Convergence of the Smoothed Particle Hydrodynamics Method for Barotropic Flows: Constructive Kernel Theory

Von der Universität Bayreuth
zur Erlangung des Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
genehmigte Abhandlung

von

Tino Franz

aus Hof

1. Gutachter: Prof. Dr. Holger Wendland
2. Gutachter: Prof. Dr. Christian Rieger

Tag der Einreichung: 13.12.2019
Tag des Kolloquiums: 24.07.2020

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Abstract

Particle methods are meshless methods for solving partial differential equations. These methods are particularly popular for fluid flow simulations. The fluid is spatially discretized into a set of particles which move along the Lagrangian trajectories of the fluid over time. The observed quantities of the fluid are thus transported with the flow. In contrast to classical grid-based methods, such as the finite volume or finite element method, the time derivatives of the quantities do not contain a convective component and are therefore easier to calculate. Due to the Lagrangian approach, these methods are particularly suitable for problems with a high velocity or free surfaces. A further advantage is that some conservation laws, for example mass conservation, are automatically satisfied.

One of the best known and oldest particle methods is the Smoothed Particle Hydrodynamics (SPH) method. This method is particularly popular because of its simple discretization technique as it uses besides the particles only a fixed kernel function which smooths the particles. Despite a multitude of applications and some impressive results, the mathematical understanding of this method is still limited.

This thesis deals with the convergence of the SPH method. For this, the method is applied to the Euler equations with a specific barotropic equation of state. The main focus lies on the convergence of the semi-discrete problem and the development of suitable kernel functions. The goal is to give a first step towards a general convergence theory for the SPH method.

Using specific conditions on the kernel function used by the SPH method, error bounds for an energy error term are derived, from which the pointwise convergence of the method is then deduced. A careful distinction is made between the smoothing parameter and the discretization parameter in order to derive an explicit relationship between the two parameters. Based on the Wendland functions, a class of radial kernel functions with compact support is developed, which satisfies both theoretical and numerical requirements. The theory is then extended to include a time discretization, demonstrating the convergence of the fully discretized system. Finally, numerical experiments are shown to verify the theoretical results.

Zusammenfassung

Partikelmethoden sind gitterfreie Verfahren zur Lösung partieller Differentialgleichungen. Diese Methoden sind besonders in der numerischen Strömungsmechanik beliebt. Dabei wird das Fluid zunächst räumlich in eine Menge an Partikeln diskretisiert, welche sich mit der Zeit entlang der Lagrangeschen Trajektorien des Fluids bewegen. Die beobachteten Größen des Fluids werden somit mit dem Fluss transportiert. Im Gegensatz zu klassischen, gitterbasierten Verfahren, wie etwa der Finite-Volumen- oder Finite-Elemente-Methode, enthalten die Zeitableitungen der Größen keinen konvektiven Anteil und sind daher einfacher zu berechnen. Aufgrund der Lagrangschen Betrachtungsweise eignen sich diese Methoden im Besonderen für Problemstellungen mit hohen Geschwindigkeiten oder freien Oberflächen. Ein weiterer Vorteil ist, dass einige Erhaltungsgleichungen, wie zum Beispiel die Massenerhaltung, automatisch erfüllt sind.
Eines der bekanntesten und ältesten Partikelverfahren ist das Smoothed Particle Hydrodynamics (SPH) Verfahren. Diese Methode ist aufgrund ihrer einfachen Diskretisierungstechnik sehr beliebt, da sie neben den Partikeln nur eine festgelegte Kernfunktion verwendet, welche die Partikel glättet. Trotz einer Vielzahl an Anwendungen und teilweise beeindruckenden Resultaten ist das mathematische Verständnis dieser Methode jedoch noch beschränkt.
In dieser Arbeit wird die Konvergenz des SPH Verfahrens untersucht. Hierfür wird das Verfahren auf die Eulergleichungen mit einer speziellen, barotropischen Zustandsgleichung angewendet. Es steht vor allem die Konvergenz des semi-diskreten Problems und die Entwicklung passender Kernfunktionen im Vordergrund. Das Ziel ist es, einen ersten Schritt in Richtung einer allgemeinen Konvergenztheorie für das SPH Verfahren zu geben.
Unter Verwendung spezieller Bedingungen an die von dem SPH Verfahren verwendeten Kernfunktion werden Fehlerschranken für einen Energiefehlerterm hergeleitet, aus dem dann die punktweise Konvergenz des Verfahrens gefolgert wird. Dabei wird besonders zwischen dem Glättungsparameter und dem Diskretisierungsparameter unterschieden, um einen expliziten Zu-sammenhang beider Parameter herzuleiten. Aufbauend auf den Wendland Funktionen wird eine Klasse radialer Kernfunktionen mit kompaktem Träger entwickelt, die sowohl die Bedingungen der Theorie als auch numerische Anforderungen erfüllt. Die Theorie wird dann um eine Zeitdiskretisierung erweitert, die die Konvergenz des vollständig diskretisierten Systems demonstrieren soll. Numerische Experimente sollen schließlich die theoretischen Resultate verifizieren.

Contents

Introduction

In the last few decades, numerical simulations in fluid dynamics have become an increasingly larger field of research. The fact that computing capacities had significantly increased has led to simulations replacing experiments in industry and research. Hence, various numerical methods have been established.
Many of these methods are grid-based methods, such as the finite difference, volume or element method. These methods use an Eulerian point of view, which means that the quantities of the fluid are calculated on fixed grid points. Those methods are all well studied and used in a wide range of applications. However, these methods have their disadvantages if complex or varying topologies are used.
Another approach are particle based meshfree methods which use a Lagrangian discretization for simulating the fluid flow. In these methods, the fluid is divided into a discrete set of particles which move along the Lagrangian trajectories. The quantities of the fluid are then transported with the flow. Through the observation of Lagrangian particle trajectories one directly satisfies conservation properties and has a natural treatment of free boundary conditions. In addition, these methods are often characterized by a simple implementation.
The smoothed particle hydrodynamics (SPH) method is one of the most popular particle based meshfree methods for fluid flow problems. The SPH method was first introduced by Lucy [Luc77] and by Gingold and Monoghan [GM77] in 1977. This method is based on a very simple discretization technique, employing besides the particles only a fixed kernel function. Quantities evolve over time by simple particle to particle interactions. Hence, using a kernel function with compact support, the SPH method is, additionally, very fast in computations. Due to its simplicity, the SPH method has many applications in multiple fields, for example in hydrodynamics, bioengineering or astrophysics. A detailed list of applications can be found in [SOLT16].
Despite the fact that SPH was introduced quite some time ago and despite the fact that it has shown remarkable results in practical applications, the theoretical understanding of the method is still limited.
There exist a few papers which already deal with the convergence of the SPH method. There is the work by Ben Moussa and Vila, see [BMV00, BM06, Vil99]. They follow the ideas of Mas-Gallic and Raviart [MGR87] about weighted particle methods for linear conservation laws. Their papers deal with smoothed particle approximation for conservation laws and their connection to the SPH method. Unfortunately, they require knowledge about the exact particle trajectories, which makes their results less interesting in applications. Then, there is the work by Di Lisio, Grenier, and Pulvirenti [DL95, DLGP97, DLGP98]. They show that the SPH method is converging if first the discretization parameter and after that the smoothing parameter go to zero. However, their theory does not hold if both parameters are sent to zero at the same time. This, unfortunately, is again only of limited use in practical applications.
Finally, there exists the work of Oelschläger [Oel91], which is widely neglected in the SPH community. As a main difference to other works, Oelschläger suggested that the kernel function has to be a convolution kernel, a function which is the convolution of a root kernel with itself. Then, by stating new conditions on the root kernel, in particular a condition we will call the approximation condition, Oelschläger proves convergence of an energy-like error term for a simplified SPH approx-

imation system of the Euler equations. Unfortunately, these new conditions are rather technical and have, so far, only been shown to hold for kernels which are globally supported and which are hence not particularly appealing from a computational point of view. Interestingly, supposing the kernel to be a convolution kernel, Oelschläger automatically supposed the kernel function to be positive definite. A property, which has recently been proven to be useful also in computations as it avoids so-called tensile instabilities; see [DA12].
This work is partly based on the ideas of Oelschläger. In addition to the kernel conditions of Oelschläger, we assume the kernel function to satisfy a moment condition. We are therefore able to extend the result of Oelschläger to a convergence result of arbitrary order. Moreover, we distinguish carefully between the smoothing and the discretization parameters and give explicit relations between both of them to guarantee convergence. Using the resulting higher convergence order, we are able to extend the convergence result to a first pointwise convergence result for the SPH method. Much of the work then consists of deriving a new class of kernel functions, which on the one hand satisfy the new conditions, but on the other hand are also easy to evaluate and have a compact support. First, we will set conditions for the kernel function, which ensures that it has a root kernel, which in turn satisfies the required properties. Then, we derive the class of kernel functions based on the radial basis Wendland functions, see [Wen95], which are already frequently used in the SPH community. Using Wendland functions for higher spatial dimensions, we can show that they have a root kernel which satisfies the proposed conditions.
This thesis is organized as follows. Chapter 1 introduces very basic concepts of analysis and approximation theory, which are required throughout the scope of this work. Chapter 2, the basic concepts of fluid dynamics from a Lagrangian point of view and the Euler equations will be introduced. After that, a general form of SPH approximation is derived and applied to the Euler equations. Most of the theory of the SPH approximation is introduced more generally than we will need it for the convergence result. However, we use this general derivation to conclude a general existence theorem for the SPH discretization of the Euler equations. In Chapter 3, we consider the Euler equations for a specific equation of state. We will provide conditions for the SPH kernel function which leads to a convergence result. Then, this result will be extended to derive a first point-wise convergence theorem for the SPH method. In Chapter 4 we will derive kernel functions based on the Wendland functions that fit into the theory of Chapter 3 as mentioned above. Chapter 5 discusses an explicit and an implicit time discretization scheme for the SPH method. For both time discretization schemes, a convergence result is derived. In Chapter 6, we present numerical results for a one-dimensional test case using the kernels derived in Chapter 4. These results are compared with the theory from Chapter 3.

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Section 3.1 to Section 3.4 as well as Section 4.1 and Section 4.2 were first published in "Convergence of the Smoothed Particle Hydrodynamics Method for a Specific Barotropic Fluid Flow: Constructive Kernel Theory" in Vol. 50, No. 5, Journal on Mathematical Analysis, published by the Society for Industrial and Applied Mathematics (SIAM), see [FW18].
The paper "An Improved Convergence Result for the Smoothed Particle Hydrodynamics Method.", including parts of Chapter 4, especially Section 4.3, was submitted for publication, see [FW19].

Notation and Terminology

Most part of the notation in this work is standard and chosen in a self-explanatory way. In addition, the less frequently used notation is introduced or repeated at the respective point. Nevertheless, we will give a small overview of the notation here, in case there should be any ambiguities at some points.

As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all natural, integer, real and complex numbers, respectively. We will follow the convention $0^0 = 1$, $0 \notin \mathbb{N}$ and denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. i is the imaginary unit and for a complex number $z \in \mathbb{C}$, $\overline{z}$ denotes its complex conjugate. The Cartesian product of two (or more) sets X and Y will be denoted by $X \times Y$, while, as usual, a superscript of a respective set denotes the dimension of the Cartesian power, for example $\mathbb{R}^d$ is the d-dimensional real space and $\mathbb{C}^{d\times n}$ is the space of all complex-valued matrices with d rows and n coloums for $d, n \in \mathbb{N}$. Moreover, the letter d will be fixed as a natural number and denotes the spatial dimension throughout this work.

For a better readability, vectors are written in bold, while the j-th entry of the vector $\mathbf{x} \in \mathbb{C}^d$ is denoted by $x_j \in \mathbb{C}$ for $1 \leq j \leq d$. For two vectors $\mathbf{x}$ and $\mathbf{y}$ from $\mathbb{C}^d$, $\|\mathbf{x}\|_2$ denotes the Euclidean length of $\mathbf{x}$ and $\mathbf{x}\boldsymbol{\cdot}\mathbf{y}$ the inner product of both elements. For a multiindex $\boldsymbol{\alpha} \in \mathbb{N}_0^d$, $|\boldsymbol{\alpha}|$ denotes the 1-norm of $\boldsymbol{\alpha}$, i.e. $|\boldsymbol{\alpha}| = \sum_{j=1}^{d} |\alpha_j|$.

For a subset $\Omega \subset \mathbb{R}^d$ and a $k \in \mathbb{N}_0$, $C^k(\Omega)$ denotes the space of all k-times continuously differentiable functions $f : \Omega \to \mathbb{R}$. If $k = 0$, we will simply write $C(\Omega)$. $\Pi_k(\Omega)$ denotes the space of polynomials with order k. We will use the convention that if k is a negative integer, then $\Pi_k(\Omega)$ only contains the zero function. For a $1 \leq p \leq \infty$, $L^p(\Omega)$ denotes the Lebesgue space. The Schwartz space, the space of rapidly decreasing $C^\infty(\mathbb{R}^d)$ functions, is denoted by $\mathcal{S}(\mathbb{R}^d)$.

For a function $g : \mathbb{R} \to \mathbb{R}$ we define the set of functions $\Theta(g)$ as all functions which grow exactly like g. More precisely, $f = \Theta(g)$ for $x \to \infty$ if and only if there exist two constants $0 < c \leq C < \infty$ and an $x_0 > 0$ such that $c|g(x)| \leq |f(x)| \leq C|g(x)|$ for all $x \geq x_0$.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be continuously or weakly differentiable. The j-th partial (weak) derivative of f is denoted by $\partial_j f$. For a multiindex $\boldsymbol{\alpha} \in \mathbb{N}_0^d$, the $\boldsymbol{\alpha}$-th (weak) derivative of f is denoted by $D^{\boldsymbol{\alpha}} f = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d} f$. The Laplacian is denoted by $\Delta f = \sum_{j=1}^{d} \partial_j^2 f$ and the gradient by $\nabla f = (\partial_1 f, \ldots, \partial_d f)^T$. For two vector fields $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^d$ and $\mathbf{v} : \mathbb{R}^d \to \mathbb{R}^d$ the divergence is given by $\operatorname{div} \mathbf{u} = \sum_{j=1}^{d} \partial_j u_j$ and we also encounter the non-linear operator $(\mathbf{u} \cdot \nabla)\,\mathbf{v} = \sum_{j=1}^{d} u_j \partial_j \mathbf{v}$.

We also encounter mappings where we keep a certain argument fixed. As an example, given a function $f : \mathbb{R}^d \times [0, \infty[\to \mathbb{R}$ depending on time and space, for some fixed $\mathbf{x} \in \mathbb{R}^d$, $f(\mathbf{x}, \cdot)$ is seen as a mapping $[0, \infty[\to \mathbb{R}$, $t \mapsto f(\mathbf{x}, t)$. Here, the dot symbol is the free argument.

Finally, the letter $c > 0$ denotes an arbitrary constant which can vary from line to line within each calculation.

Chapter 1

Auxiliary tools

In this chapter, we will introduce some basic concepts from analysis and approximation theory. These concepts will be needed for the construction of the SPH-method and the convergence result later on.

1.1 Tools From Analysis

In this section, we will recall some basic concepts of analysis.

1.1.1 The Fourier Transform

We begin with the very basic theory of Fourier transformation. Fourier transformation is a commonly used tool from analysis. It has the advantage of converting the operations of differentiation and convolution into multiplication operations. Using these concepts, we can consider partial differential equations as ordinary differential equations or as algebraic ones. In our context, it will mainly be used to simplify conditions we will give for functions and will help us to construct our kernels later on.

Let f be a function in $L^1(\mathbb{R}^d)$. The Fourier transform of f on $\mathbb{R}^d$, denoted by $\widehat{f}$, is the function given by

$$\widehat{f}(\boldsymbol{\omega}) = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{-i\boldsymbol{\omega}\cdot\mathbf{x}} f(\mathbf{x}) d\mathbf{x}, \quad \boldsymbol{\omega} \in \mathbb{R}^d.$$

The inverse Fourier transform of f on $\mathbb{R}^d$, denoted by $f^\vee$, is the function given by

$$f^\vee(\boldsymbol{\omega}) = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{i\boldsymbol{\omega}\cdot\mathbf{x}} f(\mathbf{x}) d\mathbf{x}, \quad \boldsymbol{\omega} \in \mathbb{R}^d.$$

Note that besides this symmetric definition, there are other definitions of the Fourier transform, which differ in the way of how the $(2\pi)^{-d}$ term is distributed. In this work, we will always use the given, symmetric version.

We now want to give some basic but useful properties of the Fourier transform, which can be deduced from the definition.

Lemma 1.1
Let f be a function in $L^1(\mathbb{R}^d)$. Then, the following properties hold:

i) The map $f \mapsto \widehat{f}$ is linear in f.

ii) $\widehat{f}$ is continuous.

iii) $\widehat{f}$ *belongs to* $L^\infty(\mathbb{R}^d)$*, and* $\|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} \leq (2\pi)^{-\frac{d}{2}}\|f\|_{L^1(\mathbb{R}^d)}$*.*

Proof. The first one is a conclusion of the linearity of the integral. The second one follows from the dominated convergence theorem. For the third one, we derive

$$\|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} = \left\|(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{-i\cdot\cdot\mathbf{x}}f(\mathbf{x})d\mathbf{x}\right\|_{L^\infty(\mathbb{R}^d)} \leq (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d}|f(\mathbf{x})|d\mathbf{x} = (2\pi)^{-d/2}\|f\|_{L^1(\mathbb{R}^d)},$$

which completes the proof. □

Besides these basic properties, we also have the following useful algebraic rules.

Lemma 1.2
Let f and g be functions in $L^1(\mathbb{R}^d)$. Then, the following algebraic properties hold:

i) $\int\limits_{\mathbb{R}^d} f(\mathbf{x})\widehat{g}(\mathbf{x})d\mathbf{x} = \int\limits_{\mathbb{R}^d}\widehat{f}(\mathbf{x})g(\mathbf{x})d\mathbf{x}$.

ii) For $\boldsymbol{\gamma} \in \mathbb{R}^d$ *we have* $\widehat{f(\cdot - \boldsymbol{\gamma})}(\boldsymbol{\omega}) = e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\gamma}}\widehat{f}(\boldsymbol{\omega})$ *for all* $\boldsymbol{\omega} \in \mathbb{R}^d$*.*

iii) For $f_\varepsilon := \frac{1}{\varepsilon^d}f\left(\frac{\cdot}{\varepsilon}\right)$ *with* $\varepsilon > 0$*, we have* $\widehat{f_\varepsilon}(\boldsymbol{\omega}) = \widehat{f}(\varepsilon\boldsymbol{\omega})$ *for all* $\boldsymbol{\omega} \in \mathbb{R}^d$*.*

iv) Let $1 \leq j \leq d$*. If, in addition,* $\mathbf{x} \mapsto x_j f(\mathbf{x})$ *is in* $L^1(\mathbb{R}^d)$*, then* $\widehat{f}$ *is differentiable with respect to* ω_j *and*

$$\frac{\partial\widehat{f}}{\partial\omega_j}(\boldsymbol{\omega}) = \left(-ip_{\mathbf{e}_j}f\right)^\wedge(\boldsymbol{\omega}), \quad \boldsymbol{\omega} \in \mathbb{R}^d,$$

where $p_{\mathbf{e}_j}(\mathbf{x}) := x_j$ *for* $\mathbf{x} \in \mathbb{R}^d$*.*
If $\frac{\partial f}{\partial x_j}$ *is also in* $L^1(\mathbb{R}^d)$*, then*

$$\widehat{\frac{\partial f}{\partial x_j}}(\boldsymbol{\omega}) = i\omega_j\widehat{f}(\boldsymbol{\omega}), \quad \boldsymbol{\omega} \in \mathbb{R}^d.$$

Proof. The first one follows by changing the order of integration with Fubini. The second and third one can be achieved by substitution. Finally, the last part follows with the dominated convergence theorem and partial integration. □

It is a well-known fact that the Fourier transform can be extended to the space of square integrable functions $L^2(\mathbb{R}^d)$, where it is an isometry. Since the proof can be found in i.e. [LL01] or [Wen04], we will only present a short sketch of it.

