Computing Aumann's Integral

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Abstract. Quadrature formulae for the numerical approximation of Aumann's integral are investigated, which are set-valued analogues of ordinary quadrature formulae with nonnegative weights, like certain Newton-Cotes formulae or Romberg integration.
Essentially, the approach consists in the numerical approximation of the support functional of Aumann's integral by ordinary quadrature formulae. For set-valued integrands which are smooth in an appropriate sense, this approach yields higher order methods, for set-valued integrands which are not smooth enough, it yields further insight into well-known order reduction phenomena.
The results are used to define higher order methods for the approximation of reachable sets of certain classes of linear control problems.

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1 Introduction

The main objective of this paper is to investigate higher order methods for the numerical approximation of Aumann's integral and the reachable set of linear differential inclusions. We choose an approach based essentially on the numerical approximation of the support functional of Aumann's integral by ordinary quadrature formulae, cp. in this connection [3]. But contrary to [3], we restrict our outline of basic error estimates from the very beginning to quadrature formulae with nonnegative weights, moreover, we use the weak regularity assumptions in the spirit of [17] to get higher order of convergence. Thus, we follow a direct approach to higher order quadrature formulae for set-valued mappings, avoiding the use of embedding theorems for spaces of convex sets. In this respect, our presentation differs from the indirect approach indicated in [10], which is based on [16], [12], and [4].

The underlying ideas are outlined in Section 2 resulting in the fundamental error estimate of Theorem 2.6. Moreover, the set-valued analogues of closed Newton-Cotes formulae with nonnegative weights resp. Romberg method are given in Proposition 2.7 resp. 2.8 together with all regularity and smoothness assumptions required for higher order convergence. Applying these results to smooth set-valued integrands, as in Example 4.1, in principle arbitrarily high order of convergence can be achieved, e.g. by the set-valued analogue of Romberg's method. In addition, for set-valued integrands which are not smooth enough, as in Example 4.2, we get further insight into the order barrier noticed in [19].

In Section 3 we will apply our results to the approximation of reachable sets for linear control systems and get higher order methods at least for certain problem classes. In fact, smoothness in the sense of Section 2 of the fundamental solution multiplied by the set-valued inhomogeneity is the crucial property, lack of it results in order reduction phenomena. Thus, we get at least a partial answer to some open questions discussed in [9]. These effects are illustrated by Example 4.3, which is not smooth enough in the above sense, hence giving additional insight into the order barrier described in [20], and by Example 4.4, which is arbitrarily smooth, thus admitting numerical approximations of the reachable set of arbitrarily high order.

Note that there is a theoretical approach described in [8] which results in order of convergence greater than two, if the control region is a compact convex polyhedron. But, converting these conceptual ideas into a numerical algorithm is until now only possible for order of convergence equal to three.

All test examples are treated by the above methods by means of a dual approach, explained more precisely in Section 4. The development of efficient algorithms for higher dimensional problems, following this dual approach or computing directly sums of sets according to the presented set-valued quadrature formulae, is an interesting and challenging field of research.

In the following, we describe briefly the connection between Aumann's integral and linear differential inclusions.

Problem 1.1 (Linear Initial Value Problem) *Let the $n \times n$-matrix function $A(\cdot)$ be integrable on $I = [a,b]$ and*

$$G : \ I \Longrightarrow \mathbb{R}^n$$

be a set-valued mapping.

Find an absolutely continuous function $y(\cdot): I \to \mathbb{R}^n$ *with*

(1.1) $$y'(t) \in A(t)y(t) + G(t)$$

for almost all $t \in I$ *and*

(1.2) $$y(a) = y_0 \, .$$

Such problems arise from a wide range of applications, e.g. from optimal control problems or from perturbed dynamical systems with unknown, but bounded perturbations.
The only property really needed is that all solutions of (1.1),(1.2) can be equivalently represented in the form

$$y(t) = \phi(t,a)y(a) + \int\limits_a^t \phi(t,\tau)g(\tau)d\tau$$

for all $t \in I$ with a fundamental solution $\phi(t,\tau)$ of the homogeneous system

$$\frac{d}{dt}\phi(t,\tau) = A(t)\phi(t,\tau)$$

for almost all $t \in I$, satisfying the initial condition

$$\phi(\tau,\tau) = E_n$$

for a fixed $\tau \in I$, and with an integrable selection $g(\cdot)$ of $G(\cdot)$ on I.
Hence, the **reachable set** at time $t \in I$

$$\mathcal{R}(t,a,y_0) = \big\{ z \in \mathbb{R}^n : \text{ there exists a solution } y(\cdot) \text{ on } [a,t] \text{ of } (1.1),(1.2) \text{ with } z = y(t) \,\big\}$$

can be represented by means of Aumann's integral as

$$\mathcal{R}(t,a,y_0) = \phi(t,a)y_0 + \int\limits_a^t \phi(t,\tau)G(\tau)d\tau$$

for all $t \in I$, where the integral is defined according to

Definition 1.2 (cp. [2]) *Let*

$$F : I \Longrightarrow \mathbb{R}^n$$

be a set-valued mapping. Define

$$\int\limits_I F(\tau)d\tau = \big\{ z \in \mathbb{R}^n : \textit{ there exists an integrable selection } f(\cdot) \textit{ of } F(\cdot) \textit{ on } I \textit{ with } z = \int\limits_I f(\tau)d\tau \,\big\}$$

as **Aumann's integral** *of* $F(\cdot)$ *over* I.

Consequently, our first step towards higher order difference methods for linear differential inclusions should consist in the investigation of higher order numerical methods for the computation of Aumann's integral. This is the central subject of Section 2.

