CONSTRUCTION OF LYAPUNOV FUNCTIONS ON THE DOMAIN OF ASYMPTOTIC NULLCONTROLLABILITY: NUMERICS

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Abstract: We present a numerical method for the computation of control Lyapunov functions on the domain of nullcontrollability of a nonlinear system. We apply an adaptive semi–Lagrangian discretization technique to a generalized version of the Zubov equation whose solutions provide such Lyapunov functions. In particular, we address regularization issues which need to be resolved before the scheme is applicable and discuss an adaptive space discretization technique.

Keywords: Asymptotic nullcontrollability, domain, Zubov's method, control Lyapunov function, semi–Lagrangian discretization, dynamic programming

1. INTRODUCTION

In this paper we continue the theoretical work presented in the companion paper (Camilli *et al.*, 2004) on the construction of Lyapunov functions on the domain of nullcontrollability. We consider finite-dimensional nonlinear control systems that are asymptotically nullcontrollable in a neighborhood of the origin. In (Camilli *et al.*, 2004) it was shown that the desired Lyapunov functions can be characterized as (i) optimal value functions to suitable optimal control problems and (ii) as viscosity solutions to a suitable Hamilton–Jacobi PDE, which is a generalization of Zubov's equation.

In this paper we will use both characterizations as we apply a semi–Lagrangian discretization technique to the PDE relying on ideas from optimal control and dynamic programming (Falcone, 1997). In the past, Zubov's equation has already been used as the basis for numerical computations, e.g. in (Dubljevič and Kazantsis, 2002), where the solutions to Zubov's equation for a fixed control value are approximated by truncation of series solutions and the resulting Lyapunov function is used for controller design. This series approximation resembles earlier works, e.g. (Kirin *et al.*, 1982; Vannelli and Vidyasagar, 1985). The method we present here is closer to classical finite element and finite difference techniques for the numerical solution of PDEs and can in particular deal with non smooth solutions which naturally appear in the framework of control Lyapunov functions.

This paper is organized as follows. In the Section 2 we define the class of systems under consideration and introduce the problem.In Section 3 we show

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that our original problem can be approximated by a problem with restricted control range. In Section 4 we present the scheme and show that the original equation needs to be regularized in order to guarantee convergence, an appropriate regularization technique is presented in Section 5. In Section 6 we present the final scheme and in Section 7 we discuss a numerical example.

2. SETUP

We consider the domain of asymptotic nullcontrollability

$$\mathcal{D}_0 := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{there exists } u \in \mathcal{U} \text{ with} \\ \|\varphi(t,x,u)\| \to 0 \text{ for } t \to \infty \end{array} \right\},$$

where φ denotes the solutions of a general nonlinear control system

$$\dot{x}(t) = f(x(t), u(t)) \quad (1)$$

with possibly unbounded control value space, i.e.,

$$u \in \mathcal{U} := L^\infty([0,\infty), U)$$

for some closed set $U \subset \mathbb{R}^m$, assuming that the system is locally asymptotically controllable to the origin. In (Camilli *et al.*, 2004) we have shown that $\mathcal{D}_0$ is characterized by the solution v of the generalized Zubov equation

$$\sup_{u \in U} \{-Dv(x) f(x,u) - (1 - v(x)) g(x,u)\} = 0. \quad (2)$$

More precisely, under mild local Lipschitz conditions on f and g and when $g : \mathbb{R}^n \to \mathbb{R}$ is nonnegative, vanishes at 0 and satisfies appropriate growth properties, then (2) admits a unique viscosity solution v which

(i) characterizes the domain of nullcontrollability via $\mathcal{D}_0 := \{x \in \mathbb{R}^n \,|\, v(x) \le 1\}$
(ii) is a control Lyapunov function for (1) on $\mathcal{D}_0$.

For details on the conditions on f and g we refer to (Camilli *et al.*, 2004).

In this paper we show how this equation can be solved numerically using an adaptive semi–Lagrangian scheme developed in (Capuzzo Dolcetta, 1983), (Falcone, 1987) and (Grüne, 1997), see also (Falcone, 1997).

