High order approximations by sampled–data feedback

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***Abstract*— Given a continuous time nonlinear closed loop system, we give explicit constructions for sampled–data feedback laws for which the trajectories of the sampled–data closed loop system converge to the continuous time trajectories with a prescribed rate of convergence as the sampling interval length tends to zero. In particular, we investigate necessary and sufficient conditions under which such sampled–data feedbacks exist. We give analytic solutions to the problem for local orders of convergence ≤ 4 and present a MAPLE code for general orders.**

***Keywords*— nonlinear sampled–data control, high–order convergence rate, Taylor expansion, MAPLE**

I. INTRODUCTION

A common method for sampled–data controller design is the construction of a continous–time controller followed by a discretization step [3], [4], [9]. This so called emulation method is attractive since the controller design is carried out in two relatively simple steps. The first design step is done in continuous–time, completely ignoring sampling, which is easier than the design that takes sampling into account. The second step involves the discretization of the controller and there are many methods that can be used for this purpose. Simple methods, however, may not perform well in practice since the required sampling rate may exceed the hardware limitations even for linear systems [7], [1]. This has led to a range of more advanced controller discretization techniques for linear systems, see, e.g., [1], [3].

In the nonlinear case, the survey paper [10] gives an overview about a number of methods, which show that under suitable control theoretic assumptions (involving, e.g., the relative degree of the system) an exact sampled–data reproduction of the continuous time input–output behavior is possible. An important special case is the analysis of the possibility of feedback linearization with sampled feedback control which was studied during the 1980s (see, e.g., [2] and the references therein). Our approach in this paper is on the one hand less demanding, because we only aim at an approximate reproduction of the continuous time response, on the other hand it is more demanding than the input–output behavior analysis because we want to approximately reproduce the full state trajectory.

The present paper builds on the Lyapunov function based results from [13], avoiding, however, the use of control Lyapunov functions. The purpose of our sampled–data feedback construction lies in reducing the local difference between the continuous time system and the sampled–data system, i.e, the difference after one sampling interval. If this local difference is small for each component of the state vector, then a straightforward induction allows to conclude a rate of convergence for the trajectories at sampling instances for each compact time interval. As an alternative to the numerical optimal control approach presented in [14], here we reduce the local difference by analyzing its asymptotic behavior for vanishing sampling interval length. This amounts to estimating the order of convergence $O(T^k)$ of the sampled–data solution to the continuous time solution as the sampling interval length T tends to 0. In particular, we try to find feedback laws yielding a fast order of convergence, i.e., a large k in this estimate.

For $k \leq 4$ it turns out that we can give a complete analytic answer to the problem, formulating a necessary and sufficient condition and deriving analytic formulas for the feedback law. For $k \geq 5$ both the general formulas for the feedback laws and the necessary and sufficient conditions become very involved, which is why instead of an analytic answer we present a MAPLE code which checks the condition and computes the formula for the feedback law for general $k \in \mathbb{N}$.

The paper is organized as follows. In Section II we present the setting and the preliminary results from [13]. In Section III we state and prove our main results, describing both the necessary and sufficient conditions and the formulas for the resulting feedback laws for $k = 4$. Since conditions for $k \geq 5$ become very complicated, we only comment on the condition for $k = 5$ and instead present a MAPLE program for checking these conditions and computing the corresponding sampled–data controllers in Section IV. Finally, in Section V we illustrate our results by two examples.

II. SETUP

We consider nonlinear control affine systems of the form

$$\dot{x}(t) = f(x(t), u(t)) := g_0(x(t)) + g_1(x(t))u(t) \qquad (2.1)$$

with vector fields $g_0, g_1 : \mathbb{R}^n \to \mathbb{R}^n$ and control functions $u : \mathbb{R} \to \mathbb{R}$. For simplicity of exposition we consider single input systems (i.e., $u(t) \in \mathbb{R}$), because for the multi input case the computations and expressions become much more involved.

We assume that a static state feedback $u_0 : \mathbb{R}^n \to \mathbb{R}$ has been designed which solves some control task for the

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continuous time closed loop system

$$\dot{x}(t) = f(x(t), u_0(x(t)))\,. \tag{2.2}$$

The solutions of (2.2) with initial value x_0 at initial time $t_0 = 0$ will be denoted by $\phi(t, x_0)$. We assume that all functions involved are smooth with sufficiently high degree of smoothness such that the derivatives taken in what follows are well defined and continuous.

Our goal is to find a feedback $u_T(x)$ such that the solution trajectories of the sampled data closed loop system

$$\dot{x}(t) = f(x(t), u_T(x(t_k))),\;\; t \in [t_k, t_{k+1}),\;\; k = 0, 1, \ldots \tag{2.3}$$

for the sampling sequence $t_k = kT$ and sampling period $T > 0$ are close to those of the continuous time closed loop system (2.2). More precisely, denoting solutions of (2.3) by $\phi_T(t, x_0, u_T)$, we want to find u_T such that the difference after one sampling time step

$$\Delta\phi(T, x_0, u_T) := \|\phi(T, x_0) - \phi_T(T, x_0, u_T)\|_\infty \tag{2.4}$$

becomes small, with $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$ denoting the maximum norm in $\mathbb{R}^n$. We call $\Delta\phi$ the *local difference* between ϕ and ϕ_T.

For the feedback u_T we consider the following general class of functions.

