On Hyperelliptic Manifolds

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Abstract

Hyperelliptic surfaces arise classically in the Enriques-Kodaira classification of compact complex surfaces as the surfaces S, which are uniquely determined through the invariants $\mathrm{kod}(S) = 0$, $p_g(S) = 1$, $q(S) = 0$ and $12K_S \equiv 0$. Due to the work of Enriques-Severi and Bagnera-de Franchis, these surfaces are very well understood and are all isomorphic to the quotient of an Abelian surface A by a non-trivial finite group G, which acts freely on A and contains no translations. They showed that A is isogenous to a product of two elliptic curves, which allows an explicit classification of hyperelliptic surfaces. In particular, hyperelliptic surfaces are always projective. In the '90s, Herbert Lange studied higher-dimensional analogues of hyperelliptic surfaces and in 1999, he published an article, which is dedicated to the classification of projective hyperelliptic threefolds. As it turns out, Lange's classification is incomplete, and in collaboration with Fabrizio Catanese, we describe the missing case(s) of Lange's classification. More precisely, we prove the existence of a unique complete 2-dimensional hyperelliptic threefolds A/D_4, where D_4 is the dihedral group of order 8.
Motivated by the 3-dimensional case, we decided to investigate in this thesis the case of dimension 4 in more detail as well. Using group-theoretic methods, we work out the list of exactly those abstract finite groups, which admit an embedding in the group of biholomorphic self-maps of some Abelian fourfold A in such a way that the image contains no translations and acts freely on A. We will say that such a group is *associated with a hyperelliptic fourfold.*
The question if there exist complete families of hyperelliptic threefolds (or, more generally, hyperelliptic manifolds of arbitrary dimension), which do not contain a projective manifold, remained open in Lange's article. This is studied in more detail in this thesis: we show, together with Fabrizio Catanese and Benoît Claudon, that every hyperelliptic manifold admits arbitrarily small deformations which are projective. Furthermore, we discuss in detail a special case of this result, namely the case, in which the group action on the complex torus is rigid: in this case, we construct explicitly a polarization on the complex torus coming from a direct sum of Hodge structures on CM-fields. This is a result obtained by Torsten Ekedahl around 1999.

German Abstract

In der Enriques-Kodaira Klassifikation kompakter komplexer Flächen treten minimale hyperelliptische Flächen klassisch als diejenigen Flächen S auf, die durch die Invarianten $\mathrm{kod}(S) = 0$, $p_g(S) = 0$, $q(S) = 1$ und $12K_S \equiv 0$ eindeutig festgelegt sind. Durch die Arbeit von Enriques-Severi und Bagnera-de Franchis sind diese Flächen sehr gut verstanden und sind allesamt isomorph zu Quotienten einer abelschen Fläche A nach der Wirkung einer nicht-trivial endlichen Gruppe G, die frei auf A operiert und keine Translationen enthält. Es lässt sich zeigen, dass A isogen zu einem Produkt zweier elliptischer Kurven ist, was eine explizite Klassifikation hyperelliptischer Flächen ermöglicht. Insbesondere sind hyperelliptische Flächen stets projektiv. Herbert Lange untersuchte in den 90er Jahren höherdimensionale Analoga hyperelliptischer Flächen und veröffentlichte 1999 eine Arbeit, die der Klassifikation projektiver hyperelliptischer Dreifaltigkeiten gewidmet war. Wie sich später herausstellte, war Langes Klassifikation unvollständig, und in Kollaboration mit Fabrizio Catanese beschreiben wir die fehlenden Fälle in Langes Klassifikation. Genauer gesagt beweisen wir die Existenz und die Eindeutigkeit einer vollständigen 2-dimensionalen Familie hyperelliptischer Dreifaltigkeiten A/D_4.
Durch den dreidimensionalen Fall motiviert fiel der Entschluss, den vierdimensionalen Fall ebenfalls genauer in dieser Dissertation zu beleuchten. Mit gruppentheoretischen Methoden erarbeiten wir die Liste an genau denjenigen abstrakten endlichen Gruppen, für die eine Einbettung in die Biholomorphismengruppe einer vierdimensionalen abelschen Varietät A exisiert, sodass das Bild keine Translationen enthält und frei auf A operiert.
Offen blieb schon in Langes Arbeit die Frage, ob vollständige Familien dreidimensionaler hyperelliptischer Mannigfaltigkeiten (oder allgemeiner, hyperelliptischer Manifaltigkeiten beliebiger Dimension) existieren, die keine projektive Mannigfaltigkeit enthalten. Dies wird in dieser Arbeit genauer untersucht: Zusammen mit Fabrizio Catanese und Benoît Claudon wird gezeigt, dass jede hyperelliptische Mannigfaltigkeit beliebig kleine Deformationen besitzt, die projektiv sind. Darüber hinaus diskutieren wir ausführlich den Spezialfall dieses Resultats, in dem die Gruppenwirkung auf dem komplexen Torus starr ist: In diesem Fall konstruieren wir explizit eine Polarisierung auf dem komplexen Torus, die von einer direkten Summe von Hodge-Strukturen auf CM-Körpern kommt. Dabei handelt es sich um ein Ergebnis von Torsten Ekedahl (um 1999).

Introduction

The thesis at hand covers various topics related to *(generalized) hyperelliptic manifolds*: a (generalized) hyperelliptic manifold[1] is the quotient $X = T/G$ of a complex torus $T = V/\Lambda$ by a non-trivial finite group

$$G \subset \mathrm{Bihol}(T) := \{f : T \to T \mid f \text{ is biholomorphic}\},$$

which acts freely on T and does not contain any non-trivial translations. If the complex torus T is an Abelian variety, we call X a *(generalized) hyperelliptic variety.* While it follows from the Riemann-Hurwitz formula that there are no hyperelliptic manifolds of complex dimension one, the case of dimension two was first considered and classified by Enriques-Severi [ES09], who where awarded the Bordin prize in 1907 and withdrew their article after having discussed with Bagnera-de Franchis [BdF08], who gave a simpler proof. The classification of hyperelliptic surfaces (for which Bagnera and de Franchis were then awarded the Bordin prize in 1909) shows that there are exactly seven complete families of hyperelliptic surfaces $X = T/G$, and that in each case, G is cyclic and T is isogenous to a product of two elliptic curves. In particular, T is an Abelian variety. For this reason, we call a hyperelliptic manifold (resp. a hyperelliptic variety) $X = T/G$ of arbitrary dimension a *Bagnera-de Franchis manifold* (resp. a *Bagnera-de Franchis variety*), if G is cyclic. A natural problem arises.

Problem 1. *Given a positive integer $n \geq 3$, classify all locally complete families of hyperelliptic manifolds (resp. hyperelliptic varieties) of dimension n.*

In 1976, Uchida-Yoshihara [UY76] used group-theoretic arguments to determine a list of possible groups G associated with a hyperelliptic threefold. Later, in 1999, Lange [La01] worked on the classification of hyperelliptic threefolds. His results rely heavily on the cited paper of Uchida-Yoshihara and the table of (linear) automorphisms of 2-dimensional complex tori obtained by Fujiki [Fu88], which he used in the following ways. Uchida-Yoshihara's result tells us that the group G associated with a hyperelliptic threefold is in particular either a product of two (possibly trivial) cyclic groups,

$$G = C_{d_1} \times C_{d_2}, \quad d_1 | d_2,$$

where the possibilities for d_1, d_2 are given explicitly, or the group G is the dihedral group D_4 of order 8. Lange used this to show that if G is Abelian, then the complex torus T is isogenous to a product of an elliptic curve and a 2-dimensional complex torus, and then used the tables by Fujiki to obtain information about the automorphisms of this 2-dimensional complex torus. Furthermore, Lange suggested a proof for the statement that there are no hyperelliptic threefolds with group $G \cong D_4$. However, his proof contains an (identifiable) mistake, since – as we will see in Part II of this thesis – there indeed exists a unique complete 2-dimensional family of hyperelliptic threefolds with group $G \cong D_4$.

Note that the phrasing of Problem 1 is subtle: it is not obvious that each family of hyperelliptic manifolds contains a projective member, i.e., a hyperelliptic variety. This leads to the following natural

[1] We will usually drop the word 'generalized', which refers to the generalization of hyperelliptic surfaces to higher dimensions.

Question 2. *Does every hyperelliptic manifold have arbitrary small deformations which are projective?*

This question has an interesting origin on its own: the work of Kodaira lead to the question whether any compact Kähler manifold admits an arbitrarily small deformation which is projective (for a historical account see the introduction of Part I). Motivated by this problem, Fabrizio Catanese asked Torsten Ekedahl whether there exists a rigid group action of a finite group $G \subset \mathrm{Bihol}(T)$ on a complex torus T, which is not projective. If such an example existed, it would serve as a counterexample to Kodaira's problem (see Chapter 5 of Part I for the relation to Kodaira's problem). Ekedahl answered this question negatively (i.e., the existence of a rigid group action on a complex torus implies that the torus is an Abelian variety) and sketched a proof. Part I of this thesis is the content of the article [CD17] (accepted for publication in *Commun. Contemp. Math.*) by Fabrizio Catanese and the author of this thesis. It gives a detailed proof of Ekedahl's result, which was – up to the article [CD17] – not yet contained in the literature:

Main Theorem 1. [Ekedahl, = Theorem I.1.1]
Let (T, G) be a rigid group action of a finite group $G \subset \mathrm{Bihol}(T)$ on a complex torus T. Then T (or, equivalently, T/G) is projective. Moreover, if we write $T = V^{1,0}/\Lambda$, then

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_j W_j^{n_j},$$

where W_j is a Hodge structure on a CM-field F_j and where $\bigoplus_j F_j$ is a subalgebra of the center of the group algebra $\mathbb{Q}[G]$.

After having reviewed the necessary notions of deformations of group actions in Part I, Chapter 2, we discuss Ekedahl's approach in Part I, Chapter 3, which allows a rather explicit description of rigid actions on complex tori in terms of orders in CM-fields, hence providing explicit polarizations on them. This answers Question 2 positively in the case, where the action of G on the torus is rigid.
The general version of Question 2 is dealt with in Chapter 6 of Part I, which is an appendix coauthored with Fabrizio Catanese and Benoît Claudon: we show that each group action (T, G) of a finite group G on a complex torus T admits arbitrarily small deformations, which are projective:

Main Theorem 2. [= Theorem I.6.1]
Let (T, G) be a group action of a finite group G on a complex torus T. Then there are arbitrarily small deformations (T_t, G) of the action where T_t is projective.

Since the methods of the proof of Ekedahl's result and the above Theorem are quite different (the proof of Ekedahl's Theorem being constructive), we decided to include proofs of both results.
In conclusion, we give a positive answer to Question 2 and thus can restrict to the classification of hyperelliptic *varieties* in a given dimension.

As we have already mentioned, the classification of hyperelliptic threefolds turned out to be incomplete. Together with Fabrizio Catanese, we fully completed this classification: in Part II (which is the content of the paper [CD18-2], accepted for publication in *Groups Geom. Dyn.*), we explicitly describe the missing hyperelliptic threefolds with group $G \cong D_4$, the dihedral group of order 8. As it turns out (the requirement for G to act freely on a 3-dimensional complex torus being a strong assumption) there exists a unique complete family of such hyperelliptic threefolds:

Main Theorem 3. [= Theorem II.1.1]
Let T be a complex torus of dimension 3 admitting a fixed point free action of the dihedral group

$$G := D_4 := \langle r, s \,|\, r^4 = s^2 = (rs)^2 = 1 \rangle,$$

such that $G = D_4$ contains no translations.
Then T is algebraic. More precisely, there are two elliptic curves E, E' such that:

(I) T is a quotient $T := T'/H,\ H \cong C_2$, where

$$T' := E \times E \times E' =: E_1 \times E_2 \times E_3,$$

$$H := \langle \omega \rangle, \quad \omega := (h+k, h+k, 0) \in T'[2],$$

and h, k are 2-torsion elements in E, such that $h, k \neq 0$ and $h + k \neq 0$;

(II) there is an element $h' \in E'$ of order precisely 4, such that, for $z = (z_1, z_2, z_3) \in T'$:

$$r(z) = (z_2, -z_1, z_3 + h') =: R(z_1, z_2, z_3) + \left(0, 0, h'\right),$$
$$s(z) = (z_1 + h, -z_2 + k, -z_3) =: S(z_1, z_2, z_3) + (h, k, 0).$$

Conversely, the above formulae give a fixed point free action of the dihedral group $G = D_4$ which contains no translations.
In particular, we obtain the following normal form:

$$E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \quad E' = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau'), \quad \tau, \tau' \in \mathcal{H} := \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\},$$

$$h = 1/2, k = \tau/2, h' = 1/4$$

$$r(z_1, z_2, z_3) := (z_2, -z_1, z_3 + 1/4)$$

$$s(z_1, z_2, z_3) := (z_1 + 1/2, -z_2 + \tau/2, -z_3).$$

In particular, the Teichmüller space of hyperelliptic threefolds with group D_4 is isomorphic to the product $\mathcal{H}^2$ of two upper halfplanes.

Motivated by Uchida-Yoshihara's work and having now completed the description of hyperelliptic threefolds, we went a step further to investigate hyperelliptic fourfolds. It quickly turned out that a full classification seems not feasible, since there would be too many cases to classify. However, in Part III, we were able to generalize the group theoretic methods of Uchida-Yoshihara to the fourfold case and give a list of all groups associated with some hyperelliptic fourfold. As we have noted above, we can restrict to the case of projective hyperelliptic fourfolds (for instance, one of the advantages of working with Abelian varieties is Poincaré's Complete Reducibility Theorem [Mum70,

p. 174, Corollary 1]). With quite some effort, we obtain the following classification result:

Main Theorem 4. [= Theorem III.11.2]
For each of the 77 *groups* G *contained in* TABLE 6 *on p. 183 there exists a hyperelliptic fourfold with group* G*, i.e., an Abelian fourfold* A *and an embedding* $G \hookrightarrow \mathrm{Bihol}(A)$ *such that the image contains no translations and acts freely on* A*. Conversely, if* $X = A'/G'$ *is a hyperelliptic fourfold, then the isomorphism type of* G' *is contained in* TABLE 6*.*

Among the 77 groups contained in TABLE 6, 16 are cyclic, 28 are non-cyclic Abelian and 33 are non-Abelian. Moreover, all groups in TABLE 6 whose order is divided by 5 or 7 are Abelian, and the largest group contained in the table has order 144, whereas the largest non-Abelian one has order 108.

To put it simply, our strategy of proof of Main Theorem 4 consists of determining the possible prime numbers p which can divide the order of G (these are 2, 3, 5 and 7 by Lemma 2.5 (b)) and then determining a practical bound for the order of the p-Sylow subgroups of G. This is easily done in the cases $p \in \{5, 7\}$, but – as expected – much more involved for the primes 2 and 3. After having obtained the bounds for the Sylow subgroups of G, we may go through all possibilities for G and investigate which ones are associated with a hyperelliptic fourfold; see Chapter 4 of Part III for a more detailed description of the strategy of the proof of Main Theorem 4.
As a byproduct of Main Theorem 4, we obtain a positive answer to a conjecture of Amerik-Rovinsky-Van de Ven in dimension 4, namely that there is no hyperellipic fourfold X such that $b_2(X) = 1$, see p. 188.

Moreover, we investigate further the strategy of the arguments of the very recent article of Catanese [Cat19], in which the data needed to define a $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$-module Λ, which is also a free Abelian group of finite rank, is explicitly given. This is applied to obtain an explicit description of Bagnera-de Franchis manifolds. In III.3.2, we give a similar description to Catanese's, in the sense that we describe the data needed to define a $\mathbb{Z}[G]$-module Λ, which is a free Abelian group (see Proposition 3.22) and G is a finite Abelian group. This description is obtained by embedding $\mathbb{Z}[G]$ into a direct sum of cyclotomic rings, with finite cokernel. We obtain, as in Catanese's article, an up-to-isogeny decomposition of a real torus T into a product of certain real subtori $T_{d,i}$. However, some questions, including describing explicitly hyperelliptic manifolds with Abelian groups, remain open.

Notation and Conventions

This thesis follows the following organizational conventions.

- The thesis is divided into the parts

  I Rigid Group Actions on Complex Tori are Projective (after Ekedahl)

  II The Classification of Hyperelliptic Threefolds

  III On the Classification of Hyperelliptic Fourfolds and Hyperelliptic Varieties with Abelian Group

  IV Further Remarks and Questions

Whenever we refer to a result, which was stated in a different part of the thesis, we cite it using the form 'Theorem I.5.8', which refers to Theorem 5.8 of Part I. If we however refer to a result stated in the same part as the reference, we omit the number of the part in the citation: for instance, 'Lemma 1.1' in Part II means Lemma II.1.1.

- Each Part contains a small introduction of its own.
- There is a common bibliography for all three parts at the very end of the thesis (p. 190 and the following ones).

Moreover, we use the following mathematical conventions and notations.

- By C_d, we denote the cyclic group of order d, which we usually write additively.
- For a group G, we write

$$Z(G) := \{g \in G \,|\, \forall h \in G\colon \; gh = hg\}$$

for the *center of* G. The *derived subgroup* (or *commutator subgroup*) of G is denoted by $[G, G]$ and is defined to be the normal subgroup of G generated by all elements of the form $ghg^{-1}h^{-1}$, where $g, h \in G$.

- *Conjugating of the group element* $h \in G$ *by* $g \in G$ means $g^{-1}hg$ for us.
- We denote by S_n (resp. D_n) the *symmetric group on* n *letters* (resp. *the dihedral group of order* $2n$).
- Given a positive integer d, we write $\zeta_d := \exp\left(\frac{2\pi\sqrt{-1}}{d}\right)$.
- By a representation of a finite group G, we mean a group homomorphism

$$\rho\colon G \to \mathrm{GL}(n, \mathbb{C}).$$

The number n is called the *dimension* (or the *degree*) of the representation ρ. If $n = 1$, we sometimes call ρ a *character* of G.

- For a complex torus T, we write

$$\mathrm{Bihol}(T) := \{g\colon T \to T \,|\, g \text{ is biholomorphic}\}$$

for the group of biholomorphic self-maps of T, and

$$\mathrm{Aut}(T) := \{f \in \mathrm{Bihol}(T) \,|\, f(0) = 0\}$$

for the (group) automorphisms of T.

- Whenever we write an elliptic curve E in the form $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, we assume that it is given in its standard form, i.e., τ is a complex number with positive imaginary part.
- We usually consider a group G as an abstract group. By writing "*$X = T/G$ is a hyperelliptic manifold with group G*", we mean that there exists an embedding $i\colon G \hookrightarrow \mathrm{Bihol}(T)$ such that $i(G)$ contains no translations and acts freely on T. Similarly, by writing "*G is the group associated with a hyperelliptic manifold*" or "*G is the group of a hyperelliptic manifold*", we mean that there exists a comples torus T such that $X = T/G$ is a hyperelliptic manifold with group G.

- Let $f\colon T \to T'$ be a group homomorphism of complex tori. Then the image

$$\operatorname{im}(f) \subset T'$$

is a complex subtorus of T'. Since the kernel of f is in general disconnected, we have to take the connected component of 0 of this kernel to obtain a complex subtorus of T, indicated by a superscript zero:

$$\ker(f)^0 \subset T.$$

- If N and H are finite groups, the notation $G = N \rtimes H$ will mean implicitly, that G is not the direct product of N and H. Moreover, the symbol "$\rtimes$" is often ambiguous, since usually, there are several non-equivalent actions of H on N. However, we sometimes do not specify this action, but give the ID of the group G in GAP's Database of Small Group [GAP] (coinciding with the MAGMA's database [MAGMA]).

Part I

Rigid Group Actions on Complex Tori are Projective (after Ekedahl)

Chapter 1

Introduction

This Part of the thesis is the content of the article [CD18] (accepted for publication in Commun. Contemp. Math.), which is coauthored with Fabrizio Catanese (and Benoît Claudon in the appendix).
The work of Kodaira [Kod54] [Kod60] lead to the question whether any compact Kähler manifold enjoys the property of admitting arbitrarily small deformations which are projective (Kodaira settled in [Kod60] the case of surfaces).

Motivated by Kodaira's problem (see the final section and the appendix) the first author asked Torsten Ekedahl at an Oberwolfach conference around 1999 if there exists a rigid group action of a finite group $G \subset \mathrm{Bihol}(T)$ on a complex torus T (see section 2 for definitions regarding deformations of group actions) which is not projective. T. Ekedahl answered this question and sketched a strategy of proof for the statement that the rigidity of the action (T, G) implies that T is projective (i.e., T is an Abelian variety).

Later Claire Voisin gave a counterexample to the general Kodaira problem showing in [V04] the existence of a rigid compact Kähler manifold which is not projective (and later in [V06] she even gave counterexamples which are not bimeromorphic to a projective manifold). Kodaira's property still remains a very interesting theme of research: understanding which compact Kähler manifolds or Kähler spaces with klt singularities satisfy Kodaira's property (see [Graf17] for quite recent progress).

On the other hand Ekedahl's approach allows a rather explicit description of rigid actions on complex tori in terms of orders in CM-fields, hence providing explicitly given polarizations on them. Therefore his result turned out to be quite interesting and useful for other purposes (see [Dem16] for applications to the classification theory of quotient manifolds of complex tori), and for this reason we find it important to publish here a complete proof.

Theorem 1.1 (Ekedahl)**.** *Let (T, G) be a rigid group action of a finite group $G \subset$ $\mathrm{Bihol}(T)$ on a complex torus T. Then T (or, equivalently, T/G) is projective. Moreover, if we write $T = V^{1,0}/\Lambda$, then*

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_j W_j^{n_j},$$

where W_j is a Hodge structure on a CM-field F_j and where $\oplus_j F_j$ is a subalgebra of the center of the group algebra $\mathbb{Q}[G]$.

The contents of the paper are as follows.

In Section 2, we briefly discuss deformations of group actions on complex manifolds.

Then, in the subsequent Section 3, we develop the tools used in the proof of Theorem 1.1, mainly based on Hodge theory and representation theory.

The main ideas of the proof are the following: if $\mathcal{A}$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra, the rigidity of the action of $\mathcal{A}$ (cf. Definition 3.2) on a rational Hodge structure V of weight 1 can be determined by looking at the simple summands of $\mathcal{A}\otimes_{\mathbb{Q}}\mathbb{C}$ appearing in $V^{1,0}$, respectively in $V^{0,1}$. A second ingredient is that, for $\mathcal{A} = \mathbb{Q}[G]$ with G finite (and also in a more general situation), we show that rigidity is equivalent to having a rigid action of the commutative subalgebra given by the center $Z(\mathbb{Q}[G])$.

Then we apply Proposition 4.5, stating that, if $\mathcal{A} = Z(\mathbb{Q}[G])$ is the center of the group algebra and the action of $\mathcal{A}$ on V is rigid, then the Hodge structure V is polarizable.

Finally, in the Appendix, we show that every group action (T, G) on a complex torus admits arbitrarily small deformations which are projective.

Chapter 2

Deformations of group actions

Let X be a compact complex manifold. Let $G \subset \mathrm{Bihol}(X)$ be a finite group, and denote by $\alpha\colon G \times X \to X$ the corresponding group action of G on X.

Definition 2.1. 1) A *deformation* (p, α') of the group action α of G on X consists of a deformation $p\colon (\mathfrak{X}, \mathfrak{X}_0) \to (B, t_0)$ of X (i.e., $\mathfrak{X}_0 := p^{-1}(t_0)$ and $X \cong \mathfrak{X}_0$) given together with $\alpha'\colon G \times \mathfrak{X} \to \mathfrak{X}$, a holomorphic group action commuting with p (here we let G act trivially on the base), such that the action on $\mathfrak{X}_0 \cong X$ induces the initially given action α.

2) A deformation (p, α') is said to be *trivial* if its germ is isomorphic to the trivial deformation $X \times B \to B$, endowed with the action $\alpha \times \mathrm{id}_B$.

3) The action α is said to be *rigid* if every deformation of α is trivial.

Kuranishi theory leads to an easy characterization of rigidity of an action α of a group G on X, see [Cat88, p. 23], [Cat11, Ch. 4], [Li17].

Denote by $p\colon \mathfrak{X} \to \mathrm{Def}(X)$ the Kuranishi family of X; then this characterization is related to the question: which condition on $t \in \mathrm{Def}(X)$ guarantees that G is a subgroup of $\mathrm{Aut}(\mathfrak{X}_t)$? It turns out (cf. [Cat88, p. 23]) that $G \subset \mathrm{Bihol}(\mathfrak{X}_t)$ if and only if $g_* t = t$ for any $g \in G$, so that $t \in \mathrm{Def}(X) \cap H^1(X, \Theta_X)^G$.

We then have (see proposition 4.5 of [Cat11]):

Proposition 2.2. *Set* $\mathrm{Def}(X)^G := \mathrm{Def}(X) \cap H^1(X, \Theta_X)^G$. *The group action α of G on X is rigid if and only if* $\mathrm{Def}(X)^G = 0$ *(as a set). A fortiori the action is rigid if* $H^1(X, \Theta_X)^G = 0$ *(in this latter case we say that the action is infinitesimally rigid).*

In the upcoming chapter we shall consider the case where $X = T$ is a complex torus: the rigidity of (T, G), amounting to the fact that the representation of G on $H^1(X, \Theta_X)$ contains no trivial summand, can then be read off explicitly from the action of G on the tangent bundle.

Chapter 3

Rigid actions on rational Hodge structures

Denote by $\mathcal{H}^1$ the category of rational Hodge structures of type $((1,0),(0,1))$. An object of $\mathcal{H}^1$ is a finite-dimensional $\mathbb{Q}$-vector space V endowed with a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = U \oplus \overline{U} =: V^{1,0} \oplus V^{0,1}.$$

The elements of $\mathcal{H}^1$ can be viewed as isogeny classes of complex tori

$$T := (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})/(\Lambda \oplus V^{0,1}),$$

where $\Lambda \subset V$ is an *order*, i.e. a free subgroup of maximal rank (by abuse of notation we shall also say that Λ is a lattice in V, observe that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$).

We have isogeny classes of Abelian varieties when a rational Hodge structure is polarizable, according to the following

Definition 3.1. Let $V \in \mathcal{H}^1$ and write for short $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$.

A *polarization on* V is an alternating form $E\colon V \times V \to \mathbb{Q}$ satisfying the two Hodge-Riemann Bilinear Relations:

i) The complexification $E_{\mathbb{C}}\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ satisfies $E_{\mathbb{C}}(V^{1,0}, V^{1,0}) = 0$ (hence also $E_{\mathbb{C}}(V^{0,1}, V^{0,1}) = 0$)

ii) For any non-zero vector $v \in V^{1,0}$, we have $-\, i \cdot E_{\mathbb{C}}(v, \overline{v}) > 0$

Equivalently, setting $E_{\mathbb{R}}\colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$, we have:

I) $E_{\mathbb{R}}(Jx, Jy) = E_{\mathbb{R}}(x, y)$

II) the symmetric bilinear form $E_{\mathbb{R}}(x, Jy)$ is positive definite.

Here, if $x = u + \bar{u}$, $Jx := iu - i\bar{u}$ ($J^2 = -Id$).

Let $\mathcal{A}$ be a semisimple and finite-dimensional $\mathbb{Q}$-algebra (for example the group algebra $\mathcal{A} = \mathbb{Q}[G]$ for a finite group G). We denote an action $r\colon \mathcal{A} \to \mathrm{End}_{\mathcal{H}^1}(V)$ for $V \in \mathcal{H}^1$ by a triple $(V, \mathcal{A}, r)$.

If $\Lambda \subset V$ is a lattice and $T = (V \otimes_{\mathbb{Q}} \mathbb{C})/(\Lambda \oplus V^{0,1})$ is the corresponding complex torus then $\mathcal{A}$ maps to $\mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 3.2. An action $(V, \mathcal{A}, r)$ is called *rigid*, if

$$\mathrm{Hom}_{\mathcal{A}}(V^{0,1}, V^{1,0}) = 0. \tag{3.1}$$

Rigidity 3.1 means, in view of what we saw in the previous section, and in view of

$$H^1(\Theta_T) = H^1(\mathcal{O}_T) \otimes_{\mathbb{C}} H^0(\Omega^1_T)^\vee = \overline{U}^\vee \otimes_{\mathbb{C}} U = \mathrm{Hom}_{\mathcal{A}}(V^{0,1}, V^{1,0}),$$

that there are no deformations of T preserving the $\mathcal{A}$-action.

We consider now some examples of the above notion.

Example 3.3. Let $\mathcal{A}$ be a totally imaginary number field F. This means that $[F : \mathbb{Q}] = 2k$ and F possesses $2k$ different embeddings $\sigma_j : F \to \mathbb{C}$, none of which is *real* (this means: $\sigma_j(F) \subset \mathbb{R}$).

Hence each σ_j is different from the complex conjugate, $\sigma_j \neq \overline{\sigma_j}$, and if we set $V := F$, with the obvious action of F, all the Hodge structures on V are rigid and correspond to the finite set of partitions of the set $\mathcal{E}$ of embeddings of F into two conjugate sets $\{\sigma_1, \dots, \sigma_k\}$ and $\{\overline{\sigma_1}, \dots, \overline{\sigma_k}\}$.

Since the F-module $F \otimes_{\mathbb{Q}} \mathbb{C}$ is the direct sum

$$F \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{\sigma_j \in \mathcal{E}} \, \mathbb{C}_{\sigma_j},$$

where $\mathbb{C}_{\sigma_j}$ is the vector space $\mathbb{C}$ with left action of F given by:

$$x \cdot z := \sigma_j(x) \cdot z, \ \forall x \in F, z \in \mathbb{C},$$

and choosing such a partition amounts to choosing $V^{1,0} := \oplus_{j=1,\dots k} \mathbb{C}_{\sigma_j}$.

A particular case is given by the class of CM-fields.

Example 3.4. Recall that a *CM-field* is a totally imaginary quadratic extension F of a totally real number field K.

Equivalently, (cf. [Shi71, Proposition 5.11]) F is a CM-field if it carries a non-trivial involution ρ such that $\sigma \circ \rho = \overline{\sigma}$ for all embeddings $\sigma \colon F \hookrightarrow \mathbb{C}$. In particular F is totally imaginary.

In this case any Hodge structure on $V := F$ is polarizable.

Let indeed $\sigma_1, ..., \sigma_k \colon F \hookrightarrow \mathbb{C}$ be the embeddings of F occurring in $V^{1,0}$. Following [Shi71, p. 128] choose $\zeta \in F$ satisfying the following conditions:

a) ζ is imaginary, i.e., $\rho(\zeta) = -\zeta$,

b) $\sigma_j(\zeta)$ is imaginary with positive imaginary part for each $j = 1, ..., k$.

A polarization on V of F is then given, if we set $x_j := \sigma_j(x), y_j := \sigma_j(y)$, by the skew symmetric form (we set here $\sigma_{k+j} := \overline{\sigma_j}$)

$$E(x, y) := tr_{F/\mathbb{Q}}(\zeta x \rho(y)) = \sum_{j=1}^{2k} \sigma_j(\zeta) x_j \overline{y_j} = \sum_{j=1}^{k} \sigma_j(\zeta)(x_j \overline{y_j} - \overline{x_j} y_j).$$

In fact, the first Riemann bilinear relation amounts to $E(Jx, Jy) = E(x, y)$, which is clearly satisfied, since $(Jx)_j = ix_j$, for $j = 1, \dots, k$, and the real part of the associated Hermitian form is the symmetric form

$$E(x, Jy) = \sum_{j=1}^{k} (-i)\sigma_j(\zeta)(x_j\overline{y_j} + \overline{x_j}y_j),$$

which is positive definite since

$$E(x, Jx) = \sum_{j=1}^{k} 2 \ \mathrm{Im}(\sigma_j(\zeta))|x_j|^2 > 0$$

for $x \neq 0$.

Let us now proceed towards the proof of the main theorem.

An important step towards the main Theorem is that in the case where

$$\mathcal{A} = \mathbb{Q}[G] \tag{3.2}$$

rigidity can be reduced to rigidity of the action restricted to the centre of the group algebra.

Proposition 3.5. *Let $\mathcal{A} = \mathbb{Q}[G]$ be the group algebra of a finite group G over the rationals.*

Then the triple $(V, \mathcal{A}, r)$ is rigid if and only if $(V, Z(\mathcal{A}), r')$ is rigid, where $Z(\mathcal{A})$ is the centre of $\mathcal{A}$ and r' is the restriction of r to $Z(\mathcal{A})$.

Proof. For each field K, $\mathbb{Q} \subset K \subset \mathbb{C}$, $\mathcal{A} \otimes_{\mathbb{Q}} K = K[G]$ has as center $Z_K := Z(K[G])$, the vector space with basis $v_{\mathcal{C}}$, indexed by the conjugacy classes $\mathcal{C}$ of G, and where

$$v_{\mathcal{C}} := \sum_{g \in \mathcal{C}} g.$$

For $K = \mathbb{C}$, another more useful basis is indexed by the irreducible complex representations W_χ of G, and their characters χ (these form an orthonormal basis for the space of class functions, i.e. the space $Z_{\mathbb{C}}$ if we identify the element g to its characteristic function).

For each irreducible χ, the element

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) \cdot g \in \mathbb{C}[G]$$

is an idempotent in $Z(\mathbb{C}[G])$. Indeed, we even have that

$$Z(\mathbb{C}[G]) = \bigoplus_{\chi} \mathbb{C} \cdot e_\chi,$$

and the idempotents e_χ satisfy the orthogonality relations $e_{\chi'} \cdot e_\chi = 0$ for $\chi \neq \chi'$.

This leads directly to the decomposition

$$\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}[G] = \bigoplus_{\chi \in Irr} A_\chi, \;\; A_\chi := e_\chi \mathbb{C}[G] \cong \mathrm{End}(W_\chi),$$

where χ runs over all irreducible characters of G, and to the semisimplicity of the group algebra. Notice that e_χ acts as the identity on W_χ, and as 0 on $W_{\chi'}$ for $\chi' \neq \chi$.

In fact, we shall prove the stronger statement that for any two finitely generated $\mathbb{C}[G]$-modules M and N (note that $\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}[G]$)

$$\mathrm{Hom}_{\mathbb{C}[G]}(M, N) = 0 \iff \mathrm{Hom}_{Z(\mathbb{C}[G])}(M, N) = 0.$$

The right hand side $\mathrm{Hom}_{Z(\mathcal{A}\otimes_{\mathbb{Q}}\mathbb{C})}(M, N)$ clearly contains the left hand side.

By semisimplicity, each representation M splits as a direct sum of irreducible representations,

$$M = \sum_{\chi \in Irr} M_\chi, M_\chi = W_\chi \otimes_{\mathbb{C}[G]} (\mathbb{C}^r),$$

where $\mathbb{C}^r$ is a trivial representation of G.

By bilinearity we may assume that $M = W_\chi$ and $N = W_{\chi'}$ are simple modules associated to irreducible characters χ, χ' of G.

Then, by the Lemma of Schur, the left hand side $\mathrm{Hom}_{\mathcal{A}\otimes_{\mathbb{Q}}\mathbb{C}}(M, N)$ is $= 0$ for $\chi' \neq \chi$, and isomorphic to $\mathbb{C}$ for $\chi' = \chi$.

For the right hand side, it suffices to prove that $\mathrm{Hom}_{Z(\mathcal{A}\otimes_{\mathbb{Q}}\mathbb{C})}(M, N) = 0$ for $\chi' \neq \chi$, when $M = W_\chi, N = W_{\chi'}$.

However, e_χ acts as the identity on M and as zero on N, hence $\psi \in \mathrm{Hom}_{Z(\mathcal{A}\otimes_{\mathbb{Q}}\mathbb{C})}(M, N)$ implies

$$\psi(v) = \psi(e_\chi v) = e_\chi(\psi(v)) = 0,$$

as we wanted to show.

This shows the statement. □

We have more generally:

Proposition 3.6. *Let $\mathcal{A}$ be a semisimple $\mathbb{Q}$-algebra of finite dimension, and let $(V, \mathcal{A}, r)$ be an action on a rational Hodge structure V, Then r is rigid if and only if $(V, Z(\mathcal{A}), r')$ is rigid; here $Z(\mathcal{A})$ is the center of $\mathcal{A}$ and r' is the restriction of r.*

Proof. More generally, if M, N are $\mathcal{A} \otimes \mathbb{C}$-modules, then we claim that

$$\mathrm{Hom}_{\mathcal{A}\otimes\mathbb{C}}(M, N) = 0 \Leftrightarrow \mathrm{Hom}_{Z(\mathcal{A}\otimes\mathbb{C})}(M, N) = 0.$$

By bilinearity of both sides, and by semisimplicity (each module splits as a direct sum of irreducibles) we can assume that M, N are simple modules and that $\mathcal{A}$ is a simple algebra.

By Schur's Lemma the left hand side is non zero exactly when M and N are isomorphic. The left hand side is contained in the right hand side, so it suffices to show that the right hand side is nonzero exactly when M and N are isomorphic. But ([Ja80-II], Lemma 1, page 205) any two irreducible modules over a simple Artininian ring are isomorphic. □

Remark 3.7. We have $\mathbb{C}[G] = \bigoplus_\chi \mathbb{C}[G] \cdot e_\chi$.

Working instead over a field K of characteristic 0, an algebraic extension of $\mathbb{Q}$ (so $\mathbb{Q} \subset K \subset \mathbb{C}$), the decomposition of $K[G]$ into simple summands is (see [Y74], Proposition 1.1) again provided by central idempotents in $K[G]$,

$$K[G] = \bigoplus_{[\chi]} K[G]e_K(\chi), \quad e_K(\chi) := \sum_{\chi^\sigma \in [\chi]} e_{\chi^\sigma},$$

where the first sum runs over the set of Γ-orbits $[\chi]$ in the set all irreducible characters χ of G; here Γ is the Galois group $\mathrm{Gal}(K(\chi)/K)$ of the field extension $K(\chi)$ of K, generated by the values of all the characters χ, i.e., by $\{\chi(g) \mid g \in G, \chi \in \mathrm{Irr}(G)\}$.

And the centre of $K[G]$ is a direct sum of fields

$$Z(K[G]) = \bigoplus_{[\chi]} F_{[\chi]},$$

where the field $F_{[\chi]}$ is the centre (for the last isomorphism, see [Y74], Proposition 1.4)

$$F_{[\chi]} := Z(K[G])e_K(\chi) \cong K(\{\chi(g)|g \in G\})$$

of the algebra $K[G]e_K(\chi)$, and enjoys the property that $F_{[\chi]} \otimes_K \mathbb{C} = \bigoplus_{\chi \in [\chi]} \mathbb{C} e_{\chi^\sigma}$.

The next lemma explains the relation occurring between finite groups and CM-fields.

Lemma 3.8. *The center of the group algebra $Z(\mathbb{Q}[G])$ splits as a direct sum of number fields, $Z(\mathbb{Q}[G]) = F_1 \oplus ... \oplus F_l$ which are either totally real, or CM-fields.*

Proof. Write $m := |G|$, let ζ_m be a primitive m-th root of unity and let d be the number of conjugacy classes in G, which equals the number of irreducible representations of G. Then

$$F_j \subset Z(\mathbb{Q}[G]) \subset Z(\mathbb{Q}(\zeta_m)[G]) \cong_{\mathbb{Q}-\mathrm{alg.}} \mathbb{Q}(\zeta_m)^d,$$

where we used in the last isomorphism that every complex representation of G is defined over $\mathbb{Q}(\zeta_m)$. Hence F_j embeds into the cyclotomic field $\mathbb{Q}(\zeta_m)$. The extension $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ is Galois with group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$ (the isomorphism maps $\varphi_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, such that $\varphi_a(\zeta_m) = \zeta_m^a$, to $a \in (\mathbb{Z}/m\mathbb{Z})^*$), so by the Main Theorem of Galois Theory, there is a subgroup H of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, such that $F_j \cong \mathbb{Q}(\zeta_m)^H$ (the subfield of $\mathbb{Q}(\zeta_m)$ fixed by the action of H). If $-1 \in H$ (which corresponds to φ_{-1}, the complex conjugation), the field F_j is totally real, otherwise F_j is a CM-field. $\square$

Chapter 4

Proof of Ekedahl's Theorem

Fix now an action $(V, \mathcal{A}, r)$ and assume that

$$\mathcal{A} \text{ is commutative.} \tag{4.1}$$

Since $\mathcal{A}$ is commutative, $\mathcal{A}$ is a direct sum of number fields,

$$\mathcal{A} = F_1 \oplus ... \oplus F_l.$$

Assume that we have a homomorphism of algebras $\sigma : \mathcal{A} \to \mathbb{C}$. For each idempotent e of $\mathcal{A}$, $\sigma(e)$ is an idempotent of $\mathbb{C}$, hence $\sigma(e) = 1$ or $\sigma(e) = 0$. In $\mathcal{A}$, the identity element 1 is a sum of idempotents

$$1 = 1_{F_1} + \cdots + 1_{F_l},$$

and if $\sigma \neq 0$, then $\sigma(1) = 1$. This implies that for such a homomorphism σ there is exactly one $j \in \{1, \ldots, l\}$, such that $\sigma(1_{F_j}) = 1$, and, for $i \neq j$, we have $\sigma(1_{F_i}) = 0$.

Let then $\mathcal{C} = \{\sigma_1, ..., \sigma_k\}$ be the set of all the distinct $\mathbb{Q}$-algebra homomorphisms $\mathcal{A} \to \mathbb{C}$: then these homomorphisms $\sigma_j : \mathcal{A} \to \mathbb{C}$ are obtained as the composition of one of the projections $\mathcal{A} \to F_h$ with an embedding $F_h \hookrightarrow \mathbb{C}$ (hence $k = \sum_h [F_h : \mathbb{Q}] = \dim_{\mathbb{Q}} \mathcal{A}$).

Define now (as in Example 3.3) the $\mathcal{A}$-module $\mathbb{C}_{\sigma_j}$ as the vector space $\mathbb{C}$ endowed with the action of $\mathcal{A}$ such that

$$x \cdot z := \sigma_j(x) \cdot z.$$

Hence we have a splitting of $\mathcal{A}$-modules

$$\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{j=1}^{l} (F_j \otimes_{\mathbb{Q}} \mathbb{C}) = \bigoplus_{j=1}^{k} \mathbb{C}_{\sigma_j}.$$

We now show that we have a splitting in the category of rational Hodge structures

$$V = V_1 \oplus ... \oplus V_l,$$

where V_i is an F_i-module, and an $\mathcal{A}$-module via the surjection $\mathcal{A} \to F_i$.

We simply define $V_j := 1_{F_j} \cdot V$. We have a splitting of modules

$$V = V_1 \oplus ... \oplus V_l,$$

since for $i \neq j$, $1_{F_i} 1_{F_j} = 0$, and

$$v = 1 \cdot v = (1_{F_1} + \cdots + 1_{F_l})v =: v_1 + \cdots + v_l.$$

It is a splitting in the category of rational Hodge structures because each element of $\mathcal{A}$ preserves the Hodge decomposition, hence V_j is a sub-Hodge structure of V.

Therefore the action r is a direct sum of actions

$$r_j : F_j \to \mathrm{End}_{\mathcal{H}^1}(V_j)$$

Each r_j induces, by tensor product, a homomorphism of rings

$$F_j \otimes_{\mathbb{Q}} \mathbb{C} \to \mathrm{End}(V_j \otimes_{\mathbb{Q}} \mathbb{C}) = \mathrm{End}(V_j^{1,0} \oplus V_j^{0,1}),$$

and a splitting of $\mathcal{A}$-modules

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1} = \bigoplus_{j=1}^{k} (V_{\sigma_j}^{1,0} \oplus V_{\sigma_j}^{0,1})$$

where V_{σ_j} is the character subspace on which $\mathcal{A}$ acts via $x \cdot v := \sigma_j(x) \cdot v$. This holds for the following reason: each V_j is an F_j module; and since F_j is a number field, then $F_j = \mathbb{Q}[x]/P(x)$, where P is irreducible, and $r_j(x)$ is an endomorphism a_j of V_j with minimal polynomial P (a polynomial with distinct roots). In particular, a_j is diagonalizable over $V_j \otimes_{\mathbb{Q}} \mathbb{C}$, and each diagonal entry yields some embedding σ_h of F_j into $\mathbb{C}$.

Remark 4.1. The rigidity of $(V, \mathcal{A}, r)$ is equivalent to the fact that for each $\sigma_j \in \mathcal{C}$ either $V_{\sigma_j}^{1,0}$ or $V_{\sigma_j}^{0,1}$ is zero, in particular, since $\overline{V_{\sigma_j}^{1,0}} = V_{\overline{\sigma_j}}^{0,1}$, no real σ_j appears either in $V^{1,0}$ or in $V^{0,1}$.

Following a terminology similar to the one introduced in [Cat14], we define the notion of Hodge-type.

Definition 4.2. Define the *Hodge-type* of an action of $\mathcal{A}$ by the function $\tau_V : \mathcal{C} \to \mathbb{N}$, such that

$$\tau_V(\sigma) := \dim_{\mathbb{C}} V_{\sigma}^{1,0}.$$

Hodge symmetry translates into

$$(HS)\ \tau_V(\sigma) + \tau_V(\bar{\sigma}) = \dim_{\mathbb{C}} V_{\sigma},$$

which implies in particular that if we have a real embedding, i.e. $\sigma = \overline{\sigma}$, then $\tau_V(\sigma) = \frac{1}{2} \dim_{\mathbb{C}} V_{\sigma}$.

Moreover, if Hodge symmetry holds, the action is rigid if and only if

$$(R)\ \tau_V(\sigma) \cdot \tau_V(\bar{\sigma}) = 0, \quad \forall \sigma.$$

Proposition 4.3. *If $(V, \mathcal{A}, r)$ is rigid, then it is determined by the $\mathcal{A}$-module V and by the Hodge-type.*

Conversely, if V is an $\mathcal{A}$-module, and there is a Hodge structure such that

$$(HS)\ \tau_V(j) + \tau_V(\bar{j}) = \dim_{\mathbb{C}} V_{\sigma_j},$$

whenever $\sigma_{\bar{j}} = \overline{\sigma_j}$, and moreover

$$(R)\ \tau_V(j) \cdot \tau_V(\bar{j}) = 0 \ \ \forall j,$$

then this Hodge structure determines a rigid action $(V, \mathcal{A}, r)$.

Proof. In one direction, the Hodge-type determines $V^{0,1}, V^{1,0}$, since, $\mathcal{A}$ being commutative, V splits into character spaces V_{σ_j}, and the function τ_V determines whether $V_{\sigma_j} \subset V^{0,1}$, or $V_{\sigma_j} \subset V^{1,0}$.

In the other direction, the given Hodge structure is preserved by the action of $\mathcal{A}$ hence we have an action in the category of rational Hodge structures. □

Lemma 4.4. *Assume that we have a rigid action $(V, \mathcal{A}, r)$ of split type, where*

$$\mathcal{A} = F_1 \oplus ... \oplus F_l$$

is commutative and each F_i is a field.

i) If $l = 1$ (so $\mathcal{A} =: F$ is a field), $V \cong W^n$ in $\mathcal{H}^1$, where W is a Hodge structure on F.

ii) the rational Hodge structure V splits as a direct sum

$$V = W_1^{n_i} \oplus ... \oplus W_l^{n_l},$$

where W_j is a Hodge structure on F_j and $n_j \geq 0$.

Proof. Assertion i): here V is an F-vector space, and so $f \colon V \xrightarrow{\sim} F^n$ as vector spaces. As we observed the rigidity of (V, F, r) implies that all embeddings of F into $\mathbb{C}$ appear in either $V^{1,0}$ or $V^{0,1}$, hence F has no real ones. Let $\sigma_1, ..., \sigma_d$ be the embeddings of F appearing in $V^{1,0}$, so that $\overline{\sigma_1}, ..., \overline{\sigma_d}$ are the ones appearing in $V^{0,1}$. Define a Hodge structure W on F according to the type of V, i.e. as follows:

$$W \otimes_{\mathbb{Q}} \mathbb{C} = W^{1,0} \oplus W^{0,1}, \text{ where } W^{1,0} = \bigoplus_{j=1}^{d} \mathbb{C}_{\sigma_j}, \ \ W^{0,1} = \bigoplus_{j=1}^{d} \mathbb{C}_{\overline{\sigma_j}},$$

Then $f_{\mathbb{C}} \colon V \otimes_{\mathbb{Q}} \mathbb{C} \to (W \otimes_{\mathbb{Q}} \mathbb{C})^n$ is an isomorphism of $\mathbb{C}$-vector spaces together with an F-action.

Assertion ii) follows immediately from assertion i), since we have the splittings $\mathcal{A} = F_1 \oplus ... \oplus F_l$ and $V = V_1 \oplus \cdots \oplus V_l$, and the $\mathcal{A}$-rigidity of V implies the F_j-rigidity of V_j for all $j = 1, \ldots, l$, hence we can apply step i) to each V_j. □

The crucial Proposition from which the proof of Theorem 1.1 follows is now

Proposition 4.5. *If $(V, \mathbb{Q}[G], r)$ is rigid, then V polarizable.*

Proof. First of all, if $(V, \mathbb{Q}[G], r)$ is rigid, then $(V, Z(\mathbb{Q}[G]), r)$ is rigid by Proposition 3.5. The assumption that $(V, Z(\mathbb{Q}[G]), r)$ is rigid implies now that if some field F_j does not act as 0 on V, then F_j is necessarily a CM-field by Lemma 3.8 and the previous remarks. By Lemma 4.4, the rational Hodge structure V splits as a direct sum $W_1^{n_i} \oplus ... \oplus W_l^{n_l}$, where W_j is a Hodge structure on F_j and $n_j \geq 0$.
To give a polarization on V, it therefore suffices to show the existence of a polarization for a Hodge structure W_j on a CM-field F_j. But this was shown in Example 3.4. □

Ekedahl's Theorem is therefore proven.

Chapter 5

Final remarks

Assume that $X := T$ is a complex torus of dimension ≥ 3, and that $Y = T/G$ has only isolated singularities.

Schlessinger showed in [Sch71, Theorem 3] that every deformation of the analytic germ of Y at each singular point of Y is trivial.

Hence for every deformation $\mathcal{Y} \to B$ of Y (we write informally $\mathcal{Y}$ as $\{Y_t\}_{t\in B}$) Y_t has the same singularities as Y, and in particular it follows easily that $Y_t \setminus \mathrm{Sing}(Y_t)$ and $Y \setminus \mathrm{Sing}(Y)$ are diffeomorphic and a fortiori one has an isomorphism

$$\pi_1(Y_t \setminus \mathrm{Sing}(Y_t)) \cong \pi_1(Y \setminus \mathrm{Sing}(Y)) \cong \pi_1(\mathcal{Y} \setminus \mathrm{Sing}(\mathcal{Y})).$$

Therefore the surjection $\pi_1(Y \setminus \mathrm{Sing}(Y)) \to G$ induces a surjection $\pi_1(\mathcal{Y} \setminus \mathrm{Sing}(\mathcal{Y})) \to G$.

Whence, by Grauert's and Remmert's extension of Riemann's Existence Theorem, cf. [GR58, Satz 32], Y_t and $\mathcal{Y}$ have respective Galois covers X_t and $\mathcal{X}$ with group G. Hence, the action of G extends to the family $\mathcal{X}$, and each deformation of Y yields a deformation of the pair (T, G).

The conclusion is that Y is rigid if and only if the action of G on T is rigid. On the other hand, Ekedahl's theorem implies then that if Y is rigid, then Y is projective.

Therefore in this case one cannot get a counterexample to the Kodaira property via rigidity. We show more generally in the appendix that any such a quotient $Y = T/G$ with only isolated singularities satisfies the Kodaira property, since any action can be approximated by a projective one.

An interesting question is: in the case where Y is rigid, is it true that a minimal resolution of Y is also rigid?

Chapter 6

Appendix (with F. Catanese and B. Claudon)

Ekedahl's theorem has the advantage of elucidating the structure of (rigid and non rigid) actions of a finite group G on a complex torus.

The method of period mappings, used by Green and Voisin (see proposition 17.20 and Lemma 17.21 of [V02]) for showing the density of algebraic tori (non constructive, since it uses the implicit functions theorem), was used by Graf in [Graf17] to obtain a general criterion, from which follows the following theorem.

Theorem 6.1. *Let (T, G) be a group action on a complex torus. Then there are arbitrarily small deformations (T_t, G) of the action where T_t is projective.*

Proof. Given a complex torus

$$T := (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})/(\Lambda \oplus V^{1,0}),$$

set, as in section 2,

$$V \otimes_{\mathbb{Q}} \mathbb{C} = U \oplus \overline{U} =: V^{1,0} \oplus V^{0,1}.$$

The Teichmüller space of T is an open set $\mathcal{T}$ in the Grassmann variety $Gr(n, V \otimes_{\mathbb{Q}} \mathbb{C})$,

$$\mathcal{T} = \{U_t \,|\, U_t \oplus \overline{U_t} = V \otimes_{\mathbb{Q}} \mathbb{C}\},$$

parametrizing Hodge structures. By abuse of notation we shall use the notation $t \in \mathcal{T}$ for the points of Teichmüller space.

The deformations of the pair (T, G) are parametrized by the submanifold $\mathcal{T}^G$ of the fixed points for the action of G, which correspond to the set of the subspaces U_t which are G-invariant.

The tangent space to $\mathcal{T}^G$ at the point (T, G) is, as seen in section 2, the subspace

$$H^1(\Theta_T)^G \subset H^1(\Theta_T) = H^1(\mathcal{O}_T) \otimes_{\mathbb{C}} H^0(\Omega^1_T)^\vee = \overline{U}^\vee \otimes_{\mathbb{C}} U.$$

Over $\mathcal{T}^G$ we have the Hodge bundle

$$F^1 \subset \mathcal{T}^G \times \bigwedge^2 (V \otimes_{\mathbb{Q}} \mathbb{C})^\vee \ \text{ such that } \ F^1_t = H^{1,1}(T_t) \oplus H^{2,0}(T_t).$$

Since the family of complex tori is differentiably trivial there is a canonical isomorphism

$$\bigwedge^2 (V \otimes_{\mathbb{Q}} \mathbb{C})^\vee = H^2(T, \mathbb{C}) \cong H^2(T_t, \mathbb{C}).$$

This allows to define a holomorphic mapping $\psi : F^1 \to H^2(T, \mathbb{C})$ induced by the second projection.

We can indeed consider the subbundle (defined over $\mathcal{T}^G$)

$$(F^1)^G \subset \mathcal{T}^G \times H^2(T, \mathbb{C})^G \quad s.t. \quad (F^1)^G_t = H^{1,1}(T_t)^G \oplus H^{2,0}(T_t)^G,$$

and the corresponding holomorphic mapping $\phi : (F^1)^G \to H^2(T, \mathbb{C})^G$ induced by the second projection.

Step 1: Let η be a Kähler metric on T. By averaging, we replace η by $\sum_g g^*(\eta)$ and we can assume that η is G-invariant.

Let $\omega \in H^{1,1}(T) \cap H^2(T_t, \mathbb{R})^G$ be the corresponding Kähler class.

Step 2: Setting $T =: T_0$, the map ϕ is a submersion at the point $(0, \omega)$.

Before proving step 2, let us see how the theorem follows.

Let $\mathcal{D}$ be a sufficiently small neighbourhood of ω inside

$$H^2(T, \mathbb{C})^G = H^2(T, \mathbb{Q})^G \otimes_{\mathbb{Q}} \mathbb{C}.$$

For each class $\xi \in H^2(T, \mathbb{Q})^G \cap \mathcal{D}$, there is therefore a (t, ξ) in a small neighbourhood $\mathcal{D}'$ of $(0, \omega)$ such that

$$\xi \in (F^1)^G_t = H^{1,1}(T_t)^G \oplus H^{2,0}(T_t)^G.$$

Since ξ is real, $\xi \in H^{1,1}(T_t)^G \cap H^2(T, \mathbb{Q})^G$. Taking $\mathcal{D}$ sufficiently small, the class ξ is also positive definite, hence ξ is the class of a polarization on T_t.

Shrinking $\mathcal{D}$ and $\mathcal{D}'$, we obtain that $t \in \mathcal{T}^G$ tends to 0 (the point corresponding to the torus T). Hence the assertion of the theorem is proven.

Proof of Step 2.

The tangent space to $(F^1)^G$ at the point $(0, \omega)$ is the direct sum

$$H^1(\Theta_T)^G \oplus (F^1)^G_0 = H^1(\Theta_T)^G \oplus H^{1,1}(T)^G \oplus H^{2,0}(T)^G,$$

and the derivative of ϕ is the direct sum of $\cup\omega, \iota$, where ι is the inclusion $(F^1)^G_0 \subset H^2(T, \mathbb{C})^G$, while the cup product with $\omega \in$ yields a linear map

$$\beta \colon H^1(\Theta_T)^G \to H^2(T, \mathcal{O}_T)^G = H^{0,2}(T)^G \subset H^2(T, \mathbb{C})^G.$$

Whence ϕ is a submersion at $(0, \omega)$ if and only if β is surjective.

Now, β is surjective if the cup product with ω yields a surjection

$$\beta' \colon H^1(\Theta_T) \to H^2(T, \mathcal{O}_T)$$

(taking the subspace of G-invariants is an exact functor).

Observe that $H^2(T, \mathcal{O}_T) = \wedge^2(\overline{U}^\vee)$, while

$$H^{1,1}(T) = H^1(\Omega_T^1) = \overline{U}^\vee \otimes_{\mathbb{C}} U^\vee.$$

Cup product with ω is the composition of two linear maps

$$H^1(\Theta_T) \to H^2(\Theta_T \otimes_{\mathcal{O}_T} \Omega_T^1) \to H^2(T, \mathcal{O}_T),$$

where the second map is induced by contraction.

It can be also seen as the composition of three linear maps:

$$H^1(\Theta_T) = \overline{U}^\vee \otimes_{\mathbb{C}} U \to \left(\overline{U}^\vee \otimes_{\mathbb{C}} U\right) \otimes_{\mathbb{C}} \left(\overline{U}^\vee \otimes_{\mathbb{C}} U^\vee\right) \to \overline{U}^\vee \otimes_{\mathbb{C}} \overline{U}^\vee \to \bigwedge^2 \left(\overline{U}^\vee\right) = H^2(T, \mathcal{O}_T).$$

Since the last linear map is a surjection, it suffices to show that the composition of the first two maps yields a surjection

$$b \colon \overline{U}^\vee \otimes_{\mathbb{C}} U \to \overline{U}^\vee \otimes_{\mathbb{C}} \overline{U}^\vee.$$

Since ω is a Kähler class, there exists a basis u_i of U such that

$$\omega = \sum_i \overline{u_i^\vee} \otimes u_i^\vee.$$

Hence

$$\sum_{h,k} a_{h,k} \overline{u_h^\vee} \otimes u_k \to \sum_{h,k} a_{h,k} \overline{u_h^\vee} \otimes \overline{u_k^\vee}$$

and b is an isomorphism.

$\square$

Part II

The Classification of Hyperelliptic Threefolds

Chapter 1

Introduction

The aim of the current part (which is a joint work with Fabrizio Catanese, see [CD18-2]) is to complete the classification of generalized hyperelliptic manifolds of complex dimension three. The cases where the group G is Abelian were classified by H. Lange in [La01], using work of Fujiki [Fu88] and the classification of the possible groups G given by Uchida and Yoshihara in [UY76]: the latter authors showed that the only possible non-Abelian group is the dihedral group D_4 of order 8.

This case was first excluded but it was later found that it does indeed occur (see [CD18] for an account of the story and of the role of the paper [DHS08]). Our paper is fully self-contained and shows that the family described in [CD18] gives all the possible hyperelliptic threefolds with group D_4.

Our main theorem is the following

Theorem 1.1. *Let T be a complex torus of dimension 3 admitting a fixed point free action of the dihedral group*

$$G := D_4 := \langle r, s \,|\, r^4 = s^2 = (rs)^2 = 1 \rangle,$$

such that $G = D_4$ contains no translations.
Then T is algebraic. More precisely, there are two elliptic curves E, E' such that:

(I) T is a quotient $T := T'/H$, $H \cong C_2$, where

$$T' := E \times E \times E' =: E_1 \times E_2 \times E_3,$$

$$H := \langle \omega \rangle, \quad \omega := (h + k, h + k, 0) \in T'[2],$$

and h, k are 2-torsion elements in E, such that $h, k \neq 0$ and $h + k \neq 0$;

(II) there is an element $h' \in E'$ of order precisely 4, such that, for $z = (z_1, z_2, z_3) \in T'$:

$$r(z) = (z_2, -z_1, z_3 + h') =: R(z_1, z_2, z_3) + \big(0, 0, h'\big),$$
$$s(z) = (z_1 + h, -z_2 + k, -z_3) =: S(z_1, z_2, z_3) + (h, k, 0).$$

Conversely, the above formulae give a fixed point free action of the dihedral group $G = D_4$ which contains no translations.
In particular, we obtain the following normal form:

$$E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \quad E' = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau'), \quad \tau, \tau' \in \mathcal{H} := \{z \in \mathbb{C} |\, \mathrm{Im}(z) > 0\},$$

$$h = 1/2, k = \tau/2, h' = 1/4$$

$$r(z_1, z_2, z_3) := (z_2, -z_1, z_3 + 1/4)$$

$$s(z_1, z_2, z_3) := (z_1 + 1/2, -z_2 + \tau/2, -z_3).$$

In particular, the Teichmüller space of hyperelliptic threefolds with group D_4 is isomorphic to the product $\mathcal{H}^2$ of two upper halfplanes.

Chapter 2

Proof of the main theorem

We use the following notation: $T = V/\Lambda$ is a complex torus of dimension 3, which admits a free action of the group

$$G = \langle r, s \,|\, r^4 = s^2 = (rs)^2 = 1 \rangle \cong D_4,$$

such that the complex representation $\rho\colon G \to \mathrm{GL}(3, \mathbb{C})$ is faithful.

A first observation is that the complex representation ρ of G must contain the 2-dimensional irreducible representation V_1 of G (else, ρ would be a direct sum of 1-dimensional representations: this, by the assumption on the faithfulness of ρ, would imply that G is Abelian, a contradiction).

Hence we have a splitting

$$V = V_1 \oplus V_2,$$

where V_2 is 1-dimensional, and we can choose an appropriate basis so that, setting $R := \rho(r), S := \rho(s)$, we are left with the two cases

$$\textbf{Case 1:}\ R = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix},$$

$$\textbf{Case 2:}\ R = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

which are distinguished by the multiplicity of the eigenvalue 1 of S.

Indeed R is necessarily of the form above, since the freeness of the G-action implies that $\rho(g)$ must have eigenvalue 1 for every $g \in G$.

Lemma 2.1. *In both Cases 1 and 2, the complex torus $T = V/\Lambda$ is isogenous to a product of three elliptic curves, $T \sim_{isog.} E_1 \times E_2 \times E_3$, where $E_i \subset T$, for $i = 1, 2, 3$ and E_1 and E_2 are isomorphic elliptic curves. In other words, writing $E_j = W_j/\Lambda_j$, the complex torus T is isomorphic to*

$$(E_1 \times E_1 \times E_3)/H, \quad H = \Lambda/(\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3).$$

Proof. Let I be the identity of T.

In Case 1, we set $E_1 := \ker(S-I)^0 = \mathrm{im}(S+I)$, $E_3 := \ker(R-I)^0$ and $E_2 := R(E_1)$ (here, the superscript zero denotes the connected component of the identity). Then it is clear that $E_1 \cong E_2$, and that T is isogenous to $E_1 \times E_2 \times E_3$.
In Case 2, we define similarly $E_2 := \ker(S+I)^0 = \mathrm{im}(S-I)$, $E_3 := \ker(R-I)^0$ and $E_1 := R(E_2)$. We obtain again $E_1 \cong E_2$, and that T is isogenous to $E_1 \times E_2 \times E_3$.

□

Lemma 2.2. *Writing $E_j = W_j/\Lambda_j$, the following statements hold.*

(1) In Case 1, the lattice Λ_2 is equal to $W_2 \cap \Lambda$.

(2) In Case 2, the lattice Λ_1 is equal to $W_1 \cap \Lambda$.

Proof. (1) Obviously, $E_2 = R(E_1) = W_2/R(\Lambda_1)$, i.e., $\Lambda_2 = R(\Lambda_1) \subset W_2 \cap \Lambda$. On the other hand, $R(W_2 \cap \Lambda) \subset W_1 \cap \Lambda = \Lambda_1$, and applying the automorphism R of Λ gives $W_2 \cap \Lambda \subset R(\Lambda_1) = \Lambda_2$.

(2) Here, $E_1 = R(E_2) = W_1/R(\Lambda_2)$, i.e., $\Lambda_1 = R(\Lambda_2) \subset W_1 \cap \Lambda$. For the converse inclusion, observe $R(W_1 \cap \Lambda) \subset W_2 \cap \Lambda = \Lambda_2$, and applying R yields again the result.

□

We can now choose coordinates on V such that r is induced by a transformation of the form

$$r(z_1, z_2, z_3) = (z_2, -z_1, z_3 + c_3),$$

by choosing as the origin in V_1 a fixed point of the restriction of r to V_1.

