-1})^{-N+1}}{1-(\gamma+1)(1+\gamma^{-1})^{-N+1}}$$

holds for the relative difference between $V_\infty^{\mu_N}$ and V_∞.

Proof: From Proposition 4.3 we obtain the inequality

$$(1-(1+\gamma^{-1})^{-N+1})V_N(x) \le V_{N-1}(x)$$

which implies

$$\begin{aligned}
&V_N(x) - V_{N-1}(x) \le (1+\gamma^{-1})^{-N+1} V_N(x) \\
&\le \ (\gamma+1)(1+\gamma^{-1})^{-N+1} l(x,\mu_N(x)),
\end{aligned}$$

where we used Lemma 4.2(ii) for the last inequality. Hence, we obtain (3.2) with $\alpha = 1-(\gamma+1)(1+\gamma^{-1})^{-N+1}$ which implies

$$\begin{aligned}
V_\infty^{\mu_N}(x) &\le \frac{1}{\alpha} V_N(x) \le \frac{1}{\alpha} V_\infty(x) \\
&= \frac{1}{1-(\gamma+1)(1+\gamma^{-1})^{-N+1}} V_\infty(x).
\end{aligned}$$

∎

The condition in Assumption 4.1 is somewhat difficult to check because it involves the RHC controllers μ_k. The simplified sufficient condition from Lemma 4.2(ii) avoids this but at the cost of a condition for all $u \in U$ which may not be satisfied even for simple problems, cf. Example 4.13, below. Thus, we will now try to establish results similar to Proposition 4.3 and Theorem 4.4 under a weaker condition. More precisely, we will use the inequality from Lemma 4.2(i) as stated in the following assumption.

Assumption 4.5: For a given $N \in \mathbb{N}$ there exists $\widetilde{X} \subseteq X$, optimally invariant for V_N and V_{N-1}, and $\gamma > 0$ such that the inequality

$$V_k(x) \le (\gamma+1) l(x,\mu_k(x))$$

holds for all $k = 1,\dots,N$ and all $x \in \widetilde{X}_k$ using the sets $\widetilde{X}_k$ from Remark 3.3. □

Remark 4.6: If $\widetilde{X}$ is optimally invariant for V_N and V_{N-1} and the inequality $V_N(x) \le \gamma l(x,u)$ holds for all $x \in \widetilde{X}$ and all $u \in U$, then (4.1) *immediately implies Assumption 4.5.* □

Proposition 4.7: Let $N \ge 2$ and assume that Assumption 4.5 holds for this N on a set $\widetilde{X} \subseteq X$. Then the inequality

$$\frac{(\gamma+1)^{N-2}}{(\gamma+1)^{N-2}+\gamma^{N-1}} V_N(x) \le V_{N-1}(x)$$

holds for $x \in \widetilde{X}$.

Proof: We first show that Assumption 4.5 implies the estimate

$$V_{k-1}(f(x,\mu_k(x)) \le \gamma l(x,\mu_k(x)) \qquad (4.2)$$

for all $k = 1,\dots,N$ and all $x \in \widetilde{X}_k$. In order to prove (4.2), we use the optimality principle

$$V_k(x) = V_{k-1}(f(x,\mu_k(x)) + l(x,\mu_k(x)).$$

Now (4.1) and Assumption 4.5 imply

$$\begin{aligned}
V_{k-1}(f(x,\mu_k(x)) &= V_k(x) - l(x,\mu_k(x)) \\
&\le V_N(x) - l(x,\mu_k(x)) \\
&\le (\gamma+1) l(x,\mu_k(x)) - l(x,\mu_k(x)) \\
&= \gamma l(x,\mu_k(x)),
\end{aligned}$$

which shows (4.2).

By induction over $k = 2,\dots,N$ we prove

$$\frac{(\gamma+1)^{k-2}}{(\gamma+1)^{k-2}+\gamma^{k-1}} V_k(x) \le V_{k-1}(x) \qquad (4.3)$$

for $x \in \widetilde{X}_k$, using the sets $\widetilde{X}_k \subseteq X$ from Remark 3.3 which under the optimal invariance assumption satisfy $\widetilde{X}_k \subset \widetilde{X}$.