Theorem 1.3 (Plancherel's Theorem)
The map $f \mapsto \widehat{f}$ *has a unique extension to a continuous, linear map from* $L^2(\mathbb{R}^d)$ *onto* $L^2(\mathbb{R}^d)$ *which is an isometry, i.e.* $\|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$ *for all* $f \in L^2(\mathbb{R}^d)$*. Moreover, for* $f \in L^2(\mathbb{R}^d)$*, we have* $(\widehat{f})^\vee = f$*.*

Proof. First, the relation $\|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$ can be proved for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ via approximation by C^∞-functions and the monotone convergence theorem. The proof can then be completed using the density of $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ and the completeness of the $L^p(\mathbb{R}^d)$ spaces. □

The identity $\|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$ is also called Parseval's identity. We extended the Fourier transform from the $L^1(\mathbb{R}^d)$ space to the $L^2(\mathbb{R}^d)$ space. But even more is true. With the same arguments, it is possible to extend the Fourier transformation to all $L^p(\mathbb{R}^d)$ with $1 < p < 2$. Unlike in the case of $p = 2$, this map is not invertible any more.
A simple conclusion of Parseval's identity is the extension of this formula from the norm to its inducing inner product.

Corollary 1.4 (Parseval's Formula)
Let f and g be both in $L^2(\mathbb{R}^d)$. Then, the following formula of Parseval holds

$$\int\limits_{\mathbb{R}^d} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x} = \int\limits_{\mathbb{R}^d} \widehat{f}(\boldsymbol{\omega})\overline{\widehat{g}(\boldsymbol{\omega})}d\boldsymbol{\omega}.$$

Proof. This relation follows from Theorem 1.3 using the polarization identity, i.e.

$$\int\limits_{\mathbb{R}^d} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x} = \frac{1}{4}\left(\|f+g\|^2_{L^2(\mathbb{R}^d)} - \|f-g\|^2_{L^2(\mathbb{R}^d)} + i\left(\|f+ig\|^2_{L^2(\mathbb{R}^d)} - \|f-ig\|^2_{L^2(\mathbb{R}^d)}\right)\right). \quad \square$$

There are two important concepts we will define and translate using the Fourier transformation: The convolution and the concept of positive definite functions.

1.1.2 The Convolution

The convolution of two functions will be one of our main tools to derive our numerical scheme. The convolution of the two functions f and g, denoted by $f * g$, is given by

$$f * g(\mathbf{x}) := \int\limits_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^d.$$

One has to be careful to make sure that this definition makes sense. In the case of $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$, Hölder's inequality assures that the integral is well defined for all $\mathbf{x} \in \mathbb{R}^d$. Moreover, Young's inequality guarantees the integrability of the convolution.

Theorem 1.5 (Young's inequality for Convolutions)
Let $1 \le p, q, r \le \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ and $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. Then $f * g \in L^r(\mathbb{R}^d)$ and we have

$$\|f * g\|_{L^r(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

Proof. The proof is a conclusion of Hölder's inequality. $\square$

Even if this inequality is true for all $1 \le p, q \le 2$, the cases of our interests are for $p = 1$ and $q \in \{1, 2\}$. In this case, the theorem guarantees that the convolution of two functions is in $L^q(\mathbb{R}^d)$ itself. This allows us to apply the Fourier transform to the convolution. The connection between the convolution and the Fourier transform is a very important and valuable property. It helps us to convert a convolution to a multiplication in Fourier space.

Theorem 1.6
Let $f, g \in L^1(\mathbb{R}^d)$. Then we have

$$\widehat{f * g}(\boldsymbol{\omega}) = (2\pi)^{d/2}\widehat{f}(\boldsymbol{\omega})\widehat{g}(\boldsymbol{\omega}), \quad \boldsymbol{\omega} \in \mathbb{R}^d.$$

Proof. From Theorem 1.5 we know that $f * g \in L^1(\mathbb{R}^d)$ and therefore $\widehat{f * g}$ is continuous. Hence,

$$\begin{aligned}\widehat{f * g}(\boldsymbol{\omega}) &= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}\int\limits_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})d\mathbf{y}d\mathbf{x} \\ &= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{-i\boldsymbol{\omega}\cdot(\mathbf{x}-\mathbf{y})}e^{-i\boldsymbol{\omega}\cdot\mathbf{y}}f(\mathbf{x}-\mathbf{y})g(\mathbf{y})d\mathbf{x}d\mathbf{y},\end{aligned}$$

where we switched the order of integration using Fubini's theorem. The substitution $\mathbf{z} = \mathbf{x} - \mathbf{y}$ implies

$$\widehat{f * g}(\boldsymbol{\omega}) = (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{-i\boldsymbol{\omega}\cdot\mathbf{z}}f(\mathbf{z})d\mathbf{z}\int\limits_{\mathbb{R}^d} e^{-i\boldsymbol{\omega}\cdot\mathbf{y}}g(\mathbf{y})d\mathbf{y} = (2\pi)^{d/2}\widehat{f}(\boldsymbol{\omega})\widehat{g}(\boldsymbol{\omega}),$$

which completes the proof. $\square$

This result can be generalized to the extended Fourier transformation on $L^r(\mathbb{R}^d)$ for $1 \le r \le 2$. If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ for $1 \le p, q \le 2$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, this result remains true in the $L^{r/(r-1)}(\mathbb{R}^d)$ sense. For more details, see [LL01].
Finally, we are interested in under which conditions the convolution of two functions is continuous. To prove this result, we will need following auxiliary result.

Lemma 1.7
Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^d)$. Then we have $\lim_{\mathbf{x}\to\mathbf{0}} \|f - f(\cdot - \mathbf{x})\|_{L^p(\mathbb{R}^d)} = 0$.

The proof of this lemma can be found in [Wen04] and is based on the density of $C_0(\mathbb{R}^d)$ functions in $L^p(\mathbb{R}^d)$.

Lemma 1.8
Let $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. Then, $f * g$ is continuous.

Proof. Let $\mathbf{x}_0 \in \mathbb{R}^d$ and assume $p \le q$ without loss of generality, which means in particular that $p < \infty$. For $\mathbf{x} \in \mathbb{R}^d$, using Hölder's inequality, we have

$$\begin{aligned} |f * g(\mathbf{x}) - f * g(\mathbf{x}_0)| &\le \int_{\mathbb{R}^d} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x}_0 - \mathbf{y})||g(\mathbf{y})| d\mathbf{y} \\ &\le \|f(\mathbf{x} - \cdot) - f(\mathbf{x}_0 - \cdot)\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}. \end{aligned}$$

By substitution, Lemma 1.7 states that the first norm tends to zero if $\mathbf{x}$ tends to $\mathbf{x}_0$, which completes the proof. □

1.1.3 Positive Definite Functions

Another essential concept, we want to express in the Fourier space, is the class of positive definite functions. Positive definite functions play an important role in interpolation in a multivariate setting. In our case, we are interested in the nonnegative Fourier transformation these functions must have. Before we show this, we give the definition of a positive definite function.

Definition 1.9
A continuous function $\Phi : \mathbb{R}^d \to \mathbb{C}$ is called positive definite if for all $N \in \mathbb{N}$, all sets of pairwise distinct centers $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbb{R}^d$, and all $\boldsymbol{\alpha} \in \mathbb{C}^N \setminus \{\mathbf{0}\}$, the quadric form

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \overline{\alpha_k} \Phi(\mathbf{x}_j - \mathbf{x}_k)$$

is positive. Φ is called positive semi-definite if the quadric form is nonnegative.

Since the definition is unwieldy for our purpose, we will transform this condition into the Fourier space using Bochner's characterization of positive (semi-)definite functions.

Theorem 1.10 (Bochner)
A function $f : \mathbb{R}^d \mapsto \mathbb{C}$ is positive semi-definite if and only if it is the Fourier transform of a finite nonnegative Borel measure μ on $\mathbb{R}^d$, i.e.

$$f(\mathbf{x}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\mathbf{y}\cdot\mathbf{x}} d\mu(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^d.$$

f is positive definite if the carrier of the measure μ contains an open subset.

For our purpose, we suppose that μ has a non-zero Lebesgue density. In this case, the following corollary will be useful in checking whether a function is positive definite.

Corollary 1.11
Let $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. If its Fourier transform $\widehat{\Phi} \in L^1(\mathbb{R}^d)$ is continuous, nonnegative and nonvanishing, then Φ is positive definite.

Finally, a kind of inversion of Corollary 1.11 also holds true.

Lemma 1.12
Let $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be positive definite. Then, its nonnegative Fourier transform $\widehat{\Phi}$ is in $L^1(\mathbb{R}^d)$.

Proof. The proof is based on the use of Friedrich's mollifiers, see [Wen04]. □

For more details regarding positive definite functions, see e.g. [Wen04].

1.2 Basics on Sobolev Spaces

Weakly differentiable functions play an important role in the theory of partial differential equations. Like many other analytical results on numerical methods, our analysis also requires that functions are weakly differentiable, or, to be more precise, belong to a certain Sobolev space. For this, we will recall the definition of Sobolev spaces.

Definition 1.13
Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) \,|\, D^{\boldsymbol{\alpha}} f \in L^p(\Omega) \text{ for all } 1 \leq |\boldsymbol{\alpha}| \leq k\}\,.$$

Its norm is given by

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}| \leq k} \|D^{\boldsymbol{\alpha}} f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, $f \in W^{k,p}(\Omega)$ and

$$\|f\|_{W^{k,\infty}(\Omega)} := \sum_{|\boldsymbol{\alpha}| \leq k} \|D^{\boldsymbol{\alpha}} f\|_{L^\infty(\Omega)}$$

for $f \in W^{k,\infty}(\Omega)$.

Thus, the Sobolev space $W^{k,p}(\Omega)$ contains all functions in $L^p(\Omega)$ with weak derivatives up to an order $|\boldsymbol{\alpha}| \leq k$ in $L^p(\Omega)$. Beside the norm we will also define the semi-norm

$$|f|_{W^{k,p}(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}| = k} \|D^{\boldsymbol{\alpha}} f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, $f \in W^{k,p}(\Omega)$ and

$$|f|_{W^{k,\infty}(\Omega)} := \sum_{|\boldsymbol{\alpha}| = k} \|D^{\boldsymbol{\alpha}} f\|_{L^\infty(\Omega)}$$

for $f \in W^{k,\infty}(\Omega)$.
In the following, we will concentrate on the case $\Omega = \mathbb{R}^d$. A very useful lemma will help us to handle Sobolev functions as the limit of a sequence of classical differentiable functions.

Lemma 1.14
Let $1 \leq p < \infty$. Then, the function space $C^\infty(\mathbb{R}^d) \cap W^{k,p}(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$.

Proof. The proof is based on Friedrichs mollifiers. See, for example, [Wen04]. □

Note that the Lemma is not true for $p = \infty$ since this would mean that every function from $L^\infty(\mathbb{R}^d)$ has to be continuous.
If $p = 2$, the Sobolev space $W^{k,2}(\mathbb{R}^d)$ is also a Hilbert space. In this case, we can use the Fourier transform to give an alternative characterization of the space $W^{k,2}(\mathbb{R}^d)$, which even holds for fractional k. For this, we will use the following definition.

Definition 1.15
Assume $0 \le s < \infty$*. The space* $H^s(\mathbb{R}^d)$ *is defined by*

$$H^s(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) \,|\, \boldsymbol{\omega} \mapsto \left(1 + \|\boldsymbol{\omega}\|_2^2\right)^{s/2} |\widehat{f}(\boldsymbol{\omega})| \in L^2(\mathbb{R}^d) \right\}.$$

Its norm is given by

$$\|f\|_{H^s(\mathbb{R}^d)} := \left(\int\limits_{\mathbb{R}^d} \left(1 + \|\boldsymbol{\omega}\|_2^2\right)^s |\widehat{f}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} \right)^{1/2}$$

for $f \in H^s(\mathbb{R}^d)$*.*

For an integer $k \in \mathbb{N}$, it is a well known fact that $H^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ and that the two norms are equivalent, i.e. that there exists a constant $C > 0$ such that

$$\frac{1}{C}\|f\|_{H^k(\mathbb{R}^d)} \le \|f\|_{W^{k,2}(\mathbb{R}^d)} \le C\|f\|_{H^k(\mathbb{R}^d)}, \quad f \in W^{k,2}(\mathbb{R}^d).$$

We will implicitly use the inequality above, and also that if a statement is true for $f \in H^s(\mathbb{R}^d)$ with $0 < s < \infty$, it stays true for $f \in W^{k,2}(\mathbb{R}^d)$ with $k \in \mathbb{N}$, $k \le s$. For more details about Sobolev spaces, see for example [Eva10].
With this equivalence, we can switch between these two norms. Some useful lemmas can be derived for Sobolev spaces. First of all, we can formulate a condition on f, so that its Fourier transformation $\widehat{f}$ is in $L^1(\mathbb{R}^d)$.

Lemma 1.16
Let $f \in H^s(\mathbb{R}^d)$ *for* $s > d/2$*. Then,* $\widehat{f} \in L^1(\mathbb{R}^d)$ *and there exists a constant* $C_{s,d} > 0$ *only depending on* s *and* d*, such that*

$$\|\widehat{f}\|_{L^1(\mathbb{R}^d)} \le C_{s,d}\|f\|_{H^s(\mathbb{R}^d)}.$$

Proof. Using the Cauchy-Schwarz inequality, we can bound the $L^1(\mathbb{R}^d)$ norm by

$$\begin{aligned}
\|\widehat{f}\|_{L^1(\mathbb{R}^d)} &= \int\limits_{\mathbb{R}^d} \left(1 + \|\boldsymbol{\omega}\|^2\right)^{-s/2} \left(1 + \|\boldsymbol{\omega}\|^2\right)^{s/2} |\widehat{f}(\boldsymbol{\omega})| d\boldsymbol{\omega} \\
&\le \left(\int\limits_{\mathbb{R}^d} \left(1 + \|\boldsymbol{\omega}\|^2\right)^{-s} d\boldsymbol{\omega} \right)^{1/2} \left(\int\limits_{\mathbb{R}^d} \left(1 + \|\boldsymbol{\omega}\|^2\right)^{s} |\widehat{f}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} \right)^{1/2} \\
&=: C_{s,d}\|f\|_{H^s(\mathbb{R}^d)},
\end{aligned}$$

where the first integral in the second line is finite since $s > d/2$. □

With this result we are able to check whether the Fourier transform of a function is integrable. This will help us later, together with Corollary 1.11, to check if a function is positive definite. A useful conclusion is that $H^s(\mathbb{R}^d)$ is closed regarding the product of two functions if s is sufficiently large. To prove this, we will need the following lemma.

Lemma 1.17
Let $a, b \geq 0$. Then we have

$$(a+b)^s \leq \max(1, 2^{s-1})(a^s + b^s)$$

for all $s > 0$.

Proof. First, assume that $s \geq 1$. In this case, the map $t \mapsto t^s$ is convex and we have

$$(a+b)^s = 2^s \left(\frac{a+b}{2}\right)^s \leq 2^s \frac{a^s + b^s}{2} = 2^{s-1}(a^s + b^s).$$

If $0 < s < 1$, we have $s - 1 < 0$ and hence

$$(a+b)^s - a^s = \int_0^b s(a+t)^{s-1} dt \leq \int_0^b s t^{s-1} dt = b^s,$$

which completes the proof. □

Hence, we are able to prove the following theorem.

Theorem 1.18
Let $s > \frac{d}{2}$ and $f, g \in H^s(\mathbb{R}^d)$. Then, the product fg belongs also to $H^s(\mathbb{R}^d)$ and there exists a constant $C_{d,s} > 0$, only depending on s and d, such that

$$\|fg\|_{H^s(\mathbb{R}^d)} \leq C_{d,s} \|f\|_{H^s(\mathbb{R}^d)} \|g\|_{H^s(\mathbb{R}^d)}.$$

Proof. Analogously to Theorem 1.6, the Fourier transform of a product is the convolution of the single Fourier transforms

$$\widehat{fg}(\boldsymbol{\xi}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \widehat{f}(\boldsymbol{\xi} - \boldsymbol{\omega}) \widehat{g}(\boldsymbol{\omega}) d\boldsymbol{\omega}, \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

Furthermore, using the triangle inequality, we have

$$\|\boldsymbol{\xi}\|_2^2 = \|\boldsymbol{\xi} - \boldsymbol{\omega} + \boldsymbol{\omega}\|_2^2 \leq (\boldsymbol{\xi} - \boldsymbol{\omega}\|_2 + \|\boldsymbol{\omega}\|_2)^2 \leq 2\|\boldsymbol{\xi} - \boldsymbol{\omega}\|_2^2 + 2\|\boldsymbol{\omega}\|_2^2$$

for every $\boldsymbol{\xi}$, $\boldsymbol{\omega} \in \mathbb{R}^d$. Using this inequality and Lemma 1.17, we can derive the estimate

$$\begin{aligned}
\left(1 + \|\boldsymbol{\xi}\|_2^2\right)^{s/2} &\leq \left(1 + 2\|\boldsymbol{\xi} - \boldsymbol{\omega}\|_2^2 + 2\|\boldsymbol{\omega}\|_2^2\right)^{s/2} \\
&\leq 2^{s/2} \left(\left(1 + \|\boldsymbol{\xi} - \boldsymbol{\omega}\|_2^2\right) + \left(1 + \|\boldsymbol{\omega}\|_2^2\right)\right)^{s/2} \\
&\leq 2^{s/2} \max(2^{s/2-1}, 1) \left(\left(1 + \|\boldsymbol{\xi} - \boldsymbol{\omega}\|_2^2\right)^{s/2} + \left(1 + \|\boldsymbol{\omega}\|_2^2\right)^{s/2}\right).
\end{aligned}$$

Hence, with $C_{s,d} := (2\pi)^{-d/2} 2^{s/2} \max(2^{s/2-1}, 1)$ we have

$$\begin{aligned}
\left(1 + \|\boldsymbol{\xi}\|_2^2\right)^{s/2} |\widehat{fg}(\boldsymbol{\xi})| \leq & C_{s,d} \left| \int_{\mathbb{R}^d} \left(1 + \|\boldsymbol{\xi} - \boldsymbol{\omega}\|_2^2\right)^{s/2} \widehat{f}(\boldsymbol{\xi} - \boldsymbol{\omega}) \widehat{g}(\boldsymbol{\omega}) d\boldsymbol{\omega} \right. \\
& \left. + \int_{\mathbb{R}^d} \left(1 + \|\boldsymbol{\omega}\|_2^2\right)^{s/2} \widehat{f}(\boldsymbol{\xi} - \boldsymbol{\omega}) \widehat{g}(\boldsymbol{\omega}) d\boldsymbol{\omega} \right| \\
= & C_{d,s} \left(\left[\left(1 + \|\cdot\|_2^2\right)^{s/2} \widehat{f} \right] * \widehat{g}(\boldsymbol{\xi}) + \widehat{f} * \left[\left(1 + \|\cdot\|_2^2\right)^{s/2} \widehat{g} \right] (\boldsymbol{\xi}) \right).
\end{aligned}$$

Since this is the sum of two convolutions, Young's inequality yields

$$\begin{aligned}\|fg\|_{H^s(\mathbb{R}^d)} =& \| \left(1+\|\cdot\|_2^2\right)^{s/2} \widehat{fg}\|_{L^2(\mathbb{R}^d)} \\ \leq& \widetilde{C}_{d,s}\| \left(1+\|\cdot\|_2^2\right)^{s/2} \widehat{f}\|_{L^2(\mathbb{R}^d)}\|\widehat{g}\|_{L^1(\mathbb{R}^d)} \\ &+ \widetilde{C}_{d,s}\| \left(1+\|\cdot\|_2^2\right)^{s/2} \widehat{g}\|_{L^2(\mathbb{R}^d)}\|\widehat{f}\|_{L^1(\mathbb{R}^d)} \\ \leq& C_{d,s}\left(\|f\|_{H^s(\mathbb{R}^d)}\|\widehat{g}\|_{L^1(\mathbb{R}^d)} + \|g\|_{H^s(\mathbb{R}^d)}\|\widehat{f}\|_{L^1(\mathbb{R}^d)}\right).\end{aligned}$$

Using Lemma 1.16 completes the proof. □

Finally, we prove the following lemma, which we need later in this chapter. It is a generalization of the fact that the integral over the derivative of a $W^{1,1}(\mathbb{R}^d)$ function vanishes.

