ANALYTICAL AND NUMERICAL ESTIMATES OF REACHABLE SETS IN A SUBDIVISION SCHEME

WOLFGANG RIEDL, ROBERT BAIER, AND MATTHIAS GERDTS

Abstract. Reachable sets for (discrete) nonlinear control problems can be described by feasible sets of nonlinear optimization problems. The objective function for this problem is set to minimize the distance from an arbitrary grid point of a bounding box to the reachable set.

To avoid the high computational costs of starting the optimizer for all points in an equidistant grid, an adaptive version based on the subdivision framework known in the computation of attractors and invariant measures is studied. The generated box collections provide over-approximations which shrink to the reachable set for a decreasing maximal diameter of the boxes in the collection and, if the bounding box is too pessimistic, do not lead to an exploding number of boxes as examples show. Analytical approaches for the bounding box of a 3d funnel are gained via the Gronwall-Filippov-Wazewski theorem for differential inclusions or by choosing good reference solutions. An alternative self-finding algorithm for the bounding box is applied to a higher-dimensional kinematic car model.

1. Introduction

1.1. **Reachability analysis.** Reachable sets and their (approximative) knowledge is a research area of high interest, especially with models in engineering (see e.g., [1, Chap. 5], [28], the ARCH benchmarks[1] and references mentioned there). Motivating examples are e.g., the computation of safety zones for collision avoidance of cars and planes [9, 1, 19, 28] or areospace maneuver with multi-boost launchers, satellites, . . . ([18, 24, **?**]).

Safety verification via reachability analysis as e.g., in [1, Chap. 1], [23] considers a computational decision whether a single target point[1] or a set of states (usually a strict subset of the reachable set) can be possibly visited or definitely avoided by the underlying control system. Instead we set up a method that depends on time and state space discretization and delivers an approximation of the whole *discrete reachable set*, i.e., the reachable set of the discretized system, for which error estimates for the state space discretization (for the discrete reachable set) and for the time discretization (for the reachable set in continuous time) are available. With a coupled refinement in time and state space the convergence of the computed sets is guaranteed and wrapping effects of over-approximations or

Date: 3 May 2017.

2010 *Mathematics Subject Classification.* 93B03 34A60 (49M25 49J53 65L07 93D23 93C10).

Key words and phrases. reachable sets, subdivision, direct discretization of optimal control, Filippov's theorem, nonlinear optimization.

[1]http://cps-vo.org/group/ARCH

simulations of trajectories via sampling techniques (see e.g., [1, Chap. 3 and 4], [15]) can be avoided.

Although restricted to low space dimensions the solution of partial differential equations via the Hamilton-Jacobi-Bellman (HJB) approach, semi-Lagrangian schemes and level sets [25, 12, 22], is one method to approximate reachable sets. Another class of methods [23], [1, Chap. 2], [14] uses sets with simple structure (polytopes, ellipsoids, zonotopes, ...) to over-approximate reachable sets. Further methods use linearization and hybridization approaches with guaranteed bounds to rewrite nonlinear systems to linear ones, where the control set contains the error bounds [15, 14, 2]. Most of these methods are also applied for linear control problems, but often no convergence analysis to the (discrete/continuous-time) reachable set is available for (non)linear systems.

For the rich literature on approximation of reachable sets see the exemplarily cited articles below and references contained there and the discussion of methods in [1, 7, 27, 8].

Some mentioned methods are rooted in optimization (e.g., in dynamic programming principles, minimum time function, indirect methods based on optimality conditions). Our method uses optimal control solvers based on direct discretization with the use of available nonlinear optimization solvers (e.g., WORHP[2]) and differs from those approaches ([7]).

In the non-adaptive approach of [6, 7] a given bounding box containing the reachable set is discretized with an equidistant grid. Usually the adaptive approach in [26, 27], based on the framework [16] of subdivision, is more efficient. Both algorithms are not designed for real-time computations in contrast to several of the above mentioned approaches. In [28] it was used by a car company for the verification of safety zones computed by other methods based on simpler car models.

Although this approach also contains a few heuristic ideas in the practical implementation (test points, local optimizer), it can be applied rigorously [27] and delivers rather precise results even for higher-dimensional systems, e.g., a realistic single-track car model (7 states, 2 controls) in [28, § II.1] or a docking maneuver for a satellite (13 states, 6 controls) in [24], both calculated with the non-adaptive version of this algorithm. Especially, if the number of relevant states for the higher-dimensional system is small, this method allows to discretize only the lower-dimensional projection space (e.g., a position in 2d or 3d) and not the high-dimensional state space of the whole system. A nonlinear 4d model for an industrial robot with 2 controls and boundary conditions is already studied in [27] with the adaptive approach. Here, we first present a 3d system with a 2d reachable set which creates practical (and theoretical) problems for HJB solvers and level set methods. The second example with more than 4 states seems out of reach for these methods and exceeds problem dimensions in state space and controls from [27]. In both cases a rather tight over-approximation of the discrete reachable sets is calculated which can be improved by smaller time and state space discretization.

[2] `www.worhp.de, DOI:10.1007/978-1-4614-4469-5_4`

The discussion of determining the initial bounding box analytically or numerically and a study how the computational effort is influenced by the bounding box has not been done up to now.

1.2. **Preliminaries.** $\|x\|$ denotes the *Euclidean norm*, $\langle x, y\rangle$ the corresponding *scalar product* for $x, y \in \mathbb{R}^n$ and the closed *Euclidean ball* with radius $r > 0$ and center $z \in \mathbb{R}^n$ will be called $B_r(z)$.