For $k=2$ (4.3) follows directly from Assumption 4.5 because

$$V_2(x) \le (\gamma+1) l(x,\mu_1(x)) = (\gamma+1) V_1(x),$$

which is exactly (4.3). For the induction step $k \to k+1$ we abbreviate

$$\eta_k = \frac{(\gamma+1)^{k-2}}{(\gamma+1)^{k-2}+\gamma^{k-1}}.$$

Then for $x \in \widetilde{X}_k$ we obtain

$$\begin{aligned}
V_k(x) &= V_{k-1}(\underbrace{f(x,\mu_k(x))}_{\in \widetilde{X}_{k-1}}) + l(x,\mu_k(x)) \\
&\geq \left(1+\frac{1-\eta_k}{\gamma+\eta_k}\right) V_{k-1}(f(x,\mu_k(x))) \\
&\quad + \left(1-\gamma\frac{1-\eta_k}{\gamma+\eta_k}\right) l(x,\mu_k(x)) \\
&\geq \eta_k \left(1+\frac{1-\eta_k}{\gamma+\eta_k}\right) V_k(f(x,\mu_k(x))) \\
&\quad + \left(1-\gamma\frac{1-\eta_k}{\gamma+\eta_k}\right) l(x,\mu_k(x)) \\
&= \eta_k \frac{\gamma+1}{\gamma+\eta_k}\{V_k(f(x,\mu_k(x))) + l(x,\mu_k(x))\} \\
&\geq \eta_k \frac{\gamma+1}{\gamma+\eta_k}\min_{u\in U}\{V_k(f(x,u)) + l(x,u)\} \\
&= \eta_k \frac{\gamma+1}{\gamma+\eta_k} V_{k+1}(x),
\end{aligned}$$

where we have used (4.2) in the first inequality and the induction assumption in the second inequality. This implies (4.3) because

$$\begin{aligned}
\eta_k \frac{\gamma+1}{\gamma+\eta_k} &= \frac{(\gamma+1)^{k-2}}{(\gamma+1)^{k-2}+\gamma^{k-1}} \frac{\gamma+1}{\gamma+\frac{(\gamma+1)^{k-2}}{(\gamma+1)^{k-2}+\gamma^{k-1}}} \\
&= \frac{(\gamma+1)^{k-1}}{(\gamma+1)^{k-1}+\gamma^{k}} = \eta_{k+1}.
\end{aligned}$$

□

Remark 4.8: Note that inequality (4.2) *used in this proof is in general weaker than Assumption 4.1 used in the proof of Proposition 4.3, because it imposes an upper bound for the (in general smaller) function V_{k-1} instead of V_k. In return, also the resulting estimate obtained from Proposition 4.7 is weaker than the estimate obtained from Proposition 4.3.*

This is most easily seen by looking at the iterations used in the proofs. The estimate $\nu_k = (1-(1+\gamma^{-1})^{-k+1})$ from Proposition 4.3 is obtained from the iteration

$$\nu_2 = \frac{1}{1+\gamma}, \quad \nu_{k+1} = \frac{\nu_k\gamma+1}{\gamma+1}$$

while the estimate $\eta_k = \frac{(\gamma+1)^{k-2}}{(\gamma+1)^{k-2}+\gamma^{k-1}}$ from Proposition 4.7 is obtained from the iteration

$$\eta_2 = \frac{1}{1+\gamma}, \quad \eta_{k+1} = \frac{\eta_k\gamma+\eta_k}{\gamma+\eta_k}.$$

