A UNIFORM EXPONENTIAL SPECTRUM FOR LINEAR FLOWS ON VECTOR BUNDLES

Lars Grüne*
Fachbereich Mathematik
Johann Wolfgang Goethe-Universität
Postfach 11 19 32
60054 Frankfurt am Main, Germany
Tel.: (+49) 69/798-23620
E-Mail: gruene@math.uni-frankfurt.de

Abstract: For linear flows on vector bundles we define a uniform exponential spectrum. For a compact invariant set for the projected flow we obtain this spectrum by taking all accumulation points for the time tending to infinity of the union over the finite time exponential growth rates for all initial values in this set. Using direct arguments we show that for a connected compact invariant set this spectrum is a closed interval whose boundary points are Lyapunov exponents. For a compact invariant set on which the flow is chain transitive we show that this spectrum coincides with the Morse spectrum. In particular this approach admits a straightforward analytic proof for the regularity and continuity properties of the Morse spectrum without using cohomology or ergodicity results.

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1 Introduction

For linear flows $\Phi : \mathbb{R} \times E \to E$ on vector bundles $\pi : E \to S$ with compact base space S several spectral concepts have been developed during the last decades. These can roughly be divided into two classes: One using exponential growth rates and the other using topological characterizations of the flow projected onto the projective bundle.

The growth rate approach forms the basis of the exponential dichotomy theory (see for instance Daleckii and Krein [8], Coppel [7], Sacker and Sell [14] and Sell [17]) and the Oseledets spectrum [13], whereas the topological approach has been used e.g. by Selgrade [16], Salamon and Zehnder [15].

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In this paper we introduce a spectral concept that lies somewhat in between these approaches. By its very definition the uniform exponential spectrum assigns a collection of exponential growth rates to any compact connected invariant set for the projected flow: We consider the set of all possible exponential growth rates in some finite time $T > 0$ with initial values in this set and define the spectrum to consist of all accumulation points as $T \to \infty$.

The motivation for this spectral concept is to describe the possible exponential behavior of long term trajectories of a flow: Whenever exponential growth rates are obtained by long term observation or (numerical) simulation of trajectories, the observed values lie close to the uniform exponential spectrum, cp. Proposition 3.2. Conversely, for any value in this spectrum and any (arbitrary large) time $t > 0$ there exists an initial value such that this exponential growth rate is attained by the corresponding trajectory at the time t, cp. Remark 3.4. Therefore the knowledge of this spectrum helps the interpretation of experimental or simulation results and the derivation of convergence results as described in [6].

One of the main results in this paper concerns the relation between the uniform exponential spectrum spectrum and the Lyapunov spectrum (see e.g. [4]). Certain extremal values in this spectrum have recently turned out to characterize null controllability and stabilizability of certain control systems (cp. e.g. [3], [4], [9] and [10]), which can be embedded into the linear flow context using the results from [2]. Although in general the Lyapunov spectrum is smaller that the corresponding uniform exponential spectrum (cp. Remark 4.5), it turns out that the boundary points of the uniform exponential spectrum are contained in the Lyapunov spectrum. Hence a strong relation between these spectral concepts can be established using the results in this paper.

As already mentioned the uniform exponential spectrum is defined for arbitrary compact invariant sets. Hence in order to obtain a meaningful spectrum for the flow on the whole state space we have to choose sets with certain topological properties. By a suitable choice we obtain equivalence to the spectral concept defined by Colonius and Kliemann in [5].

There the connected components of the chain recurrent set of the projected flow over some connected chain recurrent set in the base space are used in order to define a spectrum via the growth rates of (finite time) chains that lie in these components. Since these components correspond to a Morse decomposition — and therefore are Morse sets — this spectrum is named *Morse spectrum*. Two essential properties of the Morse spectrum are proved in [5], i.e. that its boundary points are actually Lyapunov exponents and that it consists of finitely many bounded intervals. However, the proofs given there could only be achieved by a heavy mathematical machinery, namely by the analysis of the Morse spectrum under cohomology and by results from ergodic theory. Although these techniques provide interesting results in itself (e.g. the integral expression of the growth rates and the relation to the Oseledets spectrum for which ergodic theory is of course essential) they admit only an indirect proof; a direct (or even constructive) proof of the properties of the Morse spectrum seemed to be missing up to now.

The closure of this gap gives another motivation for our analysis. For the uniform exponential spectrum the properties mentioned can be shown by direct analytic arguments. Hence this admits a new — and considerably shorter — approach for the proof of the properties of the Morse spectrum. In this context the proof rather than the final theorem can be regarded as the main contribution of the present paper.