We can now view r, s as affine self maps of T induced by affine self maps of $E_1 \times E_2 \times E_3$ of the form

$$r(z_1, z_2, z_3) = (z_2, -z_1, z_3 + c_3),$$
$$s(z_1, z_2, z_3) := (z_1 + a_1, -z_2 + a_2, \pm z_3 + a_3),$$

and sending the subgroup H to itself.

Lemma 2.3. *The freeness of the action of the powers of r is equivalent to: H contains no element with last coordinate equal to c_3, or $2c_3$.*

Moreover, $(0, 0, 4c_3) \in H$.

Proof. $r(z) = z$ is equivalent to $(z_1 - z_2, z_1 + z_2, -c_3) \in H$. However, the endomorphism

$$(z_1, z_2) \mapsto (z_1 - z_2, z_1 + z_2)$$

of $E_1 \times E_2$ is surjective, hence H cannot contain any element with last coordinate equal to c_3.

Since $r^2(z) = (-z_1, -z_2, z_3 + 2c_3)$, $r^2(z) = z$ is equivalent to $(-2z_1, -2z_2, 2c_3) \in H$, and we reach the similar conclusion that H cannot contain any element with last coordinate equal to $2c_3$.

Finally, the condition that r^4 is the identity is equivalent to $(0, 0, 4c_3) \in H$.

□

Proposition 2.4. *Case 2 does not occur.*

Proof. Since we assume that

$$s(z_1, z_2, z_3) := (z_1 + a_1,\ -z_2 + a_2,\ z_3 + a_3),$$

and that s^2 is the identity, it must be

$$(2a_1,\ 0,\ 2a_3) \in H.$$

Consider now rs:

$$rs(z) = (-z_2 + a_2,\ -z_1 - a_1,\ z_3 + a_3 + c_3).$$

The condition that $(rs)^2$ is the identity is equivalent to:

$$(a_1 + a_2,\ -(a_1 + a_2),\ 2(a_3 + c_3)) \in H.$$

This condition, plus the previous one, imply that

$$(a_2 - a_1,\ -(a_1 + a_2),\ 2c_3) \in H,$$

contradicting Lemma 2.3.

□

Henceforth we shall assume that we are in Case 1, and we can choose the origin in E_3 so that

$$s(z_1, z_2, z_3) := (z_1 + a_1,\ -z_2 + a_2,\ -z_3).$$

Lemma 2.5. *If*

$$s(z_1, z_2, z_3) := (z_1 + a_1,\ -z_2 + a_2,\ -z_3),$$

then

$$(2a_1,\ 0,\ 0) \in H$$

and H contains no element of the form

$$(a_1,\ w_2,\ w_3).$$

Proof. The first condition is equivalent to s^2 being the identity, while the second is equivalent to the condition that s acts freely, since $s(z) = z$ is equivalent to $(a_1,\ -2z_2 + a_2,\ -2z_3) \in H$.

□

Proposition 2.6. *For each $\lambda \in \Lambda$ there exist $\lambda' \in \Lambda, \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2, \lambda_3 \in \Lambda_3$, such that*

$$2\lambda = \lambda_1 + \lambda',\quad 2\lambda' = \lambda_2 + \lambda_3$$

More precisely, we even have:

$$\Lambda \subset \frac{1}{2}\Lambda_1 + \frac{1}{2}\Lambda_2 + \frac{1}{4}\Lambda_3.$$

Proof. Let $\lambda \in \Lambda$: we can write

$$2\lambda = \underbrace{(I+S)\lambda}_{=:\lambda_1 \in \Lambda_1} + \underbrace{(I-S)\lambda}_{=:\lambda' \in \Lambda}.$$

Furthermore, since $\lambda' \in \operatorname{im}(I-S)$, we obtain

$$2\lambda' = \underbrace{(I+R^2)\lambda'}_{=:\lambda_3 \in \Lambda_3} + \underbrace{(I-R^2)\lambda'}_{=:\lambda_2 \in \Lambda \cap W_2 = \Lambda_2}.$$

Hence, $\lambda = \frac{\lambda_1}{2} + \frac{\lambda_2}{4} + \frac{\lambda_3}{4}$ for unique $\lambda_j \in \Lambda_j$.

Applying the automorphism R of Λ and the unicity of the λ_j yields the result, since R exchanges Λ_1 and Λ_2.

$\square$

Proposition 2.7. *We have*

$$\Lambda \subset \frac{1}{2}\Lambda_1 + \frac{1}{2}\Lambda_2 + \frac{1}{2}\Lambda_3.$$

Proof. For $\lambda \in \Lambda$ we can write $\lambda = \frac{\lambda_1}{2} + \frac{\lambda_2}{2} + \frac{\lambda_3}{4}$ for unique $\lambda_j \in \Lambda_j$.

We now use the property

$$E_i \hookrightarrow T \Rightarrow \forall\ (0,0,d) \in H, \text{ we have } d = 0.$$

Indeed, $2\lambda = \lambda_1 + \lambda_2 + \frac{\lambda_3}{2}$, hence $(0,0,[\frac{\lambda_3}{2}]) \in H$ and $\frac{\lambda_3}{2} = 0$ in E_3. Equivalently, there is an element $\lambda_3' \in \Lambda_3$ with

$$\frac{\lambda_3}{4} = \frac{\lambda_3'}{2}.$$

$\square$

Lemma 2.8. *Consider the transformation rs:*

$$rs(z) = (-z_2 + a_2,\ -z_1 - a_1,\ -z_3 + c_3).$$

The condition that its square is the identity amounts to

$$(a_1 + a_2,\ -(a_1 + a_2),\ 0) \in H,$$

while the freeness of its action is equivalent to the fact that H contains no element of the form

$$(w_1 - a_2,\ w_1 + a_1,\ w_3).$$

This last condition is equivalent to

$$\forall\ (d_1, d_2, d_3) \in H:\quad d_1 + a_2 \neq d_2 - a_1.$$

Proof. The first condition is straightforward, while the freeness of the action is equivalent to the non existence of (z_1, z_2, z_3) such that

$$(z_1 + z_2 - a_2,\ z_2 + z_1 + a_1,\ 2z_3 - c_3) \in H.$$

As usual, we observe that for each w_1, w_3 there exist z_1, z_2, z_3 with $z_1+z_2 = w_1, 2z_3-c_3 = w_3$.

□

We put together the conclusions of Lemmas 2.3, 2.5, 2.8,

(i) $(0,\ 0,\ 4c_3) \in H$

(ii) $(2a_1,\ 0,\ 0) \in H$

(iii) $(a_1 + a_2,\ -a_1 - a_2,\ 0) \in H$, hence also $(a_1 - a_2,\ a_1 + a_2,\ 0) \in H$.

(1) H contains no element of the form $(w_1,\ w_2,\ c_3)$,

(2) nor of the form $(w_1,\ w_2,\ 2c_3)$

(3) nor of the form $(a_1,\ w_2,\ w_3)$

(4) nor of the form $(w_1,\ w_2,\ w_3)$ with $w_1 + a_2 = w_2 - a_1$.

It follows from (iii) and (3) that $a_2 \neq 0$. While the condition that each element of H which has two coordinates equal to zero is indeed zero (since E_i embeds in T!) imply

$$2a_1 = 0, 4c_3 = 0.$$

By conditions (1), (2), (3) the elements a_1, c_3 have respective orders exactly $2, 4$. Moreover:

- (4) and (i) imply that $a_1 + a_2 \neq 0$
- (ii), (iii) and the fact that H has exponent 2 implies $2a_2 = 2a_1 = 0$, $2a_1 + 2a_2 = 0$. Hence $a_1 \neq a_2$ are nontrivial 2-torsion elements.

We have thus obtained the desired elements

$$h := a_1, k := a_2, h' := c_3.$$

It suffices to show that H is generated by $\omega := (h + k,\ h + k,\ 0) = (a_1 + a_2,\ a_1 + a_2,\ 0)$.

Observe first that $\omega \in H$, by condition (iii).

Condition (4) implies that the first coordinate of an element of H must be a multiple of $a_1 + a_2$: since it cannot equal a_1, by condition (3), and if it equals a_2, we can add ω and obtain an element of H with first coordinate a_1. Using R, we infer that both coordinates must be a multiple of $(a_1 + a_2)$. Possibly adding ω, we may assume that $w_1 = 0$: then by (4) we conclude that also $w_2 = 0$. Finally, the condition that each element of H which has two coordinates equal to zero is indeed zero, shows that H is then generated by ω, as we wanted to show.

The last assertions of the main theorem follow now in a straightforward way (see [CC17] concerning general properties of Teichmüller spaces of hyperelliptic manifolds).

Part III

On the Classification of Hyperelliptic Fourfolds and Hyperelliptic Varieties with Abelian Group

Chapter 1

Introduction

Recall the following definition:

Definition. A *(generalized) hyperelliptic manifold* is the quotient $X = T/G$ of a complex torus T by the action of a finite, non-trivial group $G \subset \mathrm{Bihol}(T)$ such that G acts freely on T and contains no translations. If X is projective (i.e., T is an Abelian variety), we call X a *(generalized) hyperelliptic variety.*

The current part of the thesis generalizes the results of Uchida-Yoshihara [UY76] (who gave the list of groups which may possibly occur as a group attached to a hyperelliptic threefold) to dimension 4 and to describe in a general way the hyperelliptic varieties with Abelian groups. Let G be a group occurring as a group of a hyperelliptic fourfold. As we will see in Lemma 2.5, the only primes which possibly divide $|G|$ are 2, 3, 5 and 7. The rough idea for obtaining all possibilities for G is now to bound the orders of the Sylow subgroups of G. We easily obtain that the 5- and 7-Sylow subgroups of G are elementary Abelian of rank at most 1, see Corollary 2.13 and Lemma 2.14. Bounding the order of the 2- and 3-Sylow subgroups is more involved. Fortunately, we can prove that their respective orders are bounded by 32 and 27, respectively. Roughly speaking, we make use of the computer algebra system GAP [GAP] to run through all groups with orders in the above range and thus finding all possibilities for G. For a more detailed description of our strategy of proof, we refer to Chapter 4. TABLE 6 in Theorem 11.2 constitutes our main result, and contains exactly those finite groups G, such that there exists a hyperelliptic fourfold A/G.
In Chapter 3, we investigate hyperelliptic varieties $X = A/G$ of dimension n and Abelian group G and give a new decomposition of A up to isogeny (following [Cat19], which deals with the cyclic case).

Remark. The notations and conventions introduced on p. 5 will be extensively used throughout the current part of the thesis.

Chapter 2

General Results

We collect several general prerequisites for the upcoming classification of groups associated with hyperelliptic fourfolds in the next sections. Let $T = V/\Lambda$ be a complex torus of dimension n, and let $\{1\} \neq G \subset \mathrm{Bihol}(T)$ be a finite group. Recall that holomorphic maps between complex tori are affine, i.e., each $g \in G$ is of the form $g([z]) = [\alpha z + b]$ for some $\alpha \in \mathrm{GL}(V)$ such that $\alpha(\Lambda) = \Lambda$ and $b \in V$[1]. Suppose furthermore that G acts freely on T and contains no translations: then the *complex representation* $\rho\colon G \to \mathrm{GL}(n, \mathbb{C})$, which maps a group element to its linear part, is faithful. The second representation we consider here is the rational representation

$$\rho_\Lambda\colon G \to \mathrm{Aut}(\Lambda) \cong \mathrm{GL}(2n, \mathbb{Z}).$$

In fact, these two representations are related by

$$\rho_\Lambda \otimes_{\mathbb{Z}} \mathbb{C} \sim \rho \oplus \overline{\rho},$$

where $\sim$ denotes equivalence of representations.

Firstly, by Theorem I.6.1, we may assume that $T = A$ is an Abelian variety, which we shall do from now on.

Remark 2.1. An element $g(z) = \alpha z + b \in G$ acts freely on T if and only if the equation

$$(\alpha - I_n)z = \lambda - b$$

has no solution in $z \in V$, $\lambda \in \Lambda$. This implies that $\alpha = \rho(g)$ has the eigenvalue 1.

Denote by φ the *Euler totient function*, which maps an integer $d \geq 2$ to the number of integers from 1 to $d-1$ which are coprime to d (and $\varphi(1) := 1$). It is well-known that the minimal polynomial of a primitive d-th root of unity over $\mathbb{Q}$ has degree $\varphi(d)$. The previous remarks have the following elementary consequences.

Lemma 2.2. *Suppose $\alpha \in \mathrm{Aut}(A)$ is a linear automorphism of finite order $d \geq 3$. Suppose furthermore that all eigenvalues of α are primitive d-th roots of unity. Consider*

[1] We will drop the square brackets to denote equivalence classes modulo Λ from now on.

the integer valued function mult: $\mu_d^* \to \mathbb{Z}$ *which assigns to a primitive d-th root of unity $\zeta \in \mu_d^*$ the multiplicity of ζ as an eigenvalue of α. Then the function*

$$\zeta \mapsto \mathrm{mult}(\zeta) + \mathrm{mult}(\overline{\zeta})$$

is constant.

Proof. The lemma follows from the observation that $\rho \oplus \overline{\rho}$ is a rational/integral representation and the previous remarks regarding the minimal polynomial of a primitive d-th root of unity over $\mathbb{Q}$: indeed, since $\rho(\alpha) \oplus \overline{\rho(\alpha)}$ is (similar to) an integral matrix, the characteristic polynomial of $\rho(\alpha) \oplus \overline{\rho(\alpha)}$ is a power of the minimal polynomial of a primitive d-th root of unity over $\mathbb{Q}$. Since the roots of this minimal polynomial are all the primitive d-th roots of unity, we are done. $\square$

Remark 2.3. A typical situation in which we will apply Lemma 2.2 is the following: suppose that $g(x) = \beta(x) + b$ is a biholomorphism of A, such that $\mathrm{ord}(g) = \mathrm{ord}(\beta) = d \geq 3$ (we do not require $\mathrm{Eig}(\beta)$ to contain a primitive d-th root of unity). According to [Cat14, Section 5.4], the Abelian variety A is isogenous to a product of Abelian subvarieties A_k, indexed by the positive divisors k of d, such that each A_k is stable under the action of β and β acts on A_k with eigenvalues of order k. Now, the Lemma restricts the possible dimensions of the A_k and the eigenvalues of β. Of particular interest for us is the case in which the action of $\langle g \rangle$ on A is free and $\dim(A) = 4$: by dimension reasons, since $\dim(A_1) > 0$ by Remark 2.1, we obtain the implication

$$A_k \neq 0 \text{ and } k \notin \{1,2,3,4,6\} \implies \dim(A_k) = \frac{\varphi(k)}{2}.$$

The set of eigenvalues of $\beta|_{A_k}$ (where A_k, k are as in the above implication) consists therefore of exactly $\varphi(k)/2$ primitive, pairwise non-conjugate k-th roots of unity. In Lemma 2.12 below, we will list the $\mathrm{Gal}(\mathbb{Q}(\zeta_k)/\mathbb{Q})$-orbits of sets consisting of $\varphi(k)/2$ primitive and pairwise non-conjugate k-th roots of unity in the cases $\varphi(k) \in \{4,6\}$ (or, equivalently, $k \in \{5,7,8,9,10,12,14,18\}$).

We state the following result, which we will use rather frequently.

Proposition 2.4. *[CaCi93, Proposition 5.7]*
Suppose that A is an Abelian variety of dimension n such that $\zeta_d \cdot \mathrm{id}_A \in \mathrm{Aut}(A)$ for some $d \in \{3,4,6\}$. Then A is isomorphic to E^n, where E is the unique elliptic curve which admits an automorphism of order d.

Recall that the elliptic curve $E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ (resp. $F = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$) admits an automorphism of order 4 (resp. 6) and is called the *harmonic* (resp. *equianharmonic*) elliptic curve.

The Lemma stated below gives a bound on the orders of the group elements contained in G.

Lemma 2.5. *The following statements hold:*

(a) *Let $g \in G$. If an eigenvalue of $\rho(g)$ is a primitive* $\mathrm{ord}(g)$*-th root of unity, then* $\varphi(\mathrm{ord}(g)) \leq 2(n-1)$.

(b) If $n = 4$, then

$$\operatorname{ord}(g) \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30\}.$$

Proof. Assertion (a) follows from the observation that the characteristic polynomial of ρ_Λ is an integral polynomial of degree $2n$, which has a root of even order ≥ 2 in 1 by the previous remarks. To show part (b), we observe that $\varphi(\operatorname{ord}(g)) \leq 6$ implies that $\operatorname{ord}(g) \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}$. If $\varphi(\operatorname{ord}(g)) > 6$, then by (a) and Remark 2.3, g is a product of elements with coprime orders d, d' satisfying $\varphi(d) = 2, \varphi(d') = 4$. This gives the remaining possible orders in the list. □

Example 2.6. The previous Lemma 2.5 allows us to describe easy examples of hyperelliptic fourfolds, namely ones with cyclic group: these are called *Bagnera-de Franchis manifolds* (or, in the projective case, *Bagnera-de Franchis varieties*). Suppose that

$$d \in \{3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30\}$$

and let T_d be a complex torus of dimension $\frac{\varphi(d)}{2}$ admitting a linear automorphism α of order d. In fact, such an Abelian variety exists and can be constructed as a CM-Abelian variety: one simply takes $\frac{\varphi(d)}{2}$ different, pairwise non-conjugate embeddings $\mathbb{Q}(\zeta_d) \hookrightarrow \mathbb{C}$: these yield a complex structure on the torus $T_d := (\mathbb{Q}(\zeta_d) \otimes_{\mathbb{Q}} \mathbb{R})/\mathbb{Z}[\zeta_d]$. (We can then show as in Part I, Example 3.4 that T_d is indeed an Abelian variety.)
Now we simply take any complex torus T' of dimension $4 - \frac{\varphi(d)}{2}$. By our choice of d, we have $\dim(T) > 0$. Set $T := T_d \times T'$ and define an action of $\langle g \rangle = C_d$ on T as follows:

$$(z, z') \mapsto g(z, z') := (\alpha z,\ z' + h), \text{ where } h \in T' \text{ is an element of order } d.$$

It is clear by construction that $\langle g \rangle$ acts freely on T and contains no translations. Hence, $T/\langle g \rangle$ is a Bagnera-de Franchis manifold.
Of course, not all Bagnera-de Franchis fourfolds occur in exactly this way. For more complete descriptions, see [Cat19] and [Dem16].

From now on, we shall make the following meta-assumptions, which we will often not explicitly refer to.

The letter G will always denote a finite subgroup of $\operatorname{Bihol}(A)$, where A is an Abelian variety of dimension n, such that the following properties hold:
(1) G is embedded into $\operatorname{GL}(n, \mathbb{C})$ via some faithful representation $\rho \colon G \hookrightarrow \operatorname{GL}(n, \mathbb{C})$ (this is equivalent to requiring that G does not contain any translations).
(2) The matrix $\rho(g)$ has the eigenvalue 1 for any $g \in G$.
(3) The associated complex representation of the embedding $G \subset \operatorname{Bihol}(A)$ is ρ.

Lemma 2.7. *The group G does not have a subgroup isomorphic to C_d^n, $d > 1$.*

Proof. If G had a subgroup U isomorphic to C_d^n (which we shall identify with the additive group C_d^n by an isomorphism), we can choose a suitable basis such that $\rho|_U$ is given by

$$(k_1, ..., k_n) \mapsto \operatorname{diag}\left(\zeta_d^{k_1}, ..., \zeta_d^{k_n}\right).$$

In particular, $\rho(1, ..., 1)$ does not have the eigenvalue 1. □

Proposition 2.8. *[Dem16, Proposition 1.3] Assume that α is an (linear) automorphism of the n-dimensional Abelian variety A, whose eigenvalues are all primitive d-th roots of unity. Then*

$$\mathrm{Fix}(\alpha) := \{z \mid \alpha(z) = z\} \cong \begin{cases} C_p^{2n/\varphi(d)}, & \text{if } d \text{ is a power of the prime } p \\ 0, & \text{else.} \end{cases}$$

The rest of this section deals with the special case $n = 4$. By Lemma 2.5,

$$|G| = 2^a \cdot 3^b \cdot 5^c \cdot 7^d.$$

We take a closer look at the Sylow groups of G in order to get practical bounds for the exponents a, b, c, d (or even better, isomorphism types for the Sylow groups of G). In order to achieve this, we will make frequent use of the following two representation-theoretic results.

Theorem 2.9. *[Hu98, 7.2 Theorem]*
A p-group G has a faithful irreducible representation if and only if its center $Z(G)$ is cyclic.

Theorem 2.10. *[Hu98, 19.9 Theorem]*
Let G be a finite group, N an Abelian normal subgroup of G and let χ be an irreducible character of G. Then $\chi(1)$ divides the index $(G : N)$.

The latter of these two results allows us to prove:

Lemma 2.11. *The 5-Sylow subgroups and the 7-Sylow subgroups of G are Abelian.*

Proof. Let S be a 5- or 7-Sylow subgroup of G. By Theorem 2.10, $\rho|_S$ is a direct sum of four 1-dimensional representations. The proof is finished because S embeds into $\mathrm{GL}(4, \mathbb{C})$ via ρ. □

To obtain the possible isomorphism types for 5- and 7-Sylow subgroups of G, we invoke Lemma 2.12 below: its content is a table of possible sets of eigenvalues of linear automorphisms of order d (where $\varphi(d) \in \{4, 6\}$ and all eigenvalues of this automorphism are primitive d-th roots of unity) of an Abelian variety of dimension $\frac{\varphi(d)}{2}$. To ensure better readability, we will identify the multiplicative group of d-th roots of unity with the additive group C_d by the obvious isomorphism.

Lemma 2.12. *[CaCi93], [Dem16, Sections 3.1 & 3.2]*
Let $d \in \mathbb{N}$ such that $\varphi(d) \in \{4, 6\}$ and let α be an automorphism of an Abelian variety A of dimension $\frac{\varphi(d)}{2}$, such that $\mathrm{Eig}(\alpha)$ contains only primitive d-th roots of unity. Then $(d, \mathrm{Eig}(\alpha))$ is, up to the action of $\mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$, contained in the following table:

d	5	7	8	9
Possibilities for $\mathrm{Eig}(\alpha)$	$\{1,2\}$	$\{1,2,3\}$, $\{1,2,4\}$	$\{1,3\}$, $\{1,5\}$	$\{1,2,4\}$, $\{1,4,7\}$
d	10	12	14	18
Possibilities for $\mathrm{Eig}(\alpha)$	$\{1,3\}$	$\{1,5\}$, $\{1,7\}$	$\{1,3,5\}$, $\{1,5,11\}$	$\{1,5,7\}$, $\{1,7,13\}$

Sketch of Proof. The cases $d = 5, 8, 10, 12$ are dealt with in [CaCi93], and the cases $d = 7, 9$ are contained in [Dem16]. Thus, the cases $d = 14, 18$ are missing. We prove the assertion for $d = 18$, the other case being similar. The units in C_{18} form a cyclic group of order 6. There are several cases which are clearly distinguished by the number of generators of $C_{18}^* = \{1, 5, 7, 11, 13, 17\}$ in the triple, and also by the property whether there is the neutral element 1 or not. The generators of C_{18}^* are 5 and 11.
Hence, there are just the following cases:

$\{1, 5, 7\}$: one generator and 1.
$\{5, 7, 17\}$: one generator and not 1. This is 5 times $\{1, 5, 7\}$.
$\{7, 13, 17\}$: no generators and not 1. This is 7 times $\{1, 5, 17\}$.
$\{1, 5, 11\}$: two generators and 1. This is 11 times $\{1, 5, 7\}$.
$\{1, 11, 13\}$: one generator and 1. This is 13 times $\{1, 5, 7\}$.
$\{11, 13, 17\}$: one generator and not 1. This is 17 times $\{1, 5, 7\}$.
$\{1, 7, 13\}$: no generators and 1. This is never a multiple of $\{1, 5, 7\}$.
$\{5, 11, 17\}$: two generators and not 1. This is 5 times $\{1, 7, 13\}$.

Hence, up to Galois automorphisms, the only possibilities are $\{1, 5, 7\}$ and $\{1, 7, 13\}$. □

Consider the exact sequence

$$1 \to K \to G \stackrel{\det}{\to} C_m \to 0, \tag{2.1}$$

where $\det(g) := \det(\rho(g))$. We obtain immediately the following Corollary from Remark 2.3 and Lemma 2.12.

Corollary 2.13. *Suppose that $g \in G$ is an element of order 5. Then $\det(g)$ has order 5, and every 5-Sylow subgroup of G is isomorphic to C_5.*

Proof. The first assertion directly follows from Remark 2.3 and Lemma 2.12. The second assertion follows from the exact sequence (2.1). □

Lemma 2.14. *The 7-Sylow subgroups of G are either $\{0\}$ or cyclic of order 7.*

Proof. By Lemmas 2.5 and 2.11, it suffices to prove that G cannot have a subgroup isomorphic to $C_7 \times C_7$. Suppose $g_1, g_2 \in G$ span a subgroup isomorphic to $C_7 \times C_7$. We may assume that $\rho(g_1)$, $\rho(g_2)$ are diagonal matrices, and, by possibly replacing g_1 by an appropriate power, that

$$\rho(g_1) = \mathrm{diag}(1,\ \zeta_7,\ \zeta_7^r,\ \zeta_7^s).$$

If $\rho(g_2)$ is of the form

$$\rho(g_2) = \mathrm{diag}(1,\ \zeta_7,\ \zeta_7^t,\ \zeta_7^u),$$

then $\rho(g_1 g_2^{-1})$ has an eigenvalue of order 7 (since g_1 and g_2 do not span the same subgroup). However, $\rho(g_1 g_2^{-1})$ has at most two eigenvalues of order 7, contradicting Lemma 2.2.
This proves that after replacing g_2 by an appropriate power, we may assume that

$$\rho(g_2) = \mathrm{diag}(\zeta_7,\ \zeta_7^t,\ \zeta_7^u,\ \zeta_7^v),$$

where exactly one of t, u, v is divisible by 7. Without loss of generality, we can assume that t is divisible by 7. Then one of the three matrices

$$\rho(g_1 g_2), \quad \rho(g_1 g_2^2), \quad \rho(g_1 g_2^3)$$

does not have the eigenvalue 1, a contradiction. $\square$

As expected, the study of the 2- and 3-Sylow-subgroups of G becomes much more involved (cf.Chapters 6 and 7).

Corollary 2.15. *Suppose that both the* 5- *and the* 7*-Sylow subgroups of* G *are not trivial. Then* G *is not solvable.*

Proof. First of all $c = d = 1$ by the previous Lemmas. According to Hall's Theorem [Ha59, Theorem 9.3.1], solvability of G implies that it has a subgroup of order $5 \cdot 7$. By Sylow's Theorems, such a group is necessarily cyclic. By Lemma 2.5, G cannot have an element of order 35. $\square$

Corollary 2.16. *Suppose that* $|G| = 2^a \cdot 3^b \cdot 5$. *Then* G *is solvable.*

Proof. By Corollary 2.13 and exact sequence (2.1), G is an extension of a group of order $2^{a'} \cdot 3^{b'}$ (which is solvable by Burnside's $p^a q^b$-Theorem, cf. [Ha59, Theorem 9.3.2]) by a cyclic group. $\square$

Chapter 3

Hyperelliptic Manifolds with Abelian Group

Let G be a finite Abelian group, embedded into $\mathrm{GL}(n, \mathbb{C})$. We shall write G in the Frobenius normal form

$$G = C_{d_1} \times ... \times C_{d_k}, \quad \forall i\colon\ d_i | d_{i+1}, \quad d_1 > 1. \tag{3.1}$$

This chapter's purpose is twofold. In the first subsection, we will deduce restrictions on the structure of G (if G occurs as a group of a hyperelliptic n-fold), the main results being Theorem 3.5 and Proposition 3.9. Moreover, we will give examples of hyperelliptic fourfolds with certain Abelian groups as a very first step towards our main result Theorem 11.2.
The second subsection generalizes the very recent [Cat19, Proposition 3.1]: Catanese described explicitly the data needed to define a $\mathbb{Z}[G]$-module Λ, which is also a free Abelian group of finite rank in the case where G is cyclic. We generalize his results and consider Abelian groups G as well. However, some questions answered in the cyclic case remain open in the Abelian case (see Remark 3.23).

3.1 The Structure of G

Let G be as in (3.1), suppose that our meta-assumptions hold and that the embedding $G \subset \mathrm{GL}(n, \mathbb{C})$ coincides with the associated complex representation ρ. By Lemma 2.7, $k \leq n - 1$. The aim of this section is to investigate the structure of G in more detail. In particular, we investigate the boundary cases $k = n - 1$ and $k = 2$ (the case $k = 1$ being dealt with in Lemma 2.5) in the following ways:

- If $k = n - 1$ and G is the group of a hyperelliptic n-fold, what are the possibilities for $d_1, ..., d_{n-1}$ (see Corollary 3.2)? For which values of $d_1, ..., d_{n-1}$ do all group elements share a common eigenvector for the eigenvalue 1 (see Theorem 3.5)?
- Suppose that $k = 2$. Under which hypothesis on d_1 and d_2 do there exist hyperelliptic n-folds with group $G = C_{d_1} \times C_{d_2}$ (see Lemma 3.11)?

We start by investigating the boundary case $k = n - 1$.

Lemma 3.1. *Suppose that $G \subset \mathrm{GL}(n, \mathbb{C})$ is isomorphic to C_d^{n-1}, $d \geq 3$ such that any $g \in G$ has the eigenvalue 1. Then all $g \in G$ have a common eigenvector for 1, i.e., $\bigcap_{g \in G} \ker(g - \mathrm{id}) \neq \{0\}$.*

Proof. Since G is finite and Abelian, we can assume that the elements of G are diagonal matrices. Suppose that $g_1, ..., g_{n-1}$ are elements of order d generating G.

Step 1: Assume first that d is a prime power, $d = p^k \geq 3$. Then, after permuting coordinates, there are generators $\tilde{g}_1, ..., \tilde{g}_{n-1}$ of G, which take the form

$$\tilde{g}_j = \mathrm{diag}(1, ..., 1, \underbrace{\zeta_{p^k},}_{j\text{-th entry}} 1, ..., 1, \zeta_{p^k}^{a_j}).$$

In order to show that all elements of G share a common eigenvector for the eigenvalue 1, we will therefore have to prove that $a_j = 0$ for all j.
In the following, we assume that $(a_1, ..., a_{n-1}) \neq 0$ and construct an element of G without the eigenvalue 1. Suppose that $a_1 \neq 0$. Since $d = p^k \geq 3$, the element 2 is non-zero modulo $d = p^k$. Our construction implies that one of the elements

$$\tilde{g}_1 \cdot ... \cdot \tilde{g}_{n-1} \quad \text{or} \quad \tilde{g}_1^2 \cdot ... \cdot \tilde{g}_{n-1}$$

does not have the eigenvalue 1.

Step 2: We now deal with the general case. Write $d = q \cdot m$, where $q = p^k \geq 3$ is a prime power coprime to the integer m. After permuting coordinates and by Step 1, we can assume that the g_j^m are given by

$$g_j^m = \mathrm{diag}(1, ..., 1, \underbrace{\zeta_{p^k},}_{j\text{-th entry}} 1, ..., 1, 1).$$

Since G is generated by the g_j^m and the g_i^q, we will have to prove that the last diagonal entry of the g_i^q is equal to 1. If this was not the case for some index i, the element

$$g_1^m \cdot ... \cdot g_{n-1}^m \cdot g_i^q$$

does not have the eigenvalue 1, since q is coprime to m. This completes the proof of the statement. □

We will see in Example 3.6 below that the statement of Lemma 3.1 does not hold for $d = 2$.

Corollary 3.2. *Suppose that G is isomorphic to C_d^{n-1}, $d \geq 2$. If G is associated with a hyperelliptic manifold of dimension n, then $d \in \{2, 3, 4, 6\}$ (in other words, $\varphi(d) \leq 2$).*

Proof. Let $G = C_d^{n-1}$ be associated with a hyperelliptic manifold of dimension n and assume the contrary, i.e., $\varphi(d) \geq 4$. We can then assume that G is embedded in $\mathrm{GL}(n, \mathbb{C})$ by the complex representation ρ and that the elements of G are diagonal matrices. By Lemma 3.1, all elements of G share a common eigenvector for the eigenvalue 1. In other words, we can assume that $G = C_d^{n-1}$ is embedded in $\mathrm{GL}(n, \mathbb{C})$ by the rule

$$(a_1, ..., a_{n-1}) \mapsto \mathrm{diag}(\zeta_d^{a_1}, ..., \zeta_d^{a_{n-1}}, 1).$$

The element $(1, ..., 1)$ maps to $\mathrm{diag}(\zeta_d, ..., \zeta_d, 1)$, and since $\varphi(d) \geq 4$, we obtain a contradiction to Lemma 2.2. □

In fact, we can prove the following stronger result, which shows that the statement of Lemma 3.1 holds as well for $G = C_d^{n-2}$ under the stronger hypothesis $d \in \{4, 6\}$.

Proposition 3.3. *Suppose that $G \subset \mathrm{GL}(n, \mathbb{C})$ is isomorphic to C_d^{n-2}, where $n \geq 2$ and $d \in \{4, 6\}$ such that any $g \in G$ has the eigenvalue 1. Then all $g \in G$ have a common eigenvector for 1, i.e., $\bigcap_{g \in G} \ker(g - \mathrm{id}) \neq \{0\}$.*

Proof. Suppose that G is generated by the diagonal matrices $g_1, ..., g_{n-2}$ of order d. We treat the cases $d = 4$ and $d = 6$ separately.

The case $d = 4$: As in the proof of Lemma 3.1, we can assume that the generators g_j are given by

$$g_j = \mathrm{diag}(1, ..., 1, \underbrace{i,}_{j\text{-th entry}} 1, ..., 1, i^{a_{1j}}, i^{a_{2j}}), \quad \text{where } i = \zeta_4.$$

Assume that $(a_{n-1,1}, ..., a_{n-1,n-2}) \neq 0$ and $(a_{n1}, ..., a_{n,n-2}) \neq 0$. We construct an element of G without the eigenvalue 1. If the two sets

$$M_{n-1} := \{j \,|\, a_{n-1,j} \not\equiv 0 \pmod 4\} \text{ and } M_n := \{j \,|\, a_{nj} \not\equiv 0 \pmod 4\}$$

are disjoint, choose $b_1, ..., b_{n-2} \not\equiv 0 \pmod 4$ such that

$$\sum_{j \in M_{n-1}} b_j a_{n-1,j} \not\equiv 0 \pmod 4 \text{ and } \sum_{j \in M_n} b_j a_{nj} \not\equiv 0 \pmod 4.$$

(It is clear that this is possible, since M_{n-1} and M_n are disjoint and non-empty.) Then the element

$$g_1^{b_1} \cdot ... \cdot g_{n-2}^{b_{n-2}}$$

does not have the eigenvalue 1. Thus, we may assume that M_{n-1} and M_n intersect, say $a_{n-1,1}$, a_{n1} are both not congruent to 0 modulo 4. We distinguish two cases:

(I) If at least one of $a_{n-1,1}$, a_{n1} is a unit modulo 4 (the other one being possibly equal to the zero divisor 2), we prove that we are able to choose $b_1, ..., b_{n-2} \not\equiv 0 \pmod 4$ such that

$$\sum_{j=1}^{n-2} b_j a_{n-1,j} \not\equiv 0 \pmod 4 \text{ and } \sum_{j=1}^{n-2} b_j a_{nj} \not\equiv 0 \pmod 4.$$

Such a tuple $(b_1, ..., b_{n-2})$ then corresponds to the element

$$g_1^{b_1} \cdot ... \cdot g_{n-2}^{b_{n-2}}$$

without the eigenvalue 1.
Indeed, choosing $b_2, ..., b_{n-2}$ non-zero modulo 4 excludes at most two non-zero values for b_1. Thus, we are able to choose $b_1 \not\equiv 0$ as desired.

(II) If $2a_{n-1,1} \equiv 2a_{n1} \equiv 0 \pmod 4$, we argue as follows. If one of the sets M_{n-1} and M_n is a singleton, say $M_{n-1} = \{1\}$, we claim that it is possible to choose $b_2, ..., b_n \not\equiv 0$ such that

$$\sum_{j=2}^{n-2} b_j a_{nj} \equiv 0 \pmod 4 \text{ or } \sum_{j=2}^{n-2} b_j a_{nj} \equiv 1 \pmod 4.$$

(Then the element

$$g_1 \cdot g_2^{b_2} \cdot ... \cdot g_{n-2}^{b_{n-2}}$$

does not have the eigenvalue 1, a contradiction.)
In fact, if all a_{nj}, $j \geq 2$ are zero or zero divisors, it suffices to choose

$$b_2 = ... = b_{n-2} = 2.$$

If only one a_{nj}, $j \geq 2$ is different from zero and is a unit, we define $b_k := 1$ for all $k \geq 2$. Finally, if at least two a_{nj}, $j \geq 2$ are non-zero and one of them is a unit, it is clearly possible to choose the b_j as desired.

Therefore, we may assume that both M_{n-1} and M_n have cardinality ≥ 2. In the current situation, we now prove the following statement by induction:

$$\begin{aligned} &g_1, ..., g_{n-2} \text{ do not share an eigenvector for the eigenvalue 1} \\ \Longrightarrow\ &G \text{ contains an element without the eigenvalue 1.} \end{aligned}$$

The statement is clear for $n = 3$, so let us assume that $n \geq 4$. We consider the embedding ι of $\langle g_2, ..., g_{n-2} \rangle$ into $\mathrm{GL}(n-1, \mathbb{C})$ by forgetting the first diagonal entry. By induction, since $M_{n-1} \setminus \{1\}$ and $M_n \setminus \{1\}$ are non-empty, there is an element $g \in \langle g_2, ..., g_{n-2} \rangle$ such that $\iota(g)$ does not have the eigenvalue 1. Then the element $g_1^2 \cdot g \in G$ does not have the eigenvalue 1, since $2a_{n-1} \equiv 2a_{n1} \equiv 0 \pmod 4$.

This settles the case $d = 4$.

<u>The case $d = 6$:</u> We can assume that g_j^2, g_j^3 are given by

$$\begin{aligned} g_j^2 &= \mathrm{diag}(1, ..., 1, \underbrace{\zeta_3,}_{j\text{-th entry}} 1, ..., 1, \zeta_3^{a_{n-1,j}}, \zeta_3^{a_{nj}}), \\ g_j^3 &= \mathrm{diag}((-1)^{b_{1j}}, ..., (-1)^{b_{nj}}). \end{aligned}$$

First of all, as in the case $d = 4$, we observe that we can assume without loss of generality that both $a_{n-1,1}$ and a_{n1} are non-zero modulo 3. If the element $g := g_1^2 \cdot ... \cdot g_{n-2}^2$ does not have the eigenvalue 1, we are done. Otherwise, (say) the $(n-1)$-st diagonal entry of g is 1. Since $a_{n-1,1} \neq 0$, the $(n-1)$-st diagonal entry of $g \cdot g_1^2$ is different from 1; we are done, unless the n-th diagonal entry of $g \cdot g_1^2$ is equal to 1.
If there existed a j such that $b_{n-1,j} \neq 0$ or $b_{nj} \neq 0$ modulo 2, one of the elements $g \cdot g_j^3$ or $g \cdot g_1^2 \cdot g_j^3$ does not have the eigenvalue 1. Thus, we may assume that $b_{n-1,j} = b_{nj} = 0$ for all indices j. Since the subgroup spanned by $g_1^3, ..., g_{n-2}^3$ is isomorphic to C_2^{n-2} and

embeds into $\mathrm{GL}(n-2,\mathbb{C})$ (by forgetting the last two diagonal entries), the group G contains the matrix

$$\tilde{g} = \mathrm{diag}(\underbrace{-1,...,-1}_{n-2 \text{ entries}}, 1, 1).$$

But then the element $g_1^2 \cdot \tilde{g} = \mathrm{diag}(-\zeta_3, -1, ..., -1, \zeta_3^{a_{n-1,1}}, \zeta_3^{a_{n1}})$ does not have the eigenvalue 1. $\square$

Remark 3.4. The statement of Proposition 3.3 is wrong for $d = 3$ and $n \geq 4$. In fact, consider the $(n \times n)$-matrices

$$\begin{aligned} g_1 &= (\zeta_3, 1, ..., 1, \zeta_3^2, \zeta_3), \\ g_2 &= (1, \zeta_3, 1, ..., 1, \zeta_3, \zeta_3), \\ g_j &= \mathrm{diag}(1, ..., 1, \underbrace{\zeta_3,}_{j\text{-th entry}} 1, ..., 1), \quad \text{for } 3 \leq j \leq n-2. \end{aligned}$$

Then $g_1, ..., g_{n-2}$ span a subgroup G of $\mathrm{GL}(n,\mathbb{C})$ isomorphic to C_3^{n-2}, and because

- the $(n-1)$-st diagonal entry of $g_1 g_2$ is 1, while the n-th diagonal entry is different from 1, and
- the $(n-1)$-st diagonal entry of $g_1 g_2^2$ is different from 1, while the n-th diagonal entry of $g_1 g_2^2$ is 1,

each element in G has the eigenvalue 1, but $g_1, ..., g_{n-2}$ do not have a common eigenvector for the eigenvalue 1.
However, the author expects Proposition 3.3 to hold true for all $d \geq 4$. Since we are interested in applications of said Proposition to hyperelliptic manifolds, we will only deal with the case $d \in \{4, 6\}$ here.

Proposition 3.3 is the most important step in the proof of the following Theorem, which generalizes [La01, Lemma 6.5] to arbitrary dimension (Lange proved it in dimension 3).

Theorem 3.5. *Suppose that $k = n-1$ and that $G = C_{d_1} \times ... \times C_{d_{n-1}} \subset \mathrm{GL}(n,\mathbb{C})$. Assume furthermore that g has the eigenvalue 1 for any $g \in G$. Then (at least) one of the two following possibilities holds:*

(a) All $g \in G$ share a common eigenvector for the eigenvalue 1, i.e.,

$$\bigcap_{g \in G} \ker(g - \mathrm{id}) \neq \{0\}.$$

(b) G is, up to isomorphism, a subgroup of $C_2^2 \times C_d^{n-3}$, where $d \in \{2, 4, 6\}$.

Proof. It remains to show that $G = C_2 \times C_d^{n-2}$ ($d \in \{4, 6\}$) necessarily satisfies property (a).
We first deal with the case $d = 4$. According to Proposition 3.3 and the proof of Lemma 3.1, we can assume that C_4^{n-2} is embedded in $\mathrm{GL}(n,\mathbb{C})$ via

$$(a_1, ..., a_{n-2}) \mapsto \mathrm{diag}(\zeta_4^{a_1}, ..., \zeta_4^{a_{n-2}}, \zeta_4^{f(a_1,...,a_{n-2})}, 1)$$

for some homomorphism $f\colon C_4^{n-2} \to C_4$. Suppose that C_2 is embedded in $\mathrm{GL}(n,\mathbb{C})$ by a diagonal matrix $g = \mathrm{diag}((-1)^{k_1}, ..., (-1)^{k_n})$.
The statement is clear if k_n is even, so let us assume that k_n is odd. Moreover, after possibly multiplying g by a suitable element of C_4^{n-2}, we may assume that

$$k_1 = ... = k_{n-2} = 0.$$

The hypothesis that every group element has the eigenvalue 1 implies that

$$\forall a_1, ..., a_{n-2} \not\equiv 0 \pmod 4\colon \qquad f(a_1, ..., a_{n-2}) + 2\delta \equiv 0 \pmod 4,$$

where

$$\delta = \begin{cases} 1, & \text{if } k_{n-1} = 1 \\ 0, & \text{if } k_{n-1} = 0 \end{cases}.$$

It follows that $f \equiv 0$ and $\delta = 0$. This proves the statement for $d = 4$.

For $d = 6$, by Proposition 3.3, the elements in C_6^{n-2} share a common eigenvector for the eigenvalue 1, say the last unit vector $e_n = {}^t(0, ..., 0, 1)$. The squares of the generators $g_1, ..., g_{n-2}$ of C_6^{n-2} generate a subgroup isomorphic to C_3^{n-2}, which we can therefore assume to be embedded in $\mathrm{GL}(n,\mathbb{C})$ via

$$(g_1^2)^{a_1} \cdot ... \cdot (g_{n-2}^2)^{a_{n-2}} \mapsto \mathrm{diag}(\zeta_3^{a_1},\ ...,\ \zeta_3^{a_{n-2}},\ \zeta_3^{f(a_1,...,a_{n-2})}, 1)$$

for some linear form $f\colon C_3^{n-2} \to C_3$. Assume furthermore that G is generated by $g_1, ..., g_{n-2}$ and the element g of order 2, which we may assume to be embedded in $\mathrm{GL}(n,\mathbb{C})$ by

$$g = \mathrm{diag}((-1)^{k_1}, ..., (-1)^{k_n}).$$

We can without loss of generality assume that k_n is odd: if k_n were even and the last diagonal entries of all $g \cdot g_j^3$ are even, we are done. Otherwise, the last diagonal entry of one of the elements $g \cdot g_j^3$ is -1, and we can replace g by this element to assume without loss of generality that k_n is odd. Hence, if k_n is odd, the goal is to show that $f \equiv 0$ and that k_{n-1} is even.
The assumption that the group element $g_1^{a_1} \cdot ... \cdot g_{n-2}^{a_{n-2}} \cdot g$ has the eigenvalue 1 for any $a_1, ..., a_{n-2} \not\equiv 0 \pmod 3$ implies that

$$\forall a_1, ..., a_{n-2} \not\equiv 0 \pmod 3\colon \qquad 2f(a_1, ..., a_{n-2}) + 3\delta \equiv 0 \pmod 6,$$

where

$$\delta = \begin{cases} 1, & \text{if } k_{n-1} = 1 \\ 0, & \text{if } k_{n-1} = 0 \end{cases}.$$

Again, this implies $f \equiv 0$ and $\delta = 0$ as desired. □

Example 3.6. There exist hyperelliptic varieties with group $G = C_2^2 \times C_4^{n-3}$ in dimension $n \geq 4$, which do not satisfy property (a) in Theorem 3.5. We give an example of a hyperelliptic fourfold with group $C_2 \times C_2 \times C_4$, which does not satisfy property (a) in

Theorem 3.5.
Suppose that $A := A'/H := (E_1 \times E_2 \times E_3 \times E_i)/H$, where the $E_j = \mathbb{C}/(\mathbb{Z} + \tau_j\mathbb{Z})$ are arbitrary elliptic curves, $E_i = \mathbb{C}/\mathbb{Z}[i]$ is the harmonic elliptic curve and $H = \langle h_1, h_2 \rangle$, where

$$h_1 = \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ \frac{i-1}{2}\right),$$
$$h_2 = \left(\frac{1}{2},\ 0,\ \frac{1}{2},\ \frac{i-1}{2}\right).$$

Define biholomorphic self-maps of A' as follows:

$$a(z) = \left(-z_1 + \frac{\tau_1}{2},\ -z_2,\ z_3 + \frac{\tau_3}{2},\ z_4 + \frac{1}{2}\right),$$
$$b(z) = \left(-z_1,\ z_2 + \frac{\tau_2}{2},\ -z_3,\ z_4 + \frac{1}{2}\right),$$
$$c(z) = \left(z_1 + \frac{1}{4},\ z_2 + \frac{1}{4},\ z_3 + \frac{1}{4},\ iz_4\right).$$

Since the linear parts of a, b and c map H to H, the maps a, b, c descend to biholomorphic self-maps of $A = A'/H$. We claim that $G := \langle a, b, c \rangle$, viewed as a subgroup of $\mathrm{Bihol}(A)$ is isomorphic to $C_2 \times C_2 \times C_4$ and that G contains no translations:
In fact, it is clear that a, b and c have respective orders 2, 2 and 4. Moreover,

- $ab = ba$ is satisfied unconditionally.
- $ac = ca$ holds if and only if $\left(\frac{1}{2}, \frac{1}{2}, 0, \frac{i-1}{2}\right) = 0$ in A. This is the case by our definition of H.
- $bc = cb$ holds if and only if $\left(\frac{1}{2}, 0, \frac{1}{2}, \frac{i-1}{2}\right) = 0$ in A. Again, this is satisfied by our definition of H.

Thus, $G = \langle a, b, c \rangle$ is isomorphic to $C_2 \times C_2 \times C_4$.
To prove that G does not contain any translations, it suffices to observe that

$$a \mapsto \mathrm{diag}(-1,\ -1,\ 1,\ 1), \quad b \mapsto \mathrm{diag}(-1,\ 1,\ -1,\ 1), \quad c \mapsto \mathrm{diag}(1,\ 1,\ 1,\ i)$$

is a faithful representation of G.
We will now verify that G acts freely on A. The elements

$$a^j b^k(z) = \left((-1)^{j+k} z_1 + j\frac{\tau_1}{2},\ (-1)^j z_2 + k\frac{\tau_2}{2},\ (-1)^k z_3 + j\frac{\tau_3}{2},\ z_4 + \frac{j+k}{2}\right)$$

act freely on A if $j + k$ is odd, since then $\frac{j+k}{2} = \frac{1}{2}$, and H does not contain an element with last coordinate equal to $\frac{1}{2}$. If $j + k$ is even, we distinguish between the cases where j is even and j is odd: if j is odd, then the first coordinate of $a^j b^k$ is a translation by $\frac{\tau_1}{2}$, and thus $a^j b^k$ acts freely in this case (since H does not contain an element with first coordinate equal to $\frac{\tau_1}{2}$). If j is even, then k is even as well, and thus $a^j b^k$ is the identity.
We prove in the same way that the elements

$$c^\ell a^j b^k(z) = \Bigg((-1)^{j+k} z_1 + \frac{2j\tau_1 + \ell}{4},\ (-1)^j z_2 + \frac{2k\tau_2 + \ell}{4},$$
$$(-1)^k z_3 + \frac{2j\tau_3 + \ell}{4},\ i^\ell \left(z_4 + \frac{j+k}{2}\right)\Bigg)$$

act freely on A: suppose that $l \not\equiv 0 \pmod 4$. At least one of the numbers $j+k$, j and k is even. We distinguish between the partity of these numbers:

- If j is even and k is odd, the second coordinate of $c^\ell a^j b^k$ is a translation by $\frac{2k\tau_2+\ell}{4}$, and H does not contain an element with second coordinate equal to $\frac{2k\tau_2+\ell}{4}$. Thus $c^\ell a^j b^k$ acts freely in this case.
- If k is even and j is odd, we can argue as in the previous bullet point by looking at the third coordinate of $c^\ell a^j b^k$.
- If both j and k are odd, since $j+k$ is even, we can argue as in the first bullet point by considering the first coordinate instead of the second one.
- If both j and k are even, $c^\ell a^j b^k = c^\ell$, and c^ℓ acts freely on A if and only if H contains no element of the form $\left(\frac{\ell}{4}, \frac{\ell}{4}, \frac{\ell}{4}, w_4\right)$. Thus c^ℓ acts freely by our definition of H.

This proves that $G \cong C_2 \times C_2 \times C_4$ acts freely on A, so that A/G is a hyperelliptic fourfold and $a,\ b,\ c$ do not share a common eigenvector for the eigenvalue 1.

Example 3.7. In contrast to the previous example, we show that there are no hyperelliptic fourfolds with group $C_2 \times C_2 \times C_6$, which do not satisfy condition (a) of Theorem 3.5. Suppose that $a,\ b,\ c$ are of respective orders $2,\ 2,\ 6$ and generate a group G isomorphic to $C_2 \times C_2 \times C_6$.

> Claim: G contains an element g of order 6 such that $\rho(g)$ has an eigenvalue of order 6.
>
> Proof of the Claim: Assume the contrary. Then $\rho(c)$ does not have an eigenvalue of order 6, so that (after possibly replacing c by its inverse) $\rho(c)$ has -1 and ζ_3 as eigenvalues.
> Thus, we may write
>
> $$\rho(a) = \operatorname{diag}(*,*,*,1), \qquad \rho(b) = \operatorname{diag}(*,*,*,1) \qquad \rho(c) = \operatorname{diag}(1,*,-1,\zeta_3).$$
>
> By assumption, $a,\ b,\ c^3$ span a subgroup isomorphic to C_2^3, which embeds (by forgetting the last diagonal entry) into $\mathrm{GL}(3,\mathbb{C})$. Thus, $\langle a,\ b,\ c^3\rangle$ contains an element h such that $\rho(h) = \operatorname{diag}(-1,-1,-1,1)$: but then $\rho(hc^2)$ does not have the eigenvalue 1, a contradiction.

We thus may assume that $\rho(c)$ has an eigenvalue of order 6,

$$\rho(c) = \operatorname{diag}(1, c_2, c_3, -\zeta_3),$$

and that the last diagonal entry of $\rho(a)$ and $\rho(b)$ is 1:

$$\rho(a) = \operatorname{diag}(*,*,*,1), \qquad \rho(b) = \operatorname{diag}(*,*,*,1).$$

> Claim: We may assume that $\rho(c) = \operatorname{diag}(1,1,1,-\zeta_3)$.
>
> Proof of the Claim: It is possible to choose (not pairwise equal) group elements $g_j \in \langle a, b\rangle$, $j = 1,2,3$ of order 2 such that the j-th diagonal entry of $\rho(g_j)$ is -1.

(In fact, if w.l.o.g. the second diagonal entry of both a and b is 1, then $c_2 \neq 0$, so that one of $\rho(ac)$, $\rho(bc)$ and $\rho(abc)$ does not have the eigenvalue 1.)
We now distinguish two cases:

- Suppose first that the order of c_3 is divisible by 3. Then a suitable product h of g_1 and g_2 is of the form $\mathrm{diag}(-1,-1,\pm 1,1)$. This implies that $\rho(hc^2)$ does not have the eigenvalue 1.
- Thus, we may assume that the orders of c_2 and c_3 are 1 or 2, respectively. Assume that $c_2 = 1$ and $c_3 = -1$. Then, as in the previous bullet point, a suitable product h of g_1 and g_2 has the form $\mathrm{diag}(-1,-1,\pm 1,1)$: then $\rho(hc^2)$ does not have the eigenvalue 1.
  Assume now that $c_2 = c_3 = -1$. It is impossible for $\langle a, b\rangle$ to contain the matrix $d := \mathrm{diag}(-1,-1,-1,1)$, since then dc^2 does not have the eigenvalue 1. Therefore a suitable product h of g_1, g_2 and g_3 is of the form $\mathrm{diag}(1,-1,-1,1)$. We thus are allowed to replace c by ch to obtain the claim in the case where $c_2 = c_3 = -1$.

The Claim shows that we may assume

$$\rho(a) = \mathrm{diag}(-1,-1,1,1), \quad \rho(b) = \mathrm{diag}(1,-1,-1,1), \quad \rho(c) = \mathrm{diag}(1,1,1,-\zeta_3).$$

Assume now that there is a hyperelliptic fourfold A/G with group $G = \langle a,\ b,\ c\rangle$ such that the associated complex representation is ρ.
We write

$$\begin{aligned} a(z) &= (-z_1 + a_1,\ -z_2 + a_2,\ z_3 + a_3,\ z_4 + a_4), \\ b(z) &= (z_1 + b_1,\ -z_2 + b_2,\ -z_3 + b_3,\ z_4 + b_4), \\ c(z) &= (z_1 + c_1,\ z_2 + c_2,\ z_3 + c_3,\ -\zeta_3 z_4 + c_4). \end{aligned}$$

The conditions that $ac = ca$ and $bc = cb$ mean that the elements

$$v_1 := (2c_1,\ 2c_2,\ 0, (-\zeta_3 - 1)a_4), \quad v_2 := (0,\ 2c_2,\ 2c_3, (-\zeta_3 - 1)b_4).$$

are equal to 0 in A. Thus,

$$\begin{aligned} (\mathrm{id} - \rho(ab)) \cdot v_1 &= (4c_1,\ 0,\ 0, 0) = 0 \text{ in } A, \\ (\mathrm{id} - \rho(b)) \cdot v_1 &= (0,\ 4c_2,\ 0, 0) = 0 \text{ in } A, \\ (\mathrm{id} - \rho(b)) \cdot v_2 &= (0,\ 0,\ 4c_3, 0) = 0 \text{ in } A, \end{aligned}$$

This proves that $(4c_1,\ 4c_2,\ 4c_3,\ 0) = 0$ in A, and we obtain that

$$c^4(z) = (z_1,\ z_2,\ z_3,\ \zeta_3^{-2} z_4 + \tilde{c}_4)$$

does not act freely on A. This proves that there is no hyperelliptic fourfold with group $C_2 \times C_2 \times C_6$ which does not satisfy property (a) of Theorem 3.5.

In view of the two examples given above, we pose the

Question 3.8. Let $d \in \{4, 6\}$. For which $n \geq 3$ and $0 \leq r \leq n-3$ do there exist hyperelliptic manifolds of dimension n with group $C_2^{n-1-r} \times C_d^r$, which do not satisfy condition (a) of Theorem 3.5?

We leave the question open in this thesis and hope to return to it in a forthcoming paper.

A similar question can of course be asked if we require the group to fulfill property (a) of Theorem 3.5. The following Proposition gives the negative side of the answer to this second question. The idea of proof simply relies on the fact that the subgroup of m-torsion points of an elliptic curve E is isomorphic to C_m^2, and thus it is impossible to choose three m-torsion points of E, which are C_m-linearly independent.

Proposition 3.9. *Let $n \geq 4$. Suppose that $G = C_3^{n-4} \times C_6^3$, or that G falls under Case (a) of Theorem 3.5 and is one of the groups*

$$C_2^{n-4} \times C_4^3, \quad C_2^{n-4} \times C_6^3.$$

Then there is no hyperelliptic fourfold of dimension n with group G.

Proof. Necessarily, $C_3^{n-4} \times C_6^3$, satisfies (a) of Theorem 3.5. Write $G = C_{d'}^{n-4} \times C_d^3$, where $(d', d) \in \{(3,6),\ (2,4),\ (2,6)\}$. We divide the proof into several steps.

Step 1: Consider the subgroup $U := C_{d'}^{n-1}$ of $C_{d'}^{n-4} \times C_d^3$. Then, since (a) of Theorem 3.5 holds, we may assume that U is embedded into $\mathrm{GL}(n, \mathbb{C})$ by

$$(a_1,\ ...,\ a_{n-1}) \mapsto \mathrm{diag}\left(\zeta_{d'}^{a_1},\ ...,\ \zeta_{d'}^{a_{n-1}},\ 1\right).$$

Assume that A is an Abelian variety, on which a group isomorphic to G acts freely and such that the associated complex representation $\rho|_U$ is the above embedding. Choosing

$$a_1 = ... = a_{n-1} = 1$$

corresponds to $g := \mathrm{diag}(\zeta_{d'}, ..., \zeta_{d'}, 1) \in G$. Thus, $E_n := \ker(g - \mathrm{id}_A)^0$ is an elliptic curve, and A is isogenous to $A' \times E$, where $A' := \mathrm{im}(g - \mathrm{id}_A)$. Continuing in a similar way, we see that A' is isogenous to a product of elliptic curves as well. In total, we obtain that A is isogenous to a product of n elliptic curves $E_j \subset A$:

$$A \cong (E_1 \times ... \times E_n)/H.$$

Step 2: Denote generators of the subgroup $U \subset \mathrm{Bihol}(A)$ by

$$g_j(z) := (z_1 + a_{1j},\ ...,\ z_{j-1} + a_{j-1,j},\ \zeta_{d'} z_j + a_{jj},\ z_{j+1} + a_{j+1,j},\ ...,\ z_n + a_{nj}),$$

$1 \leq j \leq n-1$.
Then the relations $\left(g_1^{\delta_1} \cdot ... \cdot g_{n-1}^{\delta_{n-1}}\right)^{d'} = \mathrm{id}_A$ for $\delta_j \in \{1, d'-1\}$ imply that

$$(0,\ ...,\ 0,\ 0,\ 0,\ ...,\ \delta_1 d' a_{n1} + ... + \delta_1 d' a_{n,n-1}) \in H.$$

Since E_n embeds into A, we obtain that

$$\delta_1 d' a_{n1} + ... + \delta_{n-1} d' a_{n,n-1} = 0 \text{ in } E_n.$$

Suppose now that $h_1,\ h_2,\ h_3 \in G$ span a subgroup isomorphic to C_d^3. By replacing the h_j by the product of h_j and a suitable product of the $g_1, ..., g_{n-1}$, we may assume that

the eigenvalues of h_j are only 1 and primitive d-th roots of unity for $j = 1, 2, 3$. Suppose furthermore that h_j acts on E_n by a translation $z_n \mapsto z_n + b_j$.
We claim that $b_j \in E_n[d]$: in fact, since

$$\tilde{h}_j := h_j \cdot g_1 \cdot ... \cdot g_{n-1}$$

acts on E_n by $z_n \mapsto z_n + b_j + a_{n1} + ... + a_{n,n-1}$, and because only the last diagonal entry of $\rho(\tilde{h}_j)$ is different from 1, the condition that $\tilde{h}_j$ has order d implies that

$$(0, ..., 0, db_j + \underbrace{da_{n1} + ... + da_{n,n-1}}_{=0}) = (0, ..., 0, db_j) \in H.$$

Again, this implies that $db_j = 0$ in E_n, as claimed.

Step 3: Since h_1, h_2, h_3 span a subgroup isomorphic to C_d^3, but $E_n[d] \cong C_d^2$, there is an element $h \in G$ of order d such that h acts on E_n by a translation of order $\leq d'$.
The element h can be chosen to be one of

$$h_1^a \cdot h_2^b \cdot h_3^c$$

where $a, b, c \in \{0, ..., d\}$ are furthermore chosen such that $h = h_1^a \cdot h_2^b \cdot h_3^c$ has order d. We prove that we can assume without loss of generality that no eigenvalue of h is a primitive d'-th root of unity:
Suppose that exactly the first r diagonal entries of h are primitive d'-th roots of unity: then we may choose $\delta_1, ..., \delta_r \in \{1, d' - 1\}$ such that no eigenvalue of

$$h' := h \cdot g_1^{\delta_1} \cdot ... \cdot g_r^{\delta_r}$$

is a primitive d'-root of unity. Then $h = h' \cdot g_1^{d-\delta_1} \cdot ... \cdot g_r^{d-\delta_r}$ acts on E_n by a translation of order $\leq d'$. Consider the element

$$h \cdot g_{r+1} \cdot ... \cdot g_{n-1} = \overbrace{h' \cdot \underbrace{g_1^{d-\delta_1} \cdot ... \cdot g_r^{d-\delta_r}}_{} }^{\text{transl. of order } \leq d' \text{ on } E_n} \cdot g_{r+1} \cdot ... \cdot g_{n-1}.$$

$$\underbrace{g_1^{d-\delta_1} \cdot ... \cdot g_r^{d-\delta_r} \cdot g_{r+1} \cdot ... \cdot g_{n-1}}_{\text{acts by translation of order } d' \text{ on } E_n \text{ (Step 2)}}$$

We obtain that both $h' \cdot g_{r+1} \cdot ... \cdot g_{n-1}$ and $g_1^{d-\delta_1} \cdot ... \cdot g_r^{d-\delta_r}$ act on E_n by a translation of order $\leq d'$. Thus also the element h' of order d acts on E_n by a translation of order $\leq d'$.
This proves that we can modify h accordingly so that we may assume that h no eigenvalue of h is a primitive d'-th roots of unity. By Step 2, the element $h \cdot g_1 \cdot ... \cdot g_{n-1}$ of order d acts on E_n by a translation of order $\leq d'$, and by construction, the d'-th power of this element acts trivially on A, a contradiction. □

We obtain the following Corollary as a consequence of the previous Proposition and Theorem 3.5.

Corollary 3.10. *There is no hyperelliptic variety of dimension $n \in \{4, 5\}$ whose group is isomorphic to one of the following groups:*

$$C_2^{n-4} \times C_4^3, \quad C_2^{n-4} \times C_6^3, \quad C_3^{n-4} \times C_6^4.$$

We will see later in Section 3.1.1 that Corollary 3.10 is indeed sharp.

From now on, we consider Abelian groups with only two factors, and show that as long as the dimension n is large enough, we can construct hyperelliptic n-folds $X = A/G$ with group $G = C_{e_1} \times C_{e_2}$:

Lemma 3.11. *Suppose that $G = C_{e_1} \times C_{e_2}$ (where e_1 is not necessarily a divisor of e_2). If $e_1, e_2 \geq 3$ and $n > \frac{\varphi(e_1)+\varphi(e_2)}{2}$, then there exists a hyperelliptic n-fold with group G.*

Proof. Write $n_j := \frac{\varphi(e_j)}{2}$, which is an integer because $e_j \geq 3$. By the method described in see [Cat14, Sections 5.4], there exist Abelian varieties A_j of dimension n_j, which admit a linear automorphism α_j of order e_j. Let A' be any Abelian variety of dimension $n - n_1 - n_2 > 0$, and define an action of G on $A := A_1 \times A_2 \times A'$ (with coordinates (z_1, z_2, z')) as follows:

$$g_1(z) = \left(\alpha_1 z_1,\ z_2,\ z' + h_1\right),$$
$$g_2(z) = \left(z_1,\ \alpha_2 z_2,\ z' + h_2\right),$$

where h_j is an e_j-torsion element of A', such that $\langle h_1 \rangle \cap \langle h_2 \rangle = \{0\}$. It follows immediately from the construction that $\langle g_1, g_2 \rangle \cong G$, and that G acts freely on A. $\square$

As a consequence, we obtain the existence of hyperelliptic fourfolds with groups

$$C_6 \times C_{10} \cong C_2 \times C_{30}, \quad C_4 \times C_{10} \cong C_2 \times C_{20}, \quad C_6 \times C_8 \cong C_2 \times C_{24}.$$

Remark 3.12. (a) The above Lemma is not true in general if we allow the e_j to be 2: it has been known since Bagnera-de Franchis [BdF08] and Enriques-Severi [ES09] that there is no hyperelliptic surface with group $C_2 \times C_2$ (i.e., not containing any translation).

