The Metric Average of 1D Compact Sets

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Abstract. We study properties of a binary operation between two compact sets depending on a weight in $[0,1]$, termed metric average. The metric average is used in spline subdivision schemes for compact sets in $\mathbb{R}^n$, instead of the Minkowski convex combination of sets, to retain non-convexity [3]. Some properties of the metric average of sets in $\mathbb{R}$, like the cancellation property, and the linear behavior of the Lebesgue measure of the metric average with respect to the weight, are proven. We present an algorithm for computing the metric average of two compact sets in $\mathbb{R}$, which are finite unions of intervals, as well as an algorithm for reconstructing one of the metric average's operands, given the second operand, the metric average and the weight.

§1. Introduction

In this paper we study properties of a binary operation, termed metric average, between two compact sets $A, B \subset \mathbb{R}$.

For two compact sets A, B in $\mathbb{R}^n$ and a weight $t \in [0,1]$, the metric average of A and B with weight t is given by

$$A \oplus_t B = \big\{ t\{a\} + (1-t)\Pi_B(a) \ : \ a \in A \big\} \cup \big\{ t\Pi_A(b) + (1-t)\{b\} \ : \ b \in B \big\},$$

where $\Pi_A(b)$ is the set of all closest points to b from the set A, and the addition above is the Minkowski addition of sets.

The metric average is introduced in [1] for piecewise linear approximation of set-valued functions. It is used in spline subdivision schemes for compact sets, to replace the average between numbers [3]. With this binary average, the limit set-valued function of a spline subdivision scheme operating on initial data consisting of samples of a univariate Lipschitz continuous set-valued function, approximates the sampled function with error of the order of $O(h)$, for samples h distance apart [3]. Thus the limit set-valued functions of the spline subdivision schemes retain the non-convexity nature of the approximated set-valued functions, while if we use the Minkowski average instead of

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the metric average, any limit is convex, and the spline subdivision schemes fail to approximate set-valued functions with non-convex images [4].

The metric average has many important properties [1],[3]. It is a subset of the Minkowski average $tA + (1-t)B$, generally non-convex, recovering the set A for $t = 1$, and B for $t = 0$. Here we consider the metric average as an operation between compact sets in $\mathbb{R}$. In this setting the metric average has several more important properties, such as the cancellation property which guarantees that for a given weight t, if the metric average and one of its operands are known, then the second operand is determined uniquely. Such a property is valid for Minkowski sums, only for convex sets, and not valid for non-convex ones. While redundant convexifying parts may appear in the Minkowski average of non-convex sets, there are no redundancies in the metric average of compacts in $\mathbb{R}$. In this sense the metric average of sets in $\mathbb{R}$ is optimal.

We also show that the computation of the metric average is not costly and does not require the computation of distances. By presenting an algorithm for the calculation of the metric average, we prove that the number of operations required is linear in the sum of the numbers of closed intervals in the two operands, independently of the weight parameter.

The metric average for sets in $\mathbb{R}$ is important in the reconstruction of 2D sets from their cross-sections, and more generally, in approximating set-valued functions with images in $\mathbb{R}$.

Here is an outline of the paper: Definitions and notation are presented in Section 2. Properties of the metric average for 1D sets are presented in Section 3, without proofs. An algorithm for calculating the metric average is given in Section 4. The cancellation property is derived from an algorithm for the reconstruction of the set A from the sets B, C and the weight $t \in (0,1)$ when $C = A \oplus_t B$. This is done in Section 5, where a central theorem for the validity of the cancellation algorithm is stated. The main proofs are postponed to the last section.

§2. Definitions and Notation

Denote by $\mathcal{K}(\mathbb{R}^n)$ the set of all compact, nonempty subsets of $\mathbb{R}^n$, by $\mathcal{C}(\mathbb{R}^n)$ the set of all compact, convex, nonempty subsets of $\mathbb{R}^n$ and by $\mathcal{K}_{\mathcal{F}}(\mathbb{R})$ the set of all compact, nonempty subsets of $\mathbb{R}$ which are finite unions of nonempty intervals.

The Lebesgue measure of the set A is denoted by $\mu(A)$. The Hausdorff distance between the sets $A, B \in \mathcal{K}(\mathbb{R}^n)$ is $\mathrm{haus}(A,B)$. The Euclidean distance from a point a to a set $B \in \mathcal{K}(\mathbb{R}^n)$ is $\mathrm{dist}(a,B) = \inf_{b \in B} \|a-b\|_2$. The set of all projections of $a \in \mathbb{R}^n$ on the set $B \in \mathcal{K}(\mathbb{R}^n)$ is denoted as

$$\Pi_B(a) := \big\{ b \in B \,:\, \|a-b\|_2 = \mathrm{dist}(a,B) \big\}.$$

The set difference of $A, B \in \mathcal{K}(\mathbb{R}^n)$ is $A \setminus B = \{\, a \;:\; a \in A,\; a \notin B \,\}$.