Definition 2.1: An admissible sampled data feedback law u_T *is a family of maps* $u_T : \mathbb{R}^n \to \mathbb{R}$*, parameterized by the sampling period* $T \in (0, T^*]$ *for some maximal sampling period* T^**, such that for each compact set* $K \subset \mathbb{R}^n$ *the inequality*

$$\sup_{x \in K,\, T \in (0, T^*]} |u_T(x)| < \infty$$

holds. □

Note that for existence and uniqueness of the solutions to (2.3) we do not need any continuity assumptions on u_T. Boundedness is, however, imposed, because from a practical point of view unbounded feedback laws are physically impossible to implement and from a theoretical point of view they often lead to closed loop systems which are very sensitive to modeling or approximation errors, cf., e.g., the examples in [5], [11]. A special class of these admissible feedback laws which was proposed in [13] is given by

$$u_T(x) = \sum_{i=0}^{M} T^i u_i(x), \tag{2.5}$$

with u_0 from (2.2) and $u_1, \ldots, u_M : \mathbb{R}^n \to \mathbb{R}$ being continuous functions.

In the present paper, we are in particular interested in asymptotic estimates, i.e., in the behavior of the difference (2.4) for $T \to 0$. For this purpose we use the following definition.

Definition 2.2: Let $k \in \mathbb{N}$.

(i) For some compact set $K \subset \mathbb{R}^n$ *we write*

$$\Delta\phi(T, x_0, u_T) = O(T^k) \text{ on } K$$

if there exists $C > 0$ *such that the inequality* $\Delta\phi(T, x_0, u_T) \le CT^k$ *holds for all* $x_0 \in K$.

(ii) We write

$$\Delta\phi(T, x_0, u_T) = O(T^k)$$

if $\Delta\phi(T, x_0, u_T) = O(T^k)$ *on* K *for each compact subset* $K \subset \mathbb{R}^n$*, where the constant* C *in (i) may depend on the choice of* K. □

If we are able to establish $\Delta\phi(T, x_0, u_T) = O(T^k)$, then it follows by a standard induction argument that on each interval $[0, t^*]$ we obtain

$$\|\phi(t, x_0) - \phi_T(t, x_0, u_T)\|_\infty \le O(T^{k-1}) \tag{2.6}$$

for all times $t = iT$, $i \in \mathbb{N}$ with $t \in [0, t^*]$. In particular, this "closeness of trajectories" allows to prove that several stability concepts carry over from ϕ to ϕ_T in a semiglobal practical sense, see [12].

In order to establish estimates for (2.4) we consider a smooth real valued function

$$h : \mathbb{R}^n \to \mathbb{R}$$

and derive estimates for the local differences

$$\Delta h(T, x_0, u_T) := |h(\phi(T, x_0)) - h(\phi_T(T, x_0, u_T))|. \tag{2.7}$$

Applying the respective results to the specific functions

$$h_j(x) := x_j,\;\; j = 1, \ldots, n$$

and the respective local differences

$$\Delta h_j(T, x_0, u_T),\;\; j = 1, \ldots, n. \tag{2.8}$$

defined by (2.7) with $h = h_j$, we are able to conclude the desired estimate for $\Delta\phi$, because if $\Delta h_j(T, x_0, u_T) \le C$ holds for some constant $C > 0$ and all $j = 1, \ldots, n$, then $\Delta\Phi(T, x_0, u_T) \le C$ follows.

For a problem similar to the one posed in this paper, in [13] the feedback law

$$u_T^M(x) = \sum_{i=0}^{M} \frac{T^i}{(i+1)!} \left. \frac{d^i u(\phi(t, x))}{dt^i} \right|_{t=0}, \tag{2.9}$$

i.e., (2.5) with

$$u_i(x) = \frac{1}{(i+1)!} \left. \frac{d^i u(\phi(t, x))}{dt^i} \right|_{t=0} \tag{2.10}$$

was discussed. Note that the results in [13] were formulated for Lyapunov functions V instead of general functions h, however, the usual Lyapunov function properties were only needed for the interpretation of the results and not for the proofs. Hence, we can in particular apply Theorem 4.11 of this reference to our setting which shows that for $M = 0$ (note that $u_T^0 = u_0$) the estimate

$$\Delta h(T, x_0, u_T^0) = O(T^2)$$

holds while for $M = 1$ the estimate

$$\Delta h(T, x_0, u_T^1) = O(T^3)$$

holds.

In Remark 4.13 of [13][1] it was observed that the above estimates for Δh using (2.9) do not hold in general for $M \geq 2$. It is the purpose of the present paper to construct sampled–data controllers for which it is possible to generalize these results to larger M.

Our analysis is based on Theorem 3.1 from [13]. In order to state this theorem we need to introduce some notation: for a vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ and a scalar function $h : \mathbb{R}^n \to \mathbb{R}$ we denote the directional derivative of h in the direction of g by

$$L_g h(x) := \frac{d}{dx} h(x) \cdot g(x),$$

cf. Isidori [6]. Furthermore, we define multinomial coefficients $\binom{n}{n_0 \ \ldots \ n_M} := \frac{n!}{n_0! n_1! \ldots n_M!}$ as well as multi-indices $\nu := (n_0, n_1, \ldots, n_M)$ and we use the notation $|\nu| := \sum_{j=0}^M n_j$ and $\|\nu\| := \sum_{j=0}^M j \cdot n_j$.

Now we can state [13, Theorem 3.1].