(b) The converse of Lemma 3.11 is false in general, since we do not require elements $g \in G$ to act with primitive $\mathrm{ord}(g)$-th roots of unity. Keeping this in mind, it is easy to construct an example of a hyperelliptic 5-fold with group $C_6 \times C_{30}$, but $4 + 1 \geq 5 - 1 = 4$. However, we can give a partial converse in dimension 4 by checking manually that certain groups do not occur:

Lemma 3.13. *There are no hyperelliptic fourfolds whose group is contained in the following list:*

$$C_3 \times C_{24}, \quad C_4 \times C_{24}, \quad C_8 \times C_8, \quad C_{12} \times C_{12},$$
$$C_3 \times C_{15}, \quad C_2 \times C_{14}, \quad C_2 \times C_{18}, \quad C_4 \times C_{20}.$$

Proof. Suppose that $G = C_{d_1} \times C_{d_2}$ is generated by g_1, g_2 of respective orders d_1, d_2. Recall that if G occurs as a group of a hyperelliptic fourfold, then any $g \in G$ must have the eigenvalue 1. We deal with the cases separately:

To $C_m \times C_{24}$, $m \in \{3, 4\}$: Lemma 2.5 (a) implies that that g_2 cannot have eigenvalues of order 24. Thus we can assume that

$$g_2 = \mathrm{diag}(1, \zeta_3, \zeta_8, \zeta_8^b).$$

Suppose first that $m = 3$. Then, by possibly replacing g_1 by $g_1 g_2^8$ or $g_1 g_2^{16}$, we may assume that

$$g_1 = \operatorname{diag}(\zeta_3^a, 1, 1, 1),$$

because $g_1 g_2$ must not have eigenvalues of order 24: then $g_1 g_2$ does not have the eigenvalue 1, a contradiction.
If $m = 4$, we may apply Lemma 2.2 to assume that

$$g_1 = \operatorname{diag}(i^{a_1}, 1, i^{a_3}, i^{a_4}), \text{ where } i = \zeta_4 = \zeta_8^2.$$

Since $g_1 g_2$ must have the eigenvalue 1, we obtain that $a_1 \equiv 0 \pmod 4$. By Lemma 2.12, $b \in \{3, 5\}$. We can assume without loss of generality that $a_3 = 1$. Because g_1 is not contained in the subgroup generated by g_2, we obtain $2a_4 \not\equiv 2b \pmod 8$, and therefore

$$2a_4 \equiv 2b + k \pmod 8, \qquad \text{where } k \in \{2, 4, 6\}.$$

Now, one checks that

$$g_1 g_2 = \operatorname{diag}\left(1, \zeta_3, \zeta_8^3, \zeta_8^{3b+k}\right) \text{ or } g_1^2 g_2 = \operatorname{diag}\left(1, \zeta_3, \zeta_8^5, \zeta_8^{5b+2k}\right)$$

has multiple or conjugate eigenvalues for any $b \in \{3, 5\}$ and $k \in \{2, 4, 6\}$:

b	k	$3b + k \pmod 8$	$5b + 2k \pmod 8$
3	2	3	-
3	4	5	-
3	6	-	5
5	2	-	3
5	4	3	-
5	6	5	-

<u>To $C_8 \times C_8$:</u> By Remark 2.3, any element of order 8 must have exactly two eigenvalues of order 8. For this reason, and because the element $g_1 g_2$ of order 8 must have the eigenvalue 1, we may assume that g_1 and g_2 are given as follows:

$$g_1 = \operatorname{diag}\left(1,\ i^a,\ \zeta_8,\ \zeta_8^b\right),$$
$$g_2 = \operatorname{diag}\left(1,\ i^c,\ \zeta_8,\ \zeta_8^d\right),$$

where $b, d \in \{3, 5\}$. The element $g_1^4 g_2^4$ is then trivial, contradicting the assumption that the subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ intersect trivially.
(One could also prove the statement by noting that $g_1 g_2^{-1}$ would have only one eigenvalue of order 8.)

<u>To $C_{12} \times C_{12}$:</u> We treat two cases separately.

Case 1: If g_1 has eigenvalues of order 12, we may assume that

$$g_1 = \operatorname{diag}(1,\ *,\ \zeta_{12},\ \zeta_{12}^a)$$

for some a coprime to 12 and some second diagonal entry, which is not a primitive 12-th root of unity (cf. Lemma 2.2). We claim that g_2 does not have eigenvalues of order 12: if it would have eigenvalues of order 12, Lemma 2.2 would imply that

$$g_2 = \mathrm{diag}(*,\ *,\ \zeta_{12},\ \zeta_{12}^b)$$

for some b coprime to 12 satisfying $b \not\equiv a \pmod{12}$ and some first and second diagonal entries, which are not primitive 12-th roots of unity. By Lemma 2.12 we can assume that $a = 5$ and $b = 7$. Consider the element $g_1 g_2^4$ of order 12: by construction, it has exactly one eigenvalue of order 12, contradicting Lemma 2.2.
Thus g_2 does not have eigenvalues of order 12. Then we may assume that g_2 has (at least) the eigenvalues i, 1 and one of $\pm\zeta_3$. In addition to that, we may assume that at most one of the third and fourth diagonal entry of g_2 is different from 1 (else, replace g_2 by $g_2 g_1^j$ for a suitable j). Then the last two diagonal entries of g_2 fall, up to symmetry and taking powers of g_2, into one of the following cases:

(i) $(-1, 1)$: in this case, $g_1 g_2$ has two times the same (if $a = 7$) or conjugate (if $a = 5$) eigenvalues of order 12, so that this case cannot occur.

(ii) $(\zeta_3, 1)$: here, the element $g_1^{-1} g_2$ of order 12 only has one eigenvalue of order 12, contradicting Lemma 2.2.

Hence the last two diagonal entries of g_2 are 1 and thus the first two diagonal entries of g_2 are both different from 1. This implies that one of the elements $g_1^6 g_2$, $g_1^4 g_2$ does not have the eigenvalue 1, and finally that Case 1 does not occur.

Case 2: We are left with the case in which both g_1 and g_2 do not have eigenvalues of order 12. Then the order of the eigenvalues of the elements g_j^3 (resp. g_j^4) divides 4 (resp. 3). Since g_1^3 and g_2^3 span a subgroup isomorphic to $C_4 \times C_4$, after possibly replacing g_j^3 by g_j^9, we may assume that $g_1^3 g_2^3$ has the eigenvalue i with multiplicity at least 2. Similarly, we may assume that $g_1^4 g_2^4$ has the eigenvalue ζ_3 with multiplicity at least 2. But then $g_1^7 g_2^7$ either does not have the eigenvalue 1 (which is not possible) or has the eigenvalue ζ_{12} with multiplicity 1 or 2 (which is not possible either, since it contradicts Lemma 2.2). Henceforth Case 2 does not occur either.

<u>To $C_3 \times C_{15}$</u>: By Lemma 2.5 (a), g_2 must not have eigenvalues of order 15. Thus, we may assume that

$$g_2 = \mathrm{diag}\left(1, \zeta_3, \zeta_5, \zeta_5^a\right).$$

for some a coprime to 5. Since $g_1 g_2$ must have the eigenvalue 1 and must not have eigenvalues of order 15, we obtain that $g_1 = \mathrm{diag}\left(1, \zeta_3^b, 1, 1\right) \in \langle g_2 \rangle$, a contradiction, since then $\langle g_1, g_2 \rangle$ would be cyclic.

<u>To $C_2 \times C_m$, $m \in \{14, 18\}$</u>: By Lemma 2.2, we may assume that

$$g_2 = \mathrm{diag}\left(1, \zeta_m, \zeta_m^a, \zeta_m^b\right).$$

Since g_1 is not contained in the subgroup spanned by g_2,

$$g_1 \neq \mathrm{diag}(1, -1, -1, -1).$$

Moreover, the first diagonal entry of g_1 has to be 1, since $g_1 g_2$ must have the eigenvalue 1: but then the element $g_1 g_2$ of order m has at most two eigenvalues of order m, contradicting Lemma 2.2.

To $C_4 \times C_{20}$: By Lemma 2.5 (a), g_2 does not have eigenvalues of order 20. We may therefore assume that

$$g_2 = \operatorname{diag}(1, i, \zeta_5, \zeta_5^a) \text{ or } g_2 = \operatorname{diag}(1, i, -\zeta_5, -\zeta_5^a).$$

Using in addition Lemma 2.2, it suffices to exclude the case

$$g_1 = \operatorname{diag}\Big(1,\ i,\ (-1)^\delta,\ (-1)^\delta\Big),$$

where $\delta \in \{0, 1\}$: in this case, g_1^2 is contained in the subgroup spanned by g_2, a contradiction. □

Remark 3.14. There exists no hyperelliptic fourfold whose group is one of

- $C_3 \times C_3 \times C_{12}$ (GAP ID [108,35]); this follows essentially from Theorem 3.5: all elements in the group would have to share a common eigenvector for the eigenvalue 1, so that a matrix similar to $\operatorname{diag}(\zeta_3,\ \zeta_3,\ \zeta_{12},\ 1)$ would be contained in the group, contradicting Lemma 2.2.
- $C_2 \times C_2 \times C_8$ (GAP ID [32,36]); that this group cannot occur follows essentially from Lemma 2.2 and Lemma 2.12: assume that $g_1 = \operatorname{diag}(\zeta_8,\ \zeta_8^a,\ *,\ 1)$, $a \in \{3, 5\}$ spans a subgroup of order 8. Since all matrices in the group must have the eigenvalue 1, necessarily we find a matrix of order 2 of the form $g_2 = \operatorname{diag}(-1,\ 1,\ *,\ *)$. But then $g_1 g_2$ has conjugate or multiple eigenvalues of order 8, contradicting Lemma 2.2.

3.1.1 Some examples

In this section, we will describe certain examples of hyperelliptic fourfolds with Abelian groups as a step towards obtaining a complete list of possible groups associated with hyperelliptic fourfolds, cf. Theorem 11.2.

The following two examples show that Corollary 3.10 cannot be improved for $n = 4$, i.e., we give an example of hyperelliptic fourfolds with group $C_3 \times C_6 \times C_6$. Example 3.17 shows the existence of a hyperelliptic fourfold with group $C_2 \times C_6 \times C_6$.

Example 3.15. Before we state an example of a hyperelliptic fourfold A/G with group $G = C_3 \times C_6 \times C_6$, we observe that Theorem 3.5 implies that A necessarily is isogenous to $F \times F \times F \times E$, where F is the equianharmonic elliptic curve and E is an arbitrary elliptic curve.
Now we state the example. Define

$$A := F \times F \times F \times E, \text{ where}$$
$$E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \text{ and } F = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z}).$$

Define moreover $g_1,\ g_2,\ g_3 \in \mathrm{Bihol}(A)$ as follows:

$$g_1(z) = \left(\zeta_3 z_1,\ z_2,\ z_3,\ z_4 + \frac{\tau}{3}\right),$$
$$g_2(z) = \left(z_1 + \frac{\zeta_3 - 1}{3},\ \zeta_6 z_2,\ z_3,\ z_4 + \frac{1}{6}\right),$$
$$g_3(z) = \left(z_1,\ z_2,\ \zeta_6 z_3,\ z_4 + \frac{1}{6} + \frac{\tau}{2}\right).$$

Then $\mathrm{ord}(g_1) = 3$ and $\mathrm{ord}(g_2) = \mathrm{ord}(g_3) = 6$. Moreover, it is immediate to see that g_i and g_j commute for each i and j (the maps g_1 and g_2 commute because $\frac{\zeta_3-1}{3}$ is fixed by multiplication by ζ_3). This implies that $G := \langle g_1,\ g_2,\ g_3\rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $C_3 \times C_6 \times C_6$. It remains to prove that G acts freely on A.
Again, the element

$$g_1^a \cdot g_2^b \cdot g_3^c, \quad a \in \{0,1,2\},\ b,c \in \{0,...,5\},\ (a,b,c) \neq (0,0,0) \tag{3.2}$$

acts freely on A, except possibly when it acts trivially on E. The element (3.2) acts trivially on E if and only if $a = 0$, $c \in \{2,4\}$ and $b = 6 - c$. In this case, the above element is

$$g_2^{6-c} g_3^c = \left(z_1 + \underbrace{(6-c) \cdot \frac{\zeta_3 - 1}{3}}_{\neq 0 \text{ in } F \text{ for } c \in \{2,4\}},\ \zeta_6^{6-c} z_2,\ \zeta_6^c z_3, z_4\right),$$

which acts on the first elliptic curve by a non-trivial translation. This proves that A/G is an example of a hyperelliptic fourfold with group $C_3 \times C_6 \times C_6$.

The last two examples we will describe are hyperelliptic manifolds with group $C_2 \times C_4 \times C_{12}$ resp. $C_2 \times C_6 \times C_{12}$.

Example 3.16. As in the previous example, we first determine the isogeny type of an Abelian fourfold whose automorphism group contains a subgroup isomorphic to $C_2 \times C_4 \times C_{12}$. Let $g_1,\ g_2,\ g_3$ generate the group $C_2 \times C_4 \times C_{12}$, so that $g_1,\ g_2,\ g_3$ have respective orders 2, 4, 12. If $\rho(g_3)$ had eigenvalues of order 12, we may assume that $\rho(g_3) = \mathrm{diag}(\zeta_{12},\ \zeta_{12}^a,\ *,\ 1)$ for some a coprime to 12. It follows that after possibly replacing g_1 by $g_1 g_3^6$, we may assume that the first two diagonal entries of $\rho(g_1)$ are 1 (else, if this was not the case, $\rho(g_1 g_3)$ would have multiple or conjugate eigenvalues of order 12). By a similar reasoning, we obtain that we may assume that the first two diagonal entries of $\rho(g_2)$ are 1. We obtain a contradiction, since one of the matrices $\rho(g_2 g_3)$, $\rho(g_1 g_2 g_3)$ does not have the eigenvalue 1 (note that g_1 is not contained in the subgroup spanned by g_2).
Hence $\rho(g_3)$ does not have eigenvalues of order 12: then, up to taking powers, it has the eigenvalues 1, i and one of $\pm\zeta_3$:

$$\rho(g_3) = \mathrm{diag}(\ \zeta_3,\ i,\ *,\ 1).$$

As we have seen, no matrix in $\rho(G)$ can have eigenvalues of order 12, so that we may assume that the first two diagonal entries of g_2 are both equal to ± 1, respectively, and the third one is i. The fourth one, however, cannot be different from 1, since the two matrices $\rho(g_2 g_3)$, $\rho(g_2 g_3^2)$ must have the eigenvalue 1. Similarly, the fourth diagonal entry of $\rho(g_1)$ is 1. It follows that A is necessarily isogenous to a product of four elliptic curves, one of them being isomorphic to the equianharmonic elliptic curve and two of them being isomorphic to the harmonic elliptic curve. The fourth elliptic curve underlies no further restrictions, since we act on it only by a translation.

We now give an example of a hyperelliptic fourfold with group

$$G := C_2 \times C_4 \times C_{12}$$

to show its existence for our main result, Theorem 11.2. Suppose that

$$A = F \times E_i \times E_i \times E,$$

where $F = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$, $E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, and define the maps $g_1, g_2, g_3 \in \mathrm{Bihol}(A)$ as follows:

$$\begin{aligned} g_1(z) &= \left(-z_1,\ z_2,\ z_3 + \frac{1+i}{2},\ z_4\right), \\ g_2(z) &= \left(z_1,\ z_2,\ iz_3,\ z_4 + \frac{\tau}{4}\right), \\ g_3(z) &= \left(\zeta_3 z_1,\ iz_2,\ z_3,\ z_4 + \frac{1}{12}\right). \end{aligned}$$

It is clear that g_1, g_2, g_3 have respective orders 2, 4 and 12. Moreover, $g_2g_3 = g_3g_2$, $g_1g_3 = g_3g_1$ hold trivially. Finally, the relation $g_1g_2 = g_2g_1$ holds because $\frac{1+i}{2}$ is fixed by multiplication by i. Thus, $G := \langle g_1, g_2, g_3\rangle$ is a subgroup of $\mathrm{Bihol}(A)$, which is isomorphic to $C_2 \times C_4 \times C_{12}$. It is moreover clear that G contains no translations.
We prove that G acts freely on A: indeed, the element

$$g_1^a \cdot g_2^b \cdot g_3^c(z) = \left((-1)^a\zeta_3^c z_1,\ i^c z_2,\ i^b z_3 + a \cdot \frac{1+i}{2},\ z_4 + b \cdot \frac{\tau}{4} + c \cdot \frac{1}{12}\right), \tag{3.3}$$

$$a \in \{0,1\},\ b \in \{0,...,3\},\ c \in \{0,...,11\},\ (a,b,c) \neq (0,0,0)$$

acts freely on A if

$$b \cdot \frac{\tau}{4} + c \cdot \frac{1}{12} \neq 0 \text{ in } E.$$

If $b \cdot \frac{\tau}{4} + c \cdot \frac{1}{12} = 0$, then $b = c = 0$ and (3.3) equals g_1, which acts freely since its third coordinate is a non-trivial translation. In total, we have proved that A/G is a hyperelliptic fourfold with group $G = C_2 \times C_4 \times C_{12}$.

Example 3.17. By similar arguments as in the previous example, we show that any Abelian fourfold, whose automorphism group contains a subgroup isomorphic to $C_2 \times C_6 \times C_{12}$, is isogenous to the product of four elliptic curves, one of them being the harmonic elliptic curve, two of them being the equianharmonic elliptic curve. On the fourth elliptic curve, we act by translation, so we obtain no further restrictions.

We shall now describe a hyperelliptic fourfold with group

$$G := C_2 \times C_6 \times C_{12}.$$

Define

$$A = F \times E_i \times F \times E,$$

where $F = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$, $E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, and define the maps $g_1, g_2, g_3 \in \mathrm{Bihol}(A)$ as follows:

$$\begin{aligned}
g_1(z) &= \left(-z_1,\ z_2 + \frac{1+i}{2},\ z_3,\ z_4\right), \\
g_2(z) &= \left(z_1,\ z_2,\ -\zeta_3 z_3,\ z_4 + \frac{\tau}{6}\right), \\
g_3(z) &= \left(\zeta_3 z_1,\ iz_2,\ z_3,\ z_4 + \frac{1}{12}\right).
\end{aligned}$$

Clearly, by definition, g_1, g_2, g_3 have respective orders 2, 6, 12. Moreover, we immediately see that g_1 and g_2 as well as g_2 and g_3 commute. Moreover, g_1 and g_3 commute, since $\frac{1+i}{2}$ is fixed by multiplication by i. This proves that $G := \langle a, b, c, \rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $C_2 \times C_6 \times C_{12}$. Clearly, G contains no translations. To prove that G acts freely on A, consider the action of $g_1^a \cdot g_2^b \cdot g_3^c \neq \mathrm{id}_A$. This element acts on the fourth elliptic curve E by

$$z_4 \mapsto b \cdot \frac{\tau}{6} + c \cdot \frac{1}{12}.$$

Therefore $g_1^a \cdot g_2^b \cdot g_3^c$ acts freely on A if $b \cdot \frac{\tau}{4} + c \cdot \frac{1}{12} \neq 0$ in E. This is the case if and only if the element in discussion is different from g_1. However, g_1 acts freely on A as well, since it acts on the second elliptic curve by a non-trivial translation.
Thus, A/G is a hyperelliptic fourfold with group $G = C_2 \times C_6 \times C_{12}$.

Remark 3.18. We finish this section by determining the isogeny type of hyperelliptic fourfolds with respective groups $C_2 \times C_{30}$, $C_2 \times C_{20}$ and $C_2 \times C_{24}$, whose existences were proved on p. 54. Each of these three groups is isomorphic to exactly one group $C_{e_1} \times C_{e_2}$, $e_1, e_2 \geq 3$ satisfying

$$\frac{\varphi(e_1) + \varphi(e_2)}{2} < 4$$

as in Lemma 3.11:

$$\begin{aligned}
C_6 \times C_{10} &\cong C_2 \times C_{30}, \\
C_4 \times C_{10} &\cong C_2 \times C_{20}, \\
C_6 \times C_8 &\cong C_2 \times C_{24}.
\end{aligned}$$

Thus, the construction in the proof of Lemma 3.11 is the only possibility for these groups to act linearly on an Abelian fourfold, if we require any matrix to have the eigenvalue 1. We investigate which elliptic curves resp. Abelian surfaces admit automorphisms of order 4, 6 resp. 10, 8:

(i) elliptic curves admitting an automorphism of order 4: clearly, there is a unique such elliptic curve, the harmonic elliptic curve $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

(ii) elliptic curves admitting an automorphism of order 6: there is a unique such elliptic curve as well, the equianharmonic elliptic curve $\mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$.

(iii) Abelian surfaces admitting an automorphism of order 10: there is a unique such Abelian surface, see [CaCi93, Proposition 5.8 (i)]. (This surface is in fact simple: this follows from [ST61, Proposition 17].)

(iv) Abelian surfaces admitting an automorphism of order 8: there are exactly two isomorphism classes of such Abelian surfaces, see [CaCi93, Proposition 5.8 (ii)]. (By the above Proposition, these two Abelian surfaces are not simple. In fact, one can show that they are isomorphic to $E_{\sqrt{2}i} \times E_{\sqrt{2}i}$ and $E_i \times E_i$, respectively.)

Fix one of the above groups. It follows that A is isogenous to $E \times E' \times S$, where the group acts on the elliptic curv $E \subset A$ by translation, and on the elliptic curve $E' \subset A$ (resp. the Abelian surface $S \subset A$) by linear automorphisms of order e_1 (resp. e_2).

3.2 Abelian Symmetry on Complex Tori

Assume that G is a finite Abelian group,

$$G = C_{d_1} \times ... \times C_{d_k}, \quad \forall i\colon\ d_i | d_{i+1}, \quad d_1 > 1,$$

and that Λ is a $\mathbb{Z}[G]$-module, which is also a free Abelian group (not necessarily of even rank). The cases we are mostly interested in is the one where T/G, $T = V/\Lambda$ is a hyperelliptic manifold: in this case Λ is a $\mathbb{Z}[G]$-module. The aim of this section is to describe the data needed to define a $\mathbb{Z}[G]$-module Λ, which is also a free Abelian group in terms of certain cyclotomic rings. Our description is obtained by embedding $\mathbb{Z}[G]$ into the direct sum of these cyclotomic rings, such that the image has finite index in the target. This is a generalization of [Cat19, Proposition 3.1], which dealt with the cyclic case.
As a byproduct, we obtain a new way of decomposing a complex torus T as a product T' of certain complex subtori of T, where the decomposition is a decomposition up to isogeny, $T \cong T'/\Lambda^0$.

Definition 3.19. We set $R(d_1, ..., d_k) := \mathbb{Z}[G]$. With this notation, we have

$$R(d_1, ..., d_k) = R(d_1) \otimes ... \otimes R(d_k).$$

Moreover, we define $R_d := \mathbb{Z}[X]/(\Phi_d)$, where Φ_d is the d-th cyclotomic polynomial.

Proposition 3.20. *[AA69, pp. 247-250]*
There are positive integers r_d indexed by the positive divisors d of d_k, such that we have an inclusion

$$\iota\colon R(d_1, ..., d_k) \hookrightarrow \bigoplus_{d|d_k} R_d^{r_d}$$

and $\operatorname{coker}(\iota)$ *is finite.*

Proof. We divide the proof into several steps.

Step 1: We observe that $R(d_1, ..., d_k)$ embeds into

$$\bigotimes_{j=1}^{k} \bigoplus_{d|d_j} R_{d_j} =: \bigotimes_{j=1}^{k} R'(d_j) =: R'(d_1, ..., d_k),$$

since the $R(d_j)$ are flat $\mathbb{Z}$-modules and because each $R(d_j)$ embeds into $\bigoplus_{d|d_j} R_{d_j}$ by the Chinese Remainder Theorem.

Step 2: The cokernel of the embedding $R(d_1, ..., d_k) \hookrightarrow R'(d_1, ..., d_k)$ is finite, since the cokernels of all the embeddings $R(d_j) \hookrightarrow R'(d_j)$ are finite.

Step 3: We prove that there is an embedding of $R'(d_1, ..., d_k)$ into $\bigoplus_{d|d_k} R_d^{r_d}$ such that the cokernel is finite, if the r_d are chosen appropriately.
We shall write $\mathbb{Q}_d := R_d \otimes_{\mathbb{Z}} \mathbb{Q}$.

Claim: Let $i \leq j$. Then $\mathbb{Q}_{d_i} \otimes_{\mathbb{Q}} \mathbb{Q}_{d_j}$ is isomorphic to $\mathbb{Q}_{d_j}^{\varphi(d_i)}$, where φ is the Euler totient function.

Proof of the Claim: Let $\Phi_{d_j} = h_1 \cdot ... \cdot h_l$ be the irreducible decomposition of Φ_{d_j} in $\mathbb{Q}_{d_i}[X]$. Since $\mathbb{Q}_{d_i}(\zeta_{d_j}) = \mathbb{Q}_{d_j}$, all h_j's have the same degree, which is equal to $\varphi(d_j)/\varphi(d_i)$. This proves that $l = \varphi(d_i)$.
Now, by the Chinese Remainder Theorem, we have an isomorphism

$$\begin{aligned}\mathbb{Q}_{d_i} \otimes_{\mathbb{Q}} \mathbb{Q}_{d_j} &= \mathbb{Q}_{d_i} \otimes_{\mathbb{Q}} \mathbb{Q}[X]/(\Phi_{d_j}) = \mathbb{Q}_{d_i}[X]/(\Phi_{d_j}) \\ &\cong \bigoplus_{j=1}^{\varphi(d_i)} \mathbb{Q}_{d_i}[X]/(h_j) \qquad \cong \mathbb{Q}_{d_j}^{\varphi(d_i)}.\end{aligned}$$

This proves the Claim.

Since, by Gauß' Lemma [Lan02, Corollary IV.2.2], the irreducible decomposition of Φ_{d_j} in $\mathbb{Q}_{d_i}[X]$ is the same as the one in $R_{d_i}[X]$, the Claim tells us that $R_{d_i} \otimes R_{d_j}$ embeds into $R_{d_j}^{\varphi(d_i)}$ as a submodule of finite index. Using the distributive law of the tensor product and direct sum and consequence of the Claim multiple times, we obtain an embedding as claimed in Step 3. □

Remark 3.21. According to [AA69, Theorem 2], the number r_d is equal to the number of cyclic subgroups of order d of G.

Since $\Lambda \otimes \mathbb{Q}$ is a $\bigoplus_{d|d_k}(R_d^{r_d} \otimes \mathbb{Q})$-module, we get, by [Cat14, Lemma 24], a splitting

$$\Lambda \otimes \mathbb{Q} = \bigoplus_{d|d_k} \bigoplus_{i=1}^{r_d} \Lambda_{d,i,\mathbb{Q}},$$

where $\Lambda_{d,i,\mathbb{Q}}$ is an $R_d \otimes \mathbb{Q}$-module.

Defining $\Lambda_{d,i} := \Lambda_{d,i,\mathbb{Q}} \cap \Lambda$, we obtain the exact sequence

$$0 \to \bigoplus_{d|d_k} \bigoplus_{i=1}^{r_d} \Lambda_{d,i} \to \Lambda \to \Lambda^0 \to 0,$$

where Λ^0 is a finite Abelian group.

Now define the real tori $T := (\Lambda \otimes_{\mathbb{Z}} \mathbb{R})/\Lambda$ and $T_{d,i} := (\Lambda_{d,i} \otimes \mathbb{R})/\Lambda_{d,i}$, so that we have an exact sequence

$$0 \to \Lambda^0 \to T' := \bigoplus_{d|d_k} \bigoplus_{i=1}^{r_d} T_{d,i} \to T \to 0. \tag{3.4}$$

We are now in the situation to prove a structure theorem for $R(d_1, ..., d_k)$-modules, which are free Abelian groups.

Proposition 3.22. *The datum of an $R(d_1, ..., d_k)$-module Λ, which is a free Abelian group of finite rank is equivalent to giving*

(I) for each positive divisor d of d_k modules $\Lambda_{d,1}, ..., \Lambda_{d,r_d}$ over R_d, where r_d is the number of cyclic subgroups of G of order d, and

(II) a finite subgroup $\Lambda^0 \subset T' := \bigoplus_{d|d_k} \bigoplus_{i=1}^{r_d} T_{d,i}$, where $T_{d,i} := (\Lambda_{d,i} \otimes_{\mathbb{Z}} \mathbb{R})/\Lambda_{d,i}$

such that (a) and (b) hold:

(a) The module Λ^0 is stable under multiplication by elements of the ring $R(d_1, ..., d_k)$.

(b) $\forall d|d_k, \forall i:\ \Lambda_0 \cap T_{d,i} = \{0\}$.

Proof. The lattice Λ is determined by (I), (II) and the property (a) that Λ^0 is stabilized by $R(d_1, ..., d_k)$, since we can define $T := T'/\Lambda^0$. Now, (b) implies that Λ intersects $\Lambda_{d,i} \otimes_{\mathbb{Z}} \mathbb{R}$ exactly in $\Lambda_{d,i}$. □

We apply the previous result to a complex torus $T = V/\Lambda$ of dimension n, whose group of biholomorphic self-maps contains a finite Abelian subgroup G isomorphic to (3.1). Assume furthermore that G contains no translations. The group G acts on Λ by conjugation, that is, linearly. This turns Λ into a $\mathbb{Z}[G]$-module. By Proposition 3.22, we have an isogeny

$$\bigoplus_{d|d_k} \bigoplus_{i=1}^{r_d} T_{d,i} \to T$$

with kernel Λ^0.

Remark 3.23. Given a hyperelliptic manifold $X = T/G$ of dimension n with group G as in (3.1) and suppose that g_i generates C_{d_i}. One could as well define the 2^k Abelian subvarieties T_I indexed by subsets $I \subset \{1, ..., k\}$ of A as follows:

$$T_I := \left(\bigcap_{i \in I} \ker(g_i - \mathrm{id}_T) \cap \bigcap_{j \notin I} \mathrm{im}(g_j - \mathrm{id}_T) \right)^0.$$

It follows that T is isogenous to $\prod_{I \subset \{1,...,k\}} T_I$. This decomposition of T was used by Lange in [La01]. As stated by Lange in Remark 4.7 (c) of *loc. cit.*, it seems to be complicated to describe the kernel of the addition map $\prod_{I \subset \{1,...,k\}} T_I \to T$.
This seems to be the case for the decomposition $\bigoplus_{d|d_k} \bigoplus_{i=1}^{r_d} T_{d,i}$ given in this thesis, too. However, we hope that our module-theoretic description helps to describe the torsion group Λ^0 and the decomposition of T up to isogeny in more detail, for instance:

(i) In [Cat19, Proposition 3.1] it was proved that the number of such Λ^0 is finite, if G is cyclic. This result does not seem to easily generalize to the case where G is Abelian. However, we have no counterexample either.

(ii) Observe that $r_1 = 1$, and that $T_{1,1} \subset T$ is the subtorus on with G acts by translation. Under which hypothesis on (the action of) G is $T_{1,1} \neq 0$, i.e., when does the complex representation $\rho\colon G \to \mathrm{GL}(V)$ contain the trivial representation? This is related to Question 3.8.

We will deal with these problems in a future project.

Chapter 4

Strategy for the Classification in the Non-Abelian Case

Our goal is to determine all isomorphism types of finite groups G which admit an embedding into $\mathrm{Bihol}(A)$ for some Abelian fourfold A such that the image contains no translations and acts freely on A. It is worthwhile to describe the strategy used for determining these groups G, which will be done in the next sections (see Chapter 11 for our results).
Since we have already described the 5- and the 7-Sylow subgroups of G (cf. Lemmas 2.13 and 2.14), we will proceed by describing the possible 2- and 3-Sylow subgroups of G. We will sketch below the strategy for finding all possible (non-Abelian) 2- and 3-Sylow subgroups.

- Regarding the 3-Sylow subgroups S of G (cf. Section 9.2), it turns out that if S is non-Abelian, the complex representation ρ splits as a direct sum of a 1-dimensional and an irreducible representation of dimension 3. We prove that S has a *faithful* irreducible representation ρ_3' of dimension 3 (see Proposition 7.2) and consider the exact sequence

  $$1 \to K \to S \overset{\det(\rho_3'(\cdot))}{\to} C_m \to 1$$

  Using the classification of subgroups of $\mathrm{SL}(3, \mathbb{C})$ by Miller, Blichfeldt, and Dickson [MBD16, Chapter XII] as well as Yau and Yu [YY93, p.2 f.] we find that K is (up to isomorphism) either a subgroup of $C_3 \times C_3$ or the Heisenberg group of order 27 (see page 83). Since $m \in \{3, 9\}$, a full classification of 3-Sylow subgroups of G is then possible:
  We first consider the case where S is non-Abelian and contains an element of order 9. The main result is Proposition 7.6, which tells us that if S is non-Abelian and contains an element of order 9, then S contains a subgroup isomorphic to M_{27}, the unique non-Abelian group of order 27 and exponent 9. In Proposition 7.7 we then prove that there is no hyperelliptic fourfold with group M_{27}. This shows that if S is non-Abelian, it has exponent 3.
  Example 8.17 shows that the Heisenberg group of order 27 can indeed occur as a group associated with a hyperelliptic fourfold. However, no group of order 81 with exponent 3 can occur, cf. Section 7.3.

- As expected, the classification of 2-Sylow subgroups S' of G is much more involved. Lemma 6.1 describes all possibilities in the case where S' is Abelian, hence we assume in the sequel that S' is non-Abelian: this means that the complex representation ρ splits as a direct sum of two representations of degree 2, where we can assume one of them to be irreducible. One of the main difference to the investigation of the 3-Sylow subgroups is that S' does not need to have a *faithful* irreducible representation. Assuming first that S' has a faithful irreducible representation ρ_2 of dimension 2, we may again use the exact sequence

$$1 \to K \to S' \overset{\det(\rho_2(\cdot))}{\to} C_m \to 1.$$

The kernel K contains a unique element of order 2 (which corresponds to the matrix $\mathrm{diag}(-1,-1)$). It is implied by [Ha59, Theorem 12.5.2] that K is either cyclic or a generalized quaternion group. We easily see that $|K| \leq 8$ (where equality is achieved only if K is the quaternion group of order 8), and by Lemma 2.12, we obtain $m \leq 4$. Thus $|S'| \leq 32$.
If S' does not have a faithful irreducible representation, we consider the quotient $S'/\ker(\rho_2)$ instead, so that induced representation on this group is faithful. It is not difficult to see that $\ker(\rho_2)$ is cyclic of order 2 or 4 (Corollary 6.5), so that $|S'| \leq 128$.
After having carried out the classification of all possible 2-Sylow groups S', it turns out that in fact $|S'| \leq 32$ even if S' does not have a faithful irreducible representation (or is Abelian), see Theorem 9.1.

Having obtained sufficient information obout the Sylow subgroups of G, we put the puzzle pieces together and investigate groups of order $|G| = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$.

- If $|G| = 2^a \cdot 5$, $2^a \cdot 7$, $3^b \cdot 5$ or $3^b \cdot 7$, it turns out that G is indeed Abelian, cf. Chapter 10. We prove the statements by gradually increasing the exponents a and b, first showing e.g. that G cannot have a non-Abelian subgroup of order $14 = 2 \cdot 7$ and then moving on to consider groups of order $28 = 2^2 \cdot 7$, and so on.
- If $2^a \cdot 3^b \cdot 7$ divides the order of G, it suffices to consider only non-solvable groups (else, by Hall's Theorem [Ha59, Theorem 9.3.1], G has a subgroup of order $3^b \cdot 7$, which implies $b = 0$ as mentioned in the previous bullet point). We can therefore assume that both a and b are different from 0, and that G is not solvable. This is dealt with in Section 10.6: we first exclude Klein's simple group of order 168. We gradually increase a and b; for G to be non-solvable, it is necessary and sufficient for

$$K = \ker(g \mapsto \det(\rho(g)))$$

to be non-solvable. The classification of the 2- and 3-Sylow groups of G tells us that groups of order 27 or 32 do not embed into $\mathrm{SL}(4, \mathbb{C})$ if they are associated with a hyperelliptic fourfold. Hence, K cannot contain a subgroup of order 27 or 32 (see Lemma 10.23). In all other cases, we show that G does not act freely on any Abelian fourfold (Lemmas 10.25 and 10.28).

In total, we show that if G contains an element of order 5 or 7, then G is Abelian (Theorem 10.1).
The most involved part is the study of groups G of order $2^a \cdot 3^b$. We will use our results about the Sylow subgroups of G and our some necessary conditions G has to fulfill to obtain a list of groups of order $2^a \cdot 3^b$ associated with a hyperelliptic fourfold. In order to simplify the computations, we concentrate on *maximal* such groups: we call a finite group G *maximal* if it satisfies the following properties:

(i) G occurs as a group of a hyperelliptic fourfold.

(ii) If G' occurs as a group of a hyperelliptic fourfold and contains G, then $G = G'$.

We produce a list of maximal groups of order $2^a \cdot 3^b$, see Section 8.1 for a detailed description and a list of these maximal groups. In order to prove that the groups listed in the output of the script are indeed maximal, we prove that quite a number of groups cannot occur as groups of hyperelliptic fourfolds. These non-examples are then added to a list of 'forbidden groups' contained in GAP Script `maximal_groups.g`. If the list of forbidden groups is large enough, the output of the script will only consist of maximal groups (which by definition occur as groups of hyperelliptic fourfolds), and we are done.

Remark 4.1. GAP Script `maximal_groups.g` only finds the groups which are maximal among the groups of order $2^a \cdot 3^b$. However, this is not a problem, since, as explained, the existence of an element of order 5 or 7 forces the group to be Abelian.

By this point, we hope to have convinced the reader that even a full classification of hyperelliptic fourfolds $X = A/G$, where G is *maximal* of order $2^a \cdot 3^b$, seems not feasible, since there would be too many different group actions of the same group G on an Abelian fourfold A. For this reason, we decided to give for each maximal group of order $2^a \cdot 3^b$ only *one* example for each possible combination of irreducible representations occurring in the complex representation ρ (up to automorphisms of G). However, we shall always deduce in general the isogeny type of the Abelian variety A and hope that our 'step-by-step' approach sheds light into the general procedure.

Throughout the classification, we shall make use of the following trivialities, often without explicitly mentioning them.

Remark 4.2. The following statements hold:

(a) In order to verify that a finite group G acts freely on a given set X, it suffices to prove that *only one representative* of each non-trivial conjugacy class acts freely on X.

(b) Let $T = V/\Lambda$ be a complex torus and $T_j = V_j/\Lambda_j \subset T$ $(j = 1, ..., n)$ subtori such that T is isogenous to $T_1 \times ... \times T_n$. By definition, there is a finite subgroup $H \subset T_1 \times ... \times T_n$ such that

$$T \cong (T_1 \times ... \times T_n)/H.$$

Since every T_i embeds into T, we obtain the following important observation:

$$\forall h = (h_1, ..., h_n) \in H\colon \quad \text{if } n-1 \text{ of the } h_j \text{ are zero, then } h = 0.$$

Moreover, suppose that $\alpha \in \mathrm{Aut}(V)$ induces automorphisms of both T and the product $T_1 \times ... \times T_n$. Then α acts on $H = \Lambda/(\Lambda_1 \oplus ... \oplus \Lambda_n)$; in particular, $\alpha(H) = H$.

(c) Let A be an Abelian variety and $g \in \mathrm{End}(A)$ a semisimple endomorphism of A. Then the addition map

$$\ker(g)^0 \times \mathrm{im}(g) \to A$$

is an isogeny. The usual case of application in this thesis is $g = f - \lambda\,\mathrm{id}_A$, where f is an automorphism of finite order of A and λ is an eigenvalue of f.

(d) Let ρ be an irreducible representation of a finite group G. If $g \in Z(G)$, then $\rho(g)$ is a multiple of the identity matrix (this follows from Schur's Lemma [Hu98, 2.5 Theorem, c)]).

(e) If χ is a 1-dimensional representation of a finite group G, then $[G, G] \subset \ker(\chi)$.

Chapter 5

Further Restrictions on the Structure of G

Recall our meta-hypotheses:

> The letter G will always denote a finite subgroup of $\mathrm{Bihol}(A)$, where A is an Abelian variety of dimension $n = 4$, such that the following properties hold:
> (1) G is embedded into $\mathrm{GL}(4, \mathbb{C})$ via some faithful representation $\rho\colon G \hookrightarrow \mathrm{GL}(4, \mathbb{C})$ (this is equivalent to requiring that G does not contain any translations).
> (2) The matrix $\rho(g)$ has the eigenvalue 1 for any $g \in G$.
> (3) The associated complex representation of the embedding $G \subset \mathrm{Bihol}(A)$ is ρ.
>
> Moreover, the condition that $\rho(G)$ is a subgroup of of the (linear) automorphism group of an Abelian variety implies the integrality condition (cf. p. 37):
>
> (4) $\rho \oplus \overline{\rho}$ is (equivalent to) an integral representation.

We collect in this section more properties our group G has to fulfill, if we require that G acts freely on A.
The first result is a relation between the Sylow subgroups of G and the center of G:

Proposition 5.1. *Assume that G is a non-Abelian group of order $2^a \cdot 3^b$ such that either*

(a) The 2-Sylow group is central in G and isomorphic to C_2^2 or C_2^3, or

(b) The 3-Sylow group is central in G and isomorphic to C_3^3.

Then the group G does not occur as a group associated with a hyperelliptic fourfold.

Proof. We prove (a), statement (b) is dealt with similarly. Let S be the 2-Sylow group of G, and assume that S is isomorphic to C_2^2 or C_2^3. Since S is normal in G and G is non-Abelian, the 4-dimensional faithful representation ρ of G splits into the direct sum of a irreducible representations of respective degrees 3 and 1 (cf. Theorem 2.10). Let $s_1, s_2 \in S$ generate a subgroup isomorphic to C_2^2. After choosing a suitable basis, since ρ is faithful and S is central in G, we can assume that

$$\rho(s_1) = \mathrm{diag}(-1, -1, -1, 1), \quad \rho(s_2) = \mathrm{diag}(1, 1, 1, -1).$$

Then $\rho(s_1 s_2)$ does not have the eigenvalue 1. $\square$

A similar argument as in the above proposition shows the following two statements:

Lemma 5.2. *(a) Suppose that G is a non-Abelian group associated with a hyperelliptic fourfold, and that $Z(G)$ contains a subgroup isomorphic to C_2^d, $d \geq 2$. Then the complex representation ρ splits as the direct sum of three irreducible representations of respective dimensions* 2, 1, 1.

(b) If G is a non-Abelian group associated with a hyperelliptic fourfold and has a central subgroup isomorphic to C_d^k ($d \geq 2$), then $k \leq 2$.

Proof. (a) Since every matrix in the image of ρ must have the eigenvalue 1 and $Z(G)$ is non-trivial, ρ cannot be irreducible. Let $d \geq 2$. If ρ were the direct sum of two irreducible representations of dimensions 3 and 1, we find two central elements g_1, g_2 such that

$$\rho(g_1) = (\zeta_d,\ \zeta_d,\ \zeta_d,\ 1) \text{ and } \rho(g_2) = (1,\ 1,\ 1,\ \zeta_d),$$

Then $\rho(g_1 g_2)$ does not have the eigenvalue 1. We exclude the possibility that ρ is the direct sum of two irreducible representations of dimension 2 in the same way.

(b) We prove the statement by contradiction. By part (a), ρ is the direct sum of three irreducible representations of dimensions 2, 1, 1. If $k \geq 3$, we find find three central elements g_1, g_2, g_3 such that

$$\rho(g_1) = (\zeta_d,\ \zeta_d,\ 1,\ 1) \text{ and } \rho(g_2) = (1,\ 1,\ \zeta_d,\ 1) \text{ and } \rho(g_3) = \mathrm{diag}(1,\ 1,\ 1,\ \zeta_d).$$

Then $\rho(g_1 g_2 g_3)$ does not have the eigenvalue 1. □

The upcoming Lemma proves that the center $Z(G)$ can only contain elements of certain orders, if G is non-Abelian.

Lemma 5.3. *Let G be a non-Abelian group associated with a hyperelliptic fourfold. Then $Z(G)$ cannot contain elements of order* 8 *and* 9.

Proof. We first prove the statement in the case where ρ is the direct sum of two irreducible representations (of respective dimensions 2, 2 or 3, 1), $\rho = \rho' \oplus \rho''$. For a contradiction, assume that G contains a central element g of order 8 or 9. Then, since g is central, $\rho'(g)$ and $\rho''(g)$ are multiples of the identity matrix. Since ρ is faithful and $\rho(g)$ has to have the eigenvalue 1, the matrix $\rho(g)$ either has

- exactly one eigenvalue of order 8 or 9 and the eigenvalue 1 with multiplicity 3, or
- an eigenvalue of order 8 or 9 with multiplicity 2 resp. 3 and the eigenvalue 1 with multiplicity 2 resp. 1.

In both of the above cases, we arrive at a contradiction to Lemma 2.2. Moreover, if ρ is irreducible, then $Z(G) = \{1\}$, because ρ is faithful.

Now, let us consider the much more difficult case in which ρ is the direct sum of three irreducible representations,

$$\rho = \rho_2 \oplus \rho_1 \oplus \rho_1',$$

which necessarily have respective dimensions 2, 1, 1. Lemma 2.2 immediately implies that $Z(G)$ does not contain an element of order 9. It remains to prove that $Z(G)$ cannot contain an element of order 8. For a contradiction, assume that $g \in G$ is a central element of order 8. Invoking Lemma 2.2 again, g must be mapped to λI_2, $\lambda \in \{\pm 1, \pm i\}$ by ρ_2 and to primitive 8-th roots of unity by ρ_1, ρ_1'. Since $\rho(g)$ must have the eigenvalue 1, we obtain that $\rho_2(g) = I_2$.
We define $K := \ker(\rho_1) \cap \ker(\rho_1')$. The idea is now to investigate the structure of K to eventually reach a contradiction. We divide our arguments into several steps.

Step 1: We claim that K is non-trivial.
To prove this statement, let $g' \in G \setminus (K \cup \langle g \rangle)$. Then, by definition of K,

$$(\rho_1(g'), \rho_1'(g')) \neq (1, 1).$$

Since $\rho_1(g)$ and $\rho_1'(g)$ are primitive 8-th roots of unity, Lemma 2.5 (a) shows that $\mathrm{ord}(g')$ is a power of 2. By Lemma 2.12, we are allowed to write

$$(\rho_1(g), \rho_1'(g)) = (\zeta_8, \zeta_8^r), \qquad \text{where } r \in \{3, 5\}.$$

A crucial observation is that if $\mathrm{ord}(g') \in \{2, 4\}$, then $(\rho_1(gg'), \rho_1'(gg'))$ must consist of two different, non-conjugate eigenvalues of order 8. After replacing g' by an appropriate power, we are left with investigating the following cases:

- If $\mathrm{ord}(g') = 2$, assume that $\rho_1(g') = -1$. The above observation implies that $\rho_1'(g') = -1$ as well. Then $g'g^4 \in K$ is a non-trivial element.
- Suppose now that $\mathrm{ord}(g') = 4$. If both $\rho_1(g')$ and $\rho_1'(g')$ have order ≤ 2, we may conclude as in the previous bullet point. Therefore, we assume that $\rho_1(g') = i = \zeta_8^2$. The above observation implies that $\rho_1'(g') \neq 1$. Since $\rho(g(g')^2)$ must have exactly two different, non-conjugate eigenvalues of order 8, we obtain $\rho_1'(g') \neq -1$.
Hence $\rho_1'(g') \in \{i = \zeta_8^2, -i = \zeta_8^6\}$ is a primitive fourth root of unity. Now,

  $$(\rho_1(gg'), \rho_1'(gg')) = (\zeta_8^3, \zeta_8^{r+s}), \qquad \text{where } s \in \{2, 6\}.$$

  One easily checks that the possibilities $(r, s) \in \{(3, 2), (5, 6)\}$ lead to a contradiction to Lemma 2.2. Hence $(r, s) \in \{(3, 6), (5, 2)\}$. We obtain $\rho_1(g^2) = \rho_1(g')$ and $\rho_1'(g^2) = \rho_1'(g')$: hence $g'g^6 \in K$ is a non-trivial element.
- Assume now that $\mathrm{ord}(g') = 8$. If both $\rho_1(g')$ and $\rho_1'(g')$ have order ≤ 4, the element $\rho(g'g)$ does not have the eigenvalue 1 (since $\rho_2(g')$ must have two eigenvalues of order 8). Hence we may assume that $\rho_1(g') = \zeta_8$. Assume that $\mathrm{ord}(\rho_1'(g')) \leq 4$. In this case, the matrix $\rho(g(g')^4)$ has exactly two eigenvalues of order 8, which are either the same (if $r = 5$) or complex conjugate (if $r = 3$), a contradiction.
Hence $\rho_1'(g') = \zeta_8^s$ for some $s \in \{3, 5\}$. If $s \neq r$, the number of eigenvalues of order 8 of $\rho(gg'^{-1})$ would be 1 or 3, a contradiction to Lemma 2.2. Thus $r = s$, and $g'g^{-1} \in K$ is a non-trivial element.

We have thus shown that K is non-trivial.

Step 2: K is non-Abelian.
Take $k \in K \setminus \{1\}$ (which is possible by Step 1) and observe that $k \notin Z(G)$, since $\rho(gk)$ must have the eigenvalue 1. Thus we can find an element $g'' \in G$, which does not commute with k. By a similar argument as in Step 1, we can multiply g'' by an appropriate

power of g to assume without loss of generality that $g'' \in K$.

Step 3: The center of every non-Abelian subgroup of K is trivial. In particular, the center of K is trivial.
Let K' be a non-Abelian subgroup of K. Since K' is non-Abelian, the restriction $\rho_2|_{K'}$ remains irreducible. Hence, if $k' \in Z(K')$, the matrix $\rho_2|_{K'}(k')$ is a multiple of the identity matrix. Since $\rho(gk')$ must have the eigenvalue 1, we obtain that $\rho_2|_{K'}(k') = \mathrm{diag}(1,1)$. Since $\rho_2|_{K'}$ is a faithful representation of K', we obtain $k' = 1$.

Step 4: If $k \in K$, then $\mathrm{ord}(k) \in \{1,2,3,4,6\}$.
By Lemma 2.5, it suffices to prove that K does not contain elements of order 5, 7, 8, 9 and 12. Indeed, any element of order 8 or 12 contained in K would be mapped to a matrix without the eigenvalue 1 by ρ_2: thus the product of an element of order 8 or 12 with g is mapped to a matrix without the eigenvalue 1 by ρ. Similarly, we show that K cannot contain elements of orders 5, 7 or 9, since then we find elements of order 40, 56 or 72 in G, which is impossible in dimension 4, see Lemma 2.5.

Step 5: K does not contain a subgroup which is isomorphic to $C_d \times C_d$, $d \in \{2,3\}$.
For a contradiction, assume that K contains a subgroup U which is isomorphic to $C_d \times C_d$, where $d \in \{2,3\}$. Since ρ is faithful, we can find an element $u \in U$ such that $\rho_2(u) = \mathrm{diag}(\zeta_d, \zeta_d)$. This implies that $\rho(gu)$ does not have the eigenvalue 1, a contradiction.

Step 6: The 3-Sylow subgroups of K are either trivial or isomorphic to C_3.
Since ρ_2 has dimension 2, its restriction to a 3-Sylow subgroup of K splits into a direct sum of two characters (see Theorem 2.10). In addition to that, $\rho_2|_K$ is faithful, so we conclude that the 3-Sylow subgroups of K are Abelian. By Step 4, they do not contain any element of order 9, so by Step 5, they are either trivial or cyclic of order 3.

Step 7: The 2-Sylow subgroups of K are cyclic of order ≤ 4.
Recall the well-known group-theoretic fact that a 2-group (or, more generally, a p-group) contains subgroups of all possible orders. Therefore, Step 7 follows from Step 5 if we prove that K does not contain a subgroup of order 8. Indeed, by Steps 4 and 5, a subgroup $U \subset K$ of order 8 would be non-Abelian (hence isomorphic to D_4 or Q_8): but Step 3 tells us that $Z(U)$ is trivial, a contradiction.

Step 8: Both the 2- and 3-Sylow subgroups of K are non-trivial. In particular, the 3-Sylow subgroups of K are cyclic of order 3.
By Steps 6 and 7, the 2- and 3-Sylow subgroups of K are Abelian. By Step 4, $|K| = 2^{a'} \cdot 3^{b'}$ for some a', b'. Hence we obtain the result, because K is non-Abelian by Step 2.

Step 9: The 2-Sylow subgroups of K are isomorphic to C_2.
If the 2-Sylow groups of K were cyclic of order 4, Steps 2 and 8 imply that K is non-Abelian of order 12 with cyclic Sylow subgroups. By the classification of groups of order

12, the only possibility is that K is the dicyclic group

$$K = \langle a, b \,|\, a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

However, this group has a non-trivial center: the relations imply that a^3 commutes with b, a contradiction to Step 3.

Step 10: Finishing the proof.
By Steps 6 and 9 and since K is non-Abelian, the 2- and 3-Sylow subgroups of K are isomorphic to C_2 and C_3, respectively. Hence $K \cong S_3 = \langle \sigma, \tau \,|\, \sigma^3 = \tau^2 = (\tau\sigma)^2 = 1 \rangle$. Here, the 3-cycle σ is necessarily mapped to a matrix without the eigenvalue 1 by ρ_2 (since $\tau^{-1}\sigma\tau = \sigma^2$ implies that $\rho_2(\sigma)$ and $\rho_2(\sigma)^2$ have the same eigenvalues). We observe that $\rho(\sigma g)$ does not have the eigenvalue 1, a contradiction. $\square$

Lemma 5.4. *Let G be a non-Abelian group associated with a hyperelliptic fourfold. If $Z(G)$ is not cyclic, then $Z(G)$ is, up to isomorphism, a subgroup of one of the groups $C_2 \times C_{12}$ and $C_d \times C_d$, where $d \in \{2, 3, 4, 6\}$.*

Proof. In view of the previous results, it remains to prove that G does not have a central subgroup isomorphic to $C_3 \times C_{12}$ or $C_4 \times C_{12}$.
For a contradiction, assume that $\langle g, h \rangle$ is a central subgroup isomorphic to $C_3 \times C_{12}$ or $C_4 \times C_{12}$, where $\mathrm{ord}(g) = 12$ and $\mathrm{ord}(h) \in \{3, 4\}$. We first prove the statement in the case where ρ is the direct sum of two irreducible representations (of respective dimensions 2, 2 or 3, 1), $\rho = \rho' \oplus \rho''$. Since $\mathrm{ord}(g) = 12$, ρ is faithful and $\rho(g)$ must have the eigenvalue 1, the only possibility is that $\rho(g)$ has exactly one eigenvalue of order 12 (with multiplicity 1, 2, or 3): hence we obtain a contradiction to Lemma 2.2. Moreover, if ρ is irreducible, then $Z(G) = \{1\}$, because ρ is faithful.
This proves the statement in the case where ρ is irreducible or the sum of two irreducible representations. It remains to prove the statement in the case where ρ is the direct sum of a 2-dimensional irreducible and two 1-dimensional representations,

$$\rho = \rho_2 \oplus \rho_1 \oplus \rho_1'.$$

Since every matrix in $\rho(G)$ must have the eigenvalue 1 and ρ is faithful, we obtain that $\rho_2(g) \neq I_2$ (else, $\rho(gh)$ would not have the eigenvalue 1, since then $\rho_1(g)$, $\rho_1'(g) \neq 1$). Hence, Lemma 2.2 shows that

$$\rho_2(g) = \mu I_2, \quad \text{where } \mu \in \{\pm i, \pm\zeta_3\}$$

by ρ_2, and we may assume that

$$\rho_1(g) = \begin{cases} \pm\zeta_3, & \text{if } \mu = \pm i, \\ i, & \text{if } \mu = \pm\zeta_3 \end{cases}.$$

Thus, if $h \in Z(G) \setminus \langle g \rangle$ is an element of order 3 or 4, we can assume that it is mapped to $\pm I_2$ by ρ_2 (else, $\rho(gh)$ would contain two times the same eigenvalue of order 12): but then $\mathrm{Eig}(\rho(gh))$ either contains a single primitive 12-th root of unity (contradicting Lemma 2.2), or does not contain 1. $\square$

The following Lemma is a generalization of [UY76, Lemma 3] to complex dimension 4, for which there certainly are more general versions than what we prove. However, the version below suffices for our purposes.

Lemma 5.5. *Let G be finite group satisfying the following properties*

(i) G is non-Abelian.

(ii) G is generated by two elements g, h of respective orders r and s satisfying

$$h^{-1}gh = g^l$$

for some $l \not\equiv 1 \pmod r$ (and possibly other relations).

Then, if G occurs as the group of a hyperelliptic forufold, the following statements hold:

(a) Assume that $|G| = 2^a \cdot 3^b$, where $r \in \{4, 8\}$. Then ρ is the direct sum of three irreducible representations of respective dimensions 2, 1, 1.

(b) Assume that ρ contains a 1-dimensional representation. Then there is a non-trivial 1-dimensional representation of G which is contained in ρ.

Proof. We first show the statement in the case $b = 0$. Since $Z(G)$ is non-trivial if G is a 2-group, ρ cannot be irreducible of dimension 4. Thus, ρ splits as a direct sum of two representations ρ_2, ρ_2' of dimension 2, where we can assume ρ_2 to be irreducible. Assume that both ρ_2, ρ_2' are irreducible. The relation $h^{-1}gh = g^l$ implies that $\rho_2(g)$ and $\rho_2(g)^l$ (resp. $\rho_2'(g)$ and $\rho_2'(g)^l$) have the same eigenvalues. Since $r \in \{4, 8\}$ and $l \not\equiv 1 \pmod r$, the possible sets of eigenvalues for ρ_2 resp. ρ_2' are

$$\{\zeta_r, \zeta_r^l\}, \qquad \{1\}, \qquad \{-1\}, \qquad \{1, -1\}.$$

Since ρ is faithful and $\rho(g)$ must have the eigenvalue 1, we can assume that

$$\rho(g) = \begin{pmatrix} \rho_2(g) & \\ & \rho_2'(g) \end{pmatrix} = \mathrm{diag}(\zeta_r, \zeta_r^l, 1, \pm 1).$$

If $\rho_2'(g) = \mathrm{diag}(1, 1)$, we obtain that $\rho_2'(g)$ commutes with $\rho_2'(h)$, hence these two matrices share a common eigenvector, which implies that ρ_2' is reducible. We may therefore concentrate on the case where $\rho_2'(g) = \mathrm{diag}(1, -1)$. In this case, $\rho_2'(g) = \rho_2'(g^l) = \mathrm{diag}(1, -1)$, and thus $\rho_2'(g)$ and $\rho_2'(h)$ commute. We conclude as above, noting that then ρ_2' is not irreducible.
Since the above arguments are independent of the chosen basis, this proves the statement in the case where $b = 0$, so let us assume $b > 0$ in the following. This means that 3 divides the order of G, and thus by assumption the order s of h.

Claim: $h^2 \in Z(G)$.

Proof of the Claim: Since $r \in \{4, 8\}$, we obtain that the pair $(r, l \pmod r)$ is one of $(4, -1)$, $(8, 3)$ or $(8, 5)$ (see also Lemma 2.12). In particular, the order of l in the multiplicative group C_r^* is 2. Now, the Claim follows from

$$h^{-2}gh^2 = h^{-1}g^lh = g^{l^2} = g,$$

or, equivalently, $gh^2 = h^2g$, as desired.

The claim implies that $s \notin \{3, 9\}$ (else, G would be Abelian, contradicting (i)). Consequently, 6 divides s. Since the 3-Sylow subgroup of G is contained in $\langle h \rangle$ and acts trivially on the normal subgroup $\langle g \rangle$ of order $r \in \{4, 8\}$, we have proved that the 3-Sylow subgroup of G is central. Hence, using Theorem 2.10, it suffices to prove the remaining statements of (a) only for the 2-Sylow subgroup of G, which was already done.

We first prove part (b) in the case where ρ is the direct sum of a 2-dimensional and two 1-dimensional representations, $\rho = \rho_2 \oplus \rho_1 \oplus \rho_1'$. Assume that the 1-dimensional representations ρ_1, ρ_1' are both trivial. Since G is non-Abelian by (i), it follows from (ii) that $\mathrm{ord}(g) > 2$. In particular, $\rho_2(g)$ does not have the eigenvalue 1, since $\rho_2(g)$ and $\rho_2(g)^l$ have the same eigenvalues. This proves that the Abelian variety A is isogenous to a direct sum of two Abelian surfaces,

$$A \cong (A_1 \times A_2)/H,$$
$$A_1 := \mathrm{im}(\rho(g) - I), \quad A_2 := \ker(\rho(g) - I)^0.$$

Let us write $g(z_1, z_2) = (\rho_2(g)z_1 + t_1,\ z_2 + t_2)$, $h(z_1, z_2) = (\rho_2(h)z_1 + \tau_1,\ z_2 + \tau_2)$. The relation $h^{-1}gh = g^l$ implies that

$$(a_1, (l-1)t_2) \in H \text{ for some } a_1 \in A_1.$$

Thus, $g^{l-1} \neq \mathrm{id}_A$ does not act freely on A if we prove the following

Claim: $\rho_2(g^{l-1})$ does not have the eigenvalue 1.

Proof of the Claim: As explained above, the matrices $\rho_2(g)$ and $\rho_2(g)^l$ have the same eigenvalues, none of them being 1. By choosing an appropriate basis, we may assume that these matrices are diagonal. Then

$$\rho_2(g) = \mathrm{diag}(\zeta_r^a,\ \zeta_r^b)$$

for some $a \not\equiv b \pmod r$. Since ρ is faithful,

$$\rho_2(g)^l = \mathrm{diag}(\zeta_r^{al},\ \zeta_r^{bl}) = \mathrm{diag}(\zeta_r^b, \zeta_r^a)$$

and finally

$$\rho_2(g^{l-1}) = \mathrm{diag}(\zeta_r^{b-a},\ \zeta_r^{a-b}).$$

This proves the Claim.

If ρ is the direct sum of an irreducible representation ρ_3 of dimension 3 and a character, we prove the statement in a similar way: since ρ_3 is irreducible of dimension 3, the order s of h is a multiple of 3 (see Theorem 2.10). Indeed, take the maximal $1 < m < s$, such that h^m commutes with g. Write $s = m \cdot k$ and define N to be the Abelian normal subgroup spanned by h^m and g. Then, by Theorem 2.10, 3 divides the index of N in G, hence 3 divides n. Since $\mathrm{Aut}(C_r^*)$ contains a subgroup of order n, necessarily 3 divides the order of $\mathrm{Aut}(C_r^*)$, i.e., $3|\varphi(r)$. By Lemma 2.5 (b), we obtain that

$$r \in \{7, 9, 14, 18\}. \tag{5.1}$$

Assume now that the character contained in ρ is the trivial one. By Lemma 2.2 and by assumption, none of the eigenvalues of $\rho_3(g)$ can be equal to 1. The Abelian variety A is therefore isogenous to the product of an Abelian threefold A_1 and an elliptic curve A_2. More precisely,

$$A \cong (A_1 \times A_2)/H,$$
$$A' := \mathrm{im}(\rho(g) - I), \quad A_2 := \ker(\rho(g) - I)^0.$$

Writing as above $g(z_1, z_2) = (\rho_3(g)z_1 + t_1,\ z_2 + t_2)$, $h(z_1, z_2) = (\rho_3(h)z_1 + \tau_1,\ z_2 + \tau_2)$, the relation $h^{-1}gh = g^l$ implies that

$$(a_1, (l-1)t_2) \in H \text{ for some } a_1 \in A_1.$$

Similarly to the previous case, it suffices to prove the following

Claim: $\rho_3(g^{l-1})$ does not have the eigenvalue 1.