A *set-valued map* G, mapping the argument $x \in X$ to the image set $G(x) \subset Y$ for two spaces X, Y is denoted by $G : X \rightrightarrows Y$. For measuring the *distance* from a point $x \in \mathbb{R}^n$ to a set $Y \subset \mathbb{R}^n$ we set $\operatorname{dist}(x, Y) := \inf_{y \in Y} \|x - y\|$. The *collection of compact, nonempty sets* is denoted by $\mathcal{K}(\mathbb{R}^n)$ and by $\mathcal{C}(\mathbb{R}^n)$ if the sets are additionally convex. The (one-sided) *Hausdorff distance* for $X, Y \in \mathcal{K}(\mathbb{R}^n)$ is defined as

$$\begin{aligned} \mathrm{d}(X, Y) &:= \min\{\varepsilon > 0 \,|\, X \subset Y + \varepsilon B_1(0)\}, \\ \mathrm{d}_{\mathrm{H}}(X, Y) &:= \max\{\mathrm{d}(X, Y), \mathrm{d}(Y, X)\}. \end{aligned}$$

1.3. **Control problems and differential inclusions.**

Definition 1.1. The (nonlinear) control problem on a time interval $I = [t_0, T]$ is given by

$$\text{(CP)} \quad \begin{cases} x'(t) = f(t, x(t), u(t)), \;\; u(t) \in U \;\; (\text{a.e. } t \in I), \\ \psi(x(t_0)) \le 0 \end{cases}$$

with a control set $U \in \mathcal{K}(\mathbb{R}^m)$, a vector-valued function $f : I \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{R}^p$ defining the component-wise constraints for the initial value.

Definition 1.2. For a given set-valued map $F : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with nonempty images and a nonempty initial set $X_0 \in \mathcal{K}(\mathbb{R}^n)$ we consider the *differential inclusion*

$$\text{(DI)} \quad \begin{cases} x'(t) \in F(t, x(t)) \quad (\text{a.e. } t \in I), \\ x(t_0) \in X_0 \end{cases}$$

with absolutely continuous solutions $x : I \to \mathbb{R}^n$.
The associated differential inclusion to (CP) is given by

$$F(t, x) := \bigcup_{u \in U} \{f(t, x, u)\}, \;\; X_0 := \{x_0 \in \mathbb{R}^n \,|\, \psi(x_0) \le 0\},$$

the *set of solutions* and the *reachable set at time* T by

$$\begin{aligned} \mathcal{X}(t_0, X_0) &:= \{x(\cdot) \,|\, x(\cdot) \text{ solution of (DI) on } I\}, \\ \mathcal{R} := \mathcal{R}(T, t_0, X_0) &:= \{x(T) \,|\, x(\cdot) \in \mathcal{X}(t_0, X_0)\}. \end{aligned}$$

We consider the following *basic assumptions* for (DI):

(A1) F has images in $\mathcal{C}(\mathbb{R}^n)$
(A2) $F(\cdot, x)$ is Lipschitz on I uniformly in $x \in \mathcal{V} \subset \mathbb{R}^n$
(A3) $F(t, \cdot)$ is *locally Lipschitz* on $\mathcal{V} \subset \mathbb{R}^n$ uniformly in t, i.e., there exists a Lipschitz constant $L \ge 0$ such that

$$\mathrm{d}_{\mathrm{H}}(F(t, x), F(t, z)) \le L\|x - z\| \quad (t \in I, x, z \in \mathcal{V})$$

and the following *weakened conditions*

(A1') F has images in $\mathcal{K}(\mathbb{R}^n)$

(A2') $F(\cdot, x)$ is *measurable* for every $x \in \mathbb{R}^n$ in the sense of [4, Definition 8.1.1]

(A3') $F(t, \cdot)$ is *one-sided Lipschitz (OSL)* uniformly in t, i.e., there exists an *OSL constant* $\mu \in \mathbb{R}$ so that for all $t \in I$, $x, z \in \mathbb{R}^n$, $\xi \in F(t, x)$ there exists $\zeta \in F(t, z)$ with

$$\langle x - z, \xi - \zeta \rangle \leq \mu \|x - z\|^2 .$$

1.4. **Direct discretization via set-valued Runge-Kutta methods.** A set-valued Runge-Kutta method is defined on an equidistant time grid for $I = [t_0, T]$ with step size $h = \frac{T - t_0}{N}$ and a given $N \in \mathbb{N}$. The grid points are denoted by $t_j := t_0 + jh$, $j = 0, 1, \ldots, N$. Let $X_0^h \in \mathcal{K}(\mathbb{R}^n)$ be given.

The set iteration of the *set-valued implicit Euler's method* ([11]) can be defined via implicit discrete inclusions

$$X_{j+1}^h = \bigcup_{\eta_j^h \in X_j^h} \Big\{ \eta_{j+1}^h \in \mathbb{R}^n \,|\, \eta_{j+1}^h \in \eta_j^h + hF(t_{j+1}, \eta_{j+1}^h) \Big\}.$$

A similar iteration can be defined for other methods, e.g., the explicit Euler with $F(t_j, \eta_j^h)$ in the right-hand side.

Definition 1.3. For a given set-valued discretization method we denote the *solution set* by

$$\begin{aligned} \mathcal{X}_h(t_0, X_0) := \big\{ (\eta_j^h)_{j=0,\ldots,N} \,|\, & (\eta_j^h)_j \text{ discrete solution,} \\ & \text{i.e., } \eta_j^h \in X_j^h \text{ for } j = 0, 1, \ldots, N \big\}, \end{aligned}$$

and the *discrete reachable set at time T* by

$$\mathcal{R}_h := \mathcal{R}_h(T, t_0, X_0) := \big\{ \eta_N^h \,|\, (\eta_j^h)_j \in \mathcal{X}_h(t_0, X_0) \big\}$$

which coincides with the set X_N^h in the iterative scheme.

2. Subdivision Algorithm for Reachable Sets and Its Convergence

Approximations for compact reachable sets can be gained by solving parametric optimal control problems.

2.1. Non-adaptive and adaptive algorithm.

Algorithm 2.1. *(non-adaptive version from [6, 7])*

(i) *Choose a compact bounding box $G \supset \mathcal{R}_h$.*

(ii) *Choose a finite grid $\mathbb{G}_\rho \subset G$ with $\mathrm{d}_H(\mathbb{G}_\rho, G) \leq \rho$.*

(iii) *Solve the (discrete) optimization problem*

$$(OPT(g_\rho)) \qquad \begin{aligned} &\min \quad \tfrac{1}{2} \cdot \|g_\rho - \eta_N^h\|^2 \\ &\text{s.t.} \quad \eta_j^h \in X_j^h \qquad (j = 0, 1, \ldots, N) \end{aligned}$$

with problem (DI) as constraints for each $g_\rho \in \mathbb{G}_\rho$ with your favorite set-valued discretization method $(X_j^h)_j$.