Theorem 2.3: Consider the system (2.1)*, a smooth function $h : \mathbb{R}^n \to \mathbb{R}$, the continuous closed loop system* (2.2) *and the sampled data closed loop system* (2.3) *with controller u_T given by* (2.5)*. Then, for sufficiently small T, we can write:*

$$\begin{aligned} &\frac{h(\phi_T(T, x, u_T)) - h(x)}{T} \qquad (2.11) \\ &= \sum_{s=0}^{M} T^s [L_{g_1} h \cdot u_s + p_s(x, u_0, \ldots, u_{s-1})] \\ &\quad + O(T^{M+1}), \end{aligned}$$

where $p_0(x) = L_{g_0} h(x)$ and

$$\begin{aligned} &p_s(x, u_0, \ldots, u_{s-1}) \qquad (2.12) \\ &= \sum_{k=1}^{s} \sum_{i_0=0,\ldots,i_k=0}^{1} \frac{L_{g_{i_0}} \cdots L_{g_{i_k}} h(x)}{(k+1)!} \\ &\quad \cdot \left(\sum_{\substack{|\nu|=|I_k| \\ \|\nu\|=s-k}} \binom{|I_k|}{n_0 \ n_1 \ \ldots \ n_M} \prod_{j=0}^{s-1} u_j^{n_j} \right) \end{aligned}$$

for $s \geq 1$ with I_k denoting the multi index $I_k = (i_0, i_1, \ldots, i_k)$.

Note that p_s is independent of the value of M appearing in (2.12) provided $M \geq s-1$, because the condition $\|\nu\| = s-k \leq s-1$ implies $n_s = \ldots = n_M = 0$.

III. Analytic computation of the controllers

In this section we investigate a necessary and sufficient condition for the existence of an admissible feedback law u_T which achieves

$$\Delta h(T, x, u_T) = O(T^k) \qquad (3.1)$$

or

$$\Delta \phi(T, x, u_T) = O(T^k). \qquad (3.2)$$

[1] In fact, the main formula in [13, Remark 4.13] is flawed because the factor “2” in the term $-2L_{g_1} L_{g_0} V \cdot u_1/3!$ should not be there. Still, the assertion of the remark remains true.

and provide a formula for this feedback law. Since the necessary and sufficient condition turns out to be quite involved for $k \geq 5$ we restrict our rigorous analytic computations to the case $k = 4$ and comment only briefly on the case $k = 5$. The general case will be covered by an algorithmic approach in the following section. Note that $k = 4$ is the first nontrivial case given that (3.1) and thus (3.2) for $k \leq 3$ are always achievable by (2.9) without any further conditions, cf. [13, Theorem 4.11].

For the necessary and sufficient condition it turns out that the cases (3.1) and (3.2) require different conditions which is why we state them in two separate theorems. We start with (3.1).

Theorem 3.1: Consider the system (2.1)*, the continuous closed loop system* (2.2)*, a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ and a compact set $K \subset \mathbb{R}^n$.*

If the condition

$$\left| L_{[g_0, g_1]} h(x) \cdot L_{g_0 + g_1 u_0} u_0(x) \right| \leq c |L_{g_1} h(x)| \qquad (3.3)$$

holds for some constant $c \geq 0$ and all $x \in K$, then there exists an admissible feedback law $u_T : \mathbb{R}^n \to \mathbb{R}$ satisfying (3.1) *on K with $k = 4$ given by In this case, any feedback $u_T : \mathbb{R}^n \to \mathbb{R}$ of the form*

$$u_T(x) = \begin{cases} u_T^2(x) + \frac{T^2}{12} \frac{L_{[g_0,g_1]} h(x) \cdot L_{g_0+g_1 u_0} u_0(x)}{L_{g_1} h(x)} \\ \qquad + \ O(T^3), \qquad x \in \widetilde{K} \\ u_T^1(x) + O(T^2), \qquad x \notin \widetilde{K} \end{cases} \qquad (3.4)$$

with u_T^1 and u_T^2 from (2.9) *and*

$$\widetilde{K} := \{x \in K \mid L_{g_1} h(x) \neq 0\}$$

solves (3.1) *with $k = 4$.*

Conversely, if there exists an admissible feedback law $u_T : \mathbb{R}^n \to \mathbb{R}$ satisfying (3.1) *on $\widetilde{K}$ with $k = 4$, then* (3.3) *holds for all $x \in \mathrm{cl}\, \widetilde{K}$. In this case, this feedback u_T must be of the form* (3.4) *for all $x \in \widetilde{K}$.*

Proof: From the Taylor expansion of $h(\phi(t, x))$ in $t = 0$ we obtain the identity

$$\begin{aligned} h(\phi(t,x)) \ = \ & h(x) + \sum_{i=0}^{2} T^{i+1} \big[L_{g_1} h(x) u_i(x) \qquad (3.5) \\ & \qquad + \ p_i(x, u_0, \ldots, u_{i-1}) \big] \\ & + \ \frac{T^3}{12} \Big((L_{g_1} L_{g_0} h(x) - L_{g_0} L_{g_1} h(x)) \\ & \qquad \cdot \ L_{g_0 + g_1 u_0} u_0(x) \Big) + O(T^4) \end{aligned}$$

with u_i from (2.10) and p_i from (2.12). We use the identity

$$L_{g_1} L_{g_0} h(x) - L_{g_0} L_{g_1} h(x) = -L_{[g_0, g_1]} h(x)$$

and compare the coefficients of (3.5) with (2.11) inductively for $i = 0, 1, 2$. For $x \notin \mathrm{cl}\, \widetilde{K}$ this yields that the proposed feedback realizes (3.1) with $M = 2$ provided (3.3) holds.