Lemma 1.19
Let $k \in \mathbb{N}$ and $f \in W^{k,1}(\mathbb{R}^d)$. Then,

$$\int_{\mathbb{R}^d} D^{\boldsymbol{\alpha}} f(\mathbf{x}) d\mathbf{x} = 0$$

for all $1 \leq |\boldsymbol{\alpha}| \leq k$.

Proof. Let $k = 1$, $|\boldsymbol{\alpha}| = 1$ and suppose that $f \in C_c^1(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$. Without loss of generality we assume that $\boldsymbol{\alpha} = \mathbf{e}_1$. Hence, we can write

$$\int_{\mathbb{R}^d} D^{\boldsymbol{\alpha}} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \partial_{x_1} f(x_1, \widetilde{\mathbf{x}}) dx_1 d\widetilde{\mathbf{x}},$$

with $\widetilde{\mathbf{x}} \in \mathbb{R}^{d-1}$. Since f has compact support, there exists an $r > 0$, such that $\operatorname{supp}(f) \subset B_r(0)$. Calculating the one dimensional integral gives

$$\int_{\mathbb{R}} \partial_{x_1} f(x_1, \widetilde{\mathbf{x}}) dx_1 = \int_{-r}^{r} \partial_{x_1} f(x_1, \widetilde{\mathbf{x}}) = f(r, \widetilde{\mathbf{x}}) - f(-r, \widetilde{\mathbf{x}}) = 0.$$

Since $C_c^\infty(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$ is dense in $W^{1,1}(\mathbb{R}^d)$, the proof is finished for $k = 1$.
For $k > 1$, we assume that $|\boldsymbol{\alpha}| = k$ with $\boldsymbol{\alpha} = \boldsymbol{\beta} + \mathbf{e}_1$ for a $|\boldsymbol{\beta}| = k - 1$ without loss of generality. Hence, we can apply the proof above to $D^{\boldsymbol{\beta}} f \in W^{1,1}(\mathbb{R}^d)$. The fact that $W^{k,1}(\mathbb{R}^d) \subset W^{l,1}(\mathbb{R}^d)$ for all $0 \leq l \leq k$ finishes the proof. □

1.3 Spaces of Time-depended Functions

In this section, we want to study functions from a given time interval I to a Banach space X, which will later be identified as a Lebesgue space or a Sobolev space. More precisely, we want to briefly recall the generalization of the concepts of continuity, strong differentiation and integrability for Banach space valued functions. For the first two, we can simply generalize the definition from real-valued function, this means for example that, for a given Banach space X, a function $f : [a, b] \to X$ is differentiable in $t \in [a, b]$ if the limit $f'(t) := \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$ exists.

Definition 1.20
Let X be a Banach space and $a, b \in \mathbb{R}$ with $a < b$.

i) The space $C([a,b];X)$ *of continuous Banach space valued functions is defined by*

$$C([a,b];X) := \{f : [a,b] \to X \mid f \text{ is continuous on } [a,b]\}.$$

Its norm is given by

$$\|f\|_{C([a,b];X)} := \max_{t\in[a,b]} \|f(t)\|_X.$$

ii) Let $k \in \mathbb{N}$. *The space* $C^k([a,b];X)$ *of* k*-times continuous differentiable Banach space valued functions is defined by*

$$C^k([a,b];X) := \{f : [a,b] \to X \mid f^{(j)} \in C([a,b];X) \text{ for all } 0 \le j \le k\}.$$

Its norm is given by

$$\|f\|_{C^k([a,b];X)} := \max_{0\le j\le k} \|f^{(j)}\|_{C([a,b];X)}.$$

For a concept of integration for Banach space valued functions, we will follow the common idea of using Bochner's integral. Therefore we will require the definition of simple and strongly measurable functions.

Definition 1.21
Let X *be a Banach space and* $a, b \in \mathbb{R}$ *with* $a < b$.

i) A function $f : [a,b] \to X$ *is called a simple function if it has the form*

$$f(t) = \sum_{j=1}^{n} \chi_{B_j}(t)c_j,$$

with elements $c_j \in X$ *and disjoint Lebesgue measurable sets* $B_j \subset [a,b]$, $1 \le j \le n$, *such that* $[a,b] = \bigcup_{j=1}^{n} B_j$. *The function* $\chi_B : [a,b] \to \{0,1\}$ *denotes the characteristic function of a subset* $B \subset [a,b]$.

ii) A function $f : [a,b] \to X$ *is called strongly measurable if there exists a sequence* $(f_k)_{k\in\mathbb{N}}$ *of simple functions* $f_k : [a,b] \to X$, $k \in \mathbb{N}$, *such that*

$$\lim_{k\to\infty} \|f(t) - f_k(t)\|_X = 0$$

for almost every $t \in [a,b]$.

The idea of simple functions will give us an easy way to define an integral for Banach space valued functions. This definition can then be generalized to strongly measurable functions, which are the limit of simple functions.

Definition 1.22
Let X *be a Banach space and* $a, b \in \mathbb{R}$ *with* $a < b$.

i) For a simple function $f : [a,b] \to X$, $t \mapsto \sum\limits_{j=1}^{n} \chi_{B_j}(t)c_j$, *the Bochner integral is defined by*

$$\int_a^b f(t)dt = \sum_{j=1}^{n} |B_j|c_j \in X,$$

where $|B_j|$ *denotes the Lebesgue measure of the set* B_j.

ii) A function $f : [a, b] \to X$ is Bochner integrable if there exists a sequence $(f_k)_{k\in\mathbb{N}}$ of simple functions $f_k : [a, b] \to X$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} \int_a^b \|f_k(t) - f(t)\|_X dt = 0.$$

In this case, the Bochner integral is defined by

$$\int_a^b f(t)dt = \lim_{k\to\infty} \int_a^b f_k(t)dt.$$

The Bochner integral is well defined since it is independent of the choice of the approximating sequence, see [Yos95]. Hence, the integral is linear in f. Moreover, it follows from the definition that a Bochner integrable function is strongly measurable.
The next result provides an important connection between the Bochner integral and the Lebesgue integral.

Theorem 1.23 (Bochner)
Let X be a Banach space and $a, b \in \mathbb{R}$ with $a < b$. A strongly measurable function $f : [a, b] \to X$ is Bochner integrable if and only if $t \mapsto \|f(t)\|_X$ is Lebesgue integrable. In this case, the integral is bounded by

$$\left\| \int_a^b f(t)dt \right\|_X \leq \int_a^b \|f(t)\|_X \, dt.$$

For a proof see [Yos95]. Now we are able to generalize the concept of integrable functions to the space of Bochner integrable functions.

Definition 1.24
Let $1 \leq p \leq \infty$. The space $L^p(a, b; X)$ is defined by

$$L^p(a, b; X) := \big\{ f : [a, b] \to X \mid f \text{ is strongly measurable and } t \mapsto \|f(t)\|_X \in L^p(a, b) \big\}.$$

Its norm is given by

$$\|f\|_{L^p(a,b;X)} := \left(\int_a^b \|f(t)\|_X^p dt \right)^{1/p}$$

for $1 \leq p < \infty$ and

$$\|f\|_{L^\infty(a,b;X)} := \operatorname*{ess\,sup}_{0\leq t\leq T} \|f(t)\|_X.$$

As a direct consequence that we will need later, we see that continuous functions on a compact interval are always in $L^\infty(\mathbb{R}^d)$.

Lemma 1.25
The space $C([a, b], X)$ is a subset of $L^\infty(a, b; X)$.

Proof. Since the essential supremum of a set is always lower or equal the supremum, we can derive

$$\|f\|_{L^\infty(a,b;X)} = \operatorname*{ess\,sup}_{0\leq t\leq T} \|f(t)\|_X \leq \sup_{0\leq t\leq T} \|f(t)\|_X = \|f\|_{C([a,b],X)}. \qquad \square$$

The concept of weak differentiation and Sobolev spaces can also be transferred to Banach space-valued functions. Since this is not of interest for the following theory, we refer to e.g. [Eva10].

1.4 Tools from Approximation Theory

In this section, we want to derive an approximation scheme following the ideas of Raviart in [Rav85]. First, we want to introduce two approximation results for convolution and discretization. Then, we want to combine these two approximations to derive the approximation scheme.

1.4.1 Approximation via Convolution

The approximation of a function $f \in L^p(\mathbb{R}^d)$ via convolution is a frequently used tool in analysis. The idea is to convolute f with a scaled kernel function $\Phi_\varepsilon = \varepsilon^{-d}\Phi(\cdot/\varepsilon)$, where $\Phi \in L^1(\mathbb{R}^d)$ has some useful properties like smoothness. The convolution $f * \Phi_\varepsilon = \int_{\mathbb{R}^d} f(\mathbf{x})\Phi_\varepsilon(\cdot - \mathbf{x})d\mathbf{x}$ is then converging to f in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ if $\varepsilon \to 0$.
The advantage of this method is that the convolution $f * \Phi_\varepsilon$ will inherit the properties of the kernel function Φ, for example differentiability, so that $f * \Phi_\varepsilon$ is more regular than f. Moreover, according to Young's inequality, the convolution is in $L^p(\mathbb{R}^d)$ itself, so the method will preserve the integrability of f.
To guarantee that the convergence of $f * \Phi_\varepsilon$ to f is sufficiently fast, the kernel function has to satisfy the moment condition.

Definition 1.26
A kernel $\Phi \in L^1(\mathbb{R}^d)$ satisfies the moment condition of order $m \in \mathbb{N}$ if it satisfies

$$\int_{\mathbb{R}^d} \Phi(\mathbf{x})d\mathbf{x} = 1, \tag{1.1}$$

$$\int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}}\Phi(\mathbf{x})d\mathbf{x} = 0, \qquad \boldsymbol{\alpha} \in \mathbb{N}_0^d \text{ with } 1 \leq |\boldsymbol{\alpha}| < m, \tag{1.2}$$

$$\int_{\mathbb{R}^d} \|\mathbf{x}\|_2^m |\Phi(\mathbf{x})|d\mathbf{x} < \infty, \tag{1.3}$$

where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^d$.

A function that satisfies (1.1) is called normalized. Moreover, the integrals $\int_{\mathbb{R}^d} \mathbf{x}^\alpha \Phi(\mathbf{x})d\mathbf{x}$ are also called the moments of the function Φ. Note that if Φ satisfies a moment condition of order m, so does its scaled version $\Phi_\varepsilon = \varepsilon^{-d}\Phi(1/\varepsilon)$ for any $\varepsilon > 0$.

Lemma 1.27
Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Then, $\Phi \in L^1(\mathbb{R}^d)$ satisfies the moment condition of order m if and only if its scaled version $\Phi_\varepsilon = \varepsilon^{-d}\Phi(1/\varepsilon)$ satisfies the moment condition of order m.

Proof. Let $0 \leq |\boldsymbol{\alpha}| < m$. With a simple substitution, the first and second properties (1.1), (1.2) follow from

$$\int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}}\Phi_\varepsilon(\mathbf{x})d\mathbf{x} = \varepsilon^{-d}\int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}}\Phi(\mathbf{x}/\varepsilon)d\mathbf{x} = \varepsilon^{|\boldsymbol{\alpha}|}\int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}}\Phi(\mathbf{y})d\mathbf{y}.$$

The third property (1.3) follows from

$$\int_{\mathbb{R}^d} \|\mathbf{x}\|_2^m|\Phi_\varepsilon(\mathbf{x})|d\mathbf{x} = \varepsilon^{-d}\int_{\mathbb{R}^d} \|\mathbf{x}\|_2^m|\Phi(\mathbf{x}/\varepsilon)|d\mathbf{x} = \varepsilon^m\int_{\mathbb{R}^d} \|\mathbf{y}\|_2^m|\Phi(\mathbf{y})|d\mathbf{y},$$

where we used the same substitution. □

With the moment condition, we can now calculate the convergence rate of the approximation by convolution.

Theorem 1.28 (Approximation via convolution)
Assume $\Phi \in L^1(\mathbb{R}^d)$ satisfies the moment condition of order $m \in \mathbb{N}$. Then, if $f \in W^{m,p}(\mathbb{R}^d)$ with $1 \le p < \infty$ or $f \in C^m(\mathbb{R}^d) \cap W^{m,\infty}(\mathbb{R}^d)$, there exists a constant $C > 0$ such that

$$\|f - f * \Phi_\varepsilon\|_{L^p(\mathbb{R}^d)} \le C\varepsilon^m |f|_{W^{m,p}(\mathbb{R}^d)}.$$

Proof. For $1 \le p < \infty$, we first assume that $f \in C^\infty(\mathbb{R}^d) \cap W^{m,p}(\mathbb{R}^d)$. Hence, in both cases, $1 \le p < \infty$ and $p = \infty$, we can take the Taylor expansion of f up to order m

$$\begin{aligned} f(\mathbf{x} - \mathbf{y}) = f(\mathbf{x}) &+ \sum_{1 \le |\boldsymbol{\alpha}| < m} \frac{D^{\boldsymbol{\alpha}} f(\mathbf{x})}{\boldsymbol{\alpha}!} (-\mathbf{y})^{\boldsymbol{\alpha}} \\ &+ m \int_0^1 \sum_{|\boldsymbol{\alpha}| = m} \frac{(1-s)^{m-1}(-\mathbf{y})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}} f(\mathbf{x} - s\mathbf{y}) ds. \end{aligned}$$

Because Φ_ε has also integral 1, we achieve

$$\begin{aligned} f(\mathbf{x}) - f * \Phi_\varepsilon(\mathbf{x}) &= \int_{\mathbb{R}^d} (f(\mathbf{x}) - f(\mathbf{x} - \mathbf{y}))\Phi_\varepsilon(\mathbf{y}) d\mathbf{y} \\ &= - \sum_{1 \le |\boldsymbol{\alpha}| < m} \frac{D^{\boldsymbol{\alpha}} f(\mathbf{x})}{\boldsymbol{\alpha}!} (-1)^{|\boldsymbol{\alpha}|} \int_{\mathbb{R}^d} \mathbf{y}^{\boldsymbol{\alpha}} \Phi_\varepsilon(\mathbf{y}) d\mathbf{y} \\ &\quad - \int_{\mathbb{R}^d} m \int_0^1 \sum_{|\boldsymbol{\alpha}| = m} \frac{(1-s)^{m-1}(-\mathbf{y})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}} f(\mathbf{x} - s\mathbf{y}) \Phi_\varepsilon(\mathbf{y}) ds d\mathbf{y}, \end{aligned}$$

where the first part vanishes since the moments of Φ_ε vanish. With this equation, the triangle inequality and the fact that $\|D^{\boldsymbol{\alpha}} f(\cdot - s\mathbf{y})\|_{L^p(\mathbb{R}^d)} = \|D^{\boldsymbol{\alpha}} f\|_{L^p(\mathbb{R}^d)}$ for all $s \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}^d$, we conclude

$$\begin{aligned} \|f - f * \Phi_\varepsilon\|_{L^p(\mathbb{R}^d)} &\le \left\| \int_{\mathbb{R}^d} m \int_0^1 \sum_{|\boldsymbol{\alpha}| = m} \frac{(1-s)^{m-1}(-\mathbf{y})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}} f(\cdot - s\mathbf{y}) \Phi_\varepsilon(\mathbf{y}) ds d\mathbf{y} \right\|_{L^p(\mathbb{R}^d)} \\ &\le \sum_{|\boldsymbol{\alpha}| = m} \frac{1}{\boldsymbol{\alpha}!} \|D^{\boldsymbol{\alpha}} f\|_{L^p(\mathbb{R}^d)} \int_0^1 m(1-s)^{m-1} ds \int_{\mathbb{R}^d} \|\mathbf{y}\|_2^m |\Phi_\varepsilon(\mathbf{y})| d\mathbf{y}. \end{aligned}$$

The integral over s is bounded by 1. For the integral over $\mathbf{y}$, we use the moment condition of the function Φ to find

$$\int_{\mathbb{R}^d} \|\mathbf{y}\|_2^m |\Phi_\varepsilon(\mathbf{y})| d\mathbf{y} = \varepsilon^m \int_{\mathbb{R}^d} \|\mathbf{y}\|_2^m |\Phi(\mathbf{y})| d\mathbf{y} \le c\varepsilon^m.$$

For $p = \infty$, the proof is finished. For $1 \le p < \infty$, we can us Lemma 1.17 multiple times to bound the remaining sum by

$$\left(\sum_{|\boldsymbol{\alpha}| = m} \|D^{\boldsymbol{\alpha}} f\|_{L^p(\mathbb{R}^d)} \right)^p \le c \sum_{|\boldsymbol{\alpha}| = m} \|D^{\boldsymbol{\alpha}} f\|_{L^p(\mathbb{R}^d)}^p = c\, |D^{\boldsymbol{\alpha}} f|_{W^{m,p}(\mathbb{R}^d)}^p.$$

The fact that $C^\infty(\mathbb{R}^d) \cap W^{m,p}(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$ for $1 \le p < \infty$ completes the proof. □

The approximation by convolution will be frequently used in the numerical scheme later on. Nevertheless, for a numerical scheme, we have to do another approximation step to calculate the convolution as a discrete sum instead of the integral.