A linear Minkowski combination of two sets A and B is

$$\lambda A + \nu B = \{ \lambda a + \nu b \;:\; a \in A,\; b \in B \},$$

for $A, B \in \mathcal{K}(\mathbb{R}^n)$ and $\lambda, \nu \in \mathbb{R}$.
The Minkowski sum $A+B$ corresponds to a linear Minkowski combination with $\lambda = \nu = 1$. The linear Minkowski combination with $\lambda, \nu \in [0,1]$, $\lambda + \nu = 1$, is termed Minkowski average or Minkowski convex combination.
A segment is denoted by $[c,d] = \{\lambda c + (1-\lambda)d \ : \ 0 \le \lambda \le 1\}$, for $c, d \in \mathbb{R}^n$.

Definition 1. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$ and $0 \le t \le 1$. The t-weighted metric average of A and B is*

$$A \oplus_t B = \big\{ t\{a\} + (1-t)\Pi_B(a) \ : \ a \in A \big\} \cup \big\{ t\Pi_A(b) + (1-t)\{b\} \ : \ b \in B \big\} \quad (1)$$

where the linear combinations in (1) are in the Minkowski sense.

The sets $A, B \in \mathcal{K}_{\mathcal{F}}(\mathbb{R})$ are given as

$$A = \bigcup_{i=1}^{M} [a_i^l, a_i^r] \qquad B = \bigcup_{j=1}^{N} [b_j^l, b_j^r]. \quad (2)$$

Each interval is proper, i.e. the left endpoint is not bigger than the right one, equality is possible which stands for a so-called point (or degenerate) interval. The intervals are ordered in an increasing order, i.e. $a_i^r < a_{i+1}^l$ and $b_j^r < b_{j+1}^l$ for all relevant i, j.

We extend the representation (2) to a common closed interval containing the considered sets by adding point intervals to the left and to the right of the sets:

$$A = \bigcup_{i=0}^{M+1} [a_i^l, a_i^r], \qquad B = \bigcup_{j=0}^{N+1} [b_j^l, b_j^r], \quad (3)$$

where

$$x_{\min} = a_0^l = a_0^r = b_0^l = b_0^r < \min\{a_1^l, b_1^l\} - |a_1^l - b_1^l|, \quad (4)$$

$$x_{\max} = a_{M+1}^l = a_{M+1}^r = b_{N+1}^l = b_{N+1}^r > \max\{a_M^r, b_N^r\} + |a_M^r - b_N^r|. \quad (5)$$

This choice guarantees that $C = A \oplus_t B$ is changed only by the addition of the point intervals $\{x_{\min}\}$ and $\{x_{\max}\}$ for any $t \in [0,1]$. Denote $X = [x_{\min}, x_{\max}]$.

The "holes" of each set, namely the maximal open intervals in X, which do not intersect the set, play an important role, as well as their centers. Denote the holes by

$$H_i^A := (a_i^r, a_{i+1}^l), \quad i = 0, \ldots, M, \qquad H_j^B := (b_j^r, b_{j+1}^l), \quad j = 0, \ldots, N, \quad (6)$$

and their centers by

$$a_i^* := \frac{a_i^r + a_{i+1}^l}{2} \quad i = 0, \ldots, M, \qquad b_j^* := \frac{b_j^r + b_{j+1}^l}{2}, \quad j = 0, \ldots, N. \quad (7)$$

The dual representation of A and B is:

$$A = X \setminus \bar{A}, \quad \text{where} \quad \bar{A} = \bigcup_{i=0}^{M} H_i^A, \tag{8}$$

$$B = X \setminus \bar{B}, \quad \text{where} \quad \bar{B} = \bigcup_{j=0}^{N} H_j^B. \tag{9}$$

The t-weighted metric average $A \oplus_t B$ is denoted by C,

$$C = A \oplus_t B = \bigcup_{k=0}^{L+1} [c_k^l, c_k^r], \tag{10}$$

where $c_0^l = c_0^r = x_{\min}$, $c_{L+1}^l = c_{L+1}^r = x_{\max}$, and the center of the k-th hole $H_k^C = (c_k^r, c_{k+1}^l)$ is denoted by c_k^*.

§3. Properties of the Metric Average

The following properties of the metric average are known [3]:

Let $A, B, C \in \mathcal{K}(\mathbb{R}^n)$ and $0 \le t \le 1$, $0 \le s \le 1$. Then

1. $A \oplus_0 B = B, \qquad A \oplus_1 B = A, \qquad A \oplus_t B = B \oplus_{1-t} A.$
2. $A \oplus_t A = A.$
3. $A \cap B \subseteq A \oplus_t B \subseteq tA + (1-t)B \subseteq co(A \cup B).$
4. $\text{haus}(A \oplus_t B, A \oplus_s B) = |t-s| \text{haus}(A, B).$

The following properties are valid for sets in $\mathbb{R}$.

Proposition 2. *Let $A, B \in \mathcal{K}(\mathbb{R})$, $C, D \in \mathcal{C}(\mathbb{R})$, $t \in [0,1]$. Then*
(a) $C \oplus_t D = tC + (1-t)D,$
(b) $\mu(A \oplus_t B) = t\mu(A) + (1-t)\mu(B).$
(c) $\mu(\overline{A \oplus_t B}) = t\mu(\bar{A}) + (1-t)\mu(\bar{B}).$

The proof of the first assertion follows trivially from the definition. The third assertion is proven in the last section. The second one follows directly from the third.