3. RESTRICTION TO COMPACT U

The numerical scheme we want to apply needs stronger regularity conditions than posed in the theoretical results in (Camilli *et al.*, 2004). More precisely, here we need Lipschitz continuity of f and g uniformly in x and u. The Lipschitz assumptions in (Camilli *et al.*, 2004), however, are uniformly only on compact subsets for the state x and the control variable u. The desired uniformity in x is achieved in a natural way since for our numerical solution we will always have to restrict ourselves to a compact subset $\Omega \subset \mathbb{R}^n$; we will discuss appropriate numerical boundary conditions in the numerical examples Section 7. In order to achieve uniformity in u we restrict ourselves to a compact subset of the control range U, because in this way the global Lipschitz property follows from the assumptions in (Camilli *et al.*, 2004). The following proposition shows that if we choose this compact subset of U large enough then we will end up with a good approximation for the original domain of null controllability. For its formulation, recall the definition of set limits, which for a sequence of sets X_k are given by

$$\limsup_{k\to\infty} X_k := \bigcap_{k\in\mathbb{N}} \bigcup_{m\ge k} X_m$$

and

$$\liminf_{k\to\infty} X_k := \bigcup_{k\in\mathbb{N}} \bigcap_{m\ge k} X_m$$

and, if these two sets coincide,

$$\lim_{k\to\infty} X_k := \limsup_{k\to\infty} X_k = \liminf_{k\to\infty} X_k.$$

Proposition 1. Consider a possibly unbounded closed set $U \subset \mathbb{R}^m$ of control values and its approximation by the compact sets $U_k = \{u \in U \,|\, \|u\| \le k\}$ for $k \in \mathbb{N}$. For the associated spaces of control functions $\mathcal{U}_k$ consider the domains of nullcontrollability $\mathcal{D}_k$ for $U = U_k$. Then the set limit $\lim_{k\to\infty} \mathcal{D}_k$ exists and satisfies

$$\mathcal{D}_0 = \lim_{k\to\infty} \mathcal{D}_k.$$

Proof: We first show that that the solutions v_k of equation (2) with $U = U_k$ satisfy $v(x) = \inf_{k\in\mathbb{N}} v_k(x)$. In order to prove this property, observe that v_k satisfies

$$v_k(x) = \inf_{u\in\mathcal{U}_k} 1 - e^{J(x,u)}$$

with

$$J(x,u) = \int_0^\infty g(\varphi(t,x,u), u(t))dt,$$

see (Camilli *et al.*, 2004, Section 3). Since $\mathcal{U}_k \subseteq \mathcal{U}$ we obviously have the inequality $v_k(x) \ge v(x)$. Now let $x \in \mathcal{D}_0$ and $u \in \mathcal{U}$ be such that

$$1 - e^{J(x,u)} \le v(x) + \varepsilon$$

for some $\varepsilon > 0$. Since $u \in \mathcal{U}$ there exists $k_0 \in \mathbb{N}$ such that $\|u\|_\infty \leq k_0$, hence $u \in \mathcal{U}_{k_0}$. This implies

$$\inf_{k\in\mathbb{N}} v_k(x) \leq v_{k_0}(x) \leq v(x) + \varepsilon.$$

Since ε was arbitrary this shows the claim on $\mathcal{D}_0$. For $x \notin \mathcal{D}_0$ we have $v_k(x) = v(x) = 1$ which shows the claim also in this case.

Using the inf–property we now show the claim of the proposition. Since we have that $v \leq \ldots \leq v_{k+1} \leq v_k$ we obtain the inclusion

$$\mathcal{D}_k \subseteq \mathcal{D}_{k+1} \subseteq \ldots \subseteq \mathcal{D}_0.$$

It follows that $\bigcup_{m\geq k} \mathcal{D}_m \subseteq \mathcal{D}_0$ for each k and hence

$$\limsup_{k\to\infty} \mathcal{D}_k = \bigcap_{k\in\mathbb{N}} \bigcup_{m\geq k} \mathcal{D}_m \subseteq \mathcal{D}_0.$$