For $x \in \operatorname{cl}\widetilde{K}$ this coefficient analysis yields that any feedback $\tilde{u}_T$ of the form

$$\tilde{u}_T(x) = \begin{cases} u_0(x) + Tu_1(x) + T^2\tilde{u}_2(x) \\ \qquad +O(T^3), & x \in \widetilde{K} \\ u_0(x) + Tu_1(x) + O(T^2), & x \notin \widetilde{K} \end{cases} \quad (3.6)$$

with u_0 and u_1 from (2.10), and $\tilde{u}_2(x)$ satisfying

$$\tilde{u}_2(x)L_{g_1}h(x) = \frac{1}{12}\Big(L_{[g_0,g_1]}h(x) \cdot L_{g_0+g_1u_0}u_0(x)\Big) + u_2(x)L_{g_1}h(x) \quad (3.7)$$

with $u_2(x)$ again from (2.10) realizes (3.1) with $M = 2$.

Now assume that (3.3) holds for all $x \in K$. Then it follows that u_T from (3.4) satisfies (3.6)–(3.7) and that the feedback is admissible in the sense of Definition 2.1, because it is bounded. In particular, this shows that a feedback u_T satisfying (3.1) on $\operatorname{cl}\widetilde{K}$ with $M = 2$ exists.

Conversely, assume that u_T is an admissible feedback satisfying (3.1) on K with $M = 2$. Then, this feedback must satisfy the conditions (3.6)–(3.7). Since u_T is admissible, it is in particular bounded and thus (3.7) implies (3.3) for $x \in \widetilde{K}$. Since all expressions in (3.3) are continuous in x, we also obtain (3.3) for $x \in \operatorname{cl}\widetilde{K}$. In addition, the inductive comparison of (3.5) with (2.11) shows that any feedback $\tilde{u}_T$ realizing (3.1) with $M = 2$ must satisfy (3.6)–(3.7) for $x \in \widetilde{K}$, which shows that u_T must be of the asserted form. □

Remark 3.2: Note that condition (3.3) *is necessary and sufficient on* $\operatorname{cl}\widetilde{K}$ *but only sufficient on* $K \setminus \operatorname{cl}\widetilde{K}$. *This can be verified using the approach in [10, Section 3.1 and the references therein] based on the relative degree, when we consider* h *as an output function for the system* (2.1)*:*

Assume, for instance, the existence of an open subset $O \subset K \setminus \operatorname{cl}\widetilde{K}$ *on which* (2.1) *has relative degree* $r = 2$, *i.e.,* $L_{g_1}h(x) = 0$ *and* $L_{g_1}L_{g_0}h(x) \neq 0$ *for all* $x \in O$. *Then, by straightforward computations one sees that on* O *the feedback*

$$u_T(x) = u_0(x) + T\frac{2}{3}u_1(x) \quad (3.8)$$

for u_1 *from* (2.10) *satisfies* (3.1) *with* $M = 2$ *for each* $x \in O$, *regardless of whether* (3.3) *holds, which shows that this condition is in general not necessary outside* $\operatorname{cl}\widetilde{K}$.

At the first glance, (3.8) *seems to contradict* (3.4), *because the two feedback laws are different for* $x \in O \subset K \setminus \widetilde{K}$. *However, a closer examination reveals that under condition* (3.3) *in fact for any* $\beta \in \mathbb{R}$ *the feedback*

$$u_T(x) = u_0(x) + T\beta u_1(x)$$

for u_1 *from* (2.10) *satisfies* (3.1) *with* $M = 2$ *on* O. *The advantage of specifying* $\beta = 1$ *in* (3.4) *lies in the fact that this choice will also work on* $\partial\widetilde{K}$ *(i.e., in particular on* ∂O*). In contrast to this, the choice* $\beta = 2/3$ *— which is the only correct choice on* O *if* (3.3) *is not satisfied — will not in general work on* $\partial\widetilde{K}$.

Since in what follows we do not need necessary conditions outside $\widetilde{K}$, *we will not elaborate this topic in further detail.* □

Remark 3.3: On $\widetilde{K}$, *the necessary and sufficient condition* (3.3) *can be interpreted as follows: For* $x \in \widetilde{K}$ *the control* $\tilde{u}_2$ *can always be used in order to induce any third order correction. However, if* $L_{g_1}h(x_n) \to 0$ *for some sequence* $x_n \in K$, *then the control effort needed for this purpose may be unbounded which may make the resulting feedback not admissible in the sense of Definition 2.1. Condition* (3.3) *guards against this situation.* □

Remark 3.4: From the continuity of the expressions in (3.3) *it is easily seen that the condition* (3.3) *is always satisfied if* $\widetilde{K} = K$. *In particular, in many practical examples it might be possible to choose a reasonable set* K *for which* $\widetilde{K} = K$ *holds. Then, our proposed feedback* (3.4) *will yield* $\Delta h = O(T^3)$ *on* K *and* $\Delta h = O(T^2)$ *outside* K, *i.e., we can improve the sampled data performance with respect to* h *at least in parts of the state space. It should, however, be mentioned that for arbitrary functions* h *this is of limited use, because in general it will not be possible to inductively conclude an estimate analogous to* (2.6) *for the difference* $|h(\phi) - h(\phi_T)|$. *An exception is the case where* $h = V$ *is a Lyapunov function for* (2.2), *because in this case the proposed control law renders the Lyapunov difference along the sampled data trajectories close to those of the continuous time ones. For a detailed discussion of this topic we refer to [13].* □

The reason for the fact that $\Delta h = O(T^3)$ is rather easy to obtain is due to the fact that the values $h(\phi)$ and $h(\phi_T)$ to be matched are one–dimensional. The necessary and sufficient condition becomes much more restrictive if we consider $\Delta\phi$, as the following theorem shows.

