Attraction rates, robustness and discretization of attractors

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Abstract: We investigate necessary and sufficient conditions for the convergence of attractors of discrete time dynamical systems induced by numerical one–step approximations of ordinary differential equations (ODEs) to an attractor for the approximated ODE. We show that both the existence of uniform attraction rates (i.e., uniform speed of convergence towards the attractors) and uniform robustness with respect to perturbations of the numerical attractors are necessary and sufficient for this convergence property. In addition, we can conclude estimates for the rate of convergence in the Hausdorff metric.

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1 Introduction

The long time behavior of dissipative dynamical systems is essentially determined by the attractors of these systems, since for large times its trajectories will typically stay on or near an attractor. Even for moderately complex finite dimensional systems, however, it is rarely possible to determine attractors by analytic methods. Hence numerical approximations form a natural part of a systematic analysis. It is therefore important to know about the effects of discretization errors on attractors in order to give a reasonable interpretation to numerical experiments and to justify numerical findings.

For dynamical systems induced by ordinary differential equations (ODEs), numerical one–step approximations like Runge–Kutta or Taylor schemes form an important class of schemes. It follows from a result of Kloeden and Lorenz in 1986 [17] for attracting sets, that if the ODE possesses an attractor A, then the discrete time dynamical system induced by such a numerical scheme also has an attractor contained in a neighborhood N of A, where the size of N shrinks down to A as the time–step of the discretization approaches 0.

A number of examples (see, e.g., [7, Example (2.12)], [8, Example 1.1.1] or [11] for the case of finite dimensional approximations of infinite dimensional systems) shows that the limit set for a convergent sequence of numerical attractors for vanishing time–steps may be strictly smaller than A. This fact, however, imposes a major problem for the interpretation of numerical results, since it implies that in general one cannot conclude the existence of a real attractor close to a numerical attractor. Hence it is important to derive techniques or conditions which allow to conclude convergence of numerical attractors to a real attractor.

There are three main approaches for tackling this problem: The first is to impose suitable conditions on the approximated system, which ensure a faithful numerical approximation and exclude the appearance of numerical artifacts. Typical examples of this approach are, for instance, results on the numerical approximation of Morse–Smale systems by Garay [5, 6], on the discretization of gradient systems by Stuart and Humphries [21, Section 7.7] and a result on hyperbolic attractors by the author [9, Remark 2.10(ii)].

The second approach is to design algorithms which can be shown to converge to the right objects under no or under very mild conditions on the approximated system. An example for this approach is the subdivision algorithm for the computation of attractors originally developed by Dellnitz and Hohmann [3] using a rigorous discretization as proposed by Junge [13, 14], see also [8, Section 6.3] for a description and a quantitative convergence analysis of this method based on robust Lyapunov functions.

The idea of the third approach is the formulation of conditions on the behavior of the numerical systems under which we can ensure convergence of the respective sets or the existence of respective nearby sets for the approximated system. A typical example are the sufficient conditions for the convergence of numerical attracting sets in the Galerkin approximation to Navier–Stokes equations by Kloeden [15]. For finite dimensional systems, in [9] a necessary and sufficient condition of this type was developed based on uniform attraction properties of the numerical attractors, which were characterized using (uniformly) shrinking families of neighborhoods which are mapped onto each other by suitable perturbations of the numerical flows.

The present paper follows this third approach. As in [9] we are going to formulate necessary and sufficient conditions for the convergence of numerical attractors to a real attractor in terms of uniform attraction properties. The difference to [9] lies in the type of uniformity properties used for this purpose, because (i) here we formulate the properties directly in terms of the numerical attractors instead of using auxiliary attracting sets, (ii) we use comparison functions for characterizing attraction properties instead of using geometrical characterizations by means of shrinking neighborhoods which are difficult to identify in a numerical simulation, and (iii) most importantly, instead of using a condition on the rate of attraction (i.e., the speed of convergence towards the attractor) for perturbed numerical systems, here we "decouple" the rate and the perturbation and give two different conditions, one based on the attraction rate for the unperturbed numerical systems and the other based on the robustness against perturbations.

More precisely, we prove that a sequence of numerical attractors converges to a real attractor

- if and only if the numerical attractors are attracting with uniformly bounded attraction rates, cf. Theorem 6.2(iii)
- if and only if the numerical attractors are robust against perturbation with uniformly bounded robustness gains, cf. Theorem 6.2(ii) and Theorem 6.4.

In addition, in Theorem 6.5 we give estimates for the discretization error based on the local error of the numerical scheme and the robustness gains of the respective attractors.

The tools we need in order to obtain these results are developed step by step in this paper, which is organized as follows. After defining the setup and stating some preliminary results in Section 2, in Section 3 we define a suitable robustness concept for attracting sets with respect to perturbations, describe the concept of embedding systems into each other and show how this applies to numerical one–step approximations. In Section 4 we study the relation between robustness of attracting sets and their rate of attraction. In Section 5 we prove some useful results on the relation between a continuous time system and its time–h map and finally, in Section 6 we state the main results on attractors under one–step discretization.

2 Setup and Preliminaries

We consider ordinary differential equations given by

$$\dot{x} = f(x) \tag{2.1}$$

and, for some time–step $h > 0$, discrete time systems of the form

$$x(t+h) = \Phi_h(x(t)), \tag{2.2}$$

where f and Φ_h are maps from $\mathbb{R}^d$ to $\mathbb{R}^d$, $d \in \mathbb{N}$.

For simplicity of exposition (cf. Remark 2.1, below) we assume global Lipschitz properties of the respective systems, i.e., we assume that there exists a constant $L > 0$ such that the inequalities

$$\|f(x) - f(y)\| \leq L\|x - y\| \text{ for all } x,\, y \in \mathbb{R}^d \tag{2.3}$$

and

$$\|F_h(x) - F_h(y)\| \leq L\|x - y\| \text{ for all } x,\, y \in \mathbb{R}^d \tag{2.4}$$

hold, where $F_h(x) = (\Phi_h(x) - x)/h$.

The trajectories of system (2.1) with initial value $x_0 \in \mathbb{R}^d$ for initial time $t = 0$ are denoted by $\varphi(t, x_0)$ for $t \in \mathbb{R}_0^+$ and the respective trajectories for (2.2) are denoted by $\Phi_h(t, x)$ for $t \in h\mathbb{N}_0 := \{hk \,|\, k \in \mathbb{N}_0\}$.

It will often be convenient to combine continuous and discrete time trajectories in one notation. For this we use the notation $\Phi(t, x)$ which either denotes $\varphi(t, x)$ or $\Phi_h(t, x)$, where the precise meaning will be clear from the context. Whenever we consider a discrete time system with time–step $h > 0$ the time t is implicitly assumed to be in the respective discrete time–scale $h\mathbb{N}_0$.

For sets $C \subset \mathbb{R}^d$ we use the notation

$$\Phi(t, C) = \bigcup_{x \in C} \{\Phi(t, x)\}.$$

A special type of a discrete time system is the time–h map of (2.1) which is defined by the discrete time system

$$x(t+h) = \varphi(h, x(t)). \tag{2.5}$$

The trajectories of (2.5) are denoted by $\varphi_h(t, x)$. Note that if (2.1) satisfies (2.3) for some $L > 0$ then Gronwall's Lemma implies that the time–h map (2.5) satisfies (2.4) for the Lipschitz constant $\widetilde{L} = Le^{hL} > L$.

Another special type of a discrete time system (2.2) is a numerical one–step approximation $\widetilde{\Phi}_h$ of (2.1) which is supposed to satisfy (2.4) and is such that there exist constants c, $q > 0$ with

$$\|\widetilde{\Phi}_h(x) - \varphi(h, x)\| \leq ch^{q+1} \text{ for all } x \in \mathbb{R}^d. \tag{2.6}$$

Here q is called the order of the scheme. Examples for such approximations are Taylor and Runge–Kutta schemes; for details we refer, e.g., to the textbooks [4, 12, 20].