The property of the boundary points in fact carries over to the dynamical spectrum as defined by Sacker and Sell (see e.g. [14]). Since the boundary points of the dynamical spectrum form a subset of the boundary points of the Morse spectrum (cp. Remark 4.8) the present result also gives a direct analytic proof for the fact that the boundary points of the dynamical spectrum are indeed Lyapunov exponents which has first been shown by Johnson, Palmer and Sell [11] using ergodic theory.

We will analyze the uniform exponential spectrum in three steps. We start with the definition of finite time exponential growth rates and prove some estimates along trajectories in Section 2. The main results about the uniform exponential spectrum are contained in Section 3, where we turn to the projected flow and use the projection in order to define this spectrum over connected compact invariant sets of the projected flow. Using the estimates from Section 2 we then prove the regularity properties for this spectrum and an estimate about its parameter dependence. In Section 4 we will then use these results in order to establish the relation of this spectrum to the Lyapunov and Morse spectrum.

2 Finite time exponential growth rates

We will briefly describe our setup that coincides with the one in [5].

We consider a linear flow Φ on a vector bundle $\pi : E \to S$ with base space S, which is a compact, connected metric space. Here we use the definition of (real) vector bundles from [12, Chapter I], i.e. π is a continuous surjective mapping such that the fibers $E_p := \pi^{-1}(p)$, $p \in S$ are d-dimensional real vector spaces and E is locally isomorphic to $S \times \mathbb{R}^d$. We fix a (Riemannian) metric on E and on any fiber we denote the norm by $|\cdot|$. The zero section Z in E is given by a continuous map $Z : S \to E$ defined by $Z(p) = 0 \in E_p$, i.e. $|e| = 0$ iff $e \in Z$.

A linear flow Φ on $\pi : E \to S$ is a flow on E preserving fibers such that the induced flow $\Phi(t, \cdot)_p : E_p \to E_{\pi(\Phi(t,e))}$ is linear, i.e.

$$\begin{aligned} \Phi(t, e_1 + e_2) &= \Phi(t, e_1) + \Phi(t, e_2), \quad t \in \mathbb{R}, e_1, e_2 \in E_p, \text{ and} \\ \Phi(t, \alpha e) &= \alpha \Phi(t, e), \quad t \in \mathbb{R}, \alpha \in \mathbb{R}, e \in E_p \end{aligned}$$

We will now define exponential growth rates in finite time and in the rest of this section prove some estimates for these quantities.

Definition 2.1 For any $e \in E$ and any time $t > 0$ we define the finite time exponential growth rate by

$$\lambda^t(e) := \frac{1}{t} \ln \frac{|\Phi(t, e)|}{|e|}$$

Remark 2.2 By the compactness of the base space S and the continuity and linearity of Φ there exists a constant M such that for all $e \in E \setminus Z$ the growth rates $|\lambda^t(e)|$ are bounded by M for all $t \geq 1$ and the fractions $|\ln \frac{|\Phi(t,e)|}{|e|}|$ are bounded by M for all $t < 1$.

The following lemmas show some useful properties of the finite time exponential growth rates.

Lemma 2.3 Let t_1, $t_2 > 0$ and $t := t_1 + t_2 \geq 2$. Let e_1, $e_2 \in E$ be arbitrary points. Then the following estimates hold

$$\left| \frac{1}{t} \left(\ln \frac{|\Phi(t_1, e_1)|}{|e_1|} + \ln \frac{|\Phi(t_2, e_2)|}{|e_2|} \right) - \lambda^{t_1}(e_1) \right| \leq 2M \frac{t_2}{t}$$

and

$$\left| \frac{1}{t} \left(\ln \frac{|\Phi(t_1, e_1)|}{|e_1|} + \ln \frac{|\Phi(t_2, e_2)|}{|e_2|} \right) - \lambda^{t_2}(e_2) \right| \leq 2M \frac{t_1}{t}$$

In particular for $e_1 = e$ and $e_2 = \Phi(t_1, e)$ this implies

$$|\lambda^t(e) - \lambda^{t_1}(e)| \leq 2M \frac{t_2}{t} \text{ and } |\lambda^t(e) - \lambda^{t_2}(\Phi(t_1, e))| \leq 2M \frac{t_1}{t}$$

Proof: For $t_1 \geq 1$ and $t_2 \geq 1$ the estimates follow from the equality

$$\frac{1}{t} \left(\ln \frac{|\Phi(t_1, e_1)|}{|e_1|} + \ln \frac{|\Phi(t_2, e_2)|}{|e_2|} \right) = \frac{t_1}{t} \lambda^{t_1}(e_1) + \frac{t_2}{t} \lambda^{t_2}(e_2)$$

and the boundedness of λ^{t_1} and λ^{t_2}.