On the other hand, if $x \in \mathcal{D}_0$ then for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ with $v_k(x) \leq v(x) + \varepsilon$ for all $k \geq k_0$. This implies that $x \in \mathcal{D}_k$ for all $k \geq k_0$ and consequently $x \in \bigcap_{m\geq k_0} \mathcal{D}_m$. This implies

$$x \in \bigcup_{k\in\mathbb{N}} \bigcap_{m\geq k} \mathcal{D}_m = \liminf_{k\to\infty} \mathcal{D}_k,$$

and since $x \in \mathcal{D}_0$ was arbitrary we obtain

$$\mathcal{D}_0 \subseteq \liminf_{k\to\infty} \mathcal{D}_k,$$

which shows the claim. □

Remark 2. Note that the restriction of U to some compact set often has a well defined meaning, e.g. when the effect of actuator saturation shall be investigated. Solving (2) for U being the range of possible actuator values then gives the maximal domain of nullcontrollability under saturation.

In the remainder of this paper we always assume that the control range U is a compact set.

4. THE NUMERICAL SCHEME I

With restricted control range U we can now formally apply the semi–Lagrangian discretization technique to our system. The method relies on a discretization of the dynamic programming principle, which for v reads

$$v(x) = \inf_{u\in\mathcal{U}} \{1 + G(x,h,u)(v(\varphi(h,x,u)) - 1)\} \quad (3)$$

where

$$G(x,h,u) := \exp\left(-\int_0^h g(\varphi(t,x,u),u(t))dt\right) \quad (4)$$

and $h > 0$ is arbitrary. In the first step we discretize the system in time using a time step h and a numerical one step method $\Phi_h(x,u) \approx \varphi(h,x,u)$ for control systems, see (Grüne and Kloeden, 2001) and a quadrature rule for the integrals, above. In order to keep our presentation simple, here we use a first order approximation given by

$$\varphi(h,x,u) \approx \Phi_h(x,u) := x + hf(x,u)$$

and

$$G(x,h,u) \approx 1 - hg(x,u),$$

where $u \in U$ denotes a constant control value. This way (3) becomes

$$v_h(x) = \inf_{u\in U} \{(1 - hg(x,u))v_h(\Phi_h(x,u)) + hg(x,u)\}. \quad (5)$$

In the second step we consider a closed domain $\Omega \subset \mathbb{R}^n$ and on Ω we use a rectangular grid Γ with nodes x_i where the values on the non grid points are reconstructed by multilinear interpolation. This results in solving

$$\tilde{v}(x_i) = \inf_{u\in U} \{(1 - hg(x_i,u))\tilde{v}(\Phi_h(x_i,u)) + hg(x_i,u)\}. \quad (6)$$

for each node x_i of the grid Γ, where $\tilde{v}$ is continuous and multilinear on each element in the grid (i.e., linear in $\mathbb{R}^1$, bilinear in $\mathbb{R}^2$ etc.) and satisfies $\tilde{v}(0) = 0$ (assuming, of course, that 0 is a node of the grid) and $\tilde{v}(x_i) = 1$ for all $x_i \in \partial\Omega$ with $\Phi_h(x,u) \notin \Omega$ for all $u \in U$. We refer to (6) as the discretized Zubov equation.

While this is indeed the correct formal application of the scheme from (Capuzzo Dolcetta, 1983; Falcone, 1987) which shows good results in many numerical experiments, unfortunately the discretized Zubov equation (6) has a singularity in 0 and hence the fixed point argument used in (Falcone, 1987) fails here. Thus from a theoretical point of view convergence is not guaranteed. In fact, the following counterexample of a system without controls (i.e., $f(x,u) = f(x)$ and $g(x,u) = g(x)$) shows that non convergence of this scheme can indeed happen in practice: consider the situation depicted in Figure 1 (showing one trajectory and the elements surrounding the fixed point 0 in a two-dimensional example). Here the piecewise bilinear function $\tilde{v}$ with

$$\tilde{v}(x_i) = \begin{cases} 1, & x_i \neq 0 \\ 0, & x_i = 0 \end{cases}$$

satisfies (6), since for all nodes $x_i \neq 0$ the value $x_i + hf(x_i)$ lies in an element with nodes $x_j \neq 0$, hence $\tilde{v}(x_i + hf(x_i)) = 1$ implying

$$(1 - hg(x_i))\tilde{v}(x_i + hf(x_i)) + hg(x_i) = 1 = \tilde{v}(x_i),$$

i.e. (6). As this situation may occur for arbitrarily fine grids (and also on simplicid grids, see (Camilli *et al.*, 2001)), this scheme may indeed fail to be convergent.