Remark 2.1 The global estimates in the inequalities (2.3), (2.4) and (2.6) are in general quite restrictive. However, since we are interested in the behavior on bounded subsets of the state space, one can always assume these properties without loss of generality by applying standard cutoff techniques. □

Since we are going to measure distances between different sets, we need the following definitions.

Definition 2.2 Let C, $D \subset \mathbb{R}^d$ be nonempty compact sets, $x \in \mathbb{R}^d$, and let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^d$. We define the distance from a point to a set by

$$\|x\|_D := \min_{y\in D} \|x - y\|,$$

the nonsymmetric Hausdorff distance between two compact sets by

$$\mathrm{dist}(C, D) := \max_{x\in C} \min_{y\in D} \|x - y\|,$$

and the Hausdorff metric for compact sets by

$$d_H(C, D) := \max\{\mathrm{dist}(C, D), \mathrm{dist}(D, C)\}.$$

We use these distances for arbitrary bounded sets C, $D \subset \mathbb{R}^d$ by defining

$$\|x\|_D := \|x\|_{\mathrm{cl}\, D},\ \ \mathrm{dist}(C, D) := \mathrm{dist}(\mathrm{cl}\, C, \mathrm{cl}\, D) \text{ and } d_H(C, D) := d_H(\mathrm{cl}\, C, \mathrm{cl}\, D).$$

For $\varepsilon > 0$ we denote the (open) ε-ball around C by $\mathcal{B}(\varepsilon, C) := \{y \in \mathbb{R}^d \mid \|y\|_C < \varepsilon\}$. If $C = \{x\}$ we also write $\mathcal{B}(\varepsilon, x)$. □

Note that for all bounded sets $C, D, E \subset \mathbb{R}^d$ the equivalences $\mathrm{dist}(C, D) = 0 \Leftrightarrow C \subseteq \mathrm{cl}\, D$ and $d_H(C, D) = 0 \Leftrightarrow \mathrm{cl}\, C = \mathrm{cl}\, D$ and the implication $C \subseteq E \Rightarrow \mathrm{dist}(C, D) \leq \mathrm{dist}(E, D)$ hold.

Our main objects of interest are attracting sets and attractors as given by the following definition.

Definition 2.3 Let Φ denote the trajectories of a system of type (2.1) or (2.2). Consider a compact set $A \subset \mathbb{R}^d$ and an open and bounded set $B \subset \mathbb{R}^d$ with $A \subset B$. Then A is called an *attracting set* with *attracted neighborhood* B if it is forward invariant, i.e.,

$$\Phi(t, A) \subseteq A \text{ for all } t \geq 0$$

and satisfies

$$\lim_{t\to\infty} \mathrm{dist}(\Phi(t, B), A) \to 0.$$

An attracting set is called an *attractor*, if it is invariant, i.e.,

$$\Phi(t, A) = A \text{ for all } t \geq 0.$$

□

In order to characterize quantitative properties of attracting sets and attractors we make use of comparison functions as introduced by Hahn [10].

Definition 2.4 We define the following classes of *comparison functions*.

$$\begin{aligned}
\mathcal{K} &:= \{\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \gamma \text{ is continuous, strictly increasing and } \gamma(0) = 0\} \\
\mathcal{L} &:= \{\sigma : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \sigma \text{ is continuous, strictly decreasing and } \lim_{r\to\infty} \sigma(r) = 0\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \beta(\cdot, r) \in \mathcal{K} \text{ and } \beta(r, \cdot) \in \mathcal{L} \text{ for each } r \geq 0\}.
\end{aligned}$$

□

Remark 2.5 The functions $\beta \in \mathcal{KL}$ are closely related to the usual ε–δ definition of asymptotic stability. More precisely, for any function $a : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the two properties

(i) for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $r \leq \delta$ then $a(r, t) < \varepsilon$ for all $t \geq 0$

(ii) for all $\varepsilon > 0$ and for all $R > 0$ there exists $T > 0$ such that $a(r, t) < \varepsilon$ for all $0 \leq r \leq R$ and for all $t \geq T$

there exists a function $\beta \in \mathcal{KL}$ with $a(r, t) \leq \beta(r, t)$ for all r, $t \geq 0$.

This fact was already implicitly used in Hahn's book [10]; in this form it is stated (but not proved) in Albertini and Sontag [1, Lemma 4.1] and proved (but not explicitly stated) by Lin, Sontag and Wang [18, Section 3]. □

Using class $\mathcal{KL}$ functions we can define rates of attraction for attracting sets.

Definition 2.6 Let Φ denote the trajectories of a system of type (2.1) or (2.2) and let A be an attracting set with attracted neighborhood B. Then $\beta \in \mathcal{KL}$ is called *rate of attraction* of A if the inequality

$$\|\Phi(t, x)\|_A \leq \beta(\|x\|_A, t)$$

holds for each $x \in B$ and each $t \geq 0$. □

The following lemma shows that each attracting set possesses a rate of attraction.

Lemma 2.7 Let Φ denote the trajectories of a system of type (2.1) or (2.2) and let A be an attracting set with attracted neighborhood B. Then there exists a rate of attraction $\beta \in \mathcal{KL}$ for A.

Proof: Using the forward invariance and attractivity properties of A, and the (uniform) continuous dependence of a trajectory on the initial value (as induced by Gronwall's Lemma for (2.1) or by induction for (2.2)) one easily verifies that the function $a : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $a(0,t) = 0$ for $t \in \mathbb{R}_0^+$ and

$$a(r,t) := \sup_{\tau \geq t} \operatorname{dist}\left(\Phi\left(\tau, \mathcal{B}(r,A) \cap B\right),\, A\right)$$

satisfies the properties (i) and (ii) of Remark 2.5. Hence there exists $\beta \in \mathcal{KL}$ with $a \leq \beta$ and consequently

$$\|\Phi(x,t)\|_A \leq \sup_{\tau \geq t} \operatorname{dist}\left(\Phi\left(\tau, \mathcal{B}(\|x\|_A, A) \cap B\right),\, A\right) = a(\|x\|_A, t) \leq \beta(\|x\|_A, t)$$

for all $x \in B$ and all $t \geq 0$. This shows the claim. $\square$

We end this section by stating some useful properties of attractors which we will need in what follows.

Lemma 2.8 Let Φ denote the trajectories of a system of type (2.1) or (2.2) and let A be an attracting set with attracted neighborhood B. Then A contains an attractor with attracted neighborhood B.

Proof: Verification of the desired properties shows that

$$\widetilde{A} := \bigcap_{T \geq 0} \operatorname{cl} \bigcup_{t \geq T} \Phi(t, B)$$

is the desired attractor, see [21, Theorem 2.7.4] for details. $\square$

Lemma 2.9 Let Φ denote the trajectories of a system of type (2.1) or (2.2). Then a compact forward invariant attracting set A for Φ with attracted neighborhood B is an attractor with attracted neighborhood B if and only if it is the minimal compact forward invariant attracting set (w.r.t. set inclusion) with attracted neighborhood B. In particular for each open and bounded set $B \subset \mathbb{R}^d$ there exists at most one attractor with attracted neighborhood B.

Proof: Let A be an attractor with attracted neighborhood B. Then, in particular, A is invariant. Now assume that $\tilde{A} \subset A$, $\tilde{A} \neq A$, is a forward invariant attracting set. Then there exists a neighborhood $\mathcal{N} \supset \tilde{A}$ with $A \not\subseteq \mathcal{N}$, such that $\Phi(t,B) \subset \mathcal{N}$ for some $t \geq 0$, i.e., in particular $\Phi(t,A) \neq A$ which contradicts the invariance of A.