Proof of the Claim: Again, the matrices $\rho_3(g)$ and $\rho_3(g)^l$ have the same eigenvalues, and none of them is 1. We may assume that the matrices $\rho_3(g)$ and $\rho_3(g^l)$ are diagonal. Then

$$\rho_3(g) = \mathrm{diag}(\zeta_r^a,\ \zeta_r^b,\ \zeta_r^c)$$

for some $a,\ b,\ c$, pairwise not congruent modulo r (this follows from Lemma 2.2 in view of (5.1)). Since ρ is faithful and $g \neq g^l$, one of the following holds:

$$a \not\equiv al \pmod r \quad \text{or} \quad b \not\equiv bl \pmod r \quad \text{or} \quad c \not\equiv cl \pmod r. \tag{5.2}$$

In fact, if one of (5.2) does not hold, the matrix $\rho(g^{l-1})$ would have at most 2 eigenvalues different from 1. Since $\rho(g^{l-1})$ is the power of a matrix with exactly three eigenvalues of order r, we obtain that $l \equiv 1 \pmod r$, a contradiction. Thus all three of (5.2) hold, which implies that

$$\rho_3(g^{l-1}) = \mathrm{diag}(\zeta_r^{al-a},\ \zeta_r^{bl-b},\ \zeta_r^{cl-c})$$

does not have the eigenvalue 1, as desired.

All statements are therefore proven. □

Chapter 6

The 2-Sylow Subgroups of G

The aim of this chapter is to give an upper bound for the order of 2-groups which occur as the group of a hyperelliptic fourfold. We prove that the order of such a 2-group G is bounded by 128: in the proof, we distinguish between the three cases in which

- G is Abelian (Lemma 6.1),
- G is non-Abelian and has a faithful irreducible representation of dimension 2 (see p. 78),
- G is non-Abelian and has no faithful irreducible representation (Corollary 6.9).

Lemma 6.2 guarantees that one these three cases indeed occurs.
Later, in Theorem 9.1, we will see that the order of G is actually bounded by 32.

Let G be a 2-group associated with a hyperelliptic fourfold. We summarize our results in the case where G is Abelian in the following

Lemma 6.1. *If G is an Abelian 2-group associated with a hyperelliptic fourfold, then G is isomorphic to one of the following groups:*

$$C_2, \quad C_2 \times C_2, \quad C_4, \quad C_2 \times C_2 \times C_2, \quad C_2 \times C_4, \quad C_8,$$
$$C_2 \times C_2 \times C_4, \quad C_4 \times C_4, \quad C_2 \times C_8, \quad C_2 \times C_4 \times C_4, \quad C_4 \times C_8.$$

In particular, $|G| \leq 32$.

Proof. According to Lemma 2.5, elements of G have $1, 2, 4$ and 8 as possible orders. Thus, G does not have a subgroup isomorphic to C_{16}. Moreover, by Lemma 2.7, G does not have a subgroup isomorphic to C_2^4. That G does not contain a subgroup isomorphic to C_4^3 was shown in Corollary 3.10. It remains to exclude that G contains C_8^2: this is the content of Lemma 3.13. $\square$

We shall therefore concentrate on the case where G is non-Abelian. In this case, the representation ρ splits as a direct sum of two representations ρ_2, ρ_2' of degree 2, where we can assume ρ_2 to be irreducible.

Lemma 6.2. *One of the following possibilities holds:*

(1) ρ contains a faithful, irreducible representation of degree 2.

(2) Every irreducible representation of G is non-faithful.

Proof. Assume that (2) does not hold, i.e., that G has a faithful irreducible representation. Since ρ is faithful, we obtain

$$\ker(\rho_2) \cap \ker(\rho_2') = \{1\}.$$

If $\ker(\rho_2')$ is trivial, we are done, since the assumption that G is non-Abelian then implies that ρ_2' is irreducible. Hence we can assume that ρ_2' is not faithful. Since $\ker(\rho_2')$ is normal in G and different from $\{1\}$, it intersects $Z(G)$ non-trivially[1]. The assumption that (2) does not hold means that $Z(G)$ is cyclic (see Theorem 2.9), implying that the kernel of ρ_2 is trivial. □

Thus we will have to investigate two cases. We will first treat (1), i.e., we will assume for the time being that

The representation ρ contains a faithful irreducible representation ρ_2 of degree 2.

Consider the exact sequence

$$1 \to N \to G \to C_m \to 1,$$

where $G \to C_m$ is given by $g \mapsto \det \rho_2(g)$. The only element of order 2 in the kernel N corresponds to $-I_2$, and since ρ_2 is faithful, N has only one subgroup of order 2. It follows from [Ha59, Theorem 12.5.2] that N is either cyclic of order dividing 4 or a generalized quaternion group, which necessarily has order 8.[2] We have proven that $|G| \leq 32$, achieved only if $N \cong Q_8$ and $m = 4$.

We are then left with investigating (2), i.e., we assume for the rest of this section that

The non-Abelian 2-group G does **not** have a faithful irreducible representation.

Equivalently, we may require that $Z(G)$ is non-cyclic (cf. Theorem 2.9). This leads to the following easy

Lemma 6.3. *The representation ρ_2' splits as a direct sum of two 1-dimensional representations, $\rho_2' = \rho_1 \oplus \rho_1'$.*

Proof. Assume that ρ_2' were irreducible. By what we remarked above the Lemma, we can choose subgroup $H = \langle h_1, h_2 \rangle$ of $Z(G)$ that is isomorphic to the Klein four group. Using that ρ is faithful, and because ρ_2 and ρ_2' are irreducible, one of the matrices $\rho(h_1)$, $\rho(h_2)$, $\rho(h_1h_2)$ is equal to $-I_4$, a contradiction. □

In the same manner, we prove

Lemma 6.4. *The following statements hold.*

[1]Any normal subgroup N of G is the union of conjugacy classes, which have length 2^j for some j. Since N contains the conjugacy class $\{1\}$ of length 1, and because N is a 2-group, it follows that N contains another conjugacy class of length 1, or, equivalently, a non-trivial central element of G.

[2]Lemma 2.2 implies that N has exponent ≤ 4.

(a) $\ker(\rho_2) \subset Z(G)$,

(b) $K := \ker(\rho_1 \oplus \rho_1')$ *intersects* $Z(G)$ *non-trivially.*

(c) $K \cap \ker(\rho_2) = \{1\}$.

Proof. (a) follows because ρ_2' is a direct sum of 1-dimensional representations and ρ is faithful. For part (b), it suffices to note that is K is non-trivial (because G is non-Abelian); since it is normal, it intersects the center of the 2-group G non-trivially. Part (c) is just a reformulation of the assumption that ρ is faithful. □

Corollary 6.5. *The kernel of* ρ_2 *is a cyclic group of order* 2 *or* 4.

Proof. Assume that $\ker(\rho_2)$ was non-cyclic. Then, since ρ is faithful, we can find $g, h \in G$ such that

$$\rho(g) = \mathrm{diag}(\alpha,\ 1,\ 1,\ 1), \quad \rho(h) = \mathrm{diag}(1,\ \beta,\ 1,\ 1), \quad \alpha,\ \beta \neq 1.$$

Now, $\rho(ghk)$ does not have the eigenvalue one for any $1 \neq k \in Z(G) \cap K$ (such a k exists by Lemma 6.4 (b)).
It remains to exclude that $\ker(\rho_2) \cong C_8$. Lemma 2.2 implies that the representations ρ_1, ρ_1' must map an element $g' \in \ker(\rho_2)$ of order 8 to primitive 8-th roots of unity. Hence we may again choose $1 \neq k \in Z(G) \cap K$ to obtain that $\rho(g'k)$ does not have the eigenvalue 1. □

The following Lemma will be useful to show that certain non-solvable groups do not occur as groups of hyperelliptic fourfolds, cf. Section 10.6.

Lemma 6.6. $\rho(G)$ *is not contained in* $\mathrm{SL}(4, \mathbb{C})$.

Proof. Let $1 \neq g \in \ker(\rho_2)$ and $1 \neq h \in K \cap Z(G)$. If $\rho(G) \subset \mathrm{SL}(4, \mathbb{C})$, then, because ρ is faithful,

$$\rho(g) = \begin{pmatrix} \lambda & & \\ & \lambda^{-1} & \\ & & I_2 \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -I_2 \end{pmatrix}$$

for some $\lambda \neq 1$. Then $\rho(gh)$ does not have the eigenvalue 1, a contradiction. □

We can find all possibilities for G in Case (2) in view of the next remark.

Remark 6.7. Let G be a non-Abelian 2-group associated with a hyperelliptic fourfold, which admits no faithful irreducible representation. Consider the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d] & G \arrow[r, "g \mapsto \det \rho_2(g)"] \arrow[d] & C_m \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & N/\ker(\rho_2) \arrow[r] & G/\ker(\rho_2) \arrow[r] & C_m \arrow[r] & 1 \end{tikzcd}$$

and obviously, the induced representation on $G/\ker(\rho_2)$ is faithful. We remark that the derived subgroup of G (which is non-trivial by our assumption that G is non-Abelian) must not intersect the kernel of ρ_2, since the representation ρ needs to be faithful and $[G, G]$ is contained in the kernel of every 1-dimensional representation of G. In particular, $G/\ker(\rho_2)$ is non-Abelian.

Lemma 6.8. *The order of $G/\ker(\rho_2)$ is at most 32.*

Proof. Since the induced representation on $G/\ker(\rho_2)$ is faithful, the only element of order 2 contained in $N/\ker(\rho_2)$ uniquely corresponds to $-I_2$. By the discussion in Case (1), we obtain that $N/\ker(\rho_2)$ is either cyclic or a generalized quaternion group. It remains to show that the exponent of $N/\ker(\rho_2)$ is at most 4: if $N/\ker(\rho_2)$ contained an element g of order 8, any lift $\hat{g} \in N$ of g has order 8 as well. Since $\ker(\rho_2)$ is cyclic of order ≤ 4 (cf. Lemma 6.5), the lift $\hat{g}$ is not contained in $\ker(\rho_2)$. We arrive at one of the following possibilities:

(i) If $\hat{g}$ is mapped to an element of order 8 by ρ_2, the eigenvalues of $\rho_2(g)$ are complex conjugate primitive 8-th roots of unity. Lemma 2.2 asserts that $\hat{g}$ is mapped to primitive 8-th roots of unity by ρ_1, ρ_1', as well. This means that $\rho(\hat{g})$ does not have the eigenvalue 1, a contradiction.

(ii) If $\hat{g}$ is mapped to an element of order ≤ 4 by ρ_2, the faithfulness of ρ and Lemma 2.2 imply that $\rho_1(\hat{g})$, $\rho_1'(\hat{g})$ are primitive 8-th roots of unity. Since $\hat{g}$ is not contained in the kernel of ρ_2, but in N, we obtain that $\rho_2(\hat{g})$ does not have the eigenvalue 1. We arrive again at the contradiction that $\rho(\hat{g})$ does not have the eigenvalue 1.

This completes the proof of the statement. $\square$

Together with Lemma 6.5, we now obtain the following bound for the order of G:

Corollary 6.9. *Let G be a 2-group associated with a hyperelliptic fourfold. Then*

$$|G| \leq 128.$$

Proof. If G is Abelian, Lemma 6.1 implies that $|G| \leq 32$. If (1) holds (i.e., G has a faithful irreducible representation of dimension 2), then we have seen on page 78 that $|G| \leq 32$. If (2) holds, G sits in an exact sequence

$$1 \to \ker(\rho_2) \to G \to G/\ker(\rho_2) \to 1,$$

and Lemmas 6.5 and 6.8 tell us that $|\ker(\rho_2)| \leq 4$, $|G/\ker(\rho_2)| \leq 32$. $\square$

We will see later in Theorem 9.1 that the order of the 2-Sylow subgroups of G is indeed bounded from above by 32.

Chapter 7

The 3-Sylow Subgroups of G

In this chapter, we will investigate 3-groups G associated with a hyperelliptic fourfold. In particular, we will derive an upper bound for the order of 3-Sylow subgroups: we prove that the order of G is bounded by 27, achieved exactly by C_3^3 and the Heisenberg group of order 27, see Proposition 7.8.

7.1 General results

Let G be a 3-group, which is associated with a hyperelliptic fourfold. We first deal with the case in which G is Abelian:

Lemma 7.1. *If the* 3*-group* G *is Abelian, it is isomorphic to one of*

$$C_3, \qquad C_3 \times C_3, \qquad C_9, \qquad C_3 \times C_3 \times C_3.$$

Proof. By Lemma 2.7, a 3-group associated with a hyperelliptic fourfold cannot have a subgroup isomorphic to C_3^4. Moreover, the possible orders of elements of such a group are 1, 3 and 9 by Lemma 2.5. Thus it remains to exclude that the Sylow subgroups of G have a subgroup isomorphic to $S := C_9 \times C_3$. Since S is Abelian, finite, and embedded into $\mathrm{GL}(4, \mathbb{C})$ via ρ, we can assume that the elements of S are diagonal matrices. According to Lemma 2.12, we can assume that C_9 is generated by

$$\text{(I) } g_1 = \mathrm{diag}(1, \zeta_9, \zeta_9^2, \zeta_9^4) \text{ or (II) } g_1 = \mathrm{diag}(1, \zeta_9, \zeta_9^4, \zeta_9^7).$$

Furthermore, suppose that g_1 and

$$g_2 = \mathrm{diag}(\zeta_9^a, \zeta_9^b, \zeta_9^c, \zeta_9^d)$$

span a subgroup isomorphic to $S = C_9 \times C_3$. In both cases, (I) and (II), the condition that $g_1 g_2$ must have the eigenvalue 1 implies that $a = 0$. We now treat the cases (I) and (II) separately.

(I) We prove the following

Claim: The following statements hold:

(a) $c = 0 \iff d = 0$,

(b) $b = 0 \iff d = 0$.

Proof of the Claim: (a) If $c \neq 0$, after possibly replacing g_2 by g_2^2, we can assume that $c = 3$. If d were equal to 0, the element

$$g_1 g_2 = \mathrm{diag}(1, \zeta_9^{1+b}, \zeta_9^5, \zeta_9^4)$$

has conjugate eigenvalues, which is not possible according to Lemma 2.2 and Remark 2.3. Thus $d \neq 0$. The other direction is proved similarly.
(b) Again, if $b \neq 0$, we can assume that $b = 3$. If furthermore $d = 0$, considering

$$g_1 g_2 = \mathrm{diag}(1, \zeta_9^4, \zeta_9^{2+c}, \zeta_9^4),$$

we arrive at the contradiction that $g_1 g_2$ has an eigenvalue with multiplicity at least 2. Again, the converse direction is proved similarly. This proves the Claim.

The Claim tells us that all three of b, c, d are non-zero modulo 9. Since g_2 is by assumption not contained in the group $C_9 = \langle g_1 \rangle$ and $g_1^3 = \mathrm{diag}(1, \zeta_9^3, \zeta_9^6, \zeta_9^3)$, we obtain that (b, c, d) is not a multiple of $(3, 6, 3)$. The remaining possibilities are (up to taking inverses):

$$\begin{aligned}
(b, c, d) = (3, 3, 3) &\implies g_1 g_2 = \mathrm{diag}(1, \zeta_9^4, \zeta_9^5, \zeta_9^7),\\
(b, c, d) = (3, 3, 6) &\implies g_1 g_2 = \mathrm{diag}(1, \zeta_9^4, \zeta_9^5, \zeta_9),\\
(b, c, d) = (3, 6, 6) &\implies g_1 g_2 = \mathrm{diag}(1, \zeta_9^4, \zeta_9^8, \zeta_9).
\end{aligned}$$

The element $g_1 g_2$ has conjugate eigenvalues in all three cases, thus, case (I) does not occur.

(II) Similarly to (I), we first prove

Claim: All three of b, c, d are non-zero modulo 9.

Proof of the Claim: Assume that (at least) one of b, c, d is equal to zero. The orbit containing 1 of the action of C_3 on $C_9 = \langle \overline{1} \rangle$ (given by addition by $\overline{3}$) is $\{\overline{1}, \overline{4}, \overline{7}\}$. Together with our assumption that one of b, c, d is zero, this implies that $g_1 g_2$ or $g_1 g_2^2$ has an eigenvalue with multiplicity at least 2, a contradiction.

Since $g_1^3 = \mathrm{diag}(1, \zeta_9^3, \zeta_9^3, \zeta_9^3)$, we obtain that (b, c, d) is not a multiple of $(3, 3, 3)$. The remaining possibilities are (up to taking inverses):

$$\begin{aligned}
(b, c, d) = (3, 3, 6) &\implies g_1 g_2 = \mathrm{diag}(1, \zeta_9^4, \zeta_9^7, \zeta_9^4),\\
(b, c, d) = (3, 6, 3) &\implies g_1 g_2 = \mathrm{diag}(1, \zeta_9^4, \zeta_9, \zeta_9),\\
(b, c, d) = (3, 6, 6) &\implies g_1 g_2 = \mathrm{diag}(1, \zeta_9^4, \zeta_9, \zeta_9^4).
\end{aligned}$$

In all three cases, $g_1 g_2$ has an eigenvalue with multiplicity greater than 1, a contradiction. Thus, Case (II) does not occur, and consequently there is no hyperelliptic fourfold with group $C_9 \times C_3$. $\square$

Hence we shall assume in the following that G is a non-Abelian 3-group. Since ρ is faithful, ρ splits into a direct sum of a 3-dimensional irreducible representation ρ_3 and a 1-dimensional representation ρ_1.

Proposition 7.2. *The representation ρ_3 is faithful.*

Proof. We will first prove that G has *some* faithful irreducible representation. Assume that this were not the case. Then Theorem 2.9 implies that $Z(G)$ is non-cyclic, in particular it contains a subgroup isomorphic to $C_3 \times C_3$ spanned by two elements $g, h \in Z(G)$. Since ρ is faithful, we immediately obtain that $\rho(gh)$ cannot have the eigenvalue 1, a contradiction. Hence $Z(G)$ is cyclic, and by Theorem 2.9, we obtain that G has a faithful irreducible representation.
It remains to prove that ρ_3 is faithful. Since G is a 3-group, any non-trivial normal subgroup of G intersects $Z(G)$ non-trivially (see the first footnote on page 78). Since ρ is faithful, $\ker(\rho_3) \cap \ker(\rho_1) = \{1\}$, and thus $\ker(\rho_1)$ must intersect the cyclic group $Z(G)$. Again by faithfulness of ρ, we may conclude that $\ker(\rho_3)$ is trivial. $\square$

Consider the exact sequence

$$1 \to K \to G \to C_m \to 1, \quad m \in \{1, 3, 9\}, \tag{7.1}$$

where the map $G \to C_m$ is given by $g \mapsto \det \rho_3(g)$. Since ρ_3' is faithful, K embeds into $\mathrm{SL}(3, \mathbb{C})$ via ρ_3'.

Lemma 7.3. *The group K cannot be isomorphic to $C_3 \times C_3 \times C_3$ or C_9.*

Proof. The assertion for C_9 follows from Lemma 2.12. Assume that K is isomorphic to C_3^3 (and we shall identify K with the additive group C_3^3 by some isomorphism). Then we choose a suitable basis such that $\rho|_K$ is given by

$$(a,\ b,\ c) \mapsto \mathrm{diag}(\zeta_3^a,\ \zeta_3^b,\ \zeta_3^c,\ \zeta_3^{-a-b-c}).$$

The element $(2, 1, 1)$ is then mapped to a matrix without the eigenvalue 1. $\square$

Using the classification of finite subgroups of $\mathrm{SL}(3, \mathbb{C})$, first achieved by Miller, Blichfeldt and Dickson [MBD16, Chapter XII] (which was later found to to be incomplete by Yau and Yu [YY93, p.2 f.] and completed in the cited book), we find that K is isomorphic to one of the following groups:

(a) Diagonal Abelian groups: $\{1\}$, C_3 or $C_3 \times C_3$.

(b) A group generated by a non-trivial group of type (a) and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

In particular, if K is of type (b), it is isomorphic to the Heisenberg group of order 27,

$$K \cong \mathrm{Heis}(3) := \langle g, h, k \mid g^3 = h^3 = k^3 = 1,\ ghg^{-1}h^{-1} = k,\ gk = kg,\ hk = kh \rangle.$$

7.2 The case where G contains elements of order 9

Our next goal is to prove

Theorem 7.4. *If G is a 3-group which contains an element of order 9 which is associated with a hyperelliptic fourfold, it is isomorphic to C_9.*

By Lemma 7.1, the statement of the Theorem is true if G is Abelian. We will henceforth assume in the following that G is a non-Abelian 3-group of exponent 9.

We denote by M_{27} the unique group of order 27 and exponent 9, which is a metacyclic group defined by the following presentation

$$M_{27} := \langle a, b \mid b^9 = a^3 = 1,\ a^{-1}ba = b^4 \rangle.$$

Remark 7.5. Note that if we replace a by $c := a^2$ in the above presentation, the last relation changes to $c^{-1}bc = b^7$.

We have prepared the proof of the following

Proposition 7.6. *If G is non-Abelian and contains an element of order 9, then G contains a subgroup isomorphic M_{27}.*

Proof. We prove the statement in two steps.

Step 1: We assume first that G has a cyclic normal subgroup of order 9.
Since G is non-Abelian, the representation ρ splits as a direct sum of a 1-dimensional and a 3-dimensional irreducible representation, $\rho = \rho_1 \oplus \rho_3$. Let $b \in G$ be an element of order 9 spanning a normal subgroup of G. Then, by possibly replacing b by some appropriate power and choosing an appropriate basis,

$$\rho_3(b) = \begin{pmatrix} \zeta_9 & & \\ & \zeta_9^n & \\ & & \zeta_9^m \end{pmatrix},$$

where $(n, m) \in \{(2, 4), (4, 7)\}$, cf. Lemma 2.12. Note that this implies that $b \notin Z(G)$, because otherwise, $\rho_3(b)$ would be a multiple of the identity matrix. Thus, by Lemma 7.1, we can find $a \in G$ of order 3 such that a and b do not commute. Since $\langle b \rangle$ is a normal subgroup of G, there is $1 < k < 9$ such that $a^{-1}ba = b^k$. Therefore, $\det(\rho_3(b)) = \det(\rho_3(b))^k$, and because $1 < k < 9$, we obtain that $\det(\rho_3(b))$ is a third root of unity and $k \in \{4, 7\}$.

Step 2: The case where G does not contain a cyclic normal subgroup of order 9.
Let $b \in G$ be an element of order 9. Then $\langle b \rangle$ is not a normal subgroup of G, hence the normalizer $N := N_G(\langle b \rangle)$ is a proper subgroup of G, which contains b. Indeed, $\langle b \rangle$ is a proper subgroup of N[1]. The normalizer N is then non-Abelian by Lemma 7.1 and contains the cyclic normal subgroup $\langle b \rangle$ of order 9. By Step 1, N contains a subgroup isomorphic to M_{27}. $\square$

[1]In fact, if $\langle b \rangle = N$, the center $Z(G)$ is a subgroup of $\langle b \rangle$, and by induction, the normalizer of $\langle b \rangle / Z(G)$ inside $G/Z(G)$ properly contains $\langle b \rangle / Z(G)$. Thus the latter group is normalized by some $hZ(G) \notin \langle b \rangle / Z(G)$, and one checks that h normalizes $\langle b \rangle$ as well, a contradiction.

The following Proposition serves as the last step in the proof of Theorem 7.4.

Proposition 7.7. *There is no hyperelliptic fourfold with group M_{27}.*

Proof. Assume that G is isomorphic to M_{27}, so that we can find $g, h \in G$ of respective orders 9 and 3 satisfying $h^{-1}gh = g^4$. We prove that $\rho_1(g) = \rho_1(h) = 1$: first of all, it is clear that $\rho_1(g) = 1$, since $\rho(g)$ must have the eigenvalue 1 (see also Step 1 in the proof of Proposition 7.6). Now $\rho_1(h) = 1$ follows, since otherwise $\rho(gh)$ (which is a matrix of order 9) would not have the eigenvalue 1. However, Lemma 5.5 (b) implies now that g^3 does not act freely on A. $\square$

7.3 Non-Abelian groups of order 3^b, $b \geq 4$

We prove that if G has order $3^b \geq 81$, then G does not occur as a group associated with a hyperelliptic fourfold.
Assume that G does occur as a group associated with a hyperelliptic fourfold and that $|G| = 3^b$, $b \geq 4$. First of all, by Lemma 7.1, G is non-Abelian. In view of exact sequence (7.1), by Lemma 7.3 and Propositions 7.6, 7.7, only the possibility $K = \mathrm{Heis}(3)$, $m = 3$ is left to exclude.

We use GAP Script `3-groups.g` (cf. Chapter 12) to find all 3-groups of exponent 3, which have cyclic center (this guarantees that G has a faithful irreducible representation, cf. Proposition 7.2) and contain a normal subgroup isomorphic to Heis(3) with quotient C_3. Since the GAP program has no output, our claim is proven.

7.4 Summarizing the results

In the previous sections, we established the following

Proposition 7.8. *Let G be a 3-group associated with a hyperelliptic fourfold. Then G is isomorphic to one of the following groups:*

$$C_3, \qquad C_3 \times C_3, \qquad C_9, \qquad C_3 \times C_3 \times C_3, \qquad \mathrm{Heis}(3).$$

The existence of hyperelliptic fourfolds with group $C_3 \times C_3 \times C_3$ was shown in Example 3.15, while the existence of a hyperelliptic fourfold with group C_9 is clear (it suffices to act an Abelian threefold A' by an automorphism of order 9 and on an elliptic curve E by a translation of order 9, and then consider the action of C_9 on $A = A' \times E$). We will show the existence of a hyperelliptic fourfold with group Heis(3) in Section 8.1.4.

Chapter 8

Hyperelliptic Fourfolds with Groups of Order $2^a \cdot 3^b$

This section is organized as follows. In the upcoming subsection, we shall explicitly describe examples of hyperelliptic fourfolds with groups of order $2^a \cdot 3^b$. Our goal is to show that these groups are maximal among the groups whose order is a product of a power of 2 and a power of 3 and occur as groups of hyperelliptic fourfolds (for a precise definition, see (a) and (b) below). To achieve this goal, we shall proceed as follows:

(i) Construct examples of hyperelliptic fourfolds with some (non-Abelian) groups G.

(ii) In Section 8.2, we prove that several groups do not occur as groups of hyperelliptic fourfolds.

This strategy allows us to create a list of 'forbidden groups' (consisting of the groups for which we proved that they do not occur) and then run GAP Script `maximal_groups.g` (cf. Chapter 12), whose output will then only consist of the groups in (i), i.e., the ones of order $2^a \cdot 3^b$ for which we have shown the existence of hyperelliptic fourfolds with that group.

8.1 The description of certain hyperelliptic fourfolds

Let $A = V/\Lambda$ be a 4-dimensional Abelian variety and $G \subset \text{Bihol}(A)$ a finite non-Abelian group of order $2^a \cdot 3^b$ acting freely on A and containing no translations. We shall develop a list of examples of hyperelliptic fourfolds with interesting groups G and determine the isogeny type of the corresponding Abelian variety. As it turns out in the process, the examples of groups given are *maximal* among the groups of order $2^a \cdot 3^b$, which are associated with hyperelliptic fourfolds: we call a group G *maximal*, if it is maximal among the groups of order $2^a \cdot 3^b$ occurring as groups of hyperelliptic fourfolds, i.e.,

(a) G occurs as a group of a hyperelliptic fourfold, and

(b) if $G \subset G'$ is a group occurring as a group of a hyperelliptic fourfold, then $G = G'$.

The reasons why we restrict to maximal groups twofold. First of all, if a finite, non-trivial group $G \subset \text{Bihol}(A)$ contains no translations and acts freely on A, then of course all non-trivial subgroups of G have these properties as well, so that we constructed hyperelliptic for the subgroups of G, too. Secondly, when explicitly describing all examples of hyperelliptic fourfolds with small groups, one usually has plenty of "choices for freedom": if the groups are larger, this tends not to be the case – however, it comes with the cost of producing examples with large groups being quite a difficult task in general. More precisely, we give examples of hyperelliptic fourfolds with the following groups, all of which will turn out to be maximal (see Theorem 11.2):

Isomorphism Type	GAP ID	Shown to occur in...
SD_8	[16,8]	Section 8.1.1
$D_4 \times C_2$	[16,11]	Section 8.1.2
$Q_8 \times C_3$	[24,11]	Section 8.1.3
Heis(3)	[27,3]	Section 8.1.4
$C_8 \rtimes C_4$	[32,4]	Section 8.1.5
$(C_4 \times C_4) \rtimes C_2$	[32,11]	Section 8.1.6
$(C_4 \times C_4) \rtimes C_2$	[32,24]	Section 8.1.7
$C_3 \times ((C_4 \times C_2) \rtimes C_2)$	[48,21]	Section 8.1.8
$(C_4 \rtimes C_4) \times C_3$	[48,22]	Section 8.1.9
$A_4 \times C_4$	[48,31]	Section 8.1.10
$(C_3 \rtimes C_8) \times C_3$	[72,12]	Section 8.1.11
$S_3 \times C_{12}$	[72,27]	Section 8.1.12
$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	[72,30]	Section 8.1.13
$S_3 \times C_6 \times C_3$	[108,42]	Section 8.1.14

Throughout the chapter, we make extensive use of the statements listed in Remark 4.2.

8.1.1 The group SD_8 (GAP ID [16,8])

The aim of this section is to prove the existence of a hyperelliptic fourfold with group

$$G = SD_8 := \langle a, b \,|\, a^8 = b^2 = 1,\ b^{-1}ab = a^3 \rangle.$$

The group G has the following three irreducible representations of dimension 2:

$$a \mapsto \pm \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^3 \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad b \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The last one of these is not faithful: its kernel is generated by a^4. Since ρ is faithful and the derived subgroup of G is generated by a^2, given by this non-faithful representation does not occur in ρ. The automorphism $a \mapsto a^5$, $b \mapsto b$ of G exchanges the first and the second listed representations; henceforth we shall focus only on the first one. By Lemma

5.5 (a), ρ is the direct sum of

$$a \mapsto \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^3 \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8.1}$$

and two of the representations $a \mapsto \pm 1$, $b \mapsto \pm 1$, at least one of them being non-trivial.

Lemma 8.1. *The Abelian variety A is isogenous to a product of an Abelian surface S and two elliptic curves E, E'.*

Proof. If $A' := \ker(\rho(a) - I)^0$ is an Abelian surface, Lemma 5.5 shows that

$$E := \left(A' \cap \ker(\rho(b) - I)\right)^0, \qquad E' := \left(A' \cap \operatorname{im}(\rho(b) - I)\right)^0$$

are elliptic curves. It then suffices to define $S := \operatorname{im}(\rho(a) - I)^0$.
If A' is an elliptic curve, we set

$$E := A', \qquad A'' := \operatorname{im}(\rho(a) + I).$$

Now we just define $E' := (A'' \cap \ker(\rho(a) + I))^0$ (which is an elliptic curve, since the representations of dimension 1 map a to ± 1), and $S := (A'' \cap \operatorname{im}(\rho(a) + I))^0$, so that A is isogenous to $S \times E \times E'$. $\square$

The Abelian surface S is isogenous to a product of elliptic curves as well, as we shall see in the upcoming Proposition. However, the matrix $\operatorname{diag}(\zeta_8, \zeta_8^3)$ is not an automorphism of any product of elliptic curves. This is the reason why we need to replace the irreducible representation (8.1) by an equivalent one, which we do as follows.

Proposition 8.2. *The Abelian surface S is isomorphic to $E_{\sqrt{2}i} \times E_{\sqrt{2}i}$, where*

$$E_{\sqrt{2}i} = \mathbb{C}/(\mathbb{Z} + \sqrt{2}i\mathbb{Z}).$$

A faithful action of $G = SD_8$ on $E_{\sqrt{2}i} \times E_{\sqrt{2}i}$ is given by

$$a = \frac{1}{\sqrt{2}i}\begin{pmatrix} 0 & 1 \\ 1 & \sqrt{2}i \end{pmatrix}, \quad b = \begin{pmatrix} 1 & \sqrt{2}i \\ 0 & -1 \end{pmatrix}.$$

Proof. The Abelian surface S admits the action of an automorphism of order 8 with CM-type $\{\zeta_8, \zeta_8^3\}$. By [CaCi93, Proposition 5.8], there is only one isomorphism class of Abelian surfaces with this property. We are done, since the Abelian surface in the statement of the Proposition satisfies our desired properties. $\square$

Remark 8.3. The previous Proposition can also be proved as follows. It follows from [ST61, Proposition 17] and from the table on p. 353 of [Wa82] (which shows that the class number of $\mathbb{Q}(\zeta_8)$ is 1), that there is exactly one *isomorphism class* of principally polarizable Abelian varieties of given CM-type $\{\zeta_8, \zeta_8^3\}$. This shows that the isogeny class of S is uniquely determined.

We shall henceforth assume that

$$A \sim_{\text{isog.}} E_{\sqrt{2}i} \times E_{\sqrt{2}i} \times E \times E',$$

and that

$$\rho(a) = \begin{pmatrix} 0 & 1 & & \\ 1 & \sqrt{2}i & & \\ & & 1 & \\ & & & \alpha \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & \sqrt{2}i & & \\ 0 & -1 & & \\ & & \beta & \\ & & & \gamma \end{pmatrix}.$$

Since $\rho(ab)$ must have the eigenvalue 1, we obtain

Lemma 8.4. *If $\beta = -1$, then $(\alpha, \gamma) \in \{(1,1), (-1,-1)\}$.*

We proceed by investigating the choices of the scalars α, β, γ. As mentioned above, the case $\alpha = \beta = \gamma = 1$ is excluded by Lemma 5.5. We will henceforth not take this case into consideration in the following discussion.

<u>Case 1:</u> $\alpha = \gamma = -1$, $\beta = 1$.
If we write

$$\begin{aligned} a(z) &= \rho(a)z + (a_1,\ a_2,\ a_3,\ a_4), \\ b(z) &= \rho(b)z + (b_1,\ b_2,\ b_3,\ b_4), \end{aligned}$$

the relation $b^{-1}ab = a^3$ implies that there there are w_1, w_2 such that

$$w := (w_1,\ w_2,\ 2a_3,\ 2b_4 - 2a_4)$$

is zero in A. Consequently, the element

$$v := (\rho(b) - \mathrm{id})w = (w_1 + w_2,\ w_1 + w_2,\ 0,\ 4b_4 - 4a_4)$$

is equal to zero in A as well. Now, since

$$2w - v = (w_1 - w_2,\ w_2 - w_1,\ 4a_3,\ 0) = 0 \text{ in } A,$$

we obtain that a^4 does not act freely on A.

<u>Case 2:</u> $\alpha = -1$, $\beta = \gamma = 1$.
We use the same notation as in Case 1. The relation $b^{-1}ab = a^3$ now implies that there exist w_1, w_2 such that

$$w := (w_1,\ w_2,\ 2a_3,\ 2b_4) = 0 \text{ in } A.$$

We abbreviate by $w_1',\ w_2'$ the first two coordinates of $v := (\rho(a) - \mathrm{id})w$. Since

$$v = (\rho(a) - \mathrm{id})w = (w_1',\ w_2',\ 0,\ 4b_4) = 0 \text{ in } A$$

as well, the equality

$$2w - v = (2w_1 - w_1',\ 2w_2 - w_2',\ 4a_3,\ 0) = 0$$

shows that a^4 does not act freely on A, i.e., Case 2 does not occur.

<u>Case 3:</u> $\alpha = \beta = 1$, $\gamma = -1$.

We give an example of a hyperelliptic fourfold in this case.

Example 8.5. Define

$$A := A'/H := (E_{\sqrt{2}i} \times E_{\sqrt{2}i} \times E \times E')/H, \text{ where}$$

$$E_{\sqrt{2}i} := \mathbb{C}/(\mathbb{Z} + \sqrt{2}i\mathbb{Z}),\ E := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),\ E' := \mathbb{C}/(\mathbb{Z} + \tau'\mathbb{Z}), \text{ and}$$

$$H := \left\langle \left(0, 0, \frac{1}{2}, \frac{1}{2}\right) \right\rangle.$$

Moreover, let $a, b \in \mathrm{Bihol}(A')$ be given as follows:

$$a(z) = \left(z_2,\ z_1 + \sqrt{2}i\ z_2,\ z_3 + \frac{1}{4},\ z_4 + \frac{1}{8}\right),$$
$$b(z) = \left(z_1 + \sqrt{2}i\ z_2,\ -z_2,\ z_3 + \frac{\tau}{2},\ -z_4\right).$$

Then a, b descend to biholomorphic self-maps of A, since the linear parts of a and b map H to H.
We shall now prove that $G := \langle a, b \rangle \subset \mathrm{Bihol}(A)$ is isomorphic to SD_8. The relations $a^8 = b^2 = \mathrm{id}_A$ are clear, whereas the relation $b^{-1}ab = a^3$ holds in view of our definition of H, because

$$b^{-1}ab(z) = \left(\sqrt{2}i\ z_1 - z_2,\ -z_1,\ z_3 + \frac{1}{4},\ z_4 - \frac{1}{8}\right),$$
$$a^3(z) = \left(\sqrt{2}i\ z_1 - z_2,\ -z_1,\ z_3 + \frac{3}{4},\ z_4 + \frac{3}{8}\right).$$

Let us now verify that G acts freely on A: indeed,

$$a(z) = \left(z_2,\ z_1 + \sqrt{2}i\ z_2,\ z_3 + \frac{1}{4},\ z_4 + \frac{1}{8}\right),$$
$$a^2(z) = \left(z_1 + \sqrt{2}i\ z_2,\ \sqrt{2}i\ z_1 - z_2,\ z_3 + \frac{1}{2},\ z_4 + \frac{1}{4}\right),$$
$$a^4(z) = \left(-z_1,\ -z_2,\ z_3,\ z_4 + \frac{1}{2}\right),$$
$$b(z) = \left(z_1 + \sqrt{2}i\ z_2,\ -z_2,\ z_3 + \frac{\tau}{2},\ -z_4\right)$$

act freely on A, since these elements do not act on $E \times E' \subset A'$ by the identity or by $(z_3,\ z_4) \mapsto \left(z_3 + \frac{1}{2},\ z_4 + \frac{1}{2}\right)$. Moreover,

$$ab(z) = \left(-z_2,\ z_1,\ z_3 + \frac{1}{4} + \frac{\tau}{2},\ -z_4 + \frac{1}{8}\right),$$
$$a^5b(z) = \left(z_2,\ -z_1,\ z_3 + \frac{1}{4} + \frac{\tau}{2},\ -z_4 + \frac{5}{8}\right)$$

act freely on A as well, because their respective third coordinate is a translation by an element of order 4 of E, but $H \subset A'[2]$. Since the list

$$\mathrm{id}_A, \quad a, \quad a^2, \quad a^4, \quad b, \quad ab, \quad a^5 b$$

is a system of representatives of the conjugacy classes of G, it is therefore proven that there exist hyperelliptic fourfolds with group SD_8, which fall under Case 3.

<u>Case 4:</u> $\alpha = \beta = \gamma = -1$.

We prove that there is no hyperelliptic fourfold falling under Case 4.
By a change of coordinates, we can assume that

$$\begin{aligned} a(z) &= \left(z_2,\ z_1 + \sqrt{2}i\ z_2,\ z_3 + a_3,\ -z_4 + a_4\right), \\ b(z) &= \left(z_1 + \sqrt{2}i\ z_2 + b_1,\ -z_2 + b_2,\ -z_3,\ -z_4\right). \end{aligned}$$

The relation $b^2 = \mathrm{id}_A$ implies that $(b_1 + \sqrt{2}i\ b_2,\ 0,\ 0,\ 0) \in H$. Since the first elliptic curve embeds into A, we obtain

$$b_1 + \sqrt{2}i\ b_2 = 0 \text{ in } E_{\sqrt{2}i}. \tag{8.2}$$

We will now investigate the relation $b^{-1}ab = a^3$, or, equivalently, $ab = ba^3$:

$$\begin{aligned} ab(z) &= a\left(z_1 + \sqrt{2}i\ z_2 + b_1,\ -z_2 + b_2,\ -z_3,\ -z_4\right) \\ &= \left(-z_2 + b_2,\ z_1 + b_1 + \sqrt{2}i\ b_2,\ -z_3 + a_3,\ z_4 + a_4\right) \\ &\overset{(8.2)}{=} (-z_2 + b_2,\ z_1,\ -z_3 + a_3,\ z_4 + a_4), \\ ba^3(z) &= b\left(\sqrt{2}i\ z_1 - z_2,\ -z_1,\ z_3 + 3a_3,\ -z_4 + a_4\right) \\ &= (-z_2 + b_1,\ z_1 + b_2,\ -z_3 - 3a_3,\ z_4 - a_4). \end{aligned}$$

Consequently, $ab = ba^3$ is satisfied if and only if

$$v := (b_2 - b_1,\ -b_2,\ 4a_3,\ 2a_4) \in H. \tag{8.3}$$

We calculate

$$\rho(a)v = (-b_2,\ b_2 - b_1 - \sqrt{2}i\ b_2,\ 4a_3,\ -2a_4) \overset{(8.2)}{=} (-b_2,\ b_2,\ 4a_3,\ -2a_4) = 0,$$

and

$$v + \rho(a)v = (-b_1,\ 0,\ 8a_3,\ 0) \in H.$$

Now, the relation $a^8 = \mathrm{id}_A$ implies that $(0,\ 0,\ 8a_3,\ 0) \in H$, and because $E \subset A$, we obtain $8a_3 = 0$. Plugging this into $v + \rho(a)v \in H$ implies that

$$b_1 = 0 \text{ in } E_{\sqrt{2}i},$$

since the first elliptic curve embeds into A. In view of (8.2), this means that $\sqrt{2}i\ b_2 = 0$. In particular

$$(\sqrt{2}i)^2 b_2 = -2b_2 = 0.$$

Since necessarily $b_2 \neq 0$ (else, b does not act freely on A), we obtain

$$b_2 = \frac{\sqrt{2}i}{2}$$

as the only possibility. We prove that b does not act freely in this case. In fact, the equation $b(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, z_4)$ is satisfied if and only if

$$\left(\sqrt{2}i\ z_2,\ -2z_2 + \frac{\sqrt{2}i}{2},\ -2z_3,\ -2z_4\right) = 0 \text{ in } A.$$

For arbitrary z_1, $z_2 := \frac{1}{2}$, $z_3 := 2a_3$ and $z_4 := -a_4$, the above vector is equal to the vector v given in (8.3), which is zero in A. Thus, b does not act freely on A and there is no hyperelliptic fourfold in Case 4.

To summarize everything,

Proposition 8.6. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G = SD_8 := \langle a, b \,|\, a^8 = b^2 = 1, b^{-1}ab = a^3 \rangle.$$

The Abelian variety A is necessarily isogenous to $E_{\sqrt{2}i} \times E_{\sqrt{2}i} \times E \times E'$, where $E_{\sqrt{2}i} = \mathbb{C}/(\mathbb{Z} + \sqrt{2}i\mathbb{Z})$ and E, E' are elliptic curves as well. In particular, every complete family of hyperelliptic fourfolds with group G is 2-dimensional. Moreover, up to a change of basis,

$$\rho(a) = \begin{pmatrix} 0 & 1 & & \\ 1 & \sqrt{2}i & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} 1 & \sqrt{2}i & & \\ 0 & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

8.1.2 The group $D_4 \times C_2$ (GAP ID [16,11])

Consider the group

$$G := D_4 \times C_2 = \langle r, s, k \,|\, r^4 = s^2 = (rs)^2 = k^2 = [r, k] = [s, k] = 1 \rangle.$$

The group $D_4 = \langle r, s \rangle$ has the following irreducible representation of degree 2:

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Lemma 8.7. *We can assume without loss of generality that*

$$\rho(r) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \rho(k) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

Proof. Since k is a central element, it is mapped to I_2 or $-I_2$ by the irreducible representation of dimension 2 contained in ρ. After possibly replacing k by r^2k, we can assume that it is mapped to I_2. Now, if both the third and fourth diagonal entry of $\rho(k)$ were equal to -1, the matrix $\rho(r^2k)$ would not have the eigenvalue 1. Thus we can assume that $\rho(k)$ is as stated. By possibly replacing r by rk and s by sk, we can assume that the last diagonal entry of $\rho(r)$ resp. $\rho(s)$ are 1 resp. -1. Since $\rho(rk)$ must have the eigenvalue 1, we obtain that the third diagonal entry of $\rho(r)$ is 1. It remains to show that the third diagonal entry of $\rho(s)$ is necessarily equal to -1: if it would be equal to 1, the subgroup $\langle r, sk\rangle$ of G would be isomorphic to D_4 and does not act freely on an Abelian fourfold by Lemma 5.5 (b). □

We observe that if $\rho(r),\ \rho(s)$ are given in this way, we obtain an example of a hyperelliptic fourfold with group $D_4 = \langle r, s\rangle$ in a completely analogous way to the case in dimension 3, see Part II. In particular, A is isogenous to the product of two elliptic curves $E_1 \cong E_2$ and an Abelian surface S. Using the additional element k, we can show that S is isogenous to a product of elliptic curves as well:

Lemma 8.8. *The Abelian variety A is isogenous to the product of four elliptic curves $E_j \subset A$. More precisely, $A \cong (E_1 \times E_2 \times E_3 \times E_4)/H$, where $E_1 \cong E_2$ and*

$$H \subset E_1[2] \times E_2[2] \times E_3[8] \times E_4[8].$$

Proof. The Abelian variety A is isogenous to the product of

$$S' := \operatorname{im}(\rho(r) - \mathrm{id}) \text{ and } S := \ker(\rho(r) - \mathrm{id}_A)^0.$$

Moreover, S' is isogenous to the product of

$$E_1 := \ker(\rho(s)|_{S'} - \mathrm{id}_{S'})^0 \text{ and } E_2 := \operatorname{im}(\rho(s)|_{S'} - \mathrm{id}_{S'}),$$

while S is isogenous to the product of

$$E_3 := \ker(\rho(k)|_S - \mathrm{id}_S)^0 \text{ and } E_4 := \operatorname{im}(\rho(k)|_S - \mathrm{id}_S).$$

Moreover, $E_2 = \rho(r)(E_1)$, and therefore $E_1 \cong E_2$. Now we go on proving the statement concerning H. Denote the lattices of the E_j by Λ_j. Then

$$H = \Lambda/(\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \oplus \Lambda_4).$$

We first prove the following

Claim: If we write $A = V/\Lambda$ and $E_j = V_j/\Lambda_j$ for $j = 1,\ ...,\ 4$. Then $\Lambda_1 = V_1 \cap \Lambda$ and $\Lambda_2 = V_2 \cap \Lambda$.

Proof of the Claim: As mentioned above, $E_2 = \rho(r)(E_1)$, and thus

$$\Lambda_2 = \rho(r)(\Lambda_1) = \rho(r)(\Lambda_1) \cap \Lambda \subset V_2 \cap \Lambda.$$

Conversely, $\rho(r)(V_2 \cap \Lambda) \subset \rho(r)(V_2) \cap \rho(r)(\Lambda) = V_1 \cap \Lambda = \Lambda_1$, and applying the automorphism $\rho(r)$ of Λ gives $V_2 \cap \Lambda \subset \rho(r)(\Lambda_1) = \Lambda_2$. This proves the Claim.

Suppose now that $\lambda \in \Lambda$. Since $\mathrm{im}(I+\rho(s)) = \ker(I-\rho(s))$, we can write

$$2\lambda = \underbrace{(I+\rho(s))\lambda}_{=:\lambda_1\in\Lambda_1} + \underbrace{(I-\rho(s))\lambda.}_{=:\lambda'\in\Lambda_2\oplus\Lambda_3\oplus\Lambda_4}$$

Now, we are opting to express $2\lambda'$ as an element in $\Lambda_2\oplus\Lambda_3\oplus\Lambda_4$:

$$2\lambda' = \underbrace{\left(I+\rho(r^2)\right)\lambda'}_{=:\lambda_2\in\Lambda_2} + \underbrace{\left(I-\rho(r^2)\right)\lambda'.}_{=:\lambda''\in\Lambda_3\oplus\Lambda_4}$$

Finally,

$$2\lambda'' = \underbrace{(I+\rho(k))\lambda''}_{=:\lambda_3\in\Lambda_3} + \underbrace{(I-\rho(k))\lambda''.}_{=:\lambda_4\in\Lambda_4}$$

Summarizing everything, we have shown that

$$\lambda = \frac{\lambda_1}{2} + \frac{\lambda_2}{4} + \frac{\lambda_3}{8} + \frac{\lambda_4}{8}.$$

Since the automorphism $\rho(r)$ of Λ exchanges Λ_1 and Λ_2, we obtain that λ_2 is divisible by 2 in Λ_2. We conclude by noting that $H = \Lambda/(\Lambda_1\oplus\Lambda_2\oplus\Lambda_3\oplus\Lambda_4)$. $\square$

We shall now describe an example of a hyperelliptic fourfold with group $D_4\times C_2$.

Example 8.9. Suppose that

$$A := A'/H := (E\times E\times E'\times E''), \quad \text{where}$$

$$E = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}), \quad E' = \mathbb{C}/(\mathbb{Z}+\tau'\mathbb{Z}), \quad E'' = \mathbb{C}/(\mathbb{Z}+\tau''\mathbb{Z}) \quad \text{and}$$

$$H := \left\langle \left(\frac{1+\tau}{2},\ \frac{1+\tau}{2},\ 0,\ 0\right)\right\rangle.$$

Moreover, define $r, s, k \in \mathrm{Bihol}(A')$ as follows:

$$\begin{aligned} r(z) &= \left(-z_2,\ z_1,\ z_3+\frac{1}{4},\ z_4\right), \\ s(z) &= \left(z_1+\frac{1}{2},\ -z_2+\frac{\tau}{2},\ -z_3,\ -z_4\right), \\ k(z) &= \left(z_1,\ z_2,\ z_3+\frac{\tau'}{2},\ -z_4\right). \end{aligned}$$

Then the linear parts of $r,\ s,\ k$ map H to H, and thus $r,\ s,\ k$ descend to biholomorphic self-maps of A, which we shall again denote by $r,\ s,\ k$.
Viewed as maps $A\to A$, it is immediate that $r^4 = s^2 = k^2 = [r,k] = [s,k] = \mathrm{id}_A$, and the relation $(rs)^2 = \mathrm{id}_A$ follows, exactly as in the 3-dimensional case, by our definition of H. We prove exactly as in the 3-dimensional case that $D_4 = \langle r, s\rangle$ acts freely on A. Moreover, the actions of k, rk and r^2k on A are clearly free, since these elements act on

the third elliptic curve by a non-trivial translation. The last two elements to investigate are

$$sk(z) = \left(z_1 + \frac{1}{2},\ -z_2 + \frac{\tau}{2},\ -z_3 + \frac{\tau'}{2},\ z_4\right) : \text{acts freely, since } H \text{ does not contain an element whose first coordinate is } \frac{1}{2}.$$

$$rsk(z) = \left(z_2 + \frac{\tau}{2},\ z_1 + \frac{1}{2},\ -z_3 + \frac{\tau'}{2} + \frac{1}{4},\ z_4\right) : \text{acts freely; one proves this exactly like the freeness of the action of } rs.$$

We have established that there exist indeed hyperelliptic fourfolds with group $G := D_4 \times C_2 = \langle r, s, k\rangle$. Summarizing everything,

Proposition 8.10. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G := D_4 \times C_2 = \langle r, s, k \,|\, r^4 = s^2 = (rs)^2 = k^2 = [r, k] = [s, k] = 1\rangle.$$

The Abelian variety A is isogenous to a product of elliptic curves $E \times E \times E' \times E''$, and thus every complete family of hyperelliptic fourfolds with group G is 3-dimensional. Moreover, up to a change of basis and generators,

$$\rho(r) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \rho(k) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

8.1.3 The group $Q_8 \times C_3$ (GAP ID [24,11])

This section is dedicated to proving the existence of hyperelliptic fourfolds with group

$$G := Q_8 \times C_3 = \langle a, b, k \,|\, a^4 = k^3 = 1,\ a^2 = b^2,\ ab = b^{-1}a,\ [a, k] = [b, k] = 1\rangle.$$

By Lemma 5.5 (a), ρ is the direct sum of an irreducible representation ρ_2 of dimension 2 and two characters $\rho_1,\ \rho_1'$.

Lemma 8.11. *We can assume that $\rho_1(a) = \rho_1(b) = 1$, $\rho_1'(a) = -1$, $\rho_1'(b) = 1$.*

Proof. By Lemma 5.5 (b), (at least) one of the representations $\rho_1|_{Q_8}$, $\rho_1'|_{Q_8}$ is non-trivial. If both were non-trivial, then

$$\rho_1(a) = 1, \rho_1'(a) = -1,$$
$$\rho_1(b) = -1, \rho_1'(b) = 1,$$

since $\rho(a)$, $\rho(b)$ need to have the eigenvalue 1. In this case, $\rho(ab)$ does not have the eigenvalue 1. We shall henceforth assume that $\rho_1|_{Q_8}$ is trivial. If $\rho_1'(a) = \rho_1'(b) = -1$, we can replace b by ab to obtain $\rho_1'(ab) = 1$, so we can assume $\rho_1'(a) = -1$, $\rho_1'(b) = 1$ as claimed. $\square$

From now on, we shall assume that $\rho_1|_{Q_8}$, $\rho_1'|_{Q_8}$ are given as in the statement of the Lemma above.

Lemma 8.12. *The Abelian variety $A = V/\Lambda$ is isogenous to the product of an Abelian surface $S = \mathbb{C}^2/\Lambda_S$ and two elliptic curves $E = \mathbb{C}/\Lambda_E$, $E' = \mathbb{C}/\Lambda_{E'}$. More precisely,*

$$A \cong (S \times E \times E')/H,$$

where

$$H = \Lambda/(\Lambda_S \oplus \Lambda_E \oplus \Lambda_{E'}) \subset S[2] \times E[4] \times E'[4].$$

Proof. We define

$$\begin{aligned} S &:= \operatorname{im}(\rho(b^2) - I)^0, \\ E &:= \left(\ker(\rho(b^2) - I) \cap \ker(\rho(a) - I)\right)^0, \\ E' &:= \left(\ker(\rho(b^2) - I) \cap \operatorname{im}(\rho(a) - I)\right)^0. \end{aligned}$$

Then A is isogenous to $S \times E \times E'$. It remains to prove the statement regarding H. Let $\lambda \in \Lambda$. Then

$$2\lambda = \underbrace{(I - \rho(b)^2)\lambda}_{\in \Lambda_S} \ + \ \underbrace{(I + \rho(b^2))\lambda}_{=:\lambda' \in \Lambda_E \oplus \Lambda_{E'}}.$$

Moreover, since $\operatorname{im}(\rho(b^2) + I) = \ker(\rho(b^2) - I)$ and

$$\operatorname{im}(\rho(a) + I) \cap \ker(\rho(b^2) - I) = \ker(\rho(a) - I) \cap \ker(\rho(b^2) - I),$$

the matrices being viewed as endomorphisms of $\mathbb{C}^4$, we obtain:

$$2\lambda' = \underbrace{(\rho(a) - I)\tilde{\lambda}}_{\in \Lambda_{E'}} \ + \ \underbrace{(\rho(a) + I)\tilde{\lambda}}_{\in \Lambda_E}.$$

This shows that $\lambda \in \frac{1}{2}\Lambda_S \oplus \frac{1}{4}\Lambda_E \oplus \frac{1}{4}\Lambda_{E'}$. This proves the statement. □

We now take the central element k into account.

Lemma 8.13. *The matrix $\rho(k)$ is equal to one of*

$$\operatorname{diag}(\zeta_3, \zeta_3, 1, 1) \ \textit{and} \ \rho(k) = \operatorname{diag}(\zeta_3^2, \zeta_3^2, 1, 1).$$

Proof. According to the previous Lemma, we may write $A = (S \times E \times E')/H$. Since k is central, it is mapped to a multiple of the identity matrix by any irreducible representation of $Q_8 \times C_3$. It is clear that the third diagonal entry of $\rho(k)$ is necessarily equal to 1, since $\rho(ak)$ needs to have the eigenvalue 1. Up to replacing k by its inverse, we can therefore assume that

$$\rho(k) = \operatorname{diag}(\zeta_3, \ \zeta_3, \ 1, \ \zeta_3^j) \text{ or } \rho(k) = \operatorname{diag}(1, \ 1, \ 1, \ \zeta_3),$$

where $j \in \{0, 1, 2\}$. We will only consider the first possibility and construct an element which does not act freely in the case $j \neq 0$: the reader will be convinced that the second possibility $\rho(k) = \mathrm{diag}(1,\ 1,\ 1,\ \zeta_3)$ is dealt with in precisely the same way. Thus, assume that $\rho(k) = \mathrm{diag}(\zeta_3, \zeta_3, 1, \zeta_3^j)$, $j \neq 0$ and write

$$\begin{aligned} a(z) &= (-z_2 + a_1,\ z_1 + a_2,\ z_3 + a_3,\ -z_4 + a_4), \\ b(z) &= (iz_1 + b_1,\ -iz_2 + b_2,\ z_3 + b_3,\ z_4 + b_4). \end{aligned}$$

The relation $b^4 = \mathrm{id}_A$ implies that $(0,\ 0,\ 4b_3,\ 4b_4) \in H$, and because $\rho(k)$ fixes H setwise, $(0,\ 0,\ 0,\ 4(\zeta_3^j - 1)b_4) \in H$ as well. Since E' embeds into A, we obtain that $4(\zeta_3^j - 1)b_4 = 0$ in E'. Moreover, by the previous Lemma, H is contained in the 4-torsion group of $S \times E \times E'$: thus Lemma 2.8 implies that $4b_4 = 0$.
The condition $a^2 = b^2$ implies that H contains an element of the form

$$w := (w_1,\ w_2,\ 2(b_3 - a_3),\ 2b_4).$$

Furthermore, the relation $ab = b^{-1}a$ implies that H contains an element u of the form

$$u := (u_1,\ u_2,\ 2b_3,\ 0).$$

This means that $w - u \in H$ takes the form

$$w - u = (w_1 - u_1,\ w_2 - u_2,\ -2a_3,\ 2b_4).$$

Necessarily, $2b_4 \neq 0$ (else, a^2 would not act freely).
We now consider the element

$$w' := (\rho(k) - \mathrm{id})w = ((\zeta_3 - 1)w_1,\ (\zeta_3 - 1)w_2,\ 0,\ 2(\zeta_3^j - 1)b_4) \in H.$$

Since the element $2b_4 \neq 0$ has order 2, we obtain that

$$\left\langle 2\zeta_3^j(\zeta_3^j - 1)b_4,\ 2(\zeta_3^j - 1)b_4 \right\rangle = E'[2] \subset \mathrm{im}(p\colon H \to E'),$$

where $p\colon S \times E \times E' \to E'$ is the projection. (Here, we used that $2b_4$ and $2(\zeta_3^j - 1)b_4$ are not fixed by multiplication by ζ_3^j, see Lemma 2.8.)
This means that an appropriate integral linear combination of w' and $\rho(k)w'$ takes the form $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, 0, 2b_4)$. Adding $\tilde{w}$ and w, we obtain that H contains an element of the form

$$v := \tilde{w} + w = (v_1,\ v_2,\ 2(b_3 - a_3),\ 0).$$

Then $w' - u = (w_1' - u_1,\ w_2' - u_2,\ -2a_3,\ 0) \in H$ shows that a^2 does not act freely on A, a contradiction. $\square$

After having possibly replaced k by k^2, we may (and will) assume that

$$\rho(k) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ 1,\ 1).$$

Proposition 2.4 implies

Lemma 8.14. *The Abelian surface S is isomorphic to $F \times F$, where $F = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$ is the equianharmonic elliptic curve.*

We are almost ready to give an example of a hyperelliptic fourfold with group $Q_8 \times C_3$: since S is isogenous to the square of the equianharmonic elliptic curve, $\rho_2(b)$ as described in the proof of Lemma 8.13 does not act on $F \times F$; we will need perform a suitable change of basis. Indeed,

$$\rho_2(a) = \begin{pmatrix} 1+2\zeta_3 & -1 \\ -2 & -1-2\zeta_3 \end{pmatrix}, \quad \rho_2(b) = \begin{pmatrix} -1 & \zeta_3^2 \\ -2\zeta_3 & 1 \end{pmatrix}, \quad \rho(k) = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3 \end{pmatrix}$$

still defines an irreducible representation of $Q_8 \times C_3$ and these matrices are automorphisms of $F \times F$.

We are now able to give an example of a hyperelliptic fourfold with group $Q_8 \times C_3$.

Example 8.15. Let

$$A := A'/H := (F \times F \times E \times E')/H, \text{ where}$$
$$F := \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z}),\ E := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),\ E' := \mathbb{C}/(\mathbb{Z} + \tau'\mathbb{Z}), \text{ and}$$
$$H := \left\langle \left(0,\ 0,\ \frac{1}{2},\ \frac{1}{2}\right) \right\rangle.$$

Furthermore, define automorphisms of A' as follows:

$$\begin{aligned}
a(z) &= \left((1+2\zeta_3)z_1 - z_2,\ -2z_1 - (1+2\zeta_3)z_2,\ z_3 + \frac{1}{4},\ -z_4\right), \\
b(z) &= \left(\zeta_3^2 z_2 - z_1,\ z_2 - 2\zeta_3 z_1,\ z_3,\ z_4 + \frac{1}{4}\right), \\
k(z) &= \left(\zeta_3 z_1,\ \zeta_3 z_2,\ z_3 + \frac{1}{3},\ z_4\right).
\end{aligned}$$

Since the linear parts of a, b, k map H to H, the maps a, b, k descend to automorphisms of $A = A'/H$.
It is clear that $a^4 = b^4 = k^3 = [a,k] = [b,k] = \mathrm{id}_A$. Moreover,

$$ab(z) = \left(iz_2,\ iz_1,\ z_3 + \frac{1}{4},\ -z_4 - \frac{1}{4}\right) = b^{-1}a(z).$$

The last relation to investigate is $a^2 = b^2$. In fact,

$$\begin{aligned}
a^2(z) &= \left(-z_1,\ -z_2,\ z_3 + \frac{1}{2},\ z_4\right), \\
b^2(z) &= \left(-z_1,\ -z_2,\ z_3,\ z_4 + \frac{1}{2}\right)
\end{aligned}$$

implies that $a^2 = b^2$ holds if and only if

$$\left(0,\ 0,\ \frac{1}{2},\ \frac{1}{2}\right) = 0 \text{ in } A.$$

This is the case by our definition of H. We have thus defined a faithful action of $G := \langle a, b, k \rangle \cong Q_8 \times C_3$ on A, such that G does not contain any translations (since ρ

contains a faithful irreducible representation of G).
It remains to investigate the freeness of the action of G on A. It it immediate that a, b and ab act freely on A. To prove that the element

$$a^2(z) = \left(-z_1,\ -z_2,\ z_3 + \frac{1}{2},\ z_4\right),$$

acts freely on A, observe that a^2 has a fixed point on A if and only if H contains an element of the form $\left(w_1, w_2, \frac{1}{2}, 0\right)$. This is not the case, and thus a^2 acts freely on A. Moreover,

$$k^j,\quad ak^j,\quad bk^j,\quad abk^j,\quad a^2k^j,\qquad k \in \{1,2\}$$

act freely on A, since the third coordinate of the listed elements is a translation by a non-trivial torsion element whose order is divisible by 3, and H is a subgroup of $A'[2]$. Since the above list of elements exhausts all non-trivial conjugacy classes of $G = Q_8 \times C_3$, we have defined a free action of G on A. Therefore,

Proposition 8.16. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G := Q_8 \times C_3 = \langle a, b, k \mid a^4 = k^3 = 1, a^2 = b^2, ab = b^{-1}a, [a,k] = [b,k] = 1\rangle.$$

The Abelian variety A is isogenous to $F \times F \times E \times E'$, where F is the equianharmonic elliptic curve and E, E' are arbitrary elliptic curves. In particular, every complete family of hyperelliptic fourfolds with group G is 2-dimensional. Moreover, after a change of coordinates, the linear parts $\rho(a)$, $\rho(b)$, $\rho(k)$ of a, b and k are given by

$$\rho(a) = \begin{pmatrix} 1+2\zeta_3 & -1 & & \\ -2 & -1-2\zeta_3 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} -1 & \zeta_3^2 & & \\ -2\zeta_3 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$\rho(k) = \begin{pmatrix} \zeta_3 & & & \\ & \zeta_3 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

8.1.4 The Heisenberg group of order 27 (GAP ID [27,3])

The main result of this subsection is the following.

Theorem 8.17. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G = \mathrm{Heis}(3) := \langle g, h, k \mid g^3 = h^3 = k^3 = 1, [g,k] = [h,k] = 1, ghg^{-1}h^{-1} = k\rangle.$$

Moreover, A is isogenous to the 4-fold product of the equianharmonic elliptic curve (implying that X is a rigid complex manifold). Moreover, up to a change of basis,

$$\rho(g)=\begin{pmatrix}1&&&\\&0&0&1\\&1&0&0\\&0&1&0\end{pmatrix},\quad \rho(h)=\begin{pmatrix}\zeta_3^j&&&\\&1&&\\&&\zeta_3^2&\\&&&\zeta_3\end{pmatrix},\quad \rho(k)=\begin{pmatrix}1&&&\\&\zeta_3&&\\&&\zeta_3&\\&&&\zeta_3\end{pmatrix},$$

where $j\in\{1,2\}$.