1.4.2 Approximation by Quadrature

In the next step we will investigate the approximation of an integral by a simple quadrature formula. In the following we consider cubes $(\Omega_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}$ with centers $h\mathbf{j}$ and edge length $h > 0$, i.e.

$$\Omega_{\mathbf{j}} = \left\{\mathbf{x} \in \mathbb{R}^d | h\left(j_k - \frac{1}{2}\right) \leq x_k < h\left(j_k + \frac{1}{2}\right), k = 1, \ldots, d\right\}, \quad \mathbf{j} \in \mathbb{Z}^d.$$

The parameter h is also called the spatial discretization parameter.
The idea is to decompose $\mathbb{R}^d$ into the set of cubes and to approximate the integral over a single cube by the function value at the center of the cube times the volume of the cube. To investigate the error of this step, we will split up our integration error over $\mathbb{R}^d$ into a sum over the errors of the single cubes, i.e.

$$\int\limits_{\mathbb{R}^d} f(\mathbf{x})d\mathbf{x} - h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} f(h\mathbf{j}) = \sum_{\mathbf{j}\in\mathbb{Z}^d} \left[\int\limits_{\Omega_{\mathbf{j}}} f(\mathbf{x})d\mathbf{x} - h^d f(h\mathbf{j})\right] = \sum_{\mathbf{j}\in\mathbb{Z}^d} E_{\mathbf{j}}(f),$$

where we defined the local integration error $E_{\mathbf{j}}(f) := \int_{\Omega_{\mathbf{j}}} f(\mathbf{x})d\mathbf{x} - h^d f(h\mathbf{j})$ for the single cube $\Omega_{\mathbf{j}}$.
To achieve an error bound for the quadarture formula on $\mathbb{R}^d$, we will calculate a bound for the single local integration errors. For this, we will need the well-known Lemma of Bramble and Hilbert. Its proof can be found in [Cia78].

Lemma 1.29
Let Ω be an bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary. Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, and let $L : W^{k,p}(\Omega) \to \mathbb{R}$ be a continuous, linear functional with the property that $L(f) = 0$ for all $f \in \Pi_{k-1}(\Omega)$.
Then, there exists a constant $C > 0$, depending on L and Ω, such that

$$|Lf| \leq C|f|_{W^{k,p}(\Omega)}$$

for all $f \in W^{k,p}(\Omega)$.

We will apply the Lemma of Bramble and Hilbert to our local integration error $E_{\mathbf{j}}(f)$ to achieve an error bound depending on the spatial discretization parameter h.

Lemma 1.30
Let $k \in \{1, 2\}$, $1 \leq p \leq \infty$ and $q = \frac{p}{p-1}$. Then, there exists a constant $C > 0$, independent of h, such that

$$|E_{\mathbf{j}}(f)| \leq Ch^{k+\frac{d}{q}} |f|_{W^{k,p}(\Omega_{\mathbf{j}})}$$

for all $f \in W^{k,p}(\Omega_{\mathbf{j}}) \cap C(\Omega_{\mathbf{j}})$, $\mathbf{j} \in \mathbb{Z}^d$.

Proof. Let $\mathbf{j} \in \mathbb{Z}^d$ and let $\widetilde{\Omega} = [-\frac{1}{2}, \frac{1}{2}]^d$ be the unit cube with edge length one. Transforming the cube $\Omega_{\mathbf{j}}$ to the unit cube, we can rewrite the local integration error as

$$\begin{aligned} E_{\mathbf{j}}(f) &= \int\limits_{\Omega_{\mathbf{j}}} f(\mathbf{x})d\mathbf{x} - h^d f(h\mathbf{j}) = h^d \left[\int\limits_{\widetilde{\Omega}} f(h\mathbf{j} + h\mathbf{x})d\mathbf{x} - f(h\mathbf{j})\right] \\ &= h^d \left[\int\limits_{\widetilde{\Omega}} \widetilde{f}(\mathbf{x})d\mathbf{x} - \widetilde{f}(0)\right] =: h^d \widetilde{E}(\widetilde{f}) \end{aligned}$$

with $\widetilde{f} \in W^{k,p}(\widetilde{\Omega})$ defined by $\widetilde{f}(\mathbf{x}) = f(h\mathbf{j} + h\mathbf{x})$.
Hence, the functional $\widetilde{E} : W^{k,p}(\widetilde{\Omega}) \to \mathbb{R}$ is continuous and linear. Moreover, $\widetilde{E}(g) = 0$ for all

$g \in \Pi_1(\widetilde{\Omega})$ since the approximation of the integral is exact for linear polynomials. Thus, Lemma 1.29 states that there exists a constant $c > 0$ with $|\widetilde{E}(\widetilde{f})| \leq c|\widetilde{f}|_{W^{k,p}(\widetilde{\Omega})}$.
Let $p < \infty$. Then, we have

$$\begin{aligned}
|\widetilde{f}|^p_{W^{k,p}(\widetilde{\Omega})} &= \sum_{|\boldsymbol{\alpha}|=k} \int_{\widetilde{\Omega}} |D^{\boldsymbol{\alpha}} \widetilde{f}(\mathbf{x})|^p d\mathbf{x} = \sum_{|\boldsymbol{\alpha}|=k} \int_{\widetilde{\Omega}} |D^{\boldsymbol{\alpha}} f(h\mathbf{j} + h\mathbf{x})|^p d\mathbf{x} \\
&= h^{pk} \sum_{|\boldsymbol{\alpha}|=k} \int_{\widetilde{\Omega}} |(D^{\boldsymbol{\alpha}} f)(h\mathbf{j} + h\mathbf{x})|^p d\mathbf{x} = h^{pk-d} \sum_{|\boldsymbol{\alpha}|=k} \int_{\Omega_{\mathbf{j}}} |D^{\boldsymbol{\alpha}} f(\mathbf{x})|^p d\mathbf{x} \\
&= h^{pk-d} |f|^p_{W^{k,p}(\Omega_{\mathbf{j}})}.
\end{aligned}$$

With $d - \frac{d}{p} = \frac{d}{q}$, this gives the estimate above. A similar argumentation holds if $p = \infty$, which completes the proof. □

In the next step we want to apply the Lemma of Bramble and Hilbert to smoother functions f. But since $E_{\mathbf{j}}$ vanishes only for linear polynomials, we have to modify it.

Lemma 1.31
Let $k \in \mathbb{N}$ with $k \geq 3$, $1 \leq p \leq \infty$ and $q = \frac{p}{p-1}$. Then, there exist constants $d_{\boldsymbol{\alpha}} \in \mathbb{R}$ for $2 \leq |\boldsymbol{\alpha}| \leq k-1$ and a constant $C > 0$, independent of h, such that

$$\left| E_{\mathbf{j}}(f) - \sum_{2 \leq |\boldsymbol{\alpha}| \leq k-1} d_{\boldsymbol{\alpha}} h^{|\boldsymbol{\alpha}|} \int_{\Omega_{\mathbf{j}}} D^{\alpha} f(\mathbf{x}) d\mathbf{x} \right| \leq C h^{k + \frac{d}{q}} |f|_{W^{k,p}(\Omega_{\mathbf{j}})}$$

for all $f \in W^{k,p}(\Omega_{\mathbf{j}}) \cap C(\Omega_{\mathbf{j}})$, $\mathbf{j} \in \mathbb{Z}^d$.

Proof. We will use the same notation as in the proof of Lemma 1.30, and start with the scaled version of this problem. Let

$$L_k(\widetilde{f}) = \widetilde{E}(\widetilde{f}) - \sum_{2 \leq |\boldsymbol{\alpha}| \leq k-1} \widetilde{d}_{\boldsymbol{\alpha}} \int_{\widetilde{\Omega}} D^{\boldsymbol{\alpha}} \widetilde{f}(\mathbf{x}) d\mathbf{x},$$

where we have to find constants $\widetilde{d}_{\boldsymbol{\alpha}}$ such that $L_k(\widetilde{g}) = 0$ for all $g \in \Pi_{k-1}(\widetilde{\Omega})$. We will show this by induction on k. The case for $k = 2$ has been proven in Lemma 1.30, where the sum over $\boldsymbol{\alpha}$ vanishes. For the induction step, we set

$$L_{k+1}(\widetilde{f}) = L_k(\widetilde{f}) - \sum_{|\boldsymbol{\alpha}|=k} \widetilde{d}_{\boldsymbol{\alpha}} \int_{\widetilde{\Omega}} D^{\boldsymbol{\alpha}} \widetilde{f}(\mathbf{x}) d\mathbf{x},$$

so that we only have to find the constants $\widetilde{d}_{\boldsymbol{\alpha}}$ for $|\boldsymbol{\alpha}| = k$. Since L_{k+1} vanishes on $\Pi_{k-1}(\widetilde{\Omega})$ by induction hypothesis and since $D^{\boldsymbol{\alpha}} g = 0$ for all $|\boldsymbol{\alpha}| = k$ and $g \in \Pi_{k-1}(\widetilde{\Omega})$, we only have to investigate monomials of order k. Let $|\boldsymbol{\beta}| = k$, then $D^{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\beta}} = \boldsymbol{\beta}!$ if and only if $\boldsymbol{\alpha} = \boldsymbol{\beta}$, otherwise $D^{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\beta}} = 0$. Hence, we have

$$L_{k+1}(\mathbf{x}^{\boldsymbol{\beta}}) = L_k(\mathbf{x}^{\boldsymbol{\beta}}) - \widetilde{d}_{\boldsymbol{\beta}} \boldsymbol{\beta}! = 0$$

if we choose the constants $\widetilde{d}_{\boldsymbol{\beta}} = \frac{L_k(\mathbf{x}^{\boldsymbol{\beta}})}{\boldsymbol{\beta}!}$. With this constants L_{k+1} vanishes on $\Pi_k(\widetilde{\Omega})$ which completes the induction.
Hence, for a $k \geq 3$, applying Lemma 1.29 states that there exists a constant $c > 0$ with $|L_k(\widetilde{g})| \leq c|\widetilde{g}|_{W^{k,p}(\widetilde{\Omega})}$. Analogously to the proof of Lemma 1.30, a simple scaling argument finishes the proof. □

We are now in the situation to give an error bound for the quadrature error on $\mathbb{R}^d$. This result will be of central importance for our numerical approach.

Lemma 1.32
Let $k \in \mathbb{N}$, $1 \leq p \leq \infty$ and $q = \frac{p}{p-1}$. Then, there exists a $C > 0$, such that for all $f \in W^{k,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ with

$$\sum_{\mathbf{j} \in \mathbb{Z}^d} |f|_{W^{k,p}(\Omega_{\mathbf{j}})} < \infty$$

and $f \in L^1(\mathbb{R}^d)$ if $k \in \{1, 2\}$, or $f \in W^{k-1,1}(\mathbb{R}^d)$ if $k \geq 3$, we have

$$\left| \int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x} - h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} f(h\mathbf{j}) \right| \leq C h^{k+\frac{d}{q}} \sum_{\mathbf{j} \in \mathbb{Z}^d} |f|_{W^{k,p}(\Omega_{\mathbf{j}})}.$$

Proof. First, we note that the left hand side is equal to $\sum_{\mathbf{j} \in \mathbb{Z}^d} E_{\mathbf{j}}(f)$. Lemma 1.30 completes the proof for $k \in \{1, 2\}$.
For $k \geq 3$ and $f \in W^{k-1,1}(\mathbb{R}^d)$ Lemma 1.19 states that

$$\sum_{2 \leq |\boldsymbol{\alpha}| \leq k-1} d_{\boldsymbol{\alpha}} h^{|\boldsymbol{\alpha}|} \int_{\mathbb{R}^d} D^\alpha f(\mathbf{x}) d\mathbf{x} = 0.$$

Hence, by adding a zero, we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x} - h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} f(h\mathbf{j}) &= \sum_{\mathbf{j} \in \mathbb{Z}^d} E_{\mathbf{j}}(f) + \sum_{2 \leq |\boldsymbol{\alpha}| \leq k-1} d_{\boldsymbol{\alpha}} h^{|\boldsymbol{\alpha}|} \int_{\mathbb{R}^d} D^\alpha f(\mathbf{x}) d\mathbf{x} \\ &= \sum_{\mathbf{j} \in \mathbb{Z}^d} \left(E_{\mathbf{j}}(f) - \sum_{2 \leq |\boldsymbol{\alpha}| \leq k-1} d_{\boldsymbol{\alpha}} h^{|\boldsymbol{\alpha}|} \int_{\Omega_{\mathbf{j}}} D^\alpha f(\mathbf{x}) d\mathbf{x} \right) \end{aligned}$$

Applying Lemma 1.31 completes the proof. □

Note that according to the Sobolev embedding theorem we can dispense with the condition $f \in C(\mathbb{R}^d)$ in Lemma 1.30, Lemma 1.31 and Lemma 1.32 if we require that $f \in W^{k,p}(\mathbb{R}^d)$ for $p > d/k$. The most important part of this result is for $p = 1$ since the norms over the cubes Ω_j sum up to the norm over the whole space $\mathbb{R}^d$.

Corollary 1.33
Let $k \in \mathbb{N}$. Then, there exists a $C > 0$, such that for all $f \in W^{k,1}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ we have

$$\left| \int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x} - h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} f(h\mathbf{j}) \right| \leq C h^k |f|_{W^{k,1}(\mathbb{R}^d)}.$$

In the case $p \neq 1$, the series over the semi-norms $\sum_{\mathbf{j} \in \mathbb{Z}^d} |f|_{W^{k,p}(\Omega_{\mathbf{j}})}$ is not equal to the semi-norm over the whole space $\mathbb{R}^d$. In this case, the condition $f \in W^{k,p}(\mathbb{R}^d)$ would not be sufficient.

1.4.3 An Approximation Scheme

We are now in the situation to combine the last two subsections. We will apply the quadrature formula to the approximation by convolution, to achieve a spatial discretized approximation scheme. Later, this will be our starting point for the numerical scheme we will derive. We start with the application of Corollary 1.33 to the convolution of two functions.

Lemma 1.34
Let $k \in \mathbb{N}$, and $p, q, r \in \mathbb{N}$ such that $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then, there exists a constant $C > 0$, so that

$$\left\| f * g - h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} f(h\mathbf{j}) g(\cdot - h\mathbf{j}) \right\|_{L^p(\mathbb{R}^d)} \leq C h^k \|f\|_{W^{k,q}(\mathbb{R}^d)} \|g\|_{W^{k,r}(\mathbb{R}^d)}$$

holds for all $f \in W^{k,q}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ *and* $g \in W^{k,r}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$.

Proof. Let $\mathbf{x} \in \mathbb{R}^d$. We directly apply Corollary 1.33 and conclude

$$\left| f * g(\mathbf{x}) - h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} f(h\mathbf{j}) g(\mathbf{x} - h\mathbf{j}) \right| \leq ch^k |f(\cdot)g(\mathbf{x} - \cdot)|_{W^{k,1}(\mathbb{R}^d)}.$$

The resulting semi-norm can then be interpreted as a convolution by

$$\begin{aligned}
|f(\cdot)g(\mathbf{x} - \cdot)|_{W^{k,1}(\mathbb{R}^d)} &= \sum_{|\boldsymbol{\alpha}|=k} \int_{\mathbb{R}^d} |D^{\boldsymbol{\alpha}}(f(\mathbf{y})g(\mathbf{x} - \mathbf{y}))| \, d\mathbf{y} \\
&= \sum_{|\boldsymbol{\alpha}|=k} \int_{\mathbb{R}^d} \left| \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} (-1)^{\boldsymbol{\beta}} D^{\boldsymbol{\alpha} - \boldsymbol{\beta}} f(\mathbf{y}) D^{\boldsymbol{\beta}} g(\mathbf{x} - \mathbf{y}) \right| d\mathbf{y} \\
&\leq c \sum_{|\boldsymbol{\alpha}| \leq k} \sum_{|\boldsymbol{\beta}| \leq k} \int_{\mathbb{R}^d} \left| D^{\boldsymbol{\alpha}} f(\mathbf{y}) D^{\boldsymbol{\beta}} g(\mathbf{x} - \mathbf{y}) \right| d\mathbf{y} \\
&= c \left(\sum_{|\boldsymbol{\alpha}| \leq k} |D^{\boldsymbol{\alpha}} f| * \sum_{|\boldsymbol{\beta}| \leq k} |D^{\boldsymbol{\beta}} g| \right) (\mathbf{x}).
\end{aligned}$$

Young's inequality then yields

$$\begin{aligned}
\|f * g - h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} f(h\mathbf{j}) g(\cdot - h\mathbf{j})\|_{L^p(\mathbb{R}^d)} &\leq ch^k \left\| \sum_{|\boldsymbol{\alpha}| \leq k} |D^{\boldsymbol{\alpha}} f| * \sum_{|\boldsymbol{\beta}| \leq k} |D^{\boldsymbol{\beta}} g| \right\|_{L^p(\mathbb{R}^d)} \\
&\leq ch^k \left\| \sum_{|\boldsymbol{\alpha}| \leq k} |D^{\boldsymbol{\alpha}} f| \right\|_{L^q(\mathbb{R}^d)} \left\| \sum_{|\boldsymbol{\beta}| \leq k} |D^{\boldsymbol{\beta}} g| \right\|_{L^r(\mathbb{R}^d)},
\end{aligned}$$

where, using the triangle inequality, the resulting norms can be estimate by

$$\left\| \sum_{|\boldsymbol{\alpha}| \leq k} |D^{\boldsymbol{\alpha}} f| \right\|_{L^q(\mathbb{R}^d)} \leq c(k) \, \|f\|_{W^{k,q}(\mathbb{R}^d)},$$

which completes the proof. □

In the next step we extend this result to the approximation by convolution.

Corollary 1.35
Let $k \in \mathbb{N}$, *and* $p, q, r \in \mathbb{N}$ *such that* $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *Suppose* $\Phi \in L^1(\mathbb{R}^d) \cap W^{k,q}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ *satisfies the moment condition of order* $m \in \mathbb{N}^d$. *Then, there exists a* $C > 0$ *so that*

$$\left\| f - h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} f(h\mathbf{j}) \Phi_\varepsilon(\cdot - h\mathbf{j}) \right\|_{L^p(\mathbb{R}^d)} \leq C \left(\varepsilon^m |f|_{W^{m,p}(\mathbb{R}^d)} + \frac{h^k}{\varepsilon^{k + d(q-1)/q}} \|f\|_{W^{k,r}(\mathbb{R}^d)} \right)$$

holds for all $f \in W^{m,p}(\mathbb{R}^d) \cap W^{k,r}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ *if* $1 \leq p < \infty$ *or* $f \in C^m(\mathbb{R}^d) \cap W^{m,\infty}(\mathbb{R}^d) \cap W^{k,r}(\mathbb{R}^d)$ *if* $p = \infty$.