2 Quadrature Formulae for Set-Valued Mappings

Definition 2.1 *A set-valued mapping $F: I \Longrightarrow \mathbb{R}^n$ with nonempty and closed images is* **integrably bounded***, if there exists a function $k(\cdot) \in \mathrm{L}_1(I)$ with*

$$\sup_{f(t) \in F(t)} \|f(t)\|_2 \leq k(t)$$

for almost all $t \in I$.

In fact, we intend to use the well-known method of scalarization of a set-valued situation, just exploiting the following fundamental fact.

Theorem 2.2 *Let $F: I \Longrightarrow \mathbb{R}^n$ be a measurable set-valued mapping with nonempty and closed images. Then*

$$\int_I F(\tau) d\tau$$

is convex.
If, moreover, $F(\cdot)$ is integrably bounded, then

$$\int_I F(\tau) d\tau = \int_I \mathrm{co}(F(\tau)) d\tau$$

is nonempty, compact, and convex. Here, $\mathrm{co}(\cdot)$ denotes the convex hull operation.

For the proof of convexity and compactness see [1, Theorem 8.6.3, pp. 329–330], the existence of a measurable selection $f(\cdot)$ of $F(\cdot)$ is proven in [1, Theorem 8.1.3, pp. 308]. It follows from the integrably boundedness of $F(\cdot)$ that this selection is also integrable and hence $\int_I f(\tau) d\tau$ is an element of $\int_I F(\tau) d\tau$. Compare [13, 8.2, Theorem 1, pp. 334] for the equality of both integrals.

Definition 2.3 *Let $C \subset \mathbb{R}^n$ be a nonempty set and define*

$$\delta^*(l, C) = \sup_{c \in C} \; <l, c> \in \mathbb{R} \cup \{\infty\}$$

for all $l \in \mathbb{R}^n$, where $<\cdot,\cdot>$ denotes the usual inner product in $\mathbb{R}^n$.
Then $\delta^*(\cdot, C)$ is called **support function** *of the set C.*

We list the following property of the support function, cp. [1, Table 2.1, pp. 66] which we will use below.

Lemma 2.4 *Let $l \in \mathbb{R}^n$ and $C \subset \mathbb{R}^n$ be a nonempty set.*
Then the following equality holds:

$$\delta^*(l, C) = \delta^*(l, \mathrm{cl}(\mathrm{co}(C))),$$

where $\mathrm{cl}(\mathrm{co}(C))$ *denotes the closure of the convex hull of the set C.*

Obviously (see e.g. [1, Proposition 8.6.2, pp. 327]), under the assumptions of both parts of Theorem 2.2 we have

$$\int_I F(\tau)d\tau = \Big\{ z \in \mathbb{R}^n \;:\; <l, z> \le \delta^*(l, \int_I F(\tau)d\tau) = \int_I \delta^*(l, F(\tau))d\tau \text{ for all } l \in \mathbb{R}^n \Big\}.$$

This is the basis of the scalarization, which consists in approximating

$$\int_I \delta^*(l, F(\tau))d\tau$$

by a quadrature formula $J(l, F)$

$$\int_I \delta^*(l, F(\tau))d\tau = J(l, F) + R(l, F),$$

with remainder term $R(l, F)$ depending on $l \in \mathbb{R}^n$ and $F(\cdot)$.
E.g. for (composite) Newton-Cotes formulae of open or closed type, Gauß quadrature or Romberg integration, $J(l, F)$ has the representation

$$J(l, F) = \sum_{i=0}^{N} c_i \delta^*(l, F(t_i)) \tag{2.1}$$

with $N \in \mathbb{N}$, $c_i \in \mathbb{R}$, and a grid

$$a \le t_0 \le t_1 \le \ldots \le t_N \le b. \tag{2.2}$$

This representation and Lemma 2.4 suggest the interpretation of the quadrature formula as a support function

$$J(l,F) = \delta^*(l, \sum_{i=0}^{N} c_i \mathrm{cl}(\mathrm{co}(F(t_i))))$$

of the set

$$\sum_{i=0}^{N} c_i \mathrm{cl}(\mathrm{co}(F(t_i))) \tag{2.3}$$

which is possible if all weights c_i $(i = 0, \ldots, N)$ are nonnegative.
The following lemma, relating the Hausdorff distance $\mathrm{haus}(A,B)$ between two sets A, B to the support functions, can be found in [15, Satz 14.1, pp. 148].

Lemma 2.5 *Let A and B be nonempty, compact, and convex sets in $\mathbb{R}^n$, then*

$$\mathrm{haus}(A,B) = \sup_{\|l\|_2=1} |\delta^*(l,A) - \delta^*(l,B)|.$$

Applying it to the scalarized quadrature formula with remainder term yields the following error estimate.

Theorem 2.6 *Let $F: I \Longrightarrow \mathbb{R}^n$ be a measurable and integrably bounded set-valued mapping with nonempty and compact images, and let the quadrature formula $J(\cdot,\cdot)$ have nonnegative weights c_i, nodes $t_i \in [a,b]$ $(i = 0, \ldots, N)$ and remainder term $R(\cdot,\cdot)$.*
Then the following error estimate holds

$$\mathrm{haus}(\int_I F(t)dt, \sum_{i=0}^{N} c_i \mathrm{co}(F(t_i))) \leq \sup_{\|l\|_2=1} |R(l,F)|.$$

