Stability of the King model – a coercivity-based approach

Master thesis

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1 Introduction

There is hardly any other field of physics which both fascinates and confuses humanity as much as the behaviour of our galaxy does. The aim of astromathematics is to clear up this confusion by establishing rigorous analytical results, which not only allow us to explain acts in the past, but also yield predictions for the prospective behaviour of galaxies.
In fact, since a galaxy contains up to trillions of stars, it is not feasible to model each star individually, which would lead to an N-body problem. We will instead describe the state of a galaxy for a given time $t \in \mathbb{R}$ by its non-negative density function $f = f(t, x, v)$. Here, $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ is an element of phase space, where x denotes the position and v the velocity of a star. Then, integrating the density $f(t)$ over a certain part of phase space yields the mass contained in this region at a given time $t \in \mathbb{R}$.
We restrict our model to the gravitational interaction of stars. In particular, we neglect the influence of collisions, as they are only rarely happening. Therefore the density function f is constant along particle orbits. We will describe the latter by Newton's equation of motion, i.e., an individual particle with position x, velocity v and unit mass satisfies

$$\dot{x} = v, \qquad \dot{v} = -\partial_x U(t, x),$$

where $U = U(t, x)$ is the gravitational potential of the galaxy. This conservation property leads to the **Vlasov equation**

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0.$$

By Newton's law for gravity, paired with the common boundary condition at spatial infinity, the gravitational potential obeys the **Poisson equation**

$$\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(t, x) = 0 \text{ for } t \in \mathbb{R},$$

where ρ is the spatial mass density induced by f, more precisely

$$\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v)\,\mathrm{d}v \text{ for } t \in \mathbb{R},\ x \in \mathbb{R}^3.$$

Note that we normalized the gravitational constant to unity. The latter three equations combined form the three dimensional, gravitational **Vlasov-Poisson system**, which is a closed system of non-linear partial differential equations describing the time evolution of a galaxy.

A much more detailed motivation of this system can be found in [20, 27], for physical background we refer to [4]. An overview over some systems closely related to the Vlasov-Poisson system, in particular covering the relativistic and fluid cases, is given in [23].
The aim of this thesis is to analyse steady states of this system which only depend on their self-induced particle energy and are called "isotropic", i.e., we consider a time independent density of the form

$$f_0(x,v) = \varphi(E(x,v))$$

for some appropriate function $\varphi\colon \mathbb{R} \to [0,\infty[$, where the particle energy

$$E(x,v) := \frac{1}{2}|v|^2 + U_0(x)$$

is induced by the associated stationary potential U_0. A question of particular interest is the stability of these steady states, since the stability of an equilibrium determines whether or not it appears in reality. In the last chapter of this thesis we prove a stability result for the King model, where the ansatz function φ is of the form

$$\varphi(E) = \left(e^{E_0-E} - 1\right)_+, \quad E \in \mathbb{R}$$

for some cut-off energy $E_0 < 0$. This model – named in honour of I. King [18] – is of particular interest from an astrophysics point of view, since it describes isothermal galaxies. However, for the actual proof of this stability result we need a couple of tools. These tools are also of interest by themselves.
In the first chapter following the introduction we begin by considering the Vlasov-Poisson system under the assumption of spherical symmetry. We then rigorously define the class of isotropic steady states and analyse their effective potential similar to [10, 19], which is a crucial quantity of equilibria in the radial setting. Lastly, we introduce some technicalities needed later on, more precisely homogeneous Sobolev spaces and the spherical symmetry of functions which are only defined almost everywhere.
Chapter 3 covers the linear transport operator

$$\mathcal{D} := v \cdot \partial_x - \partial_x U_0 \cdot \partial_v,$$

where U_0 is the time independent potential corresponding to a linearly stable equilibrium, i.e., φ is strictly decreasing on its support, cf. [5, 17]. As a matter of fact, this operator naturally appears when linearising the Vlasov-Poisson system about the steady state inducing U_0, see [17, 27]. Surprisingly, a weak extension of $\mathcal{D}$ is also used in [9, 19] to obtain non-linear stability results. We will therefore carefully define $\mathcal{D}$ in a weak sense on some dense subset of a radial and weighted L^2-space and prove the required properties of the resulting operator, including

the skew-adjointness and a characterisation of its kernel. In fact, the kernel of $\mathcal{D}$ has first been rigorously analysed in [3] for smooth and radial functions and the result is well known as “Jeans’ theorem”, since it was first asserted by J. Jeans [15, 16] that functions in this kernel can only depend on the particle energy of the steady state and the modulus of the angular momentum. However, the existing proof can not be directly applied to the weak extension of the transport operator, which is why we provide an alternate proof adapted to this new setting.
In the next chapter, we use $\mathcal{D}$ to investigate another operator, which we call “Guo-Lin operator” due to its appearance in [9] and which is of the form

$$\mathcal{A}_0\psi \coloneqq -\Delta\psi + 4\pi \int_{\mathbb{R}^3} |\varphi'(E(\cdot, v))| \cdot \mathcal{P}\psi(\cdot, v)\,\mathrm{d}v - 4\pi \int_{\mathbb{R}^3} |\varphi'(E(\cdot, v))|\,\mathrm{d}v \cdot \psi$$

for appropriate $\psi\colon \mathbb{R}^3 \to \mathbb{R}$, where $\mathcal{P}$ is the orthogonal projection onto $\ker(\mathcal{D})$. It turns out that the most important tool to prove the non-linear stability of the King model is the coercivity of $\mathcal{A}_0$, i.e., a bound of the following kind:

$$\inf_{\psi \in H, \|\psi\| \neq 0} \frac{\langle \mathcal{A}_0\psi, \psi\rangle_2}{\|\psi\|^2} > 0,$$

where H is some reasonable function space with norm $\|\cdot\|$. Unfortunately, Theorem 4.6 implies that this coercivity estimate does not hold true when choosing $H = H^2_r(\mathbb{R}^3)$ and $\|\cdot\| = \|\cdot\|_{H^1(\mathbb{R}^3)}$, which is the setting Guo & Lin used in [9]. Instead, we show the coercivity of the Guo-Lin operator with $H = \dot{H}^1_r(\mathbb{R}^3)$ and its semi-norm $\|\cdot\| = \|\nabla \cdot\|_2$ by using Antonov’s coercivity bound [2, 10]. Here, $\dot{H}^1_r(\mathbb{R}^3)$ is the radial homogeneous Sobolev space introduced in Chapter 2. Despite this result being slightly weaker than the one from Guo & Lin, it still suffices for the application in the final chapter.
We can then prove the stability of the King model against spherically symmetric perturbations in Chapter 5 using the tools from above. This whole chapter is based on the second part of [9], where an equal result is shown by a similar approach. Some of the techniques applied there originate in [22], where non-linear stability for the $\frac{3}{2}$-dimensional Vlasov-Maxwell system has been shown.

2 Preliminaries

2.1 The spherically symmetric Vlasov-Poisson system

As a beginning, we want to introduce the general setting in which we will work from now on. All the results in this section are quite basic and can be found in [27] in much greater detail than they are presented here.
As motivated above, we consider the three dimensional, gravitational Vlasov-Poisson system

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0,$$

$$\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(t,x) = 0 \text{ for } t \geq 0,$$

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v \text{ for } t \geq 0,\ x \in \mathbb{R}^3,$$

with initial condition

$$f(0) = \mathring{f} \text{ on } \mathbb{R}^3 \times \mathbb{R}^3$$

for a given function $\mathring{f} \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. If not stated explicitly otherwise, $\cdot$ always denotes the scalar product. We will always restrict ourselves to non-negative, compactly supported and smooth initial data, more precisely $\mathring{f} \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\mathring{f} \geq 0$, as those launch unique global classical solutions $[0,\infty[\ni t \mapsto f(t) \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ of the Vlasov-Poisson system, see [27].
In addition, we only consider the case where $\mathring{f}$ is spherically symmetric on $\mathbb{R}^3 \times \mathbb{R}^3$.

Definition 2.1: *Let $n \in \mathbb{N}$.*

*a) $g \colon \mathbb{R}^n \to \mathbb{R}$ is **spherically symmetric on** $\mathbb{R}^n$, if $g(Ay) = g(y)$ for every rotation matrix $A \in \mathrm{SO}(n)$ and $y \in \mathbb{R}^n$.*

*b) $g \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is **spherically symmetric on** $\mathbb{R}^n \times \mathbb{R}^n$, if $g(Ay, Aw) = g(y,w)$ for every rotation matrix $A \in \mathrm{SO}(n)$ and $y, w \in \mathbb{R}^n$.*

Note that the symmetry on $\mathbb{R}^3 \times \mathbb{R}^3$ differs from the one on $\mathbb{R}^6$. To be constantly reminded of this crucial difference, we will always denote the phase space by

$\mathbb{R}^3 \times \mathbb{R}^3$. Furthermore, due to the uniqueness of classical solutions of the Vlasov-Poisson system, the spherical symmetry of $\mathring{f}$ is preserved by the system, i.e., $f(t)$ is spherically symmetric on $\mathbb{R}^3 \times \mathbb{R}^3$ for any $t \geq 0$. This causes the gravitational potential $U(t)$ and the spatial density $\rho(t)$ to be spherically symmetric on $\mathbb{R}^3$ as well.
Moreover, a spherically symmetric function on $\mathbb{R}^3 \times \mathbb{R}^3$ can be expressed in the coordinates

$$r := |x|, \quad w := \frac{x \cdot v}{r}, \quad L := |x \times v|^2,$$

where r is the spatial radius, w is the radial velocity and L is the modulus of the angular momentum squared. In these coordinates, with some abuse of notation, the Vlasov-Poisson system for the unknowns

$$f(t, x, v) = f(t, r, w, L), \quad U(t, x) = U(t, r), \quad \rho(t, x) = \rho(t, r)$$

takes the form

$$\partial_t f + w \partial_r f + \left(\frac{L}{r^3} - \partial_r U \right) \partial_w f = 0,$$
$$\left(\partial_r^2 + \frac{2}{r} \partial_r \right) U = 4\pi\rho, \quad \lim_{r \to \infty} U(t, r) = 0 \text{ for } t \geq 0,$$
$$\rho(t, r) = \frac{\pi}{r^2} \int_0^\infty \int_{\mathbb{R}} f(t, r, w, L) \, \mathrm{d}w \, \mathrm{d}L \text{ for } t \geq 0, \ r > 0,$$

cf. [10].

2.2 Isotropic states and their effective potential

For our stability analysis we restrict ourselves to isotropic steady states, i.e., steady states depending only on their self-induced particle energy. We will therefore carefully define this class of steady states based on [27].

Definition 2.2: *Let* $\boldsymbol{U_0} \in C^2(\mathbb{R}^3)$ *be a time independent and spherically symmetric potential vanishing at infinity, i.e.,* $\lim_{|x| \to \infty} U_0(x) = 0$. *In addition, let*

$$\boldsymbol{E}(x, v) := \frac{1}{2}|v|^2 + U_0(x), \quad x, v \in \mathbb{R}^3$$

be its induced particle energy. Then E *is obviously constant along solutions of the characteristic system*

$$\dot{x} = v, \qquad \dot{v} = -\partial_x U_0(x).$$

This means that every function depending only on E solves the Vlasov equation for the potential U_0 (up to regularity issues), which leads to the ansatz

$$\boldsymbol{f_0}(x,v) := \varphi(E(x,v)) \text{ for } (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Here, $\varphi \in L^\infty_{loc}(\mathbb{R})$ is non-negative and there exists a negative ***cut-off energy*** $\boldsymbol{E_0} < 0$ *satisfying*

(i) $\varphi(E) = 0$ *for* $E \geq E_0$.

(ii) $\varphi \in C^1(]-\infty, E_0[)$.

(iii) There exists $\eta < E_0$ such that $\varphi > 0$ on $[\eta, E_0[$.

Then f_0 is an ***isotropic steady state of the Vlasov-Poisson system*** *(or just isotropic state), iff*

$$\Delta U_0 = 4\pi\rho_0 = 4\pi \int_{\mathbb{R}^3} \varphi(\frac{1}{2}|v|^2 + U_0)\,\mathrm{d}v \text{ on } \mathbb{R}^3,$$

where

$$\boldsymbol{\rho_0} \colon \mathbb{R}^3 \to [0,\infty[,\ \rho_0(x) := \int_{\mathbb{R}^3} f_0(x,v)\,\mathrm{d}v$$

is the time independent spatial density induced by f_0.

Before we get to the properties of such equilibria, we first present the two most important and popular classes of isotropic states. First, there are the **polytropes**, where φ is of the form

$$\varphi(E) = (E_0 - E)_+^k$$

for some $0 \leq k < \frac{7}{2}$. Here, the subscript $+$ denotes the positive part. The other important example is the so called **King model** given by

$$\varphi(E) = \left(e^{E_0 - E} - 1\right)_+.$$

Indeed, for the stability analysis in Chapter 5, we will restrict ourselves to models of the latter kind.
Now we want to note some well known, but very important properties of general isotropic states. However, we refer to [3, 23, 28] for a much more detailed discussion, in particular concerning the existence theory of these states.

Remark 2.3: *Let f_0 be an isotropic steady state of the Vlasov-Poisson system.*

a) We require U_0 to be spherical symmetric on $\mathbb{R}^3$. However, the results in [8] imply that every isotropic steady state has to be spherically symmetric anyway, which means that we do not lose steady states by making this assumption, see also [23, 26].

b) *f_0 and ρ_0 inherit the spherical symmetry of U_0, i.e., f_0 is spherically symmetric on $\mathbb{R}^3 \times \mathbb{R}^3$ and ρ_0 is spherically symmetric on $\mathbb{R}^3$. With some abuse of notation, we will denote the radial functions by the same symbols, i.e., $U_0(x) = U_0(|x|)$, $\rho_0(x) = \rho_0(|x|)$ and so on.*
This allows us to use the (r, w, L)-coordinates known from Section 2.1, in which the particle energy takes the form

$$E(r, w, L) = \frac{1}{2}w^2 + U_0(r) + \frac{L}{2r^2}.$$

c) *Due to the negative cut-off energy $E_0 < 0$ and the boundary condition of the gravitational potential U_0, the steady state is compactly supported with*

$$\operatorname{supp}(f_0) \subset \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid E(x, v) \le E_0\} \subset \mathbb{R}^3 \times \mathbb{R}^3.$$

d) *The negative cut-off energy also causes the steady state to have finite mass*

$$\boldsymbol{M_0} \coloneqq \int_{\mathbb{R}^3} \rho_0(x)\,\mathrm{d}x = 4\pi \int_0^\infty r^2 \rho_0(r)\,\mathrm{d}r = \lim_{r\to\infty} m_0(r) \in \left]0, \infty\right[,$$

where $\boldsymbol{m_0}(r) \coloneqq 4\pi \int_0^r s^2 \rho_0(s)\,\mathrm{d}s$ denotes the mass "inside" the radius $r > 0$.

e) *Integration of the radial Poisson equation yields the following explicit formula for the gravitational potential and its derivative for $r > 0$:*

$$U_0(r) = -\frac{4\pi}{r} \int_0^r s^2 \rho_0(s)\,\mathrm{d}s - 4\pi \int_r^\infty s\rho_0(s)\,\mathrm{d}s,$$
$$U_0'(r) = \partial_r U_0(r) = \frac{4\pi}{r^2} \int_0^r s^2 \rho_0(s)\,\mathrm{d}s = \frac{m_0(r)}{r^2}.$$

In fact, isotropic states are quite general and we have to restrict ourselves to a smaller class of steady states later on. However, the assumptions from Definition 2.2 suffice to show some useful properties of the "effective potential" which we need in the following:

Definition & Theorem 2.4: *For a fixed isotropic state f_0 and $L \ge 0$ we define the **effective potential** as*

$$\boldsymbol{\psi_L} \colon \left]0, \infty\right[\to \mathbb{R},\ \psi_L(r) \coloneqq U_0(r) + \frac{L}{2r^2}.$$

Note that this quantity appears in the particle energy when expressed in (r, w, L)-coordinates. We claim the following properties:

a) *For any $L > 0$ there exists a unique $r_L > 0$ such that*

$$\min_{]0,\infty[}(\psi_L) = \psi_L(r_L) < 0.$$

Moreover, the mapping $\left]0, \infty\right[\ni L \mapsto r_L$ is continuously differentiable.

b) For any $L > 0$ and $E \in]\psi_L(r_L), 0[$ there exist two unique radii $r_\pm(E, L)$ satisfying

$$0 < r_-(E, L) < r_L < r_+(E, L) < \infty$$

and such that $\psi_L(r_\pm(E, L)) = E$. In addition, the functions

$$\{(E, L) \in]-\infty, 0[\times]0, \infty[\mid \psi_L(r_L) < E\} \ni (E, L) \mapsto r_\pm(E, L)$$

are continuously differentiable.

c) For any $L > 0$ we have $\psi_L(r_L) \geq U_0(0)$ and $\psi_L(r_L) \geq -\frac{M_0^2}{2L}$.

d) For any $L > 0$ and $E \in]\psi_L(r_L), 0[$ the radii $r_\pm(E, L)$ from above satisfy

$$\frac{L}{2M_0} \leq r_-(E, L) < r_L < r_+(E, L) \leq -\frac{M_0}{E}.$$

e) For any $L > 0$, $E \in]\psi_L(r_L), 0[$ and $r \in [r_-(E, L), r_+(E, L)]$ we have the concavity estimate

$$E - \psi_L(r) \geq L \cdot \frac{(r_+(E, L) - r) \cdot (r - r_-(E, L))}{2r^2 r_-(E, L) r_+(E, L)}.$$

All these results can be found in [10, 19]. For the sake of completeness, we will also prove them here.

Proof:

a) Since $\psi_L'(r) = U_0'(r) - \frac{L}{r^3} = r^{-2}(m_0(r) - \frac{L}{r})$ by Remark 2.3 for $r > 0$, $\psi_L'(r) = 0$ is equivalent to $m_0(r) - \frac{L}{r} = 0$. Due to the mapping $]0, \infty[\ni r \mapsto m_0(r) - \frac{L}{r}$ being strictly increasing on $]0, \infty[$ and

$$\lim_{r \to 0} \left(m_0(r) - \frac{L}{r} \right) = -\infty, \ \lim_{r \to \infty} \left(m_0(r) - \frac{L}{r} \right) = M_0 > 0,$$

there exists a unique radius $r_L > 0$ with $\psi_L'(r_L) = 0$ as well as $\psi_L'(r) < 0$ iff $0 < r < r_L$ and $\psi_L'(r) > 0$ if and only if $r > r_L$. This monotonicity together with $\lim_{r \to 0} \psi_L(r) = \infty$ and $\lim_{r \to \infty} \psi_L(r) = \lim_{r \to \infty} U_0(r) = 0$ implies that $\psi_L(r_L)$ is indeed negative and the minimal value of ψ_L on $]0, \infty[$.
Furthermore, since

$$\frac{d}{dr} \left(m_0(r) - \frac{L}{r} \right) = 4\pi r^2 \rho_0(r) + \frac{L}{r^2} > 0$$

for all $r > 0$, we can obtain the continuous differentiability by the implicit function theorem.

b) The monotonicity of ψ_L from part a) together with the limit of $\psi_L(r)$ as $r \to 0$ and $r \to \infty$ directly yields the existence and uniqueness of $r_\pm(E,L)$ with the claimed properties. In particular, since $\psi_L'(r) \neq 0$ for $r \neq \psi_L(r_L)$, the implicit function theorem once again implies that the mapping $(E,L) \mapsto r_\pm(E,L)$ is indeed continuously differentiable on the set given above.

c) The first estimate is easily obtain from $U_0'(r) \geq 0$ for $r \geq 0$, since this implies $U_0(0) = \min(U_0)$.
For the second estimate, we first note that for all $r > 0$

$$U_0(r) = -\frac{m_0(r)}{r} - 4\pi \int_r^\infty s\rho_0(s)\,\mathrm{d}s \geq$$
$$\geq -\frac{1}{r}\left(m_0(r) + 4\pi \int_r^\infty s^2\rho_0(s)\,\mathrm{d}s\right) = -\frac{M_0}{r}.$$

Hence,

$$\psi_L(r_L) \geq -\frac{M_0}{r_L} + \frac{L}{2r_L^2} = -\frac{m_0(r_L)}{L}M_0 + \frac{m_0^2(r_L)}{2L} =$$
$$= -\frac{M_0^2}{2L}\left(2\frac{m_0(r_L)}{M_0} - \frac{m_0^2(r_L)}{M_0^2}\right) \geq -\frac{M_0^2}{2L},$$

where we used $M_0 \geq m_0(r_L) = \frac{L}{r_L}$.

d) Estimating $U_0(r)$ like above, we obtain that every $r > 0$ with $E - \psi_L(r) > 0$ also satisfies $E + \frac{M_0}{r} - \frac{L}{2r^2} > 0$. Solving this quadratic inequality, we obtain

$$\frac{L}{M_0 + \sqrt{M_0^2 + 2EL}} < r < \frac{L}{M_0 - \sqrt{M_0^2 + 2EL}},$$

note that $M_0^2 + 2EL > 0$ for $0 > E > \psi_L(r_L)$ by c). Therefore

$$r_-(E,L) \geq \frac{L}{M_0 + \sqrt{M_0^2 + 2EL}} > \frac{L}{2M_0},$$
$$r_+(E,L) \leq \frac{L}{M_0 - \sqrt{M_0^2 + 2EL}} = \frac{-M_0 - \sqrt{M_0^2 + 2EL}}{2E} < -\frac{M_0}{E}.$$

e) For $r \in [r_-(E,L), r_+(E,L)]$ let

$$\xi(r) \coloneqq E - \psi_L(r) - L \cdot \frac{(r_+(E,L) - r)\cdot(r - r_-(E,L))}{2r^2 r_-(E,L) r_+(E,L)}.$$

Then the radial Poisson equation yields

$$\frac{d^2}{dr^2}\left[r\xi(r)\right] = -2\psi_L'(r) - r\psi_L''(r) + \frac{L}{r^3} = -\frac{1}{r}\cdot\frac{d}{dr}\left[r^2\psi_L'(r)\right] + \frac{L}{r^3} =$$
$$= -\frac{1}{r}\cdot\frac{d}{dr}\left[r^2 U_0'(r)\right] = -4\pi r^2\rho_0(r) \leq 0.$$

Thus, the mapping $[r_-(E,L), r_+(E,L)] \ni r \mapsto r\xi(r) \in \mathbb{R}$ is concave with $\xi(r_\pm(E,L)) = 0$, which implies the non-negativity of ξ on the interval $[r_-(E,L), r_+(E,L)]$ and therefore concludes the proof. □

We will need the two radii $r_\pm$ quite often later on, since they bound the r-range of a trajectory, see Section 3.3. However, it will turn out to be very useful to define $r_\pm(E,L)$ for as many E, L as possible, in order to avoid constant distinction of cases.
For this sake, first note that there exists a unique radius $r_-(E,L) > 0$ such that $\psi_L(r_-(E,L)) = E$ in the case $L > 0$ and $E \geq 0$ as well, since $\lim_{r\to 0} \psi_L(r) = \infty$ and $\lim_{r\to\infty} \psi_L(r) = 0$. Now let

$$r_-(E,L) \coloneqq r_L \eqqcolon r_+(E,L) \text{ if } L > 0 \text{ and } E \leq \psi_L(r_L),$$
$$r_-(E,L) \eqqcolon r_+(E,L) \text{ if } L > 0 \text{ and } E \geq 0.$$

2.3 Homogeneous Sobolev spaces

In the following, we need a certain kind of Sobolev space which is not very common. We therefore explicitly define it here and prove some useful properties.

Definition 2.5: *Let*

$$\dot{\boldsymbol{H}}^{\boldsymbol{1}}(\mathbb{R}^3) \coloneqq \{f \in L^2_{loc}(\mathbb{R}^3) \mid \nabla f \in L^2(\mathbb{R}^3;\mathbb{R}^3)\}$$

be the three dimensional, ***homogeneous Sobolev space*** *of first order.*

There are several ways to define homogeneous Sobolev spaces. In fact, Definition 2.5 has the disadvantage that $\|\nabla \cdot\|_2$ is only a semi-norm on $\dot{H}^1(\mathbb{R}^3)$, since $\|\nabla f\|_2 = 0$ does not imply $f = 0$ almost everywhere. To solve this issue, one could work with equivalence classes containing functions which are a.e. equal up to the addition of a constant, i.e., sharing the same gradient. However, it would then be much more difficult to work with the function itself, since it would only be fixed up to the addition of a constant.
Another elegant way of defining the homogenous Sobolev space is completing the space $C_c^\infty(\mathbb{R}^3)$ with respect the norm $\|\nabla \cdot\|_2$. However, this also leads to the latter space and its problems, which is why we chose the definition from above.
We now prove some Poincaré type estimates by applying the ones known from regular Sobolev spaces.

Lemma 2.6: *Let $\Omega \subset \mathbb{R}^3$ be a bounded, non-empty domain with C^1 boundary. Then there exists a constant $C > 0$, only depending on Ω, such that*

$$\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)} + \lambda(\Omega)^{-\frac{1}{2}}\left|\int_\Omega f\right|$$

for every $f \in \dot{H}^1(\mathbb{R}^3)$.

Proof: This is just an easy corollary of Poincaré's inequality (see [7]), note that $f \in H^1(\Omega)$ since Ω is bounded. □

The lemma above becomes even more useful if we can assure $\int_\Omega f = 0$. This can be achieved for any function in $\dot{H}^1(\mathbb{R}^3)$ by just adding a constant, which does not change the semi-norm $\|\nabla \cdot\|_2$. In this case we even obtain the following:

Lemma 2.7: *Let $\Omega \subset \mathbb{R}^3$ be a bounded, non-empty domain with C^1 boundary. Then there exists a constant $C > 0$, only depending on Ω, such that*

$$\|f\|_{L^6(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$$

for all $f \in \dot{H}^1(\mathbb{R}^3)$ satisfying $\int_\Omega f = 0$; in particular we get $f \in L^6(\Omega)$.

Proof: Let $f \in \dot{H}^1(\mathbb{R}^3)$ with $\int_\Omega f = 0$. From Lemma 2.6 it follows that $\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$. Moreover, a basic corollary of the Gagliardo-Nirenberg-Sobolev inequality (see [7]) yields $\|f\|_{L^6(\Omega)} \leq C\|f\|_{H^1(\Omega)}$. We conclude the desired inequality by combining these two estimates. □

In addition, we get the compact embedding of $\dot{H}^1(\mathbb{R}^3)$ into $L^2(\Omega)$ if we restrict ourselves to functions with vanishing integral over Ω like in Lemma 2.7, more precisely:

Lemma 2.8: *Let $\Omega \subset \mathbb{R}^3$ be a bounded, non-empty domain with C^1 boundary. Furthermore, let $A \subset \dot{H}^1(\mathbb{R}^3)$ be bounded with respect to $\|\nabla \cdot\|_2$, that is to say $\sup_{f\in A} \|\nabla f\|_2 < \infty$, and all $f \in A$ satisfy $\int_\Omega f = 0$.*
Then A is precompact in $L^2(\Omega)$.

Proof: Since A is bounded with respect to $\|\nabla \cdot\|_2$ and for each function $f \in A$ the integral $\int_\Omega f$ vanishes, Lemma 2.6 yields the boundedness of A with respect to $\|\cdot\|_{L^2(\Omega)}$ as well. In other words, A is a bounded subset of $H^1(\Omega)$. Due to the properties of Ω, $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, cf. [1, 7]. □

2.4 Almost everywhere radial functions

Since we are interested in the spherically symmetric Vlasov-Poisson system, all the L^2-spaces appearing later on can be restricted to radial functions as well. However, the definition of this symmetry is not as straight forward as for smooth (and in particular pointwise defined) functions, since we only work with equivalence classes of functions in L^2. We therefore carefully define this symmetry and prove some intuitive and useful characterisations by reducing the problem to smooth functions:

Lemma & Definition 2.9: *Let $f \in L^1_{loc}(\mathbb{R}^3)$. Then the following statements are equivalent:*

(i) *For all $A \in \mathrm{SO}(3)$ we have $f = f(A\cdot)$ almost everywhere on $\mathbb{R}^3$, where the set of measure zero may depend on the rotation matrix A.*

(ii) *There exists $(f_k)_{k\in\mathbb{N}} \subset C_r^\infty(\mathbb{R}^3)$ such that $f_k \to f$ in $L^1(B_R(0))$ for all $R > 0$, where $C_r^\infty(\mathbb{R}^3) := \{g \in C^\infty(\mathbb{R}^3) \mid g$ is spherically symmetric on $\mathbb{R}^3\}$.*

(iii) *There exists $f^r\colon [0,\infty[\to \mathbb{R}$ measurable such that $f = f^r(|\cdot|)$ a.e. on $\mathbb{R}^3$.*

*If f has these properties, we will call it **spherically symmetric almost everywhere on** $\mathbb{R}^3$. Also note that in this case f^r is uniquely defined a.e. on $[0,\infty[$.*

Proof:

(i) $\Rightarrow$ *(ii)*: Let $J \in C_c^\infty(B_1(0))$ be a spherically symmetric mollifyer, which means that $J \geq 0$ and $\int_{\mathbb{R}^3} J = 1$. As usual, let $J_k := k^3 J(k\cdot)$ for $k \in \mathbb{N}$. Obviously, $J_k \in C_c^\infty(B_{\frac{1}{k}}(0))$ is spherically symmetric itself and $f_k := J_k * f \to f$ in $L^1(B_R(0))$ for any $R > 0$ by basic convolution theory, cf. [21]. The key of this whole proof is that f_k inherits the symmetry of f, since

$$\begin{aligned} f_k(Ax) = (J_k * f)(Ax) &= \int_{\mathbb{R}^3} J_k(Ax - y) f(y)\,\mathrm{d}y = \\ &= \int_{\mathbb{R}^3} J_k(Ax - Az) f(Az)\,\mathrm{d}z = \int_{\mathbb{R}^3} J_k(A(x-z)) f(Az)\,\mathrm{d}z = \\ &= \int_{\mathbb{R}^3} J_k(x-z) f(z)\,\mathrm{d}z = (J_k * f)(x) = f_k(x) \end{aligned}$$

for $A \in \mathrm{SO}(3)$ and $x \in \mathbb{R}^3$.

(ii) $\Rightarrow$ *(iii)*: By defining $f_k^r\colon [0,\infty[\to \mathbb{R}$, $f_k^r(r) := f_k(re_1)$, we have $f_k = f_k^r(|\cdot|)$ on $\mathbb{R}^3$ for any $k \in \mathbb{N}$. A standard change of variables then yields

$$\begin{aligned} \|f_k - f_l\|_{L^1(B_R(0))} &= \int_{B_R(0)} |f_k(x) - f_l(x)|\,\mathrm{d}x = \\ &= 4\pi \int_0^R r^2 |f_k^r(r) - f_l^r(r)|\,\mathrm{d}r \end{aligned}$$

for $k, l \in \mathbb{N}$, which means that $(f_k^r)_{k\in\mathbb{N}}$ is a Cauchy-sequence in

$$L^1_{(r\mapsto r^2)}([0,R]) := \{g\colon [0,R] \to \mathbb{R} \text{ measurable} \mid \int_0^R r^2 |g(r)| < \infty\}.$$

As a weighted, one-dimensional L^1-space, $L^1_{(r\mapsto r^2)}([0,R])$ is complete, i.e., there exists $f^r \in L^1_{(r\mapsto r^2)}([0,R])$ such that $f^r_k \to f^r$ in $L^1_{(r\mapsto r^2)}([0,R])$. Changing back into the original variables, we arrive at

$$\begin{aligned}\|f^r_k - f^r\|_{L^1_{(r\mapsto r^2)}([0,R])} &= \int_0^R r^2 |f^r_k(r) - f^r(r)|\,\mathrm{d}r = \\ &= \frac{1}{4\pi}\int_{B_R(0)} |f_k(x) - f^r(|x|)|\,\mathrm{d}x\end{aligned}$$

for $k \in \mathbb{N}$, i.e., $f_k \to f^r(|\cdot|)$ in $L^1(B_R(0))$. Since L^1-limits are unique almost everywhere, we can conclude $f = f^r(|\cdot|)$ a.e. on $\mathbb{R}^3$.

(iii) $\Rightarrow$ *(i)*: Obvious. □

In a completely analogous fashion we can also define the spherical symmetry on $\mathbb{R}^3 \times \mathbb{R}^3$ and prove similar characterisations:

Lemma & Definition 2.10: *Let $f \in L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3)$. Then the following statements are equivalent:*

(i) For all $A \in \mathrm{SO}(3)$ we have $f(x,v) = f(Ax, Av)$ for almost every $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$, where the set of measure zero may depend on the matrix A.

(ii) There exists $(f_k)_{k\in\mathbb{N}} \subset C^\infty_r$ such that $f_k \to f$ in $L^1(B^3_R(0) \times B^3_R(0))$ for every $R > 0$, where $C^\infty_r(\mathbb{R}^3 \times \mathbb{R}^3)$ denotes the space of all infinitely differentiable and spherically symmetric functions on $\mathbb{R}^3 \times \mathbb{R}^3$.

(iii) There exists $f^r\colon [0,\infty[\times\mathbb{R}\times[0,\infty[\to \mathbb{R}$ measurable such that $f(x,v) = f^r(|x|, \frac{x\cdot v}{|x|}, |x\times v|^2)$ for almost every $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$.