For $t_1 < 1$ we obtain

$$\frac{1}{t} \left(\ln \frac{|\Phi(t_1, e_1)|}{|e_1|} + \ln \frac{|\Phi(t_2, e_2)|}{|e_2|} \right) = \frac{1}{t} \ln \frac{|\Phi(t_1, e_1)|}{|e_1|} + \frac{t_2}{t} \lambda^{t_2}(e_2)$$

and since $t_2 \geq 1$ the estimates follow from the boundedness of λ^{t_2} and $\ln \frac{|\Phi(t_1, e_1)|}{|e_1|}$.

The case $t_2 < 1$ follows analogously. □

Lemma 2.4 Let $e \in E$, $t > 2$ and

$$\sigma := \lambda^t(e)$$

Then for any $\varepsilon > 0$ there exists a time $t^* \leq \frac{(2M - \varepsilon)t}{2M}$ such that

$$\lambda^s(\Phi(t^*, e)) \leq \sigma + \varepsilon$$

for all $s \in (0, t - t^*]$. Here $t - t^* \geq \frac{\varepsilon t}{2M} \to \infty$ as $t \to \infty$.

Proof: Let

$$\beta := \sup_{s \in (0,t]} \lambda^s(e)$$

and fix $\varepsilon > 0$. If $\beta \leq \sigma + \varepsilon$ the assertion follows with $t^* = 0$.

Otherwise let

$$t^* := \sup \{ s \in (0, t] \mid \lambda^s(e) \geq \sigma + \varepsilon \}$$

By the continuity of $\lambda^s(e)$ in s the equality

$$\lambda^{t^*}(e) = \sigma + \varepsilon$$

is implied. By Lemma 2.3 it follows from $\lambda^t(e) = \sigma$ that $t - t^* = t_2 \geq \frac{\varepsilon t}{2M}$ and hence $t^* \leq \frac{(2M-\varepsilon)t}{2M}$. We claim that t^* satisfies the desired property:

From the definition of t^* and λ^s it follows that

$$\frac{1}{t^*} \ln \frac{|\Phi(t^*, e)|}{|e|} = \sigma + \varepsilon \quad \text{and} \quad \frac{1}{s} \ln \frac{|\Phi(s, e)|}{|e|} < \sigma + \varepsilon$$

for all $s \in (t^*, t]$. Hence also

$$\ln \frac{|\Phi(t^*, e)|}{|e|} - t^*(\sigma + \varepsilon) = 0 \quad \text{and} \quad \ln \frac{|\Phi(s, e)|}{|e|} - s(\sigma + \varepsilon) < 0$$

holds. Since

$$\ln \frac{|\Phi(s, e)|}{|e|} - s(\sigma + \varepsilon) = \ln \frac{|\Phi(t^*, e)|}{|e|} - t^*(\sigma + \varepsilon) + \ln \frac{|\Phi(\Phi(t^*, e), s)|}{|\Phi(t^*, e)|} - (s - t^*)(\sigma + \varepsilon)$$

the inequality

$$\ln \frac{|\Phi(\Phi(t^*, e), s)|}{|\Phi(t^*, e)|} - (s - t^*)(\sigma + \varepsilon) < 0$$

follows for all $s \in (t^*, t]$ which yields the assertion. □

3 Analysis of the spectrum

For the definition and the analysis of a spectrum based on the finite time exponential growth rates we use the following projection, cp. [15, Appendix]:

We project Φ to the projective bundle $\mathbb{P}E$. This is given by $\mathbb{P}E = (E \setminus Z)/_\sim$ where $\sim$ is the equivalence relation defined by $e \sim e'$ iff $\pi(e) = \pi(e')$ and there exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that $e = \alpha e'$. The canonical projection map will be denoted by $\mathbb{P}$ and the linearity of the flow Φ implies that its projection $\mathbb{P}\Phi$ is well defined.

The definition of λ^t implies that this growth rate is well defined for values $\mathbb{P}e \in \mathbb{P}E$. Hence for $e \in E \setminus Z$ we can write $\lambda^t(\mathbb{P}e)$ instead of $\lambda^t(e)$. As a convention subsets of $\mathbb{P}E$ will be indicated by $_{\mathbb{P}}$.

For compact invariant subsets of $\mathbb{P}E$ we can now define a spectrum via the the finite time exponential growth rates.