Let conversely A be a minimal forward invariant attracting set. Then by Lemma 2.8 A contains an attractor which again is a forward invariant attracting set. Hence by minimality it coincides with A. $\square$

The next Lemma shows that the attractor is also the maximal compact invariant set contained in $\text{int}B$.

Lemma 2.10 Let Φ denote the trajectories of a system of type (2.1) or (2.2) and let A be an attractor with attracted neighborhood B for Φ. Then each compact invariant set $D \subset B$ for Φ is contained in A.

Proof: Let $D \subset B$ be an invariant set for Φ. Then $D = \Phi(t, D) \subset \Phi(t, B)$ for all $t \geq 0$. On the other hand, for each neighborhood $\mathcal{N} \supset A$ we know that $\Phi(t, B) \subset \mathcal{N}$ for all $t \geq 0$ sufficiently large. Hence $D \subset \mathcal{N}$ which implies the assertion. $\square$

3 Inflation, robustness and embedding

The main technique that we will use in this paper in order to obtain results on convergence of numerical attractors is the embedding of the numerical approximation into a perturbed continuous time system, and vice versa. In this section we define suitable perturbed systems, the corresponding attracting sets and a useful robustness concept for attracting sets. In addition, we give a mathematically precise meaning for the embedding property.

The following definition gives the appropriate perturbed systems (see also [16] for an equivalent definition using differential inclusions).

Definition 3.1 For $\alpha \in \mathbb{R}$, $\alpha \geq 0$ we define the set of perturbation values

$$W_\alpha := \{w \in \mathbb{R}^d \,|\, \|w\| \leq \alpha\}$$

and the space of measurable functions with values in W_α by

$$\mathcal{W}_\alpha := \{w : \mathbb{R} \to \mathbb{R}^d \,|\, w \text{ measurable with } w(t) \in W_\alpha \text{ for almost all } t \in \mathbb{R}\}.$$

For functions $w \in \mathcal{W}_\alpha$ and real values $a < b$ we define $\|w\|_{[a,b]} := \operatorname{ess\,sup}_{t\in[a,b]}\|w(t)\|$.

For a continuous time system (2.1) we define the α*–inflated system* by

$$\dot{x} = f(x) + w, \quad w \in W_\alpha, \tag{3.1}$$

and for a discrete time system we define it by

$$x(t+h) = \Phi_h(x(t)) + \int_t^{t+h} w(t)dt, \quad w \in \mathcal{W}_\alpha. \tag{3.2}$$

For each initial value $x \in \mathbb{R}^d$ and each $w \in \mathcal{W}_\alpha$ we denote the corresponding trajectory by $\varphi(t, x, w)$ or $\Phi_h(t, x, w)$, respectively. It should be noted that the discrete time inflation

(3.2) of the time–h map (2.5) of a continuous time system (2.1) differs from the time–h map of the continuous time inflation (3.1) of system (2.1) defined by

$$x(t+h) = \varphi(h, x(t), w(t+\cdot)). \tag{3.3}$$

Throughout this paper, the term *inflated time–h map* refers to system (3.3), whose trajectories will be denoted by $\varphi_h(t, x, w)$.

As for the unperturbed systems, we use $\Phi(t, x, w)$ to denote both discrete and continuous time trajectories, depending on the context. Furthermore, for $\alpha > 0$, $x \in \mathbb{R}^d$ and a subset $C \subseteq \mathbb{R}^d$ we use the notations

$$\Phi^\alpha(t, x) := \bigcup_{w \in \mathcal{W}_\alpha} \{\Phi(t, x, w)\} \quad \text{and} \quad \Phi^\alpha(t, C) := \bigcup_{x \in C} \Phi^\alpha(t, x).$$

□

Next we define suitable attracting sets for inflated systems and a robustness property of attracting sets.

Definition 3.2 Consider an inflated continuous time system (3.1), an inflated discrete time system (3.2) or an inflated time–h map (3.3) with trajectories denoted by $\Phi(t, x, w)$. Then a compact set A_α with open neighborhood B is called an *α–attracting set* with *attracted neighborhood* B if it is α–forward invariant, i.e.,

$$\Phi^\alpha(t, A) \subseteq A \text{ for all } t \geq 0$$

and satisfies

$$\lim_{t\to\infty} \operatorname{dist}(\Phi^\alpha(t, B), A) \to 0.$$

Let $\alpha_0 > 0$ and $\gamma \in \mathcal{K}$. Then an attracting set (or attractor) A with attracted neighborhood B for an unperturbed system (2.1) or (2.2) is called *γ–robust* for γ and α_0, if for each $\alpha \in (0, \alpha_0]$ there exists an α–attracting set A_α with attracted neighborhood B for the corresponding inflated system with $A \subseteq A_\alpha$ and

$$d_H(A, A_\alpha) \leq \gamma(\alpha).$$

Here $\gamma \in \mathcal{K}$ is called *robustness gain.* □

Remark 3.3 Analogous to Lemma 2.7 one sees that for each α–attracting set A_α with attracted neighborhood B there exists $\beta \in \mathcal{KL}$ such that

$$\|\Phi(t, x, w)\|_{A_\alpha} \leq \beta(\|x\|_{A_\alpha}, t)$$

for all $x \in B$, $t \geq 0$ and all $w \in \mathcal{W}_\alpha$. □

We now define what we mean by an embedded system. For our purpose it is sufficient to define this concept for discrete time systems.

Definition 3.4 Consider two inflated discrete time systems of type (3.2) with perturbations from $\mathcal{W}_{\tilde{\alpha}_0}$ and $\mathcal{W}_{\alpha_0}$, respectively. Denote the trajectories of the systems by Φ_h and Ψ_h, respectively, and let $\alpha \geq 0$ and $C \geq 1$. Then we say that the second system Ψ_h is *(α, C)-embedded* in the first Φ_h if for each $x \in \mathbb{R}^d$ and each $w \in \mathcal{W}_{\alpha_0}$ there exist $\tilde{w} \in \mathcal{W}_{\tilde{\alpha}_0}$ with $\|\tilde{w}\|_{[t,t+h]} \leq \alpha + C\|w\|_{[t,t+h]}$ and

$$\Phi_h(t, x, \tilde{w}) = \Psi_h(t, x, w)$$

for all $t \in h\mathbb{N}_0$.

Here we call Φ_h the *embedding system* and Ψ_h the *embedded system.* □

Lemma 3.5 Consider three discrete time inflated systems Φ_h, Ψ_h and Θ_h of type (2.2) and assume that Ψ_h is $(\alpha_1,\ C_1)$–embedded in Θ_h and Θ_h is $(\alpha_2,\ C_2)$–embedded in Φ_h. Then Ψ_h is $(\alpha_1 + C_1\alpha_2,\ C_1C_2)$–embedded in Φ_h.

Proof: Straightforward using Definition 3.4. □

The following proposition shows how the inflated numerical system

$$x(t + h) = \widetilde{\Phi}_h(t, x(t)) + \int_t^{t+h} w(s)ds, \tag{3.4}$$

with trajectories denoted by $\widetilde{\Phi}_h(t, x, w)$, can be embedded into the inflated time–h map (3.3), and vice versa.

Proposition 3.6 Consider the numerical approximation $\widetilde{\Phi}_h$ of system (2.1) for some $h > 0$. Let $\alpha_0 > 0$ and consider the constants L and c from (2.3) and (2.6).

Then the α_0–inflated numerical system $\widetilde{\Phi}_h(t, x, w)$ from (3.4) is $(ch^q,\ 1 + hL)$–embedded in the $ch^q + (1 + hL)\alpha_0$–inflated time–h map φ_h from (3.3).