Note that both iterations converge to 1. Since $\eta_2 < 1$ and $\nu_2 < 1$, from these iterations one easily verifies the inequality $\nu_k \geq \eta_k$, i.e., ν_k converges to 1 faster. Note, however, that locally around 1 both iterations are given by

$$\begin{aligned}
\mu_{k+1} - 1 &= \frac{\gamma}{\gamma+1}(\mu_k - 1) + O((\mu_{k+1}-1)^2) \\
\eta_{k+1} - 1 &= \frac{\gamma}{\gamma+1}(\eta_k - 1) + O((\eta_{k+1}-1)^2),
\end{aligned}$$

thus asymptotically for $k \to \infty$ they have the same exponential rate of convergence $\left(\frac{\gamma}{\gamma+1}\right)^k$ □

Analogous to Theorem 4.4 we can now obtain a suboptimality estimate for the receding horizon controller from Proposition 4.7.

Theorem 4.9: Consider $\gamma > 0$ and let $N \in \mathbb{N}$ be so large that $(\gamma+1)^{N-2} > \gamma^N$ holds. Assume that Assumption 4.5 holds for this N on a set $\widetilde{X} \subseteq X$. Then

$$V_\infty^{\mu_N}(x) \leq \frac{(\gamma+1)^{N-2}+\gamma^{N-1}}{(\gamma+1)^{N-2}-\gamma^N} V_\infty(x).$$

In particular, for sufficiently large $N \in \mathbb{N}$ the inequality

$$\frac{V_\infty^{\mu_N}(x) - V_\infty(x)}{V_\infty(x)} \leq \frac{\gamma^N+\gamma^{N-1}}{(\gamma+1)^{N-2}-\gamma^N}$$

holds for the relative difference between $V_\infty^{\mu_N}$ and V_∞.

Proof: Completely analogous to Theorem 4.4 using Proposition 4.7 instead of Proposition 4.3. □

Remark 4.10: From Remark 4.8 it follows that the estimate from Theorem 4.4 converges to 1 faster as $N \to \infty$. On the other hand, since the corresponding Assumptions are different, the Assumption 4.5 needed for Theorem 4.9 may be satisfied for smaller γ than the Assumption 4.1 needed for Theorem 4.4, in which case the former may nevertheless yield a sharper estimate for moderate values of N. In particular, Assumption 4.1 might be more difficult to check because it involves the dynamics f of the system while Assumption 4.5 only involves the optimal value function and the running cost. □

This fact is also reflected in the next proposition, where we present two conditions on the running cost l which ensure Assumption 4.1 and Assumption 4.5, respectively.

Proposition 4.11: Let $\widetilde{X} \subseteq X$.

(a) Assume that there exists a function $W : X \to \mathbb{R}_0^+$ and constants α, β, and $\lambda > 0$ such that for all $x \in \widetilde{X}$ the following condition holds:

(i) *$l(x,u) \geq \alpha W(f(x,u))$ for all $u \in U$*

(ii) *for each $\tilde{u} \in U$ there exists a control sequence $u^* \in \mathcal{U}$ such that the corresponding solution $x(n)$ with $x(0) = f(x,\tilde{u})$ satisfies*

$$l(x(n),u^*(n)) \leq \beta\lambda^n W(f(x,\tilde{u})).$$

Then Assumption 4.1 holds on $\widetilde{X}$ for each $N \in \mathbb{N}$ with $\gamma = \frac{\beta}{\alpha(1-\lambda)}$.

(b) Assume that there exists a function $W : X \to \mathbb{R}_0^+$ and constants $\alpha, \beta > 0$, and $0 \leq \lambda < 1$ such that for all $x \in \widetilde{X}$ the following two conditions hold:

(i) *$l(x,u) \geq \alpha W(x)$ for all $u \in U$*

(ii) *there exists a control sequence* $u^* \in \mathcal{U}$ *such that the corresponding solution* $x(n)$ *with* $x(0) = x$ *satisfies*

$$l(x(n), u^*(n)) \leq \beta\lambda^n W(x).$$

Then Assumption 4.5 holds on $\widetilde{X}$ *with* $\gamma = \frac{\beta}{\alpha(1-\lambda)} - 1$.