In the following we present two properties of the metric average which are valid for sets in $\mathcal{K}_{\mathcal{F}}(\mathbb{R})$.

Proposition 3. *Let $A, B \in \mathcal{K}_{\mathcal{F}}(\mathbb{R})$, and let $H(A)$ denote the number of holes of A. Then for every $t \in (0,1)$*
(a) $H(A \oplus_t B) \le H(A) + H(B)$
(b) *The number of operations necessary for the calculation of $A \oplus_t B$ is $\mathcal{O}(H(A) + H(B))$.*

The first assertion is proven in the last section. The second one follows from the algorithm presented in the sequel.

The cancellation property of the metric average is

Proposition 4. *Let $A', A'', B \in \mathcal{K}_{\mathcal{F}}(\mathbb{R})$. Then for any $t \in (0,1)$*

$$A' \oplus_t B = A'' \oplus_t B \Longrightarrow A' = A''. \tag{11}$$

The proof of this claim follows from the considerations in section 5. It can be extended to sets in $\mathcal{K}(\mathbb{R})$, since all relevant statements are valid for sets consisting of an infinite number of compact segments.

To understand the nature of the metric average of two sets A and B, we distinguish four types of holes in A with respect to B, and vice versa:

Definition 5. *Let H_i^A be a hole of A. According to its position with respect to B, H_i^A is called:*

1. paired with a hole of B, *if there is a hole H_j^B of B, such that $a_i^* \in H_j^B$ and $b_j^* \in H_i^A$.*
2. paired with a point in B, *if the center $a_i^* \in B$.*
3. left shadow of a hole of B, *if there is a hole H_j^B of B, such that $a_i^* \in H_j^B$, and $b_j^* \geq a_{i+1}^l$.*
4. right shadow of a hole of B, *if there is a hole H_j^B of B, such that $a_i^* \in H_j^B$, and $b_j^* \leq a_i^r$.*

Clearly, each hole of A belongs to exactly one of the above categories of holes with respect to B, and vice versa.

Note that H_0^A is paired with H_0^B by the choice of $x_{\min}$, and similarly, H_M^A is paired with H_N^B by the choice of $x_{\max}$.

When A is averaged with B and $t \in (0,1]$, each hole of A creates a "child" hole of C, which inherits the type of its parent with respect to B, as is stated below.

Proposition 6. *Let H_i^A and H_j^B be holes of A and B, respectively, $t \in [0,1]$ and $C = a \oplus_t B$.*

1. If H_i^A and H_j^B are paired, then the interval
$$H^C = tH_i^A + (1-t)H_j^B$$
is a hole of C, paired with both H_j^B and H_i^A.
2. If H_i^A is paired with a point of B, then for $t > 0$ the interval
$$H^C = tH_i^A + (1-t)\{a_i^*\}$$
is a hole of C, paired with the point $a_i^* \in B$.
3. If H_i^A is a left shadow of H_j^B, then for $t > 0$ the interval
$$H^C = tH_i^A + (1-t)\{b_j^r\}$$
is a hole of C, and a left shadow of H_j^B.
4. If H_i^A is a right shadow of H_j^B, then for $t > 0$ the interval
$$H^C = tH_i^A + (1-t)\{b_{j+1}^l\}$$

is a hole of C, and a right shadow of H_j^B.

The proof of this proposition is postponed to the last section.

Interchanging the roles of A and B and replacing t with $1-t$ in Proposition 6, we get that for $t \in (0,1)$ some holes of C are generated by holes of A, or respectively, by holes of B, by the four ways presented above. The following proposition, proved in the last section, states that every hole of C has this property.

Proposition 7. *Let $H^C = (c', c'')$ be a hole of $C = A \oplus_t B$, $t \in [0,1]$. Then H^C is obtained either from a hole of A, by one of the four ways presented in Proposition 6, or from a hole of B, in a symmetric way.*

In the next example we have plotted the one-dimensional sets A, B and the set $C_t = A \oplus_t B$ in one picture, giving B at the y-coordinate 0, A at y=1, and C_t at y= t for $t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ (see Figure 1).

The lines connecting the boundary points of A to points of B and vice versa, show which holes of A are connected with which holes or points of B, according to their type with respect to B, and similarly for the holes of B. These lines give the holes of C_t when crossed with the line $y = t$.

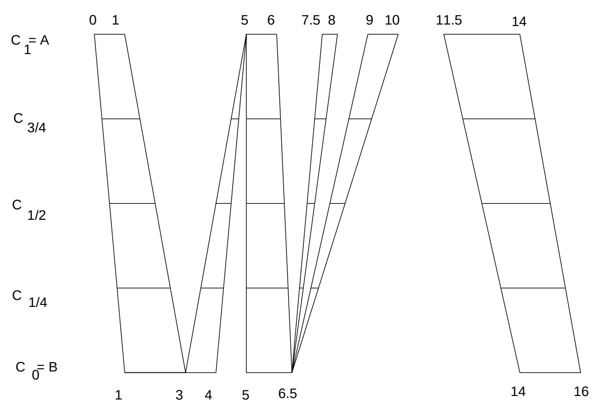

Fig. 1. The sets A, B and C_t of Example 8.