*If f has these properties, we will call it **spherically symmetric almost everywhere on** $\mathbb{R}^3 \times \mathbb{R}^3$. Also note that in this case f^r is uniquely defined a.e. on $[0,\infty[\times\mathbb{R}\times[0,\infty[$.*

3 The transport operator

In this whole chapter, let $f_0 = \varphi \circ E$ be a fixed isotropic state in the sense of Definition 2.2. In addition, we assume that φ is decreasing on its support, i.e., $\varphi'(E) < 0$ for $E < E_0$. The latter property corresponds to the linear stability of the equilibrium f_0, see [5, 17]. However, all the results below could be generalised to larger classes of steady states.
Let Ω_0 denote the set where f_0 does not vanish, that is to say

$$\boldsymbol{\Omega_0} \coloneqq \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid f_0(x, v) \neq 0\} = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid E(x, v) < E_0\}.$$

Note that $\Omega_0 \subset \mathbb{R}^3 \times \mathbb{R}^3$ is a bounded, spherically symmetric domain.
The aim of this chapter is to define and analyse the transport operator induced by the steady state f_0 given by

$$\mathcal{D}f = v \cdot \partial_x f - \partial_x U_0 \cdot \partial_v f$$

for suitable $f : \Omega_0 \to \mathbb{R}$.
In fact, it does not suffice to define $\mathcal{D}$ on classically or weakly differentiable functions for the application in the non-linear stability analysis. Instead, we define the whole transport operator $\mathcal{D}$ in a weak sense on some dense subset of a suitable L^2-space. Our approach is similar to the one for the definition of weak derivatives, see [21].
When choosing the right domain of definition, the resulting operator is not only skew-symmetric, but also skew-adjoint with respect to a properly weighted L^2-scalar product. We also provide an explicit characterisation of the kernel of $\mathcal{D}$, which is a generalisation of Jeans' theorem [3] for radial and smooth functions.
All these results have been used in [9, 19] to obtain non-linear stability for equilibria of the Vlasov-Poisson system. However, the detailed weak definition of $\mathcal{D}$ as well as the proofs of the results above have not been properly addressed yet.
We also want to note that a similar operator also appears in the stability and instability analysis in the relativistic case, i.e., when considering the Einstein-Vlasov system, cf. [11]. The skew-adjointness of the operator then follows analogously as in the non-relativistic case, see [29]. Whether or not the kernel can be characterised similarly as well is still an open question, even in the case of smooth functions.

3.1 Weak definition

The standard way to define weak derivatives is to consider the scalar product with smooth functions and integrate by parts. We will pursue this approach to define $\mathcal{D}$ in a weak sense as well. Therefore, we first have to define $\mathcal{D}$ on smooth functions:

Definition 3.1: *For $f \in C_c^1(\Omega_0)$ let*

$$\boldsymbol{\mathcal{D}} f \colon \Omega_0 \to \mathbb{R},\ (x,v) \mapsto v \cdot \partial_x f(x,v) - \partial_x U_0(x) \cdot \partial_v f(x,v).$$

Next we have to justify the "integration by parts" formula for smooth functions:

Lemma 3.2: *Let $\chi \in C(]-\infty, E_0[)$ be an energy weight function.*
Then, for any $f, g \in C_c^1(\Omega_0)$ we have

$$\int_{\Omega_0} \chi(E(x,v)) f(x,v) \mathcal{D} g(x,v)\, \mathrm{d}(x,v) = -\int_{\Omega_0} \chi(E(x,v)) \mathcal{D} f(x,v) g(x,v)\, \mathrm{d}(x,v).$$

Proof: Let $(X,V) \colon \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ be the solution of the characteristic system of f_0

$$\dot{X} = V, \quad \dot{V} = -\partial_x U_0(X)$$

satisfying the initial condition

$$X(0,x,v) = x, \quad V(0,x,v) = v$$

for $x, v \in \mathbb{R}^3$, see [27] for the global existence & uniqueness of this characteristic flow. Here, $\dot{}$ denotes ∂_t, where we will always write $X = X(t,x,v)$ etc. Applying the chain rule, we can express the transport operator as follows:

$$\mathcal{D} f(x,v) = \partial_t\big|_{t=0} \left[f(X(t,x,v), V(t,x,v)) \right], \quad (x,v) \in \Omega_0.$$

Furthermore, the particle energy E is conserved along the characteristics (X,V), i.e., $E(x,v) = E(X(t,x,v), V(t,x,v))$ for $(x,v) \in \Omega_0$ and $t \in \mathbb{R}$. In particular, $(x,v) \in \Omega_0$ is equivalent to $(X(t,x,v), V(t,x,v)) \in \Omega_0$ for any $t \in \mathbb{R}$.
In addition, since $(X,V)(t,\cdot)$ is measure preserving (cf. [27]), we obtain

$$\int_{\Omega_0} \chi(E(x,v)) \cdot f((X,V)(t,x,v)) \cdot g((X,V)(t,x,v))\, \mathrm{d}(x,v) = \\ = \int_{\Omega_0} \chi(E(x,v)) \cdot f(x,v) \cdot g(x,v)\, \mathrm{d}(x,v)$$

for every $t \in \mathbb{R}$ by a change of variables. Thus

$$
\begin{aligned}
0 = \partial_t\big|_{t=0}&\left[\int_{\Omega_0} \chi(E(x,v)) \cdot f((X,V)(t,x,v)) \cdot g((X,V)(t,x,v))\, \mathrm{d}(x,v)\right] = \\
&= \int_{\Omega_0} \chi(E(x,v)) \cdot \partial_t\big|_{t=0}\big[f((X,V)(t,x,v)) \cdot g((X,V)(t,x,v))\big]\, \mathrm{d}(x,v) = \\
&= \int_{\Omega_0} \chi(E(x,v)) \cdot \big[\mathcal{D}f(x,v)g(x,v) + f(x,v)\mathcal{D}g(x,v)\big]\, \mathrm{d}(x,v).
\end{aligned}
$$

Note that we can switch the order of differentiation and integration due to the compact support of f and g. □

Since we are dealing with the spherically symmetric Vlasov-Poisson system, we may restrict ourselves to radial function spaces. For $\mathcal{D}$ to work properly on these spaces however, it has to preserve spherical symmetry. We therefore verify this property for smooth functions first.
We call a function spherically symmetric on Ω_0, if its extension by 0 is spherically symmetric on $\mathbb{R}^3 \times \mathbb{R}^3$ in the sense of Definition 2.1. Note that the set $\Omega_0 \subset \mathbb{R}^3 \times \mathbb{R}^3$ is spherically symmetric, i.e., for any $A \in \mathrm{SO}(3)$ we have $(x,v) \in \Omega_0$ if and only if $(Ax, Av) \in \Omega_0$, since E is spherically symmetric.
In addition, it will turn out that $\mathcal{D}$ reverses v-parity. Also note that Ω_0 is obviously symmetric in v, i.e., $(x,v) \in \Omega_0$ is equivalent to $(x,-v) \in \Omega_0$.

Lemma 3.3:

a) Let $f \in C^1_c(\Omega_0)$ be spherically symmetric on Ω_0. Then $\mathcal{D}f$ is spherically symmetric on Ω_0 as well.

b) Let $f \in C^1_c(\Omega_0)$ be even in v, i.e., $f(x,-v) = f(x,v)$ for $(x,v) \in \Omega_0$. Then $\mathcal{D}f$ is odd in v, i.e., $\mathcal{D}f(x,-v) = -\mathcal{D}f(x,v)$ for $(x,v) \in \Omega_0$.

c) Let $f \in C^1_c(\Omega_0)$ be odd in v. Then $\mathcal{D}f$ is even in v.

Proof:

a) Let $A \in \mathrm{SO}(3)$ and $(x,v) \in \Omega_0$ be arbitrary. To not get ourselves confused with matrices and their transposes, $\cdot$ will denote the matrix (instead of the scalar) multiplication in this part of the proof only, i.e.,

$$\mathcal{D}f(x,v) = v^T \cdot \partial_x f(x,v) - (\partial_x U_0(x))^T \cdot \partial_v f(x,v),$$

where all vectors are interpreted as column vectors. Then

$$
\begin{aligned}
\mathcal{D}f(Ax,Av) &= (Av)^T \cdot \partial_x f(Ax,Av) - (\partial_x U_0(Ax))^T\, \partial_v f(Ax,Av) = \\
&= v^T A^T \cdot \partial_x f(Ax,Av) - (\partial_x U_0(x))^T A^T \cdot \partial_v f(Ax,Av) = \\
&= v^T \cdot \partial_x f(x,v) - (\partial_x U_0(x))^T \cdot \partial_v f(x,v) = \mathcal{D}f(x,v),
\end{aligned}
$$

where we obtained $\partial_x f(Ax,Av) = A \cdot \partial_x f(x,v)$ as well as similar statements for $\partial_v f$ and $\partial_x U_0$ from the spherical symmetry of f.

b) For $(x, v) \in \Omega_0$ and f even in v we have

$$\begin{aligned}\mathcal{D}f(x,-v) &= -v \cdot \partial_x f(x,-v) - \partial_x U_0(x) \cdot (\partial_v f)(x,-v) = \\ &= -v \cdot \partial_x f(x,-v) + \partial_x U_0(x) \cdot \partial_v \left[f(x,-v)\right] = \\ &= -v \cdot \partial_x f(x,v) + \partial_x U_0(x) \cdot \partial_v \left[f(x,v)\right] = -\mathcal{D}f(x,v).\end{aligned}$$

c) For $(x, v) \in \Omega_0$ and f odd in v we have

$$\begin{aligned}\mathcal{D}f(x,-v) &= -v \cdot \partial_x f(x,-v) - \partial_x U_0(x) \cdot (\partial_v f)(x,-v) = \\ &= -v \cdot \partial_x f(x,-v) + \partial_x U_0(x) \cdot \partial_v \left[f(x,-v)\right] = \\ &= v \cdot \partial_x f(x,v) - \partial_x U_0(x) \cdot \partial_v \left[f(x,v)\right] = \mathcal{D}f(x,v). \qquad \square\end{aligned}$$

Since $\mathcal{D}$ preserves spherical symmetry, it is convenient to define the operator $\mathcal{D}$ in (r, w, L)-coordinates as well:

Definition & Remark 3.4:

a) Let

$$\boldsymbol{\Omega_0^r} \coloneqq \{(r,w,L) \in]0,\infty[\times\mathbb{R}\times]0,\infty[\mid E(r,w,L) < E_0\},$$

where $E(r,w,L) = \frac{1}{2}w^2 + \psi_L(r)$ *as before. The idea behind* Ω_0^r *is that it expresses the set* Ω_0 *in* (r,w,L)*-coordinates. Note however that*

$$\begin{aligned}\{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (|x|, \frac{x \cdot v}{|x|}, |x \times v|^2) \in \Omega_0^r\} = \\ = \{(x,v) \in \Omega_0 \mid x \times v \neq 0\} \subsetneq \Omega_0,\end{aligned}$$

since Ω_0^r *does not contain points with* $L = 0$ *(and* $r = 0$*). Anyway, these missing points form a set of measure zero in* Ω_0 *and are therefore negligible.*

Also note that just like Ω_0*,* Ω_0^r *is a bounded set. To see this, recall that* $E(r,w,L) \geq \psi_L(r) = U_0(r) + \frac{L}{2r^2} \geq U_0(r)$ *for* $(r,w,L) \in \Omega_0^r$*, where* $\lim_{r\to\infty} U_0(r) = 0 > E_0$*. This yields the boundedness of* r*, from which we may also deduce the one of* L *and* w*, since* $U_0 \geq U_0(0)$*.*

b) For $\zeta \in C_c^1(\Omega_0^r)$ *let*

$$(\boldsymbol{\mathcal{D}^r}\zeta)(r,w,L) \coloneqq w \cdot \partial_r\zeta(r,w,L) - \psi_L'(r) \cdot \partial_w\zeta(r,w,L), \quad (r,w,L) \in \Omega_0^r.$$

$\mathcal{D}^r$ *translates* $\mathcal{D}$ *into* (r,w,L)*-coordinates, which means that for every* $\xi \in$ $\boldsymbol{C^1_{c,r}(\Omega_0)} \coloneqq \{f \in C_c^1(\Omega_0) \mid f$ *is spherically symmetric on* $\Omega_0\}$ *such that* $\xi^r \in C_c^1(\Omega_0^r)$ *we have*

$$\mathcal{D}^r(\xi^r) = (\mathcal{D}\xi)^r \text{ on } \Omega_0^r,$$

which can be easily verified by using the chain rule. Here, ξ^r and $(\mathcal{D}\xi)^r$ are defined in the sense of Lemma & Definition 2.10 (after extending ξ on $\mathbb{R}^3 \times \mathbb{R}^3$) , i.e.,

$$\xi^r(|x|, \frac{x \cdot v}{|x|}, |x \times v|^2) = \xi(x, v) \text{ for } (x, v) \in \Omega_0 \text{ s.t. } x \times v \neq 0.$$

Note that $\xi \in C^1_{c,r}(\Omega_0)$ does not imply $\xi^r \in C^1_c(\Omega_0^r)$, since the support of functions in $C^1_c(\Omega_0^r)$ have to be bounded away from the sets $\{r = 0\}$ and $\{L = 0\}$. Also, $\mathcal{D}\xi$ is spherically symmetric and Lemma 3.2 yields

$$\int_{\Omega_0^r} \chi(E(r, w, L)) \cdot \zeta_1(r, w, L) \cdot (\mathcal{D}^r \zeta_2)(r, w, L) \, \mathrm{d}(r, w, L) =$$
$$= - \int_{\Omega_0^r} \chi(E(r, w, L)) \cdot (\mathcal{D}^r \zeta_1)(r, w, L) \cdot \zeta_2(r, w, L) \, \mathrm{d}(r, w, L)$$

by a change of variables for all $\zeta_1, \zeta_2 \in C^1_c(\Omega_0^r)$ and $\chi \in C(]-\infty, E_0[)$.

It turns out that the right space for $\mathcal{D}$ to be weakly defined on is the radial subset of a weighted L^2-space. We therefore define spaces of this kind:

Definition 3.5: *For a fixed energy weight $\chi \in C(]-\infty, E_0[)$ let*

$$\boldsymbol{L^2_{|\chi|}(\Omega_0)} := \{f \colon \Omega_0 \to \mathbb{R} \text{ measurable} \mid \|f\|_{|\chi|} < \infty\},$$

where

$$\|\boldsymbol{f}\|^2_{|\chi|} := \int_{\Omega_0} |\chi(E(x, v))| \cdot |f(x, v)|^2 \, \mathrm{d}(x, v).$$

This norm is based on the (real) scalar product

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{|\chi|} := \int_{\Omega_0} |\chi(E(x, v))| \cdot f(x, v) \cdot g(x, v) \, \mathrm{d}(x, v)$$

as usual. Furthermore, let

$$\boldsymbol{L^2_{|\chi|, r}(\Omega_0)} := \{f \in L^2_{|\chi|}(\Omega_0) \mid f \text{ is spherically symmetric a.e. on } \Omega_0\}$$

be the radial subspace of $L^2_{|\chi|}(\Omega_0)$. Spherical symmetry a.e. on Ω_0 is defined similarly to Lemma & Definition 2.10, recall again that Ω_0 is a radial subset of $\mathbb{R}^3 \times \mathbb{R}^3$. Note however that when extending some element from $L^2_{|\chi|}(\Omega_0)$ by 0 on $\mathbb{R}^3 \times \mathbb{R}^3$, the resulting function does not have to be in $L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3)$ due to the additional weight. Nevertheless, since the weight depends only on the spherically symmetric particle energy E, characterisations similar to Lemma & Definition 2.10 also hold true in the weighted case.

To transfer the well known properties of regular L^2-spaces to these weighted ones, we seek a connection between the convergence in the weighted space and in the regular one. Unfortunately, since the behaviour of a weight function $\chi \in C(]-\infty, E_0[)$ is unknown near the boundary $\{E = E_0\}$, these two convergences do not need be equivalent in the general case. They are however, if we can assure a fixed compact support:

Lemma 3.6: *Let $\chi \in C(]-\infty, E_0[)$ be such that $\chi(E) \neq 0$ for $E < E_0$. Furthermore, let $f\colon \Omega_0 \to \mathbb{R}$ be measurable with compact support in Ω_0, i.e., there exists a compact subset $K \subset\subset \Omega_0$ such that $f = 0$ a.e. on $\Omega_0 \setminus K$. Then:*

a) If $f \in L^2(\Omega_0)$, then $f \in L^2_{|\chi|}(\Omega_0)$ as well and

$$\|f\|_{|\chi|} \leq C\|f\|_2,$$

where the constant $C > 0$ depends only on the steady state f_0, the weight function χ and the support K.

b) If $f \in L^2_{|\chi|}(\Omega_0)$, then $f \in L^2(\Omega_0)$ as well and

$$\|f\|_2 \leq C\|f\|_{|\chi|},$$

where the constant $C > 0$ depends only on f_0, χ and K.

Proof: Since E is continuous on $\mathbb{R}^3 \times \mathbb{R}^3$ and $K \subset\subset \Omega_0 = \{E < E_0\}$, there exists $\delta > 0$ such that $K \subset \{E < E_0 - \delta\}$. Therefore, by the continuity of χ, there exist constants $c_0, C_0 > 0$ such that $c_0 \leq |\chi(E)| \leq C_0$ for any $U_0(0) \leq E \leq E_0 - \delta$. From the latter we obtain

$$c_0 \leq |\chi(E(x,v))| \leq C_0 \text{ for } (x,v) \in K.$$

Note that the constants c_0, C_0 only depend on f_0, χ and K as required. We now conclude

a) $\|f\|^2_{|\chi|} = \displaystyle\int_K |\chi(E(x,v))| \cdot |f(x,v)|^2 \,\mathrm{d}(x,v) \leq C_0\|f\|_2^2 \,.$

b) $\|f\|_2^2 = \displaystyle\int_K \frac{|\chi(E(x,v))|}{|\chi(E(x,v))|} \cdot |f(x,v)|^2 \,\mathrm{d}(x,v) \leq \frac{1}{c_0}\|f\|^2_{|\chi|}.$ □

As mostly when working with weakly defined differential operators, it will turn out to be very useful to approximate elements from these weighted L^2-spaces by smooth functions with compact support in Ω_0. We therefore need the following density results:

Lemma 3.7: *Let $\chi \in C(]-\infty, E_0[)$ be such that $\chi(E) \neq 0$ for $E < E_0$. Then*

a) $C_c^\infty(\Omega_0)$ is dense in $L^2_{|\chi|}(\Omega_0)$ (with respect to $\|\cdot\|_{|\chi|}$).

b) $C_{c,r}^\infty(\Omega_0)$ is dense in $L^2_{|\chi|,r}(\Omega_0)$ (with respect to $\|\cdot\|_{|\chi|}$).

Proof: Let $f \in L^2_{|\chi|}(\Omega_0)$. Our aim is to approximate f by its standard mollification. To apply Lemma 3.6 and conclude the convergence of the mollifyers, we first have to restrict ourselves to a compact support.
For this sake note that $\Omega_0 = \{E < E_0\} = \bigcup_{k=1}^\infty \{E < E_0 - \frac{1}{k}\}$ as an ascending union. Therefore, by Lebesgue's dominated convergence theorem, we have

$$f \cdot \mathbb{1}_{\{E<E_0-\frac{1}{k}\}} \to f \text{ in } L^2_{|\chi|}(\Omega_0) \text{ as } k \to \infty.$$

Thus, we may assume that f has compact support in Ω_0, in particular $f \in L^2(\Omega_0)$ due to Lemma 3.6.
Now let $J \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ be a spherically symmetric mollifyer, i.e., $J \geq 0$ and $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} J = 1$. As usual, define $J_k := k^6 J(k\cdot)$ for $k \in \mathbb{N}$.
Due to the compact support of f in Ω_0, we have $f_k := J_k * f \in C_c^\infty(\Omega_0)$ for $k \in \mathbb{N}$ sufficiently large. In particular, there exists $K \subset\subset \Omega_0$ such that $\operatorname{supp}(f_k) \subset K$ for every large $k \in \mathbb{N}$. Since $f_k \to f$ in $L^2(\Omega_0)$ by basic convolution theory, we can obtain the desired convergence $f_k \to f$ in $L^2_{|\chi|}(\Omega_0)$ with the aid of Lemma 3.6. This finishes the proof of a).
For part b), note that both the multiplication with the cut-off function $\mathbb{1}_{\{E<E_0-\frac{1}{k}\}}$ and the convolution with the spherically symmetric mollifyer J_k preserve the spherical symmetry of f. □

Finally, we will define $\mathcal{D}$ weakly on a suitable and dense subset of the weighted, radial L^2-space

$$\boldsymbol{L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)} := L^2_{\left|\frac{1}{\varphi'}\right|,r}(\Omega_0),$$

where φ is the function from our fixed isotropic ansatz.

Definition 3.8: *Let $f \in L^1_{loc,r}(\Omega_0)$, i.e., f is spherically symmetric a.e. on Ω_0, which is defined similarly to Lemma & Definition 2.10. We say that **$\mathcal{D}f$ exists weakly**, if there exists $\mu \in L^1_{loc,r}(\Omega_0)$ such that*

$$\int_{\Omega_0} \frac{1}{|\varphi' \circ E|} f \cdot \mathcal{D}\xi = -\int_{\Omega_0} \frac{1}{|\varphi' \circ E|} \mu \cdot \xi$$

*for every test function $\xi \in C^1_{c,r}(\Omega_0)$. In this case **$\mathcal{D}f := \mu$ (weakly)**. Furthermore, let*

$$\mathbf{D}(\boldsymbol{\mathcal{D}}) := \{f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0) \mid \mathcal{D}f \text{ exists weakly and } \mathcal{D}f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)\}$$

denote the domain of the operator $\mathcal{D}$.

Remarks: *Let* $f \in L^1_{loc,r}(\Omega_0)$.

a) *If* $\mathcal{D}f$ *exists weakly, then it is uniquely determined almost everywhere on* Ω_0 *by a change of variables and the Du Bois-Reymond theorem, see [21].*

b) *If additionally* $f \in C^1_{c,r}(\Omega_0)$*, then the weak and "classical" definition of* $\mathcal{D}f$ *coincide due to Lemmata 3.2 and 3.3.*

c) $\mathrm{D}(\mathcal{D})$ *is a linear subspace of* $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ *and* $\mathcal{D}$ *is linear, i.e., if* $f, g \in \mathrm{D}(\mathcal{D})$ *and* $\alpha \in \mathbb{R}$*, then* $\alpha f + g \in \mathrm{D}(\mathcal{D})$ *with* $\mathcal{D}(\alpha f + g) = \alpha \mathcal{D} f + \mathcal{D} g$.

d) *Usually in weak definitions like the one above, one would choose* $C^\infty_c(\Omega_0)$ *as the class of test functions. However, we will need that functions depending only on the particle energy* E *can be considered as test functions. Since* E *is not necessarily in* C^∞*, we extend the class of test functions to* $C^1_c(\Omega_0)$*. Nonetheless, the approximation result Theorem 3.15 implies that choosing* $C^\infty_{c,r}(\Omega_0)$ *as the class of test functions would lead to the exact same operator. Furthermore, since we always work in spherically symmetric spaces, it suffices to consider radial test functions only. Indeed, for the skew-adjointness it is crucial that the space of test functions is contained in* $\mathrm{D}(\mathcal{D})$ *itself, which means that allowing non-radial test functions as well would cause some difficulties later on.*

Since $C^1_{c,r}(\Omega_0) \subset \mathrm{D}(\mathcal{D})$ by Lemmata 3.2 and 3.3, we directly obtain the following due to Lemma 3.7:

Corollary 3.9: *The unbounded linear operator* $\mathcal{D} \colon \mathrm{D}(\mathcal{D}) \to L^2_{\frac{1}{|\varphi'|},r}(\Omega_0),\ f \mapsto \mathcal{D}f$ *is densely defined.*

Also, there is a quite different approach to define $\mathcal{D}$ weakly, which was suggested in [9]. It requires some definitions and tools from functional analysis, which can all be found in [12, 14, 25].

Remark 3.10: *For* $s \in \mathbb{R}$ *and* $f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ *let* $\mathcal{U}(s)f \colon \Omega_0 \to \mathbb{R}$ *be defined by*

$$(\mathcal{U}(s)f)(x,v) \coloneqq f(X(s,x,v), V(s,x,v)), \quad (x,v) \in \Omega_0,$$

where $(X,V) \colon \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ *is the solution of the characteristic system associated with the steady state* f_0

$$\dot{X} = V, \qquad \dot{V} = -\partial_x U_0(X),$$

satisfying the initial condition $(X,V)(0,x,v) = (x,v)$ *for* $x, v \in \mathbb{R}^3$*. By using the properties of* (X,V) *from [27], one can easily verify that* $(\mathcal{U}(s))_{s\in\mathbb{R}}$ *is a unitary* C^0*-group on* $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$*, i.e.,*

(i) $\mathcal{U}(s)\colon L^2_{\frac{1}{|\varphi'|},r}(\Omega_0) \to L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ *is a linear operator such that* $\|\mathcal{U}(s)f\|_{\frac{1}{|\varphi'|}} = \|f\|_{\frac{1}{|\varphi'|}}$ *for all* $f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ *and* $s \in \mathbb{R}$.

(ii) $\mathcal{U}(0) = \mathrm{id}$ *and* $\mathcal{U}(s) \circ \mathcal{U}(t) = \mathcal{U}(s+t)$ *for* $t, s \in \mathbb{R}$.

(iii) $\lim_{s\to 0}(\mathcal{U}(s)f) = f$ *for* $f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$.

By Stone's theorem, such a unitary C^0-group has a unique skew-adjoint infinitesimal generator $\tilde{\mathcal{D}}$ defined on the dense subset

$$\mathrm{D}(\tilde{\mathcal{D}}) := \{f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0) \mid \lim_{s\to 0}\frac{\mathcal{U}(s)f - f}{s} \textit{ exists in } L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)\}$$

by

$$\tilde{\mathcal{D}}f := \lim_{s\to 0}\frac{\mathcal{U}(s)f - f}{s}, \quad f \in D(\tilde{\mathcal{D}}).$$

Since $\mathcal{D}$ is skew-adjoint on a dense subset of $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ as well (which will be shown in the following section, see Theorem 3.18) and $\mathcal{D}$ and $\tilde{\mathcal{D}}$ coincide on the dense subset $C^1_{c,r}(\Omega_0)$, we can actually show $\mathcal{D} = \tilde{\mathcal{D}}$, in particular $\mathrm{D}(\mathcal{D}) = \mathrm{D}(\tilde{\mathcal{D}})$. In the latter argument we used that each essentially skew-adjoint operator has a unique skew-adjoint extension (cf. [25]).

Since we do not need this alternate representation of $\mathcal{D}$, we omit a detailed proof.

3.2 Skew-adjointness

The aim of this section is to show that the operator $\mathcal{D}\colon \mathrm{D}(\mathcal{D}) \to L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ is skew-adjoint, which means that $\mathcal{D}^* = -\mathcal{D}$.
Since $\mathcal{D}$ is skew-symmetric on smooth functions by Lemma 3.2, the main tool for this result is to approximate a function from $\mathrm{D}(\mathcal{D})$ in a way such that the images under $\mathcal{D}$ converge as well.
For this, we first need the following properties of our domain of definition $\mathrm{D}(\mathcal{D})$ and $\mathcal{D}$:

Lemma 3.11:

a) *Let $f \in \mathrm{D}(\mathcal{D})$ and $\chi \in C^1([0,\infty[)$ be such that $f\cdot(\chi\circ L), \mathcal{D}f\cdot(\chi\circ L) \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$, where $L(x,v) := |x\times v|^2$ for $x, v \in \mathbb{R}^3$. Then $f\cdot(\chi\circ L) \in \mathrm{D}(\mathcal{D})$ with $\mathcal{D}\left(f\cdot(\chi\circ L)\right) = (\mathcal{D}f)\cdot(\chi\circ L)$ weakly.*

b) *Let $f \in \mathrm{D}(\mathcal{D})$ and $\chi \in C(]-\infty, E_0[)$ be such that $f\cdot(\chi\circ E), \mathcal{D}f\cdot(\chi\circ E) \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$. Then $f\cdot(\chi\circ E) \in \mathrm{D}(\mathcal{D})$ with $\mathcal{D}\left(f\cdot(\chi\circ E)\right) = (\mathcal{D}f)\cdot(\chi\circ E)$ weakly.*

Proof:

a) Let $\xi \in C^1_{c,r}(\Omega_0)$ be an arbitrary test function. Since $\chi \circ L \in C^1_r(\Omega_0)$, we know $\xi \cdot \chi \circ L \in C^1_{c,r}(\Omega_0)$ as well. Also, due to L being constant along characteristics, we have $\mathcal{D}L = 0$ and therefore $\mathcal{D}(\xi \cdot \chi \circ L) = (\mathcal{D}\xi) \cdot \chi \circ L$ classically. Thus, Definition 3.8 yields

$$\int_{\Omega_0} \frac{1}{|\varphi' \circ E|} f \cdot \chi \circ L \cdot (\mathcal{D}\xi) = \int_{\Omega_0} \frac{1}{|\varphi' \circ E|} f \cdot \mathcal{D}(\chi \circ L \cdot \xi) = \\ = - \int_{\Omega_0} \frac{1}{|\varphi' \circ E|} (\mathcal{D}f) \cdot \chi \circ L \cdot \xi.$$

b) This part can not be done like the first one, since χ (and therefore $\chi \circ E$) does not need to be differentiable here. We still have to show

$$\int_{\Omega_0} \frac{1}{|\varphi' \circ E|} f \cdot \chi \circ E \cdot (\mathcal{D}\xi) = - \int_{\Omega_0} \frac{1}{|\varphi' \circ E|} (\mathcal{D}f) \cdot \chi \circ E \cdot \xi$$

for a fixed but arbitrary test function $\xi \in C^1_{c,r}(\Omega_0)$. To apply similar techniques like in part a), we mollify χ:
Let $J \in C^\infty_c(]-1,1[)$ be a one-dimensional mollifyer, i.e., $\int_{\mathbb{R}} J = 1$ and $J \geq 0$. As usual, set $J_k := kJ(k\cdot) \in C^\infty_c(]-\frac{1}{k}, \frac{1}{k}[)$ and $\chi_k := J_k * \chi \in C^\infty(\mathbb{R})$ for $k \in \mathbb{N}$ (where we extend χ by 0 if necessary).
Let $\tilde{\delta} > 0$ be such that $\operatorname{supp}(\xi) \subset \{E < E_0 - \tilde{\delta}\}$. Since χ is uniformly continuous on $[U_0(0) - 1, E_0 - \frac{\tilde{\delta}}{2}]$, we obtain that for an arbitrary $\epsilon > 0$ there exists $\delta \in]0, \frac{\tilde{\delta}}{2}[$ such that for all $E, \tilde{E} \in [U_0(0) - 1, E_0 - \frac{\tilde{\delta}}{2}]$ satisfying $|E - \tilde{E}| < \delta$ we have $|\chi(E) - \chi(\tilde{E})| < \epsilon$.
If we now choose $k \in \mathbb{N}$ with $\frac{1}{k} < \delta$, this yields

$$|\chi_k(E) - \chi(E)| = |\int_{\mathbb{R}} J(E')\chi(E - \frac{1}{k}E')\,\mathrm{d}E' - \chi(E)| = \\ = |\int_{\mathbb{R}} J(E') \left(\chi(E - \frac{1}{k}E') - \chi(E)\right) \mathrm{d}E'| \leq \\ \leq \int_{-1}^{1} J(E')|\chi(E - \frac{1}{k}E') - \chi(E)|\,\mathrm{d}E' < \epsilon \int_{\mathbb{R}} J = \epsilon$$

for every $E \in [U_0(0), E_0 - \tilde{\delta}]$. Thus, we have shown that $\chi_k \to \chi$ in $L^\infty([U_0(0), E_0 - \tilde{\delta}])$ as $k \to \infty$.
Since $\xi \cdot \chi_k \circ E \in C^1_{c,r}(\Omega_0)$, $\mathcal{D}(\xi \cdot \chi_k \circ E) = (\mathcal{D}\xi) \cdot \chi_k \circ E$ for $k \in \mathbb{N}$ just like

in part a) and $\operatorname{supp}(\mathcal{D}\xi) \subset \{E < E_0 - \tilde{\delta}\}$ as well, we obtain

$$\begin{aligned}
\int_{\Omega_0} \frac{1}{|\varphi' \circ E|} f \cdot \chi \circ E \cdot (\mathcal{D}\xi) &= \int_{\{E<E_0-\tilde{\delta}\}} \frac{1}{|\varphi' \circ E|} f \cdot \chi \circ E \cdot (\mathcal{D}\xi) = \\
&= \lim_{k\to\infty} \int_{\{E<E_0-\tilde{\delta}\}} \frac{1}{|\varphi' \circ E|} f \cdot \chi_k \circ E \cdot (\mathcal{D}\xi) = \\
&= \lim_{k\to\infty} \int_{\{E<E_0-\tilde{\delta}\}} \frac{1}{|\varphi' \circ E|} f \cdot \mathcal{D}(\chi_k \circ E \cdot \xi) = \\
&= -\lim_{k\to\infty} \int_{\{E<E_0-\tilde{\delta}\}} \frac{1}{|\varphi' \circ E|} (\mathcal{D}f) \cdot \chi_k \circ E \cdot \xi = \\
&= -\int_{\{E<E_0-\tilde{\delta}\}} \frac{1}{|\varphi' \circ E|} (\mathcal{D}f) \cdot \chi \circ E \cdot \xi.
\end{aligned}$$

Note that we integrate over bounded sets only, which allows us to evaluate the limits with Lebesgue's dominated convergence theorem. □

The latter result yields the following important connection between our weak definition of $\mathcal{D}$ and the non-weighted L^2-scalar product:

Corollary 3.12: *Let $f \in \mathrm{D}(\mathcal{D})$. Then*

$$\int_{\Omega_0} f \cdot \mathcal{D}\xi = -\int_{\Omega_0} \mathcal{D}f \cdot \xi$$

for any test function $\xi \in C^1_{c,r}(\Omega_0)$.