$\mathcal{R}_\rho(T,t_0,X_0) := \bigcup_{g_\rho \in \mathbb{G}_\rho} \{\pi_{\mathcal{R}_h}(g_\rho)\}$ is the set consisting of some best approximation $\pi_{\mathcal{R}_h}(g_\rho)$ for a grid point g_ρ inside $\mathbb{G}_\rho$. Each best approximation coincides with a (global) minimizer $\widehat{\eta}_N^h$ of (OPT(g_ρ)).

Using an adaptive version often yields results of similar approximation quality in less computational time.

Algorithm 2.2. *(adaptive version from [26, 27])*

(i) *Set $k := 0$, choose a compact bounding box $G \supset \mathcal{R}_h$ and set $\mathcal{B}_0 := \{G\}$.*

(ii) refinement/"subdivision" *in* level $k+1$*:*
construct a refined *box collection $\widetilde{\mathcal{B}}_{k+1}$ with*

$$\bigcup_{\widetilde{B} \in \widetilde{\mathcal{B}}_{k+1}} \widetilde{B} = \bigcup_{B \in \mathcal{B}_k} B\,,$$

$$\max_{\widetilde{B} \in \widetilde{\mathcal{B}}_{k+1}} \operatorname{diam}(\widetilde{\mathcal{B}}) = \theta_k \cdot \max_{B \in \mathcal{B}_k} \operatorname{diam}(B)$$

and $0 < \underline{\theta} \le \theta_k \le \overline{\theta} < 1$.

(iii) selection*: define the new box collection $\mathcal{B}_{k+1}$ with*

$$\mathcal{B}_{k+1} = \left\{ B \in \widetilde{\mathcal{B}}_{k+1} \;:\; B \cap \bigcup_{g \in B} \{\pi_{\mathcal{R}_h}(g)\} \neq \emptyset \right\}, \tag{1}$$

where $\pi_{\mathcal{R}_h}(g)$ is one minimizer of (OPT(g)) in $\mathcal{R}_h$.

Since we cannot realize the union over all grid points in a box B in step (iii), we choose a few test points to decide whether the box will be kept or not ([27]). A best approximation is calculated by solving the nonlinear optimization problem (OPT(g_ρ)) and thus is costly, but the computational treatability of higher-dimensional systems is solely influenced by the upper bound on variables and constraints of the used optimization framework.

The number of boxes at level k in Algorithm 2.1 is in the worst case 2^{kn}. We can avoid the naive approach to solve 2^n optimization problems (OPT(g_ρ)) for each vertex of every box by a smart storage of results in a search tree ([27]). Fig. 1 illustrates for 2d/3d that only the $3^n - 2^n$ red grid points have to be newly solved in this step, since the computation was already done for the grid points (blue dots). We point out that the "curse of dimensionality" is not present (independently of the number of states and controls) if the projection P_d on the studied state variables is in $\mathbb{R}^d$ and $d = 2, 3$, i.e., we start Algorithm 2.2 with $G \subset \mathbb{R}^d$ and replace the objective function in (OPT(g_ρ)) by $\frac{1}{2} \cdot \|g_\rho - P_d(\eta_N^h)\|^2$.

The following plots in Fig. 3 illustrate that in contrary to the non-adaptive method a big bounding box in Algorithm 2.2 does not lead to this behavior in the first levels of the adaptive algorithm (level 1 = light blue, 2 = light pink, 3 = dark green). In the left picture in Fig. 3 the blue/red set (the reachable set of Example 4.2) is centered in the big box and only 4 out of 16 boxes are kept in levels 2–3, the others are dropped. In the right picture the set lies close to a corner of the bounding box (only 1 out of 4 boxes is kept).

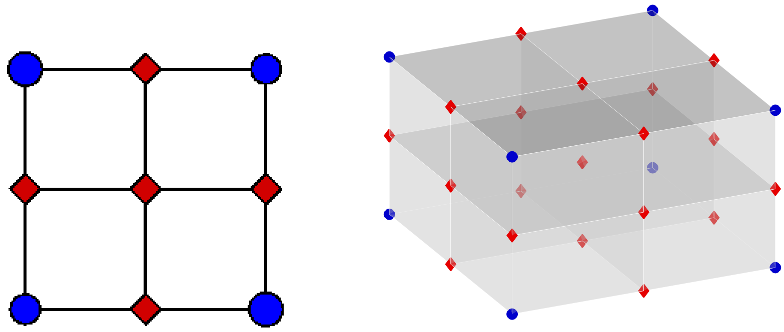

Figure 1. Selection of red test points for next level in 2d and 3d

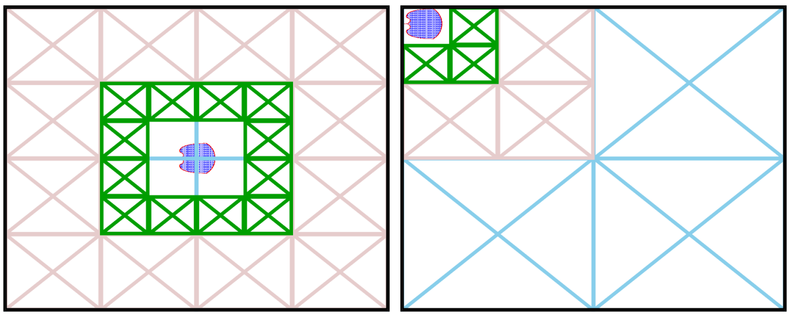

Figure 2. Dropping of boxes up to level 3 for Algorithm 2.2 during convergence to a central reachable set (red/blue) resp. a set in the corner

2.2. **Convergence study.** The convergence of the implicit Euler's method is covered by [11, Theorems 17–18]. The convergence statement is formulated for reachable sets and not in the stronger form for solution sets as in [11].

Proposition 2.3. *(implicit Euler's method) Let* $I = [t_0, T]$ *and* $F : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *fulfill the assumptions (A1)–(A3) for* $\mathcal{V}$ *containing all solution values* $x(t)$, $t \in I$ *in its interior and let additionally*

$$F(\cdot, \cdot) \text{ be jointly continuous in } (t, x) \in I \times \mathcal{V}.$$

Then, there exists $C \geq 0$ *such that for all step-sizes* $h = \frac{T-t_0}{N}$, $N \in \mathbb{N}$, *it holds that*

$$\mathrm{d}_H(\mathcal{R}(T, t_0, X_0), \mathcal{R}_h(T, t_0, X_0)) \leq Ch\,. \tag{2}$$

Here, we drop the assumption (A3') from [11] on the one-sided Lipschitz condition, since we do not emphasize its improved stability compared to the explicit Euler ([13]).