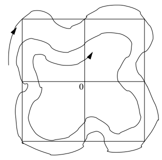

Fig. 1. A situation of non-convergence

5. REGULARIZATION OF THE EQUATION

In order to ensure convergence we will therefore have to use a regularization of (2). The main idea in this is to change (2) in such a way that the "discount rate" (i.e. the factor $g(x)$ in front of the zero order term $v(x)$) becomes strictly positive, and thus the singularity disappears. To this end consider some parameter $\varepsilon > 0$ and consider the function

$$g_\varepsilon(x, u) = \max\{g(x, u), \varepsilon\}.$$

Using this g_ε we approximate (2) by

$$\sup_{u \in U} \{-Dv(x)f(x, u) + g(x, u) - v(x)g_\varepsilon(x, u)\} = 0 \qquad (7)$$

With $DR_\varepsilon := \{x \in \mathbb{R}^n \,|\, g_\varepsilon(x, u) \neq g(x, u)$ for some $u \in U\}$ we denote the domain of regularization. The following proposition summarizes some properties of (7).

Proposition 3. Let the assumptions of (Camilli *et al.*, 2004, Theorem 10) hold and let v be the unique solution of (2) with $v(0) = 0$. Then for each $\varepsilon > 0$ equation (7) has a unique continuous viscosity solution v_ε with the following properties.

(i) $v_\varepsilon(x) \leq v(x)$ for all $x \in \mathbb{R}^n$
(ii) $v_\varepsilon \to v$ uniformly in $\mathbb{R}^n$ as $\varepsilon \to 0$
(iii) If $DR_\varepsilon \subset \mathcal{D}_0$ then the characterization $\mathcal{D}_0 = \{x \in \mathbb{R}^n \,|\, v_\varepsilon(x) < 1\}$ holds
(iv) v_ε is a control Lyapunov function on $\mathcal{D}_0 \backslash DR_\varepsilon$ in the following sense: For each $x \in \mathcal{D}_0$, each $t > 0$ and each $\delta > 0$ there exists $u \in \mathcal{U}$ such that either $\varphi(\tau, x, u) \in DR_\varepsilon$ for some $\tau \in [0, t]$ or the inequality

$$\begin{aligned} &v_\varepsilon(\varphi(t, x, u)) - v_\varepsilon(x) \\ \leq \;& (1 - G(t, x, u))(v_\varepsilon(\varphi(t, x, u)) - 1) + \delta \end{aligned}$$

holds.

Proof: Since the discount rate in (7) is strictly positive it follows by standard viscosity solution arguments (Bardi and Capuzzo Dolcetta, 1997, Chapter III) that there exists a unique continuous solution v_ε which furthermore for all $t \geq 0$ satisfies the following dynamic programming principle

$$v_\varepsilon(x) = \inf_{u \in \mathcal{U}} \{1 + G_\varepsilon(x, h, u)(v_\varepsilon(\varphi(h, x, u)) - 1)\} \qquad (8)$$

with

$$G_\varepsilon(x, h, u) := \exp\left(-\int_0^h g_\varepsilon(\varphi(t, x, a), u(t))dt\right).$$

Since v satisfies the same principle (3) with $G(x, t, u) \geq G_\varepsilon(x, t, u)$ by (4) and $g > 0$ the stated inequality (i) follows.

In order to see (ii) observe that the properties of g and v imply that for each $\delta > 0$ there exists $\varepsilon > 0$ with $DR_\varepsilon \subset \{x \in \mathbb{R}^n \,|\, v(x) \leq \delta\}$. ¿From (i) this immediately implies $DR_\varepsilon \subset \{x \in \mathbb{R}^n \,|\, v_\varepsilon(x) \leq \delta\}$. Now fix $\delta > 0$ and consider the corresponding $\varepsilon > 0$. Let $x \in \mathbb{R}^n$ and pick some $\gamma > 0$ and a control $u_\gamma \in \mathcal{U}$ such that

$$v_\varepsilon(x) \geq \int_0^\infty G_\varepsilon(x, \tau, u_\gamma)g(x(\tau, x, a_\gamma), u_\gamma(\tau))d\tau - \gamma.$$