Conversely, the α_0–inflated time–h map φ_h from (3.3) is $(e^{hL}ch^q,\ e^{hL})$–embedded in the $e^{hL}ch^q + e^{hL}\alpha_0$–inflated numerical system $\widetilde{\Phi}_h(t, x, w)$ from (3.4).

Proof: Consider the auxiliary system defined by

$$x(t + h) := \varphi(h, x(t)) + \int_t^{t+h} w(s)ds$$

for $t \in h\mathbb{N}_0$ and denote the trajectories with initial value $x \in \mathbb{R}^d$ at initial time $t = 0$ by $\tilde{\varphi}_h(t, x, w)$. It is immediate from Definition 3.4 and inequality (2.6) that $\tilde{\varphi}_h$ is $(ch^q, 1)$–embedded in $\widetilde{\Phi}_h$ and that $\widetilde{\Phi}_h$ is $(ch^q, 1)$–embedded in $\tilde{\varphi}_h$.

We claim that the system $\tilde{\varphi}_h$ is $(0,\ 1 + Lh)$–embedded in φ_h and that φ_h is $(0,\ e^{Lh})$–embedded in $\tilde{\varphi}_h$. Then the assertion follows from Lemma 3.5.

In order to prove the embedding relation between φ_h and $\tilde{\varphi}_h$ fix some $w \in \mathcal{W}_{\alpha_0}$. It is sufficient to show the embedding for $t = h$ since then we can continue by induction. We first construct $\tilde{w}$ such that $\varphi_h(h, x, \tilde{w}) = \tilde{\varphi}_h(h, x, w)$.

Consider the perturbation

$$\tilde{w}(t) = f(\varphi(t,x)) + w(t) - f\left(\varphi(t,x) + \int_0^t w(\tau)d\tau\right)$$

for $t \in [0,h]$. Then we obtain

$$\begin{aligned} \frac{d}{dt}\left(\varphi(t,x) + \int_0^t w(\tau)d\tau\right) &= f(\varphi(t,x)) + w(t) \\ &= f\left(\varphi(t,x) + \int_0^t w(\tau)d\tau\right) + \tilde{w}(t) \end{aligned}$$

and

$$\frac{d}{dt}\varphi(t,x,\tilde{w}) = f(\varphi(t,x,\tilde{w})) + \tilde{w}(t),$$

which by the uniqueness of the solution to this differential equation implies

$$\varphi_h(h,x,\tilde{w}) = \varphi(h,x,\tilde{w}) = \varphi(h,x) + \int_0^h w(\tau)d\tau = \tilde{\varphi}_h(h,x,w).$$

From the Lipschitz estimate (2.3) we obtain for almost all $\tau \in [0,h]$ the inequality

$$\|\tilde{w}(\tau)\| \le \|w(\tau)\| + L\left\|\int_0^\tau w(s)ds\right\|,$$

which implies

$$\|\tilde{w}\|_{[0,h]} \le \|w\|_{[0,h]} + Lh\|w\|_{[0,h]}$$

and thus shows the claim.

Conversely, given again $w \in \mathcal{W}_{\alpha_0}$, we now construct $\tilde{w}$ such that $\tilde{\varphi}_h(h,x,\tilde{w}) = \varphi_h(h,x,w)$. For this purpose, consider $\tilde{w}$ given by

$$\tilde{w}(t) = f(\varphi(t,x,w)) + w(t) - f(\varphi(t,x))$$

for $t \in [0,h]$. Then similar arguments as above yield the equality $\tilde{\varphi}_h(h,x,\tilde{w}) = \varphi_h(h,x,w)$. By Gronwall's Lemma one easily obtains $\|\varphi(\tau,x,w) - \varphi(\tau,x)\| \le \|w\|_{[0,\tau]}(e^{L\tau}-1)/L$ which shows that $\|f(\varphi(\tau,x,w)) - f(\varphi(\tau,x))\| \le \|w\|_{[0,\tau]}(e^{L\tau}-1)$ and thus

$$\|\tilde{w}(\tau)\| \le \|w(\tau)\| + \|w\|_{[0,\tau]}(e^{L\tau}-1)$$

for almost all $\tau \in [0,h]$, implying

$$\|\tilde{w}\|_{[0,h]} \le \|w\|_{[0,h]} + \|w\|_{[0,h]}(e^{Lh}-1) = e^{Lh}\|w\|_{[0,h]},$$

i.e., the desired estimate. □

In the following two propositions we show the relations between attracting sets of embedding and embedded systems.

Proposition 3.7 Consider a discrete time system with trajectories Ψ_h, which is (α, C)-embedded in some other discrete time system with trajectories denoted by Φ_h for some $\alpha \geq 0$, $C \geq 1$. Assume that the embedding system Φ_h has an attracting set A which is γ-robust for some $\gamma \in \mathcal{K}$ and some $\alpha_0 \geq \alpha$. Then the embedded system Ψ_h has an attracting set $\widetilde{A}$ with attracted neighborhood B which satisfies

$$d_H(\widetilde{A}, A) \leq \gamma(\alpha).$$

Proof: By the embedding property we obtain $\Psi_h(t, B) \subseteq \Phi_h^\alpha(t, B)$. Hence the α–attracting set A_α for the inflated embedding system Φ_h^α is an attracting set for the embedded system Ψ_h. Hence $\widetilde{A} = A_\alpha$ is the desired set. □

The next proposition shows that we can even conclude the existence of robust attracting sets for the embedded system, if we are willing to allow a larger distance between $\widetilde{A}$ and A.

Proposition 3.8 Consider a discrete time system with trajectories Ψ_h, which is (α, C)-embedded in some other discrete time system with trajectories denoted by Φ_h for some $\alpha \geq 0$, $C \geq 1$. Assume that the embedding system Φ_h has an attracting set A which is γ-robust for some $\gamma \in \mathcal{K}$ and some $\alpha_0 > \alpha$. Then for each $D > 1$ with $D\alpha \leq \alpha_0$ the embedded system Ψ_h has a strongly attracting set $\widetilde{A}$, which is $\gamma(CD \cdot /(D-1))$-robust for $\alpha_1 = \alpha_0(D-1)/(CD)$ and satisfies $d_H(\widetilde{A}, A) \leq \gamma(D\alpha)$.

Proof: We set $\widetilde{A} = A_{D\alpha}$. The assumption on the (α, C)-embedding implies the inclusions

$$\Psi^{\alpha'}(t, x) \subseteq \Phi^{D\alpha}(t, x) \text{ for all } \alpha' \in [0, (D-1)\alpha/C]$$

and

$$\Psi^{\alpha'}(t, x) \subseteq \Phi^{CD\alpha'/(D-1)}(t, x) \text{ for all } \alpha' \geq (D-1)\alpha/C.$$

Hence setting $\widetilde{A}_{\alpha'} = A_{D\alpha}$ for $\alpha' \in [0, (D-1)\alpha/C]$ and $\widetilde{A}_{\alpha'} = A_{CD\alpha'/(D-1)}$ for $\alpha' \geq (D-1)\alpha/C$ gives attracting sets $\widetilde{A}_{\alpha'}$ for $\Psi^{\alpha'}$ satisfying

$$d_H(\widetilde{A}_{\alpha'}, \widetilde{A}) \leq d_H(\widetilde{A}_{\alpha'}, A) \leq CD\alpha'/(D-1) \text{ for all } \alpha' \geq 0.$$

This shows the claim. □

4 Robustness and Attraction Rates

In this section we investigate the relation between the robustness gain γ and the attraction rate β. We start by showing that we can find an upper bound for the robustness gain of an attracting set A which is essentially determined by its rate of attraction.