Numerical realizations of set-valued Runge-Kutta methods can be found in [13, 6, 11, 24]. The following proposition states that the adaptive algorithm produces shrinking over-approximations of the reachable set.

Proposition 2.4. *see* [27, Proposition 1 and Theorem 2.4] *Let* $\mathcal{R}_h$ *be the discrete reachable set and* Q_k *be the compact union of boxes in the collection* $\mathcal{B}_k$ *after* k

steps of the adaptive subdivision scheme. Then, $\mathrm{d}(\mathcal{R}_h, Q_k) = 0$ *and*

$$\mathrm{d}_H(Q_k, \mathcal{R}_h) = \mathrm{d}(Q_k, \mathcal{R}_h) \le \mathrm{diam}(\mathcal{B}_k),$$

where $\mathrm{diam}(\mathcal{B}_k)$ *is the maximal diameter of boxes* $B \in \mathcal{B}_k$*.*

If the bounding box $\widetilde{G} = 2G$ is doubled in size and $\theta_k = \frac{1}{2}$ in step (ii) of Algorithm 2.2, Proposition 2.5 shows that only one additional iteration is needed to reach approximately the same Hausdorff distance to the reference set as starting the iteration with G.

We now combine the effects of the time discretization (Proposition 2.4) and the spatial discretization (Proposition 2.5) for the fully discretized scheme.

Theorem 2.5. *Let* $F : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *fulfill such assumptions that the iterative set-valued method* $(X_j^h)_j$ *with convergence order* $p > 0$*, the set* $\mathcal{R}_h(T, t_0, X_0) = X_N^h$ *for a step size* $h = \frac{T-t_0}{N}$*,* $N \in \mathbb{N}$*, allows the estimate*

$$\mathrm{d}_H(\mathcal{R}(T, t_0, X_0), \mathcal{R}_h(T, t_0, X_0)) \le C_1 h^p \,.$$

Choose a constant $C_2 > 0$*, a compact bounding box* G *with* $G \supset \mathcal{R} + \varepsilon B_1(0)$ *and compute* k *steps of Algorithm 2.2 with scaling factor* $\theta_k \le C_2 h^p$ *applied to the set* X_N^h*.*

Then, for $h > 0$ *sufficiently small we have*

$$\mathrm{d}_H(\mathcal{R}, \mathcal{R}_{h,\rho}(T, t_0, X_0)) \le (C_1 + C_2 \,\mathrm{diam}(G)) h^p, \tag{3}$$

where $\mathcal{R}_{h,\rho}(T, t_0, X_0)$ *denotes the fully discretized scheme, the union* Q_k^h *of boxes in the collection* $\mathcal{B}_k^h$*.*

Proof. Choose $N \in \mathbb{N}$ so that $h = h_N$ fulfills $C_1 h^p \le \varepsilon$. Then, $\mathcal{R}_h(T, t_0, X_0) \subset \mathcal{R} + \varepsilon B_1(0) \subset G$ so that the bounding box also contains the discrete reachable set. We now apply Proposition 2.5 for $\mathcal{R}_h(T, t_0, X_0)$ and estimate

$$\mathrm{diam}(\mathcal{B}_k^h) \le \theta_k \,\mathrm{diam}(\mathcal{B}_0^h) \le C_2 \,\mathrm{diam}(G) h^p \,.$$

The triangle inequality after inserting $\mathcal{R}_h(T, t_0, X_0)$ in (3) yields the final estimate. □

3. Analytical and Numerical Calculation of Bounding Boxes

Algorithms 2.1 and 2.2 require an initial bounding box G and we present two approaches for finding it.

3.1. **Analytical approach.** The Gronwall-Filippov-Wazewski theorem is well-known in set-valued numerical analysis. We cite it for Lipschitz set-valued right-hand sides in a version close to [20, Theorem 1].

Theorem 3.1. *(Filippov theorem) Consider* $I = [t_0, T]$ *and an absolutely continuous function* $y : I \to \mathbb{R}^n$*. Let* $F : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *fulfill the assumptions (A1'), (A2'), (A3) around* $y(\cdot)$*, i.e., on a neighborhood* $\mathcal{V} = \bigcup_{t \in I} B_r(y(t))$ *with constant radius* $r > 0$*.*

If $\delta_0 = \|x_0 - y(t_0)\|$ *and* $\delta : I \to \mathbb{R}$ *is integrable with*

$$\mathrm{dist}(y'(t), F(t, y(t))) \le \delta(t) \quad \textit{(for a.e. } t \in I\textit{),}$$

then there exists a solution $x(\cdot)$ of (DI) such that for $t \in I$

$$\|x(t) - y(t)\| \le e^{L(t-t_0)}\delta_0 + \int_{t_0}^{t} e^{L(t-s)}\delta(s)ds\,. \tag{4}$$

A weakened assumption on the right-hand side is the one-sided Lipschitz condition for which a better estimate in Filippov's theorem holds (see [20, Theorem 3]).

Theorem 3.2. *(Filippov theorem for OSL case) Let the assumptions of Theorem 3.1 hold and (A1), (A3') replace (A1'), (A3). If additionally*

- (i) *$F(t,\cdot)$ is upper semi-continuous (see* [4, Definition 1.4.1]*) for all $t \in I$,*
- (ii) *$F(t,\cdot)$ has linear growth, i.e., there exists $C \ge 0$ with*

$$\|F(t,x)\| := \sup_{\xi \in F(t,x)} \|\xi\| \le C(1 + \|x\|) \quad (x \in \mathbb{R}^n),$$

then there exists a solution $x(\cdot)$ of (DI) such that for $t \in I$

$$\|x(t) - y(t)\| \le e^{\mu(t-t_0)}\delta_0 + \int_{t_0}^{t} e^{\mu(t-s)}\delta(s)ds\,. \tag{5}$$

3.2. **Numerical approach.** The bounding box can be determined by the following

Algorithm 3.3.