Proof: Apply Lemma 3.11 b) to $\chi = |\varphi'|$, which potentially has to be cut to ensure integrability. In a similar fashion like done in the previous proof, we may restrict ourselves to a compact subset of Ω_0 due to the compact support of test functions. □

If we want to approximate functions in $\mathrm{D}(\mathcal{D})$, it is convenient to mollify them in (r, w, L)-coordinates instead of (x, v)-coordinates to ensure spherical symmetry. Therefore, we need to define the weighted L^2-spaces in (r, w, L)-coordinates and investigate how the weak version of $\mathcal{D}$ works in the transformed setting.

Definition & Remark 3.13: *For $\chi \in C(]-\infty, E_0[)$ let*

$$\boldsymbol{L^2_{|\chi|}(\Omega_0^r)} := \{f \colon \Omega_0^r \to \mathbb{R} \text{ measurable} \mid \|f\|_{|\chi|} < \infty\},$$

where

$$\|\boldsymbol{f}\|^2_{|\chi|} := 4\pi^2 \int_{\Omega_0^r} |\chi(E(r, w, L))| \cdot |f(r, w, L)|^2 \,\mathrm{d}(r, w, L).$$

This norm is based on the (real) scalar product

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{|\boldsymbol{\chi}|} := 4\pi^2 \int_{\Omega_0^r} |\chi(E(r,w,L))| \cdot f(r,w,L) \cdot g(r,w,L) \,\mathrm{d}(r,w,L)$$

as usual. Note that $L^2_{|\chi|,r}(\Omega_0) \cong L^2_{|\chi|}(\Omega_0^r)$*, since by changing variables*

$$L^2_{|\chi|,r}(\Omega_0) \ni f \mapsto f^r \in L^2_{|\chi|}(\Omega_0^r)$$

is an isometric isomorphism. Here, we made the factor $4\pi^2$ *from the change of variables directly a part of the norm.*

Lemma 3.14: *Let* $f \in \mathrm{D}(\mathcal{D})$*. Then*

$$\langle (\mathcal{D}f)^r, \zeta \rangle_{\frac{1}{|\varphi'|}} = -\langle f^r, \mathcal{D}^r \zeta \rangle_{\frac{1}{|\varphi'|}} \quad \text{and} \quad \int_{\Omega_0^r} (\mathcal{D}f)^r \cdot \zeta = -\int_{\Omega_0^r} f^r \cdot \mathcal{D}^r \zeta$$

for every $\zeta \in C^1_c(\Omega_0^r)$*.*

Proof: For $\zeta \in C^1_c(\Omega_0^r)$ let

$$\xi(x,v) := \zeta(|x|, \frac{x \cdot v}{|x|}, |x \times v|^2) \text{ for } (x,v) \in \Omega_0 \text{ s.t. } x \times v \neq 0$$

and extend ξ by 0 on Ω_0. Since the support of ζ is bounded away from the sets $\{L = 0\}$ and $\{r = 0\}$, we have $\xi \in C^1_{c,r}(\Omega_0)$. Furthermore, $\xi^r = \zeta$ by definition. Thus, a change of variables yields

$$\langle (\mathcal{D}f)^r, \zeta \rangle_{\frac{1}{|\varphi'|}} = \langle \mathcal{D}f, \xi \rangle_{\frac{1}{|\varphi'|}} = -\langle f, \mathcal{D}\xi \rangle_{\frac{1}{|\varphi'|}} = -\langle f^r, (\mathcal{D}\xi)^r \rangle_{\frac{1}{|\varphi'|}} = -\langle f^r, \mathcal{D}^r \zeta \rangle_{\frac{1}{|\varphi'|}}.$$

The equality in the non-weighted scalar product follows by the same change of variables combined with Corollary 3.12. □

We can now prove the desired approximation result:

Theorem 3.15: *Let* $f \in \mathrm{D}(\mathcal{D})$*. Then there exists a sequence* $(F_k)_{k \in \mathbb{N}} \subset C^\infty_{c,r}(\Omega_0)$ *such that* $F_k^r \in C^\infty_c(\Omega_0^r)$ *for* $k \in \mathbb{N}$ *and*

$$F_k \to f \text{ and } \mathcal{D}F_k \to \mathcal{D}f \text{ in } L^2_{\frac{1}{|\varphi'|}}(\Omega_0) \text{ as } k \to \infty.$$

Proof: We split the proof of this theorem into several steps:

1) *Reduction to a compact support:*
For each $k \in \mathbb{N}$ let $\chi_k \in C^\infty(\mathbb{R})$ be an increasing cut-off function such that

$$\chi_k(x) = 0 \text{ for } x \leq \frac{1}{2k}, \qquad \chi_k(x) = 1 \text{ for } x \geq \frac{1}{k}.$$

Now let

$$f_k(x,v) \coloneqq \chi_k(L(x,v)) \cdot \chi_k(E_0 - E(x,v)) \cdot f(x,v)$$

for $(x,v) \in \Omega_0$ and $k \in \mathbb{N}$. Since the boundedness of χ_k together with the spherical symmetry of E and L ensure $f_k \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$, Lemma 3.11 yields $f_k \in \mathrm{D}(\mathcal{D})$ with $\mathcal{D}f_k = (\chi_k \circ L) \cdot (\chi_k \circ (E_0 - E)) \cdot (\mathcal{D}f)$. Thus, by Lebesgue's dominated convergence theorem, we obtain

$$f_k \to f \text{ and } \mathcal{D}f_k \to \mathcal{D}f \text{ in } L^2_{\frac{1}{|\varphi'|}}(\Omega_0) \text{ as } k \to \infty \text{ respectively.}$$

Therefore, by applying all the following arguments to f_k for $k \in \mathbb{N}$ large enough, we may assume that f^r has compact support in Ω_0^r, i.e., there exists $m \in \mathbb{N}$ such that for a.e. $(r,w,L) \in \Omega_0^r$ with $f^r(r,w,L) \neq 0$ we can deduce

$$E(r,w,L) < E_0 - \frac{1}{m} < 0, \quad \bar{B}_{\frac{1}{m}}(r,w,L) \subset \Omega_0^r$$

To see the latter, let $k \in \mathbb{N}$ be large enough such that $U_0(0) < E_0 - \frac{1}{2k}$. Then, for each $(r,w,L) \in \Omega_0^r$ satisfying $L \geq \frac{1}{2k}$ and $E_0 - E(r,w,L) \geq \frac{1}{2k}$, we obtain

$$\frac{L}{2r^2} \leq E(r,w,L) - U_0(0) \leq E_0 - \frac{1}{2k} - U_0(0)$$

and therefore

$$r \geq \frac{1}{2}\left(kE_0 - kU_0(0) - \frac{1}{2}\right)^{-\frac{1}{2}} > 0.$$

Since E is uniformly continuous on compact sets bounded away from $\{r = 0\}$, we may conclude $\bar{B}_{\frac{1}{m}}(r,w,L) \subset \Omega_0^r$ if $m \in \mathbb{N}$ is large enough.
Note that the above also implies $\mathcal{D}f = 0$ a.e. on $\{(x,v) \in \Omega_0 \mid |x| \leq \frac{1}{m} \vee L(x,v) \leq \frac{1}{m} \vee E(x,v) \geq E_0 - \frac{1}{m}\}$ due to the Du Bois-Reymond theorem. Furthermore, $(\mathcal{D}f)^r$ has compact support in Ω_0^r.

2) *The approximation sequence:*
We want to mollify f^r to get the approximation sequence. Therefore let $J \in C_c^\infty(B_1^3(0))$ be a three-dimensional mollifyer, i.e., $\int_{\mathbb{R}^3} J = 1$ and $0 \leq J \leq 1$. As usual, define $J_k \coloneqq k^3 J(k\cdot)$ for $k \in \mathbb{N}$.
Then, due to the compact supports of f^r and $(\mathcal{D}f)^r$ in Ω_0^r, standard mollifying arguments, a change of variables and Lemma 3.6 imply

$$f^r \in L^2(\Omega_0^r) \text{ and } J_k * f^r \to f^r \text{ in } L^2(\Omega_0^r) \;\&\; L^2_{\frac{1}{|\varphi'|}}(\Omega_0^r) \text{ as } k \to \infty$$

as well as

$$(\mathcal{D}f)^r \in L^2(\Omega_0^r) \text{ and } J_k * (\mathcal{D}f)^r \to (\mathcal{D}f)^r \text{ in } L^2(\Omega_0^r) \;\&\; L^2_{\frac{1}{|\varphi'|}}(\Omega_0^r) \text{ as } k \to \infty.$$

3) *Boundedness:*
We now want to show that $\mathcal{D}^r(J_k * f^r) \rightharpoonup (\mathcal{D}f)^r$ in $L^2(\Omega_0)$, at least after extracting a subsequence.
For this sake, we first prove that $(\mathcal{D}(J_k * f^r))_{k\in\mathbb{N}} \subset L^2(\Omega_0^r)$ is bounded, from which we can obtain the weak convergence of a subsequence by the Banach-Alaoglu theorem (cf. [21]).
First, by the properties of the support of f^r and the boundedness of Ω_0^r, the mean value theorem yields the existence of some constant $C_E^r \geq 1$ such that

$$|E(z) - E(z')| \leq C_E^r |z - z'| \text{ for } z, z' \in \bar{B}_{\frac{1}{m}}(\mathrm{supp}(f^r)) \subset \Omega_0^r.$$

Now, choose $k \in \mathbb{N}$ such that $k > 2C_E^r m$. Then, for every $z = (r, w, L) \in \Omega_0^r$ with $(J_k * f^r)(z) \neq 0$, we have $\bar{B}_{\frac{1}{2m}}(z) \subset \Omega_0^r$ and $E(z) < E_0 - \frac{1}{2m}$. To see the latter, first note that $(J_k * f^r)(z) \neq 0$ yields the existence of some $z' \in B_{\frac{1}{k}}(z) \cap \mathrm{supp}(f^r)$. Thus, since $\bar{B}_{\frac{1}{m}}(z') \subset \Omega_0^r$ and $E(z') < E_0 - \frac{1}{m}$, we obtain $\bar{B}_{\frac{1}{2m}}(z) \subset \Omega_0^r$ and $E(z) < E_0 - \frac{1}{2m}$.
In particular, $J_k * f^r \in C_c^\infty(\Omega_0^r)$ for $k > 2C_E^r m$. For these k and $z = (r, w, L) \in \mathrm{supp}(J_k * f^r) \subset \Omega_0^r$ we then obtain

$$\begin{aligned}
&[\mathcal{D}^r(J_k * f^r)](z) = w \cdot [(\partial_r J_k) * f^r](z) - \psi_L'(r) \cdot [(\partial_w J_k) * f^r](z) = \\
&= \int_{\Omega_0^r} [w \cdot \partial_r J_k(z - z') - \psi_L'(r) \cdot \partial_w J_k(z - z')] f^r(z') \, \mathrm{d}z' = \\
&= \int_{B_{\frac{1}{k}}(z)} [(w - w') \cdot \partial_r J_k(z - z') - (\psi_L'(r) - \psi_L'(r')) \cdot \partial_w J_k(z - z')] f^r(z') \, \mathrm{d}z' + \\
&\qquad + \int_{\Omega_0^r} [w' \cdot \partial_r J_k(z - z') - \psi_L'(r') \cdot \partial_w J_k(z - z')] f^r(z') \, \mathrm{d}z',
\end{aligned}$$

where we used the notation $z' = (r', w', L')$. Since $\psi_L''(r) = U_0''(r) + \frac{3L}{r^4}$ is uniformly bounded on $\bar{B}_{\frac{1}{k}}(\mathrm{supp}(J_k * f^r)) \subset \bar{B}_{\frac{1}{m}}(\mathrm{supp}(f^r)) \subset \Omega_0^r$, the mean value theorem yields

$$\begin{aligned}
&\left| \int_{B_{\frac{1}{k}}(z)} [(w - w') \cdot \partial_r J_k(z - z') - (\psi_L'(r) - \psi_L'(r')) \cdot \partial_w J_k(z - z')] f^r(z') \, \mathrm{d}z' \right| \leq \\
&\quad \leq \frac{C}{k} \int_{B_{\frac{1}{k}}(z)} |DJ_k(z - z')| \cdot |f^r(z')| \, \mathrm{d}z' = \\
&\quad = \frac{C}{k} \int_{B_{\frac{1}{k}}(0)} |\mathrm{D}J_k(\tilde{z})| \cdot |f^r(z - \tilde{z})| \, \mathrm{d}\tilde{z} = \\
&\quad = Ck^3 \int_{B_{\frac{1}{k}}(0)} |(\mathrm{D}J)(k\tilde{z})| \cdot |f^r(z - \tilde{z})| \, \mathrm{d}\tilde{z} = Ck^3 \left(|(\mathrm{D}J)(k\cdot)| * |f^r| \right)(z),
\end{aligned}$$

where $C > 0$ depends on the support of f and the fixed steady state f_0. As for the second summand, note that $J_k(z - \cdot) \in C_c^1(\Omega_0^r)$ for $k > 2C_E^r m$, since $\operatorname{supp}(J_k(z-\cdot)) \subset \bar{B}_{\frac{1}{2m}}(z) \subset \Omega_0^r$. Thus, by Lemma 3.14 we conclude

$$\int_{\Omega_0^r} [w' \cdot \partial_r J_k(z-z') - \psi_L'(r') \cdot \partial_w J_k(z-z')]\, f^r(z')\, \mathrm{d}z' =$$
$$= -\langle \mathcal{D}^r [J_k(z-\cdot)], f^r\rangle_2 = \langle J_k(z-\cdot), (\mathcal{D}f)^r\rangle_2 = [J_k * (\mathcal{D}f)^r](z).$$

Altogether we get

$$\begin{aligned}\|\mathcal{D}(J_k * f^r)\|_2 &\le \|J_k * (\mathcal{D}f)^r\|_2 + Ck^3 \||(\mathrm{D}J)(k\cdot)| * |f^r|\|_2 \le \\ &\le \|J_k * (\mathcal{D}f)^r\|_2 + Ck^3 \|f^r\|_2 \cdot \|(\mathrm{D}J)(k\cdot)\|_1,\end{aligned}$$

where we used Young's inequality (see [21]). Since $J_k * (\mathcal{D}f)^r \to (\mathcal{D}f)^r$ in $L^2(\Omega_0^r)$ as $k \to \infty$, the first term is bounded. As to the second, note that

$$\|\mathrm{D}J(k\cdot)\|_1 = \int_{\mathbb{R}^3} |\mathrm{D}J(kz)|\, \mathrm{d}z = \frac{1}{k^3} \int_{\mathbb{R}^3} |\mathrm{D}J(z')|\, \mathrm{d}z' = \frac{\|\mathrm{D}J\|_1}{k^3}.$$

Overall, we obtain the desired boundedness.

4) *Weak convergence:*
Due to the previous step, there exists a subsequence $(J_{k_j} * f^r)_{j\in\mathbb{N}} \subset (J_k * f^r)_{k\in\mathbb{N}}$ and a limit $g^r \in L^2(\Omega_0^r)$ such that

$$\mathcal{D}^r(J_{k_j} * f^r) \rightharpoonup g^r \text{ in } L^2(\Omega_0^r) \text{ as } j \to \infty.$$

What remains to show is $g = \mathcal{D}f$, where

$$g(x,v) := g^r(|x|, \frac{x\cdot v}{|v|}, |x \times v|^2), \quad (x,v) \in \Omega_0 \text{ s.t. } x \times v \neq 0.$$

Let $\xi \in C_{c,r}^1(\Omega_0)$ be an arbitrary test function. We have to ensure $\xi^r \in C_c^1(\Omega_0^r)$, which can be achieved due to the compact support of f^r in Ω_0^r: From the properties of the support of $(J_{k_j} * f^r)$ shown in the third step of this proof, we obtain

$$(\chi_{2m} \circ L) \cdot (\chi_{2m} \circ (E_0 - E)) \cdot \mathcal{D}^r(J_{k_j} * f^r) = \mathcal{D}^r(J_{k_j} * f^r)$$

if $k_j > 2C_E^r m$, where χ_{2m} is known from the first step. Now let $\tilde{\xi} := (\chi_{2m} \circ L) \cdot (\chi_{2m} \circ (E_0 - E)) \cdot \xi$ and note that $\tilde{\xi}^r \in C_c^1(\Omega_0^r)$. In addition, we have $f \cdot \mathcal{D}\tilde{\xi} = f \cdot \mathcal{D}\xi$ and $g\tilde{\xi} = g\xi$ a.e. on Ω_0, where the latter follows from the properties of the supports of $(J_{k_j} * f^r)$ used above and the Du Bois-Reymond theorem. Thus, changing variables yields

$$\begin{aligned}\langle g, \xi\rangle_{\frac{1}{|\varphi'|}} &= \langle g, \tilde{\xi}\rangle_{\frac{1}{|\varphi'|}} = \langle g^r, \tilde{\xi}^r\rangle_{\frac{1}{|\varphi'|}} = \lim_{j\to\infty} \langle \mathcal{D}^r(J_{k_j} * f^r), \tilde{\xi}^r\rangle_{\frac{1}{|\varphi'|}} = \\ &= -\lim_{j\to\infty} \langle (J_{k_j} * f^r), \mathcal{D}^r\tilde{\xi}^r\rangle_{\frac{1}{|\varphi'|}} = -\langle f^r, (\mathcal{D}\tilde{\xi})^r\rangle_{\frac{1}{|\varphi'|}} = \\ &= -\langle f, \mathcal{D}\tilde{\xi}\rangle_{\frac{1}{|\varphi'|}} = -\langle f, \mathcal{D}\xi\rangle_{\frac{1}{|\varphi'|}}\end{aligned}$$

where we used the transformed version of Lemma 3.2 from Definition & Remark 3.4 as well as the fact that $\langle \cdot, \tilde{\xi}^r \rangle_{\frac{1}{|\varphi'|}} = \langle \cdot, \frac{\tilde{\xi}^r}{|\varphi' \circ E|} \rangle_2 \in [L^2(\Omega_0^r)]'$ due to the compact support of $\tilde{\xi}^r$.

5) *Strong convergence:*
A standard way to convert weak convergence into strong one is by applying Mazur's lemma (see [21]). This theorem passes to convex combinations of the original sequence to strengthen the convergence. Fortunately, these convex combinations behave very well with linear operators like $\mathcal{D}$ and also inherit regularity and properties of the support.
For brevity, we will call the weakly convergent subsequence $(f_k^r)_{k\in\mathbb{N}} := (J_{k_j} * f^r)_{j\in\mathbb{N}}$, i.e., $\mathcal{D}^r f_k^r \rightharpoonup (\mathcal{D}f)^r$ in $L^2(\Omega_0^r)$ as $k \to \infty$. Mazur's lemma now states that for every $k \in \mathbb{N}$ there exists $N_k \geq k$ and weights $c_k^k, \ldots, c_{N_k}^k \in [0,1]$ with $\sum_{j=k}^{N_k} c_j^k = 1$ such that

$$\mathcal{D}^r \left(\sum_{j=k}^{N_k} c_j^k f_j^r \right) = \sum_{j=k}^{N_k} c_j^k \mathcal{D}^r f_j^r \to (\mathcal{D}f)^r \text{ in } L^2(\Omega_0^r) \text{ as } k \to \infty.$$

Let $F_k^r := \sum_{j=k}^{N_k} c_j^k f_j^r$ for $k \in \mathbb{N}$. Since

$$\|f^r - F_k^r\|_\gamma \leq \sum_{j=k}^{N_k} c_j^k \|f^r - f_j^r\|_\gamma \text{ for } \gamma \in \{2, \frac{1}{|\varphi'|}\} \text{ and } k \in \mathbb{N},$$

we still have $F_k^r \to f^r$ in $L^2(\Omega_0^r)$ and $L^2_{\frac{1}{|\varphi'|}}(\Omega_0^r)$ as $k \to \infty$.
Also recall that $F_k^r \in C_c^\infty(\Omega_0^r)$ for $k > 2C_E^r m$. Finally, set

$$F_k(x,v) := F_k^r(|x|, \frac{x \cdot v}{|x|}, |x \times v|^2) \text{ for } (x,v) \in \Omega_0 \text{ s.t. } x \times v \neq 0$$

and extend F_k by 0 on Ω_0 for $k \in \mathbb{N}$. Then $F_k \in C_{c,r}^\infty(\Omega_0)$ for k sufficiently large due to the compact support of F_k^r and changing variables yields

$$F_k \to f \text{ in } L^2_{\frac{1}{|\varphi'|}}(\Omega_0) \text{ and } \mathcal{D}F_k \to \mathcal{D}f \text{ in } L^2(\Omega_0) \text{ as } k \to \infty.$$

Lemma 3.6 allows us to conclude $\mathcal{D}F_k \to \mathcal{D}f$ in $L^2_{\frac{1}{|\varphi'|}}(\Omega_0)$ as well, which finishes the proof of this theorem. □

Recall that Theorem 3.15 clearly implies that choosing $C_{c,r}^\infty(\Omega_0)$ as the class of test functions would lead to the exact same operator $\mathcal{D}$.
Another application of the result above is that can expand the properties of $\mathcal{D}$ from Lemmata 3.2 and 3.3 to the whole space $\mathrm{D}(\mathcal{D})$:

Corollary 3.16: *$\mathcal{D}$ is skew-symmetric on* $\mathrm{D}(\mathcal{D})$*, i.e., for all $f, g \in \mathrm{D}(\mathcal{D})$ we have*

$$\langle f, \mathcal{D}g \rangle_{\frac{1}{|\varphi'|}} = -\langle \mathcal{D}f, g \rangle_{\frac{1}{|\varphi'|}}.$$

Proof: Approximate one function like in Theorem 3.15 and use Definition 3.8. □

Corollary 3.17:

a) *Let $f \in D(\mathcal{D})$ be even in v a.e. on Ω_0, i.e., for a.e. $(x, v) \in \Omega_0$ we have $f(x, -v) = f(x, v)$. Then $\mathcal{D}f$ is odd in v a.e. on Ω_0, i.e., $\mathcal{D}f(x, -v) = -\mathcal{D}f(x, v)$ for a.e. $(x, v) \in \Omega_0$.*

b) *Let $f \in D(\mathcal{D})$ be odd in v a.e. on Ω_0. Then $\mathcal{D}f$ is even in v a.e. on Ω_0.*

Proof: What remains to show is that the sequence $(F_k)_{k\in\mathbb{N}}$ preserves the respective property of f. Then, since the even and odd subspaces are closed in $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ due to the Fischer-Riesz theorem, we may conclude the claimed results by applying Lemma 3.3 to the approximation sequence from Theorem 3.15. We will restrict ourselves to the case where f is even in v, since the other case works completely analogously.
First note that the cut-off functions from the first step of the proof of Theorem 3.15 are even in v, since they depend only on the particle energy and the modulus of the angular momentum squared. Therefore, if f is even in v, this property is preserved by the multiplication with these cut-off functions.
For the mollification to preserve this property, we have to choose the mollifyer J to be even in w, i.e., $J(r, -w, L) = J(r, w, L)$ for all $(r, w, L) \in \mathbb{R}^3$ (in the case where f is odd in v, we have to choose J to be even in w as well). Then, one can easily verify that $J_k * f^r$ is even in w as well, since f being even in v is equivalent to f^r being even in w. This property is clearly also preserved by convex combinations. □

From the skew-symmetry we can now finally obtain the desired skew-adjointness quite easily. Before that, we recall the definition of the adjoint operator of $\mathcal{D}$, see [14] for details.
Let $\mathcal{D}^* \colon \mathrm{D}(\mathcal{D}^*) \to L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ denote the adjoint of $\mathcal{D} \colon \mathrm{D}(\mathcal{D}) \to L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$, note that $\mathcal{D}$ is a densely defined operator on the Hilbert space $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$. The domain of definition of $\mathcal{D}^*$ is

$$\begin{aligned}\mathrm{D}(\mathcal{D}^*) :=& \{f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0) \mid \exists C_f > 0\ \forall g \in \mathrm{D}(\mathcal{D})\colon |\langle \mathcal{D}g, f\rangle_{\frac{1}{|\varphi'|}}| \leq C_f \|g\|_{\frac{1}{|\varphi'|}}\} = \\ =& \{f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0) \mid \exists_1 h \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)\ \forall g \in \mathrm{D}(\mathcal{D})\colon \langle \mathcal{D}g, f\rangle_{\frac{1}{|\varphi'|}} = \langle g, h\rangle_{\frac{1}{|\varphi'|}}\}.\end{aligned}$$

In case of the second definition, $\mathcal{D}^*$ is defined by $\mathcal{D}^* f := h$ for $f \in \mathrm{D}(\mathcal{D}^*)$.

Theorem 3.18: $\mathcal{D}\colon \mathrm{D}(\mathcal{D}) \to L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ *is skew-adjoint, i.e.,* $\mathcal{D}^* = -\mathcal{D}$.

Proof: We show $\mathcal{D}^* = -\mathcal{D}$ in two steps:

$\subseteq$: Let $f \in \mathrm{D}(\mathcal{D}^*)$ and $h \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ such that $\langle \mathcal{D}g, f\rangle_{\frac{1}{|\varphi'|}} = \langle g, h\rangle_{\frac{1}{|\varphi'|}}$ for all $g \in \mathrm{D}(\mathcal{D})$. In particular, since $C^1_{c,r}(\Omega_0) \subset \mathrm{D}(\mathcal{D})$, we have

$$\langle f, \mathcal{D}\xi\rangle_{\frac{1}{|\varphi'|}} = \langle h, \xi\rangle_{\frac{1}{|\varphi'|}}$$

for all test functions $\xi \in C^1_{c,r}(\Omega_0)$. However, this means $f \in D(\mathcal{D})$ with $\mathcal{D}f = -h$ by Definition 3.8.

$\supseteq$: Let $f \in \mathrm{D}(\mathcal{D})$. Corollary 3.16 immediately yields

$$\langle \mathcal{D}g, f\rangle_{\frac{1}{|\varphi'|}} = -\langle g, \mathcal{D}f\rangle_{\frac{1}{|\varphi'|}} = \langle g, -\mathcal{D}f\rangle_{\frac{1}{|\varphi'|}}$$

for all $g \in \mathrm{D}(\mathcal{D})$, i.e., $f \in \mathrm{D}(\mathcal{D}^*)$ and $\mathcal{D}^* f = -\mathcal{D}f$. □

3.3 Jeans' theorem

In this section, we want to characterise the kernel of the operator $\mathcal{D}\colon \mathrm{D}(\mathcal{D}) \to L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$. For smooth functions, it is well known that each element of this kernel depends only on the particle energy of f_0 and the modulus of the angular momentum squared, i.e., for all $f \in C^1_{c,r}(\Omega_0)$ with $\mathcal{D}f = 0$ on Ω_0 there exists $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,v) = g(E(x,v), L(x,v)) = g(\frac{1}{2}|v|^2 + U_0(x), |x \times v|^2) \text{ for } (x,v) \in \Omega_0.$$

This has first been rigorously shown in [3] and is well known as "Jeans' theorem". Indeed, this result is closely related to the fact that spherically symmetric steady states of the Vlasov-Poisson system depend only on their particle energy and the angular momentum, which was first suggested by J. Jeans [15, 16] at the beginning of the last century.
Therefore, it seems convincing that every element in the kernel of the unbounded operator $\mathcal{D}$ from the previous sections depends only on E and L as well. However, this is not as easy to show as one might think. In the Section 3.2, we could reduce some properties of $\mathcal{D}$ to the respective properties on smooth function by the approximation of Theorem 3.15, for example Corollaries 3.16 and 3.17. However, the key for those results was that the properties had been preserved by the mollification used in the proof of Theorem 3.15. Unfortunately, the mollification of a function in the kernel of $\mathcal{D}$ does not have to belong to the kernel anymore.
Nevertheless, approximating an element from the kernel similar to Theorem 3.15

will turn out to be very helpful. Indeed, we will show that the distance between elements of this approximation sequence and their projection onto the space of functions depending only on (E, L) tends to zero.
To define the projection mentioned above, we first have to analyse the solutions of the characteristic system in (r, w, L)-coordinates associated with the fixed steady state f_0. We refer to [4, 9, 10] for a much more detailed discussion. The characteristic system of $\mathcal{D}^r$ takes the form

$$\dot{r} = w,\ \dot{w} = -\psi_L'(r),\ \dot{L} = 0.$$

Let $\mathbb{R} \ni t \mapsto (r(t), w(t), L)$ be an arbitrary global solution of this system. Since the particle energy is conserved along these characteristics, there exists $E \in \mathbb{R}$ such that $E = E(r(t), w(t), L)$ for all $t \in \mathbb{R}$. We assume that the solution satisfies $L > 0$ and $E < 0$, otherwise it is not of interest. For any $t \in \mathbb{R}$ we have

$$\psi_L(r_L) \leq \psi_L(r(t)) \leq \frac{1}{2}w^2(t) + \psi_L(r(t)) = E$$

and thus $r_-(E, L) \leq r(t) \leq r_+(E, L)$ by Theorem 2.4. Furthermore, solving for w yields

$$\dot{r}(t) = w(t) = \pm\sqrt{2E - 2\psi_L(r(t))}$$

for $t \in \mathbb{R}$. Therefore, r oscillates between $r_-(E, L)$ and $r_+(E, L)$, where $\dot{r} = 0$ is equivalent to $r = r_\pm(E, L)$ and $\dot{r}$ always switches its sign when reaching $r_\pm(E, L)$. By applying the inverse function theorem and integrating, we also obtain the following explicit formula for the period of the r-motion, i.e., the time needed for r to travel from $r_-(E, L)$ to $r_+(E, L)$ and back to $r_-(E, L)$:

Definition & Remark 3.19: *For $L > 0$ and $\psi_L(r_L) \leq E < E_0$ let*

$$\boldsymbol{T(E, L)} \coloneqq 2\int_{r_-(E,L)}^{r_+(E,L)} \frac{\mathrm{d}r}{\sqrt{2E - 2\psi_L(r)}},$$

the ***period of the characteristic fixed by (E, L)****. Since $E - \psi_L(r) > 0$ for $r_-(E, L) < r < r_+(E, L)$, the expression above is well defined. The finiteness of the integral $T(E, L)$ can be verified using Theorem 2.4:*

$$\begin{aligned}
T(E, L) &= \sqrt{2}\int_{r_-(E,L)}^{r_+(E,L)} \frac{\mathrm{d}r}{\sqrt{E - \psi_L(r)}} \leq \\
&\leq 2\int_{r_-(E,L)}^{r_+(E,L)} \frac{r\sqrt{r_-(E, L)r_+(E, L)}}{\sqrt{L}\sqrt{(r_+(E, L) - r) \cdot (r - r_-(E, L))}}\,\mathrm{d}r \leq \\
&\leq 2\frac{r_+^2(E, L)}{\sqrt{L}}\int_0^1 \frac{\mathrm{d}s}{\sqrt{s(1 - s)}} = 2\pi\frac{r_+^2(E, L)}{\sqrt{L}} \leq 2\pi\frac{M_0^2}{E^2\sqrt{L}},
\end{aligned}$$

where we used the substitution $r = s(r_+(E, L) - r_-(E, L)) + r_-(E, L)$.

The projection onto the space of functions depending only on E and L can now be obtained by integrating over the trajectories fixed by (E, L). Thus, for fixed $(r, w, L) \in]0, \infty[\times\mathbb{R}\times]0, \infty[$ let $\mathbb{R} \ni t \mapsto (R, W)(t, r, w, L)$ be the unique global solution of the characteristic system

$$\dot{R} = W, \quad \dot{W} = -\psi_L'(R)$$

satisfying the initial condition $(R, W)(0, r, w, L) = (r, w)$. For the global existence of the characteristic flow (R, W) see [27].