Proof. Setting $f^{\varepsilon,h} = h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} f(h\mathbf{j}) \Phi_\varepsilon(\cdot - h\mathbf{j})$, we split up the error in a convolution error and a quadrature error

$$\|f - f^{\varepsilon,h}\|_{L^p(\mathbb{R}^d)} \leq \|f - f * \Phi_\varepsilon\|_{L^p(\mathbb{R}^d)} + \|f * \Phi_\varepsilon - f^{\varepsilon,h}\|_{L^p(\mathbb{R}^d)}.$$

The convolution error can be estimated with Theorem 1.28 as

$$\|f - f * \Phi_\varepsilon\|_{L^p(\mathbb{R}^d)} \le c\varepsilon^m |f|_{W^{m,p}(\mathbb{R}^d)}.$$

The quadrature error can be estimated with Lemma 1.34 by

$$\|f * \Phi_\varepsilon - f_\varepsilon^h\|_{L^p(\mathbb{R}^d)} \le ch^k \|f\|_{W^{k,r}(\mathbb{R}^d)} \|\Phi_\varepsilon\|_{W^{k,q}(\mathbb{R}^d)},$$

where the norm of the scaled kernel can be bounded by

$$\begin{aligned} \|\Phi_\varepsilon\|^q_{W^{k,q}(\mathbb{R}^d)} &= \sum_{|\boldsymbol{\alpha}|\le k} \int_{\mathbb{R}^d} |D^{\boldsymbol{\alpha}} \Phi_\varepsilon(\mathbf{x})|^q d\mathbf{x} \\ &= \sum_{|\boldsymbol{\alpha}|\le k} \int_{\mathbb{R}^d} |D^{\boldsymbol{\alpha}} \varepsilon^{-d} \Phi(\mathbf{x}/\varepsilon)|^q d\mathbf{x} \\ &= \sum_{|\boldsymbol{\alpha}|\le k} \varepsilon^{-q(d+|\boldsymbol{\alpha}|)} \int_{\mathbb{R}^d} |(D^{\boldsymbol{\alpha}} \Phi)(\mathbf{x}/\varepsilon)|^q d\mathbf{x} \\ &= \sum_{|\boldsymbol{\alpha}|\le k} \varepsilon^{(1-q)d-q|\boldsymbol{\alpha}|} \int_{\mathbb{R}^d} |(D^{\boldsymbol{\alpha}} \Phi)(\mathbf{x})|^q d\mathbf{x} \\ &\le \varepsilon^{(1-q)d-qk} \|\Phi\|^q_{W^{k,q}(\mathbb{R}^d)}. \end{aligned}$$

Taking the q-th root finishes the proof. □

With the result above, we obtained a discrete approximation of our function $f \in L^p(\mathbb{R}^d)$ given by $f^{\varepsilon,h} = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} f(h\mathbf{j})\Phi_\varepsilon(\cdot - h\mathbf{j})$. This approximation can be used to reconstruct the function f only from given values $f(h\mathbf{j})$ in the $L^p(\mathbb{R}^d)$-sense. It can be used directly for first order partial differential equations of evolution type, see [RW16] and is the basis of, for example, the vortex method [MB02], the smoothed particle hydrodynamic method [Mon89] and other methods for solving first-order symmetric systems [MGR87].

Chapter 2

Fluid Dynamics and Smoothed Particle Hydrodynamics

In this chapter, we want to give the basic fluid dynamic theory we will need including the Euler equations. After that, we want to derive the SPH approximation in general and the spatial discretized SPH Euler system.
From now on, we use the spatial domain $\Omega \subseteq \mathbb{R}^d$ and a time interval $[0, T] \subseteq \mathbb{R}$ for a $T > 0$. The quantities of interest, which describe the fluid, are given by the velocity $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^d$, the density $\rho(\mathbf{x}, t) \in \mathbb{R}$ and the pressure $p(\mathbf{x}, t) \in \mathbb{R}$, where $\mathbf{x} \in \Omega$ denotes the position and $t \in [0, T]$ the time. To describe these quantities, there are two different frameworks. The first one is to describe the fluid in a fixed, global coordinate system. This means, all quantities are measured at a fixed position $\mathbf{x}$ and at a time t. This description is called the *Eulerian* form.
In this work, we will mainly concentrate on the second possible framework, the description in the *Lagrangian* form. In this framework, quantities are measured in a local, time depended coordinate system which moves along the flow of the fluid. In our particular case, we are interested in the so-called particle trajectories, or particles, which represent a finite volume of the fluid and are moving along the flow.
In the first part, we will formally introduce these particle trajectories, before we will derive the Euler equations with some of their properties. Then, we will give the basic ideas of the SPH approximation and its application to the Euler equations. Finally, we derive the existence and uniqueness of the solution of the SPH approximation applied to the Euler equations.

2.1 Particle Trajectories

An important construction we will need to derive the SPH-method is the particle trajectory mapping for a given fluid velocity $\mathbf{u}$.

Definition 2.1
Let $\mathbf{u} \in C(\Omega \times [0, T])^d$ be a given velocity. The mapping $\mathbf{X}(\cdot, t_0, \cdot) : \Omega \times [0, T] \to \Omega$ is called a flow, if for the time $t_0 \in [0, T]$ it satisfies

$$\begin{aligned} \partial_t \mathbf{X}(\mathbf{x}, t_0, t) &= \mathbf{u}(\mathbf{X}(\mathbf{x}, t_0, t), t), && \mathbf{x} \in \Omega,\ t \in [0, T], \\ \mathbf{X}(\mathbf{x}, t_0, t_0) &= \mathbf{x}, && \mathbf{x} \in \Omega. \end{aligned}$$

For a fixed $\mathbf{x} \in \Omega$, the mapping $\mathbf{X}(\mathbf{x}, t_0, \cdot) : I \to \Omega$ is called particle trajectory of the particle starting at position $\mathbf{x}$ at starting time t_0. If the starting time $t_0 = 0$ we will write $\mathbf{X}(\mathbf{x}, t)$ instead of $\mathbf{X}(\mathbf{x}, 0, t)$.

According to this definition, $\mathbf{X}(\mathbf{x}, t_0, t)$ is the position of a fluid particle at the time t, initially placed at the point $\mathbf{x}$ at time t_0. The particle is moving with velocity $\mathbf{u}$, which means that the fluid velocity $\mathbf{u}$ is tangent to the particle trajectories. Note that the flow $\mathbf{X}$ is continuous as a function of the starting position $\mathbf{x}$ since we assume the velocity $\mathbf{u}$ to be continuous.

Remark 2.2
With the theorem of Picard-Lindelöff, we have the existence of the particle trajectory. If $u_j \in C([0,T]; C^p(\Omega))$ *for a* $p \in \mathbb{N}_0$ *and* $1 \leq j \leq d$, *we have* $X_j \in C^1([0,T]; C^p(\Omega))$.
Moreover, $\mathbf{X}(\cdot, t)$ *is a* C^p*-diffeomorphism in this case, which mean that* $\mathbf{X}(\cdot, t)$ *is bijective and* $\mathbf{X}(\cdot, t)$ *as well as its inverse are p-times continuously differentiable.*

The definition of the particle trajectory mapping also allows us to move the particles back in time. A direct conclusion of this fact is following lemma.

Lemma 2.3
For $t, t_0 \in [0,T]$ *and all* $\mathbf{x} \in \Omega$ *we have*

$$\mathbf{X}(\mathbf{X}(\mathbf{x}, t, t_0), t_0, t) = \mathbf{x}.$$

For our purpose, it is very important to know the time derivative of an arbitrary function $f \in C^1([0,T]; C^1(\Omega))$ along the particle trajectories. Since the particle position depends on the time, using the product rule, we derive

$$\begin{aligned}\frac{d}{dt} f(\mathbf{X}(\mathbf{x}, t_0, t), t) &= \partial_t f(\mathbf{X}(\mathbf{x}, t_0, t), t) + (\partial_t \mathbf{X}(\mathbf{x}, t_0, t) \cdot \nabla) f(\mathbf{X}(\mathbf{x}, t_0, t), t) \\ &= (\partial_t + u(\mathbf{X}(\mathbf{x}, t_0, t), t) \cdot \nabla) f(\mathbf{X}(\mathbf{x}, t_0, t), t).\end{aligned}$$

for $\mathbf{x} \in \Omega$, $t \in [0,T]$.

Definition 2.4
The material derivative of a function $f \in C^1([0,T]; C^1(\Omega))$ *with respect to a velocity* $\mathbf{u}$ *is given by*

$$\frac{Df}{Dt} := \partial_t f + (u \cdot \nabla) f.$$

Note, that we will frequently use the identity

$$\frac{Df}{Dt}(\mathbf{X}(\mathbf{x}, t_0, t), t) = \frac{d}{dt} f(\mathbf{X}(\mathbf{x}, t_0, t), t).$$

Next, we want to study the transformation of a given set moving with the flow. For this, we define the Jacobi determinant of the transformation given by the flow.

Definition 2.5
The Jacobian $J : \Omega \times [0,T] \to \mathbb{R}$ *of the particle trajectory transformation is defined by*

$$J(\mathbf{x}, t) := \det\left(\partial_i X_j(\mathbf{x}, t)\right)_{1 \leq i,j \leq d} = \det\left(\partial_i X_j(\mathbf{x}, t)\right)$$

for $\mathbf{x} \in \Omega$ *and* $t \in [0,T]$.

Since $\mathbf{X}$ is continuous differentiable in time, so is the Jacobi determinant J. The Jacobian J itself satisfies the following ODE.

Proposition 2.6
Let $\mathbf{u} \in C^1([0,T]; C^1(\Omega))^d$ *be a smooth velocity field and let* $\mathbf{X}$ *be the corresponding particle trajectory mapping. Then, the Jacobi determinant satisfies the ordinary differential equation*

$$\partial_t J(\mathbf{x}, t) - J(\mathbf{x}, t) \nabla \cdot \mathbf{u}(\mathbf{X}(\mathbf{x}, t), t) = 0, \qquad \mathbf{x} \in \mathbb{R}^d,\ t \in I, \tag{2.1}$$

$$J(\mathbf{x}, 0) = 1, \qquad \mathbf{x} \in \mathbb{R}^d. \tag{2.2}$$

Proof. For a matrix $A = (a_{i,j}) \in \mathbb{R}^{d\times d}$ the Leibniz formula for its determinant yields

$$\det(A) = \sum_{\sigma\in S_d} \operatorname{sgn}(\sigma) \prod_{i=1}^{d} a_{\sigma(i),i},$$

where S_d denotes the symmetric group with d elements. Applying this formula to the Jacobi determinant gives

$$J(\mathbf{x},t) = \det\left(\partial_i X_j(\mathbf{x},t)\right) = \sum_{\sigma\in S_d} \operatorname{sgn}(\sigma) \prod_{i=1}^{d} \partial_{\sigma(i)} X_i(\mathbf{x},t). \tag{2.3}$$

Taking the time derivative of J we find

$$\begin{aligned}\partial_t J(\mathbf{x},t) &= \sum_{\sigma\in S_d} \operatorname{sgn}(\sigma)\partial_t \left(\prod_{i=1}^{d} \partial_{\sigma(i)} X_i(\mathbf{x},t)\right)\\ &= \sum_{\sigma\in S_d} \operatorname{sgn}(\sigma) \sum_{j=1}^{d} \left(\prod_{\substack{i=1\\ i\neq j}}^{d} \partial_{\sigma(i)} X_i(\mathbf{x},t)\right) \partial_t\partial_{\sigma(j)} X_j(\mathbf{x},t).\end{aligned} \tag{2.4}$$

The theorem of Schwarz then yields

$$\partial_t\partial_{\sigma(j)} X_j(\mathbf{x},t) = \partial_{\sigma(j)}\left(u_j(\mathbf{X}(\mathbf{x},t),t)\right) = \sum_{k=1}^{d} \partial_k u_j(\mathbf{X}(\mathbf{x},t),t)\partial_{\sigma(j)} X_k(\mathbf{x},t),$$

which we can insert in (2.4) to arrive at

$$\begin{aligned}\partial_t J(\mathbf{x},t) &= \sum_{\sigma\in S_d} \operatorname{sgn}(\sigma) \sum_{j,k=1}^{d} \left(\prod_{\substack{i=1\\ i\neq j}}^{d} \partial_{\sigma(i)} X_i(\mathbf{x},t)\right) \partial_k u_j(\mathbf{X}(\mathbf{x},t),t)\partial_{\sigma(j)} X_k(\mathbf{x},t)\\ &= \sum_{j,k=1}^{d} \partial_k u_j(\mathbf{X}(\mathbf{x},t),t) \sum_{\sigma\in S_d} \operatorname{sgn}(\sigma) \left(\prod_{\substack{i=1\\ i\neq j}}^{d} \partial_{\sigma(i)} X_i(\mathbf{x},t)\right) \partial_{\sigma(j)} X_k(\mathbf{x},t).\end{aligned}$$

In the case $j = k$ the second sum is equal to J according to (2.3). For $j \neq k$, we have the determinant of a matrix of rank $d-1$, which is zero. Thus, the second sum can be simplified to

$$\sum_{\sigma\in S_d} \operatorname{sgn}(\sigma) \left(\prod_{\substack{i=1\\ i\neq j}}^{d} \partial_{\sigma(i)} X_i(\mathbf{x},t)\right) \partial_{\sigma(j)} X_k(\mathbf{x},t) = \begin{cases} J(\mathbf{x},t) & \text{if } j = k,\\ 0 & \text{if } j \neq k.\end{cases}$$

This finally gives

$$\partial_t J(\mathbf{x},t) = J(\mathbf{x},t) \sum_{j=1}^{d} \partial_j u_j(\mathbf{X}(\mathbf{x},t),t) = J(\mathbf{x},t)\nabla\cdot\mathbf{u}(\mathbf{X}(x,t),t).$$

The identity $J(\mathbf{x},0) = 1$ follows directly from Definition 2.1 and Definition 2.5. □

We see that the time derivative of the Jacobi determinant depends on the gradient of the velocity field $\mathbf{u}$. A simple conclusion is that the Jacobi determinant can be bounded by the derivatives of $\mathbf{u}$, assuming the velocity is sufficiently smooth.

Lemma 2.7
Suppose that for each $t \in [0,T]$ the map $\mathbf{X}(\cdot,t) : \Omega \to \mathbf{X}(\Omega,t)$ is a C^1-diffeomorphism. Then, we have $J(\mathbf{x},t) > 0$ for every $\mathbf{x} \in \Omega$ and every $t \in [0,T]$. Moreover, if $\mathbf{u} \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))$, then there exists a constant $C > 0$ such that

$$e^{-Ct} \leq J(\mathbf{x},t) \leq e^{Ct}$$

for all $\mathbf{x} \in \Omega$ and $t \in I$.

Proof. Since the Jacobian J is the solution of the ordinary differential equation given in (2.1)-(2.2), J can be written as

$$J(\mathbf{x},t) = \exp\left(\int_0^t \nabla\cdot\mathbf{u}(\mathbf{X}(\mathbf{x},s),s)ds\right),$$

which immediately implies $J(\mathbf{x},t) > 0$.
Moreover, if $\mathbf{u} \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))$, there exists a $C > 0$ such that

$$\int_0^t \nabla\cdot\mathbf{u}(\mathbf{X}(\mathbf{x},s),s)ds \leq \int_0^t \|\mathbf{u}(\cdot,s)\|_{W^{1,\infty}(\mathbb{R}^d)}ds \leq \|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))}t \leq Ct.$$

However, using the same arguments yields

$$\int_0^t \nabla\cdot\mathbf{u}(\mathbf{X}(\mathbf{x},s),s)ds \geq -\int_0^t \|\mathbf{u}(\cdot,s)\|_{W^{1,\infty}(\mathbb{R}^d)}ds \geq -\|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))}t \geq -Ct,$$

which completes the proof. □

Now, suppose we have a set $W \subseteq \Omega$ at time $t = 0$. Let $W(t) := \{\mathbf{y} \in \Omega \,|\, \mathbf{y} = X(\mathbf{x},t), \mathbf{x} \in W\} = X(W,t)$ be the set W transported with the flow $\mathbf{X}$ up to a time t. Then, using the transformation theorem, we can calculate the volume of the moved set by

$$\text{vol}(W(t)) = \int_{W(t)} d\mathbf{x} = \int_W J(\mathbf{x},t)d\mathbf{x},$$

which means that the Jacobi determinant gives us the change of the volume of W while it moves with the flow. Another useful property is the transport theorem.

Theorem 2.8 (Transport Theorem)
Let $W \subseteq \Omega$ be an open domain with a smooth boundary, and let $1 \leq p,q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose the velocity field $\mathbf{u} \in C([0,T];W^{1,p}(\Omega) \cap C^1(\Omega))$ and $f \in C([0,T];W^{1,q}(\Omega)) \cap C^1([0,T];L^1(\Omega))$. Then

$$\frac{d}{dt}\int_{W(t)} f(\mathbf{x},t)d\mathbf{x} = \int_{W(t)} \partial_t f(\mathbf{x},t) + \nabla\cdot(f\mathbf{u})(\mathbf{x},t)d\mathbf{x},$$

for all $t \in [0,T]$.

Proof. Let $t \in I$. First, we notice that the given integrals exist because $f(\cdot,t) \in L^1(\Omega) \cap W^{1,q}(\Omega)$ and $\mathbf{u}(\cdot,t) \in W^{1,p}(\Omega)$. Hence the transformation theorem yields

$$\begin{aligned}\frac{d}{dt}\int_{W(t)} f(\mathbf{x},t)d\mathbf{x} &= \frac{d}{dt}\int_W f(\mathbf{X}(\mathbf{x},t),t)J(\mathbf{x},t)d\mathbf{x} \\ &= \int_W \left(\partial_t f + \mathbf{u}\cdot\nabla f + f\nabla\cdot\mathbf{u}\right)(\mathbf{X}(\mathbf{x},t),t)J(\mathbf{x},t)d\mathbf{x}\end{aligned}$$

where we swaped integration and differentation due to the dominated convergence theorem and used equality (2.1) in the last line. Using the product rule, we have

$$\begin{aligned}\frac{d}{dt}\int\limits_{W(t)} f(\mathbf{x},t)d\mathbf{x} &= \int\limits_{W} \left(\partial_t f + \nabla\cdot(f\mathbf{u})\right)(\mathbf{X}(\mathbf{x},t),t)J(\mathbf{x},t)d\mathbf{x} \\ &= \int\limits_{W(t)} \partial_t f(\mathbf{x},t) + \nabla\cdot(f\mathbf{u})(\mathbf{x},t)d\mathbf{x},\end{aligned}$$

where we used the transformation theorem again. □

2.2 The Euler Equations

In the following, suppose $\Omega = \mathbb{R}^d$. Compressible flows of homogenous, non-viscous fluids without external forces in $\mathbb{R}^d$ are solutions of the system of equations

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla\cdot p, \tag{2.5}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\mathbf{u} \tag{2.6}$$

on $\mathbb{R}^d \times I$ and

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \rho|_{t=0} = \rho_0 \tag{2.7}$$

on $\mathbb{R}^d$, where (2.5) describes the conservation of the momentum of the fluid and the continuity equation (2.6) describes the conservation of mass. These equations are called the Euler equations. To complete this system of equations, we need an additional equation for the pressure p, also called an equation of state. In our case, we will look at a barotropic equation of state, where the pressure p only depends on the density ρ, i.e. $p = \frac{1}{\gamma}\rho^\gamma$ for a $\gamma \geq 1$. Note that there exists a infinite number of such equations, including other variables like, e.g. the temperature.
Since the Euler equations are an example of non-linear hyperbolic conservation laws, it is a well-known fact that, even with smooth starting values, the solution can develop shocks and can become discontinuous after a finite period of time, see, for example, [GR96]. However, the theory of quasi-linear systems states that, under appropriate initial conditions, there exists a unique local solution.