Theorem 4.1 There exist maps

$$\mu : \mathcal{KL} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathcal{K} \quad \text{and} \quad \sigma : \mathcal{KL} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$$

such that each compact attracting set $A \subset \mathbb{R}^d$ with attraction rate $\beta \in \mathcal{KL}$ and attracted neighborhood B for a system of type (2.1) or (2.2), is γ–robust for $\gamma = \mu(\beta, d_H(B, A), L)$ and all $\alpha_0 > 0$ satisfying $\alpha_0 \leq \sigma(\beta, d_H(B, A), L)$ and $\mathcal{B}(\gamma(\alpha_0), A) \subset B$, where L is the Lipschitz constant from (2.3) or (2.4), respectively.

Proof: Set $r_0 = d_H(A, B)$. For all $r \in (0, r_0]$ we can define

$$T_\beta(r) = \min\left\{ t \geq 0 \,\middle|\, \beta(s,t) \leq \frac{s}{4} \text{ for all } s \in [r, r_0] \right\}.$$

Note that T_β is finite for all $r > 0$ (because $\beta(s,t) \leq \beta(r_0, t) \to 0$ as $t \to \infty$), monotone decreasing and continuous from above, i.e., for $r_n \searrow r$ it follows $T_\beta(r_n) \to T_\beta(r)$, as a consequence of the continuity of β. This definition implies $\beta(s, T_\beta(r) + t) \leq r/4$ for all $t \geq 0$ and all $s \in [0, r]$. We set

$$\alpha_0 = \sigma(\beta, r_0, L) := e^{-LT_\beta(r_0)} \min\{r_0, \beta(r_0, 0)\}/4.$$

Now for all $\alpha \in (0, \alpha_0]$ consider the sets

$$D_\alpha := \mathrm{cl}\, \mathcal{B}(r(\alpha), A),$$

where $r(\alpha)$ is chosen minimal such that $e^{LT_\beta(r(\alpha))}\alpha \leq r(\alpha)/4$. The function $r(alpha)$ is well defined because of the continuity from above of T_β. Observe that r only depends on β, r_0 and L, and that it is is monotone increasing with $r(\alpha) \to 0$ as $\alpha \to 0$. By Gronwall's Lemma for continuous time systems or by induction for discrete time systems we obtain for $t \leq T_\beta(\|x\|_A)$

$$\|\Phi(t, x, w)\|_A \leq \beta(\|x\|_A, t) + e^{Lt}\alpha \tag{4.1}$$

for all $w \in \mathcal{W}_\alpha$, which implies that for each point $x \in D_\alpha$ we obtain

$$\Phi(T_\beta(r(\alpha)), x, w) \in D_\alpha \tag{4.2}$$

and

$$\|\Phi(t, x, w)\|_A \leq \beta(r(\alpha), 0) + r(\alpha)/4 \text{ for all } t \in [0, T_\beta(r(\alpha))]. \tag{4.3}$$

Furthermore, for any $w \in \mathcal{W}_\alpha$ and any $x \in B$ inequality (4.1) implies that the trajectory satisfies

$$\|\Phi(i\, T_\beta(r(\alpha)), x, w)\|_A \leq \max\{r_0/2^i, r(\alpha)\} \text{ for all } i \in \mathbb{N}, \tag{4.4}$$

and hence hits D_α in some uniformly bounded finite time. Now we set

$$A_\alpha := \bigcup_{t \in [0, T_\beta(r(\alpha))]} \mathrm{cl}\, \Phi^\alpha(t, D_\alpha).$$

These sets are α-forward invariant by construction and by (4.2) and α-attracting by (4.4). Furthermore they satisfy $A \subseteq A_\alpha$ for $\alpha \in (0, \alpha_0]$, $\mathcal{B}(r(\alpha), A) \subseteq A_\alpha$ and because of (4.3) one obtains $d_H(A_\alpha, A) \leq \gamma(\alpha)$ with

$$\gamma(\alpha) = \mu(\beta, r_0, L)(\alpha) := \beta(r(\alpha), 0) + r(\alpha)/4.$$

This shows the desired robustness property. □

In general, this construction of γ might not yield the best possible robustness gain for a given system and attracting set. However, the importance of this theorem is the uniformity that can be deduced from it: knowing the attraction rate, the distance between B and A and the Lipschitz constant of the system allows us to give an upper bound for the robustness gain, no matter how the geometric structure of A or the behavior of Φ around or on A look like. In particular, when uniform attraction holds for a family of systems with uniform Lipschitz properties and uniform distance between the attracting sets and their attracted neighborhoods, then also uniform robustness can be deduced.

Let us illustrate one special case, in which the proof of Theorem 4.1 yields an explicit expression for μ.

Example 4.2 Assume that A attracts exponentially, i.e., there exist constants $\rho > 0$ and $\lambda > 0$ such that $\beta(r,t) = \rho e^{-\lambda t} r$. In this case we obtain $T_\beta(r) = \ln(4\rho)/\lambda$, thus $r(\alpha) = c_1\alpha$ for $c_1 = 4(4\rho)^{L/\lambda}$, and consequently $\mu(\beta, d_H(B,A), L)(\alpha) = c_2\alpha$ for $c_2 = c_1(\rho + 1/4)$. Hence exponential attraction yields γ–robustness with linear robustness gain. □

Another interesting consequence of Theorem 4.1 is the following corollary.

Corollary 4.3 Consider an attracting set A for a system of type (2.1) or (2.2). Then there exists $\alpha_0 > 0$ and $\gamma \in \mathcal{K}$ such that A is γ–robust for γ and α_0.

Proof: By Lemma 2.7 there exists an attraction rate $\beta \in \mathcal{KL}$ for A. Hence Theorem 4.1 immediately gives the assertion. □

Knowing that any attracting set admits a robustness gain, we can easily find an upper bound for a robustness gain for nested attracting sets.

Lemma 4.4 Let A be an attracting set with attracted neighborhood B for a system of type (2.1) or (2.2) and let $\widehat{A} \supset A$ be an attracting set which is contained in B and is $\hat\gamma$–robust for the α_0–inflated system for some $\hat\gamma \in \mathcal{K}$ and some $\alpha_0 > 0$. Let $\rho := \mathrm{dist}(\widehat{A}, A)$. Then A is γ–robust for this α_0 and some $\gamma \in \mathcal{K}$ satisfying $\gamma(r) \le \hat\gamma(r) + \rho$.

Proof: By Corollary 4.3 there exists $\tilde\alpha_0 > 0$ and $\tilde\gamma \in \mathcal{K}$ such that A is $\tilde\gamma$–robust for the $\tilde\alpha_0$–inflated system. Without loss of generality we may assume $\tilde\alpha_0 \le \alpha_0$ and $\tilde\gamma(\tilde\alpha_0) \ge \hat\gamma(\tilde\alpha_0) + \rho$. Now for each $\alpha \in (0, \alpha_0]$ there exists an α–attracting set $\widehat{A}_\alpha \supseteq \widehat{A}$ with $\mathrm{dist}(\widehat{A}_\alpha, \widehat{A}) \le \hat\gamma(\alpha)$. Since this implies

$$\mathrm{dist}(\widehat{A}_\alpha, A) \le \mathrm{dist}(\widehat{A}_\alpha, \widehat{A}) + \mathrm{dist}(\widehat{A}, A) \le \hat\gamma(\alpha) + \rho$$

we can conclude that A is γ–robust with γ defined by

$$\gamma(\alpha) := \begin{cases} \min\{\tilde\gamma(\alpha), \hat\gamma(\alpha) + \rho\}, & \alpha \in [0, \tilde\alpha_0] \\ \hat\gamma(\alpha) + \rho, & \alpha \in [\tilde\alpha_0, \alpha_0] \end{cases}$$

This γ is easily verified to be of class $\mathcal{K}$, thus the assertion follows. □

We end this section by proving a "uniform attraction" property of the α–attracting sets appearing in the definition of the γ–robustness property.