To be more concrete, we use the composite closed Newton-Cotes formula of degree $k \in \mathbb{N}$ over the interval $[a,b]$ which is exact for polynomials of degree at most k. Choose the number of subintervals $N \in \mathbb{N}$ of the grid

$$t_i := a + ih \ (i = 0, \ldots, N), \ h = \frac{b-a}{N}, \tag{2.4}$$

such that $\frac{N}{k}$ is an integer, and apply the closed Newton-Cotes formula of degree k on each subinterval $[t_{ik}, t_{(i+1)k}]$, then we arrive at

$$\begin{aligned}\int\limits_a^b \delta^*(l, F(t))dt &= \sum_{i=0}^{\frac{N}{k}-1} \int\limits_{t_{ik}}^{t_{(i+1)k}} \delta^*(l, F(t))dt = \\ &= kh \sum_{i=0}^{\frac{N}{k}-1} \sum_{j=0}^{k} w_{kj} \delta^*(l, F(t_{ik+j})) + R^k_{\frac{N}{k}}(\delta^*(l, F(\cdot)))\end{aligned}$$

Using the results of [17, Theorem 3.5, pp. 52] (or the earlier results mentioned in [7]) in a slightly modified way, we could estimate the error by

$$\begin{aligned}|R^k_{\frac{N}{k}}(\delta^*(l, F(\cdot)))| &\leq (1 + \sum_{j=0}^{k} |w_{kj}|) \cdot \\ &\cdot \begin{cases} W_{k+1} \cdot \tau_{k+1}(\delta^*(l, F(\cdot)); 2\frac{k}{k+1}h)_1 \,, & \text{if } k \text{ is odd,} \\ W_{k+2} \cdot \tau_{k+2}(\delta^*(l, F(\cdot)); 2\frac{k}{k+2}h)_1 \,, & \text{if } k \text{ is even,} \end{cases}\end{aligned}$$

where $\tau_\nu(f;\delta) := \tau_\nu(f;\delta)_1$ is the averaged moduli of smoothness of order ν defined in [17, Definition 1.5, pp. 7] and W_ν denotes the ν-th Whitney constant.

Proposition 2.7 *Let $F : I \Longrightarrow \mathbb{R}^n$ be a measurable and integrably bounded set-valued mapping with nonempty, compact, and convex images. Let the closed Newton-Cotes formula of degree k have coefficients $w_{kj} \geq 0$, $j = 0, \ldots, k$. Assume that the support function $\delta^*(l, F(\cdot))$ has an absolutely continuous $(\nu-1)$-st derivative and that the ν-th derivative is of bounded variation with respect to t uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$, where*

$$\nu \in \begin{cases} \{0, 1, \ldots, k\}, & \textit{if } k \textit{ is odd,} \\ \{0, 1, \ldots, k+1\}, & \textit{if } k \textit{ is even.} \end{cases}$$

Integrating over the grid

$$t_i := a + ih \quad (i = 0, \ldots, N), \ h = \frac{b-a}{N} \,,$$

and introducing the set-valued mapping corresponding to (2.3), the following error estimates holds

$$\begin{aligned}&\mathrm{haus}(\int\limits_a^b F(t)dt, kh \sum_{i=0}^{\frac{N}{k}-1} \sum_{j=0}^{k} w_{kj} F(t_{ik+j})) \leq \\ &\leq C(k,\nu)(1 + \sum_{j=0}^{k} w_{kj}) \cdot \sup_{\|l\|_2=1} \bigvee_a^b \frac{d^\nu}{dt^\nu} \delta^*(l, F(\cdot)) \cdot h^{\nu+1} \,.\end{aligned}$$

Here, $\bigvee\limits_a^b(\cdot)$ *denotes the total variation and*

$$C(k,\nu) = \begin{cases} 2^{k+1} \dfrac{k^{\nu+1}}{\prod\limits_{j=0}^{\nu-1}(k+1-j)} \cdot W_{k+1}\,, & \textit{if } k \textit{ is odd,} \\ 2^{k+2} \dfrac{k^{\nu+1}}{\prod\limits_{j=0}^{\nu-1}(k+2-j)} \cdot W_{k+2}\,, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. Apply the error estimates in [17] mentioned before and use the estimates [17][(3),(4), pp. 8 and (7), pp. 10] for a bounded, measurable function f:

$$\begin{aligned} \tau_k(f;h) &\leq 2^{k-1}\tau_1(f;kh) \qquad (k \in \mathbb{N}), \\ \tau_k(f;h) &\leq \frac{k^\nu}{\prod\limits_{j=0}^{\nu-1}(k-j)} h^\nu \tau_{k-\nu}(f^{(\nu)}; \frac{k}{k-\nu}h) \qquad (\text{if } f^{(\nu)} \text{ exists and is} \\ &\qquad \text{bounded and measurable, } \nu \in \{0,\dots,k-1\},\ k \in \mathbb{N}), \\ \tau_1(f;h) &\leq h \bigvee_a^b f \qquad (\text{if } f \text{ is of bounded variation}) \end{aligned}$$

Q.E.D.

These results only apply if the coefficients w_{kj} $(j = 0,\dots,k)$ are nonnegative which is the case for trapezoidal rule $(k = 1)$, Simpson's rule $(k = 2)$, and for $k = 3,\dots,7,9$, so that the maximal order of convergence is 10 for the closed set-valued Newton-Cotes formulae (cp. [11, Table 6.2.1, pp. 268]). Similar results can be achieved, if we consider composite Newton-Cotes formulae of open type with degree k, where e.g. the coefficients are nonnegative for the midpoint-rule $(k = 0)$ and for $k = 1,3$ (cp. the table in [14]).
Let us briefly mention Romberg's method of extrapolation which is described in more details in [5], [11], [18]. Compute the integral

$$\int_a^b \delta^*(l, F(t))dt$$

by the composite trapezoidal rule for a sequenze of stepsizes, say

$$h_0 = b - a,\ h_1 = \frac{1}{2}h_0,\ \dots,\ h_r = \frac{1}{2^r}h_0,$$

corresponding to the sequence of grids

$$a = t_{i,0} < t_{i,1} < \ldots < t_{i,2^i} = b, \qquad t_{i,j} := a + jh_i \quad (j = 0, \ldots, 2^i),$$

and start with the first Romberg column

$$T_{i0}(l) = \frac{h_i}{2} \sum_{j=0}^{2^i-1} (\delta^*(l, F(t_{i,j})) + \delta^*(l, F(t_{i,j+1}))) \quad (i = 0, \ldots, r).$$