Definition & Remark 3.20: *For $f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ (extended by 0 on $\mathbb{R}^3 \times \mathbb{R}^3$) and $L > 0$, $\psi_L(r_L) < E < E_0$ let*

$$\begin{aligned}\boldsymbol{\mathcal{P}}f(E, L) &:= \int_0^1 f^r((R, W)(t \cdot T(E, L), r_-(E, L), 0, L), L)\,\mathrm{d}t = \\ &= \frac{1}{T(E, L)} \int_{r_-(E,L)}^{r_+(E,L)} \frac{f^r(r, \sqrt{2E - 2\psi_L(r)}, L) + f^r(r, -\sqrt{2E - 2\psi_L(r)}, L)}{\sqrt{2E - 2\psi_L(r)}}\,\mathrm{d}r,\end{aligned}$$

the ***average of f over the trajectory fixed by (E, L)****. Then $\mathcal{P}f(E, L)$ is uniquely determined for a.e. $(E, L) \in \mathbb{R}^2$ satisfying $L > 0$ and $\psi_L(r_L) < E < E_0$, since changing to (r, w, L) and from there to (t, E, L)-coordinates yields*

$$\begin{aligned}\int_{\Omega_0} f(x, v)\,\mathrm{d}(x, v) &= 4\pi^2 \int_{\Omega_0^r} f^r(r, w, L)\,\mathrm{d}(r, w, L) = \\ &= 4\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} T(E, L) \cdot \mathcal{P}f(E, L)\,\mathrm{d}E\,\mathrm{d}L.\end{aligned}$$

Further, a similar change of variables yields

$$\begin{aligned}&\int_{\Omega_0} \frac{1}{|\varphi'(E(x, v))|} \cdot \mathcal{P}f(E(x, v), L(x, v)) \cdot g(x, v)\,\mathrm{d}(x, v) = \\ &\qquad = 4\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} \frac{T(E, L)}{|\varphi'(E)|} \cdot \mathcal{P}f(E, L) \cdot \mathcal{P}g(E, L)\,\mathrm{d}E\,\mathrm{d}L = \\ &\qquad = \int_{\Omega_0} \frac{1}{|\varphi'(E(x, v))|} \cdot f(x, v) \cdot \mathcal{P}g(E(x, v), L(x, v))\,\mathrm{d}(x, v)\end{aligned}$$

for all $f, g \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$. In particular, $\langle \mathcal{P}f, f\rangle_{\frac{1}{|\varphi'|}} = \|\mathcal{P}f\|^2_{\frac{1}{|\varphi'|}}$ and therefore $\|\mathcal{P}f\|^2_{\frac{1}{|\varphi'|}} \leq \|f\|^2_{\frac{1}{|\varphi'|}}$, where we identified $\mathcal{P}f$ as $\Omega_0 \ni (x, v) \mapsto \mathcal{P}f(E(x, v), L(x, v))$ by a slight abuse of notation.
Since in addition $\mathcal{P}\left[\Omega_0 \ni (x, v) \mapsto \mathcal{P}f(E(x, v), L(x, v)) \in \mathbb{R}\right] = \mathcal{P}f$, the mapping

$$L^2_{\frac{1}{|\varphi'|},r}(\Omega_0) \ni f \mapsto \left[\Omega_0 \ni (x, v) \mapsto \mathcal{P}f(E(x, v), L(x, v)) \in \mathbb{R}\right] \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$$

is indeed the unique orthogonal projection onto the closed subspace

$$\begin{aligned}\boldsymbol{K_{\mathcal{D}}} :=& \{f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0) \mid \exists g\colon \mathbb{R}^2 \to \mathbb{R} \text{ s.t. } f(x,v) = g(E(x,v), L(x,v)) \\ & \qquad\qquad\qquad\qquad \text{for a.e. } (x,v) \in \Omega_0\} = \\ =& \{f \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0) \mid f(x,v) = \mathcal{P}f(E(x,v), L(x,v)) \text{ for a.e. } (x,v) \in \Omega_0\}\end{aligned}$$

of $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$, see [14] for details on the definition and uniqueness of orthogonal projections. Recall that $L(x,v) > 0$ and $\psi_L(r_L) < E(x,v) < E_0$ for almost every $(x,v) \in \Omega_0$, i.e., the mappings used above are a.e. defined on Ω_0.
That $K_{\mathcal{D}}$ is a closed linear subspace of $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ can be verified by changing variables and using the completeness of weighted, two dimensional L^2-spaces or alternatively by applying the Fischer-Riesz theorem.

As motivated above, we now want to show the following generalisation of Jeans' theorem:

Theorem 3.21: $\ker(\mathcal{D}) = K_{\mathcal{D}}$.

Proof: To get started, we first show the easy inclusion. Let $f \in K_{\mathcal{D}}$, i.e., there exists $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,v) = g(E(x,v), L(x,v))$ for a.e. $(x,v) \in \Omega_0$. We will show $f \in \ker(\mathcal{D})$ with similar techniques like the ones we used to prove Lemma 3.2. Let $(X,V)\colon \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ be the characteristic flow associated with the steady state f_0, see Lemma 3.2 for a detailed definition. Since E and L are conserved along characteristics, i.e.,

$$E(X(t,x,v), V(t,x,v)) = E(x,v), \quad L(X(t,x,v), V(t,x,v)) = L(x,v)$$

for $x, v \in \mathbb{R}^3$ & $t \in \mathbb{R}$, we obtain

$$\begin{aligned}&\int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \cdot f(x,v) \cdot \xi((X,V)(t,x,v))\,\mathrm{d}(x,v) = \\ &\qquad\qquad = \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \cdot f(x,v) \cdot \xi(x,v)\,\mathrm{d}(x,v)\end{aligned}$$

for every $\xi \in C^1_{c,r}(\Omega_0)$ and $t \in \mathbb{R}$ by changing variables. Note that we used $f \in K_{\mathcal{D}}$ here to deduce that f is constant along characteristics. Thus

$$\begin{aligned}0 &= \partial_t\big|_{t=0} \left[\int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \cdot f(x,v) \cdot \xi((X,V)(t,x,v))\,\mathrm{d}(x,v)\right] = \\ &= \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \cdot f(x,v) \cdot \partial_t\big|_{t=0} [\xi((X,V)(t,x,v))]\,\mathrm{d}(x,v) = \\ &= \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \cdot f(x,v) \cdot \mathcal{D}\xi(x,v)\,\mathrm{d}(x,v),\end{aligned}$$

where the compact support of the test function $\xi \in C^1_{c,r}(\Omega_0)$ allowed us to switch the order of differentiation and integration. By Definition 3.8, the latter means that $f \in \mathrm{D}(\mathcal{D})$ with $\mathcal{D}f = 0$, i.e., $f \in \ker(\mathcal{D})$.
As for the other inclusion, let $f \in \ker(\mathcal{D})$, i.e., $f \in \mathrm{D}(\mathcal{D})$ with $\mathcal{D}f = 0$. As stated above, we show $f \in K_{\mathcal{D}}$ by approximation. We split this argument into several steps:

1) *Reduction to a compact support:*
Just like in Theorem 3.15, let $\chi_k \in C^\infty(\mathbb{R})$ be a smooth, increasing cut-off function satisfying

$$\chi_k(x) = 0 \text{ for } x \leq \frac{1}{2k}, \qquad \chi_k(x) = 1 \text{ for } x \geq \frac{1}{k}$$

for each $k \in \mathbb{N}$. Now set

$$f_k(x,v) := \chi_k(L(x,v)) \cdot \chi_k(E_0 - E(x,v)) \cdot f(x,v)$$

for $(x,v) \in \Omega_0$ and $k \in \mathbb{N}$. Since $f_k \to f$ in $L^2_{\frac{1}{|\varphi'|}}(\Omega_0)$ as $k \to \infty$ and $K_{\mathcal{D}}$ is closed, it suffices to show $f_k \in K_{\mathcal{D}}$ for every $k \in \mathbb{N}$ to conclude $f \in K_{\mathcal{D}}$.
Thus, we assume that there exists $m \in \mathbb{N}$ such that for a.e. $(x,v) \in \Omega_0$ with $f(x,v) \neq 0$ we have $L(x,v) \geq \frac{1}{m}$ and $E(x,v) \leq E_0 - \frac{1}{m} < 0$.

2) *Approximation like in Theorem 3.15:*
Just like in the proof of Theorem 3.15 we can construct an approximation sequence $(F_k)_{k\in\mathbb{N}} \subset C^\infty_{c,r}(\Omega_0)$ such that

$$F_k \to f \text{ and } \mathcal{D}F_k \to \mathcal{D}f = 0 \text{ in } L^2_{\frac{1}{|\varphi'|}}(\Omega_0) \text{ respectively as } k \to \infty,$$

where the supports satisfy

$$\operatorname{supp}(F_k) \subset \{(x,v) \in \Omega_0 \mid L(x,v) \geq \frac{1}{2m}\}, \quad k \in \mathbb{N}.$$

Furthermore, we have $F_k^r \in C_c^\infty(\Omega_0^r)$ for every $k \in \mathbb{N}$.

3) *An auxiliary estimate – the heart of the proof:*
As motivated at the beginning of this section, we want to show that the distance between F_k and the projection $\mathcal{P}F_k$ tends to zero as $k \to \infty$. To prove this, we first estimate the distance between a smooth function and its projection onto the space of functions depending only on E and L in general:
Let $\xi \in C^1_{c,r}(\Omega_0)$ with $\xi^r \in C^1_c(\Omega_0^r)$ be arbitrary, but fixed. We will use the abbreviation

$$\zeta(t,E,L) := \xi^r((R,W)(t \cdot T(E,L), r_-(E,L), 0, L), L)$$

for $t \in \mathbb{R}$, $L > 0$ and $\psi_L(r_L) < E < E_0$, where (R, W) is the characteristic flow in (r, w, L)-coordinates as used in Definition & Remark 3.19. For these (E, L) we may therefore write

$$\mathcal{P}\xi(E, L) = \int_0^1 \zeta(t, E, L)\,\mathrm{d}t.$$

By a slight abuse of notation, we will call the mapping

$$(x, v) \mapsto \mathcal{P}\xi(E(x, v), L(x, v)),$$

which is defined a.e. on Ω_0, also $\mathcal{P}\xi$. Then, changing to (t, E, L)-coordinates yields

$$\begin{aligned}
&\|\xi - \mathcal{P}\xi\|^2_{\frac{1}{|\varphi'|}} = \\
&= 4\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} \frac{T(E, L)}{|\varphi'(E)|} \int_0^1 |\zeta(t, E, L) - \mathcal{P}\xi(E, L)|^2\,\mathrm{d}t\,\mathrm{d}E\,\mathrm{d}L = \\
&= 4\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} \frac{T(E, L)}{|\varphi'(E)|} \int_0^1 \left|\zeta(t, E, L) - \int_0^1 \zeta(s, E, L)\,\mathrm{d}s\right|^2 \mathrm{d}t\,\mathrm{d}E\,\mathrm{d}L \leq \\
&\leq 4\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} \frac{T(E, L)}{|\varphi'(E)|} \int_0^1 \int_0^1 |\zeta(t, E, L) - \zeta(s, E, L)|^2\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}E\,\mathrm{d}L,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last step. To estimate the inner two integrals, we first consider the case where $s \leq t$. With aid of the main theorem of calculus we arrive at

$$\begin{aligned}
\int_0^1 \int_0^t |\zeta(t, E, L) - \zeta(s, E, L)|^2\,\mathrm{d}s\,\mathrm{d}t &= \int_0^1 \int_0^t \left|\int_s^t \partial_\tau \zeta(\tau, E, L)\,\mathrm{d}\tau\right|^2 \mathrm{d}s\,\mathrm{d}t \leq \\
&\leq \int_0^1 \int_0^t (t - s) \int_s^t |\partial_\tau \zeta(\tau, E, L)|^2\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t \leq \int_0^1 |\partial_\tau \zeta(\tau, E, L)|^2\,\mathrm{d}\tau,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality once again. By estimating the part where $s > t$ in a similar way, we obtain

$$\|\xi - \mathcal{P}\xi\|^2_{\frac{1}{|\varphi'|}} \leq 8\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} \frac{T(E, L)}{|\varphi'(E)|} \int_0^1 |\partial_\tau \zeta(\tau, E, L)|^2\,\mathrm{d}\tau\,\mathrm{d}E\,\mathrm{d}L.$$

Recall

$$\partial_\tau \zeta(\tau, E, L) = T(E, L) \cdot (\mathcal{D}^r \xi^r)((R, W)(\tau \cdot T(E, L), r_-(E, L), 0, L), L)$$

for $\tau \in \mathbb{R}$, $L > 0$ and $\psi_L(r_L) < E < E_0$ by definition. Therefore, by using the estimate of $T(E, L)$ from Definition & Remark 3.19, we get

$$\begin{aligned}
&\|\xi - \mathcal{P}\xi\|^2_{\frac{1}{|\varphi'|}} \leq \\
&\qquad \leq 32\pi^4 M_0^4 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} \frac{T(E, L)}{|\varphi'(E)|} \cdot \frac{1}{E^4 L} \int_0^1 |(\mathcal{D}\xi)^r(\ldots)|^2\,\mathrm{d}\tau\,\mathrm{d}E\,\mathrm{d}L,
\end{aligned}$$

where the omitted argument of $(\mathcal{D}\xi)^r$ is $(R, W)(\tau T(E, L), r_-(E, L), 0, L), L$.

4) *Conclusion:*
We now want to apply the estimate from the previous step to the elements of the approximation sequence. Due to the properties of the support of F_k for $k \in \mathbb{N}$, we obtain the bound

$$\frac{1}{E^4 L} \leq \frac{2m}{E_0^4}$$

for all $L > 0$ and $\psi_L(r_L) \leq E < E_0$ for which there exists $\tau \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $0 \neq (\mathcal{D}F_k)^r((R,W)(\tau \cdot T(E,L), r_-(E,L), 0, L), L)$, recall again that E and L are conserved along characteristics. Using this inequality and changing back into (x,v)-coordinates, we finally arrive at

$$\begin{aligned} \|F_k - \mathcal{P}F_k\|^2_{\frac{1}{|\varphi'|}} &\leq \\ &\leq 64\pi^4 M_0^4 \frac{m}{E_0^4} \int_0^\infty \int_{\psi_L(r_L)}^{E_0} \frac{T(E,L)}{|\varphi'(E)|} \int_0^1 \left|(\mathcal{D}F_k)^r(\ldots)\right|^2 \mathrm{d}\tau \, \mathrm{d}E \, \mathrm{d}L = \\ &= 16\pi^2 M_0^4 \frac{m}{E_0^4} \|\mathcal{D}F_k\|^2_{\frac{1}{|\varphi'|}} \to 0 \text{ as } k \to \infty. \end{aligned}$$

Hence, since $F_k \to f$, we obtain $\mathcal{P}F_k \to f$ in $L^2_{\frac{1}{|\varphi'|}}(\Omega_0)$ for $k \to \infty$ as well. Since $K_\mathcal{D}$ is a closed subspace of $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ and $(\mathcal{P}F_k)_{k\in\mathbb{N}} \subset K_\mathcal{D}$, we conclude $f \in K_\mathcal{D}$. □

Remarks:

a) To prove Theorem 3.21 we did not use the already known Jeans' theorem for smooth functions. Instead, we provide a "new" proof, which also works for the weak version of $\mathcal{D}$. Note however that our proof relies on the form of the characteristics in (r,w,L)-coordinates in a crucial way (since we used them do define the projection $\mathcal{P}$). In [3], these properties of (R,W,L) were the key ingredient for the proof of Jeans' theorem as well.

b) Another possible approach to the kernel of $\mathcal{D}$ is by interpreting $\mathcal{D}$ as the infinitesimal generator of the unitary C^0-group $(\mathcal{U}(s))_{s\in\mathbb{R}}$ on $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$, see Remark 3.10 for details. Then, $\mathcal{D}f = 0$ for a function $f \in \mathrm{D}(\mathcal{D})$ can be interpreted as $\partial_s\big|_{s=0}[\mathcal{U}(s)f] = 0$ in the Bochner-sense. This instantly leads to

$$\ker(\mathcal{D}) = \{f \in \mathrm{D}(\mathcal{D}) \mid \mathcal{U}(s)f = f \text{ for every } s \in \mathbb{R}\}.$$

However, showing that the set on the right hand side equals $K_\mathcal{D}$ does not seem to be easier than what we did to prove Theorem 3.21. The latter is the main reason we did not pursue this alternate definition of $\mathcal{D}$.

c) *In a similar way as done in this chapter, one could also define the transport operator in a weak sense on non-radial functions by dropping the assumption of spherical symmetry from all function spaces involved.*
More precisely, for $f \in L^1_{loc}(\Omega_0)$, $\tilde{\mathcal{D}}f$ *exists weakly, if there exists* $\mu \in L^1_{loc}(\Omega_0)$ *such that for every test function* $\xi \in C^1_c(\Omega_0)$,

$$\int_{\Omega_0} \frac{1}{|\varphi' \circ E|} f \cdot \mathcal{D}\xi = -\int_{\Omega_0} \frac{1}{|\varphi' \circ E|} \mu \cdot \xi.$$

In this case $\tilde{\mathcal{D}}f \coloneqq \mu$ *weakly. The domain of* $\tilde{\mathcal{D}}$ *is defined as*

$$\mathrm{D}(\tilde{\mathcal{D}}) \coloneqq \{f \in L^2_{\frac{1}{|\varphi'|}}(\Omega_0) \mid \tilde{\mathcal{D}}f \text{ exists weakly and } \tilde{\mathcal{D}}f \in L^2_{\frac{1}{|\varphi'|}}(\Omega_0)\}.$$

Then the resulting operator $\tilde{\mathcal{D}} \colon \mathrm{D}(\tilde{\mathcal{D}}) \to L^2_{\frac{1}{|\varphi'|}}(\Omega_0)$ *is linear and densely defined on* $L^2_{\frac{1}{|\varphi'|}}(\Omega_0)$.
Furthermore, there holds an approximation result similar to Theorem 3.15 for $\tilde{\mathcal{D}}$ *as well, from which the skew-symmetry and skew-adjointness of* $\tilde{\mathcal{D}}$ *follow just like for* $\mathcal{D}$.
However, due to the differing classes of test functions, it is not clear whether or not $\mathcal{D}$ *is indeed the restriction of* $\tilde{\mathcal{D}}$ *on spherically symmetric functions, i.e., if*

$$\mathcal{D} \quad \text{and} \quad \tilde{\mathcal{D}}\big|_{L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)}$$

equal.
Nevertheless, for the application in the following chapters as well as in [9, 19], this uncertainty is insignificant, since the transport operator is only needed on spherically symmetric functions there.

4 The Guo-Lin operator

In this whole chapter, let $f_0 = \varphi \circ E$ be a fixed isotropic state in the sense of Definition 2.2. In addition, we assume that φ is decreasing and that the derivative is bounded near the cut-off energy E_0, i.e., $\varphi' < 0$ on $]-\infty, E_0[$ and φ' is bounded on $[\eta, E_0[$ for all $\eta < E_0$, note that φ' is continuous by definition.
This turns out to be a real restriction on the class of steady states which we can handle with our approach. For example, isotropic polytropes $(E_0 - E)_+^k$ with $0 \leq k < 1$ do not satisfy the boundedness condition. Nevertheless, isotropic polytropes with $k \geq 1$ and the equally important King model $\left(e^{E_0 - E} - 1\right)_+$ can still be considered.
Just like in the previous chapter, let Ω_0 denote the set where f_0 does not vanish, that is to say

$$\boldsymbol{\Omega_0} \coloneqq \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid f_0(x,v) \neq 0\} = \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid E(x,v) < E_0\}.$$

Note that $\Omega_0 \subset \mathbb{R}^3 \times \mathbb{R}^3$ is a bounded domain. Let $\boldsymbol{R_0},\ \boldsymbol{P_0} > 0$ be some fixed radii such that $\Omega_0 \subset B_{R_0}(0) \times B_{P_0}(0)$.
The aim of this chapter is to define a certain operator, the "Guo-Lin operator", and prove a coercivity estimate for the operator and a finite dimensional approximation. The latter will turn out to be a key tool for proving the non-linear stability of the King model in Chapter 5. Related techniques have also been used in [19] to show the non-linear stability of steady states also depending on the modulus of the angular momentum squared.

4.1 Definition

Before getting to the desired coercivity estimate, we have to define the operator in question and prove first properties of the appearing quantities. We start by defining radial Sobolev spaces, on which the operator will be defined:

Definition 4.1: *For $k \in \mathbb{N}_0$ and $0 < R \leq \infty$ let*

$$\boldsymbol{H_r^k(B_R(0))} \coloneqq \{\psi \in H^k(B_R(0)) \mid \psi \text{ is spherically symmetric a.e. on } B_R(0)\},$$

where $B_\infty(0) \coloneqq \mathbb{R}^3$ and $H^k(B_R(0))$ is the usual Sobolev space of order k over the set $B_R(0)$, see [7, 21]. Here, a function is called spherically symmetric a.e. on $B_R(0)$, if its extension by 0 is spherically symmetric a.e. on $\mathbb{R}^3$ in the sense of

Lemma & Definition 2.9. As usual, we denote $\boldsymbol{L^2_r} := H^0_r$.
Analogously, set

$$\dot{\boldsymbol{H}}^{\boldsymbol{1}}_{\boldsymbol{r}}(\mathbb{R}^3) := \{\psi \in \dot{H}^1(\mathbb{R}^3) \mid \psi \text{ is spherically symmetric a.e. on } \mathbb{R}^3\},$$

see Section 2.3 for a detailed definition of the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$ *and its properties.*

Next, we want to define a similar projection like the one from Definition & Remark 3.20 of these radial L^2-functions.

Definition 4.2: *For* $\psi \in L^2_r(B_{R_0}(0))$ *and* $L > 0$, $\psi_L(r_L) < E < E_0$ *let*

$$\boldsymbol{\mathcal{P}}\psi(E,L) := \frac{2}{T(E,L)} \int_{r_-(E,L)}^{r_+(E,L)} \frac{\psi^r(r)}{\sqrt{2E - 2\psi_L(r)}} \,\mathrm{d}r,$$

where T *is known from Definition & Remark 3.19. Also,* $r_+(E,L) < R_0$ *since* $E(r_+(E,L), 0, L) = E < E_0$.
Note the similarity to the projection defined in Definition & Remark 3.20. In particular, $\mathcal{P}\psi(E,L)$ *can be interpreted as the average of* ψ *over the trajectory fixed by* (E,L). *With slight abuse of notation, we call the mapping*

$$(x,v) \mapsto \mathcal{P}\psi(E(x,v), L(x,v))$$

$\mathcal{P}\psi$ *again. In the same way as in Definition & Remark 3.20, the latter is well defined a.e. on* Ω_0. *We will extend this function by* 0 *on the whole space* $\mathbb{R}^3 \times \mathbb{R}^3$.

Lemma 4.3: *The mapping* $\mathcal{P}$ *from above has the following properties:*

a) For any $\psi \in L^2_r(B_{R_0}(0))$ *we have* $\mathcal{P}\psi \in L^2_{|\varphi'|,r}(\Omega_0)$ *with*

$$\|\mathcal{P}\psi\|_{|\varphi'|} \le \|\psi\|_{|\varphi'|} \le C\|\psi\|_{L^2(B_{R_0}(0))},$$

where $C > 0$ *depends only on the steady state. This means that the linear operator* $\mathcal{P}\colon L^2_r(B_{R_0}(0)) \to L^2_{|\varphi'|,r}(\Omega_0)$ *is continuous. We refer to Definition 3.5 for the definition of weighted* L^2*-spaces over* Ω_0 *and their norm. Here,* $\psi \in L^2_r(B_{R_0}(0))$ *becomes a function on* Ω_0 *by simply dropping the* v*-dependency and restricting the arguments to* Ω_0, *i.e., we take* $\Omega_0 \ni (x,v) \mapsto \psi(x) \in \mathbb{R}$.

b) For any $\psi \in L^2_r(B_{R_0}(0))$ *we have*

$$(\varphi' \circ E) \cdot \mathcal{P}\psi \in \ker(\mathcal{D}) \text{ and } (\varphi' \circ E) \cdot (\psi - \mathcal{P}\psi) \perp_{\frac{1}{|\varphi'|}} \ker(\mathcal{D}),$$

where the latter means $\langle \psi - \mathcal{P}\psi, f\rangle_{\frac{1}{|\varphi'|}} = 0$ *for every* $f \in \ker(\mathcal{D})$. *Note that* $(\varphi' \circ E) = -|\varphi' \circ E|$ *and* $(\varphi' \circ E) \cdot g \in L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$ *iff* $g \in L^2_{|\varphi'|,r}(\Omega_0)$.

Proof:

a) The second inequality is a direct consequence of the boundedness of φ' on $[U_0(0), E_0[$ and the one of Ω_0 in v.
For the first estimate, we change from (x,v) to (r,E,L)-coordinates to get

$$\begin{aligned}
\langle \mathcal{P}\psi, \psi\rangle_{|\varphi'|} &= \int_{\Omega_0} |\varphi'(E(x,v))| \cdot \mathcal{P}\psi(x,v) \cdot \psi(x) \,\mathrm{d}(x,v) = \\
&= 8\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} |\varphi'(E)| \cdot \mathcal{P}\psi(E,L) \int_{r_-(E,L)}^{r_+(E,L)} \frac{\psi^r(r)}{\sqrt{2E - 2\psi_L(r)}} \,\mathrm{d}r \,\mathrm{d}E \,\mathrm{d}L = \\
&= 8\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} |\varphi'(E)| \cdot (\mathcal{P}\psi)^2(E,L) \cdot \frac{T(E,L)}{2} \,\mathrm{d}E \,\mathrm{d}L = \\
&= \int_{\Omega_0} |\varphi'(E(x,v))| \cdot (\mathcal{P}\psi)^2(x,v) \,\mathrm{d}(x,v) = \|\mathcal{P}\psi\|^2_{|\varphi'|}.
\end{aligned}$$

Thus,

$$0 \le \|\mathcal{P}\psi - \psi\|^2_{|\varphi'|} = \|\mathcal{P}\psi\|^2_{|\varphi'|} - 2\langle \mathcal{P}\psi, \psi\rangle_{|\varphi'|} + \|\psi\|^2_{|\varphi'|} = \|\psi\|^2_{|\varphi'|} - \|\mathcal{P}\psi\|^2_{|\varphi'|},$$

which implies the desired inequality.

b) From the first part it follows that $\mathcal{P}\psi \in L^2_{|\varphi'|,r}(\Omega_0)$, from which we immediately conclude $(\varphi' \circ E) \cdot \mathcal{P}\psi \in \ker(\mathcal{D})$ by the explicit characterisation of $\ker(\mathcal{D})$ from Theorem 3.21.
As for the second claim, let $f = g(E,L) \in \ker(\mathcal{D})$ be arbitrary. Then a similar change of variables like the one above yields

$$\begin{aligned}
\langle (\varphi' \circ E) \cdot \mathcal{P}\psi, f\rangle_{\frac{1}{|\varphi'|}} &= - \int_{\Omega_0} \mathcal{P}\psi(x,v) \cdot f(x,v) \,\mathrm{d}(x,v) = \\
&= -8\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} \mathcal{P}\psi(E,L) \cdot g(E,L) \cdot \frac{T(E,L)}{2} \,\mathrm{d}E \,\mathrm{d}L = \\
&= -8\pi^2 \int_0^\infty \int_{\psi_L(r_L)}^{E_0} g(E,L) \int_{r_-(E,L)}^{r_+(E,L)} \frac{\psi^r(r)}{\sqrt{2E - 2\psi_L(r)}} \,\mathrm{d}r \,\mathrm{d}E \,\mathrm{d}L = \\
&= - \int_{\Omega_0} \psi(x) \cdot f(x,v) \,\mathrm{d}(x,v) = \langle (\varphi' \circ E) \cdot \psi, f\rangle_{\frac{1}{|\varphi'|}},
\end{aligned}$$

note that all integrals are finite. □

We are now able to define the operator with its quadratic form.

Definition 4.4:

a) Let

$$\boldsymbol{\mathcal{K}_0}\colon L^2_r(\mathbb{R}^3) \to L^2_r(\mathbb{R}^3),$$
$$\psi \mapsto 4\pi \int_{\mathbb{R}^3} |\varphi'(E(\cdot,v))| \cdot \mathcal{P}\psi(\cdot,v)\,\mathrm{d}v - 4\pi \int_{\mathbb{R}^3} |\varphi'(E(\cdot,v))|\,\mathrm{d}v \cdot \psi.$$

Here, we extend φ' by 0 on $\mathbb{R}$, i.e., $\varphi'(E(x,v)) = 0$ if $(x,v) \notin \Omega_0$. Note that $\mathcal{K}_0 h \in L^2_r(\mathbb{R}^3)$ for $h \in L^2_r(\mathbb{R}^3)$ follows by the boundedness of $\varphi' \circ E$ and its support together with Lemma 4.3. Now let

$$\boldsymbol{\mathcal{A}_0}\colon H^2_r(\mathbb{R}^3) \to L^2_r(\mathbb{R}^3), \quad \mathcal{A}_0 \coloneqq -\Delta + \mathcal{K}_0.$$

*We will call $\mathcal{A}_0$ the **Guo-Lin operator** because of its appearance in [9]. Clearly $\mathcal{A}_0$ preserves spherical symmetry, i.e., $\mathcal{A}_0 h \in L^2_r(\mathbb{R}^3)$ for $h \in H^2_r(\mathbb{R}^3)$.*

b) The quadratic form associated with $\mathcal{A}_0$ can be written as

$$\langle \mathcal{A}_0\psi, \psi\rangle_2 = \|\nabla\psi\|_2^2 + 4\pi \left(\|\mathcal{P}\psi\|^2_{|\varphi'|} - \|\psi\|^2_{|\varphi'|}\right)$$

for $\psi \in H^2_r(\mathbb{R}^3)$, where we integrated by parts and used $\langle \mathcal{P}\psi, \psi\rangle_{|\varphi'|} = \|\mathcal{P}\psi\|^2_{|\varphi'|}$ from the proof of Lemma 4.3. Note that the latter expression is even defined on $\dot{H}^1_r(\mathbb{R}^3)$, since $\dot{H}^1_r(\mathbb{R}^3) \subset L^2_r(B_{R_0}(0))$ by Definition 2.5. We therefore set

$$\langle \mathcal{A}_0\psi, \psi\rangle_2 \coloneqq \|\nabla\psi\|_2^2 + 4\pi \left(\|\mathcal{P}\psi\|^2_{|\varphi'|} - \|\psi\|^2_{|\varphi'|}\right)$$

for each $\psi \in \dot{H}^1_r(\mathbb{R}^3)$.

Just like the semi-norm $\|\nabla \cdot\|_2$ on $\dot{H}^1(\mathbb{R}^3)$ itself, it turns out that the quadratic form associated with the Guo-Lin operator is invariant under addition of a constant, more precisely:

Lemma 4.5: *For any $\psi \in \dot{H}^1_r(\mathbb{R}^3)$ and $C \in \mathbb{R}$ we have*

$$\langle \mathcal{A}_0(\psi + C), \psi + C\rangle_2 = \langle \mathcal{A}_0\psi, \psi\rangle_2.$$

Proof: Clearly, $\nabla(\psi + C) = \nabla\psi$ in the weak sense. Moreover, $\mathcal{P}C = C$ and $\langle \mathcal{P}\psi, C\rangle_{|\varphi'|} = \langle \psi, C\rangle_{|\varphi'|}$ by similar computations like the ones we used to prove Lemma 4.3. Therefore, expanding the norm yields

$$\begin{aligned}\langle \mathcal{K}_0(\psi + C), \psi + C\rangle_2 &= 4\pi \left(\|\mathcal{P}(\psi + C)\|^2_{|\varphi'|} - \|\psi + C\|^2_{|\varphi'|}\right) = \\ &= 4\pi \left(\|\mathcal{P}\psi\|^2_{|\varphi'|} - \|\psi\|^2_{|\varphi'|}\right) = \langle \mathcal{K}_0\psi, \psi\rangle_2. \qquad \square\end{aligned}$$

4.2 Coercivity

The aim of this whole chapter is to prove a coercivity estimate of the following kind:

$$\inf_{\psi \in \dot{H}^1_r(\mathbb{R}^3), \|\psi\| \neq 0} \frac{\langle \mathcal{A}_0 \psi, \psi \rangle_2}{\|\psi\|^2} > 0,$$

for some reasonable norm $\| \cdot \|$. This turns out to be the most important tool to show the non-linear stability of the King model in Chapter 5.
Y. Guo & Z. Lin presented an estimate like this in [9] using the $H^1(\mathbb{R}^3)$-norm and restricting themselves to the space $H^2_r(\mathbb{R}^3)$. Unfortunately, it turns out that this result does not hold true:

Theorem 4.6:
$$\inf_{\psi \in H^2_r(\mathbb{R}^3), \|\psi\|_2 \neq 0} \frac{\langle \mathcal{A}_0 \psi, \psi \rangle_2}{\|\psi\|_2^2} \leq 0.$$

Proof: Consider a sequence of smooth and spherically symmetric cut-off functions $(\chi_k)_{k \in \mathbb{N}} \subset C^\infty_{c,r}(\mathbb{R}^3)$ with the following properties: $0 \leq \chi_k \leq 1$ on $\mathbb{R}^3$, $\chi_k = 1$ on $B_k(0)$, $\operatorname{supp}(\chi_k) \subset B_{k+1}(0)$ for all $k \in \mathbb{N}$ and $(||\nabla \chi_k||_\infty)_{k \in \mathbb{N}}$, $(||D^2 \chi_k||_\infty)_{k \in \mathbb{N}}$ are bounded.
Now, for every $k \in \mathbb{N}$ let

$$\psi_k := \frac{\chi_k}{||\chi_k||_2}.$$

Our aim is to show that $\langle \mathcal{A}_0 \psi_k, \psi_k \rangle_2 \to 0$ as $k \to \infty$, since $||\psi_k||_2 = 1$ for all $k \in \mathbb{N}$ by definition and clearly $(\psi_k)_{k \in \mathbb{N}} \subset H^2_r(\mathbb{R}^3)$.
Indeed, since $0 \leq \chi_k \leq 1$ on $\mathbb{R}^3$, $\chi_k = 1$ on $B_k(0)$ and $\operatorname{vol}(B_k(0)) = \frac{4\pi}{3} k^3$, the denominators above satisfy

$$||\chi_k||_2^2 \geq \frac{4\pi}{3} k^3$$

and thus

$$||\psi_k||_\infty \leq \left(\frac{3}{4\pi k^3} \right)^{\frac{1}{2}} \to 0 \quad \text{as } k \to \infty.$$

To show the desired convergence, we now estimate each of the three terms occurring in the sum $\langle \mathcal{A}_0 \psi, \psi \rangle_2$ separately:

1) From the boundedness of $(||\nabla \chi_k||_\infty)_{k \in \mathbb{N}}$ and $\nabla \chi_k$ vanishing outside of $B_{k+1} \setminus B_k(0)$ with $\operatorname{vol}(B_{k+1} \setminus B_k(0)) \leq Ck^2$ for some constant $C > 0$ independent of k, we easily obtain

$$||\nabla \psi_k||_2^2 = \int_{\mathbb{R}^3} \frac{|\nabla \chi_k(x)|^2}{||\chi_k||_2^2} \, \mathrm{d}x \leq \frac{C}{k} \to 0 \quad \text{as } k \to \infty.$$

2) $\int |\varphi'(E(\cdot,v))|\,\mathrm{d}v \in L^1 \cap L^\infty(\mathbb{R}^3)$ yields

$$\|\psi_k\|_{|\varphi'|}^2 \leq || \int_{\mathbb{R}^3} |\varphi'(E(\cdot,v))|\,\mathrm{d}v||_1 \cdot ||\psi_k||_\infty^2 \to 0 \quad \text{as } k \to \infty.$$

3) The convergence $\|\mathcal{P}\psi_k\|_{|\varphi'|} \to 0$ for $k \to \infty$ follows directly from the convergence of the second term together with Lemma 4.3. □

Remarks:

a) One can even show that $\mathcal{K}_0$ is relatively $(-\Delta)$-compact, where Δ is interpreted as the Laplacian on the space of spherically symmetric functions, i.e., $\Delta\colon H_r^2(\mathbb{R}^3) \to L_r^2(\mathbb{R}^3)$. By Weyl's theorem, we have

$$\sigma_{ess}(\mathcal{A}_0) = \sigma_{ess}(-\Delta) = [0, \infty[.$$

The idea behind the sequence $(\psi_k)_{k\in\mathbb{N}}$ from the proof above is that it is a Weyl sequence for $-\Delta$ and 0. Therefore, by the relative compactness, it is also a Weyl sequence for $\mathcal{A}_0$ and 0. We refer to [14] for all these rather abstract results from spectral theory.

b) In our decreasing case, i.e., $\varphi' < 0$ on $]-\infty, E_0[$, we know a priori that $\inf_{\psi\in H_r^2(\mathbb{R}^3)\setminus\{0\}} \langle \mathcal{A}_0\psi, \psi\rangle_2 \geq 0$ by the instability criterion from the first part of [9] and the well known non-existence of exponentially growing radial modes for decreasing models like f_0, see [5, 17]. Hence, we even obtain

$$\inf_{\psi\in H_r^2(\mathbb{R}^3), \|\psi\|_2\neq 0} \frac{\langle \mathcal{A}_0\psi, \psi\rangle_2}{||\psi||_2^2} = 0.$$

c) The error in the proof in [9] is that the embedding $H_r^k(\mathbb{R}^3) \subset L_r^2(\mathbb{R}^3)$ is not compact for any $k \in \mathbb{N}$, see [6] for a counterexample. Also note that we can not restrict ourselves to a bounded set, since the desired estimate will be used on gravitational potentials in Chapter 5, which do not have a compact support in general.