Definition 3.1 Let ${}_{\mathbb{P}}K \subset \mathbb{P}E$ be a compact invariant set for $\mathbb{P}\Phi$. We define the *uniform exponential spectrum over* ${}_{\mathbb{P}}K$ by

$$\Sigma_{UE}({}_{\mathbb{P}}K) := \left\{ \mu \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exist } t_k \to \infty \text{ and points } \mathbb{P}e_k \in {}_{\mathbb{P}}K \\ \text{such that } \lim_{k\to\infty} \lambda^{t_k}(e_k) = \mu \end{array} \right\}$$

If we define the limes superior of a family of sets $(B_t)_{t\in\mathbb{R}}$ in the usual way (cp. [1, p. 21]) by

$$\limsup_{t\to\infty} B_t := \bigcap_{T\geq 0} \operatorname{cl} \bigcup_{t\geq T} B_t \tag{3.1}$$

then the equality

$$\Sigma_{UE}({}_{\mathbb{P}}K) = \limsup_{t\to\infty} \lambda^t({}_{\mathbb{P}}K) \quad \text{with} \quad \lambda^t({}_{\mathbb{P}}K) := \{\lambda^t(\mathbb{P}e) \,|\, \mathbb{P}e \in {}_{\mathbb{P}}K\}$$

is obvious. Hence the uniform exponential spectrum can be interpreted as a set valued extension of the Lyapunov exponent.

The following proposition states that the finite time exponential growth rates for some fixed time T uniformly converge to $\Sigma_{UE}({}_{\mathbb{P}}K)$. This shows that this spectrum "uniformly" describes the possible behavior of long term trajectories meaning that long term evaluation or simulation of trajectories of a flow will indeed yield a value close to this spectrum independent from the initial value. In fact, this is the property which motivated the name of this spectrum.

Proposition 3.2 Let ${}_{\mathbb{P}}K \subset {}_{\mathbb{P}}E$ be a compact invariant set for the flow $\mathbb{P}\Phi$. Then for any $\varepsilon > 0$ there exists a time $T > 0$ such that

$$d(\lambda^t(\mathbb{P}e), \Sigma_{UE}({}_{\mathbb{P}}K)) < \varepsilon$$

for all $\mathbb{P}e \in {}_{\mathbb{P}}K$ and all $t \geq T$.

Proof: Fix $\varepsilon > 0$ and assume the opposite: for any $T > 0$ there exists $t > T$ and $\mathbb{P}e_t \in {}_{\mathbb{P}}K$ such that $d(\lambda^t(\mathbb{P}e_t), \Sigma_{UE}({}_{\mathbb{P}}K)) \geq \varepsilon$. Since λ^t is bounded for all $t > 1$ there exists a sequence $t_k \to \infty$ such that $\lambda^{t_k}(\mathbb{P}e_{t_k}) \to \mu \notin \Sigma_{UE}({}_{\mathbb{P}}K)$ which contradicts the definition of $\Sigma_{UE}({}_{\mathbb{P}}K)$. □

We will now turn to the analysis of $\Sigma_{UE}({}_{\mathbb{P}}K)$. On any connected compact invariant set ${}_{\mathbb{P}}K \subset {}_{\mathbb{P}}E$ we can describe the structure of $\Sigma_{UE}({}_{\mathbb{P}}K)$ by the following theorem. In contrast to the proof of these properties for the Morse spectrum in [5] here we use straightforward analytic arguments based on the estimates from the Lemmas 2.3 and 2.4.

Theorem 3.3 Let ${}_{\mathbb{P}}K \subset {}_{\mathbb{P}}E$ be a connected compact invariant set for the flow ${}_{\mathbb{P}}\Phi$. Then there exist values γ^*, $\gamma \in \mathbb{R}$ such that

$$\Sigma_{UE}({}_{\mathbb{P}}K) = [\gamma^*, \gamma]$$

Furthermore there exist points $\mathbb{P}e^*, \mathbb{P}e \in {}_{\mathbb{P}}K$ such that

$$\lambda^t(\mathbb{P}e^*) \leq \gamma^*,\ \ \lambda^t(\mathbb{P}e) \geq \gamma \ \text{ for all } \ t > 0 \ \text{ and } \ \lim_{t\to\infty} \lambda^t(\mathbb{P}e^*) = \gamma^*,\ \ \lim_{t\to\infty} \lambda^t(\mathbb{P}e) = \gamma$$

Proof: The closedness of $\Sigma_{UE}({}_{\mathbb{P}}K)$ follows from its definition via limits. Define $\gamma^* := \min \Sigma_{UE}({}_{\mathbb{P}}K)$ and $\gamma := \max \Sigma_{UE}({}_{\mathbb{P}}K)$. We first show the existence of $\mathbb{P}e^*$ and $\mathbb{P}e$. The proof is carried out for $\mathbb{P}e^*$, the existence of $\mathbb{P}e$ is proved with the same arguments.