Note that the corresponding trajectory has to enter DR_ε, because otherwise the integral would be unbounded. Hence there exists a minimal time $T \geq 0$ such that $\varphi(T, x, u_\gamma)) \in DR_\varepsilon$. Then we can conclude that

$$\begin{aligned} & v(x) - v_\varepsilon(x) - \gamma \\ \leq & \int_0^\infty (G(x, \tau, u_\gamma)g(\varphi(\tau, x, u_\gamma), u_\gamma(\tau)) \\ & \quad - \; G_\varepsilon(x, \tau, u_\gamma)g(x(\tau, x, u_\gamma), u_\gamma(\tau)))d\tau \\ \leq & \int_0^T (G(x, \tau, u_\gamma)g(x(\tau, x, u_\gamma), u_\gamma(\tau)) \\ & \quad - \; G_\varepsilon(x, \tau, u_\gamma)g(x(\tau, x, u_\gamma), u_\gamma(\tau)))d\tau \\ & \quad + G(x, T, a_\gamma)v(\varphi(T, x, u_\gamma)) \\ \leq \; & \delta. \end{aligned}$$

Since $\gamma > 0$ was arbitrary this shows $v(x) - v_\varepsilon(x) \leq \delta$ for each $x \in \mathbb{R}^n$ and each $\varepsilon > 0$ sufficiently small and hence (ii).

To prove (iii) let $\varepsilon > 0$ be so small that $DR_\varepsilon \subset \mathcal{D}_0$. Then for all $x \notin \mathcal{D}_0$, all $T > 0$ and all $u \in \mathcal{U}$ we obtain $G(x, t, u) = G_\varepsilon(x, t, u)$ for all $t \in [0, T]$ which immediately implies $v_\varepsilon(x) = v(x) = 1$. Together with (i) this yields the desired equality $\mathcal{D}_0 = \{x \in \mathbb{R}^n \,|\, v_\varepsilon(x) < 1\}$.

(iv) This property follows immediately from (8). □

Remark 4. The property (iv) can be seen as a "practical" Lyapunov function property, because it means that the optimal trajectories will eventually tend to a neighborhood of the origin whose size depends on the size of DR_ε. More precisely, defining

$$\eta_\varepsilon := \sup\{v_\varepsilon(x) \mid x \in DR_\varepsilon\}$$

we obtain that each optimal trajectory will eventually tend to the set

$$\mathcal{N}_\varepsilon := \{x \in \mathbb{R}^n \mid v_\varepsilon(x) \le \eta_\varepsilon\}$$

and stay inside this set. Since by (ii) $v_\varepsilon \to v$ uniformly on $\mathbb{R}^n$ we obtain that $\mathcal{N}_\varepsilon$ indeed shrinks down to 0 as $\varepsilon \to 0$.

6. THE NUMERICAL SCHEME II

We now apply the numerical scheme to (7). Proceeding just as in Section 4, above, we end up with the discrete regularized Zubov equation

$$\begin{aligned}\tilde{v}_\varepsilon(x_i) = \inf_{u \in U}\{&(1 - hg_\varepsilon(x_i,u))\tilde{v}_\varepsilon(\Phi_h(x_i,u)) \\ &+hg(x_i,u)\}.\end{aligned}$$

where again $\tilde{v}_\varepsilon$ is continuous and multilinear on each element of the grid Γ and satisfies $\tilde{v}_\varepsilon(0) = 0$ and $\tilde{v}_\varepsilon(x_i) = 1$ for all $x_i \in \partial\Omega$.

A straightforward modification of the arguments in (Falcone, 1997) yields that there exists a unique solution $\tilde{v}_\varepsilon$ converging to v_ε as h and the size of the grid elements tends to 0. Theoretically the convergence speed depends on the size of ε, in most numerical examples, however, this dependence could not be observed. For the solution of the resulting discrete equation (9) a number of different iterative solvers for are available, see e.g. (Grüne, 1997) and (Grüne and Semmler, 2004).