Theorem 3.5: Consider the system (2.1), *the continuous closed loop system* (2.2) *and a compact set* $K \subset \mathbb{R}^n$ *satisfying* $K = \operatorname{cl}\operatorname{int} K$. *Then there exists an admissible feedback law* $u_T : \mathbb{R}^n \to \mathbb{R}$ *satisfying* (3.2) *on* K *with* $k = 4$ *if and only if there exists a bounded function* $\alpha : K \to \mathbb{R}$ *satisfying*

$$[g_0, g_1](x)L_{g_0+g_1u_0}u_0(x) = \alpha(x)g_1(x). \quad (3.9)$$

In this case, any feedback $u_T : \mathbb{R}^n \to \mathbb{R}$ *of the form*

$$u_T(x) = \begin{cases} u_T^2(x) + \frac{T^2}{12}\alpha(x) \\ \qquad + O(T^3), & x \in \operatorname{cl}\widetilde{K} \\ \textit{arbitrary}, & x \notin \operatorname{cl}\widetilde{K} \end{cases} \quad (3.10)$$

with u_T^2 *from* (2.9) *and*

$$\widetilde{K} := \{x \in K \mid g_1(x) \neq 0\}$$

satisfies (3.2) *with* $k = 4$. *Furthermore, each feedback satisfying* (3.2) *with* $k = 4$ *is of the form* (3.10) *for* $x \in \widetilde{K}$ *and the function* α *in* (3.9) *can be chosen as* $\alpha(x) = 0$ *for* $x \notin \widetilde{K}$.

Proof: We first show that under condition (3.9) any feedback of the form (3.10) satisfies the assertion.

First note that for $x \notin \mathrm{cl}\,\widetilde{K}$ the feedback value $u_T(x)$ is indeed arbitrary. This follows since on $K \setminus \mathrm{cl}\,\widetilde{K}$ the control system is given by $\dot{x} = g_0(x)$. Thus, on the open set $\mathrm{int}\,(K \setminus \mathrm{cl}\,\widetilde{K})$ the Taylor expansions of $\phi(t,x)$ and $\phi_T(t,x,u_T)$ coincide for any order, regardless of the values of u_0 and u_T, i.e., we obtain (3.2) for any $k \in \mathbb{N}$ for arbitrary u_T. By continuity of the expressions in the Taylor expansion this property carries over to $\mathrm{cl}\,\mathrm{int}\,(K \setminus \mathrm{cl}\,\widetilde{K})$ which contains $K \setminus \mathrm{cl}\,\widetilde{K}$ because we have assumed $K = \mathrm{cl}\,\mathrm{int}\,K$.

It is hence sufficient to show that u_T satisfies the assertion for $x \in \mathrm{cl}\,\widetilde{K}$. Assume that the function α exists and is bounded. Fix $i \in \{1,\ldots,n\}$ and consider the function $h_i(x) = x_i$. Using the expression v_i for the i–th component of a vector $v \in \mathbb{R}^n$, a simple computation using the identities

$$L_{g_1}h_i(x) = g_1(x)_i \quad \text{and} \quad L_{[g_0,g_1]}h_i(x) = [g_0,g_1](x)_i$$

shows that whenever $g_1(x)_i \neq 0$, then the function α from (3.9) satisfies

$$\begin{aligned}\alpha(x) &= \frac{[g_0,g_1](x)_i L_{g_0+g_1u_0}u_0(x)}{g_1(x)_i} \\ &= \frac{L_{[g_0,g_1]}h_i(x)\cdot L_{g_0+g_1u_0}u_0(x)}{L_{g_1}h_i(x)}.\end{aligned}$$

If $g_1(x)_i = 0$ then the feedback is of the form $u_T^1 + O(T^2)$ for u_T^1 from (2.9). Thus, the feedback is of the form (3.4) for $h = h_i$ and we can use Theorem 3.1 to conclude $\Delta h_i(T,x,u_T) = O(T^4)$ for all $x \in \mathrm{cl}\,\widetilde{K}$. Since $i \in \{1,\ldots,n\}$ was arbitrary, this implies $\Delta\phi(T,x,u_T) = O(T^4)$. Furthermore, again by Theorem 3.1, any feedback yielding $\Delta\phi(T,x,u_T) = O(T^4)$ must be of the form (3.10) if $g_1(x)_i \neq 0$ and since for each $x \in \widetilde{K}$ we have $g_1(x)_i \neq 0$ for some $i \in \{1,\ldots,n\}$ it must be of the form (3.10) for all $x \in \widetilde{K}$.