Using the recursive formula for $j = 1, \ldots, i,\ j \leq s$

$$T_{ij}(l) = T_{i,j-1}(l) + \frac{T_{i,j-1}(l) - T_{i-1,j-1}(l)}{4^j - 1} \quad (i = 1, \ldots, r),$$

we are able to define the sets

$$(2.5) \quad T_{ij}(F) = \{y \in \mathbb{R}^n \mid \ <l, y> \leq T_{ij}(l) \text{ for all } l \in \mathbb{R}^n \text{ with } \|l\|_2 = 1\}$$

for $j = 0, \ldots, i,\ j \leq s,\ i = 0, \ldots, r$. It is known that each $T_{ij}(l)$ could be written in the form (2.1) with nonnegative weights (cp. [11, Theorem 8.3.1, pp. 381]) and $N = 2^i$, hence we can apply all obtained results and get

Proposition 2.8 *Let $F : I \Longrightarrow \mathbb{R}^n$ be a measurable and integrably bounded set-valued mapping with nonempty, compact, and convex images. Assume that the support function $\delta^*(l, F(\cdot))$ has an absolutely continuous $(2s)$-th derivative and that the $(2s+1)$-st derivative is of bounded variation with respect to t uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$.*
Then the estimate holds for the set-valued mapping introduced in (2.5)

$$\text{haus}(\int_a^b F(t)dt, T_{ij}(F)) \leq \prod_{\mu=1}^{j} \frac{1 + (\frac{1}{2})^{2\mu}}{1 - (\frac{1}{2})^{2\mu}} \cdot \alpha_{ij} \cdot \prod_{\nu=0}^{j} h_{i-\nu}^2$$

for all $j = 0, \ldots, i,\ j \leq s,\ i = 0, \ldots, r$, where α_{ij} can be chosen independently of all stepsizes $h_{i-j}, \ldots, h_i$.

Proof. Everything follows from a generalized Euler-Maclaurin summation formula which gives (under the stated weaker assumptions) the same asymptotic expansion of the composite trapezoidal rule for the integral

$$\int_a^b \delta^*(l, F(t))dt$$

as described in [5], [18]. Q.E.D.

Hence, each of the $s+1$ columns of Romberg's tableau defines successively integration methods of order $2, 4, 6, \ldots, 2s+2$ for the approximation of Aumann's integral, if the support function $\delta^*(l, F(\cdot))$ is sufficiently smooth.

3 Approximation of Reachable Sets for Linear Control Systems

We return to the Linear Initial Value Problem 1.1, assuming that it is given by a linear control problem of the following standard type.

Problem 3.1 *Let the $n \times n$-matrix function $A(\cdot)$ and the $n \times m$-matrix function $B(\cdot)$ be integrable on I, and the control region*

$$U \subset \mathbb{R}^m$$

be nonempty, compact, and convex.
Find an absolutely continuous function $y(\cdot): \ I \to \mathbb{R}^n$ with

$$\begin{aligned} y'(t) &\in A(t)y(t) + B(t)U \qquad \textit{for almost all } t \in I, \\ y(a) &= y_0. \end{aligned}$$

With the notations from the introduction, the reachable set at time b is

$$\mathcal{R}(b, a, y_0) = \phi(b,a)y_0 + \int\limits_a^b \phi(b,\tau)B(\tau)U d\tau.$$

Now apply a quadrature formula on an equidistant grid (2.4) of the type (2.3) with error estimate

$$\begin{aligned} &\operatorname{haus}(\int\limits_a^b \phi(b,\tau)B(\tau)U d\tau, \sum_{i=0}^N c_i \phi(b,t_i)B(t_i)U) \le \\ &\le \sup_{\|l\|_2=1} |R(l, \phi(b,\cdot)B(\cdot)U)| \le \operatorname{const}\left(\frac{1}{N}\right)^p, \end{aligned}$$

i.e. a quadrature formula of order p with respect to the discretization parameter $N \in \mathbb{N}$. Such formulae exist, if e.g.

$$\delta^*(l, \phi(b,\cdot)B(\cdot)U)$$

has an absolutely continuous $(p-2)$-nd derivative and if the $(p-1)$-st derivative is of bounded variation uniformly with respect to all $l \in \mathbb{R}^n$ with

$\|l\|_2 = 1$ (cp. Propositions 2.7 and 2.8).
Choose in addition a difference method for the computation of the fundamental system $\phi(b, \cdot)$ on the same grid (2.4) which computes approximations $\tilde{\phi}(b, t_i)$ $(i = 0, \ldots, N)$ also of order p

$$\sup_{0 \le i \le N} \|\tilde{\phi}(b, t_i) - \phi(b, t_i)\|_\infty \le \text{const} \left(\frac{1}{N}\right)^p .$$