Theorem 4.6 illustrates that every norm containing the L^2-norm, like $\|\cdot\|_{H^1(\mathbb{R}^3)}$, can not be considered to show the desired coercivity estimate. However, it turns out that $\dot{H}_r^1(\mathbb{R}^3)$ with its semi-norm $\|\nabla\cdot\|_2$ is the right space for this result.
A first evidence for this is that the quadratic form associated with the Guo-Lin operator is invariant under addition of a constant just like the semi-norm $\|\nabla\cdot\|_2$ itself by Lemma 4.5.
Also note that $\dot{H}_r^1(\mathbb{R}^3)$ is a very natural space for gravitational potentials of smooth & spherically symmetric solutions of the Vlasov-Poisson system to be part of, since their derivative is in $L^2(\mathbb{R}^3)$, but the function itself may not be quadratically integrable over the whole space $\mathbb{R}^3$.
We therefore want to show the following:

Theorem 4.7: *Let*

$$\boldsymbol{\lambda_0} := \inf_{\psi\in \dot{H}^1_r(\mathbb{R}^3), \|\nabla\psi\|_2\neq 0} \frac{\langle \mathcal{A}_0\psi,\psi\rangle_2}{\|\nabla\psi\|_2^2} \in \mathbb{R}\cup\{-\infty\}.$$

Then $\lambda_0 > 0$. In particular, for every $\psi \in \dot{H}^1_r(\mathbb{R}^3)$ we have

$$\langle \mathcal{A}_0\psi,\psi\rangle_2 \geq \lambda_0\|\nabla\psi\|_2^2.$$

We split the proof of this very important theorem into several parts. First, we show that the infimum is obtained by a non-constant function. After that, we will prove that the quadratic form of this minimizer does not vanish. It turns out that the latter result relies on the Antonov coercivity bound [2] in an essential way.

Proposition 4.8: *At least one of the following statements is true:*

(i) *The infimum λ_0 from Theorem 4.7 is obtained by a non-constant function, i.e., there exists $\psi_0 \in \dot{H}^1_r(\mathbb{R}^3)$ such that $\|\nabla\psi_0\|_2 \neq 0$ and*

$$\frac{\langle \mathcal{A}_0\psi_0,\psi_0\rangle_2}{\|\nabla\psi_0\|_2^2} = \lambda_0.$$

(ii) $\lambda_0 = 1$.

Proof: We split the proof of this result into several steps:

1) *An improved minimizing sequence:*
Let $(\chi_k)_{k\in\mathbb{N}} \subset \dot{H}^1_r(\mathbb{R}^3)$ be a minimizing sequence for the infimum in Theorem 4.7, i.e., $\|\nabla\chi_k\|_2 \neq 0$ for $k\in\mathbb{N}$ and

$$\frac{\langle \mathcal{A}_0\chi_k,\chi_k\rangle_2}{\|\nabla\chi_k\|_2^2} \to \lambda_0 \text{ as } k\to\infty.$$

We will modify this minimizing sequence such that its integral over $B_{R_0}(0)$ vanishes and that it takes a constant value in the $\dot{H}^1(\mathbb{R}^3)$ semi-norm, i.e., we consider the sequence $(\psi_k)_{k\in\mathbb{N}} \subset \dot{H}^1_r(\mathbb{R}^3)$ given by

$$\psi_k := \frac{1}{\|\nabla\chi_k\|_2}\left(\chi_k - \fint_{B_{R_0}(0)} \chi_k(x)\,\mathrm{d}x\right)$$

for $k\in\mathbb{N}$, where $\fint_\Omega := \mathrm{vol}^{-1}(\Omega)\cdot\int$ for suitable $\Omega\subset\mathbb{R}^3$. As motivated above, we clearly have $\psi_k \in \dot{H}^1_r(\mathbb{R}^3)$, $\int_{B_{R_0}(0)}\psi_k = 0$ and $\|\nabla\psi_k\|_2 = 1$ for all $k\in\mathbb{N}$. Additionally, Lemma 4.5 implies

$$\lambda_0 \leftarrow \frac{\langle \mathcal{A}_0\chi_k,\chi_k\rangle_2}{\|\nabla\chi_k\|_2^2} = \langle \mathcal{A}_0\psi_k,\psi_k\rangle_2 = 1 + 4\pi\left(\|\mathcal{P}\psi_k\|^2_{|\varphi'|} - \|\psi_k\|^2_{|\varphi'|}\right),$$

which means that $(\psi_k)_{k\in\mathbb{N}}$ is still a minimizing sequence.

2) *Convergence of the sequence:*
Since $(\|\nabla\psi_k\|_2)_{k\in\mathbb{N}}$ is bounded, we know that after extracting a subsequence, again denoted by $(\psi_k)_{k\in\mathbb{N}}$, $(\nabla\psi_k)_{k\in\mathbb{N}}$ is weakly convergent in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ due to the Banach-Alaoglu theorem, see [21]. Thanks to the compact embedding from Lemma 2.8, this causes $(\psi_k)_{k\in\mathbb{N}}$ to have a (strongly) convergent subsequence in $L^2(B_{R_0}(0))$.
However, to obtain the limit on the whole space $\mathbb{R}^3$ and show its spherical symmetry, we have to iterate Lemma 2.8:

3) *Coincidence of the limits and spherical symmetry:*
Let $\chi \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ be such that $\nabla\psi_k \rightharpoonup \chi$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ as $k\to\infty$. Furthermore, as stated above, there exists $\psi_0^1 \in L^2(B_{R_0}(0))$ and a subsequence $(\psi_k^1)_{k\in\mathbb{N}} \subset (\psi_k)_{k\in\mathbb{N}}$ such that

$$\psi_k^1 \to \psi_0^1 \text{ in } L^2(B_{R_0}(0)) \text{ as } k\to\infty.$$

For this limit we obtain $\nabla\psi_0^1 = \chi$ in $\mathcal{D}'(B_{R_0}(0))$, i.e., as a weak derivative with test functions in $C_c^\infty(B_{R_0}(0))$. The latter is a direct consequence of the two convergences. In addition, since the spherically symmetric subspace of $L^2(B_{R_0}(0))$ is closed due to the Fischer-Riesz theorem, we also obtain the spherical symmetry of the limit, i.e., $\psi_0^1 \in L_r^2(B_{R_0}(0))$.
By applying Lemma 2.8 to $B_{2R_0}(0)$ and a similar argumentation, we also get the existence of $\psi_0^2 \in L_r^2(B_{2R_0}(0))$ and a subsequence $(\psi_k^2)_{k\in\mathbb{N}} \subset (\psi_k^1)_{k\in\mathbb{N}}$ such that

$$\psi_k^2 - C_k^2 \to \psi_0^2 \text{ in } L^2(B_{2R_0}(0)) \text{ as } k\to\infty,$$

where $C_k^2 := \fint_{B_{2R_0}(0)} \psi_k^2(x)\,\mathrm{d}x$ is needed for the sequence to satisfy the conditions of Lemma 2.8. Note that the subtraction of C_k^2 does not affect the $\|\nabla\cdot\|_2$ semi-norm and therefore also not the $\|\nabla\cdot\|_2$-boundedness of the sequence. In the same way as above this yields $\psi_0^2 \in L_r^2(B_{2R_0}(0))$ and $\nabla\psi_0^2 = \chi$ in $\mathcal{D}'(B_{2R_0}(0))$. In particular, $\nabla\psi_0^1 = \nabla\psi_0^2$ in $\mathcal{D}'(B_{R_0}(0))$, which implies that there exists a constant $C^2 \in \mathbb{R}$ such that $\psi_0^1 = \psi_0^2 + C^2$ a.e. on $B_{R_0}(0)$.
We can now define the desired (global) function as

$$\psi_0 := \psi_0^2 + C^2 \text{ on } B_{2R_0}(0).$$

Iterating this argument, we get a function ψ_0 defined on the whole space $\mathbb{R}^3$ such that $\psi_0 = \psi_0^1$ a.e. on $B_{R_0}(0)$, $\psi_0 \in L_{loc,r}^2(\mathbb{R}^3)$ and $\nabla\psi_0 = \chi$ in $\mathcal{D}'(\mathbb{R}^3)$. Altogether, $\psi_0 \in \dot{H}_r^1(\mathbb{R}^3)$ and, after extracting a subsequence of $(\psi_k)_{k\in\mathbb{N}}$ which shares its name with the original sequence,

$$\psi_k \to \psi_0 \text{ in } L^2(B_{R_0}(0)) \quad\text{and}\quad \nabla\psi_k \rightharpoonup \nabla\psi_0 \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3) \text{ as } k\to\infty.$$

4) *The limit is non-constant:*
Lemma 4.3 yields $\psi_k \to \psi_0$ and $\mathcal{P}\psi_k \to \mathcal{P}\psi_0$ in $L^2_{|\varphi'|}(\Omega_0)$, from which we obtain

$$\|\mathcal{P}\psi_k\|^2_{|\varphi'|} - \|\psi_k\|^2_{|\varphi'|} \to \|\mathcal{P}\psi_0\|^2_{|\varphi'|} - \|\psi_0\|^2_{|\varphi'|} \text{ as } k \to \infty$$

and subsequently

$$\lambda_0 = 1 + 4\pi \left(\|\mathcal{P}\psi_0\|^2_{|\varphi'|} - \|\psi_0\|^2_{|\varphi'|}\right).$$

Now there are two possible cases: If $\|\mathcal{P}\psi_0\|^2_{|\varphi'|} = \|\psi_0\|^2_{|\varphi'|}$, we instantly get $\lambda_0 = 1$.
Otherwise we have $\|\mathcal{P}\psi_0\|^2_{|\varphi'|} < \|\psi_0\|^2_{|\varphi'|}$ by Lemma 4.3, which also leads to $\|\nabla\psi_0\|_2 > 0$. To see the latter, let $\tilde{\psi}_0 := \psi_0 - ⨍_{B_{R_0}(0)} \psi_0(x)\,\mathrm{d}x$, which clearly satisfies $\int_{B_{R_0}(0)} \tilde{\psi}_0(x)\,\mathrm{d}x = 0$. Moreover, Lemma 4.5 yields

$$\|\mathcal{P}\tilde{\psi}_0\|^2_{|\varphi'|} - \|\tilde{\psi}_0\|^2_{|\varphi'|} = \|\mathcal{P}\psi_0\|^2_{|\varphi'|} - \|\psi_0\|^2_{|\varphi'|} < 0,$$

and consequently

$$0 \le \|\mathcal{P}\tilde{\psi}_0\|_{|\varphi'|} < \|\tilde{\psi}_0\|_{|\varphi'|} \le C\|\tilde{\psi}_0\|_{L^2(B_{R_0}(0))} \le C\|\nabla\tilde{\psi}_0\|_2 = C\|\nabla\psi_0\|_2,$$

where we used Lemmata 2.6 and 4.3. Here, C may change with each inequality, but still depends only on the fixed steady state f_0.

5) *The infimum is obtained by the limit:*
We even get $\|\nabla\psi_0\|_2 = 1$ in the latter case. Indeed, $\psi_0 \in \dot{H}^1(\mathbb{R}^3)$ and $\|\nabla\psi_0\|_2 \neq 0$ lead to

$$\begin{aligned} 1+4\pi \left(\|\mathcal{P}\psi_0\|^2_{|\varphi'|} - \|\psi_0\|^2_{|\varphi'|}\right) = \lambda_0 \le \frac{\langle \mathcal{A}_0\psi_0, \psi_0\rangle_2}{\|\nabla\psi_0\|_2^2} = \\ = 1 + \frac{4\pi}{\|\nabla\psi_0\|_2^2}\left(\|\mathcal{P}\psi_0\|^2_{|\varphi'|} - \|\psi_0\|^2_{|\varphi'|}\right) \end{aligned}$$

by definition of the infimum λ_0. This shows $\|\nabla\psi_0\|_2 \ge 1$. On the other hand, the weak lower semicontinuity of $\|\cdot\|_2$ (cf. [21]) yields

$$\|\nabla\psi_0\|_2 \le \liminf_{k\to\infty} \|\nabla\psi_k\|_2 = 1.$$

Thus, we finally obtain

$$\lambda_0 = 1 + 4\pi \left(\|\mathcal{P}\psi_0\|^2_{|\varphi'|} - \|\psi_0\|^2_{|\varphi'|}\right) = \langle \mathcal{A}_0\psi_0, \psi_0\rangle_2 = \frac{\langle \mathcal{A}_0\psi_0, \psi_0\rangle_2}{\|\nabla\psi_0\|_2^2}. \qquad \square$$

Usually in proofs like the one above, one of the hardest steps is to show that the limit of the minimizing sequence has non-vanishing norm, i.e., $\|\nabla\psi_0\|_2 \neq 0$ in our situation. However, this has been done rather quickly, since $\|\nabla\cdot\|_2^2$ is one summand of the quadratic form and therefore a vanishing norm of the minimizer would instantly imply $\lambda_0 > 0$. Abstractly speaking, this can be interpreted as the relative $(-\Delta)$-compactness of $\mathcal{A}_0$ discussed in the remark above, cf. [14].
Since the second statement from the proposition above would instantly prove Theorem 4.7, we only have to consider the case where the first one is true. It therefore remains to show that the operator (i.e., its quadratic form) is positive, which means that for any $\psi \in \dot{H}_r^1(\mathbb{R}^3)$ with $\|\nabla\psi\|_2 \neq 0$ we have

$$\langle A_0\psi, \psi\rangle_2 > 0.$$

This has been done in [19] in a quite similar situation. Nevertheless, we present the required tools here as well.
Just like in [9], the positivity of the quadric form crucially relies on the coercivity of the Antonov functional, which has first been shown by V. Antonov in [2]. We therefore introduce this result:

Definition & Lemma 4.9: *The* ***Antonov functional*** *(on smooth functions) is given by*

$$\boldsymbol{\mathcal{A}}\colon C_c^1(\Omega_0) \to \mathbb{R},\ \mathcal{A}(f) := \frac{1}{2}\|f\|^2_{\frac{1}{|\varphi'|}} - \frac{1}{8\pi}\|\partial_x U_f\|_2^2,$$

where $U_f := -\frac{1}{|\cdot|} * \int_{\mathbb{R}^3} f(\cdot, v)\,\mathrm{d}v$ is the gravitational potential generated by f. Since the gradient of potentials induced by smooth and compactly supported functions is square integrable (see [21]), the expression above is well defined. Furthermore, the following coercivity estimate holds true:
For all $f \in C_{c,r}^\infty(\Omega_0)$ odd in v, i.e., $f(x,v) = -f(x,-v)$ for $(x,v) \in \Omega_0$, we have

$$\mathcal{A}(\mathcal{D}f) \geq \frac{1}{2}\int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \cdot \frac{\partial_r U_0(|x|)}{|x|} \cdot |f(x,v)|^2 \,\mathrm{d}(x,v),$$

where $\mathcal{D}$ is the transport operator from Definition 3.1 and $\partial_r U_0$ denotes the radial derivative of the steady state potential.

We will not prove the "Antonov coercivity bound" here, but refer to [10] for a detailed proof, where an even sharper estimate has been shown. A similar result in the case where the steady state also depends on L can be found in [19].
We have now finally collected all the tools to prove Theorem 4.7:

Proof of Theorem 4.7: Suppose that the first statement from Proposition 4.8 holds true, otherwise we could immediately conclude Theorem 4.7. Thus, let $\psi_0 \in \dot{H}_r^1(\mathbb{R}^3)$ be such that $\|\nabla\psi_0\|_2 = 1$ and

$$\lambda_0 = \langle \mathcal{A}_0\psi_0, \psi_0\rangle_2 = 1 + 4\pi\left(\|\mathcal{P}\psi_0\|^2_{|\varphi'|} - \|\psi_0\|^2_{|\varphi'|}\right).$$

It remains to show that $\langle \mathcal{A}_0\psi_0, \psi_0\rangle_2 > 0$, which we do in several steps:

1) *Approximation:*
By Lemma 4.3 we know that

$$(\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0) \in \ker(\mathcal{D})^\perp,$$

where $\perp$ works with respect to $\langle \cdot, \cdot \rangle_{\frac{1}{|\varphi'|}}$.
As a densely defined linear operator on the Hilbert space $L^2_{\frac{1}{|\varphi'|},r}(\Omega_0)$, $\mathcal{D}$ satisfies

$$\ker(\mathcal{D}^*)^\perp = \overline{\operatorname{im}(\mathcal{D})},$$

cf. [14]. Using the skew-adjointness of $\mathcal{D}$ from Theorem 3.18, we therefore obtain

$$(\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0) \in \ker(\mathcal{D})^\perp = \overline{\operatorname{im}(\mathcal{D})}.$$

We can not stress the importance of the latter statement enough. Note that we used the explicit characterisation of $\ker(\mathcal{D})$ from Theorem 3.21 to show $(\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0) \in \ker(\mathcal{D})^\perp$ and Theorem 3.18 to conclude $\mathcal{D}^* = -\mathcal{D}$. In fact, this result is the main reason why we went on the effortful journey of defining $\mathcal{D}$ in a weak sense and investigating its properties in Chapter 3. Clearly, since $(\psi_0 - \mathcal{P}\psi_0)$ depends only on r, E and L, we also know that $(\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0)$ is even in v a.e. on Ω_0, i.e., $(\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0)(x, -v) = (\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0)(x, v)$ for a.e. $(x, v) \in \Omega_0$.

2) *Improving the approximation sequence:*
From the first step of this proof we obtain the existence of a sequence $(h_k)_{k\in\mathbb{N}} \subset \mathrm{D}(\mathcal{D})$ such that $\mathcal{D}h_k \to (\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0)$ in $L^2_{\frac{1}{|\varphi'|}}(\Omega_0)$ as $k \to \infty$. By applying Theorem 3.15, we even get a sequence of smooth functions $(f_k)_{k\in\mathbb{N}} \subset C^\infty_{c,r}(\Omega_0)$ such that $\mathcal{D}f_k \to (\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0)$ in $L^2_{\frac{1}{|\varphi'|}}(\Omega_0)$ as $k \to \infty$.
Corollary 3.17 shows that $\mathcal{D}$ reverses v-parity. We therefore want to use the fact that $(\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0)$ is even in v to improve the approximation sequence even more:
For $k \in \mathbb{N}$ and $(x, v) \in \Omega_0$ we set

$$F_k(x, v) := \frac{1}{2} \left(f_k(x, v) - f_k(x, -v) \right).$$

Obviously, $F_k \in C^\infty_{c,r}(\Omega_0)$ is odd in v for each $k \in \mathbb{N}$. An easy computation yields $\mathcal{D}\left[f_k(\cdot, -\cdot)\right](x, v) = -\mathcal{D}f_k(x, -v)$ for $(x, v) \in \Omega_0$ and $k \in \mathbb{N}$, where $f_k(\cdot, -\cdot) := [\Omega_0 \ni (x, v) \mapsto f_k(x, -v)]$. Since the limit $(\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0)$ is even in v a.e. on Ω_0, we also obtain

$$\mathcal{D}F_k \to (\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0) \text{ in } L^2_{\frac{1}{|\varphi'|}}(\Omega_0) \text{ as } k \to \infty.$$

3) *The connection between λ_0 and the Antonov functional:*
Using the approximation sequence from the second step, we arrive at

$$\begin{aligned}\lambda_0 = \langle \mathcal{A}_0\psi_0, \psi_0\rangle_2 &= 1 + 4\pi\left(\|\mathcal{P}\psi_0\|^2_{|\varphi'|} - \|\psi_0\|^2_{|\varphi'|}\right) = \\ &= 1 + 4\pi\|\psi_0 - \mathcal{P}\psi_0\|^2_{|\varphi'|} - 8\pi\langle \psi_0 - \mathcal{P}\psi_0, \psi_0\rangle_{|\varphi'|} = \\ &= 1 + 4\pi\|(\varphi'\circ E)\cdot(\psi_0 - \mathcal{P}\psi_0)\|^2_{\frac{1}{|\varphi'|}} + \\ &\qquad - 8\pi\langle(\varphi'\circ E)\cdot(\psi_0 - \mathcal{P}\psi_0), (\varphi'\circ E)\cdot\psi_0\rangle_{\frac{1}{|\varphi'|}} = \\ &= \lim_{k\to\infty}\left(1 + 4\pi\|\mathcal{D}F_k\|^2_{\frac{1}{|\varphi'|}} - 8\pi\langle \mathcal{D}F_k, (\varphi'\circ E)\cdot\psi_0\rangle_{\frac{1}{|\varphi'|}}\right).\end{aligned}$$

For $k \in \mathbb{N}$ let $U_{\mathcal{D}F_k} := -\frac{1}{|\cdot|} * \int_{\mathbb{R}^3} \mathcal{D}F_k(\cdot, v)\,\mathrm{d}v$ denote the gravitational potential induced by $\mathcal{D}F_k$. Since $\mathcal{D}F_k \in C^1_c(\Omega_0)$, it follows from basic potential theory that $U_{\mathcal{D}F_k} \in C^2(\mathbb{R}^3)$ with $\nabla U_{\mathcal{D}F_k} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ and $\Delta U_{\mathcal{D}F_k} = 4\pi\int_{\mathbb{R}^3}\mathcal{D}F_k(\cdot, v)\,\mathrm{d}v$, cf. [21]. Thus

$$\begin{aligned}-8\pi\langle \mathcal{D}F_k, (\varphi'\circ E)\cdot\psi_0\rangle_{\frac{1}{|\varphi'|}} &= 8\pi\langle \mathcal{D}F_k, \psi_0\rangle_2 = \\ &= 8\pi\int_{\mathbb{R}^3}\psi_0(x)\int_{\mathbb{R}^3}\mathcal{D}F_k(x,v)\,\mathrm{d}v\,\mathrm{d}x = 2\int_{\mathbb{R}^3}\psi_0(x)\cdot\Delta U_{\mathcal{D}F_k}(x)\,\mathrm{d}x = \\ &= -2\int_{\mathbb{R}^3}\nabla\psi_0(x)\cdot\nabla U_{\mathcal{D}F_k}(x)\,\mathrm{d}x \geq -2\|\nabla\psi_0\|_2\cdot\|\nabla U_{\mathcal{D}F_k}\|_2 \geq \\ &\geq -\left(\|\nabla\psi_0\|_2^2 + \|\nabla U_{\mathcal{D}F_k}\|_2^2\right) = -\left(1 + \|\nabla U_{\mathcal{D}F_k}\|_2^2\right)\end{aligned}$$

for every $k \in \mathbb{N}$, where we used the inequalities of Cauchy-Schwarz and Cauchy.
Admittedly, one step in the computation above has not been justified yet, namely the integration by parts

$$\int_{\mathbb{R}^3}\psi_0(x)\cdot\Delta U_{\mathcal{D}F_k}(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3}\nabla\psi_0(x)\cdot\nabla U_{\mathcal{D}F_k}(x)\,\mathrm{d}x.$$

Indeed, $\psi_0 \in L^2_{loc}(\mathbb{R}^3)$, $\Delta U_{\mathcal{D}F_k} \in C_c(\mathbb{R}^3)$ and $\nabla\psi_0, \nabla U_{\mathcal{D}F_k} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, which means that both integrals exist. Since $\psi_0 \in \dot{H}^1(\mathbb{R}^3)$, it seems very convincing that this formula holds true. However, potentials of general compactly supported smooth functions do not need to have a compact support, i.e., they are not in the class of test functions used for weak derivatives. Moreover, any attempt of cutting off the functions involved fails due to the fact that ψ_0 itself does not have to be square integrable over the whole space $\mathbb{R}^3$. In fact, we need an additional property of potentials induced by the images of the transport operator $\mathcal{D}$. To give this property its well deserved appreciation, we will justify the integration by parts formula at the very end of this proof.

Inserting the estimate from above into the equation for λ_0, we now obtain

$$\lambda_0 = \lim_{k\to\infty} \left(1 + 4\pi \|\mathcal{D}F_k\|^2_{\frac{1}{|\varphi'|}} - 8\pi \langle \mathcal{D}F_k, (\varphi' \circ E) \cdot \psi_0 \rangle_{\frac{1}{|\varphi'|}}\right) \geq$$

$$\geq \limsup_{k\to\infty} \left(4\pi \|\mathcal{D}F_k\|^2_{\frac{1}{|\varphi'|}} - \|\nabla U_{\mathcal{D}F_k}\|_2^2\right) = 8\pi \limsup_{k\to\infty} \mathcal{A}(\mathcal{D}F_k),$$

where $\mathcal{A}$ is the Antonov functional known from Definition & Lemma 4.9.

4) *Conclusion by Antonov's coercivity bound:*
The Antonov coercivity bound from Definition & Lemma 4.9 yields

$$\lambda_0 \geq 4\pi \limsup_{k\to\infty} \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \cdot \frac{\partial_r U_0(|x|)}{|x|} \cdot |F_k(x,v)|^2 \,\mathrm{d}(x,v).$$

In particular $\lambda_0 \geq 0$, since $\partial_r U_0 \geq 0$ on $[0,\infty[$ by Remark 2.3.
Now suppose $\lambda_0 = 0$. Once again using the coercivity bound, this would lead to

$$\lim_{k\to\infty} \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \cdot \frac{\partial_r U_0(|x|)}{|x|} \cdot |F_k(x,v)|^2 \,\mathrm{d}(x,v) = 0.$$

Let $\xi \in C^1_{c,r}(\Omega_0)$ be a test function whose support is bounded away from $\{x = 0\}$, i.e., $\mathrm{supp}(\xi) \subset \Omega_0 \setminus (\{0\} \times \mathbb{R}^3)$. Then

$$\langle \xi, (\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0) \rangle_{\frac{1}{|\varphi'|}} = \lim_{k\to\infty} \langle \xi, \mathcal{D}F_k \rangle_{\frac{1}{|\varphi'|}} = -\lim_{k\to\infty} \langle \mathcal{D}\xi, F_k \rangle_{\frac{1}{|\varphi'|}}$$

by Lemma 3.2. Indeed, by applying the Cauchy-Schwarz inequality we even obtain

$$0 \leq |\langle \xi, (\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0) \rangle_{\frac{1}{|\varphi'|}}| = \lim_{k\to\infty} |\langle \mathcal{D}\xi, F_k \rangle_{\frac{1}{|\varphi'|}}| \leq$$

$$\leq \|\mathcal{D}\xi\|_{\frac{1}{|\varphi'|}} \lim_{k\to\infty} \left(\int_{\mathrm{supp}(\xi)} \frac{1}{|\varphi'(E(x,v))|} |F_k(x,v)|^2 \,\mathrm{d}(x,v)\right)^{\frac{1}{2}}.$$

Since $\partial_r U_0(r) = \frac{m_0(r)}{r^2} > 0$ for $r > 0$ in our case of an isotropic model, there exists a constant $c > 0$ depending on the test function ξ such that

$$\frac{\partial_r U_0(|x|)}{|x|} \geq c \text{ for } (x,v) \in \mathrm{supp}(\xi).$$

Using the Antonov coercivity inequality, we therefore conclude

$$\langle \xi, (\varphi' \circ E) \cdot (\psi_0 - \mathcal{P}\psi_0) \rangle_{\frac{1}{|\varphi'|}} = 0 \text{ for all } \xi \in C^1_{c,r}(\Omega_0 \setminus (\{0\} \times \mathbb{R}^3)).$$

Since $\psi_0 - \mathcal{P}\psi_0$ is spherically symmetric on Ω_0 just like the test function, changing to (r, w, L)-coordinates and the Du Bois-Reymond theorem yield $\psi_0 - \mathcal{P}\psi_0 = 0$ a.e. on Ω_0. However, similar to Proposition 4.8, this immediately leads to $\lambda_0 = 1$, which is the desired contradiction.
We have therefore shown $\lambda_0 > 0$.