Conversely, assume that an admissible feedback law u_T satisfying (3.2) on K with $k = 4$ exists. Then for each $x \in \widetilde{K}$ we have $g_1(x)_i \neq 0$ for some suitable $i \in \{1,\ldots,n\}$. Thus, applying Theorem 3.1 for $h = h_i$ we obtain that u_T must be of the form (3.4) for $h = h_i$ and some $i = 1,\ldots,n$, i.e., of the form (3.10). In particular, $\alpha(x)$ meeting (3.9) exists on $\widetilde{K}$ and since u_T is admissible this function α must be bounded on $\widetilde{K}$. On the open set $\mathrm{int}\,(K \setminus \mathrm{cl}\,\widetilde{K})$ we have $g_1 \equiv 0$, thus also $[g_0,g_1] \equiv 0$, which by continuity also holds on $\mathrm{cl}\,\mathrm{int}\,(K \setminus \mathrm{cl}\,\widetilde{K}) = K \setminus \mathrm{cl}\,\widetilde{K}$. Hence we can choose $\alpha(x) = 0$ for $x \in K \setminus \mathrm{cl}\,\widetilde{K}$. This defines a bounded function α for $x \in \widetilde{K} \cup (K \setminus \mathrm{cl}\,\widetilde{K}) = K \setminus (\mathrm{cl}\,\widetilde{K} \setminus \widetilde{K})$. It remains to define α on $\mathrm{cl}\,\widetilde{K} \setminus \widetilde{K}$. Since $\mathrm{cl}\,\mathrm{int}\,K = K$ and $\widetilde{K}$ is open relative to K we obtain $\mathrm{cl}\,\widetilde{K} = \mathrm{cl}\,\mathrm{int}\,\widetilde{K}$. Thus for any $x \in \mathrm{cl}\,\widetilde{K}$ we find a sequence $x_n \to x$ with $x_n \in \mathrm{int}\,\widetilde{K}$, i.e., $x_n \notin \mathrm{cl}\,\widetilde{K} \setminus \widetilde{K}$. Since α is already defined on this set, satisfies (3.9) and is bounded, by continuity we obtain

$$\begin{aligned}&\|[g_0,g_1](x)L_{g_0+g_1u_0}u_0(x)\| \\ &= \lim_{n\to\infty} \|[g_0,g_1](x_n)L_{g_0+g_1u_0}u_0(x_n)\| \\ &\leq \lim_{n\to\infty} \underbrace{|\alpha(x_n)|}_{\text{bounded}}\underbrace{\|g_1(x_n)\|}_{\to 0} = 0.\end{aligned}$$

This implies $[g_0,g_1](x)L_{g_0+g_1u_0}u_0(x) = g_1(x) = 0$ and thus we can set $\alpha(x) = 0$ on $\mathrm{cl}\,\widetilde{K} \setminus \widetilde{K}$ in order to satisfy (3.9). This finishes the proof. □

Remark 3.6: While the condition for $\Delta h = O(T^4)$ is still relatively easy to satisfy at least in parts of the state space, cf. Remark 3.4, the condition about the existence of $\alpha : K \to \mathbb{R}$ with (3.9) *is rather strong. If $L_{g_0+g_1u_0}u_0(x) \neq 0$ (i.e., if the continuous time feedback is not constant up to second order terms along the solution), it says that direction generated by the Lie bracket $[g_0,g_1]$ must be contained in the span of g_1.* □

Remark 3.7: Conditions for $k \geq 5$ can be obtained in a similar way but they become more and more involved, because the number of higher order Lie brackets to be considered grows exponentially. For instance, for $k = 5$ the analogous condition to (3.9) *is the existence of a bounded function $\beta : K \to \mathbb{R}$ satisfying*

$$\begin{aligned}&\beta(x)g_1(x)_i \\ &= \frac{1}{24}\Big(L_{g_0}^2 L_{g_1}h_i(x) - L_{g_1}L_{g_0}^2 h_i(x)\Big)L_{g_0+g_1u_0}u_0(x) \\ &\quad + \frac{1}{24}[g_0,g_1](x)_i L_{g_0+g_1u_0}^2 u_0(x) \\ &\quad + \frac{1}{24}\Big(L_{g_0}L_{g_1}^2 h_i(x) - L_{g_1}^2 L_{g_0}h_i(x)\Big) \\ &\qquad\qquad \cdot\ u_0(x)L_{g_0+g_1u_0}u_0(x) \\ &\quad - \frac{1}{24}\Big(L_{g_0}L_{g_1}h_i(x) + L_{g_1}L_{g_0}h_i(x) \\ &\qquad\qquad + 2L_{g_1}^2 h(x)u_0(x)\Big)\alpha(x)\end{aligned}$$

for each $i = 1,\ldots,n$, $h_i(x) = x_i$ and α from (3.9). □

Remark 3.8: Despite the fact that the conditions for higher order sampled–data feedback control become rather complicated, for a given continuous time closed loop system it is possible to give a recursive MAPLE *procedure which checks the conditions for arbitrary order and calculates the corresponding sampled–data feedback, if possible. The* MAPLE *code for this purpose is given in Section IV.* □

Remark 3.9: The conditions for sampled feedback linearizability derived in [2] bear some similarities with the conditions we derived here. In particular, the necessary conditions for sampled feedback linearizability derived in [2] include the condition $[g_1,[g_0,g_1]] = \alpha g_1$ for an analytic function $\alpha : \mathbb{R}^n \to \mathbb{R}$, which is similar to (3.9)*, However, apart from the fact that geometric conditions on the vector fields appear naturally in both problems, there does not seem to be a deeper connection. In fact, these two problems are different in two important points:*

on the one hand, our results give asymptotic estimates while sampled feedback linearizability is an exact property and thus more difficult to establish. On the other hand, feedback linearization allows for additional coordinate changes which add more flexibility to the problem and thus simplify it. Thus, neither problem follows from the other and hence one cannot expect that the needed conditions imply each other in one way or the other. □

IV. ALGORITHMIC COMPUTATION OF THE CONTROLLERS USING MAPLE

In this section we provide a MAPLE code, which computes the controller satisfying (3.2) for general $k \in \mathbb{N}$, provided it exists. Setting $M = k - 2$, the algorithm has the following structure