This is possible e.g. for the Adams-Bashforth method of degree $p - 1$ (cp. [17, Theorem 6.3, pp. 126]), if $A(\cdot)$ has an absolutely continuous $(p - 2)$-nd derivative and if the $(p-1)$-st derivative has bounded variation. Under slightly different assumptions one has similar results for Runge-Kutta methods as well (cp. [6], [7]).
Using the same notation $\|M\|_\infty$ for the lub-norm of a matrix M with respect to the supremum norm $\|\cdot\|_\infty$ in $\mathbb{R}^n$ and for the norm of a set $S \subset \mathbb{R}^n$

$$\|S\|_\infty = \sup_{s \in S} \|s\|_\infty ,$$

the following inequality holds

$$\begin{aligned} \text{haus}(\sum_{i=0}^{N} c_i \phi(b, t_i) B(t_i) U, \sum_{i=0}^{N} c_i \tilde{\phi}(b, t_i) B(t_i) U) &\le \\ \le \sum_{i=0}^{N} c_i \text{haus}(\phi(b, t_i) B(t_i) U, \tilde{\phi}(b, t_i) B(t_i) U) &\le \\ \le \sum_{i=0}^{N} c_i \|\phi(b, t_i) - \tilde{\phi}(b, t_i)\|_\infty \cdot \|B(t_i) U\|_\infty , \end{aligned}$$

if the weights c_i are all nonnegative. Moreover, $\sum_{i=0}^{N} c_i \|B(t_i) U\|_\infty$ is bounded uniformly for all $N \in \mathbb{N}$, if e.g. $B(\cdot)$ is bounded on I. Hence, we arrive at the following result.

Theorem 3.2 *Consider the Linear Control Problem 3.1, and assume that $A(\cdot)$ and $\delta^*(l, \phi(b, \cdot) B(\cdot) U)$ have an absolutely continuous $(p-2)$-nd derivative and that the $(p-1)$-st derivative is of bounded variation uniformly with respect to all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$.*
Assume moreover, that $\sum_{i=0}^{N} c_i \|B(t_i) U\|_\infty$ is uniformly bounded for $N \in \mathbb{N}$.
Then, combining a quadrature formula with nonnegative weights of order p with a difference method of order p in the sense described above yields a method of order p for the approximation of the reachable set at time b.

4 Test Examples

In the following, we present a series of model problems, illustrating the results of Section 2 and 3. All numerical tests were made on an HP Apollo workstation, Series 400. For every supporting hyperplane, the explicit knowledge of at least one boundary point of the set-valued integrand belonging to that hyperplane is exploited for the computation of the plots.
In all tables, we use Lemma 2.5 together with uniformly distributed points l_i $(i = 1, \ldots, \mu)$ on the boundary of the unit ball to approximate the Hausdorff distance of two nonempty, compact, and convex sets $C, D \subset \mathbb{R}^n$ in the following way:

$$\max_{i=1,\ldots,\mu} |\delta^*(l_i, C) - \delta^*(l_i, D)| \approx \mathrm{haus}(C, D).$$

In the above sense, we use a dual approach for the calculation of the presented set-valued quadrature formulae and for the verification of the corresponding error estimates.

Example 4.1 *Compute Aumann's integral*

$$\int_0^2 A(t)B_1(0)dt = \int_0^2 \begin{pmatrix} e^t & 0 \\ 0 & t^2+1 \end{pmatrix} B_1(0)dt,$$

where $B_1(0)$ denotes the closed unit ball in $\mathbb{R}^2$.

Then the support function

$$\delta^*(l, A(t)B_1(0)) = \delta^*(A(t)^*l, B_1(0)) = \|A(t)^*l\|_2$$

is arbitrarily often continuously differentiable with respect to t with bounded derivatives uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$.
In Figure 4.1 we show the boundary of the set created by the composite trapezoidal rule with stepsizes $h = 2.0$ (the biggest set), $h = 1.0, 0.5$ (the two smaller sets), and the reference set (the smallest set) computed by Romberg's method with 10 rows and columns. Figure 4.1 illustrates order 2 of the composite trapezoidal rule which is confirmed by Table 4.1 where we show the approximated Hausdorff distance between the sets calculated by different numerical integration methods and the reference set.

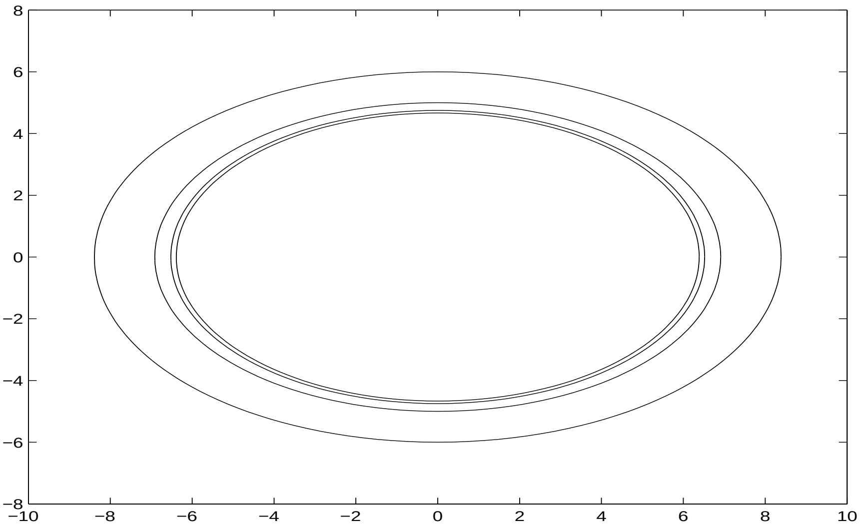

Fig. 4.1: Composite trapezoidal rule with $h = 2.0, 1.0, 0.5$ compared with the reference set

In Table 4.1, one can clearly observe convergence order 2 for the composite trapezoidal rule and order 4 for composite Simpson's rule.