- (i) *Choose an arbitrary starting box $\widetilde{B}_0$, define X as empty list of best approximations and compute for τ_0 grid points $g^\nu \in \widetilde{B}_0$ one best approximation $x^\nu := \pi_{\mathcal{R}_h}(g^\nu)$ by solving (CP)/(DI) and store each of them in X.*
- (ii) *Choose an arbitrary box B_0 that contains all the initial best approximations $x^\nu \in X$, initialize the list of boxes $L := [B_0]$ and $G_0 := B_0$, $\mathcal{B}_0 := \{B_0\}$.*
- (iii) *Start Algorithm 2.2 with $\mathcal{B}_0$ and remove B_0 from L.*
- (iv) *If there are test points $g^\nu \in B_0$ in (iii) such that $\pi_{\mathcal{R}_h}(g^\nu) \notin \operatorname{int} G_0$, collect all neighboring boxes B_μ of B_0 with $\pi_{\mathcal{R}_h}(g^\nu) \in B_\mu$ in the list L and insert all computed best approximations in the list X. G_0 is reset with the union of G_0 and all new boxes B_μ.*
- (v) *If the list L still contains an element, take out some box B_0, set $\mathcal{B}_0 := \{B_0\}$ and go to (iii). Otherwise all best approximations are stored inside X and G_0 contains a bounding box of the reachable set.*

The algorithm ends, since the reachable set is a compact and connected set under our assumptions (A1)–(A3).

4. Examples

All examples are calculated with the implicit Euler method in [11] without the therein required knowledge of the continuity modulus or the OSL/Lipschitz constant. It is also not realized as an iterative method with the need of a $\mathcal{O}(h^2)$-state space discretization in each iteration step.

4.1. **Kenderov's example.** The following problem is set up by Petar Kenderov and is discussed in [13, Example 5.2.1]. Here, the solution funnel is stated first in cylindrical coordinates. By the introduction of a virtual second control $u_\tau(\cdot)$ as in [25], the original linear 2d problem will be a bilinear 3d problem. The nonconvex reachable set with empty interior is a two-dimensional manifold which is rather problematic for methods based on level sets or the HJB approach. The adaptive method can even benefit from this situation, since a lot of boxes can be dropped and the box collection shrinks very quickly to the reachable set.

Example 4.1. *(Kenderov problem in cylindrical coordinates) We consider the following problem in cylindrical coordinates $p(t) = (r(t), \varphi(t), x_\tau(t))$ for $t \in I = [0, 8]$ and $\sigma = \frac{9}{10}$:*

$$r'(t) = u_\tau(t)(\sigma^2 - 1)r(t), \tag{6}$$

$$\varphi'(t) = u_\tau(t)\sigma\sqrt{1-\sigma^2}\, u(t), \tag{7}$$

$$x_\tau'(t) = u_\tau(t), \quad u(t) \in [-1, 3], \quad u_\tau(t) \in [0, 1], \tag{8}$$

$$p(0) = (r_0, \varphi_0, x_{\tau,0})^\top = \left(2\sqrt{2}, \frac{\pi}{4}, 0\right)^\top. \tag{9}$$

Obviously, $x_\tau(t) \in [0, t]$ and the (set-valued) graph of $x_\tau \mapsto \mathcal{R}_{(r,\varphi,x_\tau)}(T, 0, X_0)$ is contained in $\bigcup_{s\in I} \mathcal{R}_{(r,\varphi)}(s, 0, X_0) \times I$.

The previous inclusion suggests that an analytical estimate can be gained by discussion of the linear 2d sub problem (6)–(7) with $u_\tau(t) = 1$ only.

We choose $\hat{u}_\tau(t) = \hat{u}(t) = 1$ as reference control. Then, $\hat{r}(t) = r_0 e^{(\sigma^2-1)t}$ and $\hat{\varphi}(t) = \varphi_0 + \sigma\sqrt{1-\sigma^2}\, t$.

For a general feasible control $u(\cdot)$ we set $F(t, r, \varphi) = \{f(t, r, \varphi, \hat{u}_\tau(t), u(t))\}$ and apply Theorem 3.1 with $r_0 = 2\sqrt{2}$, $\varphi_0 = \frac{\pi}{4}$. The initial error is $\delta_0 = 0$ and $\delta(t)$ as error for the right-hand side can be chosen as $2\sigma\sqrt{1-\sigma^2}$, since

$$\begin{aligned} &\operatorname{dist}\Big(\big(\begin{smallmatrix}\hat{r}'(t)\\ \hat{\varphi}'(t)\end{smallmatrix}\big), F(t, \hat{r}(t), \hat{\varphi}(t))\Big) = \big\|\big(\begin{smallmatrix}\hat{r}'(t)-(\sigma^2-1)\hat{r}(t)\\ \hat{\varphi}'(t)-\sigma\sqrt{1-\sigma^2}\,u(t)\end{smallmatrix}\big)\big\| \\ = \;& \sigma\sqrt{1-\sigma^2}\;|1-u(t)| \le 2\sigma\sqrt{1-\sigma^2} \end{aligned}$$

$F(\cdot,\cdot,\cdot)$ is Lipschitz with constant $1-\sigma^2$:

$$\begin{aligned} &\mathrm{d}_H(F(t, r, \varphi), F(t, \tilde{r}, \tilde{\varphi})) \\ = \;& \big\|\big(\begin{smallmatrix}(\sigma^2-1)(r-\tilde{r})\\ \sigma\sqrt{1-\sigma^2}(u(t)-u(t))\end{smallmatrix}\big)\big\| \le (1-\sigma^2)\big\|\big(\begin{smallmatrix}r\\ \varphi\end{smallmatrix}\big) - \big(\begin{smallmatrix}\tilde{r}\\ \tilde{\varphi}\end{smallmatrix}\big)\big\|. \end{aligned}$$

The solution $x(\cdot)$ of the right-hand side is unique for the chosen $u(\cdot)$ and fulfills by (4) in Theorem 3.1:

$$\left\|\begin{pmatrix}\hat{r}(t)\\ \hat{\varphi}(t)\end{pmatrix} - \begin{pmatrix}r(t)\\ \varphi(t)\end{pmatrix}\right\| \le 2\sigma\sqrt{1-\sigma^2}\,\frac{1}{L}\left(e^{Lt}-1\right) \le 14.76$$