In higher dimensions the space discretization becomes difficult as our method — which essentially relies on the dynamic programming mechanism — is subject to the well known curse of dimensionality. It is therefore important to choose fine elements in the grid Γ only in those regions of the state space where they are really needed. In order to obtain such an efficient discretization, we apply the adaptive gridding method developed in (Grüne, 1997), see also (Grüne and Semmler, 2004). To this end we define the dynamic programming operator related to (9) by

$$\begin{aligned}T(w)(x) := \inf_{u \in U}\{&(1 - hg_\varepsilon(x,u))w(\Phi_h(x,u)) \\ &+hg(x,u)\}\end{aligned}$$

and a posteriori error estimates by

$$\eta(x) = |\tilde{v}_\varepsilon(x) - T(\tilde{v}_\varepsilon(x))|$$

for each $x \in \Omega$. It can be proved (see (Grüne, 1997, Theorem 2.2)), that there exist constants $C_1, C_2 > 0$ such that the inequality

$$C_1\|\eta\|_\infty \le \|\tilde{v}_\varepsilon - v_{\varepsilon,h}\|_\infty \le C_2\|\eta\|_\infty,$$

where $v_{\varepsilon,h}$ is the solution of the regularized version of (5). Furthermore, $\eta(x) \to 0$ as the size of the element containing x tends to 0. Thus, the values η can be used as error indicators which serve as a basis for a local refinement of the grid elements by refining those elements carrying large (relative) values of η, see (Grüne, 1997) and (Grüne and Semmler, 2004) for details. This mechanism was used in our numerical examples in the next section.

Remark 5. If we use the cost functions $\delta g(x,u)$ and $\delta g_\varepsilon(x,u)$ for $\delta > 0$ and denote the corresponding solutions by v_δ and $v_{\varepsilon,\delta}$, respectively, then it is easy to prove that the convergence property

$$v_\delta \to 1 - \chi_{\mathcal{D}_0} \quad \text{and} \quad v_{\varepsilon,\delta} \to 1 - \chi_{\mathcal{D}_0}$$

holds for $\delta \to 0$ uniformly on compact subsets $K \subset \mathbb{R}^n$ with $K \cap \partial\mathcal{D}_0 = \emptyset$. Here $\chi_{\mathcal{D}_0}$ denotes the characteristic function of $\mathcal{D}_0$.

It turns out that the numerical solutions share this behavior which can be rigorously proved using ideas from numerical dynamics and suitable robustness conditions for $\mathcal{D}_0$; for details we refer to (Grüne, 2002, Chapter 7), where also the relation to set valued numerical methods from (Grüne, 2001) is discussed.

7. A NUMERICAL EXAMPLE

We illustrate our algorithm with a simple example of an inverted pendulum with different restrictions on the control range. The model is given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin(x_1 + \pi) - x_2 + u$$

For this model we have solved equation (7) on the domain $\Omega = [-\pi,\pi] \times [-5,5]$ using periodic boundary conditions on the left and right boundary (taking into account the periodicity of the system) and using "transparent" Dirichlet boundary conditions on the upper and lower boundary. By transparent Dirichlet conditions we mean that we set $v_\varepsilon(x) = 1$ for each $x \in \partial\Omega$ for which the condition $\Phi_h(x,u) \notin \Omega$ for all $u \in \mathcal{U}$ holds. This means that we actually compute the domain of nullcontrollability relative to Ω, i.e., the set of all points which can be controlled to 0 without leaving Ω. Here we have used the control ranges $U = [-\rho,\rho]$ with $\rho = 0.5$, 0.7, 0.9 and 1. With this choice of U it turns out that that the boundary condition never becomes active because all trajectories starting in Ω remain in Ω for all future times, hence our computation really approximate the solution to (7) on $\mathbb{R}^2$ restricted to Ω. As numerical parameters for our calculations we have used the time step $h = 0.05$ and adaptively generated grids with ≈ 10000 nodes. The cost function was $g(x) = \|x\|^2/10$ and the regularization parameter was chosen as $\varepsilon = 10^{-4}$. Figure 2 shows the respective results for the different values of ρ. Note that for $\rho \leq 0.9$ the computed domain of nullcontrollability is a proper subset of Ω, while for $\rho = 1$ it coincides with Ω as the maximum of $\tilde{v}_\varepsilon$ on Ω is $0.933 < 1$.