5) *Justification of the integration by parts:*
Let $k \in \mathbb{N}$ be fixed and $\rho_{\mathcal{D}F_k} := \int_{\mathbb{R}^3} \mathcal{D}F_k(\cdot, v)\,\mathrm{d}v \in C_c^1(\mathbb{R}^3)$ be the spatial density induced by $\mathcal{D}F_k$. Then $\rho_{\mathcal{D}F_k}$ has vanishing mass, i.e.,

$$\int_{\mathbb{R}^3} \rho_{\mathcal{D}F_k}(x)\,\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} [v \cdot \partial_x F_k(x,v) - \partial_x U_0(x) \cdot \partial_v F_k(x,v)]\,\mathrm{d}v\,\mathrm{d}x = 0,$$

where we integrated by parts to get

$$\int_{\mathbb{R}^3} v \cdot \partial_x F_k(x,v)\,\mathrm{d}x = v \cdot \int_{B_{R_0}(0)} \partial_x F_k(x,v)\,\mathrm{d}x = 0 \text{ for } v \in \mathbb{R}^3,$$
$$\int_{\mathbb{R}^3} \partial_x U_0(x) \cdot \partial_v F_k(x,v)\,\mathrm{d}v = \partial_x U_0(x) \cdot \int_{B_{P_0}(0)} \partial_v F_k(x,v)\,\mathrm{d}v = 0 \text{ for } x \in \mathbb{R}^3.$$

In addition, $\mathcal{D}F_k$ inherits the spherical symmetry of F_k by Lemma 3.3. Consequently, $\rho_{\mathcal{D}F_k}$ and the potential $U_{\mathcal{D}F_k}$ are spherically symmetric on $\mathbb{R}^3$ as well. By using the radial Poisson equation for $U_{\mathcal{D}F_k}$ similar to Remark 2.3, we therefore obtain

$$\nabla U_{\mathcal{D}F_k}(x) = \frac{x}{|x|^3} \cdot \int_{B_{|x|}(0)} \rho_{\mathcal{D}F_k}(y)\,\mathrm{d}y \text{ for } x \in \mathbb{R}^3 \setminus \{0\},$$

in particular $\nabla U_{\mathcal{D}F_k}(x) = 0$ if $|x| \geq R_0$. Thus, $\nabla U_{\mathcal{D}F_k} \in C_c^1(\mathbb{R}^3;\mathbb{R}^3)$, which means that in both sides of the integration by parts formula in question we can restrict ourselves to an open & bounded set $\Omega \subset \mathbb{R}^3$, where we know that $\psi_0 \in H^1(\Omega)$. This verifies the integration by parts and concludes the proof of Theorem 4.7. □

4.3 Finite dimensional approximation

It turns out that we do not need the coercivity of the whole Guo-Lin operator for the stability result in Chapter 5, but the one of a finite dimensional approximation of the operator. In this section we therefore want to approximate the Guo-Lin operator $\mathcal{A}_0$ by replacing the projection $\mathcal{P}$ with a finite sum. We prove a coercivity result similar to Theorem 4.7 for this approximation as well.
To approximate the projection $\mathcal{P}$ finite dimensionally, we need an orthonormal basis of the projection space of $\mathcal{P}$, i.e., the space of functions depending only on the particle energy and the modulus of the angular momentum squared:

Definition & Lemma 4.10: *There exists an orthonormal basis $(\boldsymbol{b_k})_{k\in\mathbb{N}} \subset C_r^1(\overline{\Omega_0})$ of*

$$\boldsymbol{\kappa_{\mathcal{D}}} := \{f \in L^2_{|\varphi'|,r}(\Omega_0) \mid |\varphi' \circ E| \cdot f \in \ker(\mathcal{D})\}$$

with respect to the scalar product $\langle\cdot,\cdot\rangle_{|\varphi'|}$*, i.e.,* $b_k \in \kappa_{\mathcal{D}}$*,* $\langle b_k, b_l\rangle_{|\varphi'|} = \delta_k^l$ *for* $k, l \in \mathbb{N}$ *and for all* $f \in \kappa_{\mathcal{D}}$ *we have*

$$\sum_{k=1}^{n} \langle f, b_k\rangle_{|\varphi'|} b_k \to f \text{ in } L^2_{|\varphi'|}(\Omega_0) \text{ as } n \to \infty.$$

In addition, we may choose $\boldsymbol{b_1}$ *to be constant on* Ω_0*.*
Note that as a closed linear subspace of $L^2_{|\varphi'|,r}(\Omega_0)$*,* $\kappa_{\mathcal{D}}$ *is a Hilbert space with the same scalar product as well. Furthermore,* $C^1_r(\overline{\Omega_0}) \subset L^2_{|\varphi'|,r}(\Omega_0)$ *due to the boundedness of* φ' *and* Ω_0*.*

Proof: As a separable Hilbert space, $\kappa_{\mathcal{D}}$ has an orthonormal basis, see [13]. It remains to show that this orthonormal basis can be chosen to contain only smooth functions as claimed:
For this sake, we have to work in the transformed (E, L)-space. Let

$$\begin{aligned}\Omega_0^{EL} :=& \{(E(x,v), L(x,v)) \mid (x,v) \in \Omega_0,\ x \times v \neq 0\} = \\ =& \{(E(r,w,L), L) \mid r, L > 0,\ w \in \mathbb{R} \text{ with } E(r,w,L) < E_0\}\end{aligned}$$

denote the image of Ω_0 under the mapping (E, L). To investigate the shape of Ω_0^{EL}, note that $E(r,w,L) = \frac{1}{2}w^2 + \psi_L(r) \geq \psi_L(r_L) = E(r_L, 0, L)$ for $r > 0$, $w \in \mathbb{R}$ and $L > 0$ by Theorem 2.4. Thus,

$$\Omega_0^{EL} = \bigcup_{L>0} [\psi_L(r_L), E_0[\times \{L\},$$

where $[\psi_L(r_L), E_0[:= \emptyset$ if $\psi_L(r_L) \geq E_0$ for some $L > 0$. Since $]0,\infty[\ni L \mapsto \psi_L(r_L) \in \mathbb{R}$ is continuously differentiable by Theorem 2.4, we therefore obtain the measurability of $\Omega_0^{EL} \subset \mathbb{R}^2$ and that the boundary of Ω_0^{EL} is a set of measure zero in $\mathbb{R}^2$.
Now, consider the weighted two-dimensional L^2 space

$$L^2_{|\varphi'|T}(\Omega_0^{EL}) := \{g\colon \Omega_0^{EL} \to \mathbb{R} \text{ measurable} \mid \|g\|_{|\varphi'|T} < \infty\},$$

where

$$\|g\|^2_{|\varphi'|T} := 4\pi^2 \int_{\Omega_0^{EL}} |\varphi'(E)| \cdot T(E,L) \cdot g(E,L) \,\mathrm{d}(E,L)$$

and T is known from Definition & Remark 3.19 and is well defined on Ω_0^{EL}. Then $L^2_{|\varphi'|T}(\Omega_0^{EL}) = L^2_{|\varphi'|T}(\mathrm{int}(\Omega_0^{EL}))$ has a dense and countable subset containing only smooth & compactly supported functions, i.e., there exists a sequence $(\chi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathrm{int}(\Omega_0^{EL}))$ such that $\{\chi_k \mid k \in \mathbb{N}\} \subset L^2_{|\varphi'|T}(\Omega_0^{EL})$ is dense with respect to $\|\cdot\|_{|\varphi'|T}$, where $\mathrm{int}(\Omega_0^{EL})$ denotes the interior of Ω_0^{EL}. Since $\kappa_{\mathcal{D}} \cong L^2_{|\varphi'|T}(\Omega_0^{EL})$ by changing variables, we obtain the density of $\{\xi_k \mid k \in \mathbb{N}\}$ in $\kappa_{\mathcal{D}}$ with respect to

$\|\cdot\|_{|\varphi'|}$ if we set $\xi_k(x,v) := \chi_k(E(x,v), L(x,v))$ for $(x,v) \in \Omega_0$ and $k \in \mathbb{N}$. Moreover, $\xi_k \in C_c^1(\Omega_0)$ for $k \in \mathbb{N}$ due to the smoothness and support properties of χ_k. From this countable dense subset we can explicitly construct an orthonormal basis of $\kappa_{\mathcal{D}}$ using the Gram-Schmidt orthonormalisation process, see [13]. The resulting basis functions are linear combinations of elements of the sequence $(\xi_k)_{k\in\mathbb{N}}$, which means they inherit the smoothness of $(\xi_k)_{k\in\mathbb{N}}$. Starting the orthonormalisation process with a constant function completes the proof. □

In the following, we will work with one fixed orthonormal basis $(b_k)_{k\in\mathbb{N}}$ satisfying the properties from above. Since such a basis is far from unique, all the following definitions depend on this choice. However, the results we will prove hold true for every basis.
Also note that, different to Guo & Lin in [9], our orthonormal basis is not separated in (E, L). Indeed, even finding a separated orthonormal sequence is not as straight forward as one might think, due to the non-separated weight $T(E, L)$ appearing in the transformed integrals. However, for the application in Chapter 5 it is not needed for the orthonormal basis to have this property.
We will now define the approximated version of $\mathcal{P}$ and prove that it really approximates $\mathcal{P}$:

Definition & Lemma 4.11: *For $n \in \mathbb{N}$ let*

$$\boldsymbol{\mathcal{P}_n} \colon L^2_{|\varphi'|,r}(\Omega_0) \to L^2_{|\varphi'|,r}(\Omega_0),\ \mathcal{P}_n f := \sum_{k=1}^{n} \langle f, b_k\rangle_{|\varphi'|} b_k.$$

Obviously, $\mathcal{P}_n$ is a well defined, linear & continuous operator on $L^2_{|\varphi'|,r}(\Omega_0)$. Furthermore, for each $\psi \in \dot{H}^1_r(\mathbb{R}^3)$ we have

$$\mathcal{P}_n\psi \to \mathcal{P}\psi \ \text{in } L^2_{|\varphi'|}(\Omega_0) \text{ as } n \to \infty,$$

where $\mathcal{P}$ is known from Definition 4.2.

Proof: To show the claimed convergence, we first extend $\mathcal{P}$ on the whole space $L^2_{|\varphi'|,r}(\Omega_0)$ similar to Definition & Remark 3.20, i.e., let $\mathcal{P}^1 \colon L^2_{|\varphi'|,r}(\Omega_0) \to L^2_{|\varphi'|,r}(\Omega_0)$ be defined by

$$\mathcal{P}^1 f(x,v) := \int_0^1 f^r((R,W)(t\cdot T(E,L), r_-(E,L), 0, L), L)\,\mathrm{d}t = \\ = \frac{1}{T(E,L)} \int_{r_-(E,L)}^{r_+(E,L)} \frac{f^r(r, \sqrt{2E-2\psi_L(r)}, L) + f^r(r, -\sqrt{2E-2\psi_L(r)}, L)}{\sqrt{2E-2\psi_L(r)}}\,\mathrm{d}r$$

for a.e. $(x,v) \in \Omega_0$, where we used the abbreviations $E = E(x,v)$ and $L = L(x,v)$. In addition, let

$$\mathcal{P}^2 \colon L^2_{|\varphi'|,r}(\Omega_0) \to L^2_{|\varphi'|,r}(\Omega_0),\ \mathcal{P}^2 f := \lim_{n\to\infty} \mathcal{P}_n f = \sum_{k=1}^{\infty} \langle f, b_k\rangle_{|\varphi'|} b_k,$$

where the limit is taken with respect to $\|\cdot\|_{|\varphi'|}$. Since $(b_k)_{k\in\mathbb{N}}$ is an orthonormal sequence, $\mathcal{P}^2$ is well defined by Bessel's inequality (cf. [13]).
It now remains to verify that $\mathcal{P}^1$ and $\mathcal{P}^2$ are both orthogonal projections onto the closed subspace $\kappa_{\mathcal{D}}$ of $L^2_{|\varphi'|,r}(\Omega_0)$, i.e.,

$$\mathcal{P}^i\mathcal{P}^i = \mathcal{P}^i = (\mathcal{P}^i)^* \text{ and } \operatorname{im}(\mathcal{P}^i) = \kappa_{\mathcal{D}}$$

for $i = 1, 2$, since these orthogonal projections are unique, see [14].
That $\mathcal{P}^1$ is an orthogonal projection can be shown completely analogously to Definition & Remark 3.20. The respective statement for $\mathcal{P}^2$ follows quite easily by the properties of the orthonormal basis $(b_k)_{k\in\mathbb{N}}$, see [13]. □

We now also define an approximation of the Guo-Lin operator as well by replacing the projection $\mathcal{P}$ with its approximated form $\mathcal{P}_n$:

Definition 4.12: *For fixed $n \in \mathbb{N}$ let*

$$\boldsymbol{\mathcal{K}_n}\colon L^2_r(\mathbb{R}^3) \to L^2_r(\mathbb{R}^3),$$
$$\psi \mapsto 4\pi \int_{\mathbb{R}^3} |\varphi'(E(\cdot,v))| \cdot \mathcal{P}_n\psi(\cdot,v)\,\mathrm{d}v - 4\pi \int_{\mathbb{R}^3} |\varphi'(E(\cdot,v))|\,\mathrm{d}v \cdot \psi$$

and

$$\boldsymbol{\mathcal{A}_n}\colon H^2_r(\mathbb{R}^3) \to L^2_r(\mathbb{R}^3), \quad \mathcal{A}_n := -\Delta + \mathcal{K}_n.$$

Similar to Definition 4.4, $\mathcal{K}_n$ and $\mathcal{A}_n$ are well defined and we set

$$\begin{aligned}\langle \mathcal{A}_n\psi, \psi\rangle_2 :=& \|\nabla\psi\|_2^2 + 4\pi\left(\|\mathcal{P}_n\psi\|^2_{|\varphi'|} - \|\psi\|^2_{|\varphi'|}\right) = \\ =& \|\nabla\psi\|_2^2 - 4\pi\|\psi - \mathcal{P}_n\psi\|^2_{|\varphi'|}\end{aligned}$$

for each $\psi \in \dot{H}^1_r(\mathbb{R}^3)$. For the latter equality we used the projection properties of $\mathcal{P}_n$.

We now present the reason why we want b_1 from Definition & Lemma 4.10 to be constant on Ω_0. In fact, this property provides a similar result to Lemma 4.5 for the approximation as well:

Lemma 4.13: *For any $n \in \mathbb{N}$, $\psi \in \dot{H}^1_r(\mathbb{R}^3)$ and $C \in \mathbb{R}$ we have*

$$\langle \mathcal{A}_n(\psi + C), \psi + C\rangle_2 = \langle \mathcal{A}_n\psi, \psi\rangle_2.$$

Proof: Clearly, $\nabla(\psi + C) = \nabla\psi$. Moreover, $b_k \perp_{|\varphi'|} b_1$ for $k \geq 2$ by definition, and since b_1 was chosen to be constant on Ω_0, we have

$$\mathcal{P}_n C = \sum_{k=1}^{n} \langle C, b_k\rangle_{|\varphi'|} b_k = \langle C, b_1\rangle_{|\varphi'|} b_1 = C \text{ on } \Omega_0.$$

Thus, $\langle \mathcal{P}_n\psi, C\rangle_{|\varphi'|} = \langle \psi, \mathcal{P}_n C\rangle_{|\varphi'|} = \langle \psi, C\rangle_{|\varphi'|}$ by the symmetry of $\mathcal{P}_n$ and

$$\begin{aligned}\langle \mathcal{K}_n(\psi + C), (\psi + C)\rangle_2 &= 4\pi\left(\|\mathcal{P}_n(\psi + C)\|^2_{|\varphi'|} - \|\psi + C\|^2_{|\varphi'|}\right) = \\ &= 4\pi\left(\|\mathcal{P}_n\psi\|^2_{|\varphi'|} - \|\psi\|^2_{|\varphi'|}\right) = \langle \mathcal{K}_n\psi, \psi\rangle_2.\end{aligned}$$ □

Then, for $n \in \mathbb{N}$ large enough, we can obtain a similar coercivity result like the one from Section 4.2. We prove this by reducing it to Theorem 4.7 using related techniques like for the proof of Proposition 4.8.

Theorem 4.14: *For $n \in \mathbb{N}$ let*

$$\boldsymbol{\lambda_n} := \inf_{\psi \in \dot{H}^1_r(\mathbb{R}^3), \|\nabla\psi\|_2 \neq 0} \frac{\langle \mathcal{A}_n \psi, \psi \rangle_2}{\|\nabla \psi\|_2^2} \in \mathbb{R} \cup \{-\infty\}.$$

Then $\lambda_n \to \lambda_0$ as $n \to \infty$. In particular, since $\lambda_0 > 0$ by Theorem 4.7, there exists $n_0 \in \mathbb{N}$ such that $\lambda_n \geq \frac{\lambda_0}{2}$ for $n \geq n_0$, i.e.,

$$\langle \mathcal{A}_n \psi, \psi \rangle_2 \geq \frac{\lambda_0}{2} \|\nabla \psi\|_2^2$$

for all $\psi \in \dot{H}^1_r(\mathbb{R}^3)$ and $n \geq n_0$.

Proof: First note that for each $n \in \mathbb{N}$ and $\psi \in \dot{H}^1_r(\mathbb{R}^3)$ we have

$$\|\mathcal{P}_n \psi\|_{|\varphi'|}^2 = \sum_{k=1}^n |\langle \psi, b_k \rangle_{|\varphi'|}|^2 \leq \sum_{k=1}^\infty |\langle \psi, b_k \rangle_{|\varphi'|}|^2 = \|\mathcal{P}\psi\|_{|\varphi'|}^2$$

by the Pythagorean theorem and Definition & Lemma 4.11. Thus

$$\begin{aligned} \langle \mathcal{A}_n \psi, \psi \rangle_2 &= \|\nabla \psi\|_2^2 + 4\pi \|\mathcal{P}_n \psi\|_{|\varphi'|}^2 - 4\pi \|\psi\|_{|\varphi'|}^2 \leq \\ &\leq \|\nabla \psi\|_2^2 + 4\pi \|\mathcal{P}\psi\|_{|\varphi'|}^2 - 4\pi \|\psi\|_{|\varphi'|}^2 = \langle \mathcal{A}_0 \psi, \psi \rangle_2. \end{aligned}$$

In particular, this means $\lambda_n \leq \lambda_0$ for all $n \in \mathbb{N}$.
Now suppose that $\lambda_n \not\to \lambda_0$ as $n \to \infty$, i.e., there exists $0 < \epsilon < \lambda_0$, an increasing sequence $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ as well as $(\chi_k)_{k \in \mathbb{N}} \subset \dot{H}^1_r(\mathbb{R}^3)$ such that $\|\nabla \chi_k\|_2 \neq 0$ and

$$\frac{\langle \mathcal{A}_{n_k} \chi_k, \chi_k \rangle_2}{\|\nabla \chi_k\|_2^2} \leq \lambda_0 - \epsilon \text{ for } k \in \mathbb{N}.$$

Just like in the first step of the proof of Proposition 4.8, we "improve" this sequence by setting

$$\psi_k := \frac{1}{\|\nabla \chi_k\|_2} \left(\chi_k - \fint_{B_{R_0}(0)} \chi_k(x) \, \mathrm{d}x \right)$$

for $k \in \mathbb{N}$. Then $\psi_k \in \dot{H}^1_r(\mathbb{R}^3)$, $\int_{B_{R_0}(0)} \psi_k(x) \, \mathrm{d}x = 0$ and $\|\nabla \psi_k\|_2 = 1$ for any $k \in \mathbb{N}$. In addition, Lemma 4.13 implies

$$\lambda_0 - \epsilon \geq \frac{\langle \mathcal{A}_{n_k} \chi_k, \chi_k \rangle_2}{\|\nabla \chi_k\|_2^2} = \langle \mathcal{A}_{n_k} \psi_k, \psi_k \rangle_2 = 1 + 4\pi \left(\|\mathcal{P}_{n_k} \psi_k\|_{|\varphi'|}^2 - \|\psi_k\|_{|\varphi'|}^2 \right).$$

Similar to Proposition 4.8, we now obtain the existence of $\psi_0 \in \dot{H}^1_r(\mathbb{R}^3)$ such that

$$\psi_k \to \psi_0 \text{ in } L^2(B_{R_0}(0)) \quad\text{and}\quad \nabla\psi_k \rightharpoonup \nabla\psi_0 \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3) \text{ as } k \to \infty$$

by the Banach-Alaoglu theorem and the compact embedding from Lemma 2.8, where we passed to a subsequence. Due to Lemma 4.3 and the minimizing property of λ_0, we therefore obtain

$$\begin{aligned}\lambda_0 \le \lim_{k\to\infty} \frac{\langle \mathcal{A}_0\psi_k, \psi_k\rangle_2}{\|\nabla\psi_k\|_2^2} &= \lim_{k\to\infty} \left(1 + 4\pi\|\mathcal{P}\psi_k\|^2_{|\varphi'|} - 4\pi\|\psi_k\|^2_{|\varphi'|}\right) = \\ &= \left(1 + 4\pi\|\mathcal{P}\psi_0\|^2_{|\varphi'|} - 4\pi\|\psi_0\|^2_{|\varphi'|}\right) = 1 + \langle \mathcal{K}_0\psi_0, \psi_0\rangle_2.\end{aligned}$$

In addition, we have $\langle \mathcal{K}_{n_k}\psi_k, \psi_k\rangle_2 \to \langle \mathcal{K}_0\psi_0, \psi_0\rangle_2$ as $k \to \infty$. To see the latter, we split into several terms to arrive at

$$\begin{aligned}|\langle \mathcal{K}_{n_k}\psi_k,\psi_k\rangle_2 - \langle \mathcal{K}_0\psi_0, \psi_0\rangle_2| &\le |\langle \mathcal{K}_{n_k}(\psi_k - \psi_0), \psi_k - \psi_0\rangle_2| + \\ &+ |\langle \mathcal{K}_{n_k}\psi_0, \psi_k - \psi_0\rangle_2| + |\langle \mathcal{K}_{n_k}(\psi_k - \psi_0), \psi_0\rangle_2| + |\langle \mathcal{K}_{n_k}\psi_0 - \mathcal{K}_0\psi_0, \psi_0\rangle_2|.\end{aligned}$$

We now estimate each addend after the other to show the desired convergence:

1) Since $\|\mathcal{P}_{n_k}(\psi_k - \psi_0)\|^2_{|\varphi'|} \le \|\mathcal{P}(\psi_k - \psi_0)\|^2_{|\varphi'|} \le \|\psi_k - \psi_0\|^2_{|\varphi'|}$ by Lemma 4.3 and the Pythagorean theorem, we obtain

$$\begin{aligned}|\langle \mathcal{K}_{n_k}(\psi_k - \psi_0),\psi_k - \psi_0\rangle_2| &\le 4\pi\|\mathcal{P}_{n_k}(\psi_k - \psi_0)\|^2_{|\varphi'|} + 4\pi\|\psi_k - \psi_0\|^2_{|\varphi'|} \le \\ &\le 8\pi\|\psi_k - \psi_0\|^2_{|\varphi'|} \to 0 \text{ as } k \to \infty.\end{aligned}$$

2) Definition & Lemma 4.11 yields $\mathcal{P}_{n_k}\psi_0 \to \mathcal{P}\psi_0$ in $L^2_{|\varphi'|}(\Omega_0)$ and thus

$$|\langle \mathcal{K}_{n_k}\psi_0, \psi_k - \psi_0\rangle_2| \le 4\pi|\langle P_{n_k}\psi_0, \psi_k - \psi_0\rangle_{|\varphi'|}| + 4\pi|\langle \psi_0, \psi_k - \psi_0\rangle_{|\varphi'|}| \to 0.$$

3) Since $\mathcal{P}_{n_k}$ is symmetric on $L^2_{|\varphi'|,r}(\Omega_0)$, $\mathcal{K}_{n_k}$ is symmetric as well. Therefore,

$$|\langle \mathcal{K}_{n_k}(\psi_k - \psi_0), \psi_0\rangle_2| = |\langle \psi_k - \psi_0, \mathcal{K}_{n_k}\psi_0\rangle_2| \to 0 \text{ as } k \to \infty$$

just like the second term.

4) $\mathcal{K}_{n_k}\psi_0 \to \mathcal{K}_0\psi_0$ in $L^2_{|\varphi'|}(\Omega_0)$ as $k \to \infty$ by Definition & Lemma 4.11.

Overall we conclude

$$\lambda_0 \le 1 + \langle \mathcal{K}_0\psi_0, \psi_0\rangle_2 = 1 + \lim_{k\to\infty} \langle \mathcal{K}_{n_k}\psi_k, \psi_k\rangle_2 = \lim_{k\to\infty} \frac{\langle \mathcal{A}_{n_k}\psi_k, \psi_k\rangle_2}{\|\nabla\psi_k\|_2^2} \le \lambda_0 - \epsilon,$$

which is the desired contradiction and therefore completes the proof of Theorem 4.14. □

5 Stability of the King model

In this chapter we use the tools collected in the Chapters 3 and 4 to prove a non-linear stability result for the King model. This approach, in particular all the estimates in Sections 5.2 and 5.3, are extracted from the second part of [9], where a coercivity estimate similar to Theorem 4.14 is used to establish non-linear stability as well. We also want to refer to [19], where the stability of a non-isotropic model has been shown by related techniques.
Therefore, let $f_0 = \varphi \circ E$ be a fixed isotropic state of the Vlasov-Poisson system in the sense of Definition 2.2 throughout this whole chapter. In addition, we require φ to be of the form

$$\varphi(E) = \left(e^{E_0 - E} - 1\right)_+, \quad E \in \mathbb{R}$$

for some fixed negative cut-off energy $E_0 < 0$, i.e., f_0 is a King model. Again, we refer to [23, 28] for the existence theory of these models.
Note that $\varphi'(E) = -e^{E_0-E}$ for $E < E_0$, in particular, $\varphi' < 0$ on $]-\infty, E_0[$ and φ' is bounded on intervals of the form $[\eta, E_0[$ for $\eta < E_0$. This means that the fixed steady state f_0 satisfies all the general conditions from Chapters 3 and 4.
Similar to the previous chapters, let

$$\mathbf{\Omega_0} := \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid f_0(x,v) \neq 0\} = \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid E(x,v) < E_0\}$$

denote the set where f_0 does not vanish. Moreover, we need the Casimir function corresponding to the King model, which is given by

$$\mathbf{\Phi}(f) := (1+f)\ln(1+f) - f, \quad f > -1.$$

We refer to [10, 27] for a detailed motivation as well as the properties and importance of this Casimir function. However, we immediately want to note that $\Phi'(f) = \ln(1+f)$ for $f > -1$, which leads to

$$\Phi'(f_0(x,v)) = E_0 - E(x,v) \text{ for } (x,v) \in \Omega_0.$$

Furthermore, Φ is a non-negative & convex function on $[0, \infty[$.

5.1 Statement of the stability result

In this section we want to present the desired stability result and introduce all the required quantities and notations.

Theorem 5.1 (Non-linear stability of the King model):
Let $f_0 = \left(e^{E_0-E} - 1\right)_+$ be a steady state as specified above.
Then, for every $S > 0$ there exists $C > 0$ such that for every spherically symmetric, non-negative initial data $\mathring{f} \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ with

$$\|\mathring{f}\|_\infty \leq S \quad \text{and} \quad d(\mathring{f}, f_0) < \frac{1}{C},$$

the unique global & classical solution $f\colon [0,\infty[\times\mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty[$ of the Vlasov-Poisson system launched by $\mathring{f} = f(0)$ satisfies

$$d(f(t), f_0) \leq C \cdot d(\mathring{f}, f_0)$$

for all $t \geq 0$.
Here, d is a distance measure adapted to the problem and is defined in Definition 5.3.

The aim of this whole chapter is to prove Theorem 5.1. Therefore, if not stated explicitly otherwise, let $\mathring{f} \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ be some fixed spherically symmetric, non-negative initial data. Let $S > 0$ be a bound for the L^∞-norm of $\mathring{f}$ just like in Theorem 5.1, i.e., $S \geq \|\mathring{f}\|_\infty$.
In addition, let $f\colon [0,\infty[\times\mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty[$ be the unique global & classical solution of the Vlasov-Poisson system satisfying the initial condition $f(0) = \mathring{f}$. Then $f(t) \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ is spherically symmetric and non-negative for all $t \geq 0$ as well, see [27].
Before getting to the proof of Theorem 5.1 in the next sections, we first have to introduce some notations.

Definition 5.2: *Let $F \in C_c(\mathbb{R}^3 \times \mathbb{R}^3)$ be a non-negative function. As usual, let*

$$\boldsymbol{E_{kin}}(F) := \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^2 F(x,v)\,\mathrm{d}x\,\mathrm{d}v$$

be the ***kinetic energy****,*

$$\boldsymbol{E_{pot}}(F) := -\frac{1}{8\pi}\int_{\mathbb{R}^3} |\partial_x U_F(x)|^2\,\mathrm{d}x$$

the ***potential energy*** *and*

$$\boldsymbol{\mathcal{H}}(F) := E_{kin}(F) + E_{pot}(F) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^2 F(x,v)\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\partial_x U_F(x)|^2\,\mathrm{d}x$$

the ***total energy****. Here, $\boldsymbol{U_F} := -\frac{1}{|\cdot|} * \int_{\mathbb{R}^3} F(\cdot,v)\,\mathrm{d}v$ denotes the gravitational potential induced by F. By basic potential theory (cf. [21]), we know that $U_F \in C^1(\mathbb{R}^3)$ and $\partial_x U_F \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, i.e., E_{pot} is well defined. In addition, let*

$$\begin{aligned}\boldsymbol{\mathcal{C}}(F) &:= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi(F(x,v))\,\mathrm{d}x\,\mathrm{d}v = \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (1 + F(x,v)) \cdot \ln(1 + F(x,v)) - F(x,v)\,\mathrm{d}x\,\mathrm{d}v\end{aligned}$$

*be the **Casimir functional** induced by the King model. Also, let*

$$\boldsymbol{M}(F) \coloneqq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} F(x,v)\,\mathrm{d}x\,\mathrm{d}v$$

*denote the **total mass** of F. Then the sum*

$$\boldsymbol{\mathcal{H}_{\mathcal{C},M}}(F) \coloneqq \mathcal{H}(F) + \mathcal{C}(F) - E_0 \cdot M(F)$$

*is called the **energy-Casimir-mass-functional**. Note that all these quantities could be defined on a much large class of functions, which we do not need here.*

Since the fixed steady state f_0 is not infinitely smooth at the boundary $\{E = E_0\}$ of its support, it turns out to be very useful to treat $f - f_0$ on Ω_0 and $\mathbb{R}^3 \times \mathbb{R}^3 \setminus \Omega_0$ separately. We will therefore use the following abbreviations:

$$\begin{aligned} \boldsymbol{g}\colon [0,\infty[\times\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R},\ g(t,x,v) \coloneqq f(t,x,v) - f_0(x,v), \\ \boldsymbol{g_{in}} \coloneqq g \cdot \mathbb{1}_{\Omega_0} = (f - f_0)\cdot \mathbb{1}_{\Omega_0}, \quad \boldsymbol{g_{out}} \coloneqq g \cdot \mathbb{1}_{\mathbb{R}^3\times\mathbb{R}^3\setminus\Omega_0} = f \cdot \mathbb{1}_{\mathbb{R}^3\times\mathbb{R}^3\setminus\Omega_0} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\psi}\colon [0,\infty[\times\mathbb{R}^3 \to \mathbb{R},\ \psi(t,x) \coloneqq U_g(t,x) \coloneqq U_{g(t)}(x) = U_{f(t)}(x) - U_0(x), \\ \boldsymbol{\psi_{in}} \coloneqq U_{g_{in}}, \qquad \boldsymbol{\psi_{out}} \coloneqq U_{g_{out}}. \end{aligned}$$

Obviously, we have $g = g_{in} + g_{out}$ and $\psi = \psi_{in} + \psi_{out}$. Note that the potential is linear in its generating function.
As motivated above, we now want to measure the distance between the steady state f_0 and $f(t)$ for a fixed time $t \geq 0$. For this sake, we first use a Taylor expansion of the energy-Casimir-mass functional at f_0 similar to [27] to obtain

$$\begin{aligned} \mathcal{H}_{\mathcal{C},M}(f(t)) - \mathcal{H}_{\mathcal{C},M}(f_0) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi(f(t,x,v)) - \Phi(f_0(x,v))\,\mathrm{d}x\,\mathrm{d}v + \\ &+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (E(x,v) - E_0)\cdot(f(t,x,v) - f_0(x,v))\,\mathrm{d}x\,\mathrm{d}v + \\ &- \frac{1}{8\pi}\int_{\mathbb{R}^3} |\partial_x U_f(t,x) - \partial_x U_0(x)|^2\,\mathrm{d}x \end{aligned}$$

for all $t \geq 0$. Here, we integrated by parts and used the Poisson equation to get

$$\int_{\mathbb{R}^3} \big(\partial_x U_f(t,x) - \partial_x U_0(x)\big)\cdot \partial_x U_0(x)\,\mathrm{d}x = -4\pi \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g(t,x,v)\cdot U_0(x)\,\mathrm{d}x\,\mathrm{d}v,$$

note that $U_f(t), U_0 \in C^2(\mathbb{R}^3)$ and that all boundary terms vanish due to the linear decay of U_0 and the quadratic decay of $\partial_x U_g(t)$ at spatial infinity.
This leads to the following:

Definition 5.3: *For fixed $t \geq 0$ we define the distance between the steady state f_0 and $f(t)$ as follows:*

$$\begin{aligned}\boldsymbol{d}(t) \coloneqq d(f(t), f_0) \coloneqq &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi(f(t,x,v)) - \Phi(f_0(x,v))\,\mathrm{d}x\,\mathrm{d}v + \\ &+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (E(x,v) - E_0)\cdot g(t,x,v)\,\mathrm{d}x\,\mathrm{d}v + \\ &+ \frac{1}{8\pi}\int_{\mathbb{R}^3} |\partial_x \psi_{in}(t,x)|^2\,\mathrm{d}x.\end{aligned}$$

In addition, it is useful to split these integrals into their parts over Ω_0 and its complement. Therefore, let

$$\begin{aligned}\boldsymbol{d_{in}}(t) \coloneqq d_{in}(f(t), f_0) \coloneqq &\int_{\Omega_0} \Phi(f_0(x,v) + g_{in}(t,x,v)) - \Phi(f_0(x,v))\,\mathrm{d}(x,v) + \\ &+ \int_{\Omega_0} (E(x,v) - E_0)\cdot g_{in}(t,x,v)\,\mathrm{d}(x,v), \\ \boldsymbol{d_{out}}(t) \coloneqq d_{out}(f(t), f_0) \coloneqq &\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus\Omega_0} \Phi(g_{out}(t,x,v))\,\mathrm{d}(x,v) + \\ &+ \int_{\mathbb{R}^3\times\mathbb{R}^3\setminus\Omega_0} (E(x,v) - E_0)\cdot g_{out}(t,x,v)\,\mathrm{d}(x,v).\end{aligned}$$

Since $f_0 + g_{in}(t) = f(t)$ on Ω_0, we have

$$d(t) = d_{in}(t) + \frac{1}{8\pi}\int_{\mathbb{R}^3} |\partial_x \psi_{in}(t,x)|^2\,\mathrm{d}x + d_{out}(t).$$

Admittedly, d does not look like a distance measure at first glance. In fact, it is even unclear if d is non-negative. We will therefore justify the role of d as a distance measure in the following remark similar to [27].

Remark 5.4: *Let $t \geq 0$ be a fixed time. Since Φ is convex on $[0,\infty[$, we obtain*

$$d_{in}(t) \geq \int_{\Omega_0} (\Phi'(f_0(x,v)) + E(x,v) - E_0)\cdot g_{in}(t,x,v)\,\mathrm{d}(x,v) = 0,$$

where we used $\Phi'(f_0) = E_0 - E$ on Ω_0 for the last equation. Moreover, since Φ is actually strictly convex on $[0,\infty[$, we also obtain that $d_{in}(t) = 0$ is only possible if $g_{in}(t) = 0$ on Ω_0, i.e., $f(t) = f_0$ on Ω_0.
A similar computation yields

$$d_{out}(t) \geq \int_{\mathbb{R}^3\times\mathbb{R}^3\setminus\Omega_0} (\Phi'(0) + E(x,v) - E_0)\cdot g_{out}(t,x,v)\,\mathrm{d}(x,v) \geq 0,$$

since $\Phi'(0) = 0$, $E - E_0 \geq 0$ on $\mathbb{R}^3\times\mathbb{R}^3\setminus\Omega_0$ and $g_{out}(t) \geq 0$. Analogously, $d_{out}(t) = 0$ is equivalent to $g_{out}(t) = f(t) = 0 = f_0$ on $\mathbb{R}^3\times\mathbb{R}^3\setminus\Omega_0$.
Overall, we conclude $d(t) \geq 0$ and that $d(t) = 0$ is equivalent to $f(t) = f_0$ on the whole space $\mathbb{R}^3\times\mathbb{R}^3$.