By the definition of $\Sigma_{UE}({}_{\mathbb{P}}K)$ we find a sequence of points $\mathbb{P}e_k \in {}_{\mathbb{P}}K$ and times $t_k \to \infty$ as $k \to \infty$ such that $\lambda^{t_k}(\mathbb{P}e_k) < \gamma^*({}_{\mathbb{P}}K) + \varepsilon_k$ where $\varepsilon_k \to 0$ for $k \to \infty$. Defining $\tilde{\varepsilon}_k := \frac{1}{\sqrt{t_k}} \to 0$ for $k \to \infty$ we apply Lemma 2.4 to e_k and t_k with $\varepsilon = \tilde{\varepsilon}_k$ for each $k \in \mathbb{N}$ and obtain times t_k^* such that

$$\lambda^s(\mathbb{P}\Phi(t_k^*, e_k)) \leq \gamma^*({}_{\mathbb{P}}K) + \varepsilon_k + \tilde{\varepsilon}_k$$

for all $s \in (0, t_k - t_k^*]$ where $t_k - t_k^* \geq \frac{\sqrt{t_k}}{2M}$. Defining points $\mathbb{P}\tilde{e}_k := \mathbb{P}\Phi(t_k^*, e_k)$ and times $\tilde{t}_k := t_k - t_k^* \to \infty$ as $k \to \infty$ we obtain

$$\lambda^s(\mathbb{P}\tilde{e}_k) \leq \gamma^*({}_{\mathbb{P}}K) + \varepsilon_k + \tilde{\varepsilon}_k$$

for all $s \in (0, \tilde{t}_k]$.

Since ${}_{\mathbb{P}}K$ is compact we may assume w.l.o.g. that the points $\mathbb{P}\tilde{e}_k$ converge to some $\mathbb{P}\tilde{e} \in {}_{\mathbb{P}}K$. Now fix arbitrary $t > 0$ and $\varepsilon > 0$ and consider $\lambda^t(\mathbb{P}\tilde{e})$. Since λ^t is continuous we find $k_0 \in \mathbb{N}$ such that $|\lambda^t(\mathbb{P}\tilde{e}) - \lambda^t(\mathbb{P}\tilde{e}_k)| < \varepsilon$ for all $k \geq k_0$. Hence

$$\lambda^t(\mathbb{P}\tilde{e}) < \gamma^* + \varepsilon_k + \tilde{\varepsilon}_k + \varepsilon$$

follows for all $k \geq k_0$. Since $\varepsilon > 0$ was arbitrary and $\varepsilon_k + \tilde{\varepsilon}_k \to 0$ for $k \to \infty$ we can conclude

$$\lambda^t(\mathbb{P}\tilde{e}) \leq \gamma^*$$

which in particular implies $\limsup_{t\to\infty} \lambda^t(\mathbb{P}\tilde{e}) \leq \gamma^*$.

Now assume $\liminf_{t\to\infty} \lambda^t(\mathbb{P}\tilde{e}) < \gamma^*$. This implies the existence of a sequence t_k such that $\lim_{k\to\infty} \lambda^{t_k}(\mathbb{P}\tilde{e}) < \gamma^*$ which contradicts the definition of γ^*. Hence $\mathbb{P}e^* = \mathbb{P}\tilde{e}$ has the desired properties.

It remains to show that $\Sigma_{UE}({}_{\mathbb{P}}K)$ is an interval. For this purpose we will show that for each $\mu \in [\gamma^*, \gamma]$ and each $t > 0$ there exists $\mathbb{P}\tilde{e}_t \in {}_{\mathbb{P}}K$ such that $\lambda^t(\tilde{e}_t) = \mu$:

Fix $\mu \in [\gamma^*, \gamma]$ and $t > 0$. Then $\lambda^t(\mathbb{P}e^*) \leq \mu$ and $\lambda^t(\mathbb{P}e) \geq \mu$. Now since ${}_{\mathbb{P}}K$ is connected there exists a continuous path $\mathbb{P}\eta : [0,1] \to {}_{\mathbb{P}}K$ such that $\mathbb{P}\eta(0) = \mathbb{P}e^*$ and $\mathbb{P}\eta(1) = \mathbb{P}e$, thus $\lambda^t(\mathbb{P}\eta(0)) \leq \mu$ and $\lambda^t(\mathbb{P}\eta(1)) \geq \mu$. Since λ^t is continuous for each $t > 0$ also $\lambda^t(\mathbb{P}\eta(\cdot))$ is continuous and by the intermediate value theorem there exists an $s_t \in [0,1]$ such that $\lambda^t(\mathbb{P}\eta(s_t)) = \mu$. Hence the assertion follows for $\mathbb{P}\tilde{e}_t = \mathbb{P}\eta(s_t)$. □

Remark 3.4 Note that this proof also shows that for any $t > 0$ the inclusion

$$\Sigma_{UE}({}_{\mathbb{P}}K) \subset \{\lambda^t(\mathbb{P}e) \,|\, \mathbb{P}e \in {}_{\mathbb{P}}K\}$$

holds.