This estimate can be improved by the OSL condition of $F(\cdot,\cdot,\cdot)$ and Theorem 3.2. The right-hand side is OSL with constant $\mu = 0$, since for arbitray $r, \tilde{r}, \varphi, \tilde{\varphi}$ we *have*

$$\Big\langle \tbinom{r}{\varphi} - \tbinom{\tilde r}{\tilde\varphi},\ \tbinom{(\sigma^2-1)r}{\sigma\sqrt{1-\sigma^2}\,u(t)} - \tbinom{(\sigma^2-1)\tilde r}{\sigma\sqrt{1-\sigma^2}\,u(t)} \Big\rangle$$
$$= \ -(1-\sigma^2)(r-\tilde r)^2 + 0 \le 0\cdot \left\| \begin{pmatrix} r \\ \varphi \end{pmatrix} - \begin{pmatrix} \tilde r \\ \tilde\varphi \end{pmatrix} \right\|^2 .$$

We proceed with $\delta_0 = 0$ *and* $\delta(t) = 2\sigma\sqrt{1-\sigma^2}$ *as in the estimate in Theorem 3.1, but now the OSL constant* $\mu = 0$ *replaces* $L = 1-\sigma^2$ *and yields a better (even sharp) estimate:*

$$\left\| \begin{pmatrix} \hat r(t) \\ \hat\varphi(t) \end{pmatrix} - \begin{pmatrix} r(t) \\ \varphi(t) \end{pmatrix} \right\| = 2\sigma\sqrt{1-\sigma^2}\,t \le 6.277.$$

Fig. 4 shows the result of Algorithm 2.2 with bounding box $B_0 = [0,8]\times[-6,7]\times[-3,11]$*. The objective function* $\frac{1}{2}\cdot\|\widetilde g_\rho - \mathcal{T}(\eta_N^h)\|^2$ *is used in the right picture, where* $\mathcal{T}$ *is the transformation from cylindrical into Cartesian coordinates and* $\widetilde g_\rho$ *is a grid point in the bounding box* $[0,8]\times[-4,4]^2$ *with Cartesian coordinates. We note that this approach leads to better results with smaller discretization errors in comparison with the 2d model in Cartesian coordinates in [27].*

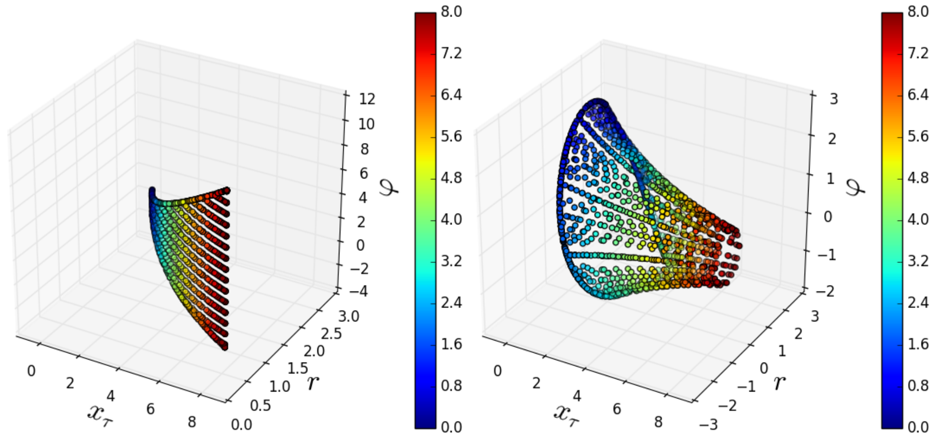

FIGURE 3. Solution funnel for Example 4.1 in cylindrical (left) and Cartesian coordinates (right) constructed by different objective functions.

TABLE 1. Convergence of Algorithm 2.2 for Example 4.1.

number of timesteps	recursion level k	distance to prev. result	est. order of convergence
8	3	–	–
16	4	0.8996	–
32	5	0.4549	0.9837
64	6	0.2341	0.9584

In Table 1 we have for the first time verified numerically the convergence order 1 for the implicit Euler in the realization with the optimization-based approach which is expected by Theorem 2.6.

4.2. **Car model.** The following problem demonstrates for the first time that the adaptive approach based on optimization with the implicit Euler yields a way to calculate the low-dimensional projection of the reachable set of a higher-dimensional nonlinear systems (see Fig. 5 for the 3d funnel and one projection), avoiding the exponential cost of discretizing the whole state space in $\mathbb{R}^6$. The following model has 6 states and 3 controls and thus is extremely difficult for level set methods and pde solvers. For the implicit Euler an approach to setup an a priori lower-dimensional bounding box and restrict computations to the area of interest has not yet been developed.

Example 4.2. *(6d kinematic car model) We consider the following nonlinear problem for* $t \in [0, 3]$*:*

$$x_1'(t) = u_\tau(t) v(t) \cos(\psi(t)), \tag{10}$$

$$x_2'(t) = u_\tau(t) v(t) \sin(\psi(t)), \tag{11}$$

$$x_\tau'(t) = u_\tau(t), \quad v'(t) = u_\tau(t) u_v(t), \tag{12}$$

$$\psi'(t) = u_\tau(t) \frac{v(t)}{l} \tan(\delta(t)), \quad \delta'(t) = u_\tau(t) u_\delta(t), \tag{13}$$

where $z(t) = (x_\tau(t), x_1(t), x_2(t), v(t), \psi(t), \delta(t))$, $l = 4$ *is the distance from the rear axle to the front axle of the car and* $z(0) = (0, 0, 0, 13, 0, 0)$ *is the initial value. The six states and three state constraints are*

x_τ	*time in s,*
x_1, x_2	*x- and y-coordinate of the midpoint of the rear axle of the car in m,*
v	*velocity of the car in m/s with* $v \in [0, 50]$*,*
δ	*steering angle in rad with* $\delta \in [-0.5, 0.5]$*,*
ψ	*yaw angle in rad, with* $\frac{v^2}{l} \tan(\delta) \in [-7, 7]$

and the three controls with control constraints are

$u_\tau \in [0, 1]$	*virtual control for the time,*
$u_v \in [-10, 5]$	*acceleration in* m/s^2*,*
$u_\delta \in [-1, 1]$	*steering angle velocity in rad/s.*

The calculation of analytical estimates for a bounding box of this model as in Example 4.1 yields very pessimistic results in the region of 10^{40}, hence a numerical approach is preferable. The subdivision algorithm can be used without knowing the bounding box G and will terminate rather quickly if the reachable set is in the interior of G. The car model allows to set up intuitive bounds for the initial box.