Let $G := \mathrm{Heis}(3)$. The complex representation $\rho\colon G\to \mathrm{GL}(4,\mathbb{C})$ splits as a direct sum of a 3-dimensional and a 1-dimensional representation,

$$\rho=\rho_1\oplus\rho_3.$$

The 3-dimensional irreducible representations of $G=\mathrm{Heis}(3)$ (i.e., the possibilities for ρ_3) are:

$$\rho_3(g)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad \rho_3(h)=\begin{pmatrix}1&&\\&\zeta_3&\\&&\zeta_3^2\end{pmatrix},\quad \rho_3(k)=\begin{pmatrix}\zeta_3^2&&\\&\zeta_3^2&\\&&\zeta_3^2\end{pmatrix},\ \text{and}$$
$$\rho_3(g)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad \rho_3(h)=\begin{pmatrix}1&&\\&\zeta_3^2&\\&&\zeta_3\end{pmatrix},\quad \rho_3(k)=\begin{pmatrix}\zeta_3&&\\&\zeta_3&\\&&\zeta_3\end{pmatrix}.$$

It suffices to consider only one of the possibilities for ρ_3, since the second one is obtained from the first one by replacing h and k by their squares. We will henceforth only focus on the second of the listed possibilities. Since every matrix in the image of ρ must have the eigenvalue 1, we are left with the possibilities

$$\rho_1(g)=\zeta_3^l,\quad \rho_1(h)=\zeta_3^j,\quad \rho_1(k)=1.$$

Proposition 8.18. *The Abelian variety A is isogenous to a product of four elliptic curves $E_j\subset A$,*

$$A\cong(E_1\times E_2\times E_3\times E_4)/H.$$

Proof. Define

$$\begin{aligned}E_1 &:= \ker(\rho(k)-I)^0,\\ E_2 &:= \left(\ker(\rho(h)-I)\cap \operatorname{im}(\rho(k)-I)\right)^0,\\ E_3 &:= \left(\ker(\rho(hk)-I)\cap \operatorname{im}(\rho(k)-I)\right)^0,\\ E_4 &:= \left(\ker(\rho(hk^2)-I)\cap \operatorname{im}(\rho(k)-I)\right)^0.\end{aligned}$$

Then A is isogenous to $E_1\times E_2\times E_3\times E_4$. $\square$

Lemma 8.19. *It is not possible that $\rho_1(g) = \rho_1(h) = 1$.*

Proof. If $\rho_1(g) = \rho_1(h) = 1$, the relation $[g, h] = k$ implies that H contains an element with first coordinate equal to c_1, which in turn means that k does not act freely on A. $\square$

Lemma 8.20. *We can assume that exactly one of $\rho_1(g)$ and $\rho_1(h)$ is equal to* 1.

Proof. If both $\rho_1(g)$ and $\rho_1(h)$ are different from 1, we have two possibilities:

(a) If we have $\rho_1(gh) = 1$, replace g by $(gh)^2$; it is then not difficult to see that the defining relations of $G = \mathrm{Heis}(3)$ are still satisfied, and we have $\rho_1((gh)^2) = 1$.

(b) If we have $\rho_1(g^2h) = 1$, replace g by g^2h; it is then easy to see that the defining relations of $G = \mathrm{Heis}(3)$ are still satisfied, and we have $\rho_1(g^2h) = 1$.

This completes the proof. $\square$

We can (and will!) therefore assume that

$$\rho_1(g) = 1, \quad \rho_1(h) = \zeta_3^j, \ j \in \{1, 2\}.$$

Corollary 8.21. *Each of the elliptic curves $E_j \subset A$ is isomorphic to the equianharmonic elliptic curve $\mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$.*

Proof. In fact, since $\rho(k)$ acts on $E_2 \times E_3 \times E_4$ by multiplication by ζ_3, the statement concerning E_2, E_3 and E_4 follows from Proposition 2.4. The elliptic curve E_1 is the equianharmonic elliptic curve as well, since $\rho(h)$ acts on it by multiplication by ζ_3^j, $j \in \{1, 2\}$. $\square$

In the following, we will assume that $j = 1$, but of course, the same discussion can be done for $j = 2$.

Remark 8.22. Note that we can change the origin in the elliptic curves E_j suitably, so that we are allowed to write

$$\begin{aligned} g(z) &= (z_1 + a_1, \ z_4 + a_2, \ z_2 + a_3, \ z_3 + a_4), \\ h(z) &= (\zeta_3 z_1, \ z_2 + b_2, \ \zeta_3^2 z_3 + b_3, \ \zeta_3 z_4 + b_4), \\ k(z) &= (z_1 + c_1, \ \zeta_3 z_2, \ \zeta_3 z_3, \ \zeta_3 z_4). \end{aligned}$$

To ensure a better readability, we write the elements of H as column vectors. The investigation of the defining relations of G gives

Lemma 8.23. *The following statements hold.*

(1) $hk = kh \iff v_1 := \begin{pmatrix} (\zeta_3 - 1)c_1 \\ (1 - \zeta_3)b_2 \\ (1 - \zeta_3)b_3 \\ (1 - \zeta_3)b_4 \end{pmatrix} \in H.$

(2) $gk = kg \iff v_2 := \begin{pmatrix} 0 \\ (1-\zeta_3)a_2 \\ (1-\zeta_3)a_3 \\ (1-\zeta_3)a_4 \end{pmatrix} \in H.$

(3) $g^3 = \mathrm{id}_A \iff v_3 := \begin{pmatrix} 3a_1 \\ a_2+a_3+a_4 \\ a_2+a_3+a_4 \\ a_2+a_3+a_4 \end{pmatrix} \in H.$

(4) $h^3 = \mathrm{id}_A \iff v_4 := \begin{pmatrix} 0 \\ 3b_2 \\ 0 \\ 0 \end{pmatrix} \in H \iff 3b_2 = 0$ *in* E_2.

(5) $k^3 = \mathrm{id}_A \iff v_5 := \begin{pmatrix} 3c_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in H \iff 3c_1 = 0$ *in* E_1.

(6) $ghg^{-1}h^{-1} = k \iff v_6 := \begin{pmatrix} (2\zeta_3-1)a_1 - c_1 \\ b_4 - \zeta_3 b_2 + a_2 + \zeta_3(a_3+a_4) \\ b_2 - \zeta_3 b_3 + a_2 + a_3 + a_4 \\ b_3 - \zeta_3 b_4 + \zeta_3^2(a_2+a_3) + a_4 \end{pmatrix} \in H.$

Proof. This is just computation. $\square$

We will now investigate the freeness of the action of G on A. Representatives of the conjugacy classes of G are given by $1,\ g,\ g^2,\ h,\ h^2,\ k,\ k^2,\ gh,\ g^2h^2,\ g^2h,\ gh^2$.

Lemma 8.24. *The following statements hold.*

(a) g *acts freely on* $A \iff H$ *contains no element of the form*

$$(a_1,\ w_2+a_2,\ w_3+a_3,\ -w_2-w_3+a_4).$$

(b) g^2 *acts freely on* $A \iff H$ *contains no element of the form*

$$(2a_1,\ w_2+a_2+a_4,\ w_3+a_2+a_3,\ -w_2-w_3+a_3+a_4).$$

(c) h *and* h^2 *act freely on* $A \iff H$ *contains no element of the form*

$$(w_1,\ 2b_2,\ w_3,\ w_4).$$

(d) k and k^2 act freely on A $\iff$ H contains no element of the form

$$(2c_1,\ w_2,\ w_3,\ w_4).$$

(e) gh acts freely on A $\iff$ H contains no element of the form

$$(w_1,\ w_2+a_2+b_4,\ w_3+a_3+b_2,\ -\zeta_3^2(w_2+w_3)+a_4+b_3).$$

(f) g^2h^2 acts freely on A $\iff$ H contains no element of the form

$$(w_1,\ w_2+a_2+a_4-\zeta_3 b_3,\ w_3+a_2+a_3-\zeta_3^2 b_4,\ -w_2-\zeta_3 w_3+a_3+a_4+2b_2).$$

(g) g^2h acts freely on A $\iff$ H contains no element of the form

$$(w_1,\ w_2+a_2+a_4+b_3,\ w_3+a_2+a_3+b_4,\ -w_2-\zeta_3^2 w_3+a_3+a_4+b_2).$$

(h) gh^2 acts freely on A $\iff$ H contains no element of the form

$$(w_1,\ w_2+a_2-\zeta_3^2 b_4,\ w_3+a_3-b_2,\ -\zeta_3(w_2+w_3)+a_4-\zeta_3 b_3).$$

Proof. (a) $g(z)=(z_1+a_1,\ z_4+a_2,\ z_2+a_3,\ z_3+a_4)$ acts freely on A if and only if there are no $z_1,\dots,z_4$ such that

$$(a_1,\ z_4-z_2+a_2,\ z_2-z_3+a_3,\ z_3-z_4+a_4)\in H.$$

The result follows if we define $w_2:=z_4-z_2$ and $w_3:=z_2-z_3$.

(b) The element $g^2(z)=(z_1+2a_1,\ z_3+a_2+a_4,\ z_4+a_2+a_3,\ z_2+a_3+a_4)$ acts freely on A if and only if there are no $z_1,\dots,z_4$ such that

$$(2a_1,\ z_3-z_2+a_2+a_4,\ z_4-z_3+a_2+a_3,\ z_2-z_4+a_3+a_4)\in H.$$

We obtain the result if we set $w_2:=z_3-z_2$ and $w_3:=z_4-z_3$.

(c) $h(z)=\big(\zeta_3 z_1,\ z_2+b_2,\ \zeta_3^2 z_3+b_3,\ \zeta_3 z_4+b_4\big)$ acts freely on A if and only if there are no $z_1,\dots,z_4$ such that

$$\big((\zeta_3-1)z_1,\ b_2,\ (\zeta_3^2-1)z_3+b_3,\ (\zeta_3-1)z_4+b_4\big)\in H.$$

Setting $u_1:=(\zeta_3-1)z_1$, $u_3:=(\zeta_3^2-1)z_3+b_3$ and $u_4:=(\zeta_3-1)z_4+b_4$, the freeness of the action of h is equivalent to requiring that H contains no element of the form

$$(u_1,\ b_2,\ u_3,\ u_4)\,.$$

Now consider $h^2(z)=\big(\zeta_3^2 z_1,\ z_2+2b_2,\ \zeta_3 z_3-\zeta_3 b_3,\ \zeta_3^2 z_4-\zeta_3^2 b_4\big)$. This element acts freely on A if and only if there are no $z_1,\dots,z_4$ such that

$$\big((\zeta_3-1)z_1,\ 2b_2,\ (\zeta_3-1)z_3-\zeta_3^2 b_3,\ (\zeta_3^2-1)z_4-\zeta_3 b_4\big)\,.$$

Defining $u_1':=(\zeta_3-1)z_1$, $u_3':=(\zeta_3-1)z_3-\zeta_3 b_3$ and $u_4':=(\zeta_3^2-1)z_4-\zeta_3^2 b_4$, we observe that h^2 acts freely if and only if H contains no element of the form

$$(u_1',\ 2b_2,\ u_3',\ u_4').$$

By Lemma 8.23 (4), $3b_2 = 0$, so that both h and h^2 act freely if and only if H contains no element of the form
$$(w_1,\ 2b_2,\ w_3,\ w_4).$$

(d) The elements
$$\begin{aligned} k(z) &= (z_1 + c_1,\ \zeta_3 z_2,\ \zeta_3 z_3,\ \zeta_3 z_4)\,, \\ k^2(z) &= \big(z_1 + 2c_2,\ \zeta_3^2 z_2,\ \zeta_3^2 z_3,\ \zeta_3^2 z_4\big)\,. \end{aligned}$$
act freely on A if and only if there are no $z_1, ..., z_4$ such that
$$\begin{aligned} &(c_1,\ (\zeta_3 - 1)z_2,\ (\zeta_3 - 1)z_3,\ (\zeta_3 - 1)z_4) \in H \text{ or} \\ &\big(2c_1,\ (\zeta_3^2 - 1)z_2,\ (\zeta_3^2 - 1)z_3,\ (\zeta_3^2 - 1)z_4\big) \in H. \end{aligned}$$
Thus, k (resp. k^2) act freely on A if and only if H contains no element with first coordinate equal to c_1 (resp. $2c_1$). Hence, since $3c_1 = 0$ by Lemma 8.23 (5), k acts freely if and only if k^2 acts freely. This shows the statement.

(e) The element
$$gh(z) = \big(\zeta_3 z_1 + a_1,\ \zeta_3 z_4 + a_2 + b_4,\ z_2 + a_3 + b_2,\ \zeta_3^2 z_3 + a_4 + b_3\big)$$
acts freely on A if and only if there are no $z_1, ..., z_4$ such that
$$\big((\zeta_3 - 1)z_1 + a_1,\ \zeta_3 z_4 - z_2 + a_2 + b_4,\ z_2 - z_3 + a_3 + b_2,\ \zeta_3^2 z_3 - z_4 + a_4 + b_3\big) \in H.$$
Setting $w_1 := (\zeta_3 - 1)z_1 + a_1$, $w_2 := \zeta_3 z_4 - z_2$ and $w_3 := z_2 - z_3$, we obtain that
$$-\zeta_3^2(w_2 + w_3) = -\zeta_3^2(\zeta_3 z_4 - z_3) = \zeta_3^2 z_3 - z_4$$
as desired.

(f) The map g^2h^2 is given by
$$g^2h^2(z) = \big(\zeta_3^2 z_1 - a_1,\ \zeta_3 z_3 + a_2 + a_4 - \zeta_3 b_3,\ \zeta_3^2 z_4 + a_2 + a_3 - \zeta_3^2 b_4,\ z_2 + a_3 + a_4 - b_2\big)$$
and acts freely on A if and only if there are no $z_1, ..., z_4$ such that
$$\begin{aligned} &\big((\zeta_3^2 - 1)z_1 - a_1,\ \zeta_3 z_3 - z_2 + a_2 + a_4 - \zeta_3 b_3, \\ &\ \zeta_3^2 z_4 - z_3 + a_2 + a_3 - \zeta_3^2 b_4,\ z_2 - z_4 + a_3 + a_4 - b_2\big) \in H. \end{aligned}$$
Defining $w_1 := (\zeta_3^2 - 1)z_1 - a_1$, $w_2 := \zeta_3 z_3 - z_2$, $w_3 := \zeta_3^2 z_4 - z_3$, we obtain that
$$-w_2 - \zeta_3 w_3 = (z_2 - \zeta_3 z_3) + \zeta_3(z_3 - z_4) = z_2 - z_4$$
as desired.

(g) The element g^2h is given by
$$g^2h(z) = \big(\zeta_3 z_1 - a_1,\ \zeta_3^2 z_3 + a_2 + a_4 + b_3,\ \zeta_3 z_4 + a_2 + a_3 + b_4,\ z_2 + a_3 + a_4 + b_2\big)$$

and acts freely on A if and only if there are no $z_1, ..., z_4$ such that

$$\Big((\zeta_3 - 1)z_1 - a_1,\ \zeta_3^2 z_3 - z_2 + a_2 + a_4 + b_3,$$
$$\zeta_3 z_4 - z_3 + a_2 + a_3 + b_4,\ z_2 - z_4 + a_3 + a_4 + b_2\Big) \in H.$$

Setting $w_1 := (\zeta_3 - 1)z_1 - a_1$, $w_2 := \zeta_3^2 z_3 - z_2$ and $w_3 := \zeta_3 z_4 - z_3$, we obtain that

$$-w_2 - \zeta_3^2 w_3 = (z_2 - \zeta_3^2 z_3) - (z_4 - \zeta_3^2 z_3) = z_2 - z_4$$

as desired.

(h) The element gh^2 is given by

$$gh^2(z) = \big(\zeta_3^2 + a_1,\ \zeta_3^2 z_4 + a_2 - \zeta_3^2 b_4,\ z_2 + a_3 - b_2,\ \zeta_3 z_3 + a_4 - \zeta_3 b_3\big)$$

and acts freely on A if and only if there are no $z_1, ..., z_4$ such that

$$\big((\zeta_3^2 - 1)z_1 + a_1,\ \zeta_3^2 z_4 - z_2 + a_2 - \zeta_3^2 b_4,\ z_2 - z_3 + a_3 - b_2,\ \zeta_3 z_3 - z_4 + a_4 - \zeta_3 b_3\big) \in H.$$

Again, defining $w_1 := (\zeta_3^2 - 1)z_1 + a_1$, $w_2 := \zeta_3^2 z_4 - z_2$ and $w_3 := z_2 - z_3$, we obtain that

$$-\zeta_3(w_2 + w_3) = (\zeta_3 z_2 - z_4) + (\zeta_3 z_3 - \zeta_3 z_2) = \zeta_3 z_3 - z_4$$

as claimed. □

We will not need directly need the following Lemma. However, it is useful if one wished to give a full classification of hyperelliptic fourfolds with group Heis(3).

Lemma 8.25. *The following statements hold:*

(i) The element a_1 is not fixed by multiplication by ζ_3.

(ii) $b_3 \neq 0$.

(iii) $b_4 \neq 0$.

Proof. (i) If a_1 was contained in $\mathrm{Fix}(\cdot\zeta_3) \subset E_1$, the property $v_6 \in H$ implied that H contains an element with first coordinate equal to $3a_1 - c_1$. Therefore, since v_3 has first coordinate equal to $3a_1$, we obtain an element in H with first coordinate equal to c_1, which is consequently contradicting the freeness of the action of k.
(ii) This is easily seen by noticing that $\rho(g^2) \cdot v_6 - v_3 \in H$ has second coordinate equal to $b_2 - \zeta_3 b_3$. Hence, if $b_3 = 0$, the element h does not act freely on A, see Lemma 8.24 (c).
(iii) We prove the statement similarly to (ii): the element $v_6 - ((\rho(g) + \rho(g^2)) \cdot v_2 - v_3$ of H has second coordinate equal to $2b_2 + b_4$. Thus, if $b_4 = 0$, the element h^2 does not act freely on A, again by Lemma 8.24 (c). □

Example 8.26. We give an example of a hyperelliptic fourfold with group $G := \mathrm{Heis}(3)$. In the above notation, choose (say) $j = 1$, and

$$a_1 = \frac{1}{3}, \quad a_2 = \frac{\zeta_3 - 1}{3}, \quad a_3 = a_4 = 0,$$
$$b_2 = b_3 = b_4 = \frac{1}{3}, \quad c_1 = \frac{1 - \zeta_3}{3},$$

and $A := (E_1 \times E_2 \times E_3 \times E_4)/H$, where each E_j is the equianharmonic elliptic curve and $H \cong C_3$ is defined to be the group generated by $v_3 = \left(0, \frac{\zeta_3-1}{3}, \frac{\zeta_3-1}{3}, \frac{\zeta_3-1}{3}\right)$. Then

$$
\begin{aligned}
g(z) &= \left(z_1 + \frac{1}{3},\ z_4 + \frac{\zeta_3 - 1}{3},\ z_2,\ z_3\right), \\
h(z) &= \left(\zeta_3 z_1,\ z_2 + \frac{1}{3},\ \zeta_3^2 z_3 + \frac{1}{3},\ \zeta_3 z_4 + \frac{1}{3}\right), \\
k(z) &= \left(z_1 + \frac{1-\zeta_3}{3},\ \zeta_3 z_2,\ \zeta_3 z_3,\ \zeta_3 z_4\right)
\end{aligned}
$$

act on A, since their linear parts map H to H (note that $\frac{1-\zeta_3}{3}$ is fixed by multiplication by ζ_3).
According to Lemma 8.23, the maps g, h, k span a subgroup $G \subset \mathrm{Bihol}(A)$ isomorphic to $\mathrm{Heis}(3)$. Since the associated complex representation ρ is by construction faithful, G contains no translations.
Making use of Lemma 8.24, it is then clear that $g,\ g^2,\ h,\ h^2,\ k,\ k^2$ act freely on A. It remains to prove the freeness of the action of $g^r h^s$, $r, s = 1, 2$. We use conditions (e) to (h) of Lemma 8.24 and that $\frac{\zeta_3-1}{3}$ is fixed by multiplication by ζ_3.

<u>To (e):</u> We investigate the equation

$$
\left(w_1,\ w_2 + \frac{\zeta_3 - 1}{3} + \frac{1}{3},\ w_3 + \frac{1}{3},\ -\zeta_3^2(w_2 + w_3) + \frac{1}{3}\right) = \delta v_2, \quad \delta \in \{0,1,2\}.
$$

Then $w_1 = 0$, $w_2 = (\delta+1)\cdot\frac{1-\zeta_3}{3} - \frac{1}{3}$, $w_3 = \delta \cdot \frac{1-\zeta_3}{3} - \frac{1}{3}$, and thus

$$
\begin{aligned}
& -\zeta_3^2(w_2 + w_3) + \frac{1}{3} \\
=& -\zeta_3^2(2\delta+1)\cdot\frac{1-\zeta_3}{3} + \frac{1+2\zeta_3^2}{3} \\
=& \,(\delta - 1)\cdot\frac{1-\zeta_3}{3} + \frac{1-\zeta_3^2}{3} \\
=& \,\delta\cdot\frac{1-\zeta_3}{3} + \frac{\zeta_3-\zeta_3^2}{3} \\
=& \,(\delta+1)\cdot\frac{1-\zeta_3}{3} \neq \delta\cdot\frac{1-\zeta_3}{3}, \qquad \forall \delta \in \{0,1,2\},
\end{aligned}
$$

which proves that gh acts freely on A.

<u>To (f):</u> Similar to (e), we investigate

$$
\left(w_1,\ w_2 + \frac{\zeta_3 - 1}{3} - \frac{\zeta_3}{3},\ w_3 + \frac{\zeta_3-1}{3} - \frac{\zeta_3^2}{3},\ -w_2 - \zeta_3 w_3 - \frac{1}{3}\right) = \delta v_2, \quad \delta \in \{0,1,2\}.
$$

This forces $w_1 = 0$, $w_2 = (\delta+1)\cdot\frac{1-\zeta_3}{3} + \frac{\zeta_3}{3}$, $w_3 = (\delta+1)\cdot\frac{1-\zeta_3}{3} + \frac{\zeta_3^2}{3}$. Then the fourth coordinate is equal to

$$
\begin{aligned}
& -w_2 - \zeta_3 w_3 - \frac{1}{3} \\
=& -2(\delta+1)\cdot\frac{1-\zeta_3}{3} - \frac{2+\zeta_3}{3} \\
=& \,(\delta-1)\cdot\frac{1-\zeta_3}{3} \neq \delta\cdot\frac{1-\zeta_3}{3}, \qquad \forall \delta \in \{0,1,2\},
\end{aligned}
$$

which proves that g^2h^2 acts freely on A.

To (g): The equation

$$\left(w_1,\ w_2 + \frac{\zeta_3 - 1}{3} + \frac{1}{3},\ w_3 + \frac{\zeta_3 - 1}{3} + \frac{1}{3},\ -w_2 - \zeta_3^2 w_3 + \frac{1}{3}\right) = \delta v_2, \quad \delta \in \{0,1,2\}$$

implies that $w_1 = 0$, $w_2 = w_3 = (\delta+1) \cdot \frac{1-\zeta_3}{3} - \frac{1}{3}$. Thus the last coordinate is equal to

$$\begin{aligned}
&-w_2 - \zeta_3^2 w_3 + \frac{1}{3} \\
=&-2(\delta+1) \cdot \frac{1-\zeta_3}{3} + \frac{2+\zeta_3^2}{3} \\
=&(\delta-1) \cdot \frac{1-\zeta_3}{3} \neq \delta \cdot \frac{1-\zeta_3}{3}, \qquad \forall \delta \in \{0,1,2\}.
\end{aligned}$$

This implies that g^2h acts freely on A.

To (h): Finally, we investigate

$$\left(w_1,\ w_2 + \frac{\zeta_3 - 1}{3} - \frac{\zeta_3^2}{3},\ w_3 - \frac{1}{3},\ -\zeta_3(w_2 + w_3) - \frac{\zeta_3}{3}\right) = \delta v_2, \quad \delta \in \{0,1,2\}.$$

Necessarily, $w_1 = 0$, $w_2 = (\delta+1) \cdot \frac{1-\zeta_3}{3} + \frac{\zeta_3^2}{3}$, $w_3 = \delta \cdot \frac{1-\zeta_3}{3} + \frac{1}{3}$, and thus

$$\begin{aligned}
&-\zeta_3(w_2 + w_3) - \frac{\zeta_3}{3} \\
=&-(2\delta+1) \cdot \frac{1-\zeta_3}{3} - \frac{1+2\zeta_3}{3} \\
=&(\delta-1) \cdot \frac{1-\zeta_3}{3} - \frac{\zeta_3 - \zeta_3^2}{3} \\
=&(\delta-2) \cdot \frac{1-\zeta_3}{3} \neq \delta \cdot \frac{1-\zeta_3}{3}, \qquad \forall \delta \in \{0,1,2\}.
\end{aligned}$$

We have established that gh^2 acts freely on A as well.

Summarizing everything, we have established that $X := A/G$ is a hyperelliptic fourfold with group $G = \mathrm{Heis}(3)$. Proposition 8.18 implies that indeed X is rigid. The proof of Theorem 8.17 is now complete.

8.1.5 The group $C_8 \rtimes C_4$ (GAP ID [32,4])

In this section, we prove the existence of hyperelliptic fourfolds with group

$$G := C_8 \rtimes C_4 = \langle a, b \,|\, a^8 = b^4 = 1, b^{-1}ab = a^5 \rangle.$$

The 2-dimensional irreducible representations of G are given by

$$a \mapsto \begin{pmatrix} 0 & \pm i \\ 1 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} i & \\ & -i \end{pmatrix},$$
$$a \mapsto \begin{pmatrix} 0 & \pm i \\ 1 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Since $a^2 \in Z(G)$, we can replace b by a^2b and the defining relations of G are still satisfied. Thus, we can focus on the first two of the above representations. Since no irreducible representation of G is faithful, we can invoke Lemma 6.3 to obtain that ρ is the direct sum of irreducible representations of respective dimensions 2, 1, 1. The kernel of the first two representations listed above is generated by a^4b^2. Thus, b is mapped to 1 and (w.l.o.g.) i by the two 1-dimensional representations occurring in ρ, respectively. Let us write

$$\rho(a) = \begin{pmatrix} 0 & \pm i & & \\ 1 & 0 & & \\ & & \alpha & \\ & & & \beta \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} i & & & \\ & -i & & \\ & & 1 & \\ & & & i \end{pmatrix},$$

where $\alpha, \beta \in \{\pm 1, \pm i\}$ and $\alpha = 1$ or $\beta = 1$. First of all, notice that if $\alpha \neq 1$, one of the matrices $\rho(ab)$, $\rho(ab^2)$ does not have the eigenvalue 1. Thus $\alpha = 1$. Since $a \mapsto ab^j$, $b \mapsto b$ are automorphisms of G for $j = 0, 1, 2, 3$, we can assume that $\beta = 1$.

Lemma 8.27. *The Abelian variety A is isogenous to $E_1 \times E_2 \times E_3 \times E_4$, where E_1, E_2, $E_4 \subset A$ are isomorphic to the harmonic elliptic curve $E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, and $E_3 \subset A$ is another elliptic curve.*

Proof. This follows by the 'standard procedure': we define

$$E_3 := \ker(\rho(b) - I)^0 \text{ and } A' := \operatorname{im}(\rho(b) - I),$$

so that A is isogenous to $E_3 \times A'$. Now, we define

$$E_4 := \left(\ker(\rho(a) - I) \cap A'\right)^0 \text{ and } A'' := \left(\operatorname{im}(\rho(a) - I) \cap A''\right)^0$$

so that A' is isogenous to $E_4 \times A''$. Observe that E_4 is the harmonic elliptic curve, since b acts on it by multiplication by i. Now, since $\rho(a^2) = \pm \operatorname{diag}(i, i, 1, 1)$, we obtain that A'' is isomorphic to $E_i \times E_i$ by Proposition 2.4. $\square$

We give an example of a hyperelliptic fourfold with group $C_8 \rtimes C_4$.

Example 8.28. Suppose that

$$A := A'/H := (E_i \times E_i \times E \times E_i)/H, \text{ where}$$
$$H := \left\langle \left(0, 0, \frac{1}{2}, \frac{1+i}{2}\right) \right\rangle, \text{ and}$$
$$E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), \quad E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}).$$

Define $a, b \in \mathrm{Bihol}(A')$ as follows:

$$a(z) = \left(\pm iz_2, z_1, z_3 + \frac{1}{8}, z_4 + \frac{1}{2}\right),$$
$$b(z) = \left(iz_1, -iz_2, z_3 + \frac{\tau}{4}, iz_4\right).$$

Since the linear parts of a and b map H to H, the maps a, b descend to biholomorphic self-maps of A, which we shall again denote by a, b. It is clear that $a^8 = b^4 = \mathrm{id}_A$. Moreover, the relation $b^{-1}ab = a^5$ is satisfied in view of our definition of H:

$$b^{-1}ab(z) = \left(\mp z_2,\ -z_1,\ z_3 + \frac{1}{8},\ z_4 + \frac{i}{2}\right),$$
$$a^5(z) = \left(\mp z_2,\ -z_1,\ z_3 + \frac{5}{8},\ z_4 + \frac{1}{2}\right).$$

We have shown that $G := \langle a, b\rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $C_8 \rtimes C_4$.
It remains to show that G acts freely on A. Consider the elements $a^j b^k$, $0 \leq j \leq 7$, $0 \leq k \leq 4$, $(j,k) \neq (0,0)$. Their third and fourth coordinates are given by

$$\left(z_3 + \frac{j}{8} + \frac{k\tau}{4},\ i^k z_4 + \frac{j}{2}\right).$$

Thus, $a^j b^k$ acts freely (in view of our definition of H), unless possibly when its third coordinate is a translation by $\frac{1}{2}$: this is the case if and only if $j = 4$ and $k = 0$. In this case, the element is given by

$$a^4(z) = \left(-z_1,\ -z_2,\ z_3 + \frac{1}{2}, z_4\right),$$

which acts freely in view of the definition of H. We have established the following

Proposition 8.29. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G := C_8 \rtimes C_4 = \langle a, b \,|\, a^8 = b^4 = 1, b^{-1}ab = a^5\rangle.$$

The Abelian variety A is isogenous to $E_i \times E_i \times E \times E_i$, where E_i is the harmonic elliptic curve and E is another elliptic curve. In particular, each complete family of hyperelliptic fourfolds with group G is 1-dimensional. Moreover, up to a change of basis and generators,

$$\rho(a) = \begin{pmatrix} 0 & \pm i & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} i & & & \\ & -i & & \\ & & 1 & \\ & & & i \end{pmatrix}.$$

Corollary 8.30. *There exist no hyperelliptic fourfolds with group $C_3 \times (C_8 \rtimes C_4)$ (GAP ID [96,47]).*

Proof. Necessarily, by Remark 2.3, an element k of order 3 is mapped to $\mathrm{diag}(1, 1, \zeta_3, 1)$ (or its square) via ρ: then the matrix $\rho(bk)$ does not have the eigenvalue 1. $\square$

8.1.6 The group $(C_4 \times C_4) \rtimes C_2$ (GAP ID [32,11])

The aim of this section is to prove the existence of hyperelliptic fourfolds with group

$$G := (C_4 \times C_4) \rtimes C_2 = \langle g, h \,|\, g^8 = h^2 = (gh)^4 = [g^2, h] = 1 \rangle.$$

Here, C_2 acts on the normal subgroup $C_4 \times C_4$ by exchanging the coordinates. The derived subgroup $[G,G]$ of G is generated by $[g,h]$, which has order 4. The four faithful irreducible representations of G are given by

$$g \mapsto \begin{pmatrix} 0 & \pm i \\ \pm 1 & 0 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The non-faithful irreducible representations of dimension 2 of G are defined by

$$g \mapsto \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

Lemma 8.31. *The representation ρ is the direct sum of a faithful irreducible and two 1-dimensional representations of G,*

$$\rho = \rho_2 \oplus \rho_1 \oplus \rho_1'.$$

Proof. First of all, we prove that ρ is not the direct sum of two irreducible representations ρ_2, ρ_2' of dimension 2. If this were the case, we can assume that ρ_2 is faithful and ρ_2' is given by

$$\rho_2'(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2'(h) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

see Lemma 6.2. But then $\rho(gh)$ does not have the eigenvalue 1.
Hence ρ is the direct sum of an irreducible representation ρ_2 of dimension 2 and two characters. It remains to show that ρ_2 is faithful. We invoke the relation $[g^2, h] = 1$ again; it implies that $g^4 \in [G,G]$ and thus g^4 is mapped to 1 by the characters contained in ρ. Hence, in order for ρ to be faithful, g has to be mapped to a matrix of order 8 by ρ_2, which is (by our description of the representations above) equivalent to requiring that ρ_2 is faithful. $\square$

Remark 8.32. The following two statements hold:

(a) It suffices to consider only the two faithful irreducible representations

$$g \mapsto \begin{pmatrix} 0 & \pm i \\ 1 & 0 \end{pmatrix}, h \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

since the other ones are obtained from these by the automorphism $g \mapsto g^5, h \mapsto h$ of G.

(b) The subgroup $M_{16} := \langle a := g,\ b := (gh)^2\rangle$ of G has the presentation

$$M_{16} = \langle a, b \,|\, a^8 = b^2 = 1,\ b^{-1}ab = a^5\rangle,$$

and thus, by Lemma 5.5 (b), one of the 1-dimensional representations contained in ρ is not trivial. More precisely,

$$g \mapsto g \cdot (gh)^2, \quad h \mapsto h$$

defines an automorphism of $G = (C_4 \times C_4) \rtimes C_2$, which restricts to the automorphism $a \mapsto a \cdot b$, $b \mapsto b$ of M_{16}. If $\rho_1(g) = \rho_1'(g) = 1$, we may therefore replace g by $g \cdot (gh)^2$ to assume without loss of generality that $\rho_1'(g) \neq 1$ (which we will do from now on).

(c) The subgroup $\langle r := [h, g],\ s := h,\ k := g^2\rangle$ of G is isomorphic to the central product of D_4 with C_4 (GAP ID [16,13]),

$$D_4 \curlyvee C_4 := \langle r, s, k \,|\, r^4 = s^2 = k^4 = [r, k] = [s, k] = (rs)^2 = 1,\ r^2 = k^2\rangle.$$

Lemma 8.33. *The Abelian variety A is isogenous to a product of an Abelian surface $S \subset A$ and two elliptic curves E, $E' \subset A$. More precisely, $A \cong (S \times E \times E')/H$, where H is contained in $S[2] \times E[4] \times E'[4]$. Furthermore, this splitting of A up to isogeny is completely determined by the element g.*

Proof. From Remark 8.32 (b) we obtain that

$$E := \ker(\rho(g) - I)^0$$

is an elliptic curve, and that A is isogenous to $E \times A'$, where $A' := \mathrm{im}(\rho(g) - I)$.
The relations $h^2 = (gh)^4 = \mathrm{id}_A$ imply that g is mapped to a fourth root of unity by the second 1-dimensional representation contained in ρ. Thus, one of $\ker(\rho(g) + I)^0$ and $\ker(\rho(g^2) + I)^0$ is an elliptic curve E', which is then contained in A'. Thus A is isogenous to the product of an Abelian surface S (the orthogonal complement of E' in A') and E, E':

$$A \cong (S \times E \times E')/H.$$

Denote the respective lattices of A, S, E, E' by Λ, Λ_S, Λ_E, $\Lambda_{E'}$. Then

$$H = \Lambda/(\Lambda_S \oplus \Lambda_E \oplus \Lambda_{E'}).$$

We prove the statement concerning H as follows. For $\lambda \in \Lambda$,

$$2\lambda = \underbrace{(\rho(g^4) + I)\lambda}_{\in \Lambda_S} + \underbrace{(\rho(g^4) - I)\lambda}_{=:\lambda' \in \Lambda}.$$

Now, if $E' = \ker(\rho(g) + I)^0$ (i.e., $\rho_1'(g) = -1$),

$$2\lambda' = \underbrace{(\rho(g) + I)\lambda'}_{\in \Lambda_E} + \underbrace{(\rho(g) - I)\lambda'}_{\in \Lambda_{E'}}.$$

In the case where $E' = \ker(\rho(g^2) + I)^0$ (i.e., $\rho_1(g) = \pm i$), we can write

$$2\lambda' = \underbrace{(\rho(g^2) + I)\lambda'}_{\in \Lambda_E} + \underbrace{(\rho(g^2) - I)\lambda'}_{\in \Lambda_{E'}}.$$

In any case, we obtain $\lambda \in \frac{1}{2}\Lambda_S \oplus \frac{1}{4}\Lambda_E \oplus \frac{1}{4}\Lambda_{E'}$. This shows that $H \subset S[2] \times E[4] \times E'[4]$. $\square$

We remark that it follows from the above proof that $\mathrm{Aut}(S)$ has a subgroup which is isomorphic to G. By Proposition 2.4, we obtain

Lemma 8.34. *The Abelian surface S is isomorphic to $E_i \times E_i$, where $E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is the harmonic elliptic curve.*

Proposition 8.35. *We can assume that $\rho_1(g) = \rho_1(h) = 1$ and $\rho_1'(g) = \rho_1'(h) = -1$.*

Proof. As mentioned in Remark 8.32, we can assume that $\rho_1'(g) \neq 1$, and hence $\rho_1(g) = 1$. According to the previous Lemmas, we shall write

$$A = (E_i \times E_i \times E \times E')/H.$$

Moreover, by a suitable change of coordinates, we are allowed to write

$$\begin{aligned} g(z) &= (\pm i z_2,\ z_1,\ z_3 + a_3,\ \rho_1'(g) z_4), \\ h(z) &= (z_2 + b_1,\ z_1 + b_2,\ \rho_1(h) z_3 + b_3,\ \rho_1'(h) z_4 + b_4). \end{aligned}$$

We distinguish several cases:

(a) First of all, we treat the case where $\rho_1(g) = 1, \rho_1'(g) = i$:

- If $\rho_1(h) = \rho_1'(h) = 1$, the condition that h has order 2 shows that

  $$(b_1 + b_2,\ b_1 + b_2,\ 2b_3,\ 2b_4) \in H,$$

  whereas the requirement $(gh)^4 = \mathrm{id}_A$ yields that there are $w_1, w_2 \in E_i$ such that

  $$(w_1,\ w_2,\ 4(a_3 + b_3),\ 0) \in H.$$

  Finally, the relation $g^2 h = h g^2$ shows that H contains an element of the form $(w_1',\ w_2',\ 0,\ 2b_4)$. In total, we have obtained that H contains an element of the form $(w_1'',\ w_2'',\ 4a_3,\ 0)$, which proves that

  $$g^4(z) = (-z_1,\ -z_2,\ z_3 + 4a_3,\ z_4)$$

  does not act freely on A.

- If $\rho_1(h) = -1$, $\rho_1'(h) = \pm 1$, we easily see that

  $$\rho(g^5 h) = \mathrm{diag}(\mp i,\ -1,\ i,\ \pm i)$$

  does not have the eigenvalue 1.

- If $\rho_1(h) = 1$ and $\rho_1'(h) = -1$, the relation $h^2 = \mathrm{id}_A$ shows that H contains $(b_1 + b_2,\ b_1 + b_2,\ 2b_3,\ 0)$. However, the condition that gh has order 4 implies that H contains an element of the form $(w_1,\ w_2,\ 4(a_3 + b_3),\ 0)$, and hence an element of the form $(w_1',\ w_2',\ 4a_3,\ 0)$. Consequently, g^4 does not act freely on A.

(b) We will now deal with the case $\rho_1(g) = 1, \rho_1'(g) = -1$:

- The case where $\rho_1(h) = \rho_1'(h) = 1$ is excluded as follows. As in the first bullet point of (a), H contains $v_1 := (b_1 + b_2,\ b_1 + b_2,\ 2b_3,\ 2b_4)$ and and element of the form $v_2 := (w_1,\ w_2,\ 4(a_3 + b_3),\ 0)$. Hence, the element

$$v_2 - 2v_1 + (\mathrm{id} - \rho(g))v_1 \in H$$

is of the form $(u_1,\ u_2,\ 4a_3,\ 0)$, which proves that a^4 does not act freely on A.

- The case where $\rho_1(h) = -1, \rho_1'(h) = 1$ is dealt with completely analogous as the case in the second bullet point of (a): then the matrix $\rho(g^5h)$ does not have the eigenvalue 1.

- The last case to exclude is $\rho_1(h) = \rho_1'(h) = -1$: here, the matrix

$$\rho(g^2h) = \begin{pmatrix} 0 & \pm i & & \\ \pm i & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

does not have the eigenvalue 1.

Hence, the only left case is the one where $\rho_1(g) = 1,\ \rho_1'(g) = -1$ and $\rho_1(h) = 1,\ \rho_1'(h) = -1$. This proves the statement. $\square$

We give an example of a hyperelliptic fourfold with group $G = (C_4 \times C_4) \rtimes C_2$.

Example 8.36. Define

$$A := A'/H := (E_i \times E_i \times E \times E')/H, \text{ where}$$
$$E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}),\ \ E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),\ \ E' = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \text{ and}$$
$$H := \left\langle \left(0,\ 0,\ \frac{1}{2},\ \frac{1}{2}\right) \right\rangle.$$

Define moreover automorphisms of A' as follows:

$$g(z) = \left(\pm i z_2,\ z_1,\ z_3 + \frac{1}{8},\ -z_4\right),$$
$$h(z) = \left(z_2,\ z_1,\ z_3 + \frac{\tau}{2},\ -z_4 + \frac{1}{8}\right).$$

Then the linear parts of g and h map H to H, hence g and h decend to automorphisms of A. It is clear that $g^8 = h^2 = \mathrm{id}_A$. From the definition of H, we obtain that

$$gh(z) = \left(\pm i z_1,\ z_2,\ z_3 + \frac{1}{8} + \frac{\tau}{2},\ z_4 - \frac{1}{8}\right)$$

has order 4 when viewed as an automorphism of A. Finally, it is immediate that g^2 and h commute.
The group $G := \langle a, b \rangle \subset \mathrm{Bihol}(A)$ is therefore isomorphic to $(C_4 \times C_4) \rtimes C_2$, and contains no translations, since the 2-dimensional irreducible representation contained in ρ is faithful. The conjugacy classes of G different from the trivial conjugacy class $\{\mathrm{id}_A\}$ are the ones containing

$$g, \quad g^2, \quad g^3, \quad g^4, \quad g^6,$$

$$[h,g] = h^{-1}g^{-1}hg, \quad g^2 \cdot [h,g],$$

$$h, \quad gh, \quad g^2h, \quad g^3h, \quad g^5h, \quad g^7h,$$

respectively. The only elements listed above whose respective third coordinates are not a translation by an element in $E[2] \setminus \{0, \frac{1}{2}\}$ are g^4 and $[h,g]$: thus, the freeness of the action of all listed elements except g^4 and $[h,g]$ is clear in view of our definition of H. Finally, the elements

$$g^4(z) = \left(-z_1, \; -z_2, \; z_3 + \frac{1}{2}, \; z_4\right)$$

and

$$[h,g](z) = \left(\mp iz_1, \; \pm iz_2, \; z_3, \; z_4 - \frac{1}{4}\right)$$

act freely on A by our definition of H as well. We have established the

Proposition 8.37. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G := (C_4 \times C_4) \rtimes C_2 = \langle g, h \,|\, g^8 = h^2 = (gh)^4 = [g^2, h] = 1 \rangle.$$

The Abelian variety A is isogenous to $E_i \times E_i \times E \times E'$, where E_i is the harmonic elliptic curve and E, E' are arbitrary elliptic curves. In particular, each complete family of hyperelliptic fourfolds with group G is 2-dimensional. Moreover, up to a change of basis and generators, the linear parts $\rho(g), \rho(h)$ of g, h are given by

$$\rho(g) = \begin{pmatrix} 0 & \pm i & & \\ 1 & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

8.1.7 The group $(C_4 \times C_4) \rtimes C_2$ (GAP ID [32,24])

We prove the existence of hyperelliptic fourfolds with group

$$G := (C_4 \times C_4) \rtimes C_2 = \langle a, b, c \,|\, a^4 = b^4 = c^2 = [a,b] = [a,c] = 1, c^{-1}bc = a^2b \rangle.$$

The four irreducible representations of degree 2 of G are given by

$$a \mapsto \pm \begin{pmatrix} i & \\ & i \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

All of these representations are non-faithful: hence, by Lemma 6.3, the representation ρ is a direct sum of a 2-dimensional irreducible representation and two characters.
Since all of these representations can be obtained from one another by replacing a by a^3 and/or b by bc (these assignments define automorphisms of G), we may restrict ourselves to only one of the above representations, say

$$a \mapsto \begin{pmatrix} i & \\ & i \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{8.4}$$

The relation $c^{-1}bc = a^2b$ implies that $a^2 \in [G, G]$. Since the kernel of (8.4) equals $\langle a^2b^2 \rangle$, we find that in order for ρ to be faithful, the element b must be mapped to $\pm i$ by a 1-dimensional representation occurring in ρ. We can therefore assume that

$$\rho(a) = \begin{pmatrix} i & & & \\ & i & & \\ & & 1 & \\ & & & \alpha \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & \beta & \\ & & & \gamma \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \delta & \\ & & & \epsilon \end{pmatrix},$$

$$\alpha \in \{\pm 1\}, \qquad (\beta, \gamma) \in \{(1, \pm i), (\pm i, 1)\}, \qquad \delta, \epsilon \in \{\pm 1\}.$$

Notice first that if $\beta = \pm i$, then $\alpha = 1$, since $\rho(ab^2)$ must have the eigenvalue 1. Up to symmetry, the only case to consider is the one where $(\beta, \gamma) = (1, i)$ (else, replace b by b^3). By replacing a by ab^2 if necessary, we can assume that $\alpha = 1$. Similarly, by replacing c by cb^2 if necessary, we can assume that $\epsilon = 1$. Finally, if $\delta = -1$, the matrix

$$\rho(abc) = \begin{pmatrix} 0 & i & & \\ i & 0 & & \\ & & -1 & \\ & & & i \end{pmatrix}$$

does not the eigenvalue 1, which proves that $\delta = 1$.

Lemma 8.38. *The Abelian variety A is isogenous to a product of elliptic curves,*

$$A \cong (E_1 \times E_2 \times E_3 \times E_4)/H,$$

where E_1, E_2, $E_4 \subset A$ are isomorphic to the harmonic elliptic curve and $E_3 \subset A$ is another elliptic curve.

Proof. It follows from the previous discussion that

$$E_3 := \ker(\rho(b) - I)^0$$

is an elliptic curve. The Abelian variety A is isogenous to $E_3 \times A'$, where $A' := \operatorname{im}(\rho(b) - I)$. Now,

$$E_4 := \left(\ker(\rho(a) - I) \cap A'\right)^0$$

is an elliptic curve, and A' is isogenous to $E_4 \times A''$, where $A'' := (\operatorname{im}(\rho(a) - I) \cap A)^0$. Now, A'' is isomorphic to the two-fold product of the harmonic elliptic curve, since $\rho(a)$ acts on the Abelian surface A'' by multiplication by i, cf. Proposition 2.4. Moreover, the elliptic curve E_4 is the harmonic elliptic curve, since $\rho(b)$ acts on it by multiplication by i. □

We give an example of a hyperelliptic fourfold with group $(C_4 \times C_4) \rtimes C_2$.

Example 8.39. Suppose that

$$A := A'/H := (E_i \times E_i \times E \times E_i)/H, \text{ where}$$
$$E_i := \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), \quad E := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \text{ and}$$
$$H := \left\langle \left(0,\ 0,\ \frac{1}{2},\ \frac{1+i}{2}\right) \right\rangle.$$

Define $a, b, c \in \mathrm{Bihol}(A')$ as follows:

$$a(z) = \left(iz_1,\ iz_2,\ z_3 + \frac{1}{4},\ z_4\right),$$
$$b(z) = \left(-z_2,\ z_1,\ z_3 + \frac{1+\tau}{4},\ iz_4\right),$$
$$c(z) = \left(z_1 + \frac{1+i}{2},\ -z_2 + \frac{1+i}{2},\ z_3 + \frac{\tau}{2},\ z_4 + \frac{1}{2}\right).$$

The linear parts of a, b, c map H to H, and thus a, b, c descend to automorphisms of A, which we shall denote again by a, b, c. Viewed as automorphisms of A, it is clear that $a^4 = b^4 = c^2 = [a, b] = [a, c] = \mathrm{id}_A$. Moreover, the relation $c^{-1}bc = a^2b$ is satisfied in view of our definition of H and

$$c^{-1}bc(z) = \left(z_2,\ -z_1,\ z_3 + \frac{1+\tau}{4},\ iz_4 + \frac{1+i}{2}\right),$$
$$a^2b(z) = \left(z_2,\ -z_1,\ z_3 + \frac{3+\tau}{4},\ iz_4\right).$$

We have proven that $G := \langle a, b, c\rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $(C_4 \times C_4) \rtimes C_2$.
We now investigate the freeness of the action of G on A. Representatives of non-trivial conjugacy classes of G are given by

$$a, \quad a^2, \quad a^3, \quad b, \quad b^2, \quad b^3, \quad c,$$
$$ab, \quad ac, \quad bc, \quad b^2c, \quad ,b^3c, \quad ab^2, \quad a^2b^2$$
$$abc, \quad ab^2c, \quad ab^2c, \quad a^2b^2c, \quad ab^3c.$$

By our definition of H, the above listed elements act freely on A, unless possibly when they act on $E \subset A'$ by $z_3 \mapsto z_3 + \frac{1}{2}$ (or trivially). We investigate these elements individually:

$$a^2(z) = \left(-z_1,\ -z_2,\ z_3 + \frac{1}{2},\ z_4\right): \text{ acts freely by definition of } H,$$
$$b^2c(z) = \left(-z_1 + \frac{1+i}{2},\ z_2 + \frac{1+i}{2},\ z_3 + \frac{1}{2},\ -z_4 + \frac{1}{2}\right): \text{ acts freely},$$
$$a^2b^2c(z) = \left(z_1 + \frac{1+i}{2},\ -z_2 + \frac{1+i}{2},\ z_3,\ -z_4\right): \text{ acts freely}.$$

Hence, G acts freely on A, and $X := A/G$ is a hyperelliptic fourfold with group $G = (C_4 \times C_4) \rtimes C_2$. We have established

Proposition 8.40. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G := (C_4 \times C_4) \rtimes C_2 = \langle a, b, c \,|\, a^4 = b^4 = c^2 = [a, b] = [a, c] = 1, c^{-1}bc = a^2b\rangle.$$

The Abelian variety A is isogenous to $E_i^3 \times E$, where E_i is the harmonic elliptic curve and E is an elliptic curve. In particular, every complete family of hyperelliptic fourfolds with group G is 1-dimensional. Moreover, up to a change of basis and generators,

$$\rho(a) = \begin{pmatrix} i & & & \\ & i & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & i \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Corollary 8.41. *There is no hyperelliptic fourfold with group $C_3 \times (C_4 \times C_4) \rtimes C_2$ (GAP ID [96,164]).*

Proof. An element k of order 3 would necessarily be mapped to $\mathrm{diag}(1, 1, \zeta_3, 1)$ or its square (cf. Remark 2.3): then the matrix $\rho(bk)$ does not have the eigenvalue 1. $\square$

8.1.8 The group $C_3 \times ((C_4 \times C_2) \rtimes C_2)$ (GAP ID [48,21])

The goal of this section is to describe an example of a hyperelliptic fourfold with group $G = C_3 \times G'$, where $G' := (C_4 \times C_2) \rtimes C_2$ has the presentation

$$G' = (C_4 \times C_2) \rtimes C_2 = \langle a, b, c \mid a^4 = b^2 = c^2 = [a, b] = [b, c] = 1,\ c^{-1}ac = ab \rangle.$$

The 2-dimensional irreducible representations of G' are given by

$$a \mapsto \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

The automorphism $a \mapsto ac, b \mapsto b, c \mapsto c$ of G' connects the two representations with each other, so that we can assume that a is mapped to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

by a 2-dimensional irreducible representation occurring in ρ. Note that ρ is the direct sum of an/this irreducible representation of dimension 2 and two 1-dimensional representations, cf. Lemma 6.3 (the group G' does not have a faithful irreducible representation, since $Z(G')$ contains $\langle a^2, b \rangle \cong C_2^2$).

The kernel of the described irreducible representation of dimension 2 is generated by $a^2 b$. Since $b \in [G, G]$ (which follows from $c^{-1}ac = ab$), we have established that the two 1-dimensional representations contained in ρ map a to 1 and (w.l.o.g.) i, respectively. Let us write

$$\rho(a) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & i \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \alpha & \\ & & & \beta \end{pmatrix},$$

where $\alpha, \beta \in \{\pm 1\}$.

We shall now consider $G := C_3 \times G'$, generated by a, b, c and an additional element k of order 3, which commutes with a, b, c.

Lemma 8.42. *$\rho(k) = \operatorname{diag}(\zeta_3, \zeta_3, 1, 1)$ or $\rho(k) = \operatorname{diag}(\zeta_3^2, \zeta_3^2, 1, 1)$.*

Proof. Since k is a central element, it is mapped to a multiple of the identity by any irreducible representation of G. By Lemma 2.2, the last diagonal entry of $\rho(k)$ is 1 (else, $\rho(ak)$ would have a single eigenvalue of order 12, contradicting the cited Lemma). If the third diagonal entry of $\rho(k)$ would be different from 1, the matrix $\rho(ak)$ would not have the eigenvalue 1. This shows that $\rho(k)$ is as asserted. □

Hence, by replacing k by its square if necessary, we may (and will) assume that

$$\rho(k) = \operatorname{diag}(\zeta_3, \zeta_3, 1, 1).$$

We return to determining α and β: we obtain $\alpha = 1$, since else, $\rho(ack)$ does not have the eigenvalue 1. By applying the automorphism $a \mapsto a$, $b \mapsto b$, $c \mapsto ca^2$, $k \mapsto k$ if necessary, we may assume that $\beta = 1$.
We determine the isogeny type of A.

Lemma 8.43. *The Abelian variety A is isogenous to a product of four elliptic curves E_1, E_2, E_3, $E_4 \subset A$ such that $E_1 \cong E_2 \cong F$ is the equianharmonic elliptic curve, $E_4 \cong E_i$ is the harmonic elliptic curve.*

Proof. Define the two Abelian surfaces $A' := \ker(\rho(k) - I)^0$, $A'' := \operatorname{im}(\rho(k) - I)$. Then A is isogenous to $A' \times A''$, and since $\operatorname{diag}(\zeta_3, \zeta_3) \in \operatorname{Aut}(A'')$, the Abelian surface A'' is isogenous to $E_1 \times E_2 \cong F \times F$, cf. Proposition 2.4. Now, A' is isomorphic to a product of elliptic curve, since A' is isogenous to the product of

$$\begin{aligned} E_3 &:= \left(\ker(\rho(a) - I) \cap A'\right)^0, \\ E_4 &:= \left(\operatorname{im}(\rho(a) - I) \cap A'\right)^0. \end{aligned}$$

Moreover, the elliptic curve E_4 is the harmonic elliptic curve, since $\rho(a)$ acts on it by multiplication by i. This shows the assertion. □

We give an example of a hyperelliptic fourfold with group $G = C_3 \times ((C_4 \times C_2) \rtimes C_2)$.

Example 8.44. Let

$$A := F \times F \times E \times E_i, \text{ where}$$

$$F := \mathbb{C}/(\mathbb{Z} + \zeta_3 x\mathbb{Z}), \ \ E := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \ \text{ and } \ E_i := \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}).$$

Define the following biholomorphic self-maps of A:

$$\begin{aligned} a(z) &= \left(-z_2, \ z_1, \ z_3 + \frac{1}{4}, \ iz_4\right), \\ b(z) &= \left(-z_1, \ -z_2, \ z_3, \ z_4 + \frac{1+i}{2}\right), \\ c(z) &= \left(z_1, \ -z_2, \ z_3 + \frac{\tau}{2}, \ z_4 + \frac{1}{2}\right), \\ k(z) &= \left(\zeta_3 z_1, \ \zeta_3 z_2, \ z_3 + \frac{1}{3}, \ z_4\right). \end{aligned}$$

Now, that the equations $a^4 = b^2 = c^2 = [a,b] = [b,c] = \mathrm{id}_A$ hold, is clear. Moreover, $k^3 = \mathrm{id}_A$, and a, b, c commute with k. The relation $c^{-1}ac = ab$ is satisfied as well, because

$$c^{-1}ac(z) = \left(z_2,\ -z_1,\ z_3 + \frac{1}{4},\ iz_4 + \frac{1+i}{2}\right) = ab(z).$$

Thus, the group $G := \langle a, b, c, k\rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $C_3 \times ((C_4 \times C_2) \rtimes C_2)$, and since ρ is faithful, G contains no translations.
It remains to investigate the freeness of the action of G on A. A system of representatives of the set of conjugacy classes $\mathcal{C}$ of $G' := \langle a, b, c\rangle$ are given by

$$\mathrm{id}_A, \quad a, \quad a^2, \quad a^3, \quad b, \quad c, \quad ab, \quad a^2b, \quad a^2c, \quad a^3c.$$

It is clear that all of the non-trivial elements listed act freely on A, since the third or fourth coordinate of each of them is a non-trivial translation. Representatives of the conjugacy classes of $G = C_3 \times G'$ are given exactly by $\mathcal{C} \cup k\mathcal{C} \cup k^2\mathcal{C}$. Again, any element in $k\mathcal{C} \cup k^2\mathcal{C}$ acts on E by a non-trivial translation. Hence, G acts freely on A. Summarizing everything,

Proposition 8.45. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G = C_3 \times ((C_4 \times C_2) \rtimes C_2) = \left\langle \begin{array}{c|c} a, b, & a^4 = b^2 = c^2 = [a,b] = [b,c] = 1, \\ c, k & c^{-1}ac = ab,\ k^3 = [a,k] = [b,k] = [c,k] = 1 \end{array} \right\rangle.$$

The Abelian variety A is isogenous to $F \times F \times E \times E'$, where F is the equianharmonic elliptic curve, E_i is the harmonic elliptic curve and E is another elliptic curve. In particular, each complete family of hyperelliptic fourfolds with group G is 1-dimensional. Moreover, up to a change of basis and generators,

$$\rho(a) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & i \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$\rho(c) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \alpha & \\ & & & \beta \end{pmatrix}, \quad \rho(k) = \begin{pmatrix} \zeta_3 & & & \\ & \zeta_3 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Proposition 8.45 and Lemma 8.42 show the following:

Corollary 8.46. *There is no hyperelliptic fourfold with group $C_3^2 \times ((C_4 \times C_2) \rtimes C_2)$ (GAP ID [144,102]).*

8.1.9 The group $(C_4 \rtimes C_4) \times C_3$ (GAP ID [48,22])

This section is dedicated to proving that there is a hyperelliptic fourfold with group

$$G := (C_4 \rtimes C_4) \times C_3 = \langle a, b, k \mid a^4 = b^4 = k^3 = [a,k] = [b,k] = 1, b^{-1}ab = a^3 \rangle.$$

The two irreducible representations of degree 2 of $C_4 \rtimes C_4$ are given by

$$(1) \quad a \mapsto \begin{pmatrix} i & \\ & -i \end{pmatrix}, \ b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad (2) \quad a \mapsto \begin{pmatrix} i & \\ & -i \end{pmatrix}, \ b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 5.5 shows that $\rho|_{C_4 \rtimes C_4}$ contains one of the above irreducible representations and two 1-dimensional ones (note that both of the above representations are not faithful; (1) has kernel equal to a^2b^2, while the kernel of (2) is generated by b^2). By applying the automorphism $a \mapsto a$, $b \mapsto ab$, $k \mapsto k$ of G if necessary, we can assume that b is mapped to a matrix with eigenvalues of order 4. In the following, we will therefore restrict our focus to (1).

Since the representation has kernel equal to a^2b^2 and the derived subgroup of $C_4 \rtimes C_4$ is generated by a^2 (one sees this from the relation $b^{-1}ab = a^3$ and by noticing that $[G,G] \subset \langle a \rangle$), we obtain that one of the 1-dimensional representations occurring in ρ must map b to a primitive fourth root of unity $\pm i$: in this case, ρ is faithful. Since $\rho(b)$ must have the eigenvalue 1, we can assume that

$$\rho(a) = \begin{pmatrix} i & & & \\ & -i & & \\ & & \alpha & \\ & & & \beta \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & i \end{pmatrix},$$

where $\alpha, \beta \in \{\pm 1\}$ and (at least) one of them is equal to 1. It follows that $\alpha = 1$, because the matrix $\rho(ab)$ needs to have the eigenvalue 1. Moreover, after possibly applying the automorphism $a \mapsto ab^2$, $b \mapsto b$ of $C_4 \rtimes C_4$, we can assume that $\beta = 1$.
We now take the central element k of order 3 into account:

Lemma 8.47. $\rho(k) = \operatorname{diag}(\zeta_3, \zeta_3, 1, 1)$ *or* $\rho(k) = \operatorname{diag}(\zeta_3^2, \zeta_3^2, 1, 1)$.

Proof. Since k is central, $\rho(k)$ is a diagonal matrix. By Remark 2.3, the matrix $\rho(bk)$ must have an even number of eigenvalues of order 12. Hence, the last diagonal entry of k is 1. If the third diagonal entry of k was different from 1, the matrix $\rho(bk)$ would not have the eigenvalue 1. Hence $\rho(k)$ is as claimed. $\square$

After possibly replacing k by k^2, we may therefore assume that

$$\rho(k) = \operatorname{diag}(\zeta_3,\ \zeta_3,\ 1,\ 1).$$

Lemma 8.48. *The Abelian variety A is isogenous a product of four elliptic curves $E_j \subset A$,*

$$A \sim_{isog.} E_1 \times E_2 \times E_3 \times E_4,$$

where E_1 and E_2 are isomorphic to the equianharmonic curve F and E_4 is isomorphic to the harmonic elliptic curve E_i.

Proof. This follows in the standard way: defining

$$E_3 := \ker(\rho(b) - \mathrm{id}_A)^0 \text{ and } A' := \mathrm{im}(\rho(b) - \mathrm{id}_A),$$

we observe that A is isogenous to $E \times A'$. Now, A' is isogenous to $S \times E_4$, where E_4 is the elliptic curve $A' \cap \ker(\rho(a) - \mathrm{id}_A)^0$ and S is the Abelian surface $(A' \cap \mathrm{im}(\rho(a) - \mathrm{id}_A))^0$. Since $\rho(b)$ acts on E_4 by multiplication by i, we obtain that E' is isomorphic to the harmonic elliptic curve E_i. Moreover, since $\rho(k)$ acts on S by multiplication by ζ_3, it follows that S is isomorphic to $F \times F$ (see Proposition 2.4). □

As already described in Section 8.1.3, we need to change the basis such that $\rho(a)$ and $\rho(b)$ act on (an Abelian variety isogenous to) $F \times F \times E \times E_i$. This leads to the following example.

Example 8.49. Let

$$A := A'/H := (F \times F \times E \times E_i)/H, \text{ where}$$

$$F = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z}), \quad E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \quad E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), \text{ and}$$

$$H = \left\langle \left(0, 0, \frac{1}{2}, \frac{1+i}{2}\right) \right\rangle.$$

Moreover, define the following biholomorphisms of A':

$$\begin{aligned}
a(z) &= \left((1 + 2\zeta_3)z_1 - z_2,\ -2z_1 - (1 + 2\zeta_3)z_2,\ z_3 + \frac{1}{4},\ z_4 + \frac{1}{2}\right),\\
b(z) &= \left(\zeta_3^2 z_2 - z_1,\ z_2 - 2\zeta_3 z_1,\ z_3 + \frac{\tau}{4},\ iz_4\right),\\
k(z) &= \left(\zeta_3 z_1,\ \zeta_3 z_2,\ z_3 + \frac{1}{3},\ z_4\right).
\end{aligned}$$

Since the linear parts of a, b, k map H to H, the above maps descend to A. Moreover, it is clear that $a^4 = b^4 = \mathrm{id}_A$, and that a and b commute with k. The relation $b^{-1}ab = a^3$ (where we view a and b as self-maps of A) is satisfied in virtue of our definition of H:

$$\begin{aligned}
b^{-1}ab(z) &= \left(z_2 + (1 + 2\zeta_3^2)z_1,\ 2z_1 + (1 + 2\zeta_3)z_2,\ z_3 + \frac{1}{4},\ z_4 - \frac{i}{2}\right),\\
a^3(z) &= \left((1 + 2\zeta_3^2)z_1 + z_2,\ 2z_1 + (1 + 2\zeta_3)z_2,\ z_3 + \frac{3}{4},\ z_4 + \frac{1}{2}\right).
\end{aligned}$$

Thus, the group $G := \langle a, b, k \rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $(C_4 \rtimes C_4) \times C_3$. The non-trivial classes of the subgroup $C_4 \rtimes C_4 = \langle a, b \rangle$ of G are given by the ones containing

$$a, \quad a^2, \quad b, \quad b^2, \quad b^3, \quad ba, \quad b^2a, \quad a^2b^2, \quad b^3a.$$

We readily check that any of the above elements acts freely on A (where we have to take in account the definition of H to check the freeness of the action of a^2). Moreover, it is clear that k, k^2 act freely on A, and that any of the elements listed above multiplied by k or k^2 act freely on A, since the third coordinate is a translation by a non-trivial torsion element whose order is a multiple of 3. We have proved that there exist hyperelliptic fourfolds with group $G = (C_4 \rtimes C_4) \times C_3$.