Remark 3.5 As already mentioned before for the Morse– as well as for the dynamical spectrum it is a well known fact that the boundary points are indeed Lyapunov exponents, cf. [5] and [11], respectively. These results can be reproduced using Theorem 3.3 as shown in Theorem 4.6 and Remark 4.8 in the following section. The proofs in [5] and [11], however, make use of arguments from ergodic theory which is avoided here. Furthermore, the possibility of chosing arbitrary invariant sets for the projected flow in our setup implies that the number of boundary points that can be obtained here is considerably larger than in the other spectral concepts. Therefore our result can be considered as an extension of this previously known property.

In the rest of this section we will discuss the parameter dependence of the uniform exponential spectrum. For this purpose we introduce a family of flows parameterized by some $\alpha \in A$ by a continuous mapping $\Phi : A \times \mathbb{R} \times E \to E$ such that

$$\Phi^\alpha(\cdot,\cdot) := \Phi(\alpha,\cdot,\cdot) : \mathbb{R} \times E \to E \tag{3.2}$$

is a linear flow for each $\alpha \in A$ where A is a compact metric space. Analogously we denote by $\lambda^{\alpha,t}$ and Σ_{UE}^α the corresponding exponential growth rates in finite time and the uniform exponential spectra, respectively.

Using this terminology the following theorem holds.

Theorem 3.6 Consider a family (3.2) of linear flows Φ^α. Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence in A satisfying $\alpha_k \to \alpha_0$ for $k \to \infty$ and some $\alpha_0 \in A$ and assume that there exist compact connected invariant sets ${}_{\mathbb{P}}K_k \subset \mathbb{P}E$ for the flows $\mathbb{P}\Phi_{\alpha_k}$ satisfying $\limsup_{k\to\infty} {}_{\mathbb{P}}K_k \subset {}_{\mathbb{P}}K_0$ Then

$$\limsup_{k\to\infty} \Sigma_{UE}^{\alpha_k}({}_{\mathbb{P}}K_k) \subset \Sigma_{UE}^{\alpha_0}({}_{\mathbb{P}}K_0)$$

i.e. the uniform exponential spectrum is upper semicontinuous. Here the limsup of sets is defined by (3.1).

Proof: Choose γ_k^*, γ_k, γ_0^* and γ_0 such that

$$\Sigma_{UE}^{\alpha_k}({}_{\mathbb{P}}K_k) = [\gamma_k^*, \gamma_k] \text{ for all } k \in \mathbb{N} \quad \text{and} \quad \Sigma_{UE}^{\alpha_0}({}_{\mathbb{P}}K_0) =: [\gamma_0^*, \gamma_0]$$

Then by Theorem 3.3 there exist points $\mathbb{P}e_k^*$ and $\mathbb{P}e_k$ such that $\lambda^{\alpha_k,t}(\mathbb{P}e_k^*) \le \gamma_k^*$ and $\lambda^{\alpha_k,t}(\mathbb{P}e_k) \ge \gamma_k$ for all $t > 0$. We may choose a subsequence $k_j \to \infty$ such that $\lim_{j\to\infty} \gamma_{k_j}^* = \liminf_{k\to\infty} \gamma_k^*$, $\lim_{j\to\infty} \gamma_{k_j} = \liminf_{k\to\infty} \gamma_k$, $\mathbb{P}e_{k_j}^* \to \mathbb{P}\tilde{e}_0^* \in {}_{\mathbb{P}}K_0$ and $\mathbb{P}e_{k_j} \to \mathbb{P}\tilde{e}_0 \in {}_{\mathbb{P}}K_0$. Continuous dependence for each fixed $t > 0$ yields

$$\lambda^{\alpha_0,t}(\mathbb{P}\tilde{e}_0^*) \le \liminf_{j\to\infty} \lambda^{\alpha_{k_j},t}(\mathbb{P}e_{k_j}^*) \le \lim_{j\to\infty} \gamma_{k_j}^*$$

and

$$\lambda^{\alpha_0,t}(\mathbb{P}\tilde{e}_0) \ge \limsup_{j\to\infty} \lambda^{\alpha_{k_j},t}(\mathbb{P}e_{k_j}) \ge \lim_{j\to\infty} \gamma_{k_j}$$

implying

$$\gamma_0^* \le \lim_{j\to\infty} \gamma_{k_j}^* \text{ and } \gamma_0 \ge \lim_{j\to\infty} \gamma_{k_j}$$

Hence

$$\limsup_{k\to\infty} \Sigma_{UE}^{\alpha_k}({}_{\mathbb{P}}K_i^k) = [\lim_{j\to\infty} \gamma_{k_j}^*, \lim_{j\to\infty} \gamma_{k_j}] \subseteq [\gamma_0^*, \gamma_0] = \Sigma_{UE}^{\alpha_0}({}_{\mathbb{P}}K_0)$$

which implies the assertion. □

4 Relation to other spectral concepts

Here we will describe the relation of Σ_{UE} to the Lyapunov- and the Morse spectrum of Φ, see [5], and conclude some results about the relation to other spectral concepts. For the convenience of the reader we recall the definitions of the Lyapunov- and the Morse spectrum.