Since we are interested in the solution funnel which translates into a free end time problem, we already know that the initial value $(x_\tau, x_1, x_2) = (0, 0, 0) \in \mathcal{R}_h$. The maximal value of x_1 can be estimated by solving the problem with $u_\delta(t) \equiv 0$

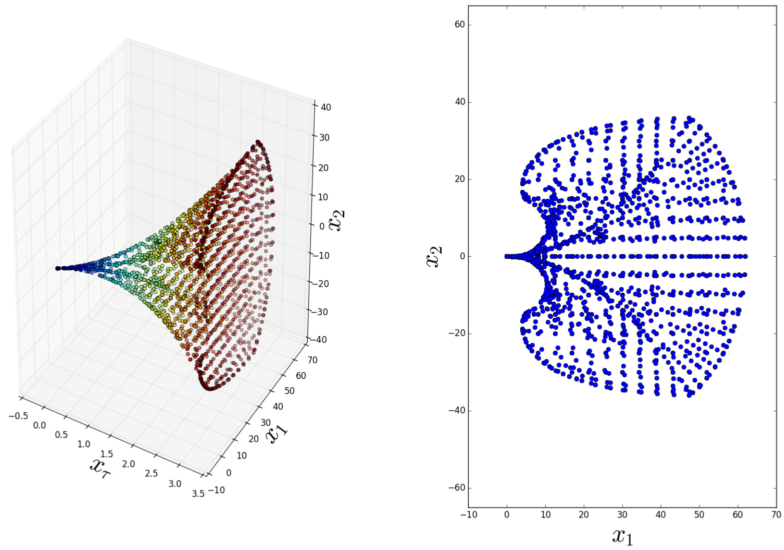

FIGURE 4. Solution funnel for Example 4.2 (left) and the projection on the (x_1, x_2)-coordinates, forming a nonconvex set (right).

and $u_v(t) \equiv 5$, i.e., full acceleration and no steering, since none of the constraints will become active for $t \in [0, 3]$. A simple integration problem for $x_1(t)$ yields on the far right the point $(x_\tau, x_1, x_2) = (3, 61.5, 0) \in \mathcal{R}_h$.

Finding an easy estimate for the distance covered on the x_2-axis is more complicated since the trajectories with maximal speed will inevitably lead to the constraint on the lateral acceleration becoming active. We can either neglect this contraint or just generate a bound with Algorithm 3.3. To illustrate the results more clearly, we only calculate the (x_1, x_2)-projection of the solution-funnel by using the differential equations for x_1, x_2, ψ, v and δ, but substitute the control u_τ with a time-independent parameter $p \in [0, 1]$.

As initial bounding box we choose $B_0 = [-5, 65] \times [-20, 20]$ and the subdivision algorithm generates the result in Fig. 6 (left). We notice that the reachable set touches the border of the chosen bounding box and the picture looks cut off. Restarting the algorithm at the boxes $B_1 = [-5, 65] \times [20, 60]$ and $B_2 = [-5, 65] \times [-20, -60]$ (L=[B_1, B_2] in step (iv)) and merging the results with those generated by using B_0 before, yields the complete set as seen in Fig. 6 (right, after restarts in step (iii) with B_1 and B_2).

In this example the efficiency of the algorithm could even be improved by exploiting the symmetry of the given system and only generating the top or bottom half of the set and mirroring the result.

5. Conclusions

The paper addresses the construction of an initial bounding box, which is required for an optimization-based subdivision approach for the approximation of reachable sets. To this end, we use analytical estimates, which are derived from Filippov theorems, or a numerical projection technique, if analytical estimates

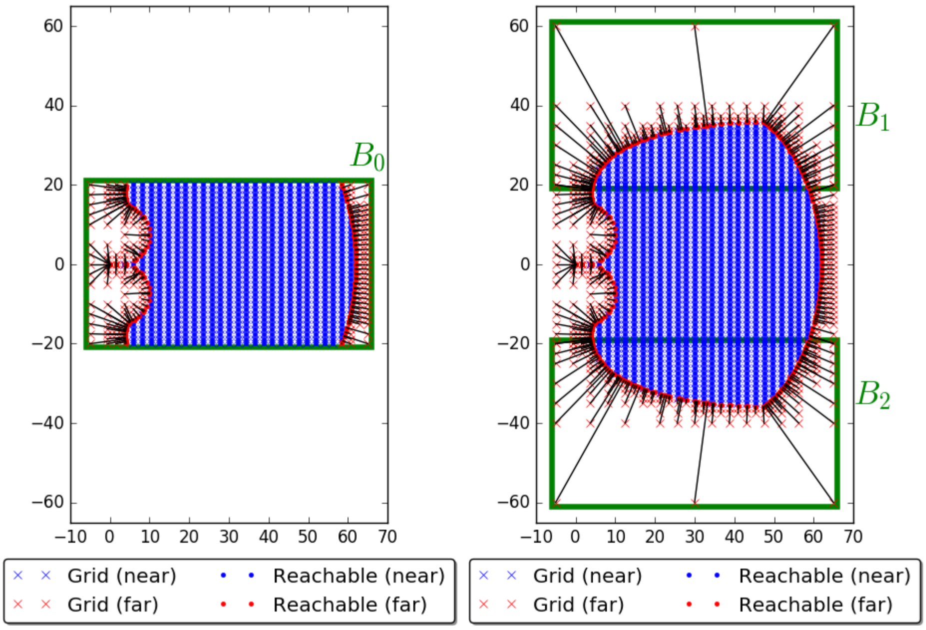

FIGURE 5. Self-finding of the bounding box by Algorithm 3.3 based on the guess B_0 resp. neighboring boxes B_1, B_2 for Example 4.2

are not available. In the latter we exploit a nice feature of the subdivision technique: its restart capability. The subdivision works efficiently even if the initial box substantially overestimates the true reachable set.

REFERENCES

[1] M. Althoff. *Reachability Analysis and its Application to the Safety Assessment of Autonomous Cars*. PhD thesis, Fakultät für Elektrotechnik und Informationstechnik, TU München, Germany, Aug. 2010.

[2] M. Althoff and B. H. Krogh. Reachability analysis of nonlinear differential-algebraic systems. *IEEE Trans. Automat. Control*, 59(2):371–383, 2014.