To summarize everything:

Proposition 8.50. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G := (C_4 \rtimes C_4) \times C_3 = \langle a, b, k \mid a^4 = b^4 = k^3 = [a,k] = [b,k] = 1, b^{-1}ab = a^3 \rangle.$$

The Abelian variety A is isogenous to $F \times F \times E \times E_i$, where F is the equianharmonic, E_i the harmonic and E is another elliptic curve. In particular, every complete family of hyperelliptic fourfolds with group G is 1-dimensional. Moreover, after a change of basis and up to automorphisms of G,

$$\rho(a) = \begin{pmatrix} 1+2\zeta_3 & -1 & & \\ -2 & 1+2\zeta_3 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} -1 & \zeta_3^2 & & \\ -2\zeta_3 & 1 & & \\ & & 1 & \\ & & & i \end{pmatrix},$$

$$\rho(k) = \begin{pmatrix} \zeta_3 & & & \\ & \zeta_3 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

It follows immediately from Proposition 8.50 and Lemma 8.47 that

Corollary 8.51. *There does not exist a hyperelliptic fourfold with group $(C_4 \rtimes C_4) \times C_3^2$ (GAP ID [144,103]) cannot be a group associated with a hyperelliptic fourfold.*

8.1.10 The group $A_4 \times C_4$ (GAP ID [48,31])

The aim of this section is to show the existence of a hyperelliptic fourfold with group [1]

$$G = A_4 \times C_4 = \langle \sigma, \tau', \kappa \mid \sigma^3 = (\tau')^2 = (\tau'\sigma)^3 = \kappa^4 = [\sigma, \kappa] = [\tau', \kappa] = 1 \rangle.$$

The complex representation $\rho|_{A_4}$ must contain the unique irreducible representation of degree 3 of A_4, which is given by

$$\rho_3(\sigma) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_3(\tau') = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}.$$

The 1-dimensional representations of A_4 are $\tau' \mapsto 1$, $\sigma \mapsto \zeta_3^j$ (the conjugacy class of 3-cycles in S_4 decomposes into two conjugacy classes of A_4). The following Lemma shows that the 1-dimensional representation occurring in ρ is necessarily the trivial one:

Lemma 8.52. *Let A' be an Abelian threefold such that* $\mathrm{Bihol}(A')$ *has a subgroup isomorphic to $A_4 = \langle \sigma, \tau' \rangle$. Then σ does not act freely on A'.*

[1] We will reserve the notation τ for transpositions and denote double transpositions by τ'.

Proof. If we write $\tau(z_1, z_2, z_3) = (z_1 + t'_1,\ -z_2 + t'_2,\ -z_3 + t'_3)$ and $\sigma(z) = (z_3 + s_1,\ z_1 + s_1,\ z_2 + s_3)$, then $\sigma^3 = \mathrm{id}_{A'}$ means that

$$(s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ s_1 + s_2 + s_3) = 0 \in A'.$$

The freeness of the action of σ is equivalent to the statement: there are no w_1, w_2 such that

$$v(w_1, w_2) := (w_1 + s_1,\ w_2 + s_2,\ -w_1 - w_2 + s_3) = 0 \text{ in } A'.$$

Setting $w_1 := s_2 + s_3$ and $w_2 := s_1 + s_3$, we obtain that $-w_1 - w_2 + s_3 = -(s_1 + s_2 + s_3)$. This means that

$$\rho(\sigma^2\tau') \cdot v(w_1, w_2) = -(s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ s_1 + s_2 + s_3) = 0 \text{ in } A'.$$

This proves the Lemma. $\square$

Lemma 8.53. *The Abelian variety A is isogenous to $E \times E \times E \times E'$, where E, $E' \subset A$ are elliptic curves.*

Proof. We define $E' := (\ker(\rho(\tau') - I) \cap \ker(\rho(\sigma) - I))^0$ and $A' := (E')^\perp$ (where "$\perp$" is to be understood with respect to the positive definite Hermitian form coming from the polarization of A). By Poincaré's Complete Reducibility Theorem [Mum70, Remark on p. 173], A is isogenous to $A' \times E'$. Setting

$$E := \left(A' \cap \ker(\rho(\tau') - I)\right)^0,$$

we obtain that the Abelian variety A' is isogenous to $E \times \rho(\sigma)E \times \rho(\sigma^2)E$. This shows the statement. $\square$

By a change of coordinates, we are allowed to write

$$\begin{aligned}
&A = (E \times E \times E \times E')/H, \text{ and} \\
&\tau'(z) = (z_1 + t'_1,\ -z_2,\ -z_3,\ z_4 + t'_4), \\
&\sigma(z) = (z_3 + s_1,\ z_1 + s_2,\ z_2 + s_3,\ z_4 + s_4).
\end{aligned}$$

Corollary 8.54. *Suppose that H contains an element whose last coordinate is equal to s_4. Then σ does not act freely on A.*

Proof. The hypothesis that H contains an element of the form $(w_1,\ w_2,\ w_3,\ s_4) \in H$ implies that σ is congruent to

$$\sigma(z) \equiv (z_3 + s_1 - w_1,\ z_1 + s_2 - w_2,\ z_2 + s_3 - w_3,\ z_4)$$

modulo H. Therefore, we can assume that σ acts trivially on E'. We conclude as in the proof of Lemma 8.52. $\square$

Remark 8.55. The defining relations of A_4 yield

$$\begin{aligned}
(\tau')^2 = \mathrm{id}_A &\iff v_1 := (2t'_1,\ 0,\ 0,\ 2t'_4), \\
\sigma^3 = \mathrm{id}_A &\iff v_2 := (s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ 3s_4) \in H, \\
(\tau'\sigma)^3 = \mathrm{id}_A &\iff v_3 := (s_1 - s_2 + s_3 + t'_1,\ -s_1 + s_2 - s_3 - t'_1, \\
&\qquad\qquad s_1 - s_2 + s_3 + t'_1,\ 3s_4 + 3t'_4) \in H.
\end{aligned}$$

We now take the additional central element κ of order 4 into account. Since $\rho(\kappa)$ must have the eigenvalue 1 and κ is central, we obtain (after possibly replacing by κ by κ^3)

$$\rho(\kappa) = \mathrm{diag}(1,1,1,i) \text{ or } \rho(\kappa) = \mathrm{diag}(i,i,i,1).$$

In the first case, Lemma 8.52 implies that $\sigma\kappa$ does not act freely on A, hence we are left with the second case. We write

$$\kappa(z) = (iz_1 + k_1,\ iz_2 + k_2,\ iz_3 + k_3,\ z_4 + k_4).$$

Remark 8.56. As in the case of A_4, the relations $\kappa^4 = \mathrm{id}_A$, $\kappa\sigma = \sigma\kappa$ and $\kappa\tau' = \tau'\kappa$ give rise to the following elements of H:

$$\begin{aligned}
&\kappa^2 = \mathrm{id}_A && \iff v_4 := (0,\ 0,\ 0,\ 4k_4) \in H \iff 4k_4 = 0,\\
&\kappa\sigma = \sigma\kappa && \iff v_5 := (k_1 - k_3 + (i-1)s_1,\ k_2 - k_1 + (i-1)s_2,\ k_3 - k_2 + (i-1)s_3,\ 0) \in H,\\
&\kappa\tau' = \tau'\kappa && \iff v_6 := ((i-1)t_1',\ 2k_2,\ 2k_3,\ 0) \in H.
\end{aligned}$$

Let us give an example to show that hyperelliptic fourfolds with group $A_4 \times C_4$ indeed exist.

Example 8.57. We define $A := A'/H := (E_i \times E_i \times E_i \times E)/H$, where $E_i = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ is the harmonic elliptic curve and E is another elliptic curve (in standard form). Moreover, we define $H \subset A'$ to be generated by

$$H := \left\langle \left(\frac{1+i}{2},\ \frac{1+i}{2},\ \frac{1+i}{2},\ 0 \right) \right\rangle.$$

The biholomorphisms

$$\begin{aligned}
\tau'(z) &= \left(z_1 + \frac{i}{2},\ -z_2,\ -z_3,\ z_4 \right),\\
\sigma(z) &= \left(z_3 + \frac{1}{4},\ z_1 + \frac{i}{4},\ z_2 + \frac{1+i}{4},\ z_4 + \frac{1}{3} \right),\\
\kappa(z) &= \left(iz_1 - \frac{1+i}{2},\ iz_2 - \frac{1+i}{4},\ iz_3 - \frac{i-1}{4},\ z_4 + \frac{1}{4} \right)
\end{aligned}$$

of A' descend to biholomorphic self-maps of A (since the linear parts of τ', σ and κ map H to H). Viewed as a subgroup of $\mathrm{Bihol}(A)$, we will show that $G := \langle \tau', \sigma, \kappa \rangle$ is isomorphic to $A_4 \times C_4$, using Remarks 8.55 and 8.56:

- The relations $(\tau')^2 = \mathrm{id}_A$ and $\kappa^4 = \mathrm{id}_A$ are clearly satisfied. (We have $v_1 = v_4 = 0$).
- $\sigma^3 = \mathrm{id}_A$ is satisfied if and only if the element

  $$v_2 = \left(\frac{1+i}{2},\ \frac{1+i}{2},\ \frac{1+i}{2},\ 0 \right)$$

  is zero in A, which is the case by our definition of H.
- The relation $(\tau'\sigma)^3 = \mathrm{id}_A$ is satisfied if and only if the element

  $$v_3 = \left(\frac{1+i}{2},\ \frac{1+i}{2},\ \frac{1+i}{2},\ 0 \right)$$

  is zero in A, which is again the case.

- σ commutes with κ, because $v_5 = 0 \in H$, too.
- τ' commutes with κ, since $v_6 = \left(\frac{1+i}{2},\ \frac{1+i}{2},\ \frac{1+i}{2},\ 0\right) \in H$.

This proves that $G \subset \mathrm{Bihol}(A)$ is indeed isomorphic to $A_4 \times C_4$. Observe that G does not contain a translation, because ρ contains a faithful irreducible representation. Consequently, we are left with showing that G acts freely on A. A system of representatives $\mathcal{C}$ of the conjugate classes of A_4 is given by

$$\mathrm{id}_A, \quad \tau', \quad \sigma, \quad \sigma^2,$$

and a system of representatives of conjugacy classes of $A_4 \times C_4$ is then given by

$$\mathcal{C} \cup \kappa\mathcal{C} \cup \kappa^2\mathcal{C} \cup \kappa^3\mathcal{C}.$$

We prove that each non-trivial element in the above list acts freely on A.

- Indeed, τ' acts freely on A if and only if H contains no element of the form
$$\left(\frac{i}{2},\ w_2,\ w_3,\ 0\right).$$
Thus, according to our definition of H, τ' acts freely on A.
- σ and σ^2 clearly act freely on A, since these act on E by a non-trivial translation of order 3, and H is a subgroup of $A'[2]$.
- the elements κ^j, $j \in \{1,2,3\}$ act freely on A if and only if H contains no element of the form
$$\left(w_1,\ w_2,\ w_3,\ \frac{j}{4}\right).$$
This is satisfied.
- The elements $\kappa^j\tau'$ act freely on A, since they act on $E \subset A'$ by a translation of order j, and the last coordinate of the non-zero element in H is 0.
- By the similar argument as in the previous bullet point, the elements $\kappa^j\sigma$ and $\kappa^j\sigma^2$ act freely on A.

We have shown the existence of a hyperelliptic fourfold with group $A_4 \times C_4$, and summarize everything in the following

Proposition 8.58. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G = A_4 \times C_4 = \langle \sigma, \tau', \kappa \,|\, \sigma^3 = (\tau')^2 = (\tau'\sigma)^3 = \kappa^4 = [\kappa,\tau] = [\kappa,\sigma] = 1\rangle.$$

The Abelian variety A is necessarily isogenous to $E \times E \times E \times E'$, where E and E' are elliptic curves. In particular, every complete family of hyperelliptic fourfolds with group G is 2-dimensional. Moreover, up to a change of basis and generators,

$$\rho(\sigma) = \begin{pmatrix} 0 & 0 & 1 & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ & & & 1 \end{pmatrix}, \quad \rho(\tau') = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad \rho(\kappa) = \begin{pmatrix} i & & & \\ & i & & \\ & & i & \\ & & & 1 \end{pmatrix}.$$

In Section 8.2.1 we show that certain groups containing A_4 are not associated with a hyperelliptic fourfold.

8.1.11 The group $(C_3 \rtimes C_8) \times C_3$ (GAP ID [72,12])

In this section, we show the existence of hyperelliptic fourfolds with group $G' \times C_3$, where G' is defined by

$$G' := C_3 \rtimes C_8 = \langle a, b \,|\, a^3 = b^8 = 1,\ b^{-1}ab = a^2 \rangle.$$

The irreducible representations of degree ≥ 2 of G' are given by

$$a \mapsto \begin{pmatrix} \zeta_3 & \\ & \zeta_3^2 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & \pm i \\ 1 & 0 \end{pmatrix},$$
$$a \mapsto \begin{pmatrix} \zeta_3 & \\ & \zeta_3^2 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}.$$

It follows immediately that ρ is the direct sum of an irreducible representation of dimension 2 and two 1-dimensional representations (else, $\rho(a)$ would not have the eigenvalue 1).
The latter two of the above representations are not faithful, and map the element b of order 8 to a matrix of order ≤ 4. If one of these two were contained in ρ, the both 1-dimensional irreducible representations contained in ρ would map b to primitive 8-th roots of unity: in this case, the matrix $\rho(ab)$ does not have the eigenvalue 1.
We may therefore concentrate only on the first two representations.

Lemma 8.59. *The Abelian variety A is isogenous to a product of four elliptic curves $E_j \subset A$. More precisely,*

$$A \sim_{isog.} E_1 \times E_2 \times E_3 \times E_4,$$

where E_1 and E_2 are isomorphic to the harmonic elliptic curve.

Proof. By Lemma 5.5 (b) and because $[G', G']$ is generated by a, we find that

$$E_3 := \ker(\rho(b) - \mathrm{id}_A)^0$$

is an elliptic curve. The Abelian variety A is isogenous to $E_3 \times A'$, where

$$A' := \mathrm{im}(\rho(b) - \mathrm{id}_A).$$

One of $(A' \cap \ker(\rho(b) + I))^0$ and $\left(A' \cap \ker(\rho(b^2) + I)\right)^0$ is an elliptic curve, which we denote by E_4, and it follows that A' is isogenous to $E_4 \times S$, where S is an Abelian surface. It remains to show that S is isomorphic to the product of E_i with itself. This follows immediately from Proposition 2.4. $\square$

Remark 8.60. Observe that $\mathrm{diag}(\zeta_3,\ \zeta_3^2)$ does not act on $E_i \times E_i$. After a change of basis, we can assume that

$$\rho(a) = \begin{pmatrix} -1 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & \pm i & & \\ \pm i - 1 & -1 & & \\ & & 1 & \\ & & & \alpha \end{pmatrix}.$$

A priori, $\alpha \in \{1, -1, i, -i\}$. However, Lemma 5.5 (b) implies that $\alpha \neq 1$. Up to automorphism, we may assume that $\alpha \in \{-1, i\}$. After a suitable change of origin, we may therefore write

$$A = (E_i \times E_i \times E \times E')/H, \tag{8.5}$$
$$a(z) = (-z_1 - z_2 + a_1,\ z_1 + a_2,\ z_3 + a_3,\ z_4 + a_4), \tag{8.6}$$
$$b(z) = (z_1 \pm iz_2,\ (\pm i - 1)z_1 - z_2,\ z_3 + b_3,\ \alpha z_4). \tag{8.7}$$

Suppose that $\alpha = i$. The relation $b^{-1}ab = a^2$ implies that H contains an element of the form $w := (w_1,\ w_2,\ a_3,\ (2-i)a_4)$. Moreover, $b^8 = \mathrm{id}_A$ shows that $8b_3 = 0$ in E.

Lemma 8.61. *Suppose that $\alpha = i$. Then there is an element of G, which does not act freely on A.*

Proof. Since b^2 is a central element of order 4 of G, the element ab^2 has order 12. This element is given by

$$ab^2(z) = (\mp i(z_1 + z_2) + a_1,\ \pm iz_1 + a_2,\ z_3 + a_3 + 2b_3,\ -z_4 + a_4).$$

The eigenvalues of $\rho(ab^2)$ are ζ_{12}^7, ζ_{12}^{11}, 1 and -1. Since $w = (w_1,\ w_2,\ a_3,\ (2-i)a_4) \in H$, the biholomorphic map ab^2 is congruent to

$$ab^2(z) \equiv (\mp i(z_1 + z_2) + a_1 - w_1,\ \pm iz_1 + a_2 - w_2,\ z_3 + 2b_3,\ -z_4 + (i-1)a_4)$$

modulo H. Raising ab^2 to the fourth power and using that b_3 is an 8-torsion element, we see that $(ab^2)^4$ does not act freely on A. $\square$

In the following, we will henceforth concentrate only on the case where $\alpha = -1$.

Up to now, we only discussed properties of the group $G' = C_3 \rtimes C_8$. Assume now that we adjoin an element k of order 3 to $G' = \langle a, b \rangle$, which commutes with a and b. Then by similar considerations as in the previous proof, we see that the only possibility is $\rho(k) = \mathrm{diag}(1, 1, 1, \zeta_3)$ (or its square): indeed, k is mapped to the identity matrix by the irreducible representation of dimension 2 occurring in ρ, because $\rho(G)$ must not contain matrices with primitive 24-th roots of unity as eigenvalues, cf. Lemma 2.5. The following possibilities remain (up to replacing k by k^2):

$\rho(k) = \mathrm{diag}(1,\ 1,\ \zeta_3,\ \zeta_3^j)$, $j \neq 0 \implies \rho(ak)$ does not have the eigenvalue 1.
$\rho(k) = \mathrm{diag}(1,\ 1,\ \zeta_3,\ 1) \implies \rho(bk)$ does have the eigenvalue 1.
$\rho(k) = \mathrm{diag}(1,\ 1,\ 1,\ \zeta_3)$: Example 8.62 shows that this is in fact possible!

We give an example of a hyperelliptic fourfold with group $(C_3 \rtimes C_8) \times C_3$.

Example 8.62. Let

$$A := E_i \times E_i \times E \times F, \text{ where}$$
$$E_i := \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}),\ E := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \text{ and } F := \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z}).$$

Define $a, b, k \in \mathrm{Bihol}(A)$ by

$$a(z) = \left(-z_1 - z_2,\ z_1,\ z_3,\ z_4 + \frac{1-\zeta_3}{3}\right),$$

$$b(z) = \left(z_1 \pm i z_2,\ (\pm i - 1)z_1 - z_2,\ z_3 + \frac{1}{8},\ -z_4\right),$$

$$k(z) = \left(z_1,\ z_2,\ z_3 + \frac{1}{3},\ \zeta_3 z_4\right).$$

That $a,\ b,\ k$ have respective orders $3,\ 8,\ 3$ is clear. Moreover,

$$b^{-1}ab(z) = \left(z_2,\ -z_1 - z_2,\ z_3, z_4 - \frac{1-\zeta_3}{3}\right) = a^2(z),$$

and $[b,k] = \mathrm{id}_A$ is clearly satisfied. Since $\frac{1-\zeta_3}{3}$ is fixed by multiplication by ζ_3, we obtain that the relation $[a,k] = \mathrm{id}_A$ is satisfied, too. Thus, the group $G := \langle a, b, k \rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $(C_3 \rtimes C_8) \times C_3$. The group G contains no translations, since ρ is by construction a faithful representation of G.
We now prove that G acts indeed freely on A:

- It is obvious that b^j, $1 \leq j \leq 7$ act freely on A, since they act on E by a non-trivial translation.
- The elements $a,\ b^2a,\ b^4a,\ b^6a$ act freely on A: since the restriction of these maps to $E \times F$ is a non-trivial translation, they indeed act freely on A.
- The restriction of

  $$k^l, \quad b^jk^l, \quad b^jk^l, \quad ak^l, \quad b^2ak^l, \quad b^4ak^l, \quad b^6ak^l \quad \text{for } 1 \leq j \leq 7,\ l = 1,2$$

  to E is a non-trivial translation, hence the above elements act freely on A as well.

The above list exhausts all (non-trivial) conjugacy classes of G, meaning that we have established the freeness of the action of G on A. Summarizing everything,

Proposition 8.63. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G := (C_3 \rtimes C_8) \times C_3 = \langle a, b, k \mid a^3 = b^8 = k^3 = [a,k] = [b,k] = 1,\ b^{-1}ab = a^2 \rangle.$$

The Abelian variety A is isogenous to $E_i \times E_i \times E \times F$, where E_i is the harmonic elliptic curve, F is the equianharmonic elliptic curve and E is some elliptic curve. In particular, every complete family of hyperelliptic fourfolds with group G is 1-dimensional, and after a suitable change of basis and generators,

$$\rho(a) = \begin{pmatrix} -1 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & \pm i & & \\ \pm i - 1 & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix},$$

$$\rho(k) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \zeta_3 \end{pmatrix}$$

Remark 8.64. It becomes apparent from the above discussion that there is no hyperelliptic fourfold with group $(C_3 \rtimes C_8) \times C_3^2$.

8.1.12 The group $S_3 \times C_{12}$ (GAP ID [72,27])

The aim of this section is to show the existence of a hyperelliptic fourfold with group $G := S_3 \times C_{12}$. This group has the presentation

$$G = S_3 \times C_{12} = \langle \tau, \sigma, \kappa \,|\, \tau^2 = \sigma^3 = (\tau\sigma)^2 = \kappa^{12} = [\tau, \kappa] = [\sigma, \kappa] = 1 \rangle.$$

The representation $\rho|_{S_3}$ needs to contain the unique irreducible representation of dimension 2 of S_3, which is given by

$$\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma \mapsto \begin{pmatrix} \zeta_3 & \\ & \zeta_3^2 \end{pmatrix}.$$

Consequently, ρ is a direct sum of an irreducible representation ρ_2 of dimension 2 and two 1-dimensional representations ρ_1, ρ_1'. By Lemma 5.5 (b), we see that (at least) one of ρ_1, ρ_1' is non-trivial. If we write

Lemma 8.65. *Let S be an Abelian surface such that* $\mathrm{Bihol}(S)$ *has a subgroup isomorphic to $S_3 = \langle \tau, \sigma \rangle$. Then τ does not act freely on S.*

Proof. If we write $\tau(z_1, z_2) = (z_2 + t_1,\ z_1 + t_2)$ and $\sigma(z) = (\zeta_3 z_1 + s_1,\ \zeta_3^2 z_2 + s_2)$, then $\tau^2 = \mathrm{id}_S$ means that $(t_1 + t_2,\ t_1 + t_2) = 0 \in S$. This yields that

$$(*) \qquad (\zeta_3^2(t_1 + t_2),\ \zeta_3(t_1 + t_2)) = 0 \in S,$$

as well. Using $\zeta_3^2 + \zeta_3 + 1 = 0$, we calculate

$$\begin{aligned} \tau(\zeta_3^2 t_2 - \zeta_3 t_1,\ 0) &= (t_1,\ -\zeta_3 t_1 + \zeta_3^2 t_2 + t_2) = (t_1,\ -\zeta_3(t_1 + t_2)) \\ &\overset{(*)}{=} (\zeta_3^2(t_1 + t_2) + t_1,\ 0) = (\zeta_3^2 t_2 - \zeta_3 t_1,\ 0). \end{aligned}$$

Thus τ has a fixed point on S. □

Since we are investigating fourfolds, τ and σ do not necessarily act on an Abelian surface. Nevertheless, the previous Lemma implies that

Corollary 8.66. *Suppose that $\tilde{G}$ is a group containing S_3 and that $\rho\colon \tilde{G} \to \mathrm{GL}(4, \mathbb{C})$ is a faithful representation. Assume furthermore that $\tilde{G}$ contains an element g, such that, up to a change of basis,*

$$\rho(g) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \alpha & \\ & & & \beta \end{pmatrix}, \quad \textit{where } \alpha, \beta \neq 1.$$

Then the element g does not act freely on any Abelian fourfold with associated complex representation ρ.

Corollary 8.67. *Up to a change of basis,*

$$\rho(\tau) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} \zeta_3 & 0 & & \\ 0 & \zeta_3^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Lemma 8.68. *The Abelian variety A is isogenous to a product of an Abelian surface $S \subset A$ and two elliptic curves $E_3 = \mathbb{C}/(\mathbb{Z} + \tau_3\mathbb{Z})$, $E_3 = \mathbb{C}/(\mathbb{Z} + \tau_3\mathbb{Z}) \subset A$.*

Proof. The Abelian variety A is isogenous to the product of the two Abelian surfaces

$$S := \ker(\rho(\sigma) - \mathrm{id}_A)^0 \text{ and } S' := \mathrm{im}(\rho(\sigma) - \mathrm{id}_A).$$

We conclude by noticing that S' is isogenous to the product of the two elliptic curves

$$E_3 := \left(S' \cap \ker(\rho(\tau) + \mathrm{id}_A)\right)^0 \text{ and } E_4 := \left(S' \cap \ker(\rho(\tau) - \mathrm{id}_A)\right)^0.$$

This proves the statement. □

Up to now, we did not investigate the additional central element κ of order 12 at all, which we shall take into account now. Up to inverses, the possibilities for $\rho(\kappa^3)$ are the following:

$\rho(\kappa^3) = \mathrm{diag}(\alpha,\ \alpha,\ i,\ 1)$, $\alpha^4 = 1 \implies \langle \sigma, \tau\kappa^6 \rangle \cong S_3$ excluded by Lemma 5.5,
$\rho(\kappa^3) = \mathrm{diag}(\alpha,\ \alpha,\ 1,\ i)$, $\alpha^4 = 1 \implies \langle \sigma, \tau\kappa^6 \rangle \cong S_3$ excluded by Corollary 8.66,
$\rho(\kappa^3) = \mathrm{diag}(i,\ i,\ 1,\ -1) \implies$ excluded, since $\rho(\tau\kappa^3)$ does not have the eigenvalue 1,
$\rho(\kappa^3) = \mathrm{diag}(i,\ i,\ \beta,\ 1)$, $\beta^2 = 1$: potentially possible, to be investigated below.

By Lemma 2.2, and because $\rho(\kappa\tau)$ must have the eigenvalue 1, we obtain that

$$\rho(\kappa) = \mathrm{diag}(i,\ i,\ \beta\zeta_3,\ 1).$$

The following Lemma follows immediately from Proposition 2.4.

Lemma 8.69. *The Abelian surface S is isomorphic to $E_i \times E_i$, where $E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is the harmonic elliptic curve.*

Moreover, since $\rho(\kappa)$ acts on E_3 by multiplication by a 6-th root of unity, we obtain that

Lemma 8.70. *The elliptic curve E_3 is isomorphic to the equianharmonic elliptic curve*

$$F = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z}).$$

A change of basis is needed so that we get a well-defined action of $G = S_3 \times C_{12}$ on $S \sim_{\text{isog.}} E_i \times E_i$. The matrices

$$\rho_2(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2(\sigma) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2(\kappa) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

do the job: written in this way, ρ_2 is again an irreducible representation of $S_3 \times C_4$, and the matrices are defined over $\mathbb{Z}[i]$. We write

$$\begin{aligned}\sigma(z) &= \rho(\sigma)z + (s_1,\ s_2,\ s_3,\ s_4),\\ \tau(z) &= \rho(\tau)z + (t_1,\ t_2,\ t_3,\ t_4).\end{aligned}$$

and

$$A \cong (E_1 \times E_2 \times E_3 \times E_4)/H.$$

Corollary 8.71. *(of Lemma 8.65)*
H contains no element whose last coordinate is equal to t_4.

Proof. If H would contain such an element $(w_1,\ w_2,\ w_3,\ t_4)$, $\tau(z)$ is congruent to

$$(\rho(\tau)z + (t_1 - w_1,\ t_2 - w_2,\ t_3 - w_3,\ 0)$$

modulo H. Applying Lemma 8.65 now implies the result. □

Lemma 8.72. $\beta = -1$ *is not possible.*

Proof. Suppose that $\beta = -1$. Investigating the defining relations of S_3, we obtain

- $\tau^2 = \mathrm{id}_A \iff (t_1 + t_2,\ t_1 + t_2,\ 0,\ 2t_4) \in H$,
- $\sigma^3 = \mathrm{id}_A \implies$ there are $w_1,\ w_2$ such that $(w_1,\ w_2,\ 3s_3,\ 3s_4) \in H$,
- $(\tau\sigma)^2 = \mathrm{id}_A \implies$ there are w_1', w_2' such that $(w_1',\ w_2',\ 0,\ 2s_4) \in H$.

Hence, H contains an element of the form $u := (u_1,\ u_2,\ 3s_3,\ s_4)$. Now, since $\beta = -1$, the relation $\kappa^3\sigma = \sigma\kappa^3$ implies the existence of v_1, v_2 such that

$$v := (v_1,\ v_2,\ 2s_3,\ 0) \in H.$$

Since $u - v = (u_1 - v_1,\ u_2 - v_2,\ s_3,\ s_4) \in H$, the element σ does not act freely on A. □

We shall now give an example of a hyperelliptic fourfold with group $S_3 \times C_{12}$.

Example 8.73. Let

$$\begin{gathered}A := E_i \times E_i \times F \times E, \text{ where}\\ E_i := \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}),\ F := \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z}),\ E := \mathbb{C}/(\mathbb{Z} + t\mathbb{Z}).\end{gathered}$$

Define biholomorphic self-maps τ, σ, κ of A as follows:

$$\begin{aligned}\tau(z) &= \left(z_2,\ z_1,\ -z_3,\ z_4 + \frac{1}{2}\right),\\ \sigma(z) &= \left(-z_1 - z_2,\ z_1,\ z_3 + \frac{\zeta_3 - 1}{3},\ z_4\right),\\ \kappa(z) &= \left(iz_1,\ iz_2,\ \zeta_3 z_3,\ z_4 + \frac{t}{12}\right).\end{aligned}$$

Then the relations $\tau^2 = \sigma^2 = \kappa^{12} = [\tau, \kappa] = \mathrm{id}_A$ are clearly satisfied. Moreover, σ and κ commute, since $\frac{\zeta_3 - 1}{3}$ is fixed by multiplication by ζ_3. Finally,

$$\tau\sigma(z) = \left(\zeta_3^2 z_2,\ \zeta_3 z_1,\ -z_3 - \frac{\zeta_3 - 1}{3},\ z_4 + \frac{1}{2}\right)$$

implies that $(\tau\sigma)^2 = \mathrm{id}_A$. We have established that $G := \langle \tau, \sigma, \kappa \rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $S_3 \times C_{12}$, and it remains to show that G acts freely on A. A system of representatives of conjugacy classes of G is given by

$$\kappa^j, \qquad \kappa^j\tau, \qquad \kappa^j\sigma, \qquad 0 \leq j \leq 11.$$

All of the above listed elements (except, of course, id_A) act freely on A, because each of them acts on F or on E by a non-trivial translation.

Proposition 8.74. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G = S_3 \times C_{12} = \langle \tau, \sigma, \kappa \,|\, \tau^2 = \sigma^3 = (\tau\sigma)^2 = \kappa^{12} = [\tau, \kappa] = [\sigma, \kappa] = 1 \rangle.$$

The Abelian variety A is necessarily isogenous to $E_i \times E_i \times F \times E'$, where E_i (resp. F) are the harmonic (resp. equianharmonic) elliptic curve and E is another elliptic curve. In particular, each complete family of hyperelliptic fourfolds with group G is 1-dimensional. Moreover, after a suitable change of coordinates, the linear parts $\rho(\tau)$, $\rho(\sigma)$, $\rho(\kappa)$ of τ, σ and κ are given by

$$\rho(\tau) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} -1 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(\kappa) = \begin{pmatrix} i & & & \\ & i & & \\ & & \zeta_3 & \\ & & & 1 \end{pmatrix}.$$

8.1.13 The group $C_3 \times ((C_6 \times C_2) \rtimes C_2)$ (GAP ID [72,30])

The group $G := C_3 \times ((C_6 \times C_2) \rtimes C_2)$ has the presentation

$$G = \langle r, s, a, k \,|\, r^4 = s^2 = a^3 = (rs)^2 = (rsa)^2 = k^3 = [s, a] = [k, r] = [k, s] = [k, a] = 1 \rangle.$$

The subgroup $G' := (C_6 \times C_2) \rtimes C_2$ of G has the following irreducible representations of dimension 2:

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s \mapsto \pm \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} \zeta_3 & \\ & \zeta_3^2 \end{pmatrix},$$

Moreover, the other irreducible representations of dimension > 1 of G' are not faithful and its kernel intersects the derived subgroup $[G', G'] = \langle r^2, a \rangle$ of G' non-trivially:

$$r \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} \zeta_3 & \\ & \zeta_3^2 \end{pmatrix},$$

$$r \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} \zeta_3 & \\ & \zeta_3^2 \end{pmatrix},$$

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

These are all irreducible representations of G', since $|G'| = 24 = 5 \cdot 2^2 + 4 \cdot 1^2$, and 1-dimensional representations of G are given by $r \mapsto \pm 1$, $s \mapsto \pm 1$, $a \mapsto 1$.

It follows that ρ is not the direct sum of two irreducible representations of dimension 2, since $\rho(r^2)$ and $\rho(ar^2)$ must have the eigenvalue 1. Consequently, ρ is the direct sum of a 2-irreducible representation and two 1-dimensional representations of G'. Since they arise from each other under the automorphism $r \mapsto r$, $s \mapsto r^2 s$, $a \mapsto a$ of G', we may concentrate on only one of them.

Remark 8.75. The subgroup $\langle r, s\rangle$ of G' is isomorphic to D_4. Thus, by Lemma 5.5 (b), one of the 1-dimensional representations occurring in ρ is not trivial.

Moreover, since $\rho(rsa)$ must have the eigenvalue 1, it follows that we can assume

$$\rho(r) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & \alpha \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \beta \end{pmatrix},$$

where $\alpha, \beta \in \{\pm 1\}$. If $\alpha = -1$, we can replace s by rs to assume that $\beta = 1$.

Lemma 8.76. *The Abelian variety A is isogenous to a product of four elliptic curves $E_j \subset A$. More precisely, $A \cong (E_1 \times E_2 \times E_3 \times E_4)/H$, where $E_1 \cong E_2$ and*

$$H \subset E_1[4] \times E_2[4] \times E_3[4] \times E_4[4].$$

Proof. The Abelian variety A is isogenous to $S \times S'$, where S, $S' \subset A$ are the Abelian surfaces

$$S := \operatorname{im}(\rho(r^2) - I), \qquad S' := \ker(\rho(r^2) - I)^0.$$

In any case, S is isogenous to $E_1 \times E_2$, where E_1, $E_2 \subset S$ are given by

$$E_1 := (\ker(\rho(s) - I) \cap S)^0, \qquad E_2 := \left(\operatorname{im}(\rho(r^2) - I) \cap S\right)^0.$$

Now, if $(\alpha, \beta) = (-1, 1)$, the Abelian surface S' is isogenous to $E_3 \times E_4$, where

$$E_3 := \left(\ker(\rho(r) - I) \cap S'\right)^0, \qquad E_4 := \left(\operatorname{im}(\rho(r) - I) \cap S'\right)^0.$$

If $(\alpha, \beta) = (1, -1)$, we obtain similarly that S' is isogenous to the product of

$$E_3 := \left(\ker(\rho(s) - I) \cap S'\right)^0, \qquad E_4 := \left(\operatorname{im}(\rho(s) - I) \cap S'\right)^0.$$

It remains to prove the statement about the torsion subgroup H.f we denote by Λ resp. Λ_j the respective lattices of A and E_j, then H is equal to $\Lambda/(\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \oplus \Lambda_4)$. Let $\lambda \in \Lambda$. Write

$$2\lambda = \underbrace{(I + \rho(r^2))\lambda}_{=:\lambda' \in \Lambda_1 \oplus \Lambda_2} \; + \; \underbrace{(I - \rho(r^2))\lambda}_{=:\lambda'' \in \Lambda_3 \oplus \Lambda_4},$$

and

$$2\lambda' = \underbrace{(I+\rho(s))\lambda'}_{\in\Lambda_1} + \underbrace{(I-\rho(r^2))\lambda'}_{\in\Lambda_2}.$$

In the case $(\alpha,\beta) = (-1,1)$, we may write

$$2\lambda'' = \underbrace{(I+\rho(r))\lambda''}_{\in\Lambda_3} + \underbrace{(I-\rho(r))\lambda''}_{\in\Lambda_4},$$

whereas in the case $(\alpha,\beta) = (1,-1)$, we write

$$2\lambda'' = \underbrace{(I+\rho(s))\lambda''}_{\in\Lambda_3} + \underbrace{(I-\rho(r))\lambda''}_{\in\Lambda_4}.$$

This proves that in any case, $4\lambda \in \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \oplus \Lambda_4$. This shows the statement. $\square$

We shall choose coordinates in each E_j, such that

$$\begin{aligned} a(z) &= (\zeta_3 z_1 + a_1,\ \zeta_3^2 z_2 + a_2,\ z_3 + a_3,\ z_4 + a_4), \\ s(z) &= (z_1 + b_1,\ -z_2 + b_2,\ z_3 + b_3,\ \beta z_4 + b_4), \\ r(z) &= (-z_2,\ z_1,\ z_3 + c_3,\ \alpha z_4 + c_4). \end{aligned}$$

$\underline{(\alpha,\beta) = (1,-1)}$: We prove that this case does not occur. By choosing the origin in E_4 to be a fixed point of $z_4 \mapsto -z_4 + b_4$, we may assume that $b_4 = 0$. We investigate some of the defining relations of G in greater detail:

(i) $(rs)^2 = \mathrm{id}_A \implies \exists v_1, v_2\colon\ (v_1,\ v_2,\ 2b_3 + 2c_3,\ 0) \in H$.

(ii) $a^3 = \mathrm{id}_A \iff h := (0,\ 0,\ 3a_3,\ 3a_4) \in H \overset{(\rho(s)-\mathrm{id}_A)h\in H}{\implies} 6a_4 = 0$,

(iii) $(rsa)^2 = \mathrm{id}_A \overset{\text{subtract (i)}}{\implies} \exists u_1, u_2\colon\ (u_1,\ u_2,\ 2a_3,\ 0) \in H$,

(iv) $[s,a] = \mathrm{id}_A \implies \exists w_1, w_2\colon\ (w_1,\ w_2,\ 0,\ 2a_4) \in H$.

Here we used in (ii) that E_4 embeds into A, and thus $(\rho(s) - \mathrm{id})h = (0,\ 0,\ 0,\ 6a_4) \in H$ implies that $6a_4 = 0$. In (iii), it was used that the relation $(rsa)^2 = \mathrm{id}_A$ produces an element of the form $(u_1',\ u_2',\ 2a_3 + 2b_3 + 2c_3,\ 0) \in H$, from which we subtracted the element in (i).
These relations help us in the following way.

- Since rsa has order 2, we can assume that a acts on E_3 as the identity, and on E_4 by translation by a_4, which is necessarily contained in $E_4[6] \setminus E_4[2]$, because a and a^2 need to act freely on A.
- Since H contains only 4-torsion points of $\prod E_j$ by Lemma 8.76, the relation $[s,a] = \mathrm{id}_A$ gives that $8a_4 = 0$, a contradiction.

We have proved that the case $(\alpha,\beta) = (1,-1)$ does not occur.

$\underline{(\alpha,\beta) = (-1,1)}$: Below, we give an example of a hyperelliptic fourfold with group

$$G = C_3 \times G' = C_3 \times ((C_6 \times C_2) \rtimes C_2),$$

where G' falls into the current case $(\alpha, \beta) = (-1, 1)$. Let us first investigate the possibilities for $\rho(k)$, where k is an additional central element of order 3, which we adjoin to G':

- Obviously, the third diagonal entry of $\rho(k)$ must be equal to 1, since $\rho(rk)$ must have the eigenvalue 1.

- Since we require r and r^2 to act freely on A, we observe that c_3 has order exactly 4 in $E_3 \subset A$. More precisely, since $s^2 = (rs)^2 = \mathrm{id}_A$, the torsion group H contains the element $v := (-b_1 - b_2,\ -b_1 - b_2,\ 2c_3,\ -2b_4)$, which, in view of

$$(\rho(s) - \mathrm{id})v = (0,\ -2(b_1 + b_2),\ 0,\ 0) \in H \overset{E_2 \subseteq A}{\Longrightarrow} 2(b_1 + b_2) = 0$$
$$\Longrightarrow\ 2v = (0,\ 0,\ 0,\ 4b_4) \in H$$

  shows that $4b_4 = 0$ and $2b_4 \neq 0$ (else, r^2 would not act freely on A).
  Hence, if the fourth diagonal entry of $\rho(k)$ is a primitive third root of unity (say without loss of generality that it is equal to ζ_3), the element $(\rho(k) - \mathrm{id})v$ takes the form

$$(\rho(k) - \mathrm{id})v = (w_1,\ w_2,\ 0, -2(\zeta_3 - 1)b_4).$$

  Since $2b_4$ is a non-trivial 2-torsion element, Lemma 2.8 implies that $2(\zeta_3 - 1)b_4 \neq 0$, and consequently

$$\langle 2(\zeta_3 - 1)b_4,\ 2\zeta_3(\zeta_3 - 1)b_4 \rangle = E_4[2].$$

  A suitable linear combination of $(\rho(k) - \mathrm{id})v$ and $\rho(k)(\rho(k) - \mathrm{id})v$ is therefore of the form $(w_1', w_2', 0, 2b_4)$. Adding this element to v, one obtains that H contains an element whose last two coordinates are equal to $2c_3$ and 0, respectively. This proves that r^2 does not act freely on A.

Since k is central, it is mapped to a multiple of the identity matrix by the 2-dimensional irreducible representation contained in ρ: henceforth, after possibly replacing k by k^2, we are allowed to assume that $\rho(k) = \mathrm{diag}(\zeta_3, \zeta_3, 1, 1)$. Proposition 2.4 shows that the elliptic curves E_1, E_2 are necessarily isomorphic to the equianharmonic elliptic curve $F = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$.

Example 8.77. We define

$$A := A'/H := (F \times F \times E \times E')/H,$$

$$H := \left\langle \left(0,\ 0,\ \frac{1}{2},\ \frac{1}{2}\right) \right\rangle,$$

where F is the equianharmonic elliptic curve and $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, $E' = \mathbb{C}/(\mathbb{Z} + \tau'\mathbb{Z})$ are arbitrary elliptic curves.

Furthermore, define biholomorphic self-maps of A' as follows:

$$\begin{aligned}
r(z) &= \left(-z_2,\ z_1,\ z_3+\frac{1}{4},\ -z_4\right),\\
s(z) &= \left(z_1,\ -z_2,\ z_3+\frac{1}{4}+\frac{\tau}{2},\ z_4+\frac{1}{4}\right),\\
a(z) &= \left(\zeta_3 z_1,\ \zeta_3^2 z_2,\ z_3,\ z_4+\frac{1}{3}\right),\\
k(z) &= \left(\zeta_3 z_1,\ \zeta_3 z_2,\ z_3+\frac{1}{3},\ z_4\right).
\end{aligned}$$

Since the linear parts of these four maps send H to H, they descend to biholomorphic self-maps of A.
We will now prove that $G := \langle r, s, a, k\rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $C_3 \times ((C_6 \times C_2) \rtimes C_2)$.

- That the relations $r^4 = a^3 = k^3 = [s,a] = [k,r] = [k,s] = [k,a] = \mathrm{id}_A$ hold is clear.
- The relation $s^2 = \mathrm{id}_A$ is satisfied in view of the given definition of H.
- The relations $(rs)^2 = (rsa)^2 = \mathrm{id}_A$ hold, because

$$\begin{aligned}
(rs)(z) &= \left(z_2,\ z_1,\ z_3+\underbrace{\frac{1+\tau}{2}}_{2-\text{torsion}},\ -z_4-\frac{1}{4}\right),\\
(rsa)(z) &= \left(\zeta_3^2 z_2,\ \zeta_3 z_1,\ z_3+\underbrace{\frac{1+\tau}{2}}_{2-\text{torsion}},-z_4-\frac{7}{12}\right).
\end{aligned}$$

We shall now comment on the freeness of the action of G on A. As mentioned at the beginning of the section, the group $(C_6 \times C_2) \rtimes C_2$ has 9 pairwise non-equivalent irreducible representations, hence this group has 9 conjugacy classes. A system of representatives is given by

$$\mathcal{C} = \{\mathrm{id}_A,\ r,\ r^2,\ s,\ rs,\ a,\ r^2a,\ sa,\ r^2sa\}.$$

Hence, a system of representatives of conjugacy classes of G is given by

$$\mathcal{C}_G \ := \ \mathcal{C} \ \cup \ k\mathcal{C} \ \cup \ k^2\mathcal{C}.$$

The element

$$r^2(z) = \left(-z_1,\ -z_2,\ z_3+\frac{1}{2},\ z_4\right)$$

acts freely on A, since the equation $r^2(z) = z$ is solvable if and only if H contains an element of the form $(2z_1,\ 2z_2,\ \frac{1}{2},\ 0)$, which is not the case.
Moreover, it is immediate that any non-trivial element in $\mathcal{C}_G$ different from r^2 acts freely on A, since their third or fourth coordinate is a non-trivial translation by an element different from $\frac{1}{2}$ (but the coordinates of the unique non-zero element in H are just 0 and $\frac{1}{2}$).

In total, this means that A/G is in fact a hyperelliptic fourfold with group $G = C_3 \times ((C_6 \times C_2) \rtimes C_2)$.

We have proved the following

Proposition 8.78. *There exist hyperelliptic fourfolds $X = A/G$ with group*

$$G = \langle r, s, a, k | r^4 = s^2 = (rs)^2 = (rsa)^2 = k^3 = [s, a] = [k, r] = [k, s] = [k, a] = 1\rangle.$$

Necessarily, the Abelian variety A is isogenous to a product of elliptic curves, $A \sim_{isog.} F \times F \times E \times E'$, where F is the equianharmonic elliptic curve and E, E' are elliptic curves. In particular, each complete family of hyperelliptic fourfolds with group G is 2-dimensional. Moreover, we can assume that

$$\rho(r) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$\rho(a) = \begin{pmatrix} \zeta_3 & & & \\ & \zeta_3^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(k) = \begin{pmatrix} \zeta_3 & & & \\ & \zeta_3 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

It follows immediately from the above discussion that

Corollary 8.79. *There is no hyperelliptic fourfold with group $C_3^2 \times ((C_6 \times C_2) \rtimes C_2)$ (GAP ID [216,139]).*

8.1.14 The group $S_3 \times C_6 \times C_3$ (GAP ID [108,42])

This section shows the existence of hyperelliptic fourfolds with group

$$G := S_3 \times C_6 \times C_3 = \left\langle \begin{array}{c|c} \sigma, \tau, & \sigma^3 = \tau^2 = (\tau\sigma)^2 = \kappa_1^6 = \kappa_2^3 = 1, \\ \kappa_1, \kappa_2 & [\sigma, \kappa_1] = [\sigma, \kappa_2] = [\tau, \kappa_1] = [\tau, \kappa_2] = [\kappa_1, \kappa_2] = 1 \end{array} \right\rangle.$$

As we have already seen in Lemma 8.67 in Section 8.1.12, we can assume that

$$\rho(\tau) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} \zeta_3 & 0 & & \\ 0 & \zeta_3^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Let us take the elements κ_1 and κ_2 of respective orders 6 and 3 into account. Since they are central, they are mapped to a multiple of the identity matrix by any irreducible representation of G. If G contains an element κ whose matrix is equal to $\text{diag}(1, 1, \zeta_3, \zeta_3^j)$ for $j \in \{1, 2\}$, the matrix $\rho(\kappa\sigma)$ does not have the eigenvalue 1. Hence, we can assume that

$$\rho(\kappa_1^2) = \text{diag}(\zeta_3,\ \zeta_3,\ 1,\ 1), \text{ and either}$$
$$\rho(\kappa_2) = \text{diag}(1,\ 1,\ 1,\ \zeta_3) \text{ or } \rho(\kappa_2) = \text{diag}(1,\ 1,\ \zeta_3,\ 1).$$

If $\rho(\kappa_2) = \mathrm{diag}(1, 1, 1, \zeta_3)$, we may apply Corollary 8.66 to see that $\kappa_2\tau$ does not act freely on A. Hence, the only possibility left is $\rho(\kappa_2) = \mathrm{diag}(1, 1, \zeta_3, 1)$.
Moreover, if the third diagonal entry of $\rho(\kappa_1)$ is -1, the fourth diagonal entry of $\rho(\kappa_1)$ is 1 (since $\rho(\kappa_1)$ must have the eigenvalue 1). In this case, we obtain that $\langle \tau\kappa_1^3, \sigma\rangle$ is a subgroup of G, isomorphic to S_3, which does not act freely on any Abelian threefold in view of Lemma 5.5 and Lemma 8.65. If instead the fourth diagonal entry of $\rho(\kappa_1)$ is -1, the matrix $\rho(\tau\kappa_1)$ does not have the eigenvalue 1. Hence, we may assume that

$$\rho(\kappa_2) = \mathrm{diag}(-\zeta_3,\ -\zeta_3,\ 1,\ 1).$$

We determine the isogeny type of A.

Lemma 8.80. *The Abelian variety A is isogenous to $F \times F \times F \times E$, where F is the equianharmonic elliptic curve and E is another elliptic curve.*

Proof. By Lemma 8.68, A is isogenous to $S \times E_1 \times E_2$, where S is an Abelian surface and $E_1,\ E_2$ are elliptic curves. Moreover, $\rho(\kappa_1)$ acts on S with multiplication by ζ_3. By Proposition 2.4, S is isomorphic to $F \times F$. Since $\rho(\kappa_2)$ acts on E_1 by multiplication by ζ_3, we obtain $E_1 = F$. Writing $E := E_2$ gives the result. □

We are now in the position to give an example of a hyperelliptic fourfold with group $G = S_3 \times C_3 \times C_3$.

Example 8.81. Let

$$A := F \times F \times F \times E, \text{ where}$$

$$F := \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z}) \text{ and } E := \mathbb{C}/(\mathbb{Z} + t\mathbb{Z}).$$

Define biholomorphic self-maps $\tau, \sigma, \kappa_1, \kappa_2$ of A as follows:

$$\begin{aligned}
\tau(z) &= \left(z_2,\ z_1,\ -z_3,\ z_4 + \frac{t}{2}\right),\\
\sigma(z) &= \left(\zeta_3 z_1,\ \zeta_3^2 z_2,\ z_3 + \frac{\zeta_3 - 1}{3},\ z_4\right),\\
\kappa_1(z) &= \left(-\zeta_3 z_1,\ -\zeta_3 z_2,\ z_3,\ z_4 + \frac{1}{6}\right),\\
\kappa_2(z) &= \left(z_1,\ z_2,\ \zeta_3 z_3,\ z_4 + \frac{t}{3}\right).
\end{aligned}$$

Since A is a product of elliptic curves and τ, σ are (up to a change of basis) as in Example 8.73, we obtain that $\langle \sigma, \tau\rangle \subset \mathrm{Bihol}(A)$ is isomorphic to S_3 and acts freely on A. Now, the relations

$$\kappa_1^6 = \kappa_2^3 = [\sigma, \kappa_1] = [\tau, \kappa_1] = [\tau, \kappa_2] = [\kappa_1, \kappa_2] = \mathrm{id}_A$$

are clear, and the remaining relation $[\sigma, \kappa_2] = \mathrm{id}_A$ follows, because $\frac{\zeta_3 - 1}{3}$ is fixed by multiplication by ζ_3. This shows that $G := \langle \sigma, \tau, \kappa_1, \kappa_2\rangle \subset \mathrm{Bihol}(A)$ is isomorphic to $S_3 \times C_6 \times C_3$. Moreover, G contains no translation, since the associated complex representation ρ is by construction faithful.

Let $\mathcal{C} = \{\mathrm{id}_A,\ \tau,\ \sigma\}$: then $\mathcal{C}$ is system of representatives of conjugacy classes of S_3. Then a system of representatives of conjugacy classes of G is given by

$$\bigcup_{i=0}^{5}\bigcup_{j=0}^{2} \kappa_1^i\kappa_2^j\mathcal{C}.$$

Each of the listed representatives not contained in $\mathcal{C}$ acts on E by a non-trivial translation, which proves that all of the representatives indeed act freely on A. Summarizing everything, we have proved

Proposition 8.82. *There exist hyperelliptic fourfolds A/G with group*

$$G := S_3 \times C_6 \times C_3 = \left\langle \begin{array}{c|c} \sigma, \tau, & \sigma^3 = \tau^2 = (\tau\sigma)^2 = \kappa_1^6 = \kappa_2^3 = 1, \\ \kappa_1, \kappa_2 & [\sigma, \kappa_1] = [\sigma, \kappa_2] = [\tau, \kappa_1] = [\tau, \kappa_2] = [\kappa_1, \kappa_2] = 1 \end{array} \right\rangle.$$

The Abelian variety A is necessarily isogenous to $F \times F \times F \times E$, where F is the equianharmonic elliptic curve and E is another elliptic curve. In particular, every family of hyperelliptic fourfolds with group G is 1*-dimensional. Moreover, after a change of coordinates and generators,*

$$\rho(\tau) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} \zeta_3 & 0 & & \\ 0 & \zeta_3^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$\rho(\kappa_1) = \begin{pmatrix} -\zeta_3 & & & \\ & -\zeta_3 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(\kappa_2) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \zeta_3 & \\ & & & 1 \end{pmatrix}.$$

8.2 Certain groups of order $2^a \cdot 3^b$ which do not occur

We shall prove that certain finite groups of order $2^a \cdot 3^b$ do *not* occur as groups of hyperelliptic fourfolds. These are added to the list of 'forbidden groups' in GAP Script `maximal_groups.g` (see Chapter 12): the output of the script will then consist only of groups which occur as groups of hyperelliptic fourfolds. Hence, the groups in the output of `maximal_groups.g` are maximal among the groups of order $2^a \cdot 3^b$ occurring as groups of hyperelliptic fourfolds.
Recall our meta-assumptions:

The letter G will always denote a finite subgroup of $\mathrm{Bihol}(A)$, where A is an Abelian fourfold, such that the following properties hold: (1) G is embedded into $\mathrm{GL}(4, \mathbb{C})$ via some faithful representation $\rho\colon G \hookrightarrow \mathrm{GL}(4, \mathbb{C})$ (this is equivalent to requiring that G does not contain any translations). (2) The matrix $\rho(g)$ has the eigenvalue 1 for any $g \in G$. (3) The associated complex representation of the embedding $G \subset \mathrm{Bihol}(A)$ is ρ.

8.2.1 Certain groups containing A_4

In this section, we prove that the four groups

(1) $A_4 \times C_3$ (GAP ID [36,11])

(2) S_4 (GAP ID [24,12])

(3) $(C_4 \times C_4) \rtimes C_3$ (GAP ID [48,3])

(4) $A_4 \rtimes C_4$ (GAP ID [48,30])

do not occur as groups of hyperelliptic fourfolds. Recall that the group A_4 has the presentation

$$A_4 = \langle \sigma, \tau' \mid \sigma^3 = (\tau')^2 = (\tau'\sigma)^3 = 1 \rangle.$$

According to Lemma 8.53, we can assume that

$$A = (E_1 \times E_1 \times E_1 \times E_2)/H,$$

where E_1, $E_2 \subset A$.
Moreover, after a suitable change of coordinates, we may write

$$\begin{aligned} \tau'(z) &= (z_1 + t_1',\ -z_2,\ -z_3,\ z_4 + t_4'), \\ \sigma(z) &= (z_3 + s_1,\ z_1 + s_2,\ z_2 + s_3,\ z_4 + s_4). \end{aligned}$$

Moreover, recall that the defining relations of A_4 yield

$$\begin{aligned} v_1 &= (2t_1',\ 0,\ 0,\ 2t_4') \in H, \\ v_2 &= (s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ 3s_4) \in H, \\ v_3 &= (s_1 - s_2 + s_3 + t_1',\ -s_1 + s_2 - s_3 - t_1',\ s_1 - s_2 + s_3 + t_1',\ 3s_4 + 3t_4') \in H, \end{aligned}$$

cf. Remark 8.55. We now exclude (1) – (4).

<u>To (1):</u> We adjoin to A_4 an element κ of order 3. After possibly replacing κ by its square, we can assume that

$$\rho(\kappa) = \mathrm{diag}(1,\ 1,\ 1,\ \zeta_3) \text{ or } \rho(\kappa) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3,\ 1),$$

because it commutes with τ' and σ, see also Remark 2.1. The first case is excluded, since then $\sigma\kappa$ does not act freely by Lemma 8.52.
Now we go on considering the second case, i.e., the one where

$$\rho(\kappa) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3,\ 1).$$

We write

$$\kappa(z) = (\zeta_3 z_1 + k_1,\ \zeta_3 z_2 + k_2,\ \zeta_3 z_3 + k_3,\ z_4 + k_4).$$

Now, since $\rho(\kappa)$ and $\rho(\tau'\sigma)$ act on H,

$$\begin{aligned} (\rho(\kappa) - \mathrm{id})v_1 &= (2(\zeta_3 - 1)t_1',\ 0,\ 0,\ 0) \in H, \\ (\rho(\tau'\sigma) - \mathrm{id})v_1 &= (0,\ 4t_1',\ 0,\ 0) \in H. \end{aligned}$$

Since E_1 embeds into A, this implies that $2(\zeta_3 - 1)t_1' = 4t_1' = 0$, and by Lemma 2.8, we obtain that

$$2t_1' = 0.$$

Now, $v_1 = (0,\ 0,\ 0,\ 2t_4') \in H$, so that $2t_4' = 0$ as well.
The relation $\tau'\kappa = \kappa\tau'$ implies that

$$((\zeta_3 - 1)t_1',\ 2k_2,\ 2k_3,\ 0) \in H. \tag{8.8}$$

We now take the elements

$$\begin{aligned} v_2 &= (s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ 3s_4) \in H, \\ v_3 &= (s_1 - s_2 + s_3 + t_1',\ -s_1 + s_2 - s_3 - t_1',\ s_1 - s_2 + s_3 + t_1',\ 3s_4 + t_4') \in H \end{aligned}$$

into consideration, and calculate

$$\begin{aligned} v_3 - v_2 &= (t_1' - 2s_2,\ -2s_1 - 2s_3 - t_1',\ t_1' - 2s_2,\ t_4') \in H, \\ \rho(\tau')(v_3 - v_2) + v_3 - v_2 &= (-4s_2,\ 0,\ 0,\ 0) \in H, \end{aligned}$$

which – since $E_1 \subset A$ – implies that $4s_2 = 0$. In view of the equivalence

$$\tau(z) = z \text{ has a solution } \iff H \text{ contains no element of the form } (t_1',\ w_2,\ w_3,\ t_4')$$

we obtain that $2s_2 \neq 0$. Adding (8.8) to $(\rho(\kappa) - \mathrm{id})(v_2 - v_3) \in H$, we obtain that H contains an element of the form

$$w := (2(\zeta_3 - 1)s_2,\ w_2,\ w_3,\ 0).$$

The crucial observation is now:

Lemma 8.83. *$2(\zeta_3 - 1)s_2$ is a non-zero 2-torsion element of E_1.*

Proof. We have observed above that $4s_2 = 0$; it remains to show that $2s_2$ is not fixed by multiplication by ζ_3. This is implied by Lemma 2.8, because $2s_2$ is a non-zero 2-torsion element. □

As a consequence, applying Lemma 2.8 again, we obtain that the two elements

$$2(\zeta_3 - 1)s_2 \text{ and } 2\zeta_3(\zeta_3 - 1)s_2$$

generate $E[2] \cong C_2^2$. This means that a suitable linear combination of w and $\rho(k)w$ is of the form

$$w' := (2s_2,\ w_2',\ w_3',\ 0) \in H.$$

Thus, H contains the element

$$v_3 - v_2 + w' = (t_1',\ w_2'',\ w_3'',\ t_4').$$

As we already observed above, this shows that τ' does not act freely on A.

To (2): The upcoming discussion shall show that there is no hyperelliptic fourfold with group S_4. The group S_4 is generated by $S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = (\sigma\tau)^2 \rangle$ and a double transposition τ'. The irreducible G-representations of dimension ≥ 2 are given by

(a) $\sigma \mapsto \begin{pmatrix} \zeta_3 & \\ & \zeta_3^2 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau' \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$

(b) $\sigma \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau' \mapsto \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix},$

(c) $\sigma \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tau' \mapsto \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$

Since the kernel of (a) contains $\tau' \in A_4 = [G, G]$, representation (a) is not contained in ρ. Thus, ρ contains a 3-dimensional irreducible representation. As for (b), restricting ρ to S_3, one obtains that $\rho|_{S_3}$ is the direct sum of a 2-dimensional irreducible and two 1-dimensional representations, the 1-dimensional representation coming from restricting the irreducible representation of dimension 3 being trivial. Thus, Lemma 5.5 (b) applied to the subgroup S_3 of G shows that τ is mapped to -1 by the 1-dimensional representation contained in ρ. Since τ' is contained in the derived subgroup A_4 of G, it is mapped to 1 by any 1-dimensional representation of G, and we obtain that

$$\rho(\tau\tau') = \begin{pmatrix} 0 & 1 & 0 & \\ -1 & 0 & 0 & \\ 0 & 0 & -1 & \\ & & & -1 \end{pmatrix}$$

does not have the eigenvalue 1.
It is left to exclude representation (c): Corollary 8.67 shows that τ is mapped to 1 by the 1-dimensional representation contained in ρ. Now, writing

$$\tau(z) = (-z_2 + t_1,\ -z_1 + t_2,\ -z_3 + t_3,\ z_4 + t_4),$$
$$\sigma(z) = (z_3 + s_1,\ z_1 + s_2,\ z_2 + s_3,\ z_4 + s_4),$$

we obtain

- $\tau^2 = \mathrm{id}_A \iff \tilde{v}_1 := (t_1 - t_2,\ t_2 - t_1,\ 0,\ 2t_4) \in H,$
- $(\tau\sigma)^2 = \mathrm{id}_A \iff \tilde{v}_2 := (0,\ -s_1 + s_3 + t_2 - t_3,\ s_1 - s_3 - t_2 + t_3,\ 2t_4 + 2s_4) \in H,$
- $\sigma^3 = \mathrm{id}_A \iff v_3 = (s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ s_1 + s_2 + s_3,\ 3s_4) \in H.$

The element $v_3 + \tilde{v}_1 - \tilde{v}_2$ takes the form

$$v_3 + \tilde{v}_1 - \tilde{v}_2 = (w_1,\ w_2,\ w_3,\ s_4) \in H,$$

and Corollary 8.54 then shows that σ does not act freely on A.
In total, we have shown that there is no hyperelliptic fourfold with group S_4.

To (3): The group

$$(C_4 \times C_4) \rtimes C_3 = \langle a, b \mid a^3 = b^4 = (ab)^3 = (ab^2)^3 = 1 \rangle$$

contains the subgroup $\langle a, b^2\rangle$, which is isomorphic to A_4. Thus, as we have seen (and repeated at the beginning of the section), the 1-dimensional representation ρ_1 contained in ρ maps a and b^2 to 1. Since

$$1 = \rho_1(1) = \rho_1((ab)^3) = \rho_1(a)^3\rho_1(b)^3 = \rho_1(b),$$

we have proved that ρ_1 is the trivial representation and that ρ contains a faithful irreducible representation. Since a corresponds to the 3-cycle of A_4, we can assume that every faithful irreducible representation of $(C_4 \times C_4) \rtimes C_3$ maps a to

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Moreover, the four faithful irreducible representations map b to the following four matrices, respectively:

$$\begin{pmatrix} \delta_1 i & & \\ & \delta_2 i & \\ & & \delta_3 \end{pmatrix},$$

where $(\delta_1, \delta_2, \delta_3) \in \{(1, 1, -1), (-1, 1, 1), (1, -1, 1), (-1, -1, -1)\}$. One checks with GAP that each of these four matrices is contained in the image of any of the four faithful irreducible representations of $(C_4 \times C_4) \rtimes C_3$. Furthermore, a and any element corresponding to one of the four matrices above generate $(C_4 \times C_4) \rtimes C_3$. Thus, we can assume without loss of generality, that

$$\rho(b) = \begin{pmatrix} i & & & \\ & i & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

Writing

$$\begin{aligned} a(z) &= (z_3 + a_1,\ z_1 + a_2,\ z_2 + a_3,\ z_4 + a_4), \\ b(z) &= (iz_1 + b_1,\ iz_2 + b_2,\ -z_3 + b_3,\ z_4 + b_4), \end{aligned}$$

and investigating the relations $a^3 = (ab)^3 = 1$ yields

- $a^3 = 1 \iff (a_1 + a_2 + a_3,\ a_1 + a_2 + a_3,\ a_1 + a_2 + a_3,\ 3a_4) \in H$,
- $(ab)^3 = 1 \implies$ there are $w_1,\ w_2,\ w_3$ such that $(w_1,\ w_2,\ w_3,\ 3a_4 + 3b_4) \in H$.