Theorem 2.9 (Local existence and uniqueness)
Let $\rho_0^{(\gamma-1)/2} \in H^s(\mathbb{R}^d)$ for a $\gamma > 1$ and $\mathbf{u}_0 \in H^s(\mathbb{R}^d)^d$ for an $s > d/2+1$. Then, there exists a $T > 0$, such that (2.5) - (2.7) has a unique solution $(\rho, \mathbf{u})$ with

$$(\rho^{(\gamma-1)/2}, \mathbf{u}) \in \left[C^1\left(0,T;H^{s-1}(\mathbb{R}^d)\right) \cap C\left(0,T;H^s(\mathbb{R}^d)\right)\right]^{d+1}.$$

Proof. The proof of this theorem is based on a special symmetrization of Euler's equations (2.5) and (2.6) given in [MUK87], which can be extended to arbitrary space dimensions. The existence of a unique local solution of quasilinear hyperbolic systems can be found in [Kat75]. □

Note that this result explicitly allows solutions with compact support. Similar results are possible if we consider an initial density ρ_0, such that $(\rho_0 - c)^{(\gamma-1)/2} \in H^s(\mathbb{R}^d)$ for a $c \in \mathbb{R}$, which means that the density does not vanish at infinity but converges to some constant c.
Having this result, we will assume from now on that the solution of the Euler equations exists up to a maximum time $T > 0$. For more details on the solvability of Euler's equations, i.e. weak solutions and non-existence of smooth solutions, see, for example, [NS04] or [CW02].
Next, we want to discuss some important properties of Euler's equations. The first property is the

conservation of mass, which is a simple conclusion of the continuity equation. The mass at time $t \in I$ inside a set $W \subset \Omega$ is defined by

$$M(W) = \int_W \rho(\mathbf{x}, t) d\mathbf{x}.$$

A useful fact is that the mass of the set W at time $t = 0$ is equal to the mass of the set $W(t)$ at time $t \in I$, which means that the mass inside of a set does not change while moving with the flow.

Lemma 2.10
Within the system of Euler's equations mass is conserved, i.e.

$$\frac{d}{dt} M(W(t)) = 0.$$

Proof. Suppose $W \subset \Omega$. Then, the transport Theorem 2.8 yields

$$\frac{d}{dt} \int_{W(t)} \rho(\mathbf{x}, t) d\mathbf{x} = \int_{W(t)} \partial_t \rho(\mathbf{x}, t) + \nabla \cdot (\rho \mathbf{u})(\mathbf{x}, t) d\mathbf{x} = 0$$

for every $t \in [0, T]$, where we used equation (2.6). □

Another important property is the conservation of energy. To derive this conservation property, we first need two auxiliary results.
The first lemma is a connection of the Jacobi determinant and the density ρ. This relation will be a fundamental increment of our numerical scheme in Section 2.3.

Lemma 2.11
Let $\mathbf{u}$ and ρ be a solution of the Euler equations. Then

$$\rho(\mathbf{X}(\mathbf{x}, t), t) J(\mathbf{x}, t) = \rho(\mathbf{x}, 0)$$

for all $\mathbf{x} \in \Omega$, $t \in I$.

Proof. Let $W \subseteq \Omega$ be arbitrary. Using the conservation of mass, we have

$$\int_W \rho(\mathbf{x}, 0) d\mathbf{x} = \int_{W(t)} \rho(\mathbf{x}, t) d\mathbf{x} = \int_W \rho(\mathbf{X}(\mathbf{x}, t), t) J(\mathbf{x}, t) d\mathbf{x}$$

Since W is arbitrary, this implies the stated equality. □

The important property of this relation is that we can calculate the density at the moved particle position $\mathbf{X}(\mathbf{x}, t)$ even without knowing this particle position. This will help us in the numerical calculations later on. Another property is that the density remains positive and bounded if the initial density ρ_0 is positive and bounded according to Lemma 2.7. Furthermore, we can conclude an alternative form of the transport theorem.

Theorem 2.12 (Alternative Transport Theorem)
Let $W \subseteq \Omega$ be an open domain with a smooth boundary, and let $(\rho, \mathbf{u})$ be the solution of the Euler equations from Theorem 2.9. Suppose that $f \in C^1([0, T]; C^1(\Omega))$. Then, for each $t \in [0, T]$, we have

$$\frac{d}{dt} \int_{W(t)} \rho(\mathbf{x}, t) f(\mathbf{x}, t) d\mathbf{x} = \int_{W(t)} \rho(\mathbf{x}, t) \frac{\mathrm{D} f}{\mathrm{D} t}(\mathbf{x}, t) d\mathbf{x}.$$

Proof. With the transformation theorem and Lemma 2.11, we have

$$\frac{d}{dt}\int\limits_{W(t)} \rho(\mathbf{x},t)f(\mathbf{x},t)d\mathbf{x} = \frac{d}{dt}\int\limits_{W} \rho_0(\mathbf{x})f(\mathbf{X}(\mathbf{x},t),t)d\mathbf{x} = \int\limits_{W} \rho_0(\mathbf{x})\frac{d}{dt}f(\mathbf{X}(\mathbf{x},t),t)d\mathbf{x}.$$

Rewriting the time derivative as the material derivative yields

$$\frac{d}{dt}\int\limits_{W(t)} \rho(\mathbf{x},t)f(\mathbf{x},t)d\mathbf{x} = \int\limits_{W} \rho_0(\mathbf{x})\frac{\mathrm{D}f}{\mathrm{D}t}(\mathbf{X}(\mathbf{x},t),t)d\mathbf{x} = \int\limits_{W(t)} \rho(\mathbf{x},t)\frac{\mathrm{D}f}{\mathrm{D}t}(\mathbf{x},t)d\mathbf{x},$$

which completes the proof. □

To derive the conservation of energy, we have to define the internal energy e of the system. Even if the internal energy is not a quantity we need for the Euler system itself, we will need it to define the energy of the system. The internal energy of the Euler equations system is given by the additional partial differential equation

$$\frac{\mathrm{D}e}{\mathrm{D}t} = -\frac{p}{\rho}\nabla\cdot\mathbf{u}, \quad \text{on } \mathbb{R}^d \times I, \tag{2.8}$$

$$e|_{t=0} = e_0, \quad \text{on } \mathbb{R}^d. \tag{2.9}$$

Given the internal energy, the energy of Euler's equations is defined by

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\int\limits_{\mathbb{R}^d} \rho(\mathbf{x},t)\|\mathbf{u}(\mathbf{x},t)\|_2^2 d\mathbf{x} + \int\limits_{\mathbb{R}^d} \rho(\mathbf{x},t)e(\mathbf{x},t)d\mathbf{x}\\ &= \mathcal{E}_{\text{kin}}(t) + \mathcal{E}_{\text{pot}}(t),\end{aligned}$$

where the first integral defines the kinetic energy $\mathcal{E}_{\text{kin}}$ of the Euler equations and the second integral the potential energy $\mathcal{E}_{\text{pot}}$.

Lemma 2.13
Within the system of Euler's equations energy is conserved, i.e.

$$\frac{d}{dt}\mathcal{E}(t) = 0.$$

Proof. Using Theorem 2.12, the time derivative of the energy yields

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}\left(\frac{1}{2}\int\limits_{\mathbb{R}^d} \rho(\mathbf{x},t)\|\mathbf{u}(\mathbf{x},t)\|_2^2 d\mathbf{x} + \int\limits_{\mathbb{R}^d} \rho(\mathbf{x},t)e(\mathbf{x},t)d\mathbf{x}\right)\\ &= \int\limits_{\mathbb{R}^d} \rho(\mathbf{x},t)\left(\mathbf{u}(\mathbf{x},t)\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(\mathbf{x},t) + \frac{\mathrm{D}e}{\mathrm{D}t}(\mathbf{x},t)\right)d\mathbf{x},\end{aligned}$$

where we used that $\frac{\mathrm{D}\|\mathbf{u}\|_2^2}{\mathrm{D}t} = 2\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$. Inserting equation (2.5) and (2.8), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= -\int\limits_{\mathbb{R}^d} \rho(\mathbf{x},t)\left(\mathbf{u}(\mathbf{x},t)\cdot\frac{1}{\rho(\mathbf{x},t)}\nabla p(\mathbf{x},t) + \frac{p(\mathbf{x},t)}{\rho(\mathbf{x},t)}\nabla\cdot\mathbf{u}(\mathbf{x},t)\right)d\mathbf{x}\\ &= -\int\limits_{\mathbb{R}^d} \left(\mathbf{u}(\mathbf{x},t)\cdot\nabla p(\mathbf{x},t) + p(\mathbf{x},t)\nabla\cdot\mathbf{u}(\mathbf{x},t)\right)d\mathbf{x}\\ &= -\int\limits_{\mathbb{R}^d} \nabla\cdot(p\mathbf{u})(\mathbf{x},t)d\mathbf{x} = 0,\end{aligned}$$

where we used the product rule in the last line. □

Obviously, the theory given here is only a small part of the existing theory about the Euler equations and general nonlinear hyperbolic equations. Much more details, including the theory of shocks, can be found in, for example, [NS04] or [GR96].

2.3 The SPH-Approximation

The SPH method was first introduced by Lucy [Luc77] and by Gingold and Monoghan [GM77] in 1977. It is a computational method used for simulating the mechanics of continuous media, primarily for fluid flows. Basically, the idea of the method is to consider the fluid as an ensemble of particles and to calculate the trajectories of these particles by a special, kernel-based approximation of the underlying equations.
First, we need a kernel function Φ, which is an even and at least one-time continuously differentiable function, and its scaled version $\Phi_\varepsilon = \varepsilon^{-d}\Phi(\cdot/\varepsilon)$ with a smoothing parameter $\varepsilon > 0$. Finally, we need a spatial discretization parameter $h > 0$, with which we decompose the space $\mathbb{R}^d$ into cubes with edge length h and centers $h\mathbf{j}$ for $\mathbf{j} \in \mathbb{Z}^d$, as we did in Section 1.4.2.
Given a flow $\mathbf{X}$ with its associated Jacobian J, the SPH approximation of an arbitrary function f can be divided into three steps: An approximation via convolution with a scaled kernel function, a transformation of the integral using the particle trajectories and an approximation of the integral via a quadrature step. Note that the second step is also an approximation if the particle trajectories are unknown. Thus, for a function f, the SPH approximation can be derived by

$$\begin{aligned} f(\mathbf{x},t) \approx& \int_{\mathbb{R}^d} f(\mathbf{y},t)\Phi_\varepsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} \\ =& \int_{\mathbb{R}^d} J(\mathbf{y},t) f(\mathbf{X}(\mathbf{y},t),t)\Phi_\varepsilon(\mathbf{x}-\mathbf{X}(\mathbf{y},t))d\mathbf{y} \\ \approx& h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} J(h\mathbf{j},t) f(\mathbf{X}(h\mathbf{j},t),t)\Phi_\varepsilon(\mathbf{x}-\mathbf{X}(h\mathbf{j},t))d\mathbf{y} \\ :=& [f]^{\varepsilon,h}(\mathbf{x},t), \end{aligned} \tag{2.10}$$

where the brackets $[\cdot]^{\varepsilon,h}$ denotes the SPH approximation of a function depending on the parameters h and ε. With this approximation method, we can build an approximation on f with the information of f along the particle trajectories.
Note that the choice of the kernel function Φ is of great importance, since it is, besides the two parameters ε and h, the only ingredient we can choose in this approximation scheme. The choice of the right kernel is frequently discussed, see for example [LL10],[LR14] or [DA12], and there exists several extensions for the use of the kernels and the method itself, see i.e. [Mon89] or [GGRDC10]. Nevertheless, there exists hardly any analysis of these extensions. For this reason, we will neglect these extensions and concentrate on the simplest form of the SPH approximation.
Finally note that this approximation scheme is very similar to the approximation scheme given at the end of Section 1.4. In (2.10), we added an additional step by transforming the integral using the particle trajectories. However, assuming that the velocity $\mathbf{u}$ of the flow is given, the particle trajectories $\mathbf{X}$ as well as the Jacobian J are known and hence, the error estimates from Corollary 1.35 can be adopted. This is the basis of several particle methods, see, for example, [MGR87], which are partly well studied. However, if the velocity $\mathbf{u}$ is unknown, the analysis of particle methods becomes much more complicated.
We will now derive an approximation scheme for the Euler equations. A consequence of the continuity equation is that $\rho(\mathbf{X}(\mathbf{x},t),t)J(\mathbf{x},t) = \rho_0(\mathbf{x})$. Writing this as an equation for the Jacobi determinant, the approximation (2.10) becomes

$$[f]^{\varepsilon,h}(\mathbf{x},t) = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \frac{\rho_0(h\mathbf{j})}{\rho(\mathbf{X}(h\mathbf{j},t),t)} f(\mathbf{X}(h\mathbf{j},t),t)\Phi_\varepsilon(\mathbf{x}-\mathbf{X}(h\mathbf{j},t)), \tag{2.11}$$

which is the basic approximation rule for the SPH method approximating a system containing the continuity equation.

Discrete Continuity Equation

One of the main ideas in our approximation lies in the approximation of the density and hence in the approximation of the continuity equation. There are two fundamental ways to give an approximation of the density. The first one is to approximate the density itself with equation (2.11), where we have to check that this approximation satisfies a kind of discrete continuity equation. The second one is to approximate the right-hand side of the continuity equation to derive a differential equation system for the density.
The discretization scheme we want to investigate later on is based upon the first option. Setting $f = \rho$ in (2.11), we note that the density at the moved particle positions cancels out and we end up with

$$[\rho]^{\varepsilon,h}(\mathbf{x},t) = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\Phi_\varepsilon(\mathbf{x} - \mathbf{X}(h\mathbf{j},t)),$$

which means that we only need the information of the density at the initial particle positions. Thus, we have an explicit formula for the approximated density instead of an additional differential equation we would have to solve.
We have to check how well this approximation satisfies the continuity equation. Let us recall that we already used the continuity equation to build this approximation since we used the equality $\rho(\mathbf{X}(\mathbf{x},t),t)J(\mathbf{x},t) = \rho_0(\mathbf{x})$. We check the compatibility of this approximation with the continuity equation. The time derivative of the approximation yields

$$\begin{aligned}\partial_t[\rho]^{\varepsilon,h}(\mathbf{x},t) &= h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}(\mathbf{X}(h\mathbf{j},t),t)\cdot\nabla\Phi_\varepsilon(\mathbf{x} - \mathbf{X}(h\mathbf{j},t)) \\ &= h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} J(h\mathbf{j},t)\rho(\mathbf{X}(h\mathbf{j},t),t)\mathbf{u}(\mathbf{X}(h\mathbf{j},t),t)\cdot\nabla\Phi_\varepsilon(\mathbf{x} - \mathbf{X}(h\mathbf{j},t)) \\ &= \nabla[\rho\mathbf{u}]^{\varepsilon,h}(\mathbf{x},t),\end{aligned}$$

which is a form of approximated discrete continuity equation.
Note that this approximation scheme of the density has not to be more accurate than the approximation mentioned in the second option. But an explicit formula will make the error analysis easier and is simpler in computations.

Discrete Momentum Equation

For the discretization of the momentum equation, we have to discretize the right-hand side of (2.5). For this, we have several possibilities. In our case, for the sake of symmetry, we write the right-hand side as

$$-\frac{1}{\rho}\nabla p = -\nabla\frac{p}{\rho} - \frac{p}{\rho^2}\nabla\rho \approx -\nabla\left[\frac{p}{\rho}\right]^{\varepsilon,h} - \frac{p}{\rho^2}\nabla[\rho]^{\varepsilon,h},$$

where the first part can be written as

$$\nabla\left[\frac{p}{\rho}\right]^{\varepsilon,h}(\mathbf{x},t) = -h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\frac{p(\mathbf{X}(h\mathbf{j},t),t)}{\rho^2(\mathbf{X}(h\mathbf{j},t),t)}\nabla\Phi_\varepsilon(\mathbf{x} - \mathbf{X}(h\mathbf{j},t))$$

and the second part as

$$\frac{p(\mathbf{x},t)}{\rho^2(\mathbf{x},t)}\nabla[\rho]^{\varepsilon,h}(\mathbf{x},t) = -h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\frac{p(\mathbf{x},t)}{\rho^2(\mathbf{x},t)}\nabla\Phi_\varepsilon(\mathbf{x} - \mathbf{X}(h\mathbf{j},t)).$$

This symmetric version of the right-hand side of (2.5) ensures that the energy of the SPH system will be conserved. Thus, we obtain

$$\begin{aligned}\frac{d}{dt}\mathbf{u}(\mathbf{X}(\mathbf{x},t),t) &= -\frac{1}{\rho(\mathbf{X}(\mathbf{x},t),t)}\nabla p(\mathbf{X}(\mathbf{x},t),t)\\ &\approx -h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\left(\frac{p(\mathbf{X}(h\mathbf{j},t),t)}{\rho^2(\mathbf{X}(h\mathbf{j},t),t)} + \frac{p(\mathbf{X}(\mathbf{x},t),t)}{\rho^2(\mathbf{X}(\mathbf{x},t),t)}\right)\nabla\Phi_\varepsilon(\mathbf{X}(\mathbf{x},t)-\mathbf{X}(h\mathbf{j},t)).\end{aligned}$$

We also have to mention here that we can not prove that this is the best approximation of the right-hand side of the momentum equation. Nevertheless, it will simplify the analysis later on.