Example 4.2 *This example was presented in [19] as a negative result for the approximation of Aumann's integral*

$$\int_0^{2\pi} A(t)[-1,1]dt = \int_0^{2\pi} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} [-1,1]dt.$$

In this example, the support function

$$\begin{aligned} \delta^*(l, A(t)[-1,1]) &= \delta^*(A(t)^* l, [-1,1]) = \\ &= |A(t)^* l| = |l_1 \sin(t) + l_2 \cos(t)| \end{aligned}$$

is only absolutely continuous, and its derivative has bounded variation uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$.

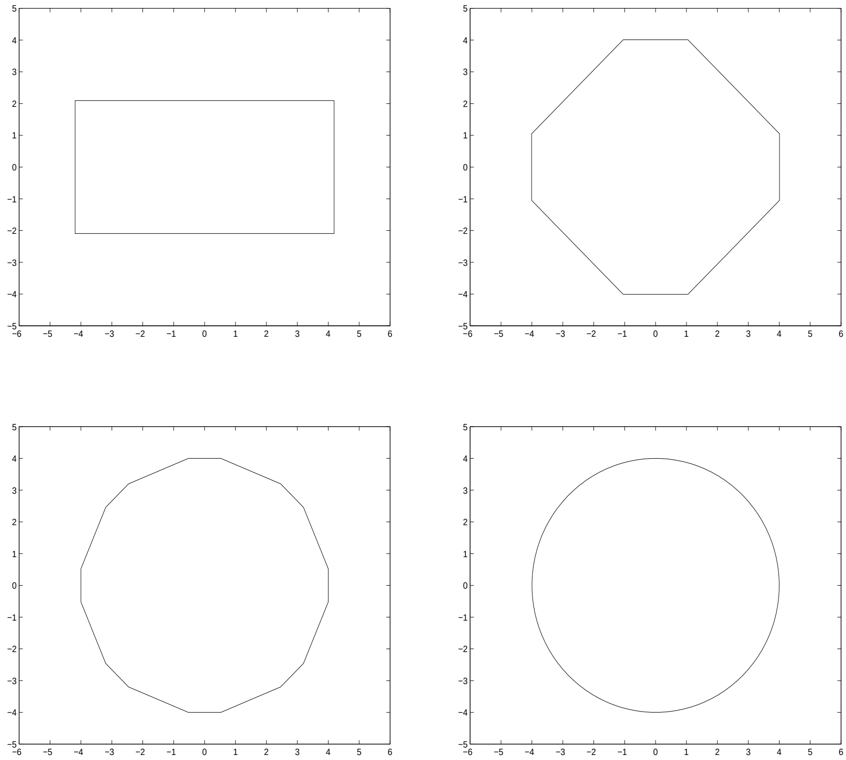

Fig. 4.2: Composite Simpson's rule with $h = \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{8}$ compared with the reference set

Table 4.2 illustrates convergence order 2 for the composite trapezoidal rule and also only convergence order 2 for the composite Simpson's rule, because lacking smoothness of the support function prevents higher order of convergence. We refer to Figure 4.2 where the results of composite Simpson's rule are plotted for stepsizes $h = 0.5\pi, 0.25\pi, 0.125\pi$ together with the result of composite trapezoidal rule with $h = 0.02\pi$ (the reference set). Simpson's rule creates polytopes with increasing number of edges. This is the geometric explanation given in [19] for the observed order reduction.

Example 4.3 *This example is also due to Veliov and was presented in [20]. Consider the linear control system*

$$\begin{aligned} y'(t) &\in \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} [-1, 1] \qquad \textit{for almost all } t \in [0, 1], \\ y(0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Then the fundamental solution is

$$\phi(t, \tau) = \begin{pmatrix} 1 & t - \tau \\ 0 & 1 \end{pmatrix},$$

and the reachable set at time $b = 1$ is

$$\int_0^1 \begin{pmatrix} 1 - \tau \\ 1 \end{pmatrix} [-1, 1] d\tau.$$

In this case, the support function

$$\delta^*(l, \phi(1, \tau) \begin{pmatrix} 0 \\ 1 \end{pmatrix} [-1, 1]) = |(1 - \tau, 1) l|$$

is only absolutely continuous, and its derivative is of bounded variation uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$.
Hence, order of convergence at most equal to 2 can be expected.
The numerical results in Table 4.3 were computed with the explicitly known fundamental solution, so that no errors occur by the approximation of the fundamental solution. Nevertheless, we observe the expected convergence order for the first method and a breakdown of the convergence order of composite Simpson's rule, which is illustrated graphically in Figure 4.3.