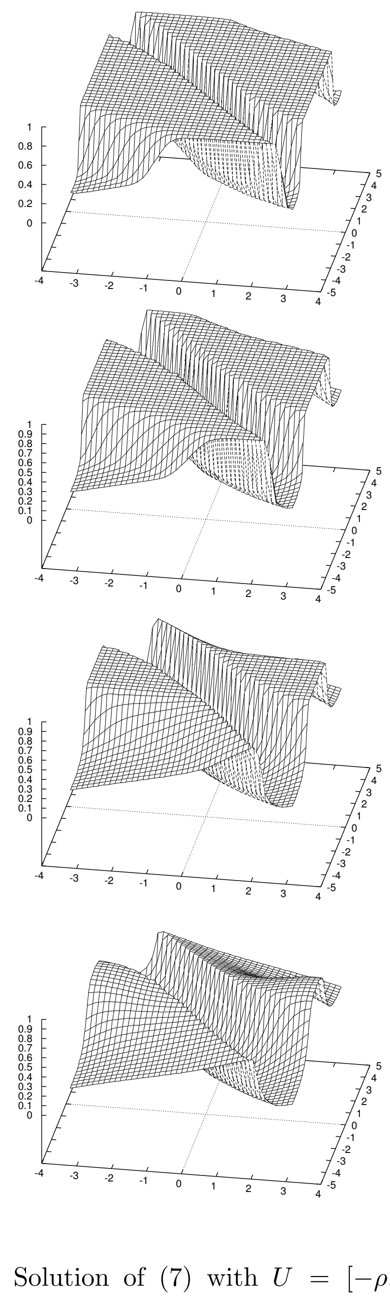

Fig. 2. Solution of (7) with $U = [-\rho, \rho]$, $\rho = 0.5, 0.7, 0.9, 1$ (top to bottom)

8. REFERENCES

Bardi, M. and I. Capuzzo Dolcetta (1997). *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc.. Boston, MA.

Camilli, F., L. Grüne and F. Wirth (2001). A regularization of Zubov's equation for robust domains of attraction. In: *Nonlinear control in the year 2000, Vol. 1 (Paris)*. Vol. 258 of *Lecture Notes in Control and Inform. Sci.*. pp. 277–289. Springer. London.

Camilli, F., L. Grüne and F. Wirth (2004). Construction of Lyapunov functions on the domain of asymptotic nullcontrollability: theory. In: *Proceedings of NOLCOS 2004, Stuttgart, Germany*. To appear.

Capuzzo Dolcetta, I. (1983). On a discrete approximation of the Hamilton-Jacobi equation of dynamic programming. *Appl. Math. Optim.* **10**, 367–377.

Dubljevič, S. and N. Kazantsis (2002). A new Lyapunov design approach for nonlinear systems based on Zubov's method. *Automatica* **34**, 1365–1391.

Falcone, M. (1987). A numerical approach to the infinite horizon problem of deterministic control theory. *Appl. Math. Optim.* **15**, 1–13. *Corrigenda*, ibid., 23 (1991), 213–214.

Falcone, M. (1997). Numerical solution of dynamic programming equations. Appendix A in (Bardi and Capuzzo Dolcetta, 1997).

Grüne, L. (1997). An adaptive grid scheme for the discrete Hamilton-Jacobi-Bellman equation. *Numer. Math.* **75**(3), 319–337.

Grüne, L. (2001). Subdivision techniques for the computation of domains of attraction and reachable sets. In: *Proceedings of NOLCOS 2001, St. Petersburg, Russia*. pp. 762–767.

Grüne, L. (2002). *Asymptotic Behavior of Dynamical and Control Systems under Perturbation and Discretization*. Lecture Notes in Mathematics, Vol. 1783. Springer–Verlag.

Grüne, L. and P. E. Kloeden (2001). Higher order numerical schemes for affinely controlled nonlinear systems. *Numer. Math.* **89**, 669–690.

Grüne, L. and W. Semmler (2004). Using dynamic programming with adaptive grid scheme for optimal control problems in economics. *Journal of Economic Dynamics and Control*. To appear.

Kirin, N. E., R. A. Nelepin and V. N. Bajdaev (1982). Construction of the attraction region by Zubov's method. *Differ. Equations* **17**, 871–880.

Vannelli, A. and M. Vidyasagar (1985). Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems. *Automatica* **21**, 69–80.