In particular, H contains an element of the form $(w_1',\ w_2',\ w_3',\ 3b_4)$, which shows that the element b^3 of order exactly 4 does not act freely on A.

To (4): We exclude $A_4 \rtimes C_4$ as follows. The group C_4 acts on $A_4 = \langle \sigma, \tau'\rangle$ by conjugation, preserving the subgroups $\langle \tau'\rangle$ and $\langle \sigma\rangle$. If we denote the generator of C_4 by γ, then

$$\gamma^{-1}\tau'\gamma = \tau',$$

$$\gamma^{-1}\sigma\gamma = \sigma^2.$$

The latter relation immediately implies that $\operatorname{ord}(\sigma\gamma) = 4$ and that γ^2 commutes with σ. Write

$$\rho = \rho_3 \oplus \rho_1,$$

where ρ_3 is a 3-dimensional irreducible and ρ_1 is a 1-dimensional representation. Assume that the group in discussion occurred as a group of a hyperelliptic fourfold with associated Abelian variety A. We distinguish two cases:

- If ρ_3 maps γ to a matrix with an eigenvalue of order 4, then $\rho_3(\gamma)$ does not have the eigenvalue 1. Hence $\rho_1(\gamma) = 1$. Now, if we write

$$\sigma(z) = (\rho_3(\sigma)z' + s',\ z_4 + s_4),$$
$$\gamma(z) = (\rho_3(\gamma)z' + s',\ z_4 + c_4),$$

  the relation $(\sigma\gamma)^4 = 1$ implies that H contains an element whose last coordinate is $4s_3 + 4c_4$. Moreover, since $\gamma^4 = 1$ implies that H contains $(0,\ 0,\ 0,\ 4c_4)$, we obtain that H contains an element whose last coordinate is $4s_3$. Finally, this and the relation $\sigma^3 = 1$ imply that H contains an element whose last coordinate is s_4. Thus, σ does not act freely on A by Corollary 8.54.

- If ρ_3 maps γ to a matrix whose eigenvalues are ± 1, then necessarily $\rho_1(\gamma) \in \{\pm i\}$ (because we require ρ to be faithful). Since γ^2 commutes with σ, we obtain that H contains an element whose last coordinate is equal to $2s_4$. Since H moreover contains an element whose last coordinate is $3s_4$, we conclude as in the previous bullet point.

This proves that $A_4 \rtimes C_4$ does not occur as a group associated with a hyperelliptic fourfold.

8.2.2 Certain groups containing M_{16}

The group M_{16} has the presentation

$$M_{16} = \langle a, b \,|\, a^8 = b^2 = 1,\ b^{-1}ab = a^5 \rangle.$$

In this section we will prove that the groups

(1) $M_{16} \times C_2$ (GAP ID [32,37])

(2) $M_{16} \times C_3$ (GAP ID [48,24])

do not occur as groups of hyperelliptic fourfolds.
By Lemma 5.5, ρ is the direct sum of an irreducible representation ρ_2 of dimension 2 and two 1-dimensional representations ρ_1, ρ_1' of M_{16}, at least one of them being non-trivial, say ρ_1'. If ρ_1 was non-trivial as well, we would w.l.o.g. have $\rho_1(b) \neq 1 \neq \rho_1'(a)$, since $\rho(a)$ must have the eigenvalue 1. However then one of the matrices $\rho(ab)$, $\rho(a^2b)$ does not have the eigenvalue 1. We will henceforth assume that ρ_1 is trivial and that ρ_1' is non-trivial.

To (1): Adjoin to M_{16} a central element k of order 2. By possibly replacing k by ka^4, we may assume that the first two diagonal entries of $\rho(k)$ are 1. Since k is central, it is mapped to a multiple of the identity by any irreducible representation of $M_{16} \times C_2$ and we are left with the following possibilities:

$$\begin{aligned}
&\rho(k) = \mathrm{diag}(1,\ 1,\ -1,\ -1) \implies \rho(a^4k) \text{ does not have the eigenvalue } 1,\\
&\rho(k) = \mathrm{diag}(1,\ 1,\ -1,\ 1) \implies \rho(ak) \text{ or } \rho(abk) \text{ does not have the eigenvalue } 1,\\
&\rho(k) = \mathrm{diag}(1,\ 1,\ 1,\ -1)\colon \text{ to be investigated further in the following.}
\end{aligned}$$

We are left with excluding the case $\rho(k) = \mathrm{diag}(1, 1, 1, -1)$.

> Claim: Neither $\rho_1(a)$ nor $\rho_1'(a)$ are a primitive fourth root of unity.
>
> Proof of the Claim: Assume that $\rho_1'(a)$ was a primitive fourth root of unity. Then, after possibly replacing b by bk, we may assume that $\rho_1'(b) = 1$. We write
>
> $$\begin{aligned} a(z) &= (\rho_2(a)z' + a',\ z_3 + a_3,\ \rho_1'(a)z_4 + a_4),\\ b(z) &= (\rho_2(b)z' + b',\ z_3 + b_3,\ z_4 + b_4), \end{aligned}$$
>
> viewed as biholomorphic self-maps of A. The relation $b^{-1}ab = a^5$ implies that there is w' such that $(w',\ 4a_3,\ 0) = 0$ in A: but then the element a^4 does not act freely on A. This proves the Claim.

We may therefore assume that $\rho_1'(a), \rho_1'(b) \in \{\pm 1\}$. By replacing a by ak and b by bk if necessary, we may assume that $\rho_1'(a) = \rho_1'(b) = 1$. By Lemma 5.5 (b), a^4 does not act freely on A. Thus $M_{16} \times C_2$ does not occur as a group of a hyperelliptic fourfold.

To (2): We show that $M_{16} \times C_3$ (GAP ID [48,24]) does not occur, either: since any irreducible representation of M_{16} of dimension 2 contained in ρ maps a to a matrix of order 8, an element k of order 3 is mapped to the identity by ρ_2. By Remark 8.32, we may assume that $\rho_1'(a) \neq 1$. Lemma 8.33 (in which the element a was called g) implies that

$$A \cong (S \times E \times E')/H,$$

where $S \subset A$ is an Abelian surface, E, $E' \subset A$ is an elliptic curve and $H \subset S[2] \times E[4] \times E'[4]$. As usual, we write

$$\begin{aligned} a(z) &= (\rho_2(a)z' + a',\ z_3 + a_3,\ \rho_1'(a)z_4 + a_4),\\ b(z) &= (\rho_2(b)z' + b', z_3 + b_3,\ \rho_1'(b)z_4 + b_4),\\ k(z) &= (z' + k',\ z_3 + k_3,\ \zeta_3 z_4 + k_4). \end{aligned}$$

The condition that a and k commute implies that H contains an element v of the form $(w'\ , 0,\ (\zeta_3 - 1)a_4) \in H$. We obtain

$$(\rho(k) - \mathrm{id})v = (0,\ 0,\ (\zeta_3 - 1)^2 a_4) \in H.$$

Since E' embeds into A, we obtain $(\zeta_3 - 1)a_4$ is fixed by multiplication by ζ_3. Lemma 2.8 implies that $(\zeta_3 - 1)a_4$ is a 3-torsion element of E'. However, since the coordinates of H are 4-torsion elements, we obtain that $4(\zeta_3 - 1)a_4 = 0$ from $v \in H$. This implies that $a_4 = 0$.

Using this, the relation $b^{-1}ab = a^5$ yields that H contains an element of the form $(u',\ 4a_4,\ 0)$: this shows that a^4 does not act freely on A, and finally, that $M_{16} \times C_3$ does not occur.

8.2.3 Certain groups containing SD_8

In this section, we prove that

(1) $SD_8 \times C_2$ (GAP ID [32,40])

(2) $SD_8 \times C_3$ (GAP ID [48,26])

do not occur as groups of hyperelliptic fourfolds. Recall that SD_8 has the presentation

$$SD_8 := \langle a, b \,|\, a^8 = b^2 = 1, b^{-1}ab = a^3 \rangle.$$

By Proposition 8.6, we can assume that

$$A = (E_{\sqrt{2}i} \times E_{\sqrt{2}i} \times E \times E')/H$$

where each of the elliptic curves embeds into A, and that

$$\rho(a) = \begin{pmatrix} 0 & 1 & & \\ 1 & \sqrt{2}i & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & \sqrt{2}i & & \\ 0 & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

To (1): Let k be the element of order 2 which commutes with a and b. Since ρ is assumed to be faithful, and since k is central, we can assume that k is mapped to one of the following matrices via ρ:

$$\mathrm{diag}(1, 1, -1, 1), \quad \mathrm{diag}(1, 1, 1, -1), \quad \mathrm{diag}(1, 1, -1, -1).$$

If $\rho(k) = \mathrm{diag}(1, 1, -1, 1)$, the group $, bk\rangle$ is isomorphic to SD_8 and is excluded, since $\rho(abk)$ must have the eigenvalue 1 (see also Lemma 8.4). Now, if $\rho(k) = \mathrm{diag}(1, 1, 1, -1)$, by Lemma 5.5 (b), the subgroup $\langle a, bk\rangle$ does not act freely on A. Finally, in the case $\rho(k) = \mathrm{diag}(1, 1, -1, -1)$, the matrix $\rho(ak)$ does not have the eigenvalue 1.
This shows that there is no hyperelliptic fourfold with group $SD_8 \times C_2$.

To (2): Write k for the element of order 3 commuting with a and b. By Lemma 2.2, we may assume that k is mapped to one of the following matrices by ρ:

$$\mathrm{diag}(1, 1, 1, \zeta_3), \quad \mathrm{diag}(1, 1, \zeta_3, 1), \quad \mathrm{diag}(1, 1, \zeta_3, \zeta_3^j),\ j \neq 0.$$

We investigate first the case where $\rho(k) = \mathrm{diag}(1, 1, 1, \zeta_3)$. We write

$$a(z) = \rho(a)z + (a_1, a_2, a_3, a_4).$$

First of all, $a^8 = \mathrm{id}_A$ means that $v := (0,\ 0,\ 8a_3,\ 8a_4) \in H$. Now,

$$(\rho(b) - \mathrm{id})v = (0,\ 0,\ 0,\ 16a_4) \in H \overset{E' \subseteq A}{\Longrightarrow} 16a_4 = 0 \text{ in } E'.$$

The relation $ak = ka$ implies that H contains an element of the form

$$w := (w_1,\ w_2,\ 0,\ (\zeta_3 - 1)a_4).$$

Now, $(\rho(k) - \mathrm{id})w \in H$ implies that $(\zeta_3 - 1)a_4$ is fixed by multiplication by ζ_3. By Lemma 2.8, since a_4 is a 16-torsion element of E', we obtain that $a_4 = 0$. Now, the relation $b^{-1}ab = a^3$ shows that there are w_1', w_2' such that $(w_1',\ w_2',\ 2a_3,\ 0) = 0$ in A. Thus a^2 does not act freely on A.
Suppose now that $\rho(k) = \mathrm{diag}(1, 1, \zeta_3, 1)$: in this case, $\rho(abk)$ does not have the eigenvalue 1. Similarly, if $\rho(k) = \mathrm{diag}(1, 1, \zeta_3, \zeta_3^j)$, $j \neq 0$, the matrix $\rho(ak)$ does not have the eigenvalue 1. Hence $SD_8 \times C_3$ does not occur as a group of a hyperelliptic fourfold.

8.2.4 Certain groups containing $D_4 \times C_2$

In this section, we prove that there are no hyperelliptic fourfolds with one of the two following groups:

(1) $D_4 \times C_4$ (GAP ID [32,25])

(2) $D_4 \times C_6$ (GAP ID [48,45])

Recall that $D_4 \times C_2$ has the presentation

$$D_4 \times C_2 = \langle r, s, k \,|\, r^4 = s^2 = (rs)^2 = k^2 = [r, k] = [s, k] = 1\rangle.$$

By Proposition 8.10, we can assume that

$$A = (E \times E \times E' \times E'')/H,$$

where each of the three elliptic curves embeds into A, and that

$$\rho(r) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \rho(k) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

To (1): We will prove that $G = D_4 \times C_4$ does not occur as a group associated with a hyperelliptic fourfold. Since $D_4 \times C_4$ contains $D_4 \times C_2$, we can assume that $\rho(r),\ \rho(s)$ are given as above. Let ℓ be an element such that $\ell^2 = k$. After possibly replacing ℓ by ℓr^2 and afterwards ℓ by ℓ^3, we can assume that

$$\rho(\ell) = \mathrm{diag}(1,\ 1,\ -1,\ i) \text{ or } \rho(\ell) = \mathrm{diag}(1,\ 1,\ 1,\ i).$$

Since $\rho(r\ell)$ needs have the eigenvalue 1, the first of the listed possibilities is excluded, and we assume in the following that $\rho(\ell) = \mathrm{diag}(1, 1, 1, i)$. After a suitable change of coordinates,

$$\begin{aligned} r(z) &= (-z_2,\ z_1,\ z_3 + c_3,\ z_4 + c_4), \\ s(z) &= (z_1 + a_1,\ -z_2 + a_2,\ -z_3,\ -z_4 + a_4), \\ \ell(z) &= (z_1 + l_1,\ z_2 + l_2,\ z_3 + l_3,\ iz_4). \end{aligned}$$

Lemma 8.84. *The elements $2a_1$ and $2a_4$ are zero in E and E'', respectively. Moreover, $2l_1 \neq 0$.*

Proof. The condition that $s\ell = \ell s$ implies that

$$h := (0,\ 2l_2,\ 2l_3,\ (i-1)a_4) \in H.$$

Now,

$$(\rho(l) - \mathrm{id})h = (0,\ 0,\ 0,\ (i-1)^2 a_4) \in H \overset{E'' \subseteq A}{\Longrightarrow} (i-1)^2 a_4 = -2ia_4 = 0 \text{ in } E''$$

and therefore $2a_4 = 0$. The assertion $2a_1 = 0$ now follows from the property that s has order 2 and because E embeds into A.
The last statement $2l_1 \neq 0$, follows, because if $2l_1 = 0$, the condition that $h \in H$ shows that $\ell^2 = k$ does not act freely on A. $\square$

The relation $\ell r = r\ell$ implies that

$$h' := (l_1 + l_2,\ l_2 - l_1,\ 0, (i-1)c_4) \in H.$$

It follows from Lemma 8.8 that $2(l_1 + l_2) = 2(l_2 - l_1) = 0$, i.e., $2l_1 = 2l_2 \neq 0$ and $4l_1 = 4l_2 = 0$. Investigating the condition $r^2 l = lr^2$, we obtain that

$$h'' := (2l_1,\ 2l_2,\ 0,\ 2(i-1)c_4) \in H,$$

as well.

We are now in the situation to prove the non-existence of hyperelliptic fourfolds with group $D_4 \times C_4$:

- If $l_1 = l_2$, the element $\ell^2 = k$ does not act freely on A, because
$$h + h' = (2l_1,\ 2l_2,\ 2l_3,\ (i-1)(a_4 + c_4)) \in H.$$
- If $l_1 = -l_2$, we consider the element
$$h + h' + h'' = (2l_1,\ 2l_2,\ 2l_3,\ (i-1)(a_4 + 3c_4))$$
and observe that $\ell^2 = k$ does not act freely.
- We are henceforth left with the case where $l_1 \neq \pm l_2$. This means that H contains elements with first coordinates equal to $l_1 + l_2$, $2l_1$, which are different and non-zero 2-torsion elements. In other words, $E[2] \subset p_1(H)$, where $p_1 \colon E \times E \times E' \times E'' \to E$ is the projection onto the first factor. Consequently, since a_1 is 2-torsion (as shown above), H contains an element of the form $(a_1,\ w_2,\ w_3,\ w_4)$, proving that s does not act freely on A.

To (2): We prove that $D_4 \times C_6$ does not occur, either. Since $D_4 \times C_6$ contains $D_4 \times C_2$, we can assume that $\rho(r),\ \rho(s)$ are given as above. Let ℓ be an element such that $\ell^3 = k$. After possibly replacing ℓ by ℓr^2 and ℓ by ℓ^5, we can assume that

$$\rho(\ell) = \mathrm{diag}(1,\ 1,\ -1,\ \zeta_3) \text{ or } \rho(\ell) = \mathrm{diag}(1,\ 1,\ 1,\ \zeta_6).$$

Again, in the first case, $\rho(r\ell)$ does not have the eigenvalue 1, and therefore we may focus on the case where $\rho(\ell) = \mathrm{diag}(1,\ 1,\ 1,\ \zeta_6)$. After a suitable change of coordinates,

$$\begin{aligned} r(z) &= (-z_2,\ z_1,\ z_3 + c_3,\ z_4 + c_4), \\ s(z) &= (z_1 + a_1,\ -z_2 + a_2,\ -z_3,\ -z_4 + a_4), \\ \ell(z) &= (z_1 + l_1,\ z_2 + l_2, z_3 + l_3,\ \zeta_6 z_4). \end{aligned}$$

Similarly to case (1), we show

Lemma 8.85. *The elements $2a_1$ and a_4 are zero in E and E'', respectively. Moreover, $2l_1 \neq 0$.*

Proof. The condition that $s\ell = \ell s$ implies that

$$h := (0,\ 2l_2,\ 2l_3,\ (\zeta_6 - 1)a_4) \in H.$$

Now,

$$(\rho(\ell) - \mathrm{id})h = (0,\ 0,\ 0,\ (\zeta_6 - 1)^2 a_4) \in H \overset{E'' \subsetneq A}{\Longrightarrow} (\zeta_6 - 1)^2 a_4 = 0.$$

Thus $(\zeta_6 - 1)a_4$ is fixed by multiplication by ζ_6. Lemma 2.8 yields that $(\zeta_6 - 1)a_4 = 0$, and applying the cited lemma again, we obtain $a_4 = 0$. The rest of the proof is completely analogous to the one of Lemma 8.84, which is the similar statement in case (1). □

Again, the relation $r\ell = \ell r$ implies that

$$h' := (l_1 + l_2,\ l_2 - l_1,\ 0,\ (\zeta_6 - 1)a_4) \in H,$$

while the relation $r^2\ell = \ell r^2$ yields

$$h'' := (2l_1,\ 2l_2,\ 0,\ 2(\zeta_6 - 1)c_4) \in H.$$

Moreover, Lemma 8.8 implies that $2(l_1 + l_2) = 2(l_2 - l_1) = 0$, i.e., $2l_1 = 2l_2 \neq 0$ and $4l_1 = 4l_2 = 0$. The rest of the proof is completely similar to case (1):

- If $l_1 = l_2$, the element ℓ^2 does not act freely on A, since

$$h + h' = (2l_1,\ 2l_2,\ 2l_3,\ (\zeta_6 - 1)(a_4 + c_4)) \in H.$$

- If $l_1 = -l_2$, the element ℓ^2 does not act freely either, because

$$h + h' + h'' = (2l_1,\ 2l_2,\ 2l_3,\ (\zeta_6 - 1)(a_4 + 3c_4)) \in H.$$

- If $l_1 \neq \pm l_2$, we prove that s does not act freely on A by the similar argument as in Case (1).

8.2.5 Certain groups containing $Q_8 \times C_3$

The aim of this section is to prove that there do not exist hyperelliptic fourfolds whose group G is one of

(1) $Q_8 \rtimes C_9$ (GAP ID [72,3])

(2) $Q_8 \times C_3^2$ (GAP ID [72,38])

Both of these groups contain a subgroup isomorphic to

$$Q_8 \times C_3 = \langle a, b, k \mid a^4 = k^3 = 1, a^2 = b^2, ab = b^{-1}a, [a,k] = [b,k] = 1 \rangle.$$

By Proposition 8.16, we may assume that

$$\rho(a) = \begin{pmatrix} 1+2\zeta_3 & -1 & & \\ -2 & -1-2\zeta_3 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} -1 & \zeta_3^2 & & \\ -2\zeta_3 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$\rho(k) = \begin{pmatrix} \zeta_3 & & & \\ & \zeta_3 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

To (1): Let $C_8 \rtimes C_9$ be generated by a, b as above and ℓ such that $\ell^3 = k$. Since $\rho(k)$ has only two eigenvalues of order 3, we observe that $\rho(\ell)$ only has two eigenvalues of order 9, contradicting Lemma 2.2. Thus $C_8 \rtimes C_9$ does not occur.

To (2): Denote the generators of C_3^2 by k and k', where $\rho(k)$ is as above. Since $\rho(ak')$ must have the eigenvalue 1, we may assume that $\rho(k') = \mathrm{diag}(1,1,1,\zeta_3)$. The proof of Lemma 8.13 now implies that $\langle a, b, k' \rangle \cong Q_8 \times C_3$ does not act freely on A.

8.2.6 Certain groups containing $\mathrm{Heis}(3)$

This section is to prove that the following two groups do not occur as groups associated with hyperelliptic fourfolds:

(1) $\mathrm{Heis}(3) \rtimes C_2$ (GAP ID [54,8])

(2) $\mathrm{Heis}(3) \times C_2$ (GAP ID [54,10])

Recall that $\mathrm{Heis}(3)$ is presented as follows:

$$\mathrm{Heis}(3) = \langle g, h, k \mid g^3 = h^3 = k^3 = [g,k] = [h,k] = 1,\ ghg^{-1}h^{-1} = k \rangle.$$

According to Theorem 8.17, we can assume that

$$\rho(g) = \begin{pmatrix} 1 & & & \\ & 0 & 0 & 1 \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} \zeta_3^j & & & \\ & 1 & & \\ & & \zeta_3^2 & \\ & & & \zeta_3 \end{pmatrix}, \quad \rho(k) = \begin{pmatrix} 1 & & & \\ & \zeta_3 & & \\ & & \zeta_3 & \\ & & & \zeta_3 \end{pmatrix},$$

where $j \in \{1, 2\}$.

To (1): Suppose $\mathrm{Heis}(3) \rtimes C_2$ is generated by g, h, k as above and an element b of order 2, which acts on $\mathrm{Heis}(3)$ as follows:

$$b^{-1}gb = g^2, \quad b^{-1}hb = h^2, \quad [b,k] = 1.$$

This implies that the quotient of G by any normal subgroup of order 9 is non-Abelian. It follows that $[G,G] = \mathrm{Heis}(3)$, and thus g, h, k are mapped to 1 by any 1-dimensional representation. Lemma 8.19 now asserts that ρ is not a direct sum of a 3-dimensional irreducible and a 1-dimensional representation. Moreover, ρ cannot contain an irreducible representation of dimension 2, since $\rho|_{\mathrm{Heis}(3)}$ necessarily is the direct sum of irreducible representations of respective dimension 1 and 3.

To (2): Suppose $\mathrm{Heis}(3) \rtimes C_2$ is generated by g, h, k as above and an element a of order 2 commuting with g, h, k. Since ρ is the direct sum of a 1-dimensional and an irreducible representation of dimension 3, we the following possibilities for $\rho(a)$:

$$\begin{aligned} \rho(a) &= \mathrm{diag}(-1,\ 1,\ 1,\ 1) \implies \rho(ak) \text{ does not have the eigenvalue } 1, \\ \rho(a) &= \mathrm{diag}(1,\ -1,\ -1,\ -1) \implies \rho(ah) \text{ does not have the eigenvalue } 1. \end{aligned}$$

This proves that there is no hyperelliptic fourfold with group $\mathrm{Heis}(3) \times C_2$.

8.2.7 Certain groups containing $C_3 \rtimes C_8$

In this section, we prove that there are no hyperelliptic fourfolds whose group is isomorphic to one of

(1) $C_{24} \rtimes C_2$ (GAP ID [48,5])

(2) $(C_3 \rtimes C_8) \times C_2$ (GAP ID [48,9])

(3) $(C_3 \rtimes C_8) \rtimes C_2$ (GAP ID [48,10])

(4) $(C_3 \times C_3) \rtimes C_8$ (GAP ID [72,13])

Each of these groups contains the group

$$C_3 \rtimes C_8 = \langle a, b \,|\, a^3 = b^8 = 1, b^{-1}ab = a^2 \rangle$$

as a subgroup. In fact, if we denote the two generators of (1) of respective orders 24 and 2 by g and h, then $\langle g^8,\, g^3h \rangle \cong C_3 \rtimes C_8$. In (4), C_8 acts diagonally on $C_3 \times C_3$, so that we can find a subgroup isomorphic to $C_3 \rtimes C_8$.
According to Remark 8.60 and Lemma 8.61, we can assume that $\rho|_{C_3 \rtimes C_8}$ is given as follows:

$$\rho(a) = \begin{pmatrix} -1 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & \pm i & & \\ \pm i - 1 & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \tag{8.9}$$

To (1): Assume that we adjoin another element k of order 2 to $\langle a, b \rangle = C_3 \rtimes C_8$, which commutes with a and b. Since ρ is faithful, we can assume that $\rho(k)$ is given by one of the following matrices:

$\rho(k) = \mathrm{diag}(1,\ 1,\ -1,\ -1) \implies \rho(ak)$ does not have the eigenvalue 1.
$\rho(k) = \mathrm{diag}(1,\ 1,\ 1,\ -1) \implies \langle a, bk \rangle \cong C_3 \rtimes C_8$ does not act freely by Lemma 5.5 (b).
$\rho(k) = \mathrm{diag}(1,\ 1,\ -1,\ 1) \implies \rho(bk)$ does have the eigenvalue 1.

Thus there is no hyperelliptic fourfold in case (1).

To (2): First of all, since the group in discussion has a normal Abelian subgroup of order 24, its dimensions of irreducible representations are 1 and 2.
As we stated above, the element $a = g^8$ of order 3 is necessarily mapped to a matrix with eigenvalues ζ_3, ζ_3^2 by some irreducible representation of dimension 2 contained in ρ. Since the element g of order 24 must not be mapped to a matrix which has primitive 24-th roots of unity as eigenvalues (see Lemma 2.5 (a)), g is mapped to a matrix with eigenvalues $\zeta_3, \zeta_3^2, \zeta_8^c, \zeta_8^d$ for some c, d coprime to 8 (cf. Lemma 2.2).
Thus, $\rho(g)$ does not have the eigenvalue 1 and $C_{24} \rtimes C_2$ does not occur as a group of a hyperelliptic fourfold.

To (3): We prove that the group

$$(C_3 \rtimes C_8) \rtimes C_2 = \langle a, b, c \,|\, a^3 = b^8 = c^2 = 1, b^{-1}ab = a^2, c^{-1}bc = b^5, ac = ca \rangle$$

does not occur either. This follows, since the element ab^2c of order 12 is conjugate to its inverse,

$$b^{-1}(ab^2c)b = a^2b^2 \underbrace{(b^{-1}cb)}_{=b^4c^{-1}} = c^{-1}b^6a^2.$$

Hence, any faithful irreducible representation of $(C_3 \rtimes C_8) \rtimes C_2$ maps ab^2c to a matrix with conjugate eigenvalues, contradicting Lemma 2.2/Remark 2.3. There is still the possibility that ρ contains a non-faithful irreducible representation ρ_2 of dimension 2. However, then one of the following possibilities occurs:

- the restriction of ρ_2 to the subgroup $C_3 \rtimes C_8$ is non-faithful, which is excluded in view of the previous discussion.
- the kernel of ρ_2 contains c. In this case, c is mapped to -1 by at least one of the 1-dimensional representations contained in ρ (since we require ρ to be faithful). Then, in any case, we arrive at a contradiction:
  - if both the third and the fourth diagonal entry of $\rho(c)$ are -1, then $\rho(ac)$ does not have the eigenvalue 1.
  - if the third, but not the fourth diagonal entry of $\rho(c)$ is -1, then $\rho(bc)$ does not have the eigenvalue 1.
  - if the fourth, but not the third diagonal entry of $\rho(c)$ is -1, then $\langle a, bc \rangle$ is isomorphic to $C_3 \rtimes C_8$ and is excluded in view of Lemma 5.5.

To (4): As explained above, C_8 acts diagonally on $C_3 \times C_3$. Hence (4) contains four subgroups $U_1, ..., U_4$ isomorphic to $C_3 \rtimes C_8$ corresponding to the four subgroups of $C_3 \times C_3$ of order 3. The restriction of ρ to U_j must be a representation similar to (8.9), in particular, the elements a_j of order 3 of U_j must be mapped to a matrix with eigenvalues ζ_3, ζ_3^2 by the 2-dimensional irreducible representation contained in $\rho|_{U_j}$. However, since the a_j commute with each other, one of them is mapped to the identity by $\rho|_{U_j}$. This proves that (4) cannot occur.

8.2.8 The groups D_8 and Q_{16}

The dihedral group and the quaternion group of order 16 are defined by the following presentations:

$$\begin{aligned} D_8 &= \langle r, s \mid r^8 = s^2 = 1, s^{-1}rs = r^{-1} \rangle, \\ Q_{16} &= \langle a, b \mid a^8 = 1, x^4 = y^2, b^{-1}ab = a^{-1} \rangle. \end{aligned}$$

According to Lemma 5.5 (a), ρ contains an irreducible representation ρ_2 of dimension 2 and two 1-dimensional representations. The relations $s^{-1}rs = r^{-1}$ (resp. $b^{-1}ab = b^{-1}$) imply that r (resp. a) is mapped to ± 1 by any 1-dimensional representation of D_8 (resp. Q_{16}). Thus, $\rho_2(r)$ (resp. $\rho_2(a)$) must have 8-th roots of unity as eigenvalues. However, the relations $s^{-1}rs = r^{-1}$ (resp. $b^{-1}ab = b^{-1}$) show that the eigenvalues of $\rho_2(r)$ (resp. $\rho_2(a)$) are conjugate eigenvalues of order 8.
Thus, D_8 and Q_{16} do not occur as groups of hyperelliptic fourfolds.

8.2.9 The group $Q_8 \times C_2$ (GAP ID [16,12])

Consider the group

$$Q_8 \times C_2 = \langle a, b, k \mid a^4 = k^2 = [a, k] = [b, k] = 1,\ a^2 = b^2,\ ab = ba^{-1} \rangle.$$

According to Lemma 8.11, we may assume that

$$\rho(a) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} i & & & \\ & -i & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

After possibly replacing k by a^2k, one of the following possibilities occurs:

$$\begin{aligned} \rho(k) &= \mathrm{diag}(1, 1, -1, -1) \implies \rho(a^2k) \text{ does not have the eigenvalue } 1, \\ \rho(k) &= \mathrm{diag}(1, 1, \ \ 1, -1) \implies \langle ak, b \rangle \cong Q_8 \text{ does not act freely by Lemma 5.5 (b)}, \\ \rho(k) &= \mathrm{diag}(1, 1, -1, \ \ 1) \implies \rho(ak) \text{ does not have the eigenvalue } 1. \end{aligned}$$

This proves that $Q_8 \times C_2$ does not occur as a group of a hyperelliptic fourfold.

8.2.10 The group D_9 (GAP ID [18,1])

We prove that there is no hyperelliptic fourfold whose group is the dihedral group D_9 of order 18.
By Theorem 2.10, the irreducible representations of D_9 have dimensions 1 and 2.
Let ρ_2 be an irreducible representation of D_9 of dimension 2, and let r be the rotation of order 9. Then the relation $s^{-1}rs = r^{-1}$ implies that $\rho_2(r)$ and $\rho_2(r)^{-1}$ have the same eigenvalues. Lemma 2.2 implies that if ρ_2 is contained in ρ, then the eigenvalues of $\rho_2(r)$ are ζ_3 and ζ_3^2. Thus, if D_9 occurred as a group associated with a hyperelliptic fourfold, the representation ρ cannot be the direct sum of two irreducible representations of dimension 2. However, this means that r is mapped to primitive 9-th roots of unity by the 1-dimensional representations contained in ρ: this is easily seen not to be possible, e.g. by considering the relation $s^{-1}rs = r^{-1}$.
This proves that D_9 does not occur as a group of a hyperelliptic fourfold.

8.2.11 The group $(C_3 \times C_3) \rtimes C_2$ (GAP ID [18,4])

This section is dedicated to proving that there is no hyperelliptic fourfold with group

$$(C_3 \times C_3) \rtimes C_2 = \langle r_1, r_2, s \,|\, r_1^3 = r_2^3 = s^2 = [r_1, r_2] = (r_1 s)^2 = (r_2 s)^2 = 1 \rangle.$$

By Theorem 2.10, the dimensions of irreducible representations of this group are 1 and 2. Denote by ρ_2 an irreducible representation of dimension 2 of the above group. The relations $s^{-1} r_j s = r_j^{-1}$ ($j \in \{1, 2\}$) imply that

- $\rho_2(r_j)$ has the eigenvalues ζ_3, ζ_3^2, and
- either $r_1 \in \ker(\rho_2)$ or $r_2 \in \ker(\rho_2)$.

Since the derived subgroup of $(C_3 \times C_3) \rtimes C_2$ is $C_3 \times C_3$, the elements r_j are mapped to 1 by every 1-dimensional representation of $(C_3 \times C_3) \rtimes C_2$. This proves that ρ is the direct sum of two irreducible representations of dimension 2. But then, making use of the two bullet points above, we observe that one of the matrices $\rho(r_1)$, $\rho(r_2)$, $\rho(r_1 r_2)$ or $\rho(r_1 r_2^2)$ does not have the eigenvalue 1.
Consequently, the group in discussion does not occur as a group of a hyperelliptic fourfold.

8.2.12 The group $SL(2, 3)$ (GAP ID [24,3])

The group $\mathrm{SL}(2, 3)$ does not occur as a group of a hyperelliptic fourfold, since its derived subgroup is isomorphic to Q_8 and thus is mapped to 1 by any 1-dimensional representation of $\mathrm{SL}(2, 3)$. We conclude by Lemma 5.5 (b).

8.2.13 The group $C_3 \rtimes Q_8$ (GAP ID [24,4])

We prove that there is no hyperelliptic fourfold with group

$$G = C_3 \rtimes Q_8 = \langle a, b, c \,|\, a^4 = c^3 = 1,\ a^2 = b^2,\ ab = b^{-1} a,\ [c, b] = 1,\ a^{-1} ca = c^{-1} \rangle.$$

Since G contains the normal subgroup $\langle b, c \rangle \cong C_{12}$, Theorem 2.10 implies that the degrees of irreducible representations of G are 1 and 2. We prove the following claim without having to make use of a computer algebra system.

Claim: Every irreducible representation of dimension 2 of G maps the element bc of order 12 to a matrix without the eigenvalue 1.

Proof of the Claim: Let ρ_2 be an irreducible representation of dimension 2 of G. The relations $[b, c] = 1$, $a^{-1} ba = b^{-1}$ and $a^{-1} ca = c^{-1}$ now imply that $a^{-1}(bc)a = (bc)^{-1}$. Thus $\rho_2(bc)$ and $\rho_2(bc)^{-1}$ have the same eigenvalues. The only possibility for $\rho_2(bc)$ to have the eigenvalue 1 is thus if $\rho_2(bc)$

(a) only has the eigenvalue 1, or

(b) has the eigenvalues 1 and -1.

We first prove that option (a) is not possible. Since b resp. c commute and have respective orders 4 and 3, (a) is only possible of $\rho_2(b) = \rho_2(c) = \mathrm{diag}(1, 1)$. But then $\rho_2(a)$, $\rho_2(b)$ and $\rho_2(c)$ share a common eigenvector, which implies that ρ_2 is

not irreducible.
If (b) occurred, we again have $\rho_2(c) = \mathrm{diag}(1, 1)$. Thus, $\rho_2|_{Q_8}$ is a 2-dimensional representation of Q_8, which maps the matrix b to a matrix with eigenvalues of order ≤ 2. Thus, $\rho_2|_{Q_8}$ is irreducible, proving that $\rho|_{Q_8}$ is a direct sum of four 1-dimensional irreducible representations. This proves the claim.

The claim immediately implies that ρ is not the direct sum of two irreducible representations of dimension 2 (else, $\rho(bc)$ would not have the eigenvalue 1).
Hence, ρ is the direct sum of two 1-dimensional and one 2-dimensional irreducible representation of G. Since the derived subgroup of G is $\langle c, b^2 \rangle = C_3 \times Z(Q_8) \cong C_6$, every 1-dimensional representation of G maps the element bc of order 12 to 1 or -1. It follows that the dimension 2 irreducible summand of ρ is faithful. The relation $a^{-1}(bc)a = (bc)^{-1}$ now implies that the element bc of order 12 is mapped to a matrix with conjugate eigenvalues of order 12 by the dimension 2 irreducible summand of ρ. This leads to the desired contradiction (see Lemma 2.2).

8.2.14 The group D_{12} (GAP ID [24,6])

We exclude D_{12} as follows. The group D_{12} has a normal cyclic subgroup of order 12, spanned by the rotation r. Denote by s the symmetry of D_{12}. The relation $s^{-1}rs = r^{-1}$ implies that $\rho(r)$ has conjugate eigenvalues of order 12, unless ρ contains a non-faithful 2-dimensional irreducible representation of D_{12}. However, the relation $s^{-1}rs = r^{-1}$ implies that r^2 is contained in the derived subgroup of D_{12}, so that r is mapped to ± 1 by any 1-dimensional representation of D_{12}. Hence, in order for $\rho(r)$ to have order 12, the matrix $\rho(r)$ must have eigenvalues of order 12. We have already seen above that this cannot be the case.

8.2.15 The group $\mathrm{Dic}_{12} \times C_2$ (GAP ID [24,7])

The group

$$\mathrm{Dic}_{12} \times C_2 = \langle a, b, k \mid a^6 = k^2 = [a, k] = [b, k] = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle$$

is excluded in the following way.
The relations $[b, k] = 1$ and $b^{-1}ab = a^{-1}$ imply that b^2 is contained in the center of $\mathrm{Dic}_{12} \times C_2$. Hence, the element b^2 of order 2 is mapped to $\mathrm{diag}(-1, -1)$ or $\mathrm{diag}(1, 1)$ by any 2-dimensional irreducible representation contained in ρ. Since $\langle b^2, k \rangle$ is a central subgroup isomorphic to the Klein four group, ρ is the direct sum of an irreducible representation of dimension 2 and two 1-dimensional representations,

$$\rho = \rho_2 \oplus \rho_1 \oplus \rho_1'.$$

By Lemma 5.5 (b), at least one of ρ_1, ρ_1' is non-trivial. If both of them were non-trivial, one of them would map b to 1 (else, $\rho(b)$ would not have the eigenvalue 1), say $\rho_1(b) = 1$. By the same reasoning, $\rho_1'(a) = 1$, and we obtain that $\rho_1'(a) \neq 1$, $\rho_1(b) \neq 1$. Then the matrix $\rho(ab)$ does not have the eigenvalue 1, since $\rho_2(ab)$ is a matrix of order 4.
Thus, w.l.o.g. ρ_1 is trivial. Now we take the central element k into account: since ρ is required to be faithful, we may assume that k is mapped to one of the following matrices:

$$\rho(k) = \mathrm{diag}(1,\ 1,\ -1,\ -1) \implies \rho(b^2k) \text{ does not have the eigenvalue 1},$$
$$\rho(k) = \mathrm{diag}(1,\ 1,\ -1,\ 1) \implies \rho(bk) \text{ does not have the eigenvalue 1},$$
$$\rho(k) = \mathrm{diag}(1,\ 1,\ 1,\ -1)\colon \text{ to be investigated in the following.}$$

Assume from now on that $\rho(k) = \mathrm{diag}(1,1,1,-1)$ and that $\rho_2(b) = \mathrm{diag}(-1,-1)$.

> <u>Claim:</u> $\rho_1'(b) \in \{1,-1\}$.
>
> <u>Proof of the Claim:</u> Assume the contrary, i.e., that $\rho_1'(b)$ is a primitive fourth root of unity. The relation $a^3 = b^2$ yields $\rho_1'(a) = -1$.
> The Abelian variety A splits according to the decomposition $\rho = \rho_2 \oplus \rho_1 \oplus \rho_1'$; more precisely,
>
> $$A \cong (S \times E \times E')/H,$$
>
> where $S \subset A$ is an Abelian surface and E, $E' \subset A$ are elliptic curves. Let us write
>
> $$\begin{aligned} a(z) &= (\rho_2(a)z' + a',\ z_3 + a_3,\ -z_4 + a_4), \\ b(z) &= (\rho_2(b)z' + b',\ z_3 + b_3,\ \pm iz_4 + b_4). \end{aligned}$$
>
> The relation $b^{-1}ab = a^{-1}$ implies that H contains an element of the form
>
> $$w := (w',\ 4a_3,\ w_4).$$
>
> Then also the element $(\mathrm{id} - \rho(k))w = (0,0,0,2w_4)$ is contained in H. Since E' embeds into A, we obtain that $2w_4 = 0$.
> Moreover, the relation $a^6 = 1$ implies that H contains $(0,0,6a_3,0)$. Since E embeds into A, we obtain $6a_3 = 0$.
> In total, these arguments show that H contains an element of the form $(w',\ 2a_3,\ 0)$, proving in fact that a^2 does not act freely on A.

The Claim implies that $\rho_1'(b) = -1$ and thus $\rho_1'(a) = 1$. The subgroup $\langle a, bk\rangle$ of $\mathrm{Dic}_{12} \times C_2$ is isomorphic to Dic_{12}, and does not act freely on any Abelian fourfold by Lemma 5.5 (b). This proves that $\mathrm{Dic}_{12} \times C_2$ does not occur as a group of a hyperelliptic fourfold.

8.2.16 The group $S_3 \times C_2^2$ (GAP ID [24,14])

In this section, we show that there is no hyperelliptic fourfold whose group is $S_3 \times C_2^2$.

Recall that the group S_3 has the presentation

$$S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = (\sigma\tau)^2 = 1 \rangle.$$

By Corollary 8.67, up to a change of basis

$$\rho(\tau) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} \zeta_3 & 0 & & \\ 0 & \zeta_3^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Moreover, by Lemma 8.68, we may write

$$A = (S \times E_1 \times E_2)/H,$$

where S is an Abelian surface and E_1, E_2 are elliptic curves.

Now we prove that $G := S_3 \times C_2^2$ does indeed not occur. Necessarily, an element an element $\tilde{\kappa}$ which is mapped to one of

$$\mathrm{diag}(1,\ 1,\ -1,\ -1) \text{ or } \mathrm{diag}(-1,\ -1,\ 1,\ -1) \text{ or } \mathrm{diag}(-1,\ -1,\ -1,\ 1)$$

by ρ is contained in G. In the first case, $\rho(\sigma\tilde{\kappa})$ does not have the eigenvalue 1. If $\rho(\tilde{\kappa}) = \mathrm{diag}(-1,-1,1,-1)$, Corollary 8.66 shows that $\tau\tilde{\kappa}$ does not act freely on A. Finally, in the last case, the subgroup $\langle \tau\tilde{\kappa}, \sigma\rangle$ of G is isomorphic to S_3 and, by Lemma 5.5, does not act freely on A.

8.2.17 The group $(C_8 \times C_2) \rtimes C_2$ (GAP ID [32,5])

The above group has the presentation

$$G := (C_8 \times C_2) \rtimes C_2 = \langle a, b, c \,|\, a^8 = b^2 = c^2 = [a,b] = 1, c^{-1}ac = ab\rangle.$$

The 2-dimensional irreducible representations of G which map a to a matrix of order 8 are given by

$$a \mapsto \begin{pmatrix} 0 & \pm i \\ 1 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

The other 2-dimensional representations of G map a to a matrix with eigenvalues of order ≤ 4. Since $C_4 \times C_2 = \langle a^4, b\rangle \subset Z(G)$, Lemma 6.3 is applicable, which shows that ρ splits into a direct sum of a 2-dimensional and two 1-dimensional representations. Thus, if one of these latter 2-dimensional representations of G was contained in ρ, the element a must be mapped to primitive 8-th roots of unity by both 1-dimensional representations contained in ρ: in this case, $\rho(ab)$ or $\rho(a^2b)$ does not have the eigenvalue 1. This proves that ρ necessarily contains one of the irreducible representations listed above.
The above representations have kernel generated by a^4b. In order for ρ to be faithful, a must be mapped to a primitive 8-th root of unity by exactly one of the irreducible representations of G, so that $\rho(a)$ has the eigenvalue 1. In this case, we obtain a contradiction to Lemma 2.2.

8.2.18 The group $(C_8 \times C_2) \rtimes C_2$ (GAP ID [32,9])

The group in the title of this section has the presentation

$$G := (C_8 \times C_2) \rtimes C_2 = \langle a, b, c \,|\, a^8 = b^2 = c^2 = [a,b] = 1, c^{-1}ac = a^3b\rangle.$$

The irreducible G-representations of degree 2, which map a to a matrix of order 8 are given by

$$a \mapsto \pm \begin{pmatrix} \zeta_8 & \\ & \zeta_8^3 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$a \mapsto \pm \begin{pmatrix} -\zeta_8 & \\ & \zeta_8^3 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Since $C_2 \times C_2 = \langle a^2, b\rangle \subset Z(G)$, Lemma 6.3 tells us that ρ is the direct sum of an irreducible representation ρ_2 of dimension 2 and two 1-dimensional representations. We can show exactly as in the previous Section 8.2.17, that ρ_2 necessarily maps a to a matrix with eigenvalues of order 8. Thus ρ_2 is one of the representations listed above.
Only the representations in the first row of the above list can potentially occur in ρ, since ones in the second row map a to an element of order 8 with conjugate eigenvalues, which is impossible by Lemma 2.2.
By replacing a by a^5 and b by a^4b if necessary, we can assume that ρ_2 is given by

$$a \mapsto \begin{pmatrix} \zeta_8 & \\ & \zeta_8^3 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The kernel of the above representation is generated by b, which implies that b must be mapped to -1 by (at least) one representation of dimension 1 occurring in ρ. We may assume that

$$\rho(b) = \operatorname{diag}(1,\ 1,\ 1,\ -1),$$

because $\rho(a^4b)$ must have the eigenvalue 1.
The relation $c^{-1}ac = a^3b$ implies that a^2b lies in the derived subgroup of G, and is thus mapped to 1 by every 1-dimensional representation of G. Since $\rho(a)$ must have the eigenvalue 1, this allows us to write

$$\rho(a) = \begin{pmatrix} \zeta_8 & & & \\ & \zeta_8^3 & & \\ & & 1 & \\ & & & i \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \alpha & \\ & & & \beta \end{pmatrix}, \alpha, \beta \in \{\pm 1\}.$$

Since $\rho(ac)$ must have the eigenvalue 1, it follows that $\alpha = 1$. Furthermore, by replacing c by bc if necessary, we can assume that $\beta = 1$.
Let us write

$$\begin{aligned} a(z) &= \rho(a)z + (a_1, a_2, a_3, a_4), \\ b(z) &= \rho(b)z + (b_1, b_2, b_3, b_4), \\ c(z) &= \rho(c)z + (c_1, c_2, c_3, c_4). \end{aligned}$$

The relation $b^2 = \mathrm{id}_A$ implies that $u := (2b_1, 2b_2, 2b_3, 0) = 0$ in A, while the relation $c^{-1}ac = a^3b$ implies the existence of w_1, w_2, w_4 such that

$$w := (w_1,\ w_2,\ 2a_3 + b_1,\ w_4) = 0 \text{ in } A.$$

Since $\rho(b) = \operatorname{diag}(1,1,1,-1)$, we obtain that $(\rho(b) - \mathrm{id})w_4 = (0,0,0,2w_4) = 0$ in A as well. Now,

$$2w - u = (2w_1 - 2b_1,\ 2w_3 - 2b_2,\ 4a_3,\ 0) = 0 \text{ in } A$$

shows that a^4 does not act freely on A, because $\rho(a^4) = \operatorname{diag}(-1,-1,1,1)$.
This discussion shows that $(C_8 \times C_2) \rtimes C_2$ does not occur as a group of a hyperelliptic fourfold.

8.2.19 The group $C_4 \rtimes C_8$ (GAP ID [32,12])

The group with the GAP ID [32,12] has the presentation

$$G := C_4 \rtimes C_8 = \langle a, b \,|\, a^4 = b^8 = 1, b^{-1}ab = a^3 \rangle.$$

First of all, we observe that $C_4 \times C_2 \cong \langle b^2, a^2 \rangle \subset Z(G)$. Thus, Lemma 6.3 tells us that ρ is a direct sum of an irreducible representation ρ_2 of dimension 2 and two 1-dimensional ones. The group G has the following two irreducible representations of dimension 2, which map b to a matrix with eigenvalues of order 8:

$$a \mapsto \begin{pmatrix} i & \\ & -i \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & \pm i \\ 1 & 0 \end{pmatrix}.$$

The other irreducible representations of dimension 2 of G map b to matrices with eigenvalues of order ≤ 4; these latter ones cannot occur, since it would imply that

- if a is mapped to a matrix with eigenvalues of order ≤ 2, then $\rho(ab^2)$ or $\rho(a^2b)$ does not have the eigenvalue 1,
- if a is mapped to a matrix with eigenvalues of order 4 (which are then necessarily i and $-i$, in view of the relation $b^{-1}ab = a^3$), then $\rho(ab^2)$ does not have the eigenvalue 1.

Hence, it remains to exclude the two representations listed above. The kernel of these is generated by a^2b^4, and the relation $b^{-1}ab = a^3$ implies that $a^2 \in [G,G]$. Consequently, in order for ρ to be faithful and since $\rho(b)$ must have the eigenvalue 1, the element b must be mapped to a primitive 8-th root of unity by exactly one of the two 1-dimensional representations contained in ρ. This contradicts Lemma 2.2.
We have therefore proved that $C_4 \rtimes C_8$ does not occur as a group associated with a hyperelliptic fourfold.

8.2.20 The group $C_8 \rtimes C_4$ (GAP ID [32,13])

We will now prove that the group G with the presentation

$$G := C_8 \rtimes C_4 = \langle a, b \,|\, a^8 = b^4 = 1, b^{-1}ab = a^3 \rangle.$$

does not occur as a group of a hyperelliptic fourfold.
Since $C_2 \times C_2 = \langle a^4, b^2 \rangle \subset Z(G)$, Lemma 6.3 is applicable, which shows that ρ splits into a direct sum of a 2-dimensional and two 1-dimensional representations.
Checking with GAP, we observe that the 2-dimensional irreducible representations of G which map a to a matrix of order 8 are given by

$$a \mapsto \pm \begin{pmatrix} \zeta_8 & \\ & \zeta_8^3 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \tag{8.10}$$

The other irreducible representations of dimension 2 of G map a to a matrix with eigenvalues of order ≤ 4. These cannot be contained in ρ, since the relation $b^{-1}ab = a^3$ implies that $a^2 \in [G,G]$, so that a must be mapped to a matrix with eigenvalues of order 8 by ρ_2 (a commutator lies in the kernel of any 1-dimensional representation, hence any 1-dimensional representation maps a to ± 1).

Now, since the kernels of the representations listed in (8.10) are generated by a^4b^2 and since we require ρ to be faithful, the two 1-dimensional representations occurring in ρ map b to 1 and i, respectively. Hence, up to group automorphisms, we may assume that

$$\rho(a) = \begin{pmatrix} \zeta_8 & & & \\ & \zeta_8^3 & & \\ & & \alpha & \\ & & & \beta \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & i \end{pmatrix},$$

where $\alpha, \beta \in \{\pm 1\}$ and at least one of them is equal to 1. If $\alpha = -1$, then $\beta = 1$ and the matrix $\rho(ab^2)$ does not have the eigenvalue 1, which means that $\alpha = 1$. By replacing a by ab^2 if necessary, we can assume that $\beta = -1$.
Let us now write

$$\begin{aligned} a(z) &= \rho(a)z + (a_1, a_2, a_3, a_4), \\ b(z) &= \rho(b)z + (b_1, b_2, b_3, b_4). \end{aligned}$$

The relation $b^{-1}ab = a^3$ implies that there are w_1, w_2, w_4 such that

$$w := (w_1,\ w_2,\ 2a_3,\ w_4) = 0 \text{ in } A.$$

We notice that $(\rho(a^4b^2) - \mathrm{id})w = (0,\ 0,\ 0,\ 2w_4) = 0$ in A as well, and thus

$$(2w_1,\ 2w_2,\ 4a_3,\ 0) = 0 \text{ in } A.$$

This shows that a^4 does not act freely on A, since a^4 is of the form

$$z \mapsto (-z_1 + u_1,\ -z_2 + u_2,\ z_3 + 4a_3,\ z_4).$$

Consequently, there does not exist any hyperelliptic fourfold with group $C_8 \rtimes C_4$.

8.2.21 The group $C_8 \rtimes C_4$ (GAP ID [32,14])

This group is presented by

$$C_8 \rtimes C_4 = \langle a, b \,|\, a^8 = b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

By Lemma 5.5 (a), ρ is a direct sum of an irreducible representation of dimension 2 and two 1-dimensional representations. It is apparent from the presentation that $a^2 \in [G, G]$, thus the element a must be mapped to an element of order exactly 8 by the irreducible representation of dimension 2 contained in ρ. The relation $b^{-1}ab = a^{-1}$ then gives a contradiction to Lemma 2.2, since $\rho(a)$ and $\rho(a^{-1})$ have the same eigenvalues.

8.2.22 The group $C_9 \rtimes C_4$ (GAP ID [36,1])

According to Theorem 2.10, possible dimensions of irreducible representations of $C_9 \rtimes C_4$ are 1, 2 and 4. Moreover, since $C_2 \subset C_4$ acts trivially on C_9, we obtain that $C_9 \times C_2 = C_{18}$ is a normal subgroup of $C_9 \rtimes C_4$, and thus $C_9 \rtimes C_4$ only has irreducible representations of dimensions 1 and 2.
The group $C_9 \rtimes C_4$ has a presentation of the form

$$C_9 \rtimes C_4 = \langle a, b \,|\, a^9 = b^4 = 1,\ b^{-1}ab = a^{\ell} \rangle$$

for some ℓ. We show first that ρ is not a direct sum of two irreducible representations of dimension 2. Let ρ_2 be such a representation. Then, since $\rho_2(a)$ and $\rho_2(a)^\ell$ have the same eigenvalues, we obtain that these are either both primitive 9-th roots of unity or both 1. In the latter case, ρ_2 would not be irreducible, since $\rho_2(a)$ and $\rho_2(b)$ would share a common eigenvector. Thus, if ρ were the direct sum of two irreducible representations of dimension 2, the matrix $\rho(a)$ would not have the eigenvalue 1.
Thus, ρ is the sum of an irreducible representation of dimension 2 and two 1-dimensional representations. The relation $b^{-1}ab = a^\ell$ implies now that a is not mapped to a primitive a-th root of unity by any 1-dimensional representation of $C_9 \rtimes C_4$. Thus, $\rho(a)$ has at most two eigenvalues of order 9, contradicting Lemma 2.2.
This discussion shows that $C_9 \rtimes C_4$ does not occur as a group associated with a hyperelliptic fourfold.

8.2.23 The group $(C_2 \times C_2) \rtimes C_9$ (GAP ID [36,3])

We prove that the group

$$(C_2 \times C_2) \rtimes C_9 = \langle a_1, a_2, b \,|\, a_1^2 = a_2^2 = b^9 = [a_1, a_2] = 1,\ b^{-1}a_1b = a_2,\ b^{-1}a_2b = a_1a_2\rangle$$

mentioned in the title of the section does not occur as a group of a hyperelliptic fourfold. It is easy to see that the non-faithful 3-dimensional irreducible representation cannot occur, since it maps elements of order 9 to matrices of order 3. Hence a faithful 3-dimensional irreducible representation ρ_3 is contained in ρ. There exist elements $g, h \in (C_2 \times C_2) \rtimes C_9$ of respective orders 6 and 9 such that

$$\rho_3(g) = \begin{pmatrix} -\zeta_3 & & \\ & \zeta_3 & \\ & & -\zeta_3 \end{pmatrix}, \quad \rho_3(h) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta_3^2 & 0 & 0 \end{pmatrix}.$$

Hence, the 1-dimensional representation contained in ρ maps g and h to 1. Write the action of g, h on A as follows:

$$\begin{aligned} g(z) &= (z_1 + a_1,\ -\zeta_3 z_2 + a_2,\ \zeta_3 z_3 + a_3,\ -\zeta_3 z_4 + a_4), \\ h(z) &= (z_1 + b_1,\ z_3 + b_2,\ z_4 + b_3,\ \zeta_3^2 z_2 + b_4). \end{aligned}$$

One calculates that $\mathrm{ord}(gh) = 9$. We observe:

(a) $g^6 = \mathrm{id}_A \iff (6a_1,\ 0,\ 0,\ 0) = 0$, and

(b) $h^9 = \mathrm{id}_A \iff (9b_1,\ 0,\ 0,\ 0) =$, and

(c) $(gh)^9 = \mathrm{id}_A \iff (9(a_1 + b_1),\ 0,\ 0,\ 0) = 0$.

In total, $(3a_1,\ 0,\ 0,\ 0) \in H$. This proves that g^3 does not act freely on A, since $\rho_3(g^3)$ does not have the eigenvalue 1. Consequently, the group in discussion does not occur.

8.2.24 The group $(C_3 \times C_3) \rtimes C_4$ (GAP ID [36,7])

This section shows that

$$\left\langle \begin{matrix} a_1, a_2, \\ b \end{matrix} \,\middle|\, \begin{matrix} \sigma^3 = a_1^6 = a_2^6 = [a_1, a_2] = b^4 = 1, \\ a_1^3 = a_2^3 = b^2,\ b^{-1}a_1b = a_1^{-1},\ b^{-1}a_2b = a_2^{-1} \end{matrix} \right\rangle$$

does not occur as a group of a hyperelliptic fourfold. Fix a 2-dimensional irreducible representation ρ_2 of the above group.
The relations $b^{-1}a_jb = a_j^{-1}$ imply that $b^{-1}a_j^2b = a_j^{-2}$, so that $\rho_2(a_j^2)$ and $\rho_2(a_j)^{-3}$ have the same eigenvalues. Since $\mathrm{ord}(a_j) = 3$, the set of eigenvalues of $\rho_2(a_j)$ is either $\{1\}$ (with multiplicity 2) or $\{\zeta_3, \zeta_3^2\}$. Since a_1 and a_2 commute, they can be simultaneously diagonalized, and a suitable product of a_1 and a_2 is mapped to $\mathrm{diag}(1,1)$ by ρ_2. Thus, ρ is not equal to $\rho_2 \oplus \rho_2$ for any 2-dimensional irreducible representation of the group $(C_3 \times C_3) \rtimes C_4$. If ρ was the direct sum of two non-equivalent irreducible representations of dimension 2, we can find elements a_1', a_2' of order 3, which span different subgroups, and are contained in the kernel of the respective representation. In this case, $\rho(a_1'a_2')$ does not have the eigenvalue 1.
Hence, ρ is the direct sum of two 1-dimensional and one 2-dimensional irreducible representation of $(C_3 \times C_3) \rtimes C_4$. As described above, the kernel of every 2-dimensional irreducible representation of $(C_3 \times C_3) \rtimes C_4$ intersects its derived subgroup $C_3 \times C_3$ non-trivially. We have shown that the group in discussion cannot occur.

8.2.25 The group $S_3 \times S_3$ (GAP ID [36,10])

We prove that $S_3 \times S_3$ does not occur. In fact, the representation ρ cannot be equal to the 4-dimensional irreducible representation of $S_3 \times S_3$ (which is the external tensor product of the unique irreducible representation of S_3 of dimension 2 with itself), since it maps a tuple of 3-cycles to a matrix without the eigenvalue 1.
We now prove that ρ is not the direct sum of two irreducible representations of degree 2. Fix a 2-dimensional irreducible representation ρ_2 of $S_3 \times S_3$. The derived subgroup of $S_3 \times S_3$ is spanned by (σ, id) and (id, σ), where σ is of order 3. Since the 2-dimensional irreducible representations of $S_3 \times S_3$ are obtained as tensor products of 2-dimensional irreducible representations of S_3 with 1-dimensional representations of S_3, there is an element of order 3, which is mapped to the identity matrix by ρ_2. Thus, ρ cannot be equal to $\rho_2 \oplus \rho_2$. Moreover, this proves that if ρ were the direct sum of two non-equivalent irreducible representations of dimension 2, we could find $\sigma_1, \sigma_2 \in S_3 \times S_3$ of order 3 such that $\rho(\sigma_1\sigma_2)$ does not have the eigenvalue 1.
It follows from the previous discussion that ρ cannot be the direct sum of three irreducible representations of respective dimensions 2, 1, 1 either, since the derived subgroup of $S_3 \times S_3$ equals $A_3 \times A_3$.

8.2.26 The group $C_3 \times (D_4 \curlyvee C_4)$ (GAP ID [48,47])

This section is dedicated to proving that $C_3 \times (D_4 \curlyvee C_4)$ does not occur as a group of a hyperelliptic fourfold. Here, $D_4 \curlyvee C_4$ is the central product of D_4 and C_4 with respect to the central subgroup of order 2 of D_4,

$$D_4 \curlyvee C_4 = \langle r, s, k \,|\, r^4 = s^2 = k^4 = [r,k] = [s,k] = (rs)^2 = 1, r^2 = k^2 \rangle.$$

It follows from the above presentation that the group $G := C_3 \times (D_4 \curlyvee C_4)$ has a central element ℓ (the product of k with an element g of order 3) of order 12. The element ℓ must be mapped to a matrix of order 4 by any 2-dimensional irreducible representation contained in ρ: indeed, since ℓ is central, Lemma 2.2 shows that ℓ cannot be mapped to a matrix with eigenvalues of order 12, whereas the relation that $k^2 = r^2$ shows that k^2 is mapped to $\mathrm{diag}(-1,-1)$ by such a 2-dimensional irreducible representation because r^2 is.

Since $\rho(\ell)$ must have the eigenvalue 1, we can therefore assume that the element g of order 3 is mapped to

$$\mathrm{diag}(1,\ 1,\ 1,\ \zeta_3)$$

by ρ. Since $\rho(kg)$ must not have eigenvalues of order 12 and $r^2 = k^2$, we may assume that

$$\rho(r) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix},$$

where we invoked Lemma 5.5 (b) to show that the fourth diagonal entry of $\rho(s)$ is -1. By Lemma 8.76, the Abelian variety A is isogenous to the product of four elliptic curves $E_j \subset A$,

$$A \cong (E_1 \times E_2 \times E_3 \times E_4)/H,$$

where $H \subset E_1[4] \times E_2[4] \times E_3[4] \times E_4[4]$ (note that in the proof Lemma 8.76 only uses elements contained in $D_4 = \langle r, s\rangle$). Now, let us write

$$\begin{aligned} s(z) &= (z_1 + b_1,\ -z_2 + b_2,\ z_3 + b_3,\ -z_4 + b_4), \\ r(z) &= (-z_2,\ z_1,\ z_3 + c_3,\ z_4 + c_4), \\ g(z) &= (z_1 + t_1,\ z_2 + z_2,\ z_3 + t_3,\ \zeta_3 z_4 + t_4). \end{aligned}$$

Since r and g commute, H contains an element w of the form $(w_1,\ w_2,\ 0,\ (\zeta_3 - 1)c_4)$. Now $(\rho(g) - \mathrm{id})w = (0, 0, 0, (\zeta_3 - 1)^2 c_4) \in H$. Since E_4 embeds into A, we obtain that $(\zeta_3 - 1)^2 c_4 = 0$ in E_4. Thus, $(\zeta_3 - 1)c_4$ is fixed by multiplication by ζ_3. However, since $(\zeta_3 - 1)c_4$ is 4-torsion, Lemma 2.8 yields that $(\zeta_3 - 1)c_4 = 0$. Applying the cited Lemma again, we obtain that c_4 is a 3-torsion element in E_4. However, the relation $r^4 = \mathrm{id}_A$ implies that $(0,\ 0,\ 4c_3,\ 4c_4) \in H$, which in turn implies that $16c_4 = 0$. In total, this shows that $c_4 = 0$.
We conclude by investigating the relations $s^2 = (rs)^2 = \mathrm{id}_A$: they yield elements $(2b_1,\ 0,\ 2b_3,\ 0)$ and $(u_1,\ u_2,\ 2b_3 + 2c_3,\ 0)$ contained in H. Thus, H contains an element whose last two coordinates are $2c_3$ and 0, respectively, which proves that r^2 does not act freely on A.

8.2.27 The group $C_8 \rtimes C_8$ (GAP ID [64,3])

We prove that the group

$$C_8 \rtimes C_8 = \langle a, b \,|\, a^8 = b^8 = 1, b^{-1}ab = a^5\rangle.$$

does not occur as a group of a hyperelliptic fourfold.