[3] J.-P. Aubin, A. M. Bayen, and P. Saint-Pierre. *Viability theory. New directions*. Springer, Heidelberg, second edition, 2011. First edition: J.-P. Aubin in *Systems & Control: Foundations & Applications*, Birkhäuser Boston Inc., Boston, MA, 2009.

[4] J.-P. Aubin and H. Frankowska. *Set-Valued Analysis*, volume 2 of *Systems & Control: Foundations & Applications*. Birkhäuser Boston Inc., Boston, MA, 1990.

[5] R. Baier, I. A. Chahma, and F. Lempio. Stability and convergence of Euler's method for state-constrained differential inclusions. *SIAM J. Optim.*, 18(3):1004–1026, 2007.

[6] R. Baier and M. Gerdts. A computational method for non-convex reachable sets using optimal control. In *Proc. of the European Control Conf. (ECC '09), Budapest (Hungary), Aug 23–26, 2009*, pages 97–102, Budapest, 2009. European Union Control Association (EUCA).

[7] R. Baier, M. Gerdts, and I. Xausa. Approximation of reachable sets using optimal control algorithms. *Numer. Algebra Control Optim.*, 3(3):519–548, 2013.

[8] R. Baier and Thuy T. T. Le. Construction of the minimum time function via reachable sets of linear control systems. Part 1: Error estimates. https://arxiv.org/abs/1512.08617, December 29, 2015.

[9] A. M. Bayen, I. M. Mitchell, M. Oishi, and C. Tomlin. Aircraft autolander safety analysis through optimal control-based reach set computation. *J. Guidance Control Dynam.*, 30(1):68–77, 2007.

[10] W.-J. Beyn and J. Rieger. Numerical fixed grid methods for differential inclusions. *Computing*, 81(1):91–106, 2007.

[11] W.-J. Beyn and J. Rieger. The implicit Euler scheme for one-sided Lipschitz differential inclusions. *Discrete Contin. Dyn. Syst. Ser. B*, 14(2):409–428, 2010.

[12] O. Bokanowski, N. Forcadel, and H. Zidani. Reachability and minimal times for state constrained nonlinear problems without any controllability assumption. *SIAM J. Control Optim.*, 48(7):4292–4316, 2010.

[13] I. A. Chahma. Set-valued discrete approximation of state-constrained differential inclusions. *Bayreuth. Math. Schr.*, 67:3–162, 2003.

[14] Xin Chen, E. Ábrahám, and S. Sankaranarayanan. Taylor model flowpipe construction for non-linear hybrid systems. In *Proc. of the 2012 IEEE 33rd Real-Time Systems Sympos. (RTSS '12) in San Juan, PR, Dec 4–7, 2012*, pages 183–192, Los Alamitos, CA, 2012. IEEE.

[15] T. Dang, O. Maler, and R. Testylier. Accurate hybridization of nonlinear systems. In *Proc. of the 13th ACM Internat. Conf. on Hybrid Systems (HSCC '10): Computation and Control in Stockholm, Sweden, Apr 12–16, 2010*, pages 11–19. ACM, New York, 2010.

[16] M. Dellnitz and A. Hohmann. A subdivision algorithm for the computation of unstable manifolds and global attractors. *Numer. Math.*, 75(3):293–317, 1997.

[17] M. Dellnitz, A. Hohmann, O. Junge, and M. Rumpf. Exploring invariant sets and invariant measures. *Chaos*, 7(2):221–228, 1997.

[18] M. Dellnitz, O. Junge, M. Post, and B. Thiere. On target for Venus — set oriented computation of energy efficient low thrust trajectories. *Celestial Mech. Dynam. Astronom.*, 95(1-4):357–370, 2006.

[19] A. Désilles, H. Zidani, and E. Crück. Collision analysis for an UAV. In *AIAA Guidance, Navigation, and Control Conf. 2012 (GNC-12), Aug 13–16, 2012 in Minneapolis, MIN*, page 23 pages. AIAA, 2012.

[20] T. Donchev and E. Farkhi. On the theorem of Filippov-Pliś and some applications. *Control and Cybernet.*, 38(4A):1251–1271, 2009.

[21] A. L. Dontchev and E. M. Farkhi. Error Estimates for Discretized Differential Inclusions. *Computing*, 41(4):349–358, 1989.

[22] M. Falcone and R. Ferretti. *Semi-Lagrangian Approximation Schemes for Linear and Hamilton-Jacobi Equations*, volume 133 of *Applied Mathematics*. SIAM, 2014.

[23] A. Girard, C. Le Guernic, and O. Maler. Efficient computation of reachable sets of linear time-invariant systems with inputs. In *Hybrid systems: computation and control*, volume 3927 of *Lecture Notes in Comput. Sci.*, pages 257–271. Springer, Berlin, 2006.

[24] T. U. Jahn. *A Feasibility Problem Approach for Reachable Set Approximation*. PhD thesis, Fakultät für Mathematik, Physik und Informatik, Univ. Bayreuth, Germany, Dez. 2014.

[25] I. M. Mitchell, A. M. Bayen, and C. J. Tomlin. A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games. *IEEE Trans. Automat. Control*, 50(7):947–957, 2005.

[26] W. Riedl. Optimization-based subdivision algorithm for reachable sets. *Proc. Appl. Math. Mech.*, 14(1):937–938, 2014.

[27] W. Riedl, R. Baier, and M. Gerdts. Optimization-based subdivision algorithm for reachable sets. University of Bayreuth, December 2016. 30 pages, available at https://epub.uni-bayreuth.de/3157/.

[28] I. Xausa. *Verification of Collision Avoidance Systems using Optimal Control and Sensitivity Analysis*. PhD thesis, Fakultät Luft- und Raumfahrttechnik, Univ. der Bundeswehr München, Germany, Dez. 2015.

(Wolfgang Riedl) Lehrstuhl für Ingenieurmathematik, Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany
Email address: `wolfgang.riedl@uni-bayreuth.de`

(Robert Baier) Lehrstuhl für Angewandte Mathematik, Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany
Email address: `robert.baier@uni-bayreuth.de`

(Matthias Gerdts) Ingenieurmathematik, Institut für Mathematik und Rechneranwendung (LRT-1), Universität der Bundeswehr München, 85577 Neubiberg/München, Germany
Email address: `matthias.gerdts@unibw.de`