Definition 4.1 For any point $e \in E$ the *Lyapunov exponent* is defined by

$$\lambda(e) = \lambda(\mathbb{P}e) := \limsup_{t\to\infty} \lambda^t(\mathbb{P}e)$$

Let ${}_{\mathbb{P}}K \subset \mathbb{P}E$ be a compact invariant set for the projected flow $\mathbb{P}\Phi$. The *Lyapunov spectrum* over ${}_{\mathbb{P}}K$ is defined by

$$\Sigma_{Ly}({}_{\mathbb{P}}K) := \{\lambda(\mathbb{P}e) \mid \mathbb{P}e \in {}_{\mathbb{P}}K\}$$

Definition 4.2 For positive parameters $\varepsilon, T > 0$ an (ε, T)*-chain* ζ is given by a number $n \in \mathbb{N}$, times $T_0, \ldots, T_{n-1} \geq T$ and points $\mathbb{P}e_0, \ldots, \mathbb{P}e_n$ such that

$$d(\mathbb{P}\Phi(T_i, \mathbb{P}e_i), \mathbb{P}e_{i+1}) < \varepsilon \text{ for all } i = 0, \ldots, n-1$$

The *(finite time) exponential growth rate* of a chain ζ is given by

$$\lambda(\zeta) := \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \sum_{i=0}^{n-1} T_i \lambda^{T_i}(\mathbb{P}e_i)$$

Let ${}_{\mathbb{P}}K \subset \mathbb{P}E$ be a compact invariant set for the projected flow $\mathbb{P}\Phi$ such that $\mathbb{P}\Phi|_{{}_{\mathbb{P}}K}$ is chain transitive. The *Morse spectrum* over ${}_{\mathbb{P}}K$ is defined by

$$\Sigma_{Mo}({}_{\mathbb{P}}K) := \left\{\mu \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exist } T_k \to \infty,\ \varepsilon_k \to 0 \text{ and} \\ (\varepsilon_k, T_k)\text{-chains } \zeta_k \text{ in } {}_{\mathbb{P}}K \text{ such that } \lim_{k\to\infty} \lambda(\zeta_k) = \mu \end{array} \right\}$$

Remark 4.3 From the definition of $\lambda(\zeta)$ for a chain ζ it immediately follows that the growth rate of a chain ζ cannot be smaller (or larger) that the minimum (or maximum) over the growth rates of the trajectory pieces in ζ.

A reinterpretation of Theorem 3.3 shows the following relation between the Lyapunov spectrum and the uniform exponential spectrum.

Theorem 4.4 Let ${}_{\mathbb{P}}K \subset \mathbb{P}E$ be a connected compact invariant set for the projected flow $\mathbb{P}\Phi$. Then

$$\Sigma_{Ly}({}_{\mathbb{P}}K) \subseteq \Sigma_{UE}({}_{\mathbb{P}}K)$$

and there exist points $\mathbb{P}e^*$ and $\mathbb{P}e \in {}_{\mathbb{P}}K$ such that

$$\lambda(\mathbb{P}e^*) = \min \Sigma_{UE}({}_{\mathbb{P}}K) \text{ and } \lambda(\mathbb{P}e) = \max \Sigma_{UE}({}_{\mathbb{P}}K)$$

For these points the Lyapunov exponents are actually limits.

Proof: The inclusion follows immediately from the definition of the spectra. The existence of $\mathbb{P}e^*$ and $\mathbb{P}e$ has been proved in Theorem 3.3. □

Remark 4.5 Note that it is easily seen that the strict inclusion $\Sigma_{Ly}({}_{\mathbb{P}}K) \subset \Sigma_{UE}({}_{\mathbb{P}}K)$ may occur. A simple example for this case is given by a flow induced by a linear ordinary differential equation $\dot{x} = Ax$ on $\mathbb{R}^d$ with $k \geq 2$ different Lyapunov exponents $\lambda_1 < \ldots < \lambda_k$ (which here coincide with the real parts of the eigenvalues of A). Choosing ${}_{\mathbb{P}}K = \mathbb{P}^{d-1}$, i.e. the whole real projective space, we obtain that

$$\Sigma_{Ly}(\mathbb{P}^{d-1}) = \{\lambda_1, \ldots, \lambda_k\} \neq [\lambda_1, \lambda_k] = \Sigma_{UE}(\mathbb{P}^{d-1}).$$

Next we show the relation to the Morse spectrum. Clearly, the value of $\Sigma_{UE}({}_{\mathbb{P}}K)$ depends in a strong way on the choice of ${}_{\mathbb{P}}K \subseteq \mathbb{P}E$. By choosing subsets on which the flow is chain transitive we can show equivalence of the Morse spectrum and the uniform exponential spectrum.