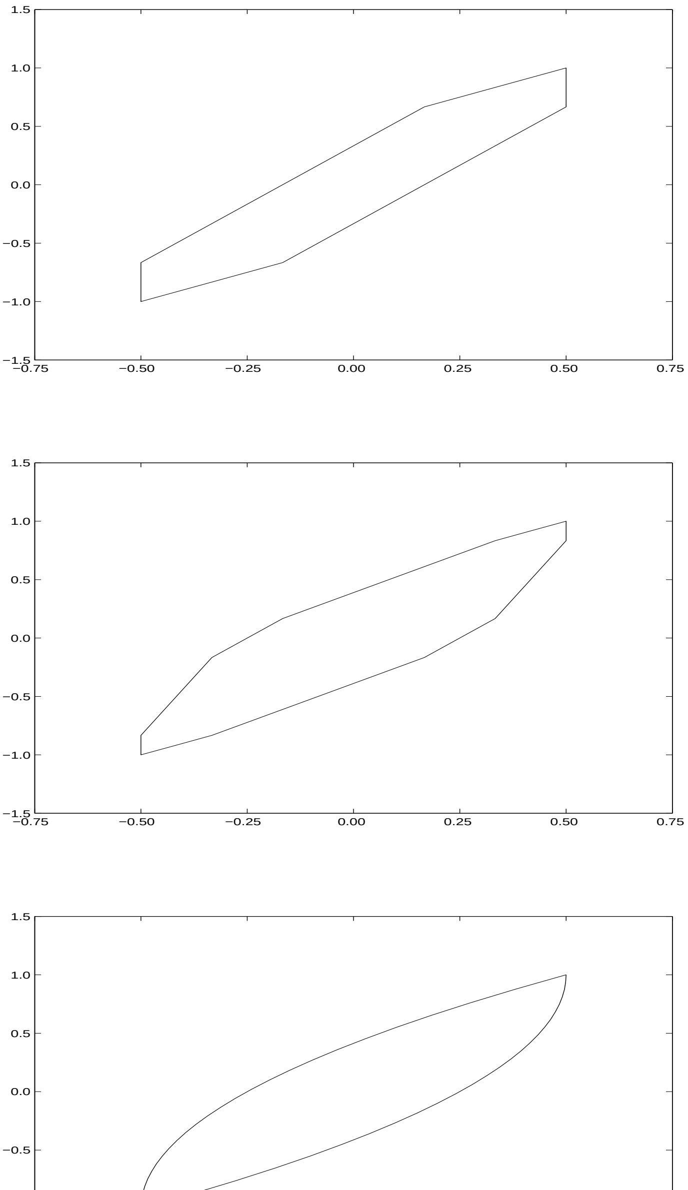

Fig. 4.3: Composite Simpson's rule with $h = 0.5, 0.25$ compared with the reference set

Example 4.4 *Consider the linear control system*

$$\begin{aligned} y'(t) &\in \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} y(t) + B_1(0) \qquad \textit{for almost all } t \in [0,2], \\ y(0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

with the closed unit ball $B_1(0) \subset \mathbb{R}^2$.

Then the fundamental solution is

$$\phi(t,\tau) = \begin{pmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{pmatrix},$$

and the reachable set at time $b = 2$ is

$$\int_0^2 \phi(2,\tau) B_1(0) d\tau.$$

In this case, the support function

$$\delta^*(l, \phi(2,\tau)B_1(0)) = \|\phi(2,\tau)^* l\|_2$$

is arbitrarily often differentiable with bounded derivatives with respect to τ uniformly on the set $\{l \in \mathbb{R}^2 : \|l\|_2 = 1\}$.

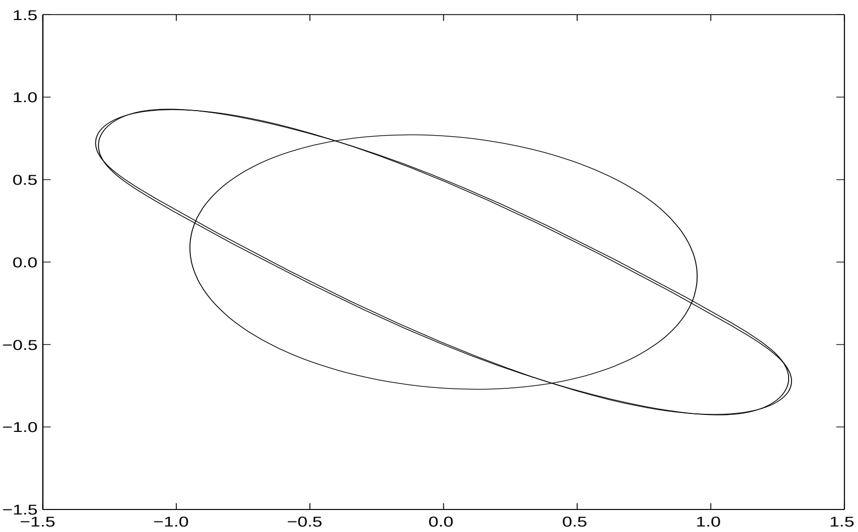

Fig. 4.4: Composite Simpson's rule combined with Runge-Kutta (4) with $h = 1.0, 0.5$ compared with the reference set

Fourth order of convergence of composite Simpson's rule is clearly indicated by Table 4.4 and illustrated by Figure 4.4, where a very rough stepsize $h = 0.5$ gives a remarkably good approximation (which nearly does not differ from the reference set within plotting precision). Comparing the results using the explicitly known fundamental solution with the combined methods using a numerical approximation of the fundamental solution, we observe the same order of convergence in Table 4.4, but higher starting errors. Notice that we have chosen appropriate methods for the computation of the fundamental solution which have the same order of convergence as the integration method.

Table 4.1: Results for Example 4.1

numerical method	stepsize	approximated Hausdorff distance
composite trapezoidal rule	2.0	1.9999999999999991
	1.0	0.5237537789937194
	0.5	0.1325540105506304
	0.25	0.0332417225019865
	1.0	0.5237537789937194
	0.1	0.0053233262580985
	0.01	0.0000532420454213
	0.001	0.0000005324213319
	0.0001	0.0000000053242024
composite Simpson's rule	1.0	0.0316717053249587
	0.5	0.0021577697845663
	0.25	0.0001401261042604
	1.0	0.0316717053249587
	0.1	0.0000036242154220
	0.01	0.0000000003630625
	0.001	0.0000000000000480

Table 4.2: Results for Example 4.2

numerical method	stepsize	approximated Hausdorff distance
composite trapezoidal rule	1.0π	3.9999999986840358
	0.5π	0.8584073450942529
	0.25π	0.2077622028099686
	0.2π	0.1324688024984382
	0.02π	0.0013160325315322
	0.002π	0.0000131581652458
composite Simpson's rule	0.5π	1.9056048962908503
	0.25π	0.4246439169047305
	0.125π	0.0876439543002339
	0.2π	0.1324688024984382
	0.02π	0.0026324128852804
	0.002π	0.0000263176810722