Proof: (a) Condition (ii) implies

$$\begin{aligned} V_N(f(x,u)) &\leq V_\infty(f(x,u)) \\ &\leq \sum_{n=0}^{\infty} l(x(n), u^*(n)) \\ &\leq \sum_{n=0}^{\infty} \beta\lambda^k W(f(x,u)) \\ &= \frac{\beta}{1-\lambda} W(f(x,u)). \end{aligned}$$

Combining this with condition (i) yields

$$V_N(f(x,u)) \leq \frac{\beta}{1-\lambda} W(f(x,u)) \leq \frac{\beta}{\alpha(1-\lambda)} l(x,u).$$

(b) follows similarly. ∎

Remark 4.12: (i) In both (a) and (b) the conditions (ii) are exponential controllability condition for the running cost l*. Note that exponentially converging cost functions can always be constructed from control Lyapunov functions, however, since such control Lyapunov functions are hard to find, this approach may not be feasible. In an RHC context, exponential controllability conditions for the running cost are discussed in [5, Section III], in particular for homogeneous systems.*

(i) The main difference between (a) and (b) is that condition (a)(i) requires information about the dynamics of the next step to be taken into account in the running cost l*. More precisely, this condition demands that steps that lead into the "wrong" direction (in the sense that* W *is increasing) must be penalized in* l*. In contrast to this, in condition (b)(i) only the current state must be appropriately penalized.* □

Note that for each of the two Theorems 4.4 and 4.9 we have three types of assumptions and conditions, which differ in the type of information used:

- assumptions involving the optimal value functions and the RHC controllers (Assumptions 4.1 and 4.5)
- sufficient conditions involving the optimal value functions (Lemma 4.2(ii) and Remark 4.6)
- sufficient conditions involving an auxiliary function W (Proposition 4.11)

The following simple example highlights the difference between these assumptions and the resulting Theorems 4.4 and 4.9.

Example 4.13: Consider the linear 1d control system

$$x(n+1) = 2x(n) + u(n) =: f(x(n), u(n))$$

with $x(n) \in \widetilde{X} = \mathbb{R}$ *and* $u(n) \in U = \mathbb{R}$*. We first consider the running cost*

$$l(x) = x^2.$$

Here it is easy to solve the infinite horizon optimal control problem, because for $\mu(x) = -2x$ *the related optimal value function*

$$V_\infty^\mu(x) = x^2$$

satisfies the optimality principle, because

$$\begin{aligned} x^2 &= V_\infty^\mu(x) = \inf_{u\in\mathbb{R}} \{l(x) + V_\infty^\mu(f(x,u))\} \\ &= \inf_{u\in\mathbb{R}} \{x^2 + (2x(n)+u)^2\} = x^2. \end{aligned}$$

Using the same argument one also sees that the finite time optimal value functions are given by

$$V_N(x) = x^2, \quad N \geq 1$$

with corresponding RHC feedback laws

$$\mu_N(x) = -2x, \quad N \geq 2.$$

Thus, for $N \geq 2$*, the RHC controller is indeed optimal for the infinite horizon problem.*

This optimality property can be obtained from both Theorem 4.4 and 4.9 using Assumptions 4.1 and 4.5, respectively. For instance, in the case of Theorem 4.9, the corresponding Assumption 4.5 is satisfied for each $N \in \mathbb{N}$ *with* $\gamma = 0$ *for* $\widetilde{X} = X = \mathbb{R}$*. Thus, for each* $N \geq 2$ *we obtain the estimate*

$$\frac{V_\infty^{\mu_N}(x) - V_\infty(x)}{V_\infty(x)} \leq \frac{\gamma^N + \gamma^{N-1}}{(\gamma+1)^{N-2} - \gamma^N} = 0,$$

i.e., a sharp estimate. The application of Theorem 4.4 works similarly.