Example 8. *Consider the two sets*

$$A = [0,1] \cup [5,6] \cup [7.5,8] \cup [9,10] \cup [11.5,14],$$
$$B = [1,4] \cup [5,6.5] \cup [14,16].$$

For these two sets a possible X is $[-2,20]$. The metric average $C_t = A \oplus_t B$ is $C_t = X \setminus \bar{C}_t$, where

$$\bar{C}_t = \big(t(-2,0) + (1-t)(-2,1)\big) \cup \big(t(1,5) + (1-t)\{3\}\big) \cup \big(t\{5\} + (1-t)(4,5)\big)$$

$$\cup \big(t(6,7.5) + (1-t)\{6.5\}\big) \cup \big(t(8,9) + (1-t)\{6.5\}\big)$$
$$\cup \big(t(10,11.5) + (1-t)(6.5,14)\big) \cup \big(t(14,20) + (1-t)(16,20)\big).$$

The end points of X and of A, B, C_t, $x_{\min} = -2$, $x_{\max} = 20$, are not present in the picture.

The holes of the set A are related to the set B as follows:

The hole $(1,5)$ is paired with a point in B, each of the holes $(6,7.5)$, $(8,9)$ is a left shadow of a hole of B, the hole $(10,11.5)$ is paired with a hole of B.

The holes of the set B are related to the set A as follows:

The hole $(4,5)$ is a right shadow of a hole of A and the hole $(6.5,14)$ is paired with a hole of A.

See Figure 1 for the way these holes induce holes in $C_t = A \oplus_t B$.

§4. Algorithm for Computing the Metric Average

In this section we propose an algorithm for calculating the metric average $C = A \oplus_t B$, of two given sets $A, B \in \mathcal{K}_{\mathcal{F}}(\mathbb{R})$, and $t \in (0,1)$.

Relying on Propositions 6 and 7, we construct the holes of the set C considering the generating holes of A and B, in an order from left to right, determining the type of each hole.

Algorithm for calculating $C = A \oplus_t B$

Given are $t \in (0,1)$, $A, B \in \mathcal{K}_{\mathcal{F}}(\mathbb{R})$ of the form (3).

1. $H_0^C := tH_0^A + (1-t)H_0^B, \quad i := 1, \quad j := 1, \quad k := 1.$
2. While $i \le M$ and $j \le N$,
   (a) If H_i^A is a right shadow of H_{j-1}^B, then
   $H_k^C := tH_i^A + (1-t)\{b_j^l\}, \quad k := k+1,\ i := i+1.$
   (b) Else, if H_j^B is a right shadow of H_{i-1}^A, then
   $H_k^C := t\{a_i^l\} + (1-t)H_j^B, \quad k := k+1,\ j := j+1.$
   (c) Else, if H_i^A is a left shadow of H_j^B, then
   $H_k^C := tH_i^A + (1-t)\{b_j^r\}, \quad k := k+1,\ i := i+1.$
   (d) Else, if H_j^B is a left shadow of H_i^A, then
   $H_k^C := t\{a_i^r\} + (1-t)H_j^B, \quad k := k+1,\ j := j+1.$
   (e) Else, if $a_i^* < b_j^r$, then
   $H_k^C := tH_i^A + (1-t)\{a_i^*\}, \quad k := k+1,\ i := i+1.$
   (f) Else, if $b_j^* < a_i^r$, then
   $H_k^C := t\{b_j^*\} + (1-t)H_j^B, \quad k := k+1,\ j := j+1.$
   (g) Else (H_i^A and H_j^B are paired)
   $H_k^C := tH_i^A + (1-t)H_j^B, \quad k := k+1,\ i := i+1,\ j := j+1.$

   End of the loop.
3. $L := k-1, \qquad C = X \setminus \Big(\bigcup_{k=0}^{L} H_k^C\Big).$

Each hole of A, B belongs exactly to one of the cases described in Step 2 of the algorithm. A hole which is a shadow hole, or paired with a point of the other set, is connected to a single point of the other set to generate a hole of C (cases (a)-(f) of Step 2). Note that the condition (e) (resp. (f)) checked after the condition (a) (resp.(b)) yields that the hole H_i^A (resp. H_j^B) is paired with a point of B (A).

Note also that the order of the holes from the left to the right yields that all the right shadow holes of a given hole are considered after it. That is why, in the cases (a),(b) of Step 2 we check for right shadows of the previously considered holes H_{i-1}^A, H_{j-1}^B.

§5. Cancellation Property

To prove the cancellation property (11), we present an algorithm which computes the set A, if $t \in (0,1)$, B and $C(= A \oplus_t B)$ are given.

The following proposition is the basis for our cancellation algorithm.