Table 4.3: Results for Example 4.3

numerical method	stepsize	approximated Hausdorff distance
composite trapezoidal rule	1.0	0.2254227652525390
	0.5	0.0604922332712965
	0.25	0.0155000388904730
	0.125	0.0038983702734540
	0.1	0.0023997874829183
	0.01	0.0000246670782419
	0.001	0.0000002481024457
	0.0001	0.0000000023598524
composite Simpson's rule	0.5	0.0686732300277554
	0.25	0.0180416645285239
	0.125	0.0049492014035796
	0.0625	0.0012651981868981
	0.1	0.0026219672162807
	0.01	0.0000263963285972
	0.001	0.0000002264152427
	0.0001	0.0000000016145246

Table 4.4: Results for Example 4.4

numerical method	stepsize	appoximated Hausdorff distance
composite trapezoidal rule	1.0	0.9433330463362816
	0.1	0.0024347876750569
	0.01	0.0000243687539239
	0.001	0.0000002436654987
	0.0001	0.0000000024125214
composite trapezoidal rule combined with the method of Euler-Cauchy	1.0	2.5354954374884917
	0.1	0.0054487041523342
	0.01	0.0000496413103789
	0.001	0.0000004919838970
	0.0001	0.0000000048984064
composite Simpson's rule	1.0	0.1335888228107664
	0.5	0.0224859067427672
	0.25	0.0016216053482911
	0.125	0.0000845026154785
	1.0	0.1335888228107664
	0.1	0.0000332142695469
	0.01	0.0000000030953542
composite Simpson's rule combined with Runge-Kutta (4)	1.0	0.5738839013456635
	0.5	0.0130316902023255
	0.25	0.0008327343054384
	0.125	0.0000457276981323
	1.0	0.5738839013456635
	0.1	0.0000180766372746
	0.01	0.0000000018105748

References

[1] J.-P. Aubin and H. Frankowska (1990) *Set-Valued Analysis.* Systems & Control: Foundations & Applications 2, Birkhäuser, Boston–Basel–Berlin.

[2] R. J. Aumann (1965) Integrals of Set-Valued Functions. *J. Math. Anal. Appl.* **12**, no. 1, pp. 1–12.

[3] E. I. Balaban (1982) On the Approximate Evaluation of the Riemann Integral of Many-Valued Mapping. *U.S.S.R. Comput. Maths. Math. Phys.* **22**, no. 2, pp. 233–238.

[4] H. T. Banks and M. Q. Jacobs (1970) A Differential Calculus for Multifunctions. *J. Math. Anal. Appl.* **29**, no. 2, pp. 246–272.

[5] R. Bulirsch (1964) Bemerkungen zur Romberg-Integration. *Numer. Math.* **6**, pp. 6–16.

[6] B. A. Chartres and R. S. Stepleman (1974) Actual Order of Convergence of Runge-Kutta Methods on Differential Equations with Discontinuities *SIAM J. Numer. Anal.* **11**, no. 6, pp. 1193–1206.

[7] B. A. Chartres and R. S. Stepleman (1976) Convergence of Linear Multistep Methods for Differential Equations with Discontinuities. *Numer. Math.* **27**, pp. 1–10.

[8] B. D. Doitchinov and V. M. Veliov (1991) Parametrizations of Integrals of Set-Valued Mappings and Applications, to appear in *J. Math. Anal. Appl.*

[9] A. Dontchev and F. Lempio (1992) Difference Methods for Differential Inclusions: A Survey. *SIAM Rev.* **34**, no. 2, 263–294.

[10] T. D. Donchev and E. M. Farkhi (1990) Moduli of Smoothness of Vector Valued Functions of a Real Variable and Applications. *Numer. Funct. Anal. Optim.* **11**, no. 5&6, pp. 497–509.

[11] H. Engels (1980) *Numerical Quadrature and Cubature. Computational mathematics and applications.* Academic Press, London–New York–Toronto–Sydney–San Francisco.

[12] P. L. Hörmander (1954) Sur la fonction d'appui des ensembles convexes dans un espace localement convexe. *Ark. Mat.* **3**, no. 12, pp. 181–186.

[13] A. D. Ioffe and V. M. Tihomirov (1979) *Theory of Extremal Problems. Studies in Mathematics and Applications, Volume 6.* North-Holland Publishing Company, Amsterdam–New York–Oxford.

[14] E. Isaacson and H. B. Keller (1966) *Analysis of Numerical Methods.* John Wiley & Sons, Inc., New York–London–Sydney.

[15] K. Leichtweiß (1980) *Konvexe Mengen.* Springer-Verlag, Berlin–Heidelberg–New York.

[16] H. Rådström (1952) An Embedding Theorem for Spaces of Convex Sets. *Proc. Amer. Math. Soc.* **3**, pp. 165–169.

[17] B. Sendov and V. A. Popov (1988) *The Averaged Moduli of Smoothness.* Applications in Numerical Methods and Approximation, John Wiley & Sons, Chichester–New York–Brisbane–Toronto–Singapore, Translation of: Usredneni Moduli na Gladkost, Bulgarian Mathematical Monographs Volume 4, Publishing House of the Bulgarian Academy of Sciences, Sofia, 1983.

[18] J. Stoer and R. Bulirsch (1980) *Introduction to Numerical Analysis*, Springer-Verlag, Berlin–Heidelberg–New York.

[19] V. M. Veliov (1989) Discrete Approximations of Integrals of Multivalued Mappings, *Comptes rendus de l'Académie bulgare des Sciences* **42**, no. 12, pp. 51–54.

[20] V. M. Veliov (1992) Second Order Discrete Approximation to Linear Differential Inclusions, *SIAM J. Numer. Anal.* **29**, no. 2, pp. 439–451.

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