We also want to note straight away that one could define the distance measure differently, in particular when it comes to the question which part of the potential to include into d. In fact, the analogous distance used for the stability results in [10, 27] slightly differs from d. However, it turns out that a similar stability result to the one from Theorem 5.1 holds true for this alternate distance. We refer to Section 5.4, where we will discuss this matter in more detail.
We now introduce two last abbreviations:

Definition 5.5: *For fixed $t \geq 0$ let*

$$\boldsymbol{I_{in}}(t) := d_{in}(t) - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\partial_x \psi_{in}(t,x)|^2 \, \mathrm{d}x,$$
$$\boldsymbol{I_{out}}(t) := d_{out}(t) - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\partial_x \psi_{out}(t,x)|^2 \, \mathrm{d}x - \frac{1}{4\pi} \int_{\mathbb{R}^3} \partial_x \psi_{in}(t,x) \cdot \partial_x \psi_{out}(t,x) \, \mathrm{d}x.$$

Using the Taylor expansion of the energy-Casimir-mass functional from above, we can now express $\mathcal{H}_{\mathcal{C},M}(f(t)) - \mathcal{H}_{\mathcal{C},M}(f_0)$ in terms of $I_{in}(t)$ and $I_{out}(t)$ as follows:

Remark 5.6: *For every $t \geq 0$ we have*

$$\begin{aligned}
&\mathcal{H}_{\mathcal{C},M}(f(t)) - \mathcal{H}_{\mathcal{C},M}(f_0) = d_{in}(t) - \frac{1}{8\pi} \|\partial_x \psi(t)\|_2^2 + d_{out}(t) = \\
&= d_{in}(t) - \frac{1}{8\pi} \left(\|\partial_x \psi_{in}(t)\|_2^2 + 2\langle \partial_x \psi_{in}(t), \partial_x \psi_{out}(t) \rangle_2 + \|\partial_x \psi_{out}(t)\|_2^2 \right) + d_{out}(t) = \\
&= I_{in}(t) + I_{out}(t).
\end{aligned}$$

In the following sections we will therefore separately estimate $I_{in}(t)$ and $I_{out}(t)$ in terms of $d(t)$.

5.2 Estimating I_{out}

The target of this section is to estimate

$$I_{out}(t) = d_{out}(t) - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\partial_x \psi_{out}(t,x)|^2 \, \mathrm{d}x - \frac{1}{4\pi} \int_{\mathbb{R}^3} \partial_x \psi_{in}(t,x) \cdot \partial_x \psi_{out}(t,x) \, \mathrm{d}x$$

in terms of $d(t)$ for $t \geq 0$. We start with $\|\partial_x \psi_{out}(t)\|_2^2$:

Lemma 5.7: *There exists a constant $C > 0$, only depending on the steady state f_0 and S, such that for all $t \geq 0$ and $0 < \gamma < \min\{-\frac{E_0}{2}, 1\}$ we have*

$$\|\partial_x \psi_{out}(t)\|_2^2 \leq C \cdot \left(\gamma \cdot d_{out}(t) + \gamma^{-\frac{5}{3}} \cdot d_{out}^{\frac{5}{3}}(t) \right).$$

Proof: Let $t \geq 0$ and $0 < \gamma < \min\{-\frac{E_0}{2}, 1\}$ be arbitrary. First, a basic corollary of the Hardy-Littlewood-Sobolev inequality (cf. [21]) yields

$$\begin{aligned}\|\partial_x \psi_{out}(t)\|_2^2 &\leq C \cdot \|\int_{\mathbb{R}^3} g_{out}(t,\cdot,v)\,\mathrm{d}v\|_{\frac{6}{5}}^2 \leq \\ &\leq C \cdot \Big(\|\int_{\mathbb{R}^3} g_{out}(t,\cdot,v)\cdot \mathbb{1}_{\{E_0 \leq E(\cdot,v) \leq E_0+\gamma\}}\,\mathrm{d}v\|_{\frac{6}{5}}^2 + \\ &\qquad + \|\int_{\mathbb{R}^3} g_{out}(t,\cdot,v)\cdot \mathbb{1}_{\{E_0+\gamma < E(\cdot,v)\}}\,\mathrm{d}v\|_{\frac{6}{5}}^2\Big),\end{aligned}$$

where we used $g_{out}(t) = g_{out}(t)\cdot(\mathbb{1}_{\{E_0 \leq E \leq E_0+\gamma\}} + \mathbb{1}_{\{E_0+\gamma<E\}})$, the triangle inequality and Cauchy's inequality. Also note that for the sake of abbreviation, we allow our constants $C > 0$ to change from line to line.
We now estimate both of these summands separately:

1) By applying Hölder's inequality twice, we obtain

$$\begin{aligned}&\|\int_{\mathbb{R}^3} g_{out}(t,\cdot,v)\cdot \mathbb{1}_{\{E_0 \leq E(\cdot,v) \leq E_0+\gamma\}}\,\mathrm{d}v\|_{\frac{6}{5}}^2 = \\ &= \left(\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} g_{out}(t,x,v)\cdot \mathbb{1}_{\{E_0 \leq E(x,v) \leq E_0+\gamma\}}\,\mathrm{d}v\right)^{\frac{6}{5}}\mathrm{d}x\right)^{\frac{5}{3}} \leq \\ &\leq \left(\int_{\mathbb{R}^3} \|g_{out}(t,x,\cdot)\|_2^{\frac{6}{5}} \cdot \|\mathbb{1}_{\{E_0 \leq E(x,\cdot) \leq E_0+\gamma\}}\|_2^{\frac{6}{5}}\,\mathrm{d}x\right)^{\frac{5}{3}} \leq \\ &\leq \int_{\mathbb{R}^3} g_{out}^2(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \cdot \left(\int_{\mathbb{R}^3} \mathrm{vol}^{\frac{3}{2}}(\{E_0 \leq E(x,\cdot) \leq E_0+\gamma\})\,\mathrm{d}x\right)^{\frac{2}{3}}.\end{aligned}$$

In addition, for every $a \in [0,S]$ the Taylor theorem yields the existence of some $\alpha \in [0,a]$ such that

$$\Phi(a) = \frac{a^2}{2}\Phi''(\alpha) = \frac{a^2}{2}\frac{1}{1+\alpha} \geq \frac{a^2}{2}\frac{1}{1+S},$$

note $\Phi(0) = 0 = \Phi'(0)$. Therefore, since $0 \leq g_{out}(t) \leq \|f(t)\|_\infty = \|\mathring{f}\|_\infty \leq S$ by the L^p-norm preservation of the Vlasov-Poisson system (cf. [27]), we arrive at

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g_{out}^2(t,x,v)\,\mathrm{d}x\,\mathrm{d}v &\leq C(S)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi(g_{out}(t,x,v))\,\mathrm{d}x\,\mathrm{d}v \leq \\ &\leq C(S)\cdot d_{out}(t).\end{aligned}$$

Next we have to identify the volume of the set

$$\{E_0 \leq E(x,\cdot) \leq E_0+\gamma\} := \{v \in \mathbb{R}^3 \mid E_0 \leq E(x,v) \leq E_0+\gamma\} \subset \mathbb{R}^3$$

for fixed $x \in \mathbb{R}^3$, recall $E(x,v) = \frac{1}{2}|v|^2 + U_0(x)$.
We only have to consider the case where $\mathrm{vol}(\{E_0 \le E(x,\cdot) \le E_0 + \gamma\}) \ne 0$, which particularly implies $U_0(x) < \frac{E_0}{2}$, since $\gamma < -\frac{E_0}{2}$. Also, due to the boundary condition $\lim_{|y|\to\infty} U_0(y) = 0$, the set $\{U_0 < \frac{E_0}{2}\} \subset \mathbb{R}^3$ is bounded. Furthermore, by applying the mean value theorem on $a \mapsto a^{\frac{3}{2}}$, we obtain

$$\begin{aligned}
&\mathrm{vol}(\{E_0 \le E(x,\cdot) \le E_0 + \gamma\}) = \\
&\quad = \mathrm{vol}\left(\{v \in \mathbb{R}^3 \mid 2(E_0 - U_0(x)) \le |v|^2 \le 2(E_0 + \gamma - U_0(x))\}\right) \le \\
&\quad \le \mathrm{vol}\left(\{v \in \mathbb{R}^3 \mid \sqrt{2(E_0 - U_0(x))_+} \le |v| \le \sqrt{2(E_0 - U_0(x))_+ + 2\gamma}\}\right) = \\
&\quad = \frac{4\pi}{3}\left((2(E_0 - U_0(x))_+ + 2\gamma)^{\frac{3}{2}} - (2(E_0 - U_0(x)))_+^{\frac{3}{2}}\right) \le \\
&\quad \le 2\pi \cdot 2\gamma \cdot \sqrt{2(E_0 - U_0(x))_+ + 2\gamma} \le C(f_0) \cdot \gamma,
\end{aligned}$$

note that $U_0(0) \le U_0 \le 0$. We therefore conclude

$$\begin{aligned}
\left(\int_{\mathbb{R}^3} \mathrm{vol}^{\frac{3}{2}}(\{E_0 \le E(x,\cdot) \le E_0 + \gamma\})\,\mathrm{d}x\right)^{\frac{2}{3}} &\le \left(\int_{\{U_0 < \frac{E_0}{2}\}} C(f_0) \cdot \gamma^{\frac{3}{2}}\,\mathrm{d}x\right)^{\frac{2}{3}} \le \\
&\le C(f_0) \cdot \gamma.
\end{aligned}$$

2) First, interpolating $L^{\frac{6}{5}}$ by L^1 and $L^{\frac{5}{3}}$ (cf. [7]) yields

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^3} g_{out}(t,\cdot,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(\cdot,v)\}}\,\mathrm{d}v \right\|_{\frac{6}{5}}^2 \le \\
&\qquad \le \left\| \int_{\mathbb{R}^3} g_{out}(t,\cdot,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(\cdot,v)\}}\,\mathrm{d}v \right\|_1^{\frac{7}{6}} \cdot \\
&\qquad\quad \cdot \left\| \int_{\mathbb{R}^3} g_{out}(t,\cdot,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(\cdot,v)\}}\,\mathrm{d}v \right\|_{\frac{5}{3}}^{\frac{5}{6}}.
\end{aligned}$$

Then, using the standard estimate

$$\begin{aligned}
&\int_{\mathbb{R}^3} g_{out}(t,x,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(x,v)\}}\,\mathrm{d}v \le \\
&\qquad \le C(S) \cdot \left(\int_{\mathbb{R}^3} |v|^2 \cdot g_{out}(t,x,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(x,v)\}}\,\mathrm{d}v\right)^{\frac{3}{5}}
\end{aligned}$$

for $x \in \mathbb{R}^3$ (cf. [27]) and observing that

$$|v|^2 = 2(E(x,v) - U_0(x)) \le 2(E(x,v) - E_0) - 2U_0(0)$$

for $x, v \in \mathbb{R}^3$, we obtain

$$\begin{aligned}
&\| \int_{\mathbb{R}^3} g_{out}(t,\cdot,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(\cdot,v)\}} \,\mathrm{d}v \|_{\frac{6}{5}}^2 \leq \\
&\leq C(S) \cdot \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g_{out}(t,x,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(x,v)\}} \,\mathrm{d}x \,\mathrm{d}v \right)^{\frac{7}{6}} \cdot \\
&\qquad \cdot \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (E(x,v) - E_0) \cdot g_{out}(t,x,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(x,v)\}} \,\mathrm{d}x \,\mathrm{d}v + \right. \\
&\qquad\qquad \left. - U_0(0) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g_{out}(t,x,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(x,v)\}} \,\mathrm{d}x \,\mathrm{d}v \right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\frac{E-E_0}{\gamma} > 1$ on $\{E_0 + \gamma < E\}$ and $\Phi \geq 0$ on $[0, \infty[$, we conclude

$$\begin{aligned}
&\| \int_{\mathbb{R}^3} g_{out}(t,\cdot,v) \cdot \mathbb{1}_{\{E_0+\gamma<E(\cdot,v)\}} \,\mathrm{d}v \|_{\frac{6}{5}}^2 \leq \\
&\qquad \leq C(S) \cdot \left(\frac{1}{\gamma} d_{out}(t) \right)^{\frac{7}{6}} \cdot \left(d_{out}(t) - \frac{1}{\gamma} U_0(0) \cdot d_{out}(t) \right)^{\frac{1}{2}} = \\
&\qquad = C(S) \cdot d_{out}^{\frac{5}{3}}(t) \cdot \gamma^{-\frac{7}{6}} \cdot \left(1 - \frac{U_0(0)}{\gamma} \right)^{\frac{1}{2}} \leq \\
&\qquad \leq C(S, f_0) \cdot \left(\gamma^{-1} \cdot d_{out}(t) \right)^{\frac{5}{3}}.
\end{aligned}$$

Combining the inequalities for the two summands from above finishes the proof of Lemma 5.7. □

Lemma 5.7 together with the Cauchy-Schwarz inequality also yields the following estimate for the mixed term $\langle \partial_x \psi_{in}(t), \partial_x \psi_{out}(t) \rangle_2$ of $I_{out}(t)$:

Corollary 5.8: *There exists a constant $C > 0$, only depending on the steady state f_0 and S, such that for all $t \geq 0$ and $0 < \gamma < \min\{-\frac{E_0}{2}, 1\}$ we have*

$$\left| \int_{\mathbb{R}^3} \partial_x \psi_{in}(t,x) \cdot \partial_x \psi_{out}(t,x) \,\mathrm{d}x \right| \leq C \cdot \left(\gamma^{\frac{1}{2}} \cdot d(t) + \gamma^{-\frac{5}{6}} \cdot d^{\frac{4}{3}}(t) \right).$$

Proof: For arbitrary $t \geq 0$ and $0 < \gamma < \min\{-\frac{E_0}{2}, 1\}$ Lemma 5.7 yields

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} \partial_x \psi_{in}(t,x) \cdot \partial_x \psi_{out}(t,x) \,\mathrm{d}x \right| \leq \| \partial_x \psi_{in}(t) \|_2 \cdot \| \partial_x \psi_{out}(t) \|_2 \leq \\
&\qquad \leq \sqrt{8\pi} \cdot d^{\frac{1}{2}}(t) \cdot C(f_0, S) \cdot \sqrt{\gamma \cdot d(t) + \gamma^{-\frac{5}{3}} \cdot d^{\frac{5}{3}}(t)} \leq \\
&\qquad \leq C(f_0, S) \cdot \left(\gamma^{\frac{1}{2}} \cdot d(t) + \gamma^{-\frac{5}{6}} \cdot d^{\frac{4}{3}}(t) \right),
\end{aligned}$$

where we used the non-negativity of d_{in} and d_{out} from Remark 5.4. □

Overall, by combining Lemma 5.7 and Corollary 5.8, we obtain the following estimate for $I_{out}(t)$:

Corollary 5.9: *There exists a constant $C > 0$, only depending on the steady state f_0 and S, such that*

$$I_{out}(t) \geq d_{out}(t) - C \cdot \left(\gamma^{\frac{1}{2}} \cdot d(t) + \gamma^{-\frac{5}{3}} \cdot d^{\frac{5}{3}}(t) + \gamma^{-\frac{5}{6}} \cdot d^{\frac{4}{3}}(t)\right).$$

for all $t \geq 0$ and $0 < \gamma < \min\{-\frac{E_0}{2}, 1\}$.

5.3 Estimating I_{in}

The target of this section is to estimate

$$I_{in}(t) = d_{in}(t) - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\partial_x \psi_{in}(t,x)|^2 \,\mathrm{d}x$$

in terms of $d(t)$ for $t \geq 0$. To this end we split $I_{in}(t)$ into three parts:

Definition 5.10: *For fixed $0 < \tau < 1$ and $t \geq 0$ let*

$$\begin{aligned}
\boldsymbol{I^{\tau}_{in,1}}(t) \coloneqq & \frac{\tau}{8\pi} \|\partial_x \psi_{in}(t)\|_2^2 + \\
& + \tau \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Big(\Phi(f_0(x,v) + g_{in}(t,x,v)) - \Phi(f_0(x,v)) + \\
& \qquad + (E(x,v) - E_0) \cdot g_{in}(t,x,v) + \psi_{in}(t,x) \cdot g_{in}(t,x,v) \Big) \,\mathrm{d}x \,\mathrm{d}v, \\
\boldsymbol{I^{\tau}_{in,2}}(t) \coloneqq & \frac{1-\tau}{8\pi} \|\partial_x \psi_{in}(t)\|_2^2 + \\
& + (1-\tau) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Big(\Phi(f_0(x,v) + g_{in}(t,x,v)) - \Phi(f_0(x,v)) + \\
& \qquad + (E(x,v) - E_0) \cdot g_{in}(t,x,v) + \\
& \qquad + \big(\psi_{in}(t,x) - \mathcal{P}_N(\psi_{in}(t))(x,v)\big) \cdot g_{in}(t,x,v) \Big) \,\mathrm{d}x \,\mathrm{d}v, \\
\boldsymbol{I^{\tau}_{in,3}}(t) \coloneqq & (1-\tau) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{P}_N(\psi_{in}(t))(x,v) \cdot g_{in}(t,x,v) \,\mathrm{d}x \,\mathrm{d}v.
\end{aligned}$$

Here, $\mathcal{P}_N$ is the finite dimensional approximation from Definition & Lemma 4.11, where $N \coloneqq n_0$ is chosen like in Theorem 4.14. Note that $\psi_{in}(t)$ is continuous & spherically symmetric on $\mathbb{R}^3$ and $\partial_x \psi_{in}(t) \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ by basic potential theory (cf. [21]), i.e., $\psi_{in}(t) \in \dot{H}^1_r(\mathbb{R}^3)$.

The following result shows that I_{in} is indeed the sum of the three parts defined above.

Lemma 5.11: *For all $t \geq 0$ and $0 < \tau < 1$ we have*

$$I_{in}(t) = I^{\tau}_{in,1}(t) + I^{\tau}_{in,2}(t) + I^{\tau}_{in,3}(t).$$

Proof: It remains to show that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot g_{in}(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = -\frac{1}{4\pi}\int_{\mathbb{R}^3} |\partial_x \psi_{in}(t,x)|^2\,\mathrm{d}x$$

for fixed $t \geq 0$. Just like in the proof of Theorem 4.7, it may seem very convincing that this integrating by parts formula holds true, since both integrals exist. However, a detailed proof is far from trivial. In particular, we do not know if $\psi_{in}(t)$ is twice continuously differentiable, since $\int_{\mathbb{R}^3} g_{in}(t,\cdot,v)\,\mathrm{d}v$ does not need to be continuous.
To actually justify the integration by parts, we denote

$$\rho_{in}(t,x) := \int_{\mathbb{R}^3} g_{in}(t,x,v)\,\mathrm{d}v \quad \text{for } x \in \mathbb{R}^3.$$

Obviously, $\rho_{in}(t) \in L^1 \cap L^\infty(\mathbb{R}^3)$. Let $(h_k)_{k\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^3)$ be an approximation sequence such that

$$h_k \to \rho_{in}(t) \text{ in } L^1(\mathbb{R}^3) \text{ and } L^{\frac{6}{5}}(\mathbb{R}^3) \text{ as } k \to \infty.$$

By potential theory (cf. [21]), we also obtain $\partial_x U_{h_k} \to \partial_x \psi_{in}(t)$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ as $k \to \infty$, where $U_{h_k} := -\frac{1}{|\cdot|} * h_k$ is the potential induced by h_k. Therefore,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot g_{in}(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \lim_{k\to\infty}\int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot h_k(x)\,\mathrm{d}x,$$
$$\int_{\mathbb{R}^3} |\partial_x \psi_{in}(t,x)|^2\,\mathrm{d}x = \lim_{k\to\infty}\int_{\mathbb{R}^3} \partial_x\psi_{in}(t,x) \cdot \partial_x U_{h_k}(x)\,\mathrm{d}x,$$

which means that it suffices to show

$$\int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot h_k(x)\,\mathrm{d}x = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \partial_x\psi_{in}(t,x) \cdot \partial_x U_{h_k}(x)\,\mathrm{d}x$$

for fixed $k \in \mathbb{N}$. To this end, note that $\psi_{in}(t) \in C^1(\mathbb{R}^3)$ with $\lim_{|x|\to\infty} \psi_{in}(t,x) = 0$ and $\partial_x \psi_{in}(t) \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, since $\rho_{in}(t) \in L^1 \cap L^\infty(\mathbb{R}^3)$. Analogous statements hold true for U_{h_k} as well. Fortunately, since h_k is smooth & compactly supported, we also get $U_{h_k} \in C^2(\mathbb{R}^3)$ with $\Delta U_{h_k} = 4\pi h_k$ and the quadratic decay of $\partial_x U_{h_k}$, i.e., the existence of some constant $C_k > 0$ such that

$$|\partial_x U_{h_k}(x)| \leq \frac{C_k}{1+|x|^2} \quad \text{for } x \in \mathbb{R}^3.$$

To show the approximated integration by parts formula we have to approximate once again. Let $\chi \in C_c^\infty(\mathbb{R}^3)$ be such that $\chi = 1$ on $B_1(0)$, $0 \le \chi \le 1$ on $\mathbb{R}^3$ and $\operatorname{supp}(\chi) \subset B_2(0)$. For $n \in \mathbb{N}$ set $\chi_n := \chi(\frac{\cdot}{n})$. Then

$$\begin{aligned}\int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot h_k(x) \,\mathrm{d}x &= \frac{1}{4\pi} \lim_{n\to\infty} \int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot \chi_n(x) \cdot \Delta U_{h_k}(x) \,\mathrm{d}x = \\ &= -\frac{1}{4\pi} \lim_{n\to\infty} \int_{\mathbb{R}^3} \partial_x(\psi_{in}(t) \cdot \chi_n)(x) \cdot \partial_x U_{h_k}(x) \,\mathrm{d}x = \\ &= -\frac{1}{4\pi} \lim_{n\to\infty} \int_{\mathbb{R}^3} \chi_n(x) \cdot \partial_x \psi_{in}(t,x) \cdot \partial_x U_{h_k}(x) \,\mathrm{d}x + \\ &\quad - \frac{1}{4\pi} \lim_{n\to\infty} \int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot \partial_x \chi_n(x) \cdot \partial_x U_{h_k}(x) \,\mathrm{d}x.\end{aligned}$$

Note that $\psi_{in}(t) \cdot \chi_n \in H^1(\mathbb{R}^3)$ for $n \in \mathbb{N}$, which justifies the performed integration by parts. Since $\partial_x \psi_{in}(t), \partial_x U_{h_k} \in L^2(\mathbb{R}^3; \mathbb{R}^3)$, Lebesgue's dominated convergence theorem together with the Cauchy-Schwarz inequality yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \chi_n(x) \cdot \partial_x \psi_{in}(t,x) \cdot \partial_x U_{h_k}(x) \,\mathrm{d}x = \int_{\mathbb{R}^3} \partial_x \psi_{in}(t,x) \cdot \partial_x U_{h_k}(x) \,\mathrm{d}x.$$

As to the other term, we use the quadratic decay of $\partial_x U_{h_k}$ to obtain

$$\begin{aligned}\left| \int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot \partial_x \chi_n(x) \cdot \partial_x U_{h_k}(x) \,\mathrm{d}x \right| &\le \\ &\le \int_{\mathbb{R}^3 \setminus B_n(0)} |\psi_{in}(t,x)| \cdot |\partial_x \chi_n(x)| \cdot |\partial_x U_{h_k}(x)| \,\mathrm{d}x \le \\ &\le \frac{C_k}{1+n^2} \cdot \|\psi_{in}(t)\|_{L^\infty(\mathbb{R}^3 \setminus B_n(0))} \cdot \int_{\mathbb{R}^3} |\partial_x \chi_n(x)| \,\mathrm{d}x\end{aligned}$$

for $n \in \mathbb{N}$. Since

$$\int_{\mathbb{R}^3} |\partial_x \chi_n(x)| \,\mathrm{d}x = \frac{1}{n} \int_{\mathbb{R}^3} |(\partial_x \chi)(\frac{x}{n})| \,\mathrm{d}x = n^2 \int_{\mathbb{R}^3} |\partial_x \chi(y)| \,\mathrm{d}y = n^2 \cdot \|\partial_x \chi\|_1$$

and $\|\psi_{in}(t)\|_{L^\infty(\mathbb{R}^3 \setminus B_n(0))} \to 0$ as $n \to \infty$ by the boundary condition of $\psi_{in}(t)$, we may conclude

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot \partial_x \chi_n(x) \cdot \partial_x U_{h_k}(x) \,\mathrm{d}x = 0,$$

which finishes the proof of Lemma 5.11. □

Note that the approach to prove the integration by parts in Theorem 4.7 differs highly from the proof above. In the latter, the key is that the potential $\psi_{in}(t)$ vanishes at infinity. In the proof of Theorem 4.7 however, we used that the potential involved is generated by a spatial density with vanishing mass, which causes the gradient of the potential to be compactly supported.
Lemma 5.11 now allows to estimate $I^\tau_{in,1}$, $I^\tau_{in,2}$ and $I^\tau_{in,3}$ separately in order to establish an estimate for I_{in}.

Lemma 5.12: *There exists a constant $C > 0$, only depending on the steady state f_0 and S, such that for all $t \geq 0$ and $0 < \tau < 1$ we have*

$$I^{\tau}_{in,1}(t) \geq \tau \cdot d_{in}(t) - C \cdot \tau \cdot \|\partial_x \psi_{in}(t)\|_2^2.$$

Proof: Let $t \geq 0$ and $0 < \tau < 1$ be fixed. First, using the integration by parts formula from the proof of Lemma 5.11, we obtain

$$I^{\tau}_{in,1}(t) = \tau \cdot d_{in}(t) + \frac{\tau}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot g_{in}(t,x,v) \,\mathrm{d}x \,\mathrm{d}v$$

by Definitions 5.3 and 5.10. Moreover,

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi_{in}(t,x) \cdot g_{in}(t,x,v) \,\mathrm{d}x \,\mathrm{d}v \right| &\leq \\ &\leq \|\psi_{in}(t)\|_{L^2(B_{R_0}(0))} \cdot \| \int_{B_{P_0}(0)} g_{in}(t,\cdot,v) \,\mathrm{d}v \|_{L^2(B_{R_0}(0))} \leq \\ &\leq C(f_0, S) \cdot \|\psi_{in}(t)\|_{L^2(B_{R_0}(0))}, \end{aligned}$$

where $R_0,\ P_0 > 0$ are chosen like in Chapter 4, i.e., $\Omega_0 \subset B_{R_0}(0) \times B_{P_0}(0)$, and where we used the Cauchy-Schwarz inequality as well as the estimate $|g_{in}(t)| \leq \|f(t)\|_\infty + \|f_0\|_\infty = \|\mathring{f}\|_\infty + \|f_0\|_\infty \leq S + \|f_0\|_\infty$.
From the spherical symmetry of $\psi_{in}(t) \in C^1(\mathbb{R}^3)$ and $\lim_{r\to\infty} \psi^r_{in}(t,r) = 0$ we now conclude the radial Sobolev inequality

$$\begin{aligned} |\psi^r_{in}(t,r)| = \Big| \int_r^\infty \frac{s}{s} \cdot \partial_r \psi^r_{in}(t,s) \,\mathrm{d}s \Big| &\leq \\ &\leq \left(\int_r^\infty \frac{\mathrm{d}s}{s^2} \right)^{\frac{1}{2}} \cdot \left(\int_0^\infty s^2 |\partial_r \psi^r_{in}(t,s)|^2 \,\mathrm{d}s \right)^{\frac{1}{2}} \leq \frac{\|\partial_x \psi_{in}(t)\|_2}{\sqrt{4\pi r}} \end{aligned}$$

for $r > 0$ by a change of variables and $\partial_x \psi_{in}(t,x) = \frac{x}{|x|} \cdot \partial_r \psi^r_{in}(t,|x|)$ for $x \neq 0$. A similar estimate has been used in [19] as well. This yields

$$\|\psi_{in}(t)\|^2_{L^2(B_{R_0}(0))} = 4\pi \int_0^{R_0} s^2 |\psi^r_{in}(t,s)|^2 \,\mathrm{d}s \leq \frac{1}{2} R_0^2 \cdot \|\partial_x \psi_{in}(t)\|_2^2$$

and therefore completes the proof of Lemma 5.12. □

To estimate $I^{\tau}_{in,2}$, we first establish some basic inequalities.

Lemma 5.13: *Let $c \in \mathbb{R}$ and $F_0 \geq 0$ be real numbers. Then*

$$\Phi(F_0 + h) - \Phi(F_0) - \Phi'(F_0) \cdot h - c \cdot h \geq (1 + F_0) \cdot (1 + c - e^c)$$

for all $h > -(1 + F_0)$, where Φ is the Casimir function of the King model defined at the start of Chapter 5.

Proof: For fixed $c \in \mathbb{R}$ and $F_0 \geq 0$ we will investigate the extrema of the left hand side, i.e., let

$$\xi \colon]-(1+F_0), \infty[\to \mathbb{R},\ \xi(h) := \Phi(F_0+h) - \Phi(F_0) - \Phi'(F_0)\cdot h - c\cdot h.$$

For $h > -(1+F_0)$ we have

$$\xi'(h) = \ln(1+F_0+h) - \ln(1+F_0) - c,$$

recall $\Phi(F) = (1+F)\cdot\ln(1+F) - F$ for $F > -1$. This implies that $\xi'(h) = 0$ is equivalent to $h = (e^c - 1)\cdot(1+F_0)$. We therefore set $h_- := (e^c-1)\cdot(1+F_0)$, note that $h_- > -(1+F_0)$. In addition, by taking account of the sign of ξ, we know that ξ obtains its global minimum in h_-. Thus

$$\begin{aligned}\min(\xi) = \xi(h_-) = \\ &= (1+F_0+h_-)\left[\ln(1+F_0+h_-) - \ln(1+F_0) - c\right] + c\cdot(1+F_0) - h_- = \\ &= c\cdot(1+F_0) - h_- = (1+F_0)\cdot(1+c-e^c).\end{aligned}$$

□

By using the series representation of the exponential function we can estimate the right hand side of the inequality from Lemma 5.13 even further:

Remark 5.14: *For every $c \in \mathbb{R}$ we have*

$$1 + c - e^c = -\sum_{k=2}^{\infty}\frac{c^k}{k!} = -\frac{c^2}{2} - c^3\sum_{k=0}^{\infty}\frac{c^k}{(k+3)!} \geq -\frac{c^2}{2} - |c|^3 e^{|c|}$$

as well as

$$1 + c - e^c = -\sum_{k=2}^{\infty}\frac{c^k}{k!} = -c^2\sum_{k=0}^{\infty}\frac{c^k}{(k+2)!} \geq -c^2 e^{|c|}.$$

We now apply Lemma 5.13 to conclude the following auxiliary inequality:

Lemma 5.15: *There exists a constant $C > 0$, only depending on the steady state f_0 and S, such that*

$$\|g_{in}(t)\|_2^2 \leq C \cdot d_{in}(t)$$

for all $t \geq 0$.

Proof: Let $0 < \lambda \leq 1$ be arbitrary for the time being. Then

$$\begin{aligned}d_{in}(t) = \int_{\Omega_0}\Big(&\Phi(f_0(x,v)+g_{in}(t,x,v)) - \Phi(f_0(x,v)) + (E(x,v)-E_0)g_{in}(t,x,v) + \\ &-\lambda\cdot g_{in}^2(t,x,v)\Big)\,\mathrm{d}(x,v) + \lambda\cdot\|g_{in}(t)\|_2^2.\end{aligned}$$

Now, applying Lemma 5.13 combined with the second estimate from Remark 5.14 pointwise to the integrand, note that $-(E-E_0) = \Phi'(f_0)$ on Ω_0, yields

$$d_{in}(t) \geq \lambda \cdot \|g_{in}(t)\|_2^2 + \int_{\Omega_0} (1+f_0(x,v)) \cdot \left(-\lambda^2 g_{in}^2(t,x,v) e^{|\lambda g_{in}(t,x,v)|}\right) \mathrm{d}(x,v) \geq$$
$$\geq \lambda \cdot \|g_{in}(t)\|_2^2 - \lambda^2 e^S \int_{\Omega_0} (1+f_0(x,v)) \cdot g_{in}^2(t,x,v) \, \mathrm{d}(x,v),$$

where we used $|g_{in}(t)| \leq S + \|f_0\|_\infty$ once again. Also, since f_0 is bounded, we conclude

$$d_{in}(t) \geq \lambda \cdot \|g_{in}(t)\|_2^2 - C(f_0,S)\lambda^2 \|g_{in}(t)\|_2^2 =$$
$$= \lambda \cdot \big(1 - \lambda \cdot C(f_0,S)\big) \cdot \|g_{in}(t)\|_2^2$$

for some constant $C(f_0,S) \geq 1$ depending on f_0 and S. Thus, setting $\lambda = \frac{1}{2C(f_0,S)}$ completes the proof of Lemma 5.15. □

We have now collected all the required tools to estimate $I_{in,2}^\tau$.

Lemma 5.16: *There exists a constant $C > 0$, only depending on the steady state f_0 and S, such that for all $t \geq 0$ and $0 < \tau < 1$ we have*

$$I_{in,2}^\tau(t) \geq \frac{1-\tau}{16\pi} \cdot \lambda_0 \cdot \|\partial_x \psi_{in}(t)\|_2^2 - C \cdot e^{C \cdot d_{in}^{\frac{1}{2}}(t)} \cdot d_{in}^{\frac{3}{2}}(t),$$

where $\lambda_0 > 0$ is the infimum from Theorem 4.7.