1 set $u_T = u_T^1$ from (2.9)
2 for p from 2 to M do
3 for k from 1 to n do
4 compute the Taylor approximations $T_c \approx \phi(T,x)$, $T_d(u) \approx \phi_T(T,x,u)$ up to order T^{p+2}
5 compute the difference $\Delta(u_{solve}) = T_c - T_d(u_T + T^p u_{solve})$ and truncate all terms $\leq O(T^{p+3})$
6 solve $\Delta(u_{solve}) = 0$, set $u_k^{test} = u_{solve}$
7 if $k \geq 2$ and $u_k^{test} \neq u_{k-1}^{test}$ **stop**
8 end of k–loop
9 set $u_T := u_T + T^p u_n^{test}$
10 end of p–loop

Starting from u_T defined in Step 1, iteratively for $p = 2, \ldots, M$ the procedure computes feedback terms u_k^{test} such that (3.1) holds for $k = p+2$, $u_T + T^p u_k^{test}$ and the functions $h_k(x) = x_k$, $k = 1, \ldots, n$. If all u_k^{test}, $k = 1, \ldots, n$, can be computed and coincide, then the new feedback $u_T + T^p u_n^{test}$ will satisfy (3.2) for $k = p+2$. Thus, after successful completion of the p–loop we obtain a feedback satisfying (3.2) for $k = M+2$. Note that the procedure takes the parameter M from (2.5) as input instead of the parameter k from Section III.

Since the equation to be solved in Step 6 is linear in u_{solve}, the algorithm will return a solution provided u_{solve} appears in this equation. During this procedure, the algorithm will not check the boundedness of u_{solve}, hence the boundedness of the resulting feedback u_T has to be checked by the user.

The following MAPLE implementation of the algorithm was tested with MAPLE 9.5 and MAPLE 8. It is available for download on `www.math.uni-bayreuth.de/~lgruene/publ/highordersampling.html`.

```
uT := proc(g0::Vector, g1::Vector, u0::algebraic,
           dim::algebraic, M::algebraic)

local uT, uTc, u, fc, fd, Lc, Tc, Ld, Td, xv,
      p, k, i, hd, hdiff, hdiffu, hdiffus,
      utest, failure, ord;

  # define the continuous and sampled-data vector
  # field for one sampling period T

fc := Vector(dim);
fd := Vector(dim);

for k from 1 to dim do
  fc[k] := g0[k] + g1[k]*u0:
  fd[k] := g0[k] + g1[k]*uTc:
od;

# define an auxiliary vector for computing
# derivatives

xv:=Vector(dim,((j)->x[j]));

# define the zeroth and first order term of
# the sampled-data controller uT

uT := simplify(u0 + T*evalm(
      jacobian(Vector([u0]),xv)&*fc)[1]/2):

ord := M;

for p from 2 to M do
  for k from 1 to dim do
    Lc[0] := [x[k]]:
    Tc[0] := Lc[0]:
    Ld[0] := [x[k]]:
    Td[0] := Ld[0]:

    # compute the coefficients Tc of the
    # Taylor expansion for the continuous
    # time system and state component k

    for i from 1 to p+1 do
      Lc[i] := evalm(jacobian(Lc[i-1],xv)&*fc);
      Tc[i] := T^i/(i!)*Lc[i];
    od;

    # compute the coefficients Td of the Taylor
    # expansion for the sampeled-data system
    # and state component k

    for i from 1 to p+1 do
      Ld[i] := evalm(jacobian(Ld[i-1],xv)&*fd);
      Td[i] := T^i/(i!)*Ld[i];
    od:

    # compute the difference hdiff of the k-th
    # component of the Taylor approximations

    hdiff := 0;
    for i from 0 to p+1 do
      hd[i]:=evalm(Td[i]-Tc[i]):
      hdiff:=evalm(hdiff+hd[i]);
    od:

    # compute the p-th component utest[k] of
    # the sampled-data feedback for h_k(x)=x_k

    hdiffu:=eval( subs(uTc=uT+T^p*usolve,
                 hdiff[1]));
    hdiffus:=simplify(convert(series(
                 hdiffu,T=0,p+2),polynom));
    utest[k]:=solve(hdiffus=0,usolve):

    # check, whether utest[k]=utest[k-1]

    failure := false;
    if (k>=2) then
      if (utest[k]<>utest[k-1]) then
        failure := true;
        break;
      end;
    end;
  od;
```

```
    # if not, print error message and stop
    # computation

    if failure then
      printf("desired order M=%d not feasible\n",
              M);
      ord := p-1;
      break
    end;

    # if yes, add p-th component to sampled-data
    # feedback uT

    uT := simplify(uT + T^p*utest[1]):
  od:

  # output of the resulting feedback

  printf("feedback computed for M=%d",ord);
  uT;
end:
```

For the two examples from Section V, the application of the procedure is given below. For Example 1, the feedback laws computed by `uT(g0,g1,u0,1);` and `uT(g0,g1,u0,2);` are given in (5.2) and (5.3), respectively.

```
# Example 1

g0 := Vector([x[1],x[2]]);
g1 := Vector([x[1]^2,x[2]^2]);
u0 := -x[1] - 1;
uT(g0,g1,u0,2,1);
uT(g0,g1,u0,2,2);
uT(g0,g1,u0,2,3);
uT(g0,g1,u0,2,4);


# Example 2

g0 := Vector([-x[2] - 3/2*x[1]^2
              - 1/2*x[1]^3, 0]);
g1 := Vector([0,1]);
u0 := -7*x[1] + 5*x[2];
uT(g0,g1,u0,2,1);
uT(g0,g1,u0,2,2);
```