Proposition 4.5 Consider an attracting set A with attracted neighborhood B for a system of type (2.1) or (2.2), which is γ–robust for some $\gamma \in \mathcal{K}$ and some $\alpha_0 > 0$. Then for each $\varepsilon > 0$ there exists a function $\beta \in \mathcal{KL}$ such that the inequality

$$\|\Phi(t,x,w)\|_A \leq \beta(\|x\|_A, t) + (1+\varepsilon)\gamma(\alpha) \tag{4.5}$$

holds for all $t \geq 0$, all $x \in B$, all $\alpha \in [0, \alpha_0]$, all $w \in \mathcal{W}_\alpha$ and the trajectories of the corresponding inflated system.

Proof: It is easily seen that there exists a monotone decreasing sequence $\alpha_n \to 0$ such that α_0 is the inflation parameter from the assumption and $\gamma(\alpha) < (1+\varepsilon)\gamma(\alpha_{n+1})$ for all $\alpha \in [\alpha_{n+1}, \alpha_n]$. We set $d_n = (1+\varepsilon)\gamma(\alpha_{n+1})$ and $r_0 = d_H(B, A)$. Now for each $r \in (0, r_0]$ we define the functions

$$\sigma_n(r,t) := \sup_{\tau \geq t} \mathrm{dist}(\Phi^{\alpha_n}(\tau, \mathcal{B}(r, A) \cap B), A)$$

and

$$\mu_n(r,t) := \max\{\sigma_n(r,t) - d_n, 0\}.$$

It is immediate that for all $r, t > 0$ the sequences $\sigma_n(r,t)$ and $\mu_n(r,t) + d_n$ are monotone decreasing in n and monotone increasing in r. From Remark 3.3 we obtain the existence of functions $\beta_n \in \mathcal{KL}$ such that

$$\mathrm{dist}(\Phi^{\alpha_n}(t,x), A_{\alpha_n}) \leq \beta_n(\|x\|_{A_{\alpha_n}}, t). \tag{4.6}$$

This implies

$$\limsup_{t\to\infty} \mathrm{dist}(\Phi^{\alpha_n}(\tau, \mathcal{B}(r_0, A) \cap B), A) \leq \gamma(\alpha_n) < d_n$$

and thus for each $n \in \mathbb{N}$ there exists $T > 0$ such that

$$\mu_k(r,t) = 0 \text{ for all } k = 1, \ldots, n \text{ all } r \in (0, r_0] \text{ and all } t \geq T. \tag{4.7}$$

Furthermore, since $A \subseteq A_\alpha$ for all $\alpha \in (0, \alpha_0]$, from (4.6) for each $n \in \mathbb{N}$ and all $r > 0$ sufficiently small (depending on n) we obtain

$$\sigma_n(r, 0) \leq \beta_n(r, 0) + \gamma(\alpha_n) \leq d_n.$$

Hence for each $n \in \mathbb{N}$ there exists $R > 0$ such that

$$\mu_k(r,t) = 0 \text{ for all } k = 1, \ldots, n, \text{ all } r \in [0, R] \text{ and all } t \geq 0. \tag{4.8}$$

Now consider the function $a(r,t) := \sup_{n \in \mathbb{N}_0} \mu_n(r,t)$. From the definition of the μ_n we obtain

$$\|\Phi(t,x,w)\|_A \leq \mu_n(\|x\|_A, t) + d_n \leq a(\|x\|_A, t) + d_n \leq a(\|x\|_A, t) + (1+\varepsilon)\gamma(\alpha)$$

for all $t \geq 0$, all $\alpha \in [\alpha_{n+1}, \alpha_n]$ and all $w \in \mathcal{W}_\alpha$. Furthermore, $a(r,t)$ is monotone increasing in r and monotone decreasing in t. If we fix $n \in \mathbb{N}$ and choose $T > 0$ such that (4.7) holds, then (4.7) and the monotonicity of $\mu_n(r,t) + d_n$ in n imply

$$a(r,t) \leq \sup_{k \in \mathbb{N}_0} \mu_k(r,t) \leq \sup_{k \geq n} \mu_k(r,t) \leq \sup_{k \geq n} \mu_k(r,t) + d_k \leq \mu_n(r,t) + d_n \leq d_n$$

for all $t \geq T$. Similarly, from (4.8) one sees that for each $n \in \mathbb{N}$ and each $t \geq 0$ there exists $R \geq 0$ such that $a(r,t) \leq d_n$ for all $r \leq R$. Thus, since $d_n \to 0$ as $n \to \infty$ we obtain $a(r,t) \to 0$ if either $r \to 0$ or $t \to \infty$. Hence by Remark 2.5 a can be bounded from above by some function $\beta \in \mathcal{KL}$ which shows the claim. □

Remark 4.6 Inequality (4.5) describes a property which in nonlinear control theory is known as *input–to–state stability (ISS)*, see, e.g., the survey [19]. For a detailed comparative study of various robustness concepts for attracting sets including their characterization via Lyapunov functions we refer to [8]. □

5 Discrete and Continuous Time Systems

By its very nature, a numerical one–step approximation with time–step $h > 0$ only gives an approximation to the time–h map φ_h (2.5) of the continuous time system φ (2.1). It is therefore necessary to obtain information on the dynamical behavior of φ from its time–h map φ_h. In this section we give two results for this purpose.

Proposition 5.1 Consider a sequence of time–steps $h_n \to 0$, and a sequence of γ_n–robust attracting sets A_n for $\gamma_n \in \mathcal{K}$ and $\alpha_n > 0$, each with the same attracted neighborhood B for the inflated time–h_n maps (3.3), where $\alpha_n \to \alpha_0 > 0$ as $n \to \infty$. Assume there exists $\gamma \in \mathcal{K}$ such that $\limsup_{n\to\infty} \gamma_n(\alpha) \leq \gamma(\alpha)$ and a compact set $A \subset B$ such that $\lim_{n\to\infty} d_H(A_n, A) = 0$.

Then A is a γ–robust attracting set for the continuous time system (2.1) for γ and each $\tilde{\alpha}_0 \in (0, \alpha_0)$.

Proof: We first show that A is an attracting set for φ. For this, fix $\varepsilon > 0$ and consider $n \in \mathbb{N}$ such that $d_H(A_n, A) < \varepsilon/3$ and $h_n M < \varepsilon/3$, where M is a bound on $\|f(x)\|$ for x in a sufficiently large neighborhood of B. Then it is easily seen that there exists $T > 0$ such that $\varphi(ih_n, B) \subset \mathcal{B}(\varepsilon/3, A_n)$ for all $i \in \mathbb{N}$ with $ih_n \geq T$. Consequently, we obtain $\varphi(t, B) \subset \mathcal{B}(\varepsilon, A)$ for all $t \geq T$, and since $\varepsilon > 0$ was arbitrary, this shows the desired convergence $\mathrm{dist}(\varphi(t,B), A) \to 0$ as $t \to \infty$.

It remains to show the γ–robustness. To this end, fix some $\alpha \in (0, \alpha_0)$ and consider the set

$$A^\alpha = \bigcap_{n\geq 0} \mathrm{cl} \bigcup_{k\geq n} A_n^\alpha$$

where the A_n^α denote the α–attracting sets for the inflated time–h_n maps (3.3). Using the fact that

$$d_H\left(\mathrm{cl} \bigcup_{k\geq n} A_n^\alpha,\, A^\alpha\right) \to 0 \text{ as } k \to \infty$$

(cf. [2, Proposition 1.1.5]), with the same argument as above one sees that this set is α–attracting for the inflated system. Since for each $\varepsilon > 0$ we find $N \in \mathbb{N}$ such that for all $n \geq N$ the inequalities

$$d_H(A_n, A) < \varepsilon/2 \quad \text{and} \quad d_H(A_n^\alpha, A_n) \leq \gamma_n(\alpha) \leq \gamma(\alpha) + \varepsilon/2$$

hold we can conclude $d_H(A_n^\alpha, A) \le \gamma(\alpha) + \varepsilon$ for all $n \ge N$ and thus

$$d_H(A_n^\alpha, A) \le \gamma(\alpha) + \varepsilon$$

which shows the desired distance since $\varepsilon > 0$ was arbitrary. $\square$

While in general an attracting set for the time–h map is not an attracting set for the continuous time system, this property is always true for attractors, as the following Lemma shows.