Discrete Internal Energy

Even though we do not require the internal energy to set up the Euler equations system, we will need a discrete internal energy to define the energy of the SPH system. Once more, we have several possibilities to discretize the equation of the internal energy. First of all, we notice that

$$\frac{\mathrm{D}e}{\mathrm{D}t}(\mathbf{x},t) = -\frac{p(\mathbf{x},t)}{\rho(\mathbf{x},t)}\nabla\cdot\mathbf{u}(\mathbf{x},t) = \frac{p(\mathbf{x},t)}{\rho^2(\mathbf{x},t)}\frac{\mathrm{D}\rho}{\mathrm{D}t}(\mathbf{x},t).$$

With the approximated density $[\rho]^{h,\varepsilon}$, we can approximate the material derivative of ρ by

$$\begin{aligned}\frac{\mathrm{D}\rho}{\mathrm{D}t}(\mathbf{x},t) &\approx \frac{\mathrm{D}[\rho]^{\varepsilon,h}}{\mathrm{D}t}(\mathbf{x},t)\\ &= (\mathbf{u}(\mathbf{x},t))\cdot\nabla)\,[\rho]^{\varepsilon,h}(\mathbf{x},t) + \partial_t[\rho]^{\varepsilon,h}(\mathbf{x},t)\\ &= h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\,(\mathbf{u}(\mathbf{x},t)) - \mathbf{u}(\mathbf{X}(h\mathbf{j},t),t))\cdot\nabla\Phi_\varepsilon(\mathbf{X}(\mathbf{x},t)-\mathbf{X}(h\mathbf{j},t)).\end{aligned}$$

Inserting this approximation leads us to

$$\begin{aligned}&\frac{d}{dt}e(\mathbf{X}(\mathbf{x},t),t)\\ &\quad\approx \frac{p(\mathbf{X}(\mathbf{x},t),t)}{\rho^2(\mathbf{X}(\mathbf{x},t),t)}h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\,(\mathbf{u}(\mathbf{X}(\mathbf{x},t),t)) - \mathbf{u}(\mathbf{X}(h\mathbf{j},t),t))\cdot\nabla\Phi_\varepsilon(\mathbf{X}(\mathbf{x},t)-\mathbf{X}(h\mathbf{j},t)).\end{aligned}$$

The SPH System for the Euler Equations

For every $\mathbf{j}\in\mathbb{Z}^d$, we will denote by $\mathbf{x}_\mathbf{j}^{\varepsilon,h} : [0,\infty[\to\mathbb{R}^d$ the approximated particle position and by $\mathbf{u}_\mathbf{j}^{\varepsilon,h} : [0,\infty[\to\mathbb{R}^d$ the approximated particle velocity. Taking the approximations for Euler's equations above, the SPH system of the Euler equations is given by the following system of ordinary differential equations:

$$\frac{d}{dt}\mathbf{x}_\mathbf{j}^{\varepsilon,h}(t) = \mathbf{u}_\mathbf{j}^{\varepsilon,h}(t), \tag{2.12}$$

$$\frac{d}{dt}\mathbf{u}_\mathbf{j}^{\varepsilon,h}(t) = -h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\left(\frac{p_\mathbf{k}^{\varepsilon,h}(t)}{\rho_\mathbf{k}^{\varepsilon,h}(t)^2} + \frac{p_\mathbf{j}^{\varepsilon,h}(t)}{\rho_\mathbf{j}^{\varepsilon,h}(t)^2}\right)\nabla\Phi_\varepsilon(\mathbf{x}_\mathbf{j}^{\varepsilon,h}(t)-\mathbf{x}_\mathbf{k}^{\varepsilon,h}(t)) \tag{2.13}$$

for all $\mathbf{j}\in\mathbb{Z}^d$ and $t\in]0,\infty[$ and

$$\mathbf{x}_\mathbf{j}^{\varepsilon,h}(0) = h\mathbf{j}, \qquad \mathbf{u}_\mathbf{j}^{\varepsilon,h}(0) = u_0(h\mathbf{j}) \tag{2.14}$$

for all $\mathbf{j} \in \mathbb{Z}^d$, where $\rho_{\mathbf{j}}^{\varepsilon,h}(t) := \rho^{\varepsilon,h}(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t), t)$ with

$$\rho^{\varepsilon,h}(\mathbf{x}, t) = h^d \sum_{\mathbf{k} \in \mathbb{Z}^d} \rho_0(h\mathbf{k}) \Phi_\varepsilon(\mathbf{x} - \mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)), \tag{2.15}$$

and $p_{\mathbf{k}}^{\varepsilon,h}$ can be calculated with a given equation of state. Note carefully that $\rho^{\varepsilon,h} \neq [\rho]^{\varepsilon,h}$, since $[\rho]^{\varepsilon,h}$ is calculated with the particle positions $(\mathbf{X}(h\mathbf{j}, \cdot))_{\mathbf{j} \in \mathbb{Z}^d}$ which result from the velocity field $\mathbf{u}$ of the Euler equations, while $\rho^{\varepsilon,h}$ is calculated with the approximated particle positions $(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h})_{\mathbf{j} \in \mathbb{Z}^d}$. In the following, we will assume the SPH system has finite discrete mass, which means that the initial density ρ_0 is summable such that

$$h^d \sum_{\mathbf{k} \in \mathbb{Z}^d} \rho_0(h\mathbf{k}) \leq M < \infty$$

for all $0 < h < h_0$. According to Corollary 1.33, this will be the case if for example $\rho_0 \in W^{1,1}(\mathbb{R}^d)$. It is important to know whether the SPH system given in (2.12) - (2.14) even has a solution and if, in which sense this solution exists. Since answering this question would be too extensive at this point, we will come back to it in Section 2.4. As we will see, there exists a unique, global solution of the SPH system.

For our purpose, it is important to know whether the conservation properties of the Euler equations also hold for the SPH discretization of the Euler equations. For this, we take a look at the mass of the system which is given by the integral over the approximated density.

Lemma 2.14

The approximated total mass of the SPH system given in (2.12) - (2.14) *is conserved, i.e.*

$$\frac{d}{dt} \int_{\mathbb{R}^d} \rho^{\varepsilon,h}(\mathbf{x}, t) d\mathbf{x} = 0.$$

Proof. For the time derivative of the approximated density $\rho^{\varepsilon,h}$ we obtain

$$\frac{d}{dt}\rho^{\varepsilon,h} = h^d \sum_{\mathbf{k} \in \mathbb{Z}^d} \rho_0(h\mathbf{k}) \frac{d}{dt} \Phi_\varepsilon(\mathbf{x} - \mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)) = -h^d \sum_{\mathbf{k} \in \mathbb{Z}^d} \rho_0(h\mathbf{k}) \mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t) \cdot \nabla \Phi_\varepsilon(\mathbf{x} - \mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)),$$

where we used the chain rule and equation (2.12). Using this for the time derivative of the integral, we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} \rho^{\varepsilon,h}(\mathbf{x}, t) d\mathbf{x} = -h^d \sum_{\mathbf{k} \in \mathbb{Z}^d} \rho_0(h\mathbf{k}) \mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) \cdot \int_{\mathbb{R}^d} \nabla \Phi_\varepsilon(\mathbf{x} - \mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)) d\mathbf{x} = 0,$$

where the integral vanishes since the kernel function is even. □

The second conservation property is energy conservation. For this, we need the inner energy $e_{\mathbf{j}}^{\varepsilon,h} : [0, \infty[\to \mathbb{R}^d$, $\mathbf{j} \in \mathbb{Z}^d$, of the discretized Euler equations, which is given by

$$\frac{d}{dt} e_{\mathbf{j}}^{\varepsilon,h}(t) = h^d \sum_{\mathbf{k} \in \mathbb{Z}^d} \rho_0(h\mathbf{k}) \frac{p_{\mathbf{j}}^{\varepsilon,h}(t)}{\rho_{\mathbf{j}}^{\varepsilon,h}(t)^2} \left(\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t) \right) \cdot \nabla \Phi_\varepsilon(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)), \tag{2.16}$$

$$e_{\mathbf{j}}^{\varepsilon,h}(0) = e_0(h\mathbf{j}), \tag{2.17}$$

for every $\mathbf{j} \in \mathbb{Z}^d$ and $t \in]0, \infty[$. Given the internal energy, the energy of the Euler system is defined by

$$\begin{aligned} \mathcal{E}_{\text{SPH}}(t) &= \frac{1}{2} h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t)\|_2^2 + h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \rho_0(h\mathbf{j}) e_{\mathbf{j}}^{\varepsilon,h}(t) \\ &= \mathcal{E}_{\text{SPH,kin}} + \mathcal{E}_{\text{SPH,pot}}, \end{aligned} \tag{2.18}$$

where the first sum can be identified as the kinetic energy $\mathcal{E}_{\text{SPH,kin}}$ of the SPH system and the second sum as the potential energy $\mathcal{E}_{\text{SPH,pot}}$.

Lemma 2.15
The energy of the SPH-System given in (2.12) - (2.14) *is conserved, i.e.*

$$\frac{d}{dt}\mathcal{E}_{SPH}(t) = 0.$$

Proof. Taking the time derivative of the first sum yields with (2.13)

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t)\|_2^2 = &- h^{2d} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\rho_0(h\mathbf{k})\frac{p_{\mathbf{k}}^{\varepsilon,h}(t)}{\rho_{\mathbf{k}}^{\varepsilon,h}(t)^2}\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t)\cdot\nabla\Phi_\varepsilon(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t))\\ &- h^{2d} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\rho_0(h\mathbf{k})\frac{p_{\mathbf{j}}^{\varepsilon,h}(t)}{\rho_{\mathbf{j}}^{\varepsilon,h}(t)^2}\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t)\cdot\nabla\Phi_\varepsilon(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)),\end{aligned}$$

where we splitted up the sum into two sums. The first sum can be written as

$$\begin{aligned}&h^{2d} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\rho_0(h\mathbf{k})\frac{p_{\mathbf{k}}^{\varepsilon,h}(t)}{\rho_{\mathbf{k}}^{\varepsilon,h}(t)^2}\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t)\cdot\nabla\Phi_\varepsilon(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t))\\ &= -h^{2d} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\rho_0(h\mathbf{k})\frac{p_{\mathbf{j}}^{\varepsilon,h}(t)}{\rho_{\mathbf{j}}^{\varepsilon,h}(t)^2}\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\cdot\nabla\Phi_\varepsilon(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)),\end{aligned}$$

where we changed the names of the indices and used that $\nabla\Phi_\varepsilon(\mathbf{x}) = -\nabla\Phi_\varepsilon(-\mathbf{x})$ for all $\mathbf{x}\in\mathbb{R}^d$. Hence, we have with (2.15)

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t)\|_2^2\\ &= -h^{2d} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\rho_0(h\mathbf{k})\frac{p_{\mathbf{j}}^{\varepsilon,h}(t)}{\rho_{\mathbf{j}}^{\varepsilon,h}(t)^2}\left(\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t)-\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\right)\cdot\nabla\Phi_\varepsilon(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t))\\ &= -\frac{d}{dt}h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})e_{\mathbf{j}}^{\varepsilon,h}(t)\end{aligned}$$

for all $t\in]0,\infty[$. Since the time derivation of the energy $\mathcal{E}_{\mathrm{SPH}}$, it is constant in time. □

For an overview of the SPH method and its applications, see, for example, [LL10], [Pri12] or [Vio12].

2.4 Existence and Uniqueness of the Solution of the SPH Equations System

In the next step we want to prove the existence of a global, unique solution of the SPH equations. Since this system consists of an infinite number of equations, we have to use the theory of ordinary differential equations in Banach spaces. Note that the right hand side of (2.12) - (2.13) does not explicitly depend on the time t, so it will be sufficient to study only the theory of autonomous equations.
Let $\mathcal{H}$ be a real Banach space, $T > 0$, $f : \mathcal{H} \to \mathcal{H}$ and consider the initial value problem

$$\frac{d}{dt}z(t) = f(z(t)), \tag{2.19}$$

$$z(0) = z_0, \tag{2.20}$$

for $t\in]0,T[$ and a $z_0\in\mathcal{H}$. The following theorem of Picard–Lindelöf states under what conditions the initial value problem (2.19) - (2.20) has a unique, global solution.

Theorem 2.16
Let $\mathcal{H}$ be a Banach space, and $f : \mathcal{H} \to \mathcal{H}$ continuous, so that f satisfies a global Lipschitz condition, i.e. there exists an $L > 0$ so that for all $x, y \in \mathcal{H}$ we have

$$\|f(x) - f(y)\|_{\mathcal{H}} \leq L\|x - y\|_{\mathcal{H}}.$$

Then, for every $z_0 \in \mathcal{H}$ there exists a global solution $z : [0, \infty[\to \mathcal{H}$ of (2.19) - (2.20).

The proof of this theorem can be found in [Dei77].
To prove the unique existence of a solution of the SPH system, we have to rewrite (2.12) - (2.14) as an ODE of the form (2.19) - (2.20). Instead of the particle positions $\mathbf{x_j}$, $\mathbf{j} \in \mathbb{Z}^d$, we will investigate the shifting $\boldsymbol{\xi}_\mathbf{j}$ of a particle, so that $\mathbf{x_j}(t) = \mathbf{x_j}(0) + \boldsymbol{\xi}_\mathbf{j}(t)$ for $t > 0$. This will simplify the choice of a suitable Banach space to apply Theorem 2.16. Note that we have $\frac{d}{dt}\boldsymbol{\xi}_\mathbf{j} = \mathbf{u_j}$.
In the following, we will denote the set of shifts by $\boldsymbol{\Xi} = (\boldsymbol{\xi}_\mathbf{j})_{j\in\mathbb{Z}^d}$ and the set of corresponding particles by $\boldsymbol{\mathcal{X}} = (\mathbf{x_j})_{j\in\mathbb{Z}^d}$. Hence, for a given initial distribution the particles $\boldsymbol{\mathcal{X}}$ only depend on the shifts $\boldsymbol{\Xi}$. Moreover, we will denote the set of particle velocities by $\boldsymbol{U} = (\mathbf{u_j})_{j\in\mathbb{Z}^d}$. Consider the real Banach space

$$\mathcal{H} = \left\{(\boldsymbol{\Xi}, \boldsymbol{U}) = (\boldsymbol{\xi}_\mathbf{j}, \mathbf{u_j})_{\mathbf{j}\in\mathbb{Z}^d} \subset \mathbb{R}^{2d} \middle| \, \|(\boldsymbol{\Xi}, \boldsymbol{U})\|_{\mathcal{H}} < \infty \right\},$$

where the norm is given by

$$\|(\boldsymbol{\Xi}, \boldsymbol{U})\|^2_{\mathcal{H}} := h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\|\boldsymbol{\xi}_\mathbf{j}\|_2^2 + \|\mathbf{u_j}\|_2^2\right)$$

for $(\boldsymbol{\Xi}, \boldsymbol{U}) \in \mathcal{H}$.
Let $f : \mathcal{H} \to \mathcal{H}$ be given by

$$f((\boldsymbol{\Xi}, \boldsymbol{U})) = \left(\mathbf{u_k}, h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_k})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_k})^2} + \frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_j})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_j})^2}\right) \nabla\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j})\right)_{\mathbf{k}\in\mathbb{Z}^d} \in \mathcal{H}, \tag{2.21}$$

for $(\boldsymbol{\Xi}, \boldsymbol{U}) \in \mathcal{H}$, where $\mathbf{x_j} := h\mathbf{j} + \boldsymbol{\xi}_\mathbf{j}$ for all $\mathbf{j} \in \mathbb{Z}^d$ and where we denoted the approximated density over the particle distribution $\boldsymbol{\mathcal{X}}$ by $\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}} = h^d \sum_{\mathbf{j}\in\mathbb{Z}} \rho_0(h\mathbf{j})\Phi_\varepsilon(\cdot - \mathbf{x_j})$ and its associated pressure by $p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}$, which is given by the underlying equation of state.
In order for f to be well-defined, i.e. $f((\boldsymbol{\Xi}, \boldsymbol{U})) \in \mathcal{H}$ for every $(\boldsymbol{\Xi}, \boldsymbol{U}) \in \mathcal{H}$, we have to assume that the quotient of $p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}$ and $(\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}})^2$ is bounded, i.e. that there exists a constant $C_1 > 0$ such that

$$\left|\frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_j})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_j})^2}\right| \leq C_1, \quad \mathbf{j} \in \mathbb{Z}^d, \tag{2.22}$$

for every particle distribution $\boldsymbol{\mathcal{X}}$ or every set of shifts $\boldsymbol{\Xi}$, respectively. Note that we suppose that the pressure $p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}$ is only a function of the particle distribution $\boldsymbol{\mathcal{X}}$ and explicilty not a function in the time t. Hence, the right hand side of (2.21) only depends on $\boldsymbol{\Xi}$ and $\boldsymbol{U}$, so that, given the Banach space $\mathcal{H}$ and f from (2.21), the initial value problem (2.19) - (2.20) is equivalent to the SPH system (2.12) - (2.14).
In order for f to satisfy a global Lipschitz condition, we have to suppose that there exists a constant $C_2 > 0$, such that

$$\left|\frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_j})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_j})^2} - \frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_j})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_j})^2}\right| \leq C_2|\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_j}) - \rho^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_j})|, \quad \mathbf{j} \in \mathbb{Z}^d, \tag{2.23}$$

for every pair of particle distributions $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}} = (\mathbf{y_j})_{\mathbf{j}\in\mathbb{Z}^d}$ or every pair of shifts $\boldsymbol{\Xi}$ and $\boldsymbol{\Theta} = (\theta_\mathbf{j})_{\mathbf{j}\in\mathbb{Z}^d}$, respectively, where $\mathbf{y_j} = h\mathbf{j} + \boldsymbol{\theta}_\mathbf{j}$ for $\mathbf{j} \in \mathbb{Z}^d$.

Note that (2.22) as well as (2.23) can be seen as conditions on the equation of state for the pressure p. For example, both conditions are satisfied if we have a barotropic equation of state, i.e. if $p = \gamma^{-1}\rho^{\gamma}$ for a $\gamma \geq 2$. Then, we also have that $p_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h} = \gamma^{-1}(\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h})^{\gamma}$. In this case, (2.22) is satisfied with $C_1 = \left(\varepsilon^{-d}M\|\Phi\|_{L^\infty(\mathbb{R}^d)}\right)^{\gamma-2}$ since we suppose ρ_0 to have finite discrete mass M. For $\gamma = 2$, (2.23) is satisfied since the left hand side vanishes. Moreover, using that $\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}$ is bounded, the mapping $\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h} \mapsto \frac{1}{\gamma}(\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h})^{\gamma-2}$ is Lipschitz continuous if $\gamma \geq 3$ and thus (2.23) is satisfied. Finally, if $2 < \gamma < 3$, we must assume that $\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}$ is bounded from below by a positive constant so that the mapping $\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h} \mapsto \frac{1}{\gamma}(\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h})^{\gamma-2}$ is Lipschitz continuous.
To prove f from (2.21) satisfies a global Lipschitz condition, so that we can apply Theorem 2.16, we will need the following auxiliary result.

Lemma 2.17
Let $N \in \mathbb{N}$ and $x_k \in \mathbb{R}$ for $1 \leq k \leq N$. Then, we have

$$\left(\sum_{k=1}^{N} x_k\right)^2 \leq N \sum_{k=1}^{N} x_k^2. \tag{2.24}$$

Proof. Applying the Cauchy-Schwarz inequality yields

$$\sum_{k=1}^{N} x_k \leq \left(\sum_{k=1}^{N} 1\right)^{1/2} \left(\sum_{k=1}^{N} x_k^2\right)^{1/2} = \sqrt{N}\left(\sum_{k=1}^{N} x_k^2\right)^{1/2}.$$

Squaring the inequality finishes the proof. □

Considering (2.22) and (2.23), we are now able to prove the following theorem.