Proposition 9. *Given are $t \in (0,1)$, $C = A \oplus_t B$ with holes H_k^C $(0 \le k \le L)$, and B with holes H_j^B $(0 \le j \le N)$.*

1. *Let H_k^C and H_j^B be paired and define $a' = a'_k = \frac{1}{t} c_k^r + (1 - \frac{1}{t}) b_j^r$, $a'' = a''_k = \frac{1}{t} c_{k+1}^l + (1 - \frac{1}{t}) b_{j+1}^l$. If $a' < a''$, then $(a', a'') \subset X \setminus A$.*
2. *Let H_k^C be paired with a point of B, and define $a' = a'_k = \frac{1}{t} c_k^r + (1 - \frac{1}{t}) c_k^*$, $a'' = a''_k = \frac{1}{t} c_{k+1}^l + (1 - \frac{1}{t}) c_k^*$. Then $(a', a'') \subset X \setminus A$.*
3. *Let H_k^C be a left shadow of the hole H_j^B, and define $a' = a'_k = \frac{1}{t} c_k^r + (1 - \frac{1}{t}) b_j^r$, $a'' = a''_k = \frac{1}{t} c_{k+1}^l + (1 - \frac{1}{t}) b_j^r$. Then $(a', a'') \subset X \setminus A$.*
4. *Let H_k^C be a right shadow hole of H_j^B, and define $a' = a'_k = \frac{1}{t} c_k^r + (1 - \frac{1}{t}) b_{j+1}^l$, $a'' = a''_k = \frac{1}{t} c_{k+1}^l + (1 - \frac{1}{t}) b_{j+1}^l$. Then $(a', a'') \subset X \setminus A$.*

Definition 10. *A hypothetic hole (a', a'') of A is any proper open interval (a', a'') constructed in one of the four ways described in the above proposition.*

Let $C = A \oplus_t B$, $t \in (0,1)$. By Proposition 9, every hypothetic hole of A is a subset of some (real) hole of A. Thus the set of all hypothetic holes is contained in the set of holes of A.

On the other hand, by Proposition 6, every hole of A generates a "child" hole of C of the same type with respect to B. The procedure described in Proposition 9 guarantees that every hole of A will be recovered by its "child" hole of C. Thus the set of all holes of A is contained in the set of all hypothetic holes of A constructed from the holes of C. Therefore the set of all holes of A is equal to the set of all hypothetic holes.

Theorem 11. *Let $J = \{k \ : \ 0 \le k \le L,\ a'_k < a''_k\}$, where a'_k, a''_k are defined in Proposition 9. Then $A = X \setminus \Big(\bigcup_{k \in J} (a'_k, a''_k) \Big)$.*

Note that Propositions 6, 9 and Theorem 11 remain true when B and C are infinite unions of compact segments, since their proofs do not use essentially the finite number of segments. Thus the cancellation property is true for sets in $\mathcal{K}(\mathbb{R})$.

Given two sets $B, C \in \mathcal{K}_{\mathcal{F}}(\mathbb{R})$, and a weight $t \in (0,1)$, we propose the following algorithm for reconstructing $A \in \mathcal{K}_{\mathcal{F}}(\mathbb{R})$, if $C = A \oplus_t B$.

Cancellation Algorithm

1. $J := \emptyset, \quad k := 0.$
2. While $k \leq L$,
   (a) Compute a'_k, a''_k according to Proposition 9.
   (b) If $a'_k \geq a''_k$, then $J := J \cup \{k\}$.
3. $A = [b^l_0, b^r_{N+1}] \setminus \Big(\bigcup_{\substack{k=0 \\ k \notin J}}^{L} (a'_k, a''_k) \Big).$

§6. Proofs

First we prove propositions 6,7, which are then used in the proof of Propositions 2(c) and 3(a).

Proof of Proposition 6:

1. Let H^A_i be paired with H^B_j. Denote $c' = ta^r_i + (1-t)b^r_j$, $c'' = ta^l_{i+1} + (1-t)b^l_{j+1}$. To prove that $H^C = (c', c'') = tH^A_i + (1-t)H^B_j$ is a hole of $C = A \oplus_t B$, we first prove that $H^C \cap C = \emptyset$. Suppose that there is $c \in H^C \cap C$. Then $c = ta' + (1-t)b'$, where either $a' \in A, b' \in \Pi_B(a')$, or $a' \in \Pi_A(b'), b' \in B$. Suppose that $b' \in \Pi_B(a')$, where $a' \in A$, and $a' \leq a^r_i$. Then since $a' < b^*_j$, it follows that $b' \leq b^r_j$, and $c = ta' + (1-t)b' \leq ta^r_i + (1-t)b^r_j = c'$, i.e. $c \notin H^C$, a contradiction. Similarly one gets contradictions if $a' \geq a^l_{i+1}, b' \in \Pi_B(a')$, or if $a' \in \Pi_A(b')$, where $b' \in B$ satisfies $b \leq b^r_j$ or $b \geq b^l_{j+1}$.
   Thus we have proven that $H^C \subset X \setminus C$. To verify that H^C is a hole of C, we have to prove that its end points are elements of C. This follows trivially from the definition of H^C and the fact that, for the left end points, either $b^r_j \in \Pi_B(a^r_i)$, or $a^r_i \in \Pi_A(b^r_j)$, and similarly, for the right end points, either $b^l_{j+1} \in \Pi_B(a^l_{i+1})$, or $a^l_{i+1} \in \Pi_A(b^l_{j+1})$.
   The proof that H^C is paired with A and B is trivial and follows from the relation $\frac{c'+c''}{2} = ta^*_i + (1-t)b^*_j \in H^A_i \cap H^B_j$.
2. Let H^A_i be paired with a point of B, i.e. $a^*_i \in B$. Then $a^r_i, a^l_{i+1} \in \Pi_A(a^*_i)$, hence $c' = ta^r_i + (1-t)a^*_i \in C$, $c'' = ta^l_{i+1} + (1-t)a^*_i \in C$. To prove that $H^C = (c', c'')$ is a hole of C, we have to prove that $H^C \cap C = \emptyset$. If there is $c \in H^C \cap C$, then $c = ta' + (1-t)b'$, where either $a' \in A, b' \in \Pi_B(a')$, or $a' \in \Pi_A(b'), b' \in B$. Suppose first that $a' \in A, b' \in \Pi_B(a')$ and $a' \leq a^r_i$. Then since $a^*_i \in B$, it follows that $b' \leq a^*_i$. Thus $c = ta' + (1-t)b' \leq ta^r_i + (1-t)a^*_i = c'$, i.e. $c \notin H^C$, a contradiction. Similarly one proves the other three cases.
   Thus (c', c'') is a hole of C with $\frac{c'+c''}{2} = a^*_i \in B$, hence H^C is paired with a point of B.
3. Let H^A_i be a left shadow of H^B_j. Denote $c' = ta^r_i + (1-t)b^r_j$, $c'' = ta^l_{i+1} + (1-t)b^r_j$, and $H^C = (c', c'')$. Clearly, $b^r_j \in \Pi_B(a^l_{i+1})$, and either