Note that for checking Assumptions 4.1 and 4.5 directly we have used information about the RHC controllers, which we cannot expect to know in general. If this information is not available, Theorem 4.4 is not applicable for this example, because both the sufficient condition given in Lemma 4.2(ii) and the sufficient condition from Proposition 4.11(a) fail, because $f(x,u)$ *grows unbounded for varying* $u \in U = \mathbb{R}$ *which is not reflected in the running cost* l*. In contrast to this, the condition from Proposition 4.11(b) for Theorem 4.9 can be checked easily with* $W(x) = x^2$ *and the control sequence* $u^* = (-2x(0), 0, 0, \ldots)$*, yielding* $\alpha = 1$*,* $\beta = 1$ *and* $\lambda = 0$ *and thus again* $\gamma = 0$*.*

The situation changes when we alter the running cost, e.g. to

$$l(x,u) = x^2 + u^2.$$

Now, both conditions (a) and (b) in Lemma 4.11 are checkable. More precisely, using $W(x) = x^2$ *one obtains (a)(i) with* $\alpha = 1/5$ *and (b)(i) with* $\alpha = 1$*. Applying again the control sequence* $u^* = (-2x(0), 0, 0, \ldots)$ *yields* $\beta = 5$ *and* $\lambda = 0$ *for both (a)(ii) and (b)(ii). This results in* $\gamma = 25$ *for (a) and* $\gamma = 4$ *for (b). Table 4.1 shows the minimal horizon length* N *needed according to Theorems 4.4 and 4.9, respectively, in order to ensure the given values for the relative accuracy. It is easily seen that for this example Theorem 4.9 yields much better results.*

	needed horizon length N	
relative accuracy	Theorem 4.4, $\gamma = 25$	Theorem 4.9, $\gamma = 4$
0.50	113	21
0.10	146	27
0.01	202	37

TABLE 4.1

COMPARISON FOR RUNNING COST $l(x,u) = x^2 + u^2$

The situation changes again when we use the running cost

$$l(x,u) = x^2 + f(x,u)^2.$$

Now, using $W(x) = x^2$ as above, (a) is satisfied for $\alpha = 1$ while all other values remain unchanged, yielding $\gamma = 5$ for (a) and $\gamma = 4$ for (b). Table 4.2 shows the minimal horizon length N for this case. Now Theorem 4.4 yields the better estimates, at least for smaller N.

	needed horizon length N	
relative accuracy	Theorem 4.4, $\gamma = 5$	Theorem 4.9, $\gamma = 4$
0.50	17	21
0.10	24	27
0.01	37	37

TABLE 4.2

COMPARISON FOR RUNNING COST $l(x,u) = x^2 + f(x,u)^2$

Note that we do not claim that these estimates are tight or even optimal. In particular, the use of other sequences u^* might lead to smaller values of γ and hence tighter estimates. We have chosen the given sequences u^* because they allow for particularly easy computations. □

V. RESULTS FOR TERMINAL COSTS BEING LYAPUNOV FUNCTIONS

Many RHC schemes make use of a suitable terminal cost in order to ensure closed loop asymptotic stability of the RHC controller. Often, in these settings the terminal costs are chosen as Lyapunov functions with respect to the running cost l, see [9] and the references therein. In this section we discuss the consequences on suboptimality of these choices. Here we make the following assumption on the terminal cost F.

Assumption 5.1: For each $x \in Y_0$ there exists $u \in U$ such that

$$f(x,u) \in X_0 \quad \text{and} \quad F(f(x,u)) \le F(x) - l(x,u).$$

□

This condition is often imposed in receding horizon schemes in order to ensure asymptotic stability of the closed loop, see [9, Section 3.3 and the references therin]. Note that Assumption 5.1 implies (3.1) for $\widetilde{V} = F$ with $\alpha = 1$. Hence, Proposition 3.4 implies $F(x) \ge V_\infty(x)$ and we can define the positive difference $\eta := \max_{x\in Y_0} F(x) - V_\infty(x) \ge 0$.