Theorem 4.6 Let ${}_{\mathbb{P}}K \subset \mathbb{P}E$ be a compact invariant set for the projected flow $\mathbb{P}\Phi$ such that $\mathbb{P}\Phi|_{{}_{\mathbb{P}}K}$ is chain transitive. Then

$$\Sigma_{Mo}({}_{\mathbb{P}}K) = \Sigma_{UE}({}_{\mathbb{P}}K)$$

Proof: Note that any finite time trajectory is also a (trivial) chain with zero jumps, hence $\Sigma_{UE}({}_{\mathbb{P}}K) \subseteq \Sigma_{Mo}({}_{\mathbb{P}}K)$ is immediately implied. Furthermore the closedness of $\Sigma_{Mo}({}_{\mathbb{P}}K)$ follows from the definition via limits. Since $\Sigma_{UE}({}_{\mathbb{P}}K)$ is an interval (note that the chain transitivity implies the connectedness of ${}_{\mathbb{P}}K$), it remains to show that $\min \Sigma_{UE}({}_{\mathbb{P}}K) \leq \min \Sigma_{Mo}({}_{\mathbb{P}}K)$ and $\max \Sigma_{UE}({}_{\mathbb{P}}K) \geq \max \Sigma_{Mo}({}_{\mathbb{P}}K)$. We show the property for the minimum; the corresponding inequality for the maximum follows by the same arguments.

Denote $\kappa^* := \min \Sigma_{Mo}({}_{\mathbb{P}}K)$ and Let ζ_k be a sequence of (ε_k, T_k)-chains in ${}_{\mathbb{P}}K$ with $\varepsilon_k \to 0$, $T_k \to \infty$ and $\lambda(\zeta_k) \to \kappa^*$ as $k \to \infty$. From the definition of $\lambda(\zeta_k)$ (cp. Remark 4.3) it follows that in each chain ζ_k there exist a trajectory piece starting in $\mathbb{P}e_k$ with time $t_k \geq T_k$ such that $\lambda^{t_k}(\mathbb{P}e_k) \leq \lambda(\zeta_k)$. Hence there exists a subsequence t_{k_n}, e_{k_n} such that $\lim_{n\to\infty} \lambda^{t_{k_n}}(\mathbb{P}e_{k_n}) \leq \kappa^*$ which yields the desired property. □

In particular this theorem states that the jumps in the chains do not contribute to the values in the Morse spectrum. However, in order to define the Morse sets topologically they are nevertheless necessary. Using the equality from this theorem we are now able to transfer our results from the last section to the Morse spectrum.

Corollary 4.7 Let ${}_{\mathbb{P}}K \subset \mathbb{P}E$ be a compact invariant set for the projected flow $\mathbb{P}\Phi$ such that $\mathbb{P}\Phi$ is chain transitive on ${}_{\mathbb{P}}K$.

Then the Morse spectrum $\Sigma_{Mo}({}_{\mathbb{P}}K)$ is a closed interval whose extremal points are actually Lyapunov exponents for some points $\mathbb{P}e^*$ and $\mathbb{P}e \in {}_{\mathbb{P}}E$. For these points the Lyapunov exponents are actually limits.

Proof: Follows immediately from the theorems in this section. □

Apart from the fact that this yields an alternative proof for the properties of the Morse spectrum one can use this equality in order to define a spectrum for Φ via the finite time exponential growth rates. By defining

$$\Sigma_{UE}(\Phi) := \bigcup \left\{ \Sigma_{UE}({}_{\mathbb{P}}K) \;\middle|\; \begin{array}{c} {}_{\mathbb{P}}K \text{ is a connected component} \\ \text{of the chain recurrent set of } \mathbb{P}\Phi \end{array} \right\} \tag{4.1}$$

we obtain a spectrum that is equivalent to the Morse spectrum but is defined using trajectory pieces instead of chains.

Remark 4.8 By defining a spectrum this way the relation to the Oseledets, topological and dynamical (or dichotomy) spectrum as stated in [5] do hold as well for the uniform exponential spectrum.

In particular [5, Inclusion (5.17)] implies that the boundary points of the dynamical spectrum form a subset of the boundary points of the uniform exponential spectrum. This enables us to reproduce a result from [11] — namely that the boundary points of the dynamical spectrum are Lyapunov exponents for the flow — by the direct analytic arguments of Theorem 3.3.

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