$b_j^r \in \Pi_B(a_i^r)$, or $a_i^r \in \Pi_A(b_j^r)$. Thus $c', c'' \in C$. If $H^C \cap C = \emptyset$, it is trivial to show that H^C is a left shadow of H_j^B.

It remains to show that $H^C \cap C = \emptyset$. Suppose that $c \in H^C \cap C$, i.e. $c = ta' + (1-t)b'$, where either $a' \in A, b' \in \Pi_B(a')$, or $a' \in \Pi_A(b'), b' \in B$. Suppose first that $a' \in A, b' \in \Pi_B(a')$ and $a' \leq a_i^r$. Then since $a_i^r < b_j^*$, it follows that $b' \leq b_j^r$. Hence $c = ta' + (1-t)b' \leq ta_i^r + (1-t)b_j^r = c'$, i.e. $c \notin H^C$, a contradiction. The other three cases are proven similarly.

4. The case that H_i^A is a right shadow of H_j^B is symmetric to the previous case and we omit the proof. $\square$

Proof of Proposition 7:

Let $c' = ta' + (1-t)b'$, where either $b' \in \Pi_B(a')$ for some $a' \in A$, or $a' \in \Pi_A(b')$ for some $b' \in B$, and let $c'' = ta'' + (1-t)b''$, where either $b'' \in \Pi_B(a'')$ for some $a'' \in A$, or $a'' \in \Pi_A(b'')$ for some $b'' \in B$.

First we prove that both inequalities $a' \leq a''$, $b' \leq b''$ hold and at least one of them is strict. Clearly, if $a' \geq a''$ and $b' \geq b''$, then $c' \geq c''$, which is impossible. Next we show that $a' > a''$, $b' < b''$ is impossible. We use the inequality

$$\max\{|b' - a'|, |b'' - a''|\} > \max\{|b' - a''|, |b'' - a'|\}, \qquad (12)$$

which is proven at the end of the present proof.

Suppose, e.g. that $|b' - a'| = \max\{|b' - a'|, |b'' - a''|\}$. It follows from (12) that $a' \notin \Pi_A(b')$ and $b' \notin \Pi_B(a')$, a contradiction. Similarly we get a contradiction if $|b' - a'| \leq |b'' - a''|$.

Thus $a' \leq a''$, $b' \leq b''$ and at least one of these inequalities is strict. To prove that $(a', a'') \subset X \setminus A$, suppose that there exists $a \in (a', a'') \cap A$. Then a belongs to one of the following ranges (some of them might be empty):

1. If $b' \leq a \leq b''$, there is $b \in \Pi_B(a) \cap [b', b'']$, hence $ta + (1-t)b \in (c', c'')$, a contradiction.
2. If $a < b'$, then there exists $a(b') \in \Pi_A(b') \cap (a', a'') \cap (a', b')$, such that $ta(b') + (1-t)b' \in (c', c'')$, a contradiction.
3. The case $b'' < a$ is symmetric to the previous one.

Thus $(a', a'') \subset X \setminus A$. Similarly one proves that $(b', b'') \subset X \setminus B$.

Next, we prove that the intervals (a', a''), (b', b'') satisfy the conditions of one of the four cases of Proposition 6.

Assume that (a', a''), (b', b'') are non-degenerate, i.e. $a' < a''$ and $b' < b''$. We will prove that they are paired. If $a^* = \frac{a'+a''}{2} \notin (b', b'')$, for instance $a^* \leq b'$, then either $a^* \leq b' \leq a''$, implying that $a'' \in \Pi_A(b')$ and $c' < ta'' + (1-t)b' < c''$, a contradiction, or $a'' < b'$, implying that $a' \notin \Pi_A(b')$, and therefore $b' \in \Pi_B(a')$, from which it is concluded that in the interval $(a' - (b' - a'), a' + (b' - a'))$ there are no points of B. Thus $b' \in \Pi_B(a'')$ and $c' < ta'' + (1-t)b' < c''$, a contradiction. The case $a^* \geq b''$ is symmetric. Similarly one proves that $b^* = \frac{b'+b''}{2} \in (a', a'')$. Therefore (a', a''), (b', b'') are paired.