Its center $Z(G)$ contains (in fact, is generated by) a^2 and b^2 and thus contains a subgroup isomorphic to $C_4 \times C_4$. By Lemma 6.3, ρ is the direct sum of an irreducible representation ρ_2 of dimension 2 and two 1-dimensional representations.
Since ρ_2 is not faithful, the kernel of ρ_2 contains a central element g of order 4. We

choose an element $h \in G$ of order 8, such that $h^2 = g$ (that such an element h exists is seen using the relations of the group, which for instance imply $a^2b^2 = (a^3b)^2$). Thus, the element h must be mapped to primitive 8-th roots of unity by the 1-dimensional representations contained in ρ.
Now, choose a central element g' which is not contained in the kernel of ρ_2. Then $\rho(g'h)$ does not have the eigenvalue 1.
This proves that $C_8 \rtimes C_8$ does not occur as a group of a hyperelliptic fourfold.

8.2.28 The group $D_4 \times C_3^2$ (GAP ID [72,37])

We prove that $D_4 \times C_3^2$ does not occur as a group of a hyperelliptic fourfold. Denote by r resp. s the rotation of order 4 resp. the symmetry of order 2 of D_4. Since $\rho(r^2) = \mathrm{diag}(-1,-1,1,1)$, no central element of order 3 is mapped to a diagonal matrix whose last two diagonal entries are different from 1.
Thus, we may assume that the group contains central elements k_1, k_2 of order 3, such that

$$\begin{aligned}\rho(k_1) &= \mathrm{diag}(\zeta_3, \zeta_3, 1, 1),\\ \rho(k_2) &= \mathrm{diag}(1, 1, \zeta_3, 1).\end{aligned}$$

By Lemma 5.5 (b), and since all matrices in the image of ρ must have the eigenvalue 1, we may write

$$\rho(r) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & \alpha & \\ & & & 1 \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \beta & \\ & & & 1 \end{pmatrix},$$

where $(\alpha, \beta) \in \{(1,-1), (-1,1)\}$.

Now, the proof of Lemma 8.8 shows that

$$A \cong (S \times E \times E')/H,$$

where $S \subset A$ is an Abelian surface and E, $E' \subset A$ are elliptic curves.

Claim: $H \subset S[4] \times E[4] \times E'[4]$.

Proof of the Claim: Denote the lattices of A, S, E and E' by Λ, Λ_S, Λ_E, $\Lambda_{E'}$, respectively. The torsion group H is isomorphic to $\Lambda/(\Lambda_S \oplus \Lambda_E \oplus \Lambda_{E'})$. Let $\lambda \in \Lambda$. We write

$$2\lambda = \underbrace{(I + \rho(r^2))\lambda}_{\in \Lambda_S} + \underbrace{(I - \rho(r^2))\lambda}_{=:\lambda' \in \Lambda_E \oplus \Lambda_{E'}}.$$

If $\alpha = -1$, we observe that

$$2\lambda' = \underbrace{(I + \rho(r))\lambda'}_{\in \Lambda_E} + \underbrace{(I - \rho(r))\lambda'}_{\in \Lambda_{E'}},$$

while in the case $\beta = -1$, we obtain

$$2\lambda' = \underbrace{(I + \rho(s))\lambda'}_{\in \Lambda_E} + \underbrace{(I - \rho(s))\lambda'}_{\in \Lambda_{E'}}.$$

In any case, $4\lambda \in \Lambda_S \oplus \Lambda_E \oplus \Lambda_{E'}$. This proves the claim.

Let us now write the elements r and s as follows:

$$r(z) = \rho(r)z + (r', r_3, r_4),$$
$$s(z) = \rho(s)z + (s', s_3, s_4),$$
$$k_2(z) = \rho(k_2)z + (\ell', \ell_3, \ell_4)$$

The relation $r^2k_2 = k_2r^2$ implies that $v := (2\ell',\ 2(\zeta_3 - 1)r_3,\ 0) \in H$. This implies that

$$(\rho(k_2) - \mathrm{id})v = (0,\ 2(\zeta_3 - 1)^2 r_3,\ 0) \in H.$$

Since $E \subset A$, this implies that $2(\zeta_3 - 1)^2 r_3 = 0$ in E. By applying Lemma 2.8 twice and by the Claim, we obtain that $2r_3 = 0$. This proves that r^2 does not act freely on A, and thus $D_4 \times C_3^2$ does not occur.

8.2.29 The group $\mathrm{Dic}_{12} \times C_3^2$ (GAP ID [108,32])

We exclude the group $\mathrm{Dic}_{12} \times C_3^2$, where

$$\mathrm{Dic}_{12} = \langle a, b \,|\, a^6 = 1,\ a^3 = b^2,\ b^{-1}ab = a^{-1}\rangle$$

is the dicyclic group of order 12. The group $\mathrm{Dic}_{12} \times C_3^2$ has a central subgroup $\langle k_1, k_2\rangle$ isomorphic to C_3^2. Thus, ρ is the direct sum of three irreducible representations of respective dimensions 2, 1, 1,

$$\rho = \rho_2 \oplus \rho_1 \oplus \rho_1'.$$

Moreover, by Lemma 5.5 (b), at least one of the representations $\rho_1|_{\mathrm{Dic}_{12}}$, $\rho_1'|_{\mathrm{Dic}_{12}}$ is non-trivial. We can assume that ρ_1 is non-trivial. Then, by the relations of Dic_{12}, we obtain that $\rho_1(b) \neq 1$.
Since $\rho(b)$ must have the eigenvalue 1, we can assume that

$$\rho(k_1) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ 1,\ 1) \text{ and } \rho(k_2) = \mathrm{diag}(1,\ 1,\ \zeta_3,\ 1).$$

Lemma 2.2 now implies $\rho_1(b) = -1$ and thus $\rho_1(a) = 1$. Since $\rho(k_1k_2) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3,\ 1)$ acts on A, Proposition 2.4 implies that

$$A \cong (E_1 \times E_2 \times E_3 \times E_4)/H,$$

where E_1, E_2, $E_3 \subset A$ are copies of the equianharmonic elliptic curve and $E \subset A$ is another elliptic curve. We write $S = E_1 \times E_2$.

<ins>Claim:</ins> The torsion group H is contained in $S[4] \times E_3[4] \times E_4[4]$.

Proof of the Claim: Denote by Λ, Λ_S, Λ_3 and Λ_4 the lattices of A, S, E_3, E_4, respectively. The torsion group H is equal to $\Lambda/(\Lambda_S \oplus \Lambda_3 \oplus \Lambda_4)$. Let $\lambda \in \Lambda$. Then

$$2\lambda = \underbrace{(I + \rho(b^2))\lambda}_{\in \Lambda_S} + \underbrace{(I - \rho(b^2))\lambda}_{=:\lambda' \in \Lambda_3 \oplus \Lambda_4}.$$

Now,

$$2\lambda' = \underbrace{(I + \rho(b))\lambda}_{\in \Lambda_3} + \underbrace{(I - \rho(b))\lambda}_{\in \Lambda_4}.$$

This proves that $4\lambda \in \Lambda_S \oplus \Lambda_3 \oplus \Lambda_4$, which implies the Claim.

Let us now write

$$a(z) = \rho(a)z + (a_1, a_2, a_3, a_4),$$
$$b(z) = \rho(b)z + (b_1, b_2, b_3, b_3).$$

The relation $b^{-1}ab = a^{-1}$ implies that H contains an element of the form

$$v := (w_1,\ w_2,\ 0,\ 4a_4).$$

On the other hand, since k_2 and a commute, we obtain that H contains an element of the form

$$(w_1',\ w_2',\ (\zeta_3 - 1)a_3,\ 0).$$

The Claim implies now that $(\zeta_3 - 1)a_3$ is 4-torsion, or, equivalently, $4a_3$ is fixed by multiplication by ζ_3. By Lemma 2.8, we obtain that $4a_3 = 0$. Together with $v \in H$ this shows that a^4 does not act freely on A, and thus $\mathrm{Dic}_{12} \times C_3^2$ does not occur.

Chapter 9

Subgroups of G, Revisited

We summarize the results of the previous sections in the following tables.

2-Sylows	GAP ID	Excluded by/Shown to occur in...
D_8	$[16, 7]$	Excluded in Section 8.2.8
SD_8	$[16, 8]$	Shown to occur in Section 8.1.1
Q_{16}	$[16, 9]$	Excluded in Section 8.2.8
$D_4 \times C_2$	$[16, 11]$	Shown to occur in Section 8.1.2
$Q_8 \times C_2$	$[16, 12]$	Excluded in Section 8.2.9
C_2^4	$[16, 14]$	Excluded by Lemma 2.7
$C_4 \times C_8$	$[32, 3]$	Shown to occur in Lemma 3.11
$C_8 \rtimes C_4$	$[32, 4]$	Shown to occur in Section 8.1.5
$(C_8 \times C_2) \rtimes C_2$	$[32, 5]$	Excluded on p. 157
$(C_8 \times C_2) \rtimes C_2$	$[32, 9]$	Excluded on p. 157
$(C_4 \times C_4) \rtimes C_2$	$[32, 11]$	Shown to occur in Section 8.1.6
$C_4 \rtimes C_8$	$[32, 12]$	Excluded on p. 159
$C_8 \rtimes C_4$	$[32, 13]$	Excluded on p. 159
$C_8 \rtimes C_4$	$[32, 14]$	Excluded in Section 8.2.21
$C_2 \times C_4 \times C_4$	$[32, 21]$	Shown to occur in Example 3.16
$(C_4 \times C_4) \rtimes C_2$	$[32, 24]$	Shown to occur in Section 8.1.7
$D_4 \times C_4$	$[32, 25]$	Excluded on p. 147
$C_2 \times C_2 \times C_8$	$[32, 36]$	Excluded in Remark 3.14
$M_{16} \times C_2$	$[32, 37]$	Excluded on p. 144
$SD_8 \times C_2$	$[32, 40]$	Excluded on p. 146
$C_8 \rtimes C_8$	$[64, 3]$	Excluded on p. 163
C_4^3	$[64, 55]$	Excluded by Corollary 3.10

TABLE 1: References to proofs of (non-)existence of hyperelliptic fourfolds with the listed 2-groups.

An analysis of these 2-groups yields

Theorem 9.1. *Let $X = A/G$ be a hyperelliptic fourfold. Then the* 2*-Sylow subgroups of* G *are isomorphic to one of the following ones:*

Structure	*GAP ID*	*Structure*	*GAP ID*
$\{1\}$	*[1,1]*	D_4	*[8,3]*
C_2	*[2,1]*	Q_8	*[8,4]*
C_4	*[4,1]*	$(C_4 \times C_2) \rtimes C_2$	*[16,3]*
$C_2 \times C_2$	*[4,2]*	$C_4 \rtimes C_4$	*[16,4]*
C_8	*[8,1]*	M_{16}	*[16,6]*
$C_2 \times C_4$	*[8,2]*	SD_8	*[16,8]*
$C_2 \times C_2 \times C_2$	*[8,5]*	$D_4 \times C_2$	*[16,11]*
$C_4 \times C_4$	*[16,2]*	$Q_8 \curlyvee C_4$	*[16,13]*
$C_2 \times C_8$	*[16,5]*	$C_8 \rtimes C_4$	*[32,4]*
$C_2 \times C_2 \times C_4$	*[16,10]*	$(C_4 \times C_4) \rtimes C_2$	*[32,11]*
$C_4 \times C_8$	*[32,3]*	$(C_4 \times C_4) \rtimes C_2$	*[32,24]*
$C_2 \times C_4 \times C_4$	*[32,21]*		

TABLE 2: *All possible* 2*-Sylow groups of a group associated with a hyperelliptic fourfold.*

Conversely, for every group $G' \neq \{1\}$ *contained in the above table, there exists an Abelian variety* A *and an action of* G' *on* A *without translations such that* A/G' *is a hyperelliptic fourfold.*

Proof. Running GAP Script `2-groups_order_32_faithful.g` (cf. Chapter 12) gives only the group with GAP ID [32,11] as output, i.e., the only non-Abelian groups with cyclic center occurring as groups associated with hyperelliptic fourfolds are the ones with GAP IDs [8,3], [8,4], [16,6], [16,8], [16,13] and [32,11]. Script `2-groups_non_faithful.g` investigates the case where the group is non-Abelian with non-cyclic center: this script has as output only the groups with the IDs [16,3], [16,4], [16,11], [32,4] and [32,24]. The left table contains only Abelian groups, which we discussed in Section 3.1. □

As should be apparent by now, the situation for the 3-Sylow groups of G is considerably easier.

Theorem 9.2. *Let $X = A/G$ be a hyperelliptic fourfold. Then the* 3*-Sylow subgroups of* G *are isomorphic to one of the following ones:*

Structure	*GAP ID*
$\{1\}$	*[1,1]*
C_3	*[3,1]*
C_9	*[9,1]*
$C_3 \times C_3$	*[9,2]*
Heis(3)	*[27,3]*
$C_3 \times C_3 \times C_3$	*[27,5]*

Table 3: *All possible 3-Sylow groups of a group associated with a hyperelliptic fourfold.*

Conversely, for every group $G' \neq \{1\}$ contained in the above table, there exists an Abelian variety A and an action of G' on A without translations such that A/G' is a hyperelliptic fourfold.

Moreover, we have shown that the following groups of order $2^a \cdot 3^b$, $a, b \neq 0$ do **not** occur as groups of hyperelliptic fourfolds:

GAP ID	Excluded in...	GAP ID	Excluded in...
$[18,1]$	Section 8.2.10	$[48,30]$	Section 8.2.1
$[18,4]$	Section 8.2.11	$[48,45]$	Page 148
$[24,3]$	Section 8.2.12	$[48,47]$	Section 8.2.26
$[24,4]$	Section 8.2.13	$[54,8]$	Page 150
$[24,6]$	Section 8.2.14	$[54,10]$	Page 151
$[24,7]$	Section 8.2.15	$[72,3]$	Section 8.2.5
$[24,12]$	Page 141	$[72,13]$	Section 8.2.7
$[24,14]$	Corollary 8.2.16	$[72,14]$	Corollary 3.13
$[36,1]$	Section 8.2.22	$[72,27]$	Corollary 8.2.16
$[36,3]$	Section 8.2.23	$[72,37]$	Section 8.2.28
$[36,5]$	Corollary 3.13	$[72,38]$	Section 8.2.5
$[36,7]$	Section 8.2.24	$[96,46]$	Corollary 3.13
$[36,10]$	Section 8.2.25	$[96,47]$	Corollary 8.30
$[36,11]$	Page 140	$[96,164]$	Corollary 8.41
$[48,3]$	Page 142	$[108,32]$	Section 8.2.29
$[48,4]$	Corollary 8.2.16	$[108,35]$	Remark 3.14
$[48,5]$	Section 8.2.7	$[144,101]$	Corollary 3.13
$[48,9]$	Section8.2.7	$[144,102]$	Corollary 8.46
$[48,10]$	Section 8.2.7	$[144,103]$	Page 122
$[48,24]$	Section 8.2.2	$[216,139]$	Corollary 8.79
$[48,26]$	Page 146	$[216,177]$	Corollary 3.10

Table 4: This table constitutes a list of certain groups which do not occur as groups of hyperelliptic fourfolds. These groups form the list of 'forbidden groups' in GAP Script `maximal_groups.g` (cf. Chapter 12).

Chapter 10

The Cases where $5 \mid |G|$ or $7 \mid |G|$

In order to obtain a full classification of groups which can occur as groups associated with hyperelliptic fourfolds, we still have to investigate the cases in which 5 or 7 divide the group order. It turns out that if this is the case, then G is necessarily Abelian, i.e., we prove the following result:

Theorem 10.1. *Let A be an Abelian fourfold and let $G \subset \mathrm{Bihol}(A)$ be a subgroup containing no translations. If $|G| = 2^a \cdot 3^b \cdot 5^c \cdot 7^c$, where $(c,d) \neq (0,0)$ and G acts freely on A, then G is Abelian and isomorphic to one of the following groups:*

Isomorphism Type of G	*GAP ID*	*Isomorphism Type of G*	*GAP ID*
C_5	*[5,1]*	C_7	*[7,1]*
C_{10}	*[10,2]*	C_{14}	*[14,2]*
C_{15}	*[15,1]*		
C_{20}	*[20,2]*		
C_{30}	*[30,4]*		
$C_2 \times C_{10}$	*[20,5]*		
$C_2 \times C_{20}$	*[40,9]*		
$C_2 \times C_{30}$	*[60,13]*		

Conversely, if G' is a group isomorphic to one of the groups contained in the above table, there exists an Abelian fourfold A' and an embedding $G' \hookrightarrow \mathrm{Bihol}(A')$, such that the image of G' does not contain any translations and acts freely on A'.

10.1 Groups of order $3^b \cdot 5$

This section is dedicated to investigating groups of order $3^b \cdot 5^c$, $c \geq 1$, which are associated with hyperelliptic fourfolds. More precisely, we show that the only groups of order $3^b \cdot 5^c$ associated with hyperelliptic fourfolds are the cyclic groups of order 5 and 15. This is done as follows:
Lemma 2.13 implies that $c = 1$. Now, $b \leq 3$ by Theorem 9.2. Then Sylow's Theorems imply that the 5-Sylow is normal. Since all elements of the 3-Sylow subgroups centralize the elements of order 5, it follows that G is the direct product of its Sylow groups. Hence,

by using Theorem 9.2 again, we obtain that if $b \geq 1$, the group G contains a subgroup isomorphic to one of

$$C_{15}, \quad C_{45}, \quad C_3 \times C_{15}.$$

By Lemma 2.5, C_{45} does not occur, whereas by Lemma 3.13, the group $C_3 \times C_{15}$ does not occur. We have established the following

Proposition 10.2. *If $|G| = 3^b \cdot 5$, then $b \leq 1$ and G is the cyclic group of order 5 or 15.*

This leads immediately to the following

Corollary 10.3. *If G is a group of order $2^a \cdot 3^b \cdot 5$, then $b \leq 1$.*

Proof. We will use exact sequence (2.1)

$$1 \to K \to G \to C_m \to 1,$$

where 5 divides m by Lemma 2.13. Thus, the only primes possibly dividing $|K|$ are 2 and 3. We conclude that G is a solvable group, so that by Hall's Theorem (cf. [Ha59, Theorem 9.3.1]), G has a subgroup of order $3^b \cdot 5$. The assertion now follows from the previous Proposition. □

10.2 Groups of order $3^b \cdot 7$

Suppose that G has order $3^b \cdot 7$. In Section 7 it was proved that $b \leq 3$. The main result of this section is

Proposition 10.4. *If $|G| = 3^b \cdot 7$, then $b = 0$ and G is the cyclic group of order 7.*

First, we shall investigate the case $b = 1$, i.e., $|G| = 21$.

Lemma 10.5. *The group G cannot have a subgroup of order 21.*

Proof. Suppose that G is of order 21. We can then assume that G is the non-Abelian group of order 21, since the cyclic group of order 21 is excluded in view of Lemma 2.5. The unique non-Abelian group of order 21 has the presentation

$$G = C_7 \rtimes C_3 = \langle g, h \,|\, g^7 = h^3 = 1,\ h^{-1}gh = g^2 \rangle.$$

Since G is non-Abelian and contains a normal subgroup of order 7, the faithful representation ρ splits as a direct sum of a character and a 3-dimensional irreducible representation, $\rho = \rho_1 \oplus \rho_3$ (see Theorem 2.10). Moreover, because g has order 7, all the eigenvalues of $\rho_3(g)$ are primitive 7-th roots of unity, and hence $\rho_1(g) = 1$ (see Lemma 2.2). It follows from the relation $h^{-1}gh = g^2$ that $\rho_1(h) \neq 1$ (otherwise, g would not act freely on A by Lemma 5.5 (b)). Thus, after possibly replacing h by h^2, we can assume that $\rho_1(h) = \zeta_3$.
The exact sequence

$$1 \to K \to G \overset{\det(\rho_3(\cdot))}{\to} C_m \to 1,$$

implies that g is contained in the kernel K. Hence, by Lemma 2.12, we can assume that the representation ρ_3 is given by

$$g \mapsto \begin{pmatrix} \zeta_7 & & \\ & \zeta_7^2 & \\ & & \zeta_7^4 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This allows us to write

$$h(z) = (\zeta_3 z_1 + a_1,\ z_4 + a_2,\ z_2 + a_3,\ z_3 + a_4).$$

The condition that $h^3 = \mathrm{id}_A$ is then equivalent to

$$(0,\ a_2 + a_3 + a_4,\ a_2 + a_3 + a_4,\ a_2 + a_3 + a_4) =: (0,\ a,\ a,\ a) = 0 \text{ in } A.$$

According to [Cat14, Section 5.4], the Abelian variety A is isogenous to a product of an elliptic curve E and an Abelian threefold A'. The linear part of h acts on E by multiplication by $\rho_1(h) = \zeta_3$, and thus E equals the equianharmonic elliptic curve $F = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$. Let us write

$$A \cong (F \times A')/H.$$

In particular, $(0, a,\ a,\ a) \in H$. The Lemma is now implied by the following

<u>Claim:</u> h^2 does not act freely on A.

<u>Proof of the Claim:</u> Since

$$h^2(z) = (\zeta_3^2 z_1 - \zeta_3^2 a_1,\ z_3 + a_2 + a_4,\ z_4 + a_2 + a_3,\ z_2 + a_3 + a_4),$$

it follows that

$$h^2(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, z_4) \iff \begin{pmatrix} (\zeta_3^2 - 1)z_1 - \zeta_3^2 a_1 \\ z_3 - z_2 + a_2 + a_4 \\ z_4 - z_3 + a_2 + a_3 \\ z_2 - z_4 + a_3 + a_4 \end{pmatrix} \in H. \tag{10.1}$$

We show that it is possible to choose $z_1, ..., z_4$ such that the above vector is contained in H. Choose

$$z_1 := \frac{\zeta_3 a_1}{2\zeta_3 + 1},$$

so that z_1 is a solution of $(\zeta_3^2 - 1)z_1 = \zeta_3^2 a_1$ in F. Now, choose z_2, z_3, z_4 such that

$$z_2 - z_4 = a_3 + (\zeta_7 + \zeta_7^3) \cdot a,$$
$$z_3 - z_2 = a_4 + (\zeta_7^2 + \zeta_7^6) \cdot a.$$

Then

$$\begin{aligned} z_4 - z_3 = -(z_3 - z_2) - (z_2 - z_4) &= -a_3 - a_4 - (\zeta_7 + \zeta_7^2 + \zeta_7^3 + \zeta_7^6) \cdot a \\ &= a_2 + (\zeta_7^4 + \zeta_7^5)a. \end{aligned}$$

Moreover, the following calculation shows that condition (10.1) is satisfied, and hence h^2 does not act freely:

$$\begin{pmatrix} (\zeta_3^2-1)z_1-\zeta_3^2a_1 \\ z_3-z_2+a_2+a_4 \\ z_4-z_3+a_2+a_3 \\ z_2-z_4+a_3+a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ (1+\zeta_7+\zeta_7^3)\cdot a \\ (1+\zeta_7^2+\zeta_7^6)\cdot a \\ (1+\zeta_7^4+\zeta_7^5)\cdot a \end{pmatrix} = \big(\mathrm{id}+\rho(g)+\rho(g^3)\big)\begin{pmatrix} 0 \\ a \\ a \\ a \end{pmatrix} \in H.$$

$\square$

Now we go one step further and investigate the case $b=2$.

Lemma 10.6. *The group G cannot have a subgroup of order $63 = 3^2 \cdot 7$.*

Proof. We prove the statement by contradiction. We then can assume without loss of generality that G is of order 63. By Sylow's Theorems, the 7-Sylow subgroup of G is normal in G. Now, since $\mathrm{Aut}(C_7) \cong C_6$ is cyclic, the 7-Sylow subgroup and an element of order 3 generate a subgroup of order 21, contradicting Lemma 10.5. $\square$

Now we can easily prove

Lemma 10.7. *The group G cannot have a subgroup of order $189 = 3^3 \cdot 7$.*

Proof. By Theorem 9.2, the 3-Sylow subgroups of G are C_3^3 or Heis(3). The group G fits into the exact sequence

$$1 \to K \to G \overset{\det \rho(\cdot)}{\to} C_3 \to 1, \quad |K| = 63$$

since the 3-Sylow groups of G are not contained in $\mathrm{SL}(4,\mathbb{C})$, cf. Lemma 7.3 for C_3^3 and Theorem 8.17 for Heis(3). This means that G has a subgroup of order 63, contradicting the previous Lemma. $\square$

Proposition 10.4 is therefore completely proven.

10.3 Groups of order $2^a \cdot 5$

In this section, we prove the following result regarding groups of order $2^a \cdot 5$ occurring as groups of hyperelliptic fourfolds.

Proposition 10.8. *If G has order $2^a \cdot 5$, then G is Abelian and isomorphic to one of the groups C_5, C_{10}, C_{20}, $C_2 \times C_{10}$ or $C_2 \times C_{20}$.*

Observe preliminarily that if $a \leq 3$, Sylow's Theorems imply that the 5-Sylow subgroup of a group of order $2^a \cdot 5$ is normal.

The first step in the proof of Proposition 10.8 is the following

Lemma 10.9. *Suppose that G is of order $2^a \cdot 5$. Then the following statements hold:*

(a) *If G contains a central element of order 5, then G is Abelian.*

(b) *If G is non-Abelian, G contains more than one 5-Sylow subgroup.*

Proof. (a) Assume that G has a central element g of order 5 and that G is non-Abelian. Then G is the direct product of $\langle g\rangle$ and its non-Abelian 2-Sylow group S. Since G is non-Abelian, the representation ρ is the direct sum of two 2-dimensional representations ρ_2, ρ_2', at least one of them being irreducible, say ρ_2 is. Because g is central, it is mapped to the identity matrix by ρ_2 (cf. Lemma 2.2). Moreover, since G is non-Abelian, its 2-Sylow subgroups must be non-Abelian. Inspecting the right column of the table of Theorem 9.1 yields that every group contained in this column falls under (at least) one of the following bullet points:

(1) it contains an element of order 8, or

(2) it contains D_4 or Q_8 as a subgroup, or

(3) its center is non-cyclic.

We will exclude these three cases, arriving then at a contradiction.

To (1): Since the eigenvalues of $\rho_2'(g)$ are of order 5, Lemma 2.5 implies that an element h of order 8 cannot be mapped to a matrix of order 8 by ρ_2'. It then follows that $\rho(gh)$ does not have the eigenvalue 1.

To (2): Similar to (1), an element h of order 4 contained in D_4 or Q_8 cannot be mapped to a matrix of order 4 by ρ_2', since by Lemma 2.5, $\rho(gh)$ cannot have eigenvalues of order 20. Hence, $\rho_2(h)$ has order 4. Inspecting the 2-dimensional irreducible representations of D_4 and Q_8 immediately yields that $\rho_2(h)$ does not have the eigenvalue 1, and hence $\rho(gh)$ does not have the eigenvalue 1.

To (3): In this case, pick a central subgroup $\langle h_1, h_2\rangle$ isomorphic to $C_2 \times C_2$. Since ρ is faithful and since every matrix in $\rho(G)$ must have either zero or two eigenvalues of order 10, we can find an element $h \in \langle h_1, h_2\rangle$ such that $\rho_2(h) = -I_2$: again, $\rho(gh)$ does not have the eigenvalue 1.

(b) Assume the contrary, i.e., that the 5-Sylow subgroup $S = \langle g\rangle$ is normal in G. Since we assumed that G is non-Abelian, $g \notin Z(G)$ by (a). If $h \in G$ is an element which does not commute with g, there is $1 < k \leq 4$ such that $h^{-1}gh = g^k$. It follows that $\rho(g)$ and $\rho(g^k)$ have the same eigenvalues. By replacing g by some power if necessary, we can assume that the eigenvalues of $\rho(g)$ are $1, 1, \zeta_5, \zeta_5^l$, $l \in \{2, 3\}$. Hence $k = l$ and $l^2 \equiv 1 \pmod 5$. Since $l \in \{2, 3\}$, this is not possible. □

By Sylow's Theorems, we immediately obtain

Corollary 10.10. *Assume that G is non-Abelian of order $2^a \cdot 5$. Then G does not have a subgroup isomorphic to D_5, the dihedral group of order 10.*

Corollary 10.11. *Assume that G is of order 20. Then G is isomorphic to $C_2 \times C_{10}$ or to C_{20}.*

Corollary 10.12. *If G is of order 40, then G is Abelian and isomorphic to $C_4 \times C_{10} \cong C_2 \times C_{20}$.*

Proof. As mentioned above, by Sylow's Theorems, the 5-Sylow of G is normal in G. Hence G is Abelian by Lemma 10.9. The last part of the statement follows immediately from Lemma 2.5. □

Note that Lemma 3.11 shows that there indeed exist hyperelliptic fourfolds with group $C_4 \times C_{10} \cong C_2 \times C_{20}$.

The final cases $a \in \{4, 5\}$ in the proof of Proposition 10.8 are dealt with in the following Lemma.

Lemma 10.13. *The group G does not contain a subgroup of order $2^a \cdot 5$, $a \geq 4$.*

Proof. Note that $a \leq 5$ by our results concerning 2-groups associated with a hyperelliptic fourfold. If G is non-Abelian, we run GAP Script `order_80_and_160.g` (cf. Chapter 12): its output only contains groups whose 2-Sylow group contains C_2^4 or $Q_8 \times C_2$. Both of these are excluded, cf. TABLE 1 on p. 167.
It remains to deal with the case where G is Abelian, and it suffices to exclude the case $|G| = 80$. Lemma 10.12 shows that G contains $G' := C_4 \times C_{10}$. By Lemma 2.5 (b), G does not contain an element of order 40. Moreover, the case $G = C_4 \times C_{20}$ is excluded by Lemma 3.13. Therefore it suffices to exclude $G = C_2 \times C_4 \times C_{10}$. By Lemma 2.5 (a), no matrix in $\rho(G')$ can have eigenvalues of order 20. Hence, since every matrix must have the eigenvalue 1 and since ρ is faithful, we can assume that the image of G' in $\mathrm{GL}(4, \mathbb{C})$ is given by

$$\rho(G') = \langle \mathrm{diag}(1, i, 1, 1),\ \mathrm{diag}(1, 1, \zeta_{10}, \zeta_{10}^a) \rangle .$$

Since ρ is faithful and the matrix of any element of order 10 must either have zero or exactly two eigenvalues of order 10 (see Lemma 2.2), we obtain that G is generated by G' and $\mathrm{diag}(-1, 1, 1, 1)$: but then we find a matrix, which does not have the eigenvalue 1. □

This completes the proof of Proposition 10.8.

10.4 Groups of order $2^a \cdot 7$

This section investigates groups of order $2^a \cdot 7$ which occur as groups of hyperelliptic fourfolds. We prove the following result:

Proposition 10.14. *If G has order $2^a \cdot 7$, then G is Abelian and isomorphic to C_7 or C_{14}.*

We will first prove the following Lemma:

Lemma 10.15. *If a non-Abelian group G of order $2^a \cdot 7$, $a \geq 1$ occurs as a group of a hyperelliptic fourfold, the 2-Sylow of G is a normal subgroup.*

Proof. Let g be an element of order 7. Then there is no $h \in G$, such that $h^{-1}gh = g^{-1}$; this is because of Lemma 2.2.
Denote now by N the normalizer of $\langle g \rangle$ in G. By what we just proved, every element in N commutes with g. Since 7^2 does not divide the order of N, the normalizer N is the direct product of its Sylow subgroups,

$$N = \langle g \rangle \times N',$$

where N' is of course the 2-Sylow subgroup of N. By Lemma 2.5 and since every element in N' commutes with g, we obtain that the exponent of N' is ≤ 2. Hence N' is Abelian and isomorphic to C_2^r for some r.
By Lemma 3.13, we have $r \in \{0, 1\}$. In particular, N is cyclic of order 7 or 14. We will now treat the cases $r = 0$ and $r = 1$ separately.

Case $r = 0$: We count the cardinality of the set

$$\bigcup_{i=1}^{6} \{h^{-1}g^i h \mid h \in G\}.$$

Indeed, if $h_1, h_2 \in G \setminus N$ and $i, j \in \{1, ..., 6\}$, we obtain

$$h_1^{-1}g^i h_1 = h_2^{-1}g^j h_2 \iff (h_1h_2^{-1})^{-1}g^i(h_1h_2^{-1}) = g^j \iff h_1h_2^{-1} \in N.$$

We obtain that the above statements can only be satisfied if $i = j$, i.e., g^i and g^j can only be conjugate for $i = j$.
Now, for fixed h_2, there are exactly $|N| = 7$ possibilities for h_1, such that $h_1h_2^{-1} \in N$. Consequently, for fixed $i \in \{1, ..., 6\}$, the length of the conjugacy class of g^i is $\frac{|G|}{7} = 2^a$. It follows that

$$\left| \bigcup_{i=1}^{6} \{h^{-1}g^i h \mid h \in G\} \right| = 6 \cdot 2^a.$$

This shows that there are exactly 2^a elements of order coprime to 7. By cardinality reasons, this implies that the 2-Sylow subgroup of G is normal.

Case $r = 1$: Denote by k a generator of N. Similarly as in the case $r = 0$, we count the cardinality of the set

$$\bigcup_{i=1}^{13} \{h^{-1}k^i h \mid h \in G\}.$$

By arguing similarly as above, it follows that this set contains $6 \cdot 2^{a-1}$ elements of order 7 and $6 \cdot 2^{a-1}$ elements of order 14. Hence, G contains exactly $2 \cdot 2^{a-1} = 2^a$ elements of order coprime to 7, which again implies that the 2-Sylow is normal in G. □

Using the above Lemma, we prove:

Corollary 10.16. *If G is non-Abelian of order $14 = 2 \cdot 7$ or $28 = 2^2 \cdot 7$, then G does not occur as a group of a hyperelliptic fourfold.*

Proof. By Sylow's Theorems, the 7-Sylow subgroup is normal in G. If G occurs, Lemma 10.15 implies that G is the direct product of its Sylow subgroups. Hence both orders 14 and 28 are ruled out, if we restrict ourselves to non-Abelian groups. $\square$

The proof of the above proposition consists of several steps, the first of which being:

Lemma 10.17. *G does not contain a subgroup isomorphic to the dihedral group of order 14,*

$$D_7 = \langle r, s \mid r^7 = s^2 = (rs)^2 = 1 \rangle.$$

Proof. If G had such a subgroup, it would follow from the relation $s^{-1}rs = r^{-1}$ that $\rho(r)$ and $\rho(r)^{-1}$ have the same eigenvalues. This means that $\rho(r)$ has complex conjugate eigenvalues, a contradiction. $\square$

Since $a \leq 5$ (see Theorem 9.1), the following Lemma completes the proof of Proposition 10.14:

Lemma 10.18. *If the order of G is one of $56 = 2^3 \cdot 7$, $112 = 2^4 \cdot 7$ or $224 = 2^5 \cdot 7$, then G does not occur as the group of a hyperelliptic fourfold.*

Proof. Suppose that G is of order 56. Then Lemma 10.15 shows that the 2-Sylow group is normal in G. Observe furthermore that G cannot be Abelian, since it cannot contain a cyclic subgroup of order 28 by Lemma 2.5 and it cannot contain the non-cyclic Abelian group of order 28 by Lemma 3.13). Hence, any element of order 7 acts non-trivially on the 2-Sylow subgroup of G. The only group of order 8 whose automorphism group has order divisible by 7 is the elementary Abelian group C_2^3: this can be checked with GAP. We conclude by observing that the degree of an irreducible representation of degree > 1 of G is 7 (see Theorem 2.10). This settles the case $|G| = 56$.
As for the cases $|G| = 112$ and 224, we observe again that an element of order 7 acts non-trivially on the 2-Sylow subgroup of G. By a GAP search, we obtain that the 2-Sylow subgroup contains a subgroup isomorphic to C_2^4. Hence G cannot occur, see Lemma 2.7. $\square$

10.5 Groups of order $2^a \cdot 3^b \cdot 5$

Suppose that G is a group of order $2^a \cdot 3^b \cdot 5$, where $a, b \geq 1$. Assume that G occurs as the group of a hyperelliptic fourfold. We will prove the following

Proposition 10.19. *If $|G| = 2^a \cdot 3^b \cdot 5$ and $b \geq 1$, then G is Abelian and isomorphic to one of C_{15}, C_{30} or $C_2 \times C_{30}$.*

Proof. The proof is a simple application of our previous results. By Corollary 10.3, we obtain that $b = 1$. The proof of the cited Corollary shows that G is solvable, in particular G contains a subgroup of order 15. By the same reasoning, G contains a subgroup of order $2^a \cdot 5$. Now, Proposition 10.8 and Lemma 2.5 (b) imply that G is contained in $C_2 \times C_2 \times C_{15} \cong C_2 \times C_{30}$. Since $C_2 \times C_{30}$ is isomorphic to $C_6 \times C_{10}$, Lemma 3.11 proves that this group indeed occurs. $\square$

10.6 Groups of order $2^a \cdot 3^b \cdot 5^c \cdot 7$

Let G be a group of order $2^a \cdot 3^b \cdot 5^c \cdot 7$, where $a, b \neq 0$ and $c \in \{0,1\}$. In this section, we prove the following result, completing the proof of Theorem 10.1.

Proposition 10.20. *Suppose that G is of order $2^a \cdot 3^b \cdot 5^c \cdot 7$, where $a, b \neq 0$. Then there is no hyperelliptic fourfold with group G.*

We prove this in several steps, the first of which being the following

Lemma 10.21. *If G is a group of order $2^a \cdot 3^b \cdot 5^c \cdot 7$ $(a, b \neq 0)$ associated with a hyperelliptic fourfold, then G is not solvable.*

Proof. Else, by Hall's Theorem [Ha59, Theorem 9.3.1], G has a subgroup of order $3^b \cdot 7$, which forces $b = 0$ (cf. Proposition 10.4). □

A direct consequence is

Corollary 10.22. *If G is a group of order $2^a \cdot 3^b \cdot 5^c \cdot 7$, $a, b \geq 1$ associated with a hyperelliptic fourfold, then G fits into the exact sequence*

$$1 \to K \to G \overset{\det\rho(\cdot)}{\to} C_m \to 1,$$

where $7 \nmid m$ and K is a non-solvable group of order $2^{\tilde{a}} \cdot 3^{\tilde{b}} \cdot 7$, $\tilde{a}, \tilde{b} \neq 0$.

Proof. Lemma 10.21 implies that K is a non-solvable group. If $|K|$ were the product of two prime powers, Burnside's $p^a q^b$-Theorem [Ha59, Theorem 9.3.2] would assert that K is solvable – since the quotient G/K is cyclic, this would mean that also G is solvable, a contradiction to Lemma 10.21. Moreover, by Lemma 2.12, the prime 5 cannot divide $|K|$. Consequently $|K| = 2^{\tilde{a}} \cdot 3^{\tilde{b}} \cdot 7$, where $\tilde{a}, \tilde{b} \neq 0$. □

Hence, in order to prove Proposition 10.20, it suffices to prove that any group G occuring as a group of a hyperelliptic fourfold cannot have a non-solvable subgroup of order $2^{\tilde{a}} \cdot 3^{\tilde{b}} \cdot 7$, where $\tilde{a}, \tilde{b} \neq 0$. We will henceforth investigate the subgroup K in more detail.

Lemma 10.23. *K does not have a subgroup of order 27 or 32.*

Proof. According to Theorem 7.8, the only 3-groups of order 27 which can be associated with hyperelliptic fourfolds are $C_3 \times C_3 \times C_3$ and $\mathrm{Heis}(3)$. Lemma 7.3 asserts that the former cannot be contained in $\mathrm{SL}(4, \mathbb{C})$, if it is associated with a hyperelliptic fourfold. Theorem 8.17 is the similar result for $\mathrm{Heis}(3)$.

We now prove the statement that K cannot have a subgroup of order 32. Since any 2-Sylow group S of K is contained in $\mathrm{SL}(4, \mathbb{C})$, the discussing preceding Lemma 6.6 asserts that $Z(S)$ is cyclic or that S is Abelian.
We claim that if S is Abelian, then $|S| \neq 32$: Lemma 6.1 asserts that if $|S| = 32$, then either $S = C_2 \times C_4 \times C_4$ or $S = C_4 \times C_8$. We deal with these two cases separately:

(i) If $S = C_2 \times C_4 \times C_4$, Theorem 3.5 asserts that all elements of S have a common eigenvector for the eigenvalue 1. This implies that there is a basis such that the subgroup C_2^3 of S is given by

$$C_2^3 = \{\operatorname{diag}((-1)^{a_1}, (-1)^{a_2}, (-1)^{a_3}, 1) \mid a_1, a_2, a_3 \in \{0,1\}\}.$$

But then S contains $\operatorname{diag}(-1,-1,-1,1)$, which is not contained in $\mathrm{SL}(4,\mathbb{C})$.

(ii) If $S = C_4 \times C_8$ is generated by g, h of respective orders 4, 8 and contained in $\mathrm{SL}(4,\mathbb{C})$, Lemma 2.12 implies that we can assume that

$$\rho(h) = \operatorname{diag}(\zeta_8, \zeta_8^k, \varepsilon, 1), \quad k \in \{3,5\}.$$

Here, ε is a fourth root of unity such that $\det \rho(h) = 1$. We can assume that at most one of the first two diagonal entries of $\rho(g)$ is different from 1 (else, we can replace g by gh^j for some appropriate j). Then:

- If exactly one of the first two diagonal entries of $\rho(g)$ is different from 1, we can assume without loss of generality that the first diagonal entry of $\rho(g)$ is equal to $i = \zeta_8^2$ or to -1. Then $\rho(gh)$ either has multiple or conjugate eigenvalues of order 8, contradicting Remark 2.3.
- If the first two diagonal entries of $\rho(g)$ are one, the latter two are i and $-i$. In this case, $\rho(gh^4)$ does not have the eigenvalue 1.

It remains to show that no non-Abelian group S of order 32 can be contained in K: the only 2-group with cyclic center occurring as a group associated with a hyperelliptic fourfold of order 32 is the one with GAP ID [32,11] (see the proof of Theorem 9.1), which does not embed into $\mathrm{SL}(4,\mathbb{C})$ via the complex representation ρ, cf. Proposition 8.37. The proof of the Lemma is therefore complete. $\square$

Corollary 10.24. $168 = 2^3 \cdot 3 \cdot 7$ *divides the order of* K.

Proof. According to Lemmas 10.21 and 10.23, K has order

$$|K| = 2^{\tilde{a}} \cdot 3^{\tilde{b}} \cdot 7, \quad \tilde{a} \in \{1,2,3,4\},\ \tilde{b} \in \{1,2\}.$$

Hence, it suffices to prove that

$$\tilde{a} \leq 2 \implies K \text{ is solvable.}$$

We give a proof of this implication without using a computer algebra system. For $\tilde{a} \in \{1,2\}$, $\tilde{b} \in \{1,2\}$, we have

$$2^{\tilde{a}} \cdot 3^{\tilde{b}} \in \{6, 12, 18, 36\}.$$

Since the proper divisors of 36 are not congruent to 1 modulo 7, by Sylow's Theorems, groups of order $6 \cdot 7$, $12 \cdot 7$, $18 \cdot 7$ contain a normal subgroup of order 7. Since groups of order 6, 12 and 18 are solvable, all groups of order $6 \cdot 7$, $12 \cdot 7$ or $18 \cdot 7$ are solvable as well.

Thus, the remaining case is $\tilde{a} = \tilde{b} = 2$. Now, let U be a group of order $252 = 36 \cdot 7$. We can assume that U contains exactly 36 subgroups of order 7 (else, the 7-Sylow subgroup of U is normal in U, and since groups of order 36 are solvable, we can argue as before).

Recall that by what we already proved and by Burnside's $p^a q^b$-Theorem, any group whose order is a proper divisor of $252 = 2^2 \cdot 3 \cdot 7$ is solvable. Moreover, by Sylow's Theorems, the number n_3 of 3-Sylow subgroups is one of 1, 4, 7, 28. We investigate these cases:

- If $n_3 = 1$, then U has a normal subgroup P of order 9, hence G/P has order 28; it follows that U is solvable.
- If $n_3 = 4$, we obtain a non-trivial homomorphism $U \to S_4$, whose kernel and the respective quotient are necessarily solvable. Hence U is solvable.
- Suppose now that $n_3 \in \{7, 28\}$. Since we have exactly $36 \cdot 6$ elements of order 7, there are only 36 elements whose orders are coprime to 7. Therefore, there must be two distinct 3-Sylow subgroups P_1, P_2, which intersect non-trivially, i.e., $P := P_1 \cap P_2$ has order 3. Consider now the centralizer $Z_U(P)$ of P: it contains the set $P_1 P_2$ consisting of 27 elements. Hence $|Z_U(P)|$ divides 252, is divisible by 9, and $|Z_U(P)| \geq 27$: therefore,
$$|Z_U(P)| \in \{36, 63, 252\}.$$
Now, if $|Z_U(P)| = 252$, the group P is normal in U, and P, U/P are both solvable. Hence U is solvable. If $|Z_U(P)| = 63$, then U acts on the set $U/Z_U(P)$ consisting of 4 elements in the usual way: hence, we again obtain a non-trivial homomorphism $U \to S_4$ and conclude as in the previous bullet point. Hence we are left with the case $|Z_U(P)| = 36$: since U contains only 36 elements whose orders are coprime to 7, we may conclude that $Z_U(P)$ is the unique subgroup of U which has order 36. Hence $Z_U(P)$ is normal in U. Again, $Z_U(P)$ and $U/Z_U(P)$ are both solvable, so U is solvable.

□

Hence, it suffices to exclude the cases where

$$K \text{ is non-solvable, and } |K| = 2^{\tilde{a}} \cdot 3^{\tilde{b}} \cdot 7, \text{ where } \tilde{a} \in \{3, 4\},\ \tilde{b} \in \{1, 2\}.$$

We proceed by a case by case analysis.

Lemma 10.25. *The group K cannot contain a non-solvable subgroup of order* 168.

Proof. As is well-known, there is one and only one non-solvable group of order 168, namely $\mathrm{GL}(3,2)$ (it is even simple). The group $\mathrm{GL}(3,2)$ contains a subgroup isomorphic to $C_7 \rtimes C_3$, for instance the group generated by

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{order 7 in GL(3,2)}} \quad \text{and} \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{\text{order 3 in GL(3,2)}}.$$

We have excluded the group $C_7 \rtimes C_3$ to be associated with hyperelliptic fourfolds in Lemma 10.5. □

To exclude the remaining orders

$$\begin{aligned}336 &= 168 \cdot 2 = 2^4 \cdot 3 \cdot 7,\\ 504 &= 168 \cdot 3 = 2^3 \cdot 3^2 \cdot 7,\\ 1008 &= 168 \cdot 6 = 2^4 \cdot 3^2 \cdot 7,\end{aligned}$$

we will make use of the following Lemma.

Lemma 10.26. *The group* $\mathrm{PSL}(2,8)$ *(GAP ID [504,156]) has no irreducible representations of dimensions* 2, 3 *or* 4.

Proof. This is checked with GAP Script `irreducible_representations.g`: the dimensions of irreducible representations of $\mathrm{PSL}(2,8)$ are 1, 7, 8 and 9. □

Remark 10.27. We can prove in general that 2 can never be the dimension of an irreducible representation of a non-Abelian finite simple group G:
Suppose that this were the case, and let ρ_2 be such a representation. Then Theorem 2.10 implies that G contains an element g of order 2. Since G is simple, ρ_2 is faithful and $\det(\rho_2(g)) = 1$. Hence g has only the eigenvalue -1 of multiplicity 2. This implies that $\rho_2(g) = -I_2$. However, since ρ_2 is faithful, we obtain that g is a central element of G, a contradiction.
Since $\mathrm{PSL}(2,8)$ is simple, this argument can be used to exclude the case of an irreducible representation of dimension 2 in Lemma 10.26.

The final step to the proof of Proposition 10.20 is to prove

Lemma 10.28. *The order of the kernel* K *cannot be equal to*

$$\begin{aligned}336 &= 168 \cdot 2 = 2^4 \cdot 3 \cdot 7, \text{ and}\\ 504 &= 168 \cdot 3 = 2^3 \cdot 3^2 \cdot 7, \text{ and}\\ 1008 &= 168 \cdot 6 = 2^4 \cdot 3^2 \cdot 7.\end{aligned}$$

Proof. The only non-solvable groups of order 336, 504 or 1008 contain a subgroup, which is isomorphic to $\mathrm{SL}(2,3)$ (excluded in Section 8.2.12), $\mathrm{GL}(3,2)$ or $\mathrm{PSL}(2,8)$, as can be checked with GAP Script `non_solv.g` (cf. Chapter 12). □

This completes the proof of Proposition 10.20.

Chapter 11

Summary of the Results

Theorem 11.1. *The following* Table 5 *contains maximal groups G such there exists a hyperelliptic fourfold with group G. Conversely,* Table 5 *exhausts all possible maximal groups which are associated with hyperelliptic fourfolds.*

G	*ID*	*# of moduli*	*G*	*ID*	*# of moduli*
A_4	*[12,3]*	1	$C_2 \times C_{20}$	*[40,9]*	1
C_{14}	*[14,2]*	1	$((C_4 \times C_2) \rtimes C_2) \times C_3$	*[48,21]*	1
SD_8	*[16,8]*	2	$(C_4 \rtimes C_4) \times C_3$	*[48,22]*	1
$D_4 \times C_2$	*[16,11]*	3	$C_2 \times C_{24}$	*[48,23]*	1
C_{18}	*[18,2]*	1	$A_4 \times C_4$	*[48,31]*	1
C_{20}	*[20,2]*	1	$C_3 \times C_3 \times C_6$	*[54,15]*	1
$S_3 \times C_4$	*[24,5]*	2	$C_2 \times C_{30}$	*[60,13]*	1
$Q_8 \times C_3$	*[24,11]*	2	$(C_3 \rtimes C_8) \times C_3$	*[72,12]*	1
$C_2 \times C_2 \times C_6$	*[24,15]*	3	$S_3 \times C_{12}$	*[72,27]*	1
Heis(3)	*[27,3]*	*rigid*	$((C_6 \times C_2) \rtimes C_2) \times C_3$	*[72,30]*	2
C_{30}	*[30,4]*	1	$C_6 \times C_{12}$	*[72,36]*	1
$C_4 \times C_8$	*[32,3]*	1	$C_2 \times C_4 \times C_{12}$	*[96,161]*	1
$C_8 \rtimes C_4$	*[32,4]*	1	$S_3 \times C_6 \times C_3$	*[108,42]*	1
$(C_4 \times C_4) \rtimes C_2$	*[32,11]*	2	$C_3 \times C_6 \times C_6$	*[108,45]*	1
$C_2 \times C_4 \times C_4$	*[32,21]*	2	$C_2 \times C_6 \times C_{12}$	*[144,178]*	1
$(C_4 \times C_4) \rtimes C_2$	*[32,24]*	1			

Table 5*: list of all possible maximal groups associated with a hyperelliptic fourfold.*

Proof. The groups of order $2^a \cdot 3^b$ are exactly the output of GAP Script `maximal_groups.g` (cf. Chapter 12). In Chapter 10 it was proved that if 5 or 7 divide $|G|$, then G is Abelian: from this it is easy to deduce the maximal Abelian groups associated with hyperelliptic fourfolds (see Lemma 3.13 for further discussion). □

By examining all subgroups of the groups listed in Table 5, we obtain our main result:

Theorem 11.2. *The following* Table 6 *contains groups G such that there exists a hyperelliptic fourfold with group G. Conversely, if $X = A/G'$ is a hyperelliptic fourfold, then G' is contained in the table below.*

G	ID	Max.?	G	ID	Max.?	G	ID	Max.?
C_2	[2,1]	No	$C_2 \times C_8$	[16,5]	No	$C_2 \times C_4 \times C_4$	[32,21]	No
C_3	[3,1]	No	M_{16}	[16,6]	No	$(C_4 \times C_4) \rtimes C_2$	[32,24]	Yes
C_4	[4,1]	No	SD_8	[16,8]	Yes	$\mathrm{Dic}_{12} \times C_3$	[36,6]	No
$C_2 \times C_2$	[4,2]	No	$C_2 \times C_2 \times C_4$	[16,10]	No	$C_3 \times C_{12}$	[36,8]	No
C_5	[5,1]	No	$D_4 \times C_2$	[16,11]	Yes	$S_3 \times C_6$	[36,12]	No
S_3	[6,1]	No	$Q_8 \curlyvee C_4$	[16,13]	No	$C_6 \times C_6$	[36,14]	No
C_6	[6,2]	No	C_{18}	[18,2]	Yes	$C_2 \times C_{20}$	[40,9]	Yes
C_7	[7,1]	No	$S_3 \times C_3$	[18,3]	No	$C_4 \times C_{12}$	[48,20]	No
C_8	[8,1]	No	$C_3 \times C_6$	[18,5]	No	$((C_4 \times C_2) \rtimes C_2) \times C_3$	[48,21]	Yes
$C_2 \times C_4$	[8,2]	No	C_{20}	[20,2]	No	$(C_4 \rtimes C_4) \times C_3$	[48,22]	Yes
D_4	[8,3]	No	$C_2 \times C_{10}$	[20,5]	No	$C_2 \times C_{24}$	[48,23]	Yes
Q_8	[8,4]	No	$C_3 \rtimes C_8$	[24,1]	No	$A_4 \times C_4$	[48,31]	Yes
$C_2 \times C_2 \times C_2$	[8,5]	No	C_{24}	[24,2]	No	$C_2 \times C_2 \times C_{12}$	[48,44]	No
C_9	[9,1]	No	$S_3 \times C_4$	[24,5]	No	$S_3 \times C_3 \times C_3$	[54,12]	No
$C_3 \times C_3$	[9,2]	No	$(C_6 \times C_2) \rtimes C_2$	[24,8]	No	$C_3 \times C_3 \times C_6$	[54,15]	No
C_{10}	[10,2]	No	$C_2 \times C_{12}$	[24,9]	No	$C_2 \times C_{30}$	[60,13]	Yes
Dic_{12}	[12,1]	No	$D_4 \times C_3$	[24,10]	No	$(C_3 \rtimes C_8) \times C_3$	[72,12]	Yes
C_{12}	[12,2]	No	$Q_8 \times C_3$	[24,11]	Yes	$S_3 \times C_{12}$	[72,27]	Yes
A_4	[12,3]	No	$A_4 \times C_2$	[24,13]	No	$((C_6 \times C_2) \rtimes C_2) \times C_3$	[72,30]	Yes
$S_3 \times C_2$	[12,4]	No	$C_2 \times C_2 \times C_6$	[24,15]	No	$C_6 \times C_{12}$	[72,36]	No
$C_2 \times C_6$	[12,5]	No	$\mathrm{Heis}(3)$	[27,3]	Yes	$C_2 \times C_6 \times C_6$	[72,50]	No
C_{14}	[14,2]	Yes	$C_3 \times C_3 \times C_3$	[27,5]	No	$C_2 \times C_4 \times C_{12}$	[96,161]	Yes
C_{15}	[15,1]	No	C_{30}	[30,4]	Yes	$S_3 \times C_6 \times C_3$	[108,42]	Yes
$C_4 \times C_4$	[16,2]	No	$C_4 \times C_8$	[32,3]	Yes	$C_3 \times C_6 \times C_6$	[108,45]	Yes
$(C_4 \times C_2) \rtimes C_2$	[16,3]	No	$C_8 \rtimes C_4$	[32,4]	Yes	$C_2 \times C_6 \times C_{12}$	[144,178]	Yes
$C_4 \rtimes C_4$	[16,4]	No	$(C_4 \times C_4) \rtimes C_2$	[32,11]	Yes			

Table 6*: list of all possible groups associated with a hyperelliptic fourfold.*

As an immediate consequence, we obtain:

Corollary 11.3. *Let $X = A/G$, $A = V/\Lambda$ be a hyperelliptic fourfold. Then V is not an irreducible representation of G.*

Proof. Every group listed in Table 6 does not have an irreducible representation of degree 4: this follows essentially from Theorem 2.10. We give an example to illustrate the procedure. Take for instance the group

$$G := (C_3 \rtimes C_8) \times C_3 = \langle a, b, k \mid a^3 = b^8 = k^3 = [a,k] = [b,k] = 1, b^{-1}ab = a^2 \rangle$$

with GAP ID [72,12]: since it contains a normal Abelian subgroup of order 9, degrees of irreducible characters of G divide 8. However, this is not quite strong enough. We notice that G has indeed an Abelian (normal!) subgroup of index 2, namely the one generated by a, b^2 and k (the relation $b^{-1}ab = a^2$ implies that b^2 is central). Thus, the degrees of irreducible characters of G divide 2. Similar arguments work for any group in TABLE 6. $\square$

Remark 11.4. Of course Corollary 11.3 can be checked with GAP as well.

Chapter 12

GAP Codes

We briefly describe the GAP codes used in the classification carried out in the previous chapters. We made use of the following GAP algorithms[1]

- `2-groups_non_faithful.g`
  This code is used to determine candidates of non-Abelian 2-groups, which do not admit a faithful irreducible representation and occur as a group attached to a hyperelliptic fourfold.

- `2-groups_order_32_faithful.g`
  We use this script to obtain the candidates 2-groups of order 32, which have a faithful irreducible representation of dimension 2 and occur as a group attached to a hyperelliptic fourfold.

- `3-groups.g`
  In order to prove Proposition 7.8, we needed to show that a 3-group associated with a hyperelliptic fourfold has order at most 27. The output of this GAP Script contains this information, showing that no group of order 81 and exponent 3 can occur as a group of a hyperelliptic fourfold.

- `irreducible_representations.g`
  This a general piece of code, whose output is, given a group G contained in the database of small groups, its irreducible matrix representations, conjugacy classes and other useful information. We used it in several places, for instance to determine the dimensions of irreducible representations of $\mathrm{PSL}(2,8)$ in Lemma 10.26.

- `maximal_groups.g`
  This code forms the heart of the classification of groups of order $2^a \cdot 3^b$ associated with hyperelliptic fourfolds. It is used to determine the maximal groups of order $2^a \cdot 3^b$, which occur as groups attached to hyperelliptic fourfolds.

- `order_80_and_160.g`
  We used this script in the proof of Lemma 10.13 to exclude groups of order 80 and 160 to occur as groups attached to hyperelliptic fourfolds.

- `non_solv.g`
  This piece of code was used in the proof of Lemma 10.28 to exclude non-solvable

[1] The file extension *.g ensures that it is readable by GAP. The files can be opened with a usual text editor.

groups of order 336 and 1008 to occur as groups attached to hyperelliptic fourfolds.

Part IV

Further Remarks and Questions

After having carried out the classification of groups associated with hyperelliptic fourfolds, we hope that we convinced the reader that a full classification of hyperelliptic fourfolds or a classification of groups associated with hyperelliptic fivefolds is probably too lengthy. However, there are still interesting open questions involving hyperelliptic manifolds (or varieties). We only discuss a few of them:

Question 1.1. [ARV99, p. 414] Is there a hyperelliptic manifold T/G, $T = V/\Lambda$, such that $V = V_\chi$ is an irreducible representation of G (with irreducible character χ)?

This question is related to the following conjecture by Amerik [Ame97, p. 196].

Conjecture 1.2. Let X and Y be smooth n-dimensional projective varieties with $b_2(X) = b_2(Y) = 1$. Assume that $Y \neq \mathbb{P}^n_{\mathbb{C}}$ and if $n = 1$, that Y is not an elliptic curve. Then the degree of surjective holomorphic maps $f\colon X \to Y$ can be bounded in terms of the discrete invariants of X and Y.

Indeed, if $Y = T/G$, $T = V/\Lambda$ is a hyperelliptic manifold with $b_2(Y) = 1$, then the complex representation $\rho\colon G \to \mathrm{GL}(V)$ is irreducible. To see this, note that

$$H^2(Y,\mathbb{C}) = H^{1,1}(Y) = H^{1,1}(T)^G = \left(H^{1,0}(T) \otimes H^{0,1}(T)\right)^G.$$

Now, if $W \subset \mathbb{C}^N$ is an irreducible representation of G, it is not difficult to see that $\dim\left((W \otimes \overline{W})^G\right) = 1$. Consequently, the G-representation $H^{1,0}(T)$ must be irreducible. If such a (projective) Y exists, it would serve as a counterexample to Amerik's conjecture, see [ARV99, Proof of Proposition 2.1].
We collect several necessary conditions such a manifold Y (or the pair (T, G)) must satisfy.

(1) By the classification of groups attached to hyperelliptic manifolds in dimension ≤ 4, such a hyperelliptic manifold must have dimension $n \geq 5$ (cf. the introduction of Part II for dimension 3 and Corollary III.11.3 for dimension 4).

(2) By Schur's Lemma, (T, G) is necessarily rigid or belongs to a 1-dimensional family, since $\dim(\mathrm{Hom}(V, \overline{V})^G) \leq 1$ (see Definition I.3.2 and Theorem I.1.1). In any case, by Theorem I.6.1, we may assume that T is an Abelian variety.

(3) Since $\rho_\Lambda \sim \rho \oplus \overline{\rho}$, we observe that $\chi + \overline{\chi}$ is integer-valued.

(4) The group G has trivial center: in fact, since V_χ is a simple $\mathbb{C}[G]$-module, a central element of G is mapped to a multiple of the identity matrix by the corresponding representation. We conclude by faithfulness of ρ. (In particular, G is not a p-group.)

(5) For any prime power p^k dividing $|G|$, by the Theorem of Minkowski-Schur [Hu98, 39.8 Theorem] (whose hypothesis is satisfied because of (3)),

$$k \leq \sum_{i=0}^{\infty} \left\lfloor \frac{2n}{p^i(p-1)} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer $\leq x$. For instance, if $n = 5$, the possible primes dividing $|G|$ are 2, 3, 5 and 7, and we obtain

$$|G| = 2^a \cdot 3^b \cdot 5^c \cdot 7^d, \qquad \text{where}$$
$$a \leq 18, \quad b \leq 6, \quad c \leq 2, \quad d \leq 1,$$

which is of course completely impractical for the primes 2 and 3, but gives good information about the exponents c and d. Since we want G to have an irreducible representation of degree 5, it follows that $c \geq 1$, and G does not have a normal 5-Sylow subgroup (see Theorem 2.10). The similar argument applies to the case where n is some prime $p \geq 5$ as well (i.e., p divides $|G|$, p^3 does not divide $|G|$, and G does not have a normal p-Sylow subgroup).

(6) The quotient $Y = T/G$ must have $b_1(Y) = 0$, since $H^{1,0}(Y) = H^{1,0}(T)^G = 0$.

(7) If B, B' are non-isogenous simple complex tori, then $\mathrm{Hom}(B, B') = 0$ (see [Mum70, Corollary 2 on p. 174]). Thus, $T = V/\Lambda$ is isogenous to $(T')^k$ for a simple complex torus T' and some $k \geq 1$ (otherwise V would not be an irreducible G-representation).Moreover, T is not simple (i.e., $k \geq 2$): indeed, $\ker(\rho(g) - \mathrm{id}_T)^0$ is a proper complex subtorus of T for any $g \in G \setminus \{\mathrm{id}_T\}$.

Remark 1.3. The fact that T is non-simple can also be proved differently – we sketch a second proof here. If T were simple, $D := \mathrm{End}_{\mathbb{Q}}(T)$ would be a division algebra (see the cited Corollary in Mumford's book). Amitsur [Ami55] studied which finite subgroups can possibly admit an embedding into the multiplicative group of a division algebra. The only groups on his list that can potentially have trivial center are groups whose Sylow-subgroups are all cyclic. The Hölder-Burnside-Zassenhaus Theorem [Ha59, Theorem 9.4.3] asserts that a group G whose Sylow-subgroups are cyclic is metacyclic, i.e., G has a presentation of the form

$$G = G_{m,n,r} := \langle a, b \,|\, a^m = b^n = 1, \; b^{-1}ab = a^r \rangle$$

for some integers $m, n, r > 1$, such that $\gcd(r, m) = 1$, $r^n \equiv 1 \pmod m$ as well as $\gcd(m, r(n-1)) = 1$. We can then slightly slightly generalize [Lam01, (3.1) Theorem] to prove that a group $G = G_{m,n,r}$ cannot be contained in the multiplicative group D^* of the division algebra D, unless possibly when $Z(G)$ is non-trivial.

Problem 1.4. Investigate further the torsion subgroup Λ^0 defined in Section 3.2.

For instance, as already mentioned in the cited section, it was proved by Catanese [Cat19, Proposition 3.3] in the case where the group G was cyclic that the number of such Λ^0 is finite. Moreover, Catanese used his results to explicitly describe the data needed to define a Bagnera-de Franchis manifold. Following his module-theoretic approach, it might be possible to prove similar results also in the case where G is Abelian.

Interesting is also the following. In [BCF15], Bauer, Catanese and Frapporti constructed a family of surfaces S of general type with $K_S^2 = 6$ and $p_g(S) = q(S) = 1$, obtained as ample divisors in a Bagnera-de Franchis threefold. Moreover, they explicitly described its moduli space, which is irreducible of dimension 4. Thanks to the very explicit description of hyperelliptic manifolds/varieties, it might be worthwhile to study in detail the more general

Problem 1.5. Construct interesting (new) submanifolds of hyperelliptic varieties, e.g. surfaces of general type with interesting invariants and moduli.

An interesting starting point is to take a divisor in the hyperelliptic threefold with group D_4 (see Part II of the thesis): we suspect that this could lead to a family of surfaces of general type with interesting symmetries and moduli. This will be investigated in future work.

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