Proof: Let $t \geq 0$ and $0 < \tau < 1$ be fixed. Recalling Definition 5.10, we have

$$I_{in,2}^\tau(t) = (1-\tau) \int_{\Omega_0} \Big(\Phi(f_0(x,v) + g_{in}(t,x,v)) - \Phi(f_0(x,v)) +$$
$$+ (E(x,v) - E_0) \cdot g_{in}(t,x,v) +$$
$$+ \big(\psi_{in}(t,x) - \mathcal{P}_N(\psi_{in}(t))(x,v)\big) \cdot g_{in}(t,x,v)\Big) \mathrm{d}(x,v) +$$
$$+ \frac{1-\tau}{8\pi} \cdot \|\partial_x \psi_{in}(t)\|_2^2.$$

We now combine Lemma 5.13 with the first estimate from Remark 5.14 and apply them pointwise to the integrand. Note that $\Phi'(f_0) = E_0 - E$ and $1 + f_0 = |\varphi' \circ E|$ on Ω_0 in the case of the King model. This leads to

$$I_{in,2}^\tau(t) \geq (1-\tau) \int_{\Omega_0} |\varphi'(E(x,v))| \cdot \Big(-\frac{1}{2}\big(\psi_{in}(t,x) - \mathcal{P}_N(\psi_{in}(t))(x,v)\big)^2 +$$
$$- \big|\psi_{in}(t,x) - \mathcal{P}_N(\psi_{in}(t))(x,v)\big|^3 e^{|\psi_{in}(t,x) - \mathcal{P}_N(\psi_{in}(t))(x,v)|}\Big) \mathrm{d}(x,v) +$$
$$+ \frac{1-\tau}{8\pi} \cdot \|\partial_x \psi_{in}(t)\|_2^2.$$

Next, observe that

$$\frac{1-\tau}{8\pi}\langle \mathcal{A}_N\psi_{in}(t),\psi_{in}(t)\rangle_2 = \frac{1-\tau}{8\pi}\cdot\|\partial_x\psi_{in}(t)\|_2^2 + \\ -\frac{1-\tau}{2}\int_{\Omega_0}|\varphi'(E(x,v))|\cdot\big(\psi_{in}(t,x)-\mathcal{P}_N(\psi_{in}(t))(x,v)\big)^2\,\mathrm{d}(x,v),$$

where $\langle\mathcal{A}_N\cdot,\cdot\rangle_2$ is the extended quadratic form associated with the finite dimensional approximation of the Guo-Lin operator from Definition 4.12. This is why we can now solemnly apply the finite dimensional coercivity result from Theorem 4.14. Recall that we have chosen $N = n_0$ in Definition 5.10. Also, $\psi_{in}(t)$ is continuous, spherically symmetric and $\partial_x\psi_{in}(t)\in L^2(\mathbb{R}^3;\mathbb{R}^3)$, in particular $\psi_{in}(t)\in\dot{H}^1_r(\mathbb{R}^3)$.
As for the remaining terms, note that Ω_0 and $\varphi'\circ E$ are bounded. Thus,

$$I^\tau_{in,2}(t)\geq\frac{1-\tau}{16\pi}\cdot\lambda_0\cdot\|\partial_x\psi_{in}(t)\|_2^2 + \\ -C(f_0)\cdot e^{\|\psi_{in}(t)-\mathcal{P}_N(\psi_{in}(t))\|_\infty}\cdot\|\psi_{in}(t)-\mathcal{P}_N(\psi_{in}(t))\|_\infty^3.$$

It therefore remains to estimate $\|\psi_{in}(t)-\mathcal{P}_N(\psi_{in}(t))\|_\infty$. First, by Definition & Lemma 4.11, we have

$$\|\mathcal{P}_N(\psi_{in}(t))\|_\infty = \|\sum_{k=1}^N\langle\psi_{in}(t),b_k\rangle_{|\varphi'|}b_k\|_\infty\leq\max_{1\leq k\leq N}\|b_k\|_\infty\cdot\sum_{k=1}^N|\langle\psi_{in}(t),b_k\rangle_{|\varphi'|}|.$$

In addition, the boundedness of $\varphi'\circ E$ and Ω_0 yields

$$|\langle\psi_{in}(t),b_k\rangle_{|\varphi'|}|\leq\int_{\Omega_0}|\varphi'(E(x,v))|\cdot|\psi_{in}(t,x)|\cdot|b_k(x,v)|\,\mathrm{d}(x,v)\leq \\ \leq C(f_0)\cdot\|b_k\|_\infty\cdot\|\psi_{in}(t)\|_\infty$$

for $k\in\mathbb{N}$. Since $n_0 = N$ from Theorem 4.14 depends only on the steady state, we conclude that there exists some constant $C(f_0)>0$ depending only on f_0 – in particular not on t – such that

$$\|\mathcal{P}_N(\psi_{in}(t))\|_\infty\leq C(f_0)\cdot\|\psi_{in}(t)\|_\infty.$$

Lastly, we need to establish an estimate for $\|\psi_{in}(t)\|_\infty$. We will do this by exploiting the spherical symmetry of $\psi_{in}(t)$. In fact, similar to the steady state in Remark 2.3, we obtain the following explicit formula for the potential:

$$\psi^r_{in}(t,r) = -\frac{4\pi}{r}\int_0^r s^2\rho^r_{in}(t,s)\,\mathrm{d}s - 4\pi\int_r^\infty s\rho^r_{in}(t,s)\,\mathrm{d}s \text{ for } r>0,$$

where $\rho_{in}(t,x) := \int_{\mathbb{R}^3} g_{in}(t,x,v)\,\mathrm{d}v$ for $x \in \mathbb{R}^3$. Now, let $R_0, P_0 > 0$ be such that $\Omega_0 \subset B_{R_0}(0) \times B_{P_0}(0)$, in particular $\operatorname{supp}(\rho_{in}(t)) \subset \bar{B}_{R_0}(0)$. Since

$$\|\rho_{in}(t)\|_2^2 = \int_{\mathbb{R}^3} \rho_{in}(t,x)^2\,\mathrm{d}x = 4\pi \int_0^{R_0} s^2 \rho_{in}^r(t,s)^2\,\mathrm{d}s$$

by a standard change of variables, the Cauchy-Schwarz inequality yields

$$\begin{aligned} |\psi_{in}^r(t,r)| &\le 8\pi \int_0^{R_0} s \cdot |\rho_{in}^r(t,s)|\,\mathrm{d}s \le \\ &\le \left(\int_0^{R_0} \mathrm{d}s\right)^{\frac{1}{2}} \cdot \left(\int_0^{R_0} s^2 \rho_{in}^r(t,s)^2\,\mathrm{d}s\right)^{\frac{1}{2}} = \\ &= 4\sqrt{\pi R_0} \cdot \|\rho_{in}(t)\|_2 \end{aligned}$$

for every $r > 0$. Since Ω_0 is bounded, we also have $\|\rho_{in}(t)\|_2 \le C(f_0) \cdot \|g_{in}(t)\|_2$. Combining this inequality with Lemma 5.15 yields

$$\|\psi_{in}(t)\|_\infty \le C(f_0,S) \cdot d_{in}^{\frac{1}{2}}(t),$$

from which we conclude

$$|\langle \psi_{in}(t), b_k\rangle_{|\varphi'|}| \le C(f_0,S) \cdot d_{in}^{\frac{1}{2}}(t). \qquad \square$$

To establish an estimate for $I_{in,3}^r$, we need the following invariants of the Vlasov-Poisson system.

Definition & Lemma 5.17: *Let $(b_k)_{k\in\mathbb{N}} \subset C^1(\overline{\Omega_0})$ be the orthonormal basis from Definition & Lemma 4.10. Since $b_k \in \kappa_{\mathcal{D}}$ for every $k \in \mathbb{N}$, there exists* $\boldsymbol{b_k^{EL}}\colon \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$b_k(x,v) = b_k^{EL}(E(x,v), L(x,v)) \text{ for } (x,v) \in \overline{\Omega_0}.$$

Due to the explicit construction of $(b_k)_{k\in\mathbb{N}}$ in Definition & Lemma 4.10, we can even choose the elements of $(b_k^{EL})_{k\in\mathbb{N}}$ such that $b_k^{EL} \in C^1(\mathbb{R}^2)$ for all $k \in \mathbb{N}$. Then, for all real numbers $F, L \ge 0$ and $k \in \mathbb{N}$ define

$$\boldsymbol{Q_k(F,L)} := \int_0^F b_k^{EL}(E_0 - \ln(1+s), L)\,\mathrm{d}s.$$

Moreover, for $k \in \mathbb{N}$ and every suitable non-negative function $F\colon \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty[$ let

$$\begin{aligned} \boldsymbol{J_k(F)} &:= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} Q_k(F(x,v), L(x,v))\,\mathrm{d}x\,\mathrm{d}v = \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^{F(x,v)} b_k^{EL}(E_0 - \ln(1+s), L(x,v))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v. \end{aligned}$$

Then the functional J_k has the following properties:

a) For $k \in \mathbb{N}$ and $t \geq 0$ we have

$$J_k(f(t)) = J_k(f_0 + g_{in}(t)) + J_k(g_{out}(t)).$$

b) J_k is constant along classical solutions of the Vlasov-Poisson system, i.e.,

$$J_k(f(t)) = J_k(\mathring{f})$$

for $k \in \mathbb{N}$ and $t \geq 0$.

Proof: Part a) follows directly by splitting the phase space integral in J_k into Ω_0 and its complement.
As to the second statement, note that

$$f(t,x,v) = \mathring{f}(X(t,x,v), V(t,x,v))$$

for $t \geq 0$, $x, v \in \mathbb{R}^3$, where (X,V) is the characteristic flow associated with f_0, see Lemma 3.2 for a detailed definition. Since L is constant along characteristics and $(X,V)(t,\cdot)$ is measure preserving, part b) can be obtained by a change of variables. We refer to [27] for details. □

Note that our definition of J_k slightly differs from the one in [9], since the orthonormal basis from Definition & Lemma 4.10 is not separated in E and L. However, it turns that this discrepancy is insignificant for the proof of the following result.

Lemma 5.18: *There exists a constant $C > 0$, only depending on the steady state f_0 and S, such that for all $t \geq 0$, $0 < \tau < 1$ and $0 < \gamma < \min\{-\frac{E_0}{2}, 1\}$ we have*

$$|I^{\tau}_{in,3}(t)| \leq C \cdot \left(d^{\frac{1}{2}}(0) + d(0) + \gamma^{\frac{1}{2}} \cdot d^{\frac{1}{2}}(t) + \gamma^{-1} \cdot d(t)\right) \cdot d^{\frac{1}{2}}(t).$$

Proof: Let $t \geq 0$ and $0 < \tau < 1$ be fixed. First, by recalling Definition 5.10 and Definition & Lemma 4.11, we get

$$\begin{aligned} |I^{\tau}_{in,3}(t)| &= (1-\tau) \cdot \Big| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{P}_N(\psi_{in}(t))(x,v) \cdot g_{in}(t,x,v)\,\mathrm{d}(x,v) \Big| \leq \\ &\leq \Big| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sum_{k=1}^{N} \langle \psi_{in}(t), b_k\rangle_{|\varphi'|} b_k(x,v) \cdot g_{in}(t,x,v)\,\mathrm{d}(x,v) \Big| \leq \\ &\leq \sum_{k=1}^{N} |\langle \psi_{in}(t), b_k\rangle_{|\varphi'|}| \cdot |\langle b_k, g_{in}(t)\rangle_2|. \end{aligned}$$

Just like in the proof of Lemma 5.16, we have

$$\max_{1\leq k\leq N} |\langle \psi_{in}(t), b_k\rangle_{|\varphi'|}| \leq C(f_0, S) \cdot d^{\frac{1}{2}}(t).$$

Thus, we arrive at

$$|I^{\tau}_{in,3}(t)| \leq C(f_0, S) \cdot d^{\frac{1}{2}}(t) \cdot \sum_{k=1}^{N} |\langle b_k, g_{in}(t)\rangle_2|.$$

We will now estimate $|\langle b_k, g_{in}(t)\rangle_2|$ for fixed $1 \leq k \leq N$ with the aid of the invariants from Definition & Lemma 5.17. Indeed,

$$J_k(f(t)) - J_k(f_0) = J_k(f_0 + g_{in}(t)) + J_k(g_{out}(t)) - J_k(f_0)$$

by the first part of the lemma above. Furthermore,

$$\begin{aligned} &J_k(f_0 + g_{in}(t)) - J_k(f_0) = \\ &\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Big(\int_0^{f_0(x,v)+g_{in}(t,x,v)} - \int_0^{f_0(x,v)} \Big)\, b_k^{EL}(E_0 - \ln(1+s), L(x,v))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v. \end{aligned}$$

For fixed $x, v \in \mathbb{R}^3$ we therefore consider the mapping

$$[0,\infty[\ni h \mapsto \int_0^h b_k^{EL}(E_0 - \ln(1+s), L(x,v))\,\mathrm{d}s.$$

Since $b_k^{EL} \in C^1(\mathbb{R}^2)$, this mapping is twice continuously differentiable. Second order Taylor expansion at $f_0(x,v) \geq 0$ then yields that for every $h \in \mathbb{R}$ with $h + f_0(x,v) \geq 0$ there exists $\sigma \in [\min\{f_0(x,v), f_0(x,v)+h\}, \max\{f_0(x,v), f_0(x,v)+h\}]$ such that

$$\begin{aligned} \Big(\int_0^{f_0(x,v)+h} - \int_0^{f_0(x,v)} \Big)\, b_k^{EL}(E_0 - \ln(1+s), L(x,v))\,\mathrm{d}s = \\ = h \cdot b_k^{EL}(E_0 - \ln(1+f_0(x,v)), L(x,v)) + \\ + \frac{h^2}{2} \cdot \partial_E b_k^{EL}(E_0 - \ln(1+\sigma), L(x,v)) \cdot \frac{-1}{1+\sigma}. \end{aligned}$$

Let $F(t,x,v) \in \mathbb{R}$ denote the σ-value we obtain in the case of $h = g_{in}(t,x,v)$. Accordingly, since $E_0 - \ln(1+f_0) = E$ on Ω_0 for the King model,

$$\begin{aligned} &J_k(f_0 + g_{in}(t)) - J_k(f_0) = \\ &\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g_{in}(t,x,v) \cdot b_k^{EL}(E_0 - \ln(1+f_0(x,v)), L(x,v))\,\mathrm{d}x\,\mathrm{d}v + \alpha_k(t) = \\ &\quad = \langle b_k, g_{in}(t)\rangle_2 + \alpha_k(t), \end{aligned}$$

where we used the abbreviation

$$\alpha_k(t) := -\frac{1}{2}\int_{\Omega_0} \frac{g_{in}^2(t,x,v)}{1+F(t,x,v)} \cdot \partial_E b_k^{EL}(E_0 - \ln(1+F(t,x,v)), L(x,v))\,\mathrm{d}(x,v)$$

for the remainder term. Thus,

$$J_k(f(t)) - J_k(f_0) = \langle b_k, g_{in}(t)\rangle_2 + \alpha_k(t) + J_k(g_{out}(t)),$$

which yields the inequality

$$|\langle b_k, g_{in}(t)\rangle_2| \leq |J_k(f(t)) - J_k(f_0)| + |\alpha_k(t)| + |J_k(g_{out}(t))|.$$

Now, since $0 \leq F(t) \leq \|f(t)\|_\infty + \|f_0\|_\infty \leq S + \|f_0\|_\infty$ and $\partial_E b_k^{EL}$ is bounded on compact sets, we obtain

$$|\alpha_k(t)| \leq C(f_0, S) \cdot \|g_{in}(t)\|_2^2 \leq C(f_0, S) \cdot d(t)$$

by applying Lemma 5.15. Again, the boundedness of $f(t)$ and b_k^{EL} on compact sets as well as the support of $g_{out}(t)$ imply

$$\begin{aligned} |J_k(g_{out}(t))| &\leq C(f_0, S) \cdot \|g_{out}(t)\|_1 = \\ &= C(f_0, S) \cdot \left(\|g_{out}(t)\|_{L^1(\{E_0 \leq E \leq E_0+\gamma\})} + \|g_{out}(t)\|_{L^1(\{E_0+\gamma < E\})}\right) \end{aligned}$$

for every $\gamma \in]0, \min\{-\frac{E_0}{2}, 1\}[$. As to the first summand, observe that the domain of x-values satisfying $\{E_0 \leq E(x,\cdot) \leq E_0+\gamma\} \neq \emptyset$ is bounded due to U_0 vanishing at spatial infinity. Thus,

$$\operatorname{vol}(\{E_0 \leq E \leq E_0 + \gamma\}) \leq C(f_0) \cdot \sup_{x \in \mathbb{R}^3} \Bigg(\operatorname{vol}(\{E_0 \leq E(x,\cdot) \leq E_0 + \gamma\})\Bigg).$$

Then, similar to the proof of Lemma 5.7, we get $\operatorname{vol}(\{E_0 \leq E(x,\cdot) \leq E_0 + \gamma\}) \leq C(f_0) \cdot \gamma$ for $x \in \mathbb{R}^3$ as well as $\|g_{out}(t)\|_2^2 \leq C(S) \cdot d_{out}(t)$, from which we conclude

$$\begin{aligned} \|g_{out}(t)\|_{L^1(\{E_0 \leq E \leq E_0+\gamma\})} &\leq \|g_{out}(t)\|_2 \cdot \operatorname{vol}^{\frac{1}{2}}(\{E_0 \leq E \leq E_0 + \gamma\}) \leq \\ &\leq C(f_0, S) \cdot d_{out}^{\frac{1}{2}}(t) \cdot \gamma^{\frac{1}{2}}, \\ \|g_{out}(t)\|_{L^1(\{E_0+\gamma < E\})} &\leq \gamma^{-1} \cdot \|g_{out}(t) \cdot (E - E_0)\|_1 \leq \gamma^{-1} \cdot d_{out}(t). \end{aligned}$$

Lastly, the invariance property of J_k from Definition & Lemma 5.17 yields

$$\begin{aligned} |J_k(f(t)) - J_k(f_0)| &= |J_k(\mathring{f}) - J_k(f_0)| \leq \\ &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Big| \int_{f_0(x,v)}^{\mathring{f}(x,v)} |b_k^{EL}(E_0 - \ln(1+s), L(x,v))| \,\mathrm{d}s \Big| \,\mathrm{d}x\,\mathrm{d}v \leq \\ &\leq C(f_0, S) \cdot \|\mathring{f} - f_0\|_1 = C(f_0, S) \cdot \|g(0)\|_1 \leq \\ &\leq C(f_0, S) \cdot (\|g_{in}(0)\|_1 + \|g_{out}(0)\|_1). \end{aligned}$$

Then, by the boundedness of Ω_0 and Lemma 5.15, we obtain

$$\|g_{in}(0)\|_1 \leq C(f_0) \cdot \|g_{in}(0)\|_2 \leq C(f_0, S) \cdot d_{in}^{\frac{1}{2}}(0).$$

On the other hand, by splitting the integral similarly to the $|J_k(g_{out}(t))|$-estimate from above for fixed γ, we see that

$$\|g_{out}(0)\|_1 \leq C(f_0, S) \cdot \left(d_{out}^{\frac{1}{2}}(0) + d_{out}(0)\right).$$

Combining all these inequalities yields

$$|\langle b_k, g_{in}(t)\rangle_2| \leq C(f_0, S) \cdot \left(d^{\frac{1}{2}}(0) + d(0) + \gamma^{\frac{1}{2}} \cdot d^{\frac{1}{2}}(t) + \gamma^{-1} d(t)\right)$$

and therefore completes the proof of Lemma 5.18. □

5.4 Proof of Theorem 5.1 and Remarks

The main target of this section is to prove Theorem 5.1 by combining the numerous estimates from Section 5.2 and Section 5.3.

Proof of Theorem 5.1: Remarks 5.4 and 5.6, Corollary 5.9 and the Lemmata 5.11, 5.12, 5.16 and 5.18 yield

$$\begin{aligned}
\mathcal{H}_{\mathcal{C},M}(f(t)) - \mathcal{H}_{\mathcal{C},M}(f_0) = I_{in}(t) + I_{out}(t) = I_{out}(t) + I^{\tau}_{in,1}(t) + I^{\tau}_{in,2}(t) + I^{\tau}_{in,3}(t) \geq \\
\geq d_{out}(t) - C \cdot \left(\gamma^{\frac{1}{2}} \cdot d(t) + \gamma^{-\frac{5}{3}} \cdot d^{\frac{5}{3}}(t) + \gamma^{-\frac{5}{6}} \cdot d^{\frac{4}{3}}(t)\right) + \\
+ \tau \cdot d_{in}(t) - C \cdot \tau \cdot \|\partial_x \psi_{in}(t)\|_2^2 + \\
+ \frac{1-\tau}{16\pi} \cdot \lambda_0 \cdot \|\partial_x \psi_{in}(t)\|_2^2 - C \cdot e^{C \cdot d^{\frac{1}{2}}(t)} \cdot d^{\frac{3}{2}}(t) + \\
- C \cdot \left(d^{\frac{1}{2}}(0) + \gamma^{\frac{1}{2}} \cdot d^{\frac{1}{2}}(t) + \gamma^{-1} \cdot d(t)\right) \cdot d^{\frac{1}{2}}(t) \geq \\
\geq \min\{1, \tau, \frac{1-\tau}{16\pi} \cdot \lambda_0 - C \cdot \tau\} \cdot d(t) - C \cdot d^{\frac{1}{2}}(0) \cdot d^{\frac{1}{2}}(t) + \\
- C \cdot \left(2\gamma^{\frac{1}{2}} \cdot d(t) + \gamma^{-\frac{5}{6}} \cdot d^{\frac{4}{3}}(t) + \left(\gamma^{-1} + e^{C \cdot d^{\frac{1}{2}}(t)}\right) \cdot d^{\frac{3}{2}}(t) + \gamma^{-\frac{5}{3}} \cdot d^{\frac{5}{3}}(t)\right)
\end{aligned}$$

for all $t \geq 0$ and arbitrary $0 < \tau < 1$, $0 < \gamma < \min\{-\frac{E_0}{2}, 1\}$, where $C > 0$ always denotes a constant depending only on the fixed steady state f_0 and the $\|\mathring{f}\|_\infty$-bound S. We assumed $d(0) \leq 1$ while applying Lemma 5.18. In the second inequality, we grouped the terms by their sign and $d(t)$-exponent.
Now, let $0 < \tau < 1$ be small enough such that $\min\{1, \tau, \frac{1-\tau}{16\pi} \cdot \lambda_0 - C \cdot \tau\} = \tau > 0$, which can be achieved since $\lambda_0 > 0$ by Theorem 4.7. After fixing τ, choose γ such that it satisfies $2\gamma^{\frac{1}{2}} < C \cdot \frac{\tau}{2}$. Then

$$\begin{aligned}
\mathcal{H}_{\mathcal{C},M}(f(t)) - \mathcal{H}_{\mathcal{C},M}(f_0) \geq \frac{1}{C} \cdot d(t) - C \cdot d^{\frac{1}{2}}(0) \cdot d^{\frac{1}{2}}(t) + \\
- C \cdot \left(d^{\frac{4}{3}}(t) + \left(1 + e^{C \cdot d^{\frac{1}{2}}(t)}\right) \cdot d^{\frac{3}{2}}(t) + d^{\frac{5}{3}}(t)\right)
\end{aligned}$$

after increasing C, note that the fixed values of τ and γ only depend on f_0, S. On the other hand, since the energy-Casimir-mass functional $\mathcal{H}_{\mathcal{C},M}$ is an invariant of the Vlasov-Poisson system (see [27]), we have

$$\mathcal{H}_{\mathcal{C},M}(f(t)) - \mathcal{H}_{\mathcal{C},M}(f_0) = \mathcal{H}_{\mathcal{C},M}(\mathring{f}) - \mathcal{H}_{\mathcal{C},M}(f_0) = d(0) - \frac{1}{4\pi}\|\partial_x \psi_{in}(0)\|_2^2 \leq d(0).$$

Altogether, we therefore arrive at

$$d(0) + C \cdot d^{\frac{1}{2}}(0) \cdot d^{\frac{1}{2}}(t) \geq \frac{1}{C} \cdot d(t) - C \cdot \left(d^{\frac{4}{3}}(t) + \left(1 + e^{C \cdot d^{\frac{1}{2}}(t)}\right) \cdot d^{\frac{3}{2}}(t) + d^{\frac{5}{3}}(t)\right)$$

for $t \geq 0$. We now interpret both sides of this inequality as a function in $\sqrt{d(t)}$, i.e., let

$$\begin{aligned} y_1(s) &\coloneqq d(0) + C \cdot d^{\frac{1}{2}}(0) \cdot s, \\ y_2(s) &\coloneqq \frac{1}{C} \cdot s^2 - C \cdot \left(s^{\frac{8}{3}} + \left(1 + e^{C \cdot s}\right) \cdot s^3 + s^{\frac{10}{3}}\right), \end{aligned}$$

for $s \geq 0$ respectively. We can then rewrite the estimate from above as

$$y_1(\sqrt{d(t)}) \geq y_2(\sqrt{d(t)}) \text{ for } t \geq 0.$$

Since $s \mapsto s^2$ is the lowest power occurring in y_2, there exists $\delta > 0$ such that $y_2(s) \geq \frac{1}{2C} \cdot s^2$ for all $s \in [0, \delta]$.
Therefore, if $y_1(s) \geq y_2(s)$ for some $s \in [0, \delta]$, we also have

$$\begin{aligned} 0 &\leq -\frac{1}{2C} s^2 + C \cdot d^{\frac{1}{2}}(0) \cdot s + d(0) = \\ &= -\frac{1}{2C} \cdot \left(s - C^2 d^{\frac{1}{2}}(0)\right)^2 + \left(1 + \frac{C^3}{2}\right) \cdot d(0), \end{aligned}$$

from which we obtain

$$s \leq \left(C^2 + \sqrt{2C + C^4}\right) \cdot d^{\frac{1}{2}}(0) \leq \tilde{C} \cdot d^{\frac{1}{2}}(0)$$

by rearranging, where $\tilde{C} \coloneqq C^2 + \sqrt{2C + C^4} + 1 > 1$.
Lastly, note that the mapping $[0, \infty[\ni t \mapsto d(t) \in [0, \infty[$ is continuous, which can be verified using $f \in C^1([0, \infty[\times \mathbb{R}^3 \times \mathbb{R}^3)$, see [27] for details. If we now assume $\tilde{C}^2 \cdot d(0) < \delta^2$, this continuity and $y_1(\sqrt{d(t)}) \geq y_2(\sqrt{d(t)})$ yield

$$d^{\frac{1}{2}}(t) \leq \tilde{C} \cdot d^{\frac{1}{2}}(0)$$

for $t \geq 0$, which completes the proof of Theorem 5.1. □

Before getting to the concluding remarks, we want to discuss our distance measure d from Definition 5.3. In fact, for the stability results in [10, 27], a slightly different distance function has been used. Expressed in our notation, it is of the form

$$\begin{aligned}\tilde{\boldsymbol{d}}(t) \coloneqq \tilde{d}(f(t), f_0) \coloneqq& d_{in}(t) + \frac{1}{8\pi}\|\partial_x\psi(t)\|_2^2 + d_{out}(t) = \\ =& d(t) - \frac{1}{8\pi}\|\partial_x\psi_{in}(t)\|_2^2 + \frac{1}{8\pi}\|\partial_x\psi(t)\|_2^2 = \\ =& d(t) + \frac{1}{4\pi}\langle\partial_x\psi_{in}(t), \partial_x\psi_{out}(t)\rangle_2 + \frac{1}{8\pi}\|\partial_x\psi_{out}(t)\|_2^2.\end{aligned}$$

Note that in [10, 27] it is required for the disturbed solution $f(t)$ to have the same mass as f_0 for every time $t \geq 0$, which is why there are additional mass-terms in our definition compared to [10, 27].
However, it turns out that a similar stability result for this alternate distance measure $\tilde{d}$ holds true:

Corollary 5.19: *Let f_0 be a steady state as specified at the start of Chapter 5. Then, for every $S > 0$ there exists $C > 0$ such that for every spherically symmetric, non-negative initial data $\mathring{f} \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ with*

$$\|\mathring{f}\|_\infty \leq S \quad \text{and} \quad \tilde{d}(\mathring{f}, f_0) < \frac{1}{C},$$

the unique global & classical solution $f\colon [0,\infty[\times\mathbb{R}^3\times\mathbb{R}^3 \to [0,\infty[$ of the Vlasov-Poisson system launched by $\mathring{f} = f(0)$ satisfies

$$\tilde{d}(f(t), f_0) \leq C \cdot \tilde{d}(\mathring{f}, f_0)$$

for all $t \geq 0$.

This result can be immediately derived from Theorem 5.1 and the following non-linear distance equivalence of d and $\tilde{d}$:

Lemma 5.20: *There exists a constant $C(f_0, S) > 0$ such that*

$$\begin{aligned}d(t) &\leq C(f_0, S) \cdot \left(\tilde{d}(t) + \tilde{d}^{\frac{5}{3}}(t)\right), \\ \tilde{d}(t) &\leq C(f_0, S) \cdot \left(d(t) + d^{\frac{4}{3}}(t) + d^{\frac{5}{3}}(t)\right)\end{aligned}$$

for all $t \geq 0$.

Proof: By Lemma 5.7 and the non-negativity of d_{in} and d_{out} from Remark 5.4, we obtain

$$\begin{aligned}\|\partial_x\psi_{in}(t)\|_2^2 &= \|\partial_x\psi(t) - \partial_x\psi_{out}(t)\|_2^2 \leq \\ &\leq 2\|\partial_x\psi(t)\|_2^2 + 2\|\partial_x\psi_{out}(t)\|_2^2 \leq \\ &\leq 2\tilde{d}(t) + 2C(f_0, S)\cdot(\tilde{d}(t) + \tilde{d}^{\frac{5}{3}}(t))\end{aligned}$$

for $t \geq 0$, which yields the first inequality.
On the other hand, Lemma 5.7 and Corollary 5.8 yield

$$\begin{aligned}\tilde{d}(t) &= d(t) + \frac{1}{4\pi}\langle \partial_x \psi_{in}(t), \partial_x \psi_{out}(t)\rangle_2 + \frac{1}{8\pi}\|\partial_x \psi_{out}(t)\|_2^2 \leq \\ &\leq C(f_0, S) \cdot \Big(d(t) + d^{\frac{4}{3}}(t) + d^{\frac{5}{3}}(t)\Big)\end{aligned}$$

for every $t \geq 0$. □

We want to end this thesis on some concluding remarks.

Remarks:

a) *Unfortunately, the proof of Theorem 5.1 is not constructive, i.e., we do not know how large the constant $C > 0$ has to be chosen. In fact, C depends on $\lambda_0 > 0$ from Theorem 4.7, which explicit value is unknown, since the existence of the minimizer in Proposition 4.8 has not been shown constructively. However, if one could provide a positive lower bound for λ_0 – for example by numerical calculations – we could derive an explicit upper bound for the constant from Theorem 5.1.*

b) *A nice feature of the stability results from Theorem 5.1 and Corollary 5.19 is that we have to bound the very quantity at $t = 0$ for which we gain control at later times. Nevertheless, despite of the properties from Remark 5.4, it is still desirable to replace d with some norm. In [27], this has been easily derived from a stability result similar to Corollary 5.19, at least after restricting the class of perturbations. However, since the general approach in [27] highly differs from ours, an analogous conclusion does not seem to work here.*

c) *The stability results presented here are slightly improved compared to [9], since we bound $d(f(t), \mathring{f})$ for all $t \geq 0$ linearly in terms of the initial deviation $d(f(0), \mathring{f})$. This has been achieved by modifying the proof of Theorem 5.1; the auxiliary estimates from Sections 5.2 and 5.3 still equal the ones from [9].*

d) *Another way of improving Theorem 5.1 is to expand the class of admissible steady states. In particular, since the results from Chapters 3 and 4 also hold true for isotropic polytropes of the form $(E_0 - E)_+^k$ for $1 \leq k < \frac{7}{2}$, it is desirable to establish the stability of these equilibria as well.*
However, some of our results rely on the explicit form of the King model in a crucial way. First, we used the properties of the Casimir function Φ corresponding to the King model to obtain the estimate in Lemma 5.13. Later, we needed the equality $1 + f_0 = |\varphi' \circ E|$ to prove Lemma 5.16, which happens to be a particular property of the King model.

Therefore, generalising the stability results can not be done in passing, but still seems possible by establishing analogous estimates in another way.

e) Compared to the stability result in [10], our class of perturbations is quite large. In fact, in the latter source only physical relevant perturbations, also known as "dynamically accessible", are admissible. Therefore, even though this being a nice analytical improvement, it is barely significant from a physics point of view.

f) The stability proven above shows that the classical solution of the Vlasov-Poisson system launched by a radially, weakly perturbed King model always stays close to the original equilibria. Unfortunately, nothing further is known about the explicit behaviour of such solutions.
The results in [24] are numerical evidence that for a large class of steady states, a radial and sufficiently weak perturbation of an equilibria leads to a "pulsating" or "oscillating" behaviour, where the period of the oscillation is given by the Eddington-Ritter relation. Hopefully, the techniques used in this thesis, in particular the coercivity of the Guo-Lin operator from Chapter 4, can be applied to prove the existence of these pulsating solutions by rigorous analysis.

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