Let one of (a', a''), (b', b'') be degenerate, for instance, $b' = b''$.

If $b'' < a^*$, then $a'' \notin \Pi_A(b'')$ and $b'' \in \Pi_B(a'')$. Hence in the interval $(a'' - |b'' - a''|, a'' + |b'' - a''|)$ there are no points of B. Let $b''' = \min\{b \in B,\ b \geq a''\}$, then (a', a'') is a left shadow of (b'', b''').

Similarly, if $b'' > a^*$, we get that (a', a'') is a right shadow of a hole of B.

If $b'' = a^*$, then obviously (a', a'') is paired with $a^* = b'' \in B$.

In a similar way, if $a' = a''$, then (b', b'') is a shadow of a hole in A, or paired with $b^* = a'' \in A$. □

Proof of (12):

The inequality (12) follows easily from the fact that in the trapezoid with vertices $(a'', 0)$, $(a', 0)$, $(b'', 1)$, $(b', 1)$, the large diagonal is longer than the sides, and the Pythagorean theorem. To prove the above geometric fact, it is sufficient to prove that if one of the sides BC, AD of the trapezoid $ABCD$ $(AB \| CD)$, is not less than one of the diagonals of $ABCD$, then it is less than the other diagonal. Suppose, for instance, that $BC \geq BD$. We prove that $BC < AC$. In the triangle BCD, the inequality of the sides yields inequality of the angles, $\angle BCD \leq \angle BDC$. Continuing to compare the angles, since $AB \| CD$, it follows that $\angle BAC = \angle ACD < \angle BCD$. On the other hand, $\angle BDC = \angle ABD < \angle ABC$. Thus we get $\angle BAC < \angle ABC$, hence, in the triangle ABC, $BC < AC$, which completes the geometric proof of (12). □

Proof of Proposition 2(c) and Proposition 3(a):

As was proven in Proposition 7, every hole in C is generated either by a hole of A connected to a single point of B (if the hole of A is a shadow of some hole of B or is paired with a point of B), or, symmetrically, by a hole of B connected to a single point of A, or by two paired holes of A and B. By Proposition 6, different holes of A (or of B) produce different holes in C, and the only case when two holes, one of A and one of B, produce one hole of C is the case of paired holes. This yields the claim of Proposition 3(a).

Denote by I_A (respectively I_B) the set of indices of holes in A (resp. B) which are connected to a single point in B (resp. A). Since for every $i \notin I_A$ there exists a unique $j(i) \notin I_B$ such that H_i^A is paired with $H_{j(i)}^B$, then Proposition 6 implies

$$\begin{aligned}\mu(\bar{C}) &= \sum_{i \in I_A} t\mu(H_i^A) + \sum_{j \in I_B} (1-t)\mu(H_j^B) + \sum_{i \notin I_A} \Big(t\mu(H_i^A) + (1-t)\mu(H_{j(i)}^B) \Big) \\ &= t\mu(\bar{A}) + (1-t)\mu(\bar{B}) \ . \quad \square\end{aligned}$$

Proof of Proposition 9:

1. Let H_k^C be paired with H_j^B. Since $C = A \oplus_t B$, by Propositions 6, 7, the only possibility for H_k^C is that $H_k^C = tH_i^A + (1-t)H_j^B$, where H_i^A is a hole of A, paired with H_j^B. Then clearly $H_i^A = (a', a'')$.
2. Let H_k^C be paired with a point of B, i.e. $c_k^* \in B$. Then $a' = \frac{1}{t}c_k^r + (1 - \frac{1}{t})c_k^*$, $a'' = \frac{1}{t}c_{k+1}^l + (1 - \frac{1}{t})c_k^*$. Suppose that $a \in A \cap (a', a'')$. Then, since a is closer to c_k^* than a' and a'', there is a point $a_0 \in (a', a'') \cap \Pi_A(c_k^*)$.

Hence $ta_0 + (1-t)c_k^* \in C \cap H_k^C$, a contradiction.

3. Let H_k^C be a left shadow of H_j^B, $a' = \frac{1}{t}c_k^r + (1-\frac{1}{t})b_j^r$, $a'' = \frac{1}{t}c_{k+1}^l + (1-\frac{1}{t})b_j^r$. Define also the point $a_s' \geq b_j^r$ such that $|a_s' - b_j^r| = |a' - b_j^r|$ (possibly $a_s' = a'$).

   We perform the proof in several steps.

Step 1 First we prove that $(a', a_s') \cap A = \emptyset$.
Assume $(a', a_s') \neq \emptyset$, i.e. $a' < b_j^r < a_s'$, and suppose that there is $a \in A \cap (a', a_s')$. Then for b_j^r there is $a(b_j^r) \in \Pi_A(b_j^r) \cap (a', a_s')$. Thus $ta(b_j^r) + (1-t)b_j^r \in (c_k^r, c_{k+1}^l) \cap C$, which is a contradiction.