Lemma 5.2 Let $h > 0$ and A_h be an attractor with attracted neighborhood B for the time–h map φ_h (2.5) of the continuous time system (2.1). Then A_h is also an attractor with attracted neighborhood B for the continuous time system (2.1).

Proof: We first show forward invariance of A_h for φ, i.e., $\varphi(t, A_h) \subseteq A_h$ for each $t \ge 0$. By invariance of A_h for φ_h for each $t \ge 0$ we know $\varphi_h(h, \varphi(t, A_h)) = \varphi(t, \varphi_h(h, A_h)) = \varphi(t, A_h)$, hence $\varphi(t, A_h)$ is a compact invariant set for φ_h, and by Lemma 2.10 it is contained in A_h.

Now we show that A_h is an attracting set for the continuous time system φ. Forward invariance of A_h and continuous dependence on the initial value imply that for each $\delta > 0$ there exists $\varepsilon > 0$ such that

$$d_H(D, A_h) < \varepsilon \quad \Rightarrow \quad d_H(\varphi(t, D), A_h) < \delta$$

for all $t \in [0, h]$ and arbitrary bounded sets $D \subset \mathbb{R}^d$. Since attractivity of A_h for φ_h implies $\lim_{i\to\infty,\, i\in\mathbb{N}} \operatorname{dist}(\varphi(ih, B), A) = 0$ we can conclude $\lim_{t\to\infty} \operatorname{dist}(\varphi(t, B), A_h) = 0$, i.e. A_h is also an attracting set for φ with attracted neighbourhood B.

It remains to show that A_h is an attractor for φ. By Lemma 2.8 there exists an attractor $A \subseteq A_h$ for φ. This, in turn, is also an attractor set for φ_h, hence by Lemma 2.9 it must coincide with A_h. Thus A_h is an attractor for φ. $\square$

6 Numerical Discretization

In this section we combine the results from the previous sections in order to derive criteria under which one can conclude the existence of attracting sets and attractors from numerical approximations. We start with sufficient conditions for the existence of attracting sets.

Proposition 6.1 Consider the continuous time system (2.1) and a numerical one–step approximation $\widetilde{\Phi}_h$ for $h > 0$ satisfying (2.6) for $c\, q > 0$. Let L denote a Lipschitz constant for both systems from (2.3) and (2.4), respectively.

(a) Let A be a γ–robust attracting set for (2.1) for $\gamma \in \mathcal{K}$ and $\alpha_0 \ge e^{hL} c h^q$. Then there exists an attracting set A_h for the discrete time system induced by the numerical approximation $\widetilde{\Phi}_h$ satisfying

$$d_H(A_h, A) \le \gamma(e^{hL} c h^q).$$

(b) Let A_h be a γ–robust attracting set for $\widetilde{\Phi}_h$ for $\gamma \in \mathcal{K}$ and $\alpha_0 \ge c h^q$. Then there exists an attracting set $\widetilde{A}_h$ for the time–h map (2.5) of the continuous time system satisfying

$$d_H(\widetilde{A}_h, A_h) \le \gamma(c h^q).$$

Proof: This follows directly from Proposition 3.6 and Proposition 3.7. □

Theorem 6.2 Consider the continuous time system (2.1) and a family of numerical one–step approximation $\widetilde{\Phi}_{h_n}$ satisfying (2.6) for a sequence of time–steps $h_n \to 0$ as $n \to \infty$. Let A_n be attractors for the discrete time systems induced by these numerical approximations, each with the same attracted neighborhood B, and assume that there exists a compact set $A \subset B$ with $d_H(A_n, A) \to$ as $n \to \infty$.

Then the following properties are equivalent

(i) A is an attractor for (2.1) with attracted neighborhood B

(ii) there exist $N \in \mathbb{N}$, $\gamma \in \mathcal{K}$, $\alpha_0 > 0$ and sequences $C_n \to 1$ and $\rho_n \to 0$ such that for each $n \geq N$ the attractor A_n is γ_n–robust for the numerical system $\widetilde{\Phi}_{h_n}$ for α_0 and $\gamma_n(r) \leq \gamma(C_n r) + \rho_n$

(iii) there exist $N \in \mathbb{N}$, $\beta \in \mathcal{KL}$ and a sequence $\varepsilon_n \to 0$ such that for each $n \geq N$ the attractor A_n for the numerical system $\widetilde{\Phi}_{h_n}$ has attraction rate $\beta_n \in \mathcal{KL}$ satisfying $\beta_n(r,t) \leq \beta(r + \varepsilon_n, t) + \varepsilon_n$.

In addition, if (ii) holds for $\gamma \in \mathcal{K}$ and $\alpha_0 > 0$ then A is γ–robust for this γ and each $\tilde{\alpha}_0 \in (0, \alpha_0)$ and if (iii) holds for $\beta \in \mathcal{KL}$ then A is attracting with this rate β for this continuous time system (2.1).

Proof: (i)⇒(ii): Since A is an attracting set by Corollary 4.3 it is also γ–robust for some suitable $\gamma \in \mathcal{K}$ and $\tilde{\alpha}_0 > 0$. Since by Proposition 3.6 the α_0–inflated numerical system (3.4) for $\alpha_0 = \tilde{\alpha}_0/2$ and n sufficiently large is embedded in the $\tilde{\alpha}_0$–inflated time–h_n map (3.3), Proposition 3.8 applied with $\alpha = ch_n^q$, $C = 1 + h_n L$ and $D = 1/\sqrt{ch_n^q}$ implies the existence of $\tilde{\gamma}_n$–robust attracting sets $\widetilde{A}_n$ with $d_H(A, \widetilde{A}_n) \leq \gamma(\sqrt{ch_n^q})$ and

$$\tilde{\gamma}_n(r) \leq \gamma\left(\frac{1 + h_n L}{1 - \sqrt{ch_n^q}} r\right).$$

Since the attractors A_n converge to A and—by minimality—are contained in the attracting sets $\widetilde{A}_n$ we can conclude that $\rho_n = \mathrm{dist}(A_n, \widetilde{A}_n) \to 0$ as $n \to \infty$, hence by Lemma 4.4 the A_n are γ_n–robust with $\gamma_n(r) \leq \tilde{\gamma}_n(r) + \rho_n$ and α_0 which shows the claim.

(i) ⇒ (iii): Since A is an attracting set by Corollary 4.3 and Proposition 4.5 it satisfies inequality (4.5) for suitable β, γ and ε. Thus by Proposition 3.6 for all sufficiently large $n \in \mathbb{N}$, all $x \in B$ and all $i \in \mathbb{N}$ we find some $w \in \mathcal{W}_{e^{h_n L} ch_n^q}$ such that

$$\varphi(ih_n, x, w) = \widetilde{\Phi}_{h_n}(ih_n, x).$$

This yields

$$\begin{aligned}
\|\widetilde{\Phi}_{h_n}(ih_n, x)\|_{A_n} &\leq \|\widetilde{\Phi}_{h_n}(ih_n, x)\|_A + d_H(A_n, A) \\
&= \|\varphi(ih_n, x, w)\|_A + d_H(A_n, A) \\
&\leq \beta(\|x\|_A, ih_n) + d_H(A_n, A) + (1+\varepsilon)\gamma(e^{h_n L} ch_n^q) \\
&\leq \beta(\|x\|_{A_n} + d_H(A_n, A), ih_n) + d_H(A_n, A) + (1+\varepsilon)\gamma(e^{h_n L} ch_n^q),
\end{aligned}$$

which shows the assertion for $\varepsilon_n = d_H(A_n, A) + (1+\varepsilon)\gamma(e^{h_n L} c h_n^q)$.