V. Examples

We illustrate our results by two examples. The first example is a simple artificial system for which (3.9) holds. It is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} u$$

with $u_0(x) = -x_1 - 1$. Here one computes

$$[g_0, g_1] = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = g_1(x),$$

which immediately implies that (3.9) holds on every compact set K with

$$\alpha(x) = L_{g_0+g_1u_0}u_0(x) = x_1^3 + x_1^2 - x_1.$$

The resulting sampled–data feedback laws for $k = 2, 3, 4$ are, respectively,

$$u_T(x) = -x_1 - 1, \qquad (5.1)$$

$$u_T(x) = -x_1 - 1 + \frac{T}{2}\left(x_1^3 + x_1^2 - x_1\right) \qquad (5.2)$$

and

$$\begin{aligned} u_T(x) &= -x_1 - 1 + \frac{T}{2}\left(x_1^3 + x_1^2 - x_1\right) \qquad (5.3) \\ &+ T^2\left(-\frac{1}{2}x_1^5 - \frac{5}{6}x_1^4 + \frac{5}{12}x_1^3 + \frac{7}{12}x_1^2 - \frac{1}{4}x_1\right). \end{aligned}$$

Figure 5.1 shows the x_1–component of the respective trajectories for $x_0 = (-1, 1)^T$ and sampling interval $T = 0.2$. Here the line without symbols is the continuous time trajectory.

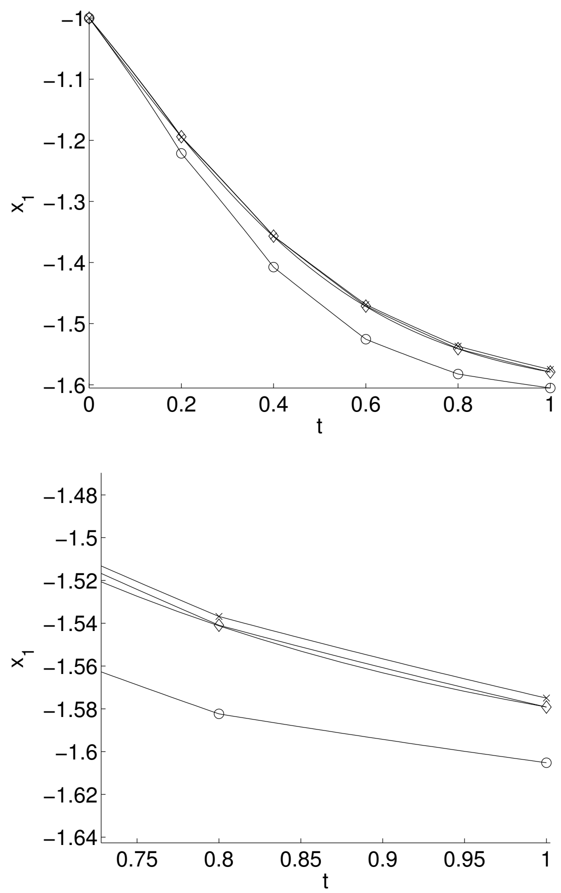

Fig. 5.1. Comparison of the sampled–data controllers (5.1, o), (5.2, x) and (5.3, ⋄), full integration interval (top) and detail (bottom)

Note that at time $t = 1$, i.e., after $1/T$ sampling intervals, we expect the difference between the continuous time solution and the sampled–data solution to be of order T^{k-1}, cf. (2.6). Figure 5.2 shows a log–log plot of these differences which confirms that the respective controllers yield this accuracy.

Our second example is a second order version of the Moore–Greitzer jet engine model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_2 - 3x_1^2/2 - x_1^3/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

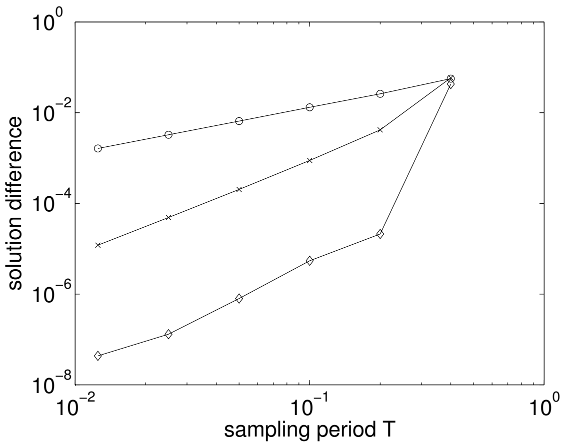

Fig. 5.2. Difference between continuous time and sampled–data solutions at $t = 1$ for sampled–data controllers (5.1, o), (5.2, x) and (5.3, $\diamond$)

Based on a continuous time stabilizing backstepping feedback law derived in [8, Section 2.4.3] given by

$$u_0(x) = -7x_1 + 5x_2,$$

several sampled–data controllers laws were derived in [13]. Despite the fact that these controllers show good performance, we can now prove that no sampled data feedback u_T can satisfy $\Delta\phi(T, x, u_T) \leq O(T^3)$. This follows, because for this system we obtain

$$[g_0, g_1](x) L_{g_0+g_1u_0} u_0(x) = \begin{pmatrix} 32x_2 + \frac{21}{2}x_1^2 + \frac{7}{2}x_1^3 - 35x_1 \\ 0 \end{pmatrix},$$

which is not of the form

$$\alpha(x)g_1(x) = \begin{pmatrix} 0 \\ \alpha(x) \end{pmatrix}$$

for any scalar function $\alpha : \mathbb{R}^n \to \mathbb{R}$. Thus condition (3.9) is violated and consequently a controller u_T yielding $\Delta\phi(T, x, u_T) \leq O(T^3)$ cannot exist.

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