Step 2 We prove now that $(a', a'') \cap [a_s', b_j^*] \cap A = \emptyset$.
Suppose that there is $a \in A \cap (a', a'') \cap [a_s', b_j^*]$. Then since $a \leq b_j^*$, it follows that $b_j^r \in \Pi_B(a)$. Hence $ta + (1-t)b_j^r < ta'' + (1-t)b_j^r = c_{k+1}^l$. On the other hand, $ta + (1-t)b_j^r > ta' + (1-t)b_j^r = c_k^r$. Thus $ta + (1-t)b_j^r \in (c_k^r, c_{k+1}^l) \cap C$, which is a contradiction.

Clearly, if $a'' \leq b_j^*$, the proof is completed. In all next steps we suppose that $b_j^* < a''$.

Note that the point $c_{k+1}^l \in C$ is obtained either by

$$c_{k+1}^l = ta(b) + (1-t)b, \qquad \text{where} \quad b \in B,\ a(b) \in \Pi_A(b), \tag{13}$$

or by

$$c_{k+1}^l = ta + (1-t)b(a), \qquad \text{where} \quad a \in A,\ b(a) \in \Pi_B(a''). \tag{14}$$

In Steps 3 and 4 we suppose that (13) holds and prove that $[b_j^*, a'') \cap A = \emptyset$, which implies $(a', a'') \cap A = \emptyset$. In Step 5 we show that (14) is impossible when $b_j^* < a''$.

Step 3 We prove that $[b_j^*, a'') \cap A = \emptyset$, in case $b \leq b_j^r$ in (13).
Indeed, since $\frac{1}{t} > 1$, then $a(b) = \frac{1}{t}c_{k+1}^l + (1-\frac{1}{t})b \geq \frac{1}{t}c_{k+1}^l + (1-\frac{1}{t})b_j^r = a''$. This yields that there are no elements of A in the interval $[b, a'') \supset [b_j^*, a'')$.

Step 4 We prove that $[b_j^*, a'') \cap A = \emptyset$, in case $b \geq b_{j+1}^l$ in (13).
Assume $b \geq b_{j+1}^l$. Define $b_j^s > c_{k+1}^l$ such that $b_j^s - c_{k+1}^l = c_{k+1}^l - b_j^r$. Such a point exists since H_k^C is a left shadow of H_j^B, i.e. $b_j^r < c_{k+1}^l < b_j^s < b_{j+1}^l$.
Denote $a_s'' = \frac{1}{t}c_{k+1}^l + (1-\frac{1}{t})b_j^s$. Then by the definition of a'' we get $a'' - c_{k+1}^l = c_{k+1}^l - a_s'' = (\frac{1}{t} - 1)(c_{k+1}^l - b_j^r)$. Since $b \geq b_{j+1}^l > b_j^s$ and $a(b) = \frac{1}{t}c_{k+1}^l + (1-\frac{1}{t})b$, it follows that $a(b) < a_s''$. Since $a(b)$ is a projection of b, it follows that there are no points of A in the interval $I = [b - (b - a_s''), b + (b - a_s'')] \supset [b_j^s - (b_j^s - a_s''), b_j^s + (b_j^s - a_s'')] = I'$. Since $a_s'' < c_{k+1}^l < b_j^*$ and $b_j^s + (b_j^s - a_s'') = 2c_{k+1}^l - b_j^r + \frac{1}{t}(c_{k+1}^l - b_j^r) = a'' + (b_j^s - b_j^r) > a''$, it is easy to see that $[b_j^*, a'') \subset I' \subset I$. Thus there are no points of A in $[b_j^*, a'')$.

Step 5 Let $c_{k+1}^l = ta + (1-t)b(a)$, where $a \in A$, $b(a) \in \Pi_B(a)$. Since $a \in A$, it follows by Steps 1,2 that $a \notin (a', b_j^*] \cap (a', a'')$, hence either $a \geq a''$, or $a > b_j^*$, or $a \leq a'$.

If $a > a''$, then since $a'' > c^l_{k+1} > b^r_j$, it follows that $b(a) \geq b^r_j$ and $ta + (1-t)b(a) > ta'' + (1-t)b^r_j = c^l_{k+1}$, a contradiction.

If $a > b^*_j$, then $b(a) \geq b^l_{j+1}$ and $ta + (1-t)b(a) > tb^*_j + (1-t)b^l_{j+1} > b^*_j$, a contradiction.

If $a \leq \min\{a', b^*_j\}$, then since $a \leq b^*_j$, $b(a) \leq b^r_j$. Hence $c^l_{k+1} \leq ta' + (1-t)b^r_j = c^r_k < c^l_{k+1}$, a contradiction.

Thus the only possibility for $a \in A$ is $a = a''$, hence $b(a) = b^r_j$. But, since $b^*_j < a''$, we obtain that $b^r_j \notin \Pi_B(a'')$, which is a contradiction.

This completes the proof of 3.

4. The case that H^C_k is a right shadow of H^B_j is symmetric to the previous case and is proven similarly. □

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