A typical situation in which F meeting Assumption 5.1 can be found is if the linearization of f is controllable to 0 and l is close to a quadratic function around the origin. In this case, F can be chosen as the optimal value function of the linear quadratic problem for a quadratic cost function $\tilde{l}$ which is strictly smaller than l. Then, the closer l and $\tilde{l}$ are and the smaller the neighborhood Y_0 is chosen, the smaller η becomes, see also the discussion after Lemma 3 [6].

In the following theorem we distinguish the case with and without terminal constraint set.

Theorem 5.2: Assume that the terminal cost in (2.5) *satisfies Assumption 5.1 on some neighborhood Y_0 of the origin and let $N \in \mathbb{N}$.*

(i) Consider the optimal receding horizon controller μ_N from (2.8) *based on V_N from* (2.4)*, i.e., without terminal constraint. Let $Y_N \subset X$ be the set of initial values for which the optimal solution $x(n)$ for the finite horizon functional* (2.5) *satisfies $x(N) \in Y_0$. Then the inequality*

$$V_\infty^{\mu_N}(x) \le V_N(x) \le V_\infty(x) + \eta$$

holds for each $x \in Y_N$.

(ii) Consider the optimal receding horizon controller μ_N from (2.8) *based on V_N from* (2.6) *with terminal constraint set $X_f = Y_0$. Then the inequality*

$$V_\infty^{\mu_N}(x) \le V_N(x)$$

holds on the feasible set X_N. Let, furthermore, $Y_\infty^N \subset X_N$ be the set of initial values for which the optimal solution $x(n)$ for the infinite horizon functional (2.2) *satisfies $x(n) \in Y_0$ for all $n \ge N$. Then the inequality*

$$V_N(x) \le V_\infty(x) + \eta$$

holds for each $x \in Y_\infty^N$.

Proof: (i) For $x \in Y_N$ abbreviate $x^+ = f(x, \mu_N(x))$. Then, from the optimality principle we obtain $x^+ \in Y_{N-1}$. Now consider an optimal control sequence $u_{N-1} \in \mathcal{U}$ for the problem (2.4) with horizon length $N-1$ and the corresponding trajectory $x_{u_{N-1}}$ with initial value $x_{u_{N-1}}(0) = x^+$. Since $x^+ \in Y_{N-1}$ we obtain $\bar{x} := x_{u_{N-1}}(N-1) \in Y_0$. Let $\bar{u}$ denote the control value from Assumption 5.1 for $\bar{x}$ and define a control sequence $\tilde{u} = (u_{N-1}(0), \ldots, u_{N-1}(N-1), \bar{u}, \ldots)$. This sequence yields

$$\begin{aligned} V_N(x^+) \le J_N(x^+, \tilde{u}) &= V_{N-1}(x^+) - F(\bar{x}) \\ &\quad + l(\bar{x}, \bar{u}) + F(f(\bar{x}, \bar{u})) \\ &\le V_{N-1}(x^+). \end{aligned}$$

Thus, (3.2) follows with $\alpha = 1$ which implies

$$V_\infty^{\mu_N}(x) \le V_N(x).$$

The inequality $V_N(x) \le V_\infty(x) + \eta$ follows immediately from the definition of J_N and J_∞ and $F \ge V_\infty$, which was observed in the discussion after Assumption 5.1.

(ii) The inequality $V_\infty^{\mu_N}(x) \le V_N(x)$ is concluded as in (i). The second inequality again follows from the definition of J_N and J_∞ and $F \ge V_\infty$, observing that for $x \in Y_\infty^N$ the optimal control sequence u for (2.2) satisfies the constraint in (2.6). □

VI. Conclusions

We have derived rigorous suboptimality estimates for the infinite horizon performance of RHC controllers. In particular, we have shown that suitable exponential controllability assumptions for the running cost allow for obtaining suboptimality estimates for RHC schemes without terminal cost and constraints, a setting which to the best of our knowledge is not covered by the existing inverse optimality results. These results are complemented by novel estimates for the case where the RHC terminal cost is a Lyapunov functions, which is the classical setting for inverse optimality results. In both cases, techniques from relaxed dynamic programming are the main tool for establishing our results.

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