(ii) $\Rightarrow$ (i): Similar to the arguments in case "(i) $\Rightarrow$ (ii)" we obtain that for sufficiently large $n \in \mathbb{N}$ there exist $\gamma_n(D_n \cdot)$–robust attracting sets $\widetilde{A}_n$ for the inflated time–h_n map (3.3) of the α_n–inflated system for suitable constants $C_n \to 1$ and $\alpha_n \to \alpha_0$, such that $d_H(\widetilde{A}_n, A) \to 0$. Hence by Proposition 5.1 we obtain that A is a γ–robust attracting set for each $\tilde{\alpha}_0 \in (0, \alpha_0)$ for (2.1). It remains to show that A is an attractor. If this was not the case, then by Lemma 2.8 and Lemma 2.9 there exists an attractor $\widetilde{A}$ for φ with $\widetilde{A} \subset A$, $\widetilde{A} \neq A$ and attracted neighborhood B. Denote $\eta := d_H(\widetilde{A}, A) > 0$. Again following the arguments from the case "(i) $\Rightarrow$ (ii)" this implies that for all sufficiently large $n \in \mathbb{N}$ the attractors A_n for the numerical systems $\widetilde{\Phi}_{h_n}$ must satisfy $\mathrm{dist}(A_n, \widetilde{A}) < \eta/2$. This implies $d_H(A_n, A) \geq \eta/2$ and hence contradicts the convergence $d_H(A_n, A) \to 0$ for $n \to \infty$.

(iii) $\Rightarrow$ (i): Fixing some $T > 0$ and some $\varepsilon > 0$, by Gronwall's Lemma for all $n > 0$ sufficiently large and all $x \in B$ we obtain the inequality

$$\|\widetilde{\Phi}_h(i_n h_n, x) - \varphi(T, x)\| \leq \varepsilon$$

where $i_n \in \mathbb{N}$ can be chosen such that $|T - i_n h_n| < \varepsilon$. Hence from the convergence of A_n to A and from the properties of β_n and β we obtain

$$\|\varphi(T, x)\|_A \leq \beta(\|x\|_A + \varepsilon, T + \varepsilon) + \varepsilon$$

and since $\varepsilon > 0$ was arbitrary by continuity of β this implies

$$\|\varphi(T, x)\|_A \leq \beta(\|x\|_A, T),$$

which implies that A is an attracting set since $T > 0$ was arbitrary. The fact that A is an attractor follows similar to the case "(ii) $\Rightarrow$ (i)", above. □

In other words, Theorem 6.2 states that a sequence of "numerical" attractors converges to a "real" attractor if and only if the elements of this sequence are either uniform robust or attracting with a uniform rate.

Remark 6.3 Note that we have used the minimality of attractors only in the proof of the implication "(i) $\Rightarrow$ (ii)". Hence the equivalence "(i) $\Leftrightarrow$ (iii)" and the implication "(ii) $\Rightarrow$ (i)" remain true for general attracting sets. □

In the next theorem we shift our attention to a sequence of uniformly robust numerical attractors (in the sense of Theorem 6.2 (ii)) without the a priori assumption about convergence of these sets. It turns out that this sequence of numerical attractors converges to a set if and only if this set is an attractor.

Theorem 6.4 Consider the continuous time system (2.1) and a family of numerical one–step approximation $\widetilde{\Phi}_{h_n}$ satisfying (2.6) for a sequence of time–steps $h_n \to 0$ as $n \to \infty$. Let A_n be attractors for the discrete time systems induced by these numerical approximations, each with the same attracted neighborhood B, assume that they are γ_n–robust for the numerical system $\widetilde{\Phi}_{h_n}$ for some $\alpha_0 > 0$ and $\gamma_n(r) \leq \gamma(C_n r) + \rho_n$ for some suitable $\gamma \in \mathcal{K}$ and sequences $C_n \to 1$ and $\rho_n \to 0$, and let $A \subset B$ be a compact set.

Then the following statements are equivalent.

(i) A is an attractor for (2.1) with attracted neighborhood B

(ii) $d_H(A_n, A) \to 0$ as $n \to \infty$

In this case, A is γ–robust for (2.1) for γ and each $\tilde{\alpha}_0 \in (0, \alpha_0)$.

Proof: (i) $\Rightarrow$ (ii): Since by Lemma 2.7 and Theorem 4.1 the attractor A is γ–robust for some suitable $\gamma \in \mathcal{K}$, by Proposition 6.1(a) and Lemma 2.8 we can conclude $\mathrm{dist}(A_n, A) \to 0$ as $n \to \infty$. For the converse "dist" estimate, by the assumption on the γ_n–robustness of the A_n, for each $\varepsilon > 0$ and all $n \in \mathbb{N}$ sufficiently large there exist attracting sets $\widetilde{A}_n$ for the time–h_n map of the continuous time system with $\mathrm{dist}(\widetilde{A}_n, A_n) \leq \varepsilon$. By Lemma 2.8 each of these sets contains an attractor for the time–h_n map (2.5) and attracted neighborhood B, which by Lemma 5.2 coincides with A. This implies $\mathrm{dist}(A, A_n) \leq \varepsilon$ and since $\varepsilon > 0$ was arbitrary we obtain $\mathrm{dist}(A, A_n) \to 0$ as $n \to \infty$. This shows the desired convergence.

(ii) $\Rightarrow$ (i): This follows from the implication "(ii) $\Rightarrow$ (i)" in Theorem 6.2. ∎

In other words, Theorem 6.4 states that a sequence of uniformly robust "numerical" attractors converges to some set A if and only if it is a "real" attractor.

Finally, we are going to investigate the rates of convergence of A_n to A under the assumptions of Theorem 6.4.

Theorem 6.5 Consider the continuous time system (2.1) and a family of numerical one–step approximation $\widetilde{\Phi}_{h_n}$ satisfying (2.6) for a sequence of time–steps $h_n \to 0$ as $n \to \infty$ and constants c, $q > 0$. Let A_n be attractors for the discrete time systems induced by these numerical approximations, each with the same attracted neighborhood B, and assume that they are γ_n–robust for the numerical system $\widetilde{\Phi}_{h_n}$ for some $\alpha_0 > 0$ and $\gamma_n(r) \leq \gamma(C_n r) + \rho_n$ for some suitable $\gamma \in \mathcal{K}$ and sequences $C_n \to 1$ and $\rho_n \to 0$. Let $A \subset B$ be a compact set and assume that one of the two following conditions is satisfied

(i) A is an attractor for (2.1) with attracted neighborhood B

(ii) $d_H(A_n, A) \to 0$ as $n \to \infty$.

Then for all sufficiently large $n \in \mathbb{N}$ we obtain the estimates

$$\mathrm{dist}(A, A_n) \leq \gamma(C_n\, e^{Lh_n} c h_n^q) + \rho_n \quad \text{and} \quad \mathrm{dist}(A_n, A) \leq \gamma(c h_n^q)$$

for the rate of convergence of A_n to A.

Proof: Under the assumptions Theorem 6.4 implies that A is γ–robust for (2.1). Hence by Proposition 6.1 we obtain the existence of attracting sets for the numerical systems and the time–h_n maps, respectively, with the desired distances. By Lemma 2.8, Lemma 2.9 and Lemma 5.2 the attractors A_n and A are contained in these attracting sets, hence the "dist" estimates remain valid. ∎

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