Theorem 2.18
Let $f : \mathcal{H} \to \mathcal{H}$ be defined as in (2.21) *and let the equation of state be given such that* (2.22) *and* (2.23) *are satisfied. Then, f satisfies a global Lipschitz condition.*

Proof. Let $(\boldsymbol{\Xi}, \boldsymbol{U}), (\boldsymbol{\Theta}, \boldsymbol{V}) \in \mathcal{H}$, and let $\mathbf{x_j} = h\mathbf{j} + \boldsymbol{\xi}_\mathbf{j}$, $\mathbf{y_j} = h\mathbf{j} + \boldsymbol{\theta}_\mathbf{j}$ for all $\mathbf{j} \in \mathbb{Z}^d$. Inserting f in the the norm of $\mathcal{H}$ gives

$$\|f((\boldsymbol{\Xi}, \boldsymbol{U})) - f((\boldsymbol{\Theta}, \boldsymbol{V}))\|_{\mathcal{H}}^2 = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\|\mathbf{u_j} - \mathbf{v_j}\|_2^2 + S,$$

where

$$\begin{aligned} S := &h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \Bigg\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(\frac{p_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}(\mathbf{x_k})}{\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}(\mathbf{x_k})^2} + \frac{p_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}(\mathbf{x_j})}{\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}(\mathbf{x_j})^2}\right) \nabla\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j}) \\ &- h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(\frac{p_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_k})}{\rho_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_k})^2} + \frac{p_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_j})}{\rho_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_j})^2}\right) \nabla\Phi_\varepsilon(\mathbf{y_k} - \mathbf{y_j}) \Bigg\|_2^2. \end{aligned}$$

By adding a zero and using Lemma 2.17 for $N = 3$, we have

$$\begin{aligned} S \leq &3h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(\frac{p_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}(\mathbf{x_k})}{\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}(\mathbf{x_k})^2} - \frac{p_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_k})}{\rho_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_k})^2}\right) \nabla\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j}) \right\|_2^2 \\ &+ 3h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(\frac{p_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}(\mathbf{x_j})}{\rho_{\boldsymbol{\mathcal{X}}}^{\varepsilon,h}(\mathbf{x_j})^2} - \frac{p_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_j})}{\rho_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_j})^2}\right) \nabla\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j}) \right\|_2^2 \\ &+ 3h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(\frac{p_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_j})}{\rho_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_j})^2} + \frac{p_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_k})}{\rho_{\boldsymbol{\mathcal{Y}}}^{\varepsilon,h}(\mathbf{y_k})^2}\right) (\nabla\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j}) - \nabla\Phi_\varepsilon(\mathbf{y_k} - \mathbf{y_j})) \right\|_2^2 \\ =: &S_1 + S_2 + S_3. \end{aligned}$$

To find an estimate for S_1, we firstly note that, using the mean value theorem, we find an $s \in [0,1]$ such that we have

$$\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j}) - \Phi_\varepsilon(\mathbf{y_k} - \mathbf{y_j}) = (\mathbf{x_k} - \mathbf{x_j} - \mathbf{y_k} + \mathbf{y_j}) \cdot \nabla\Phi_\varepsilon(\mathbf{y_k} - \mathbf{y_j} + s(\mathbf{x_k} - \mathbf{x_j} - \mathbf{y_k} + \mathbf{y_j})).$$

Taking the absolute value of this difference and using the Cauchy-Schwarz inequality yields

$$|\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j}) - \Phi_\varepsilon(\mathbf{y_k} - \mathbf{y_j})| \leq \|\mathbf{x_k} - \mathbf{x_j} - \mathbf{y_k} + \mathbf{y_j}\|_2 \|\nabla\Phi_\varepsilon\|_{L^\infty(\mathbb{R}^d)},$$

where the norm over the scaled kernel can be bounded by $\|\nabla\Phi_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon^{-(d+1)}\|\nabla\Phi\|_{L^\infty(\mathbb{R}^d)}$. Hence, the approximated density satisfies

$$\begin{aligned}
|\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_k}) - \rho^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_k})| &\leq h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\, |\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j}) - \Phi_\varepsilon(\mathbf{y_k} - \mathbf{y_j})| \\
&\leq \frac{c}{\varepsilon^{d+1}} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\, \|\mathbf{x_k} - \mathbf{x_j} - \mathbf{y_k} + \mathbf{y_j}\|_2 \\
&\leq \frac{c}{\varepsilon^{d+1}} \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\, \|\mathbf{x_j} - \mathbf{y_j}\|_2 + \|\mathbf{x_k} - \mathbf{y_k}\|_2\, h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \right) \\
&\leq \frac{c}{\varepsilon^{d+1}} \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\, \|\mathbf{x_j} - \mathbf{y_j}\|_2 + M\, \|\mathbf{x_k} - \mathbf{y_k}\|_2 \right),
\end{aligned}$$

where we used the triangle inequality and the finite mass of the SPH system. Combining the stated estimate with (2.23), we have

$$\begin{aligned}
&\left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(\frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_k})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_k})^2} - \frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_k})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_k})^2} \right) \nabla\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j}) \right\|_2 \\
&\qquad \leq h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left| \frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_k})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_k})^2} - \frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_k})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_k})^2} \right| \|\nabla\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j})\|_2 \\
&\qquad \leq \frac{c}{\varepsilon^{2d+2}} h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{x_j} - \mathbf{y_j}\|_2 + M \|\mathbf{x_k} - \mathbf{y_k}\|_2 \right) \\
&\qquad \leq \frac{2cM}{\varepsilon^{2d+2}} h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \|\mathbf{x_k} - \mathbf{y_k}\|_2,
\end{aligned}$$

where we used the triangle inequality and the estimate for the scaled kernel again. Hence, using the Cauchy-Schwarz inequality, the square of the norm yields

$$\begin{aligned}
&\left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(\frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_k})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{X}}}(\mathbf{x_k})^2} - \frac{p^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_k})}{\rho^{\varepsilon,h}_{\boldsymbol{\mathcal{Y}}}(\mathbf{y_k})^2} \right) \nabla\Phi_\varepsilon(\mathbf{x_k} - \mathbf{x_j}) \right\|_2^2 \\
&\qquad \leq c(\varepsilon, M) h^{2d} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\rho_0(h\mathbf{k})\, \|\mathbf{x_j} - \mathbf{y_j}\|_2\, \|\mathbf{x_k} - \mathbf{y_k}\|_2 \\
&\qquad \leq c(\varepsilon, M) h^{2d} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\rho_0(h\mathbf{k}) \left(\|\mathbf{x_j} - \mathbf{y_j}\|_2^2 + \|\mathbf{x_k} - \mathbf{y_k}\|_2^2 \right) \\
&\qquad \leq 2c(\varepsilon, M) M h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k})\, \|\mathbf{x_k} - \mathbf{y_k}\|_2^2,
\end{aligned}$$

such that we have

$$S_1 \leq c(\varepsilon, M) h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k})\, \|\mathbf{x_k} - \mathbf{y_k}\|_2^2.$$

Noting that $\|\mathbf{x_k} - \mathbf{y_k}\|_2 = \|\boldsymbol{\xi}_\mathbf{k} - \boldsymbol{\eta}_\mathbf{k}\|_2$, we finally arrive at

$$S_1 \leq c(\varepsilon, M) h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \, \|\boldsymbol{\xi}_\mathbf{k} - \boldsymbol{\eta}_\mathbf{k}\|_2^2 \, .$$

In a similar way, we also can give the same estimate for S_2. For S_3 we have use (2.22) and the arguments as above to derive an estimate also by $c(\varepsilon, M) h^d \sum_{\mathbf{k}\in\mathbb{Z}} \rho_0(h\mathbf{k}) \, \|\boldsymbol{\xi}_\mathbf{k} - \boldsymbol{\eta}_\mathbf{k}\|_2^2$. Hence, we have for S

$$S \leq c(\varepsilon, M) h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \, \|\boldsymbol{\xi}_\mathbf{k} - \boldsymbol{\eta}_\mathbf{k}\|_2^2 \, .$$

Altogether, we have

$$\begin{aligned} \|f((\boldsymbol{\Xi}, \boldsymbol{U})) - f((\boldsymbol{\Theta}, \boldsymbol{V}))\|_{\mathcal{H}}^2 &\leq c(\varepsilon, M) h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(\|\boldsymbol{\xi}_\mathbf{k} - \boldsymbol{\eta}_\mathbf{k}\|_2^2 + \|\mathbf{u_k} - \mathbf{v_k}\|_2^2 \right) \\ &= c(\varepsilon, M) \|(\boldsymbol{\Xi}, \boldsymbol{U}) - (\boldsymbol{\Theta}, \boldsymbol{V}))\|_{\mathcal{H}}^2 \end{aligned}$$

□

Since f satisfies a global Lipschitz condition, the initial value problem (2.19) has a unique solution according to Theorem 2.16.

Corollary 2.19
Let the equation of state be given such that (2.22) *and* (2.23) *are satisfied. Then, the SPH system* (2.12) - (2.14) *has a unique solution* $t \in [0, \infty[\mapsto (\mathbf{x_k}(t), \mathbf{u_k}(t))_{\mathbf{k}\in\mathbb{Z}^d} \subset \mathbb{R}^d \times \mathbb{R}^d$.

Hence, the solution of the SPH approximation is given globally, while, in general, the solution of the Euler equations only exists up to a limited time $T > 0$. Hence, whenever we have to restrict ourselves to time intervals where the solution of the Euler equations exists, we can conclude that the solution of the SPH approximated system also exists.

Chapter 3

Convergence Results

In this chapter, we investigate the SPH method for the Euler equations for a specific barotropic equation of state. Our goal is to derive a pointwise convergence result for the SPH method.
In 1991, Oelschläger [Oel91] investigated an SPH-like discretization method for the Euler equations. He stated conditions for the kernel function which lead to the convergence of an $L^2(\mathbb{R}^d)$ energy error term. Unfortunately, these conditions are rather technical. Moreover, the convergence result was too weak to prove pointwise convergence of the particle trajectories.
We will generalize and improve the result of Oelschläger by adding another condition to the kernel which will lead to a stronger convergence result of the $L^2(\mathbb{R}^d)$ energy error term. Under appropriate conditions, the new convergence result will be strong enough to prove a pointwise convergence result.
We will start by giving the Euler equations for the barotropic equation of state and its associated SPH system.

3.1 Euler Equations for a Specific Equation of State

We consider the Euler equations as mentioned in Section 2.2 on all of $\mathbb{R}^d$ in the specific case that the pressure $p : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ and the density $\rho : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ are connected by the barotropic equation $p = \frac{1}{2}\rho^2$. For a given initial velocity $\mathbf{u}_0 : \mathbb{R}^d \to \mathbb{R}^d$ and a initial density $\rho_0 : \mathbb{R}^d \to \mathbb{R}$, we seek the solution $\mathbf{u} : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ and $\rho : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ of

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \rho, \tag{3.1}$$

$$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}, \tag{3.2}$$

on $\mathbb{R}^d \times]0,T]$ and

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0, \quad \rho(\cdot,0) = \rho_0, \tag{3.3}$$

on $\mathbb{R}^d$.
This system is a simplification of the system given in (2.5) - (2.7) due to the specific equation of state. As a consequence, the equation for the internal energy (2.8) simplifies to

$$\frac{\mathrm{D}e}{\mathrm{D}t} = -\frac{1}{2}\rho \nabla \cdot \mathbf{u} = \frac{1}{2}\frac{\mathrm{D}\rho}{\mathrm{D}t}$$

on $\mathbb{R}^d \times]0,T]$, where we used (3.2). Hence, the internal energy can be described by the density, as we have

$$e(\mathbf{X}(\mathbf{x},t),t) = \frac{1}{2}\rho(\mathbf{X}(\mathbf{x},t),t) + c(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^d$, where $c = e_0 - \rho_0/2$. For the sake of simplicity, we choose $e_0 = \rho_0/2$ such that $c \equiv 0$. Hence, using Lemma 2.11, the potential energy at a time $t \in [0, T]$ can be rewritten as

$$\begin{aligned}\mathcal{E}_{\text{pot}}(t) &= \int_{\mathbb{R}^d} \rho(\mathbf{x}, t) e(\mathbf{x}, t) d\mathbf{x} = \int_{\mathbb{R}^d} \rho_0(\mathbf{x}) e(\mathbf{X}(\mathbf{x}, t), t) d\mathbf{x} \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \rho_0(\mathbf{x}) \rho(\mathbf{X}(\mathbf{x}, t), t) d\mathbf{x} = \frac{1}{2} \int_{\mathbb{R}^d} \rho(\mathbf{x}, t)^2 d\mathbf{x}.\end{aligned}$$

This means in particular, that we do not need the internal energy to calculate the energy of Euler's equations anymore.
For the associated SPH approximation, we remember the requirements from Section 2.3. Let $h > 0$ be the spatial discretization parameter. Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be an at least one time differentiable even function, and let $\Phi_\varepsilon := \varepsilon^{-d}\Phi(\cdot/\varepsilon)$ be its scaled version for a smoothing parameter $\varepsilon > 0$. The SPH system (2.12) - (2.14) becomes in this situation

$$\frac{d}{dt}\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) = \mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t), \tag{3.4}$$

$$\frac{d}{dt}\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) = -\nabla\rho^{\varepsilon,h}(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t), t) = -h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \nabla\Phi_\varepsilon(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)), \tag{3.5}$$

for all $\mathbf{j} \in \mathbb{Z}^d$ and all $t \in]0, T]$, and

$$\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(0) = h\mathbf{j}, \quad \mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(0) = \mathbf{u}_0(h\mathbf{j}) \tag{3.6}$$

for all $\mathbf{j} \in \mathbb{Z}^d$.
On the right-hand side of (3.5), we do not have any evaluation of the approximated density but only an evaluation of the gradient of the density. More precisely, we do not need to evaluate the kernel Φ, but only the gradient $\nabla\Phi$ of the kernel at the particle position differences, which gives us a simplification for our numerical scheme compared to (2.13).
As in the continuous case, we can rewrite the potential energy for the SPH system from (2.18). Since we know that the internal energy is given by $e = \rho/2$ on the particle trajectories, we can choose $e_{\mathbf{j}}^{\varepsilon,h} = \rho_{\mathbf{j}}^{\varepsilon,h}/2$ for all $\mathbf{j} \in \mathbb{Z}^d$. This is also what we obtain if we insert the equation of state $p = \rho^2/2$ into (2.16) and integrate over time. Hence, we note for the potential energy of the SPH system from (2.18)

$$\mathcal{E}_{\text{SPH,pot}}(t) = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) e_{\mathbf{j}}^{\varepsilon,h}(t) = \frac{1}{2} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \rho_{\mathbf{j}}^{\varepsilon,h}(t) \tag{3.7}$$

for all $t \in [0, T]$.

3.2 Properties of the Kernel

In this section, we want to introduce the properties the smoothing kernel has to satisfy in order to prove convergence. We will, in principle, have three different types of conditions. The first property is that the kernel function is a convolution kernel, which means that it can be written as the convolution of a root kernel Φ^r with itself.

Definition 3.1
A kernel function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is called a convolution kernel if there exists a function $\Phi^r : \mathbb{R}^d \to \mathbb{R}$ such that

$$\Phi = \Phi^r * \Phi^r = \int_{\mathbb{R}^d} \Phi^r(\cdot - \mathbf{y}) \Phi^r(\mathbf{y}) d\mathbf{y}.$$

The function Φ^r is called a convolution root or a root kernel of Φ.

A convolution root may not be unique as we will see in Chapter 4. Note that we did not define in which sense the integral on the right-hand side exists. If the convolution root Φ^r is in $L^1(\mathbb{R}^d)$, Φ is in $L^1(\mathbb{R}^d)$ itself according to Theorem 1.5. If, in addition, Φ^r is bounded, it is easy to see that the convolution kernel Φ has to be positive definite.

Proposition 3.2
Let $\Phi^r \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be even and suppose that its Fourier transformation $\widehat{\Phi^r}$ is nonvanishing. Then its associated convolution kernel $\Phi \in L^1(\mathbb{R}^d)$ is positive definite.

Proof. Since Φ^r is even and in $L^1(\mathbb{R}^d)$, its Fourier transformation is real and continuous. Using Theorem 1.6, we have

$$\widehat{\Phi}(\boldsymbol{\omega}) = \widehat{\Phi^r * \Phi^r}(\boldsymbol{\omega}) = (2\pi)^{d/2}\widehat{\Phi^r}(\boldsymbol{\omega})^2 \geq 0.$$

Hence, $\widehat{\Phi}$ is nonnegative. Moreover, Φ is continuous as a convolution of an $L^1(\mathbb{R}^d)$ and an $L^\infty(\mathbb{R}^d)$ function according to Lemma 1.8. Applying Corollary 1.11 finishes the proof. □

Another conclusion is that the existence of a convolution root is invariant under scaling. This will be important for our propose since we are using scaled kernel functions in the SPH method.

Lemma 3.3
Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a convolution kernel with associated root kernel $\Phi^r : \mathbb{R}^d \to \mathbb{R}$. For an $\varepsilon > 0$, the scaled kernels are given by $\Phi_\varepsilon = \varepsilon^{-d}\Phi(\cdot/\varepsilon)$ and $\Phi^r_\varepsilon = \varepsilon^{-d}\Phi^r(\cdot/\varepsilon)$. Then, Φ^r_ε is a convolution root of Φ_ε.

Proof. With a simple substitution we see this relation via

$$\Phi_\varepsilon(\mathbf{x}) = \varepsilon^{-d}\int\limits_{\mathbb{R}^d} \Phi^r(\mathbf{x}/\varepsilon - \mathbf{y})\Phi^r(\mathbf{y})d\mathbf{y} = \varepsilon^{-2d}\int\limits_{\mathbb{R}^d} \Phi^r((\mathbf{x}-\mathbf{y})/\varepsilon)\Phi^r(\mathbf{y}/\varepsilon)d\mathbf{y} = \Phi^r_\varepsilon * \Phi^r_\varepsilon(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^d$. □

The second condition we state for our kernel function Φ is that it and its root kernel Φ^r satisfy the moment condition we introduced in Definition 1.26. As the SPH approximation includes a convolution step, this is a quite natural condition.
In applications, the kernel is often used to be positive to ensure that the density remains positive, which is justified as a physical requirement. However, this restriction to the kernel function is not justified from a mathematical point of view. The moment condition guarantees us better approximations by convolution. But a function satisfying a moment condition of order $m \geq 3$ cannot be positive. Nevertheless, the cost of losing the positivity is far less compared to the gain of having significant better approximations. Moreover, a kernel function that may become negative does not have to result in a negative density approximation.
For our purpose we want both, the convolution kernel and the root kernel, to satisfy a moment condition. The following result ensures that if one of these functions satisfies a moment condition, the other function satisfies the moment condition of the same order.

Proposition 3.4
Let $\Phi^r \in L^1(\mathbb{R}^d)$ and $\Phi = \Phi^r * \Phi^r$. Then, Φ satisfies the moment condition of order m if and only if Φ^r satisfies the moment condition of order m.

Proof. From the relation

$$\int\limits_{\mathbb{R}^d} \Phi(\mathbf{x})d\mathbf{x} = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \Phi^r(\mathbf{y})\Phi^r(\mathbf{x}-\mathbf{y})d\mathbf{y}d\mathbf{x} = \int\limits_{\mathbb{R}^d} \Phi^r(\mathbf{x})d\mathbf{x}\int\limits_{\mathbb{R}^d} \Phi^r(\mathbf{y})d\mathbf{y}$$

we see that $\int \Phi^r d\mathbf{x} = 1$ implies $\int \Phi d\mathbf{x} = 1$. We also have that $\int \Phi d\mathbf{x} = 1$ implies $\int \Phi^r d\mathbf{x} = \pm 1$. In the case of a negative sign we can simply replace Φ^r by $-\Phi^r$.
For $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ with $|\boldsymbol{\alpha}| \le m$ we have by substitution

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} \Phi(\mathbf{x}) d\mathbf{x} &= \int\limits_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} \int\limits_{\mathbb{R}^d} \Phi^r(\mathbf{y}) \Phi^r(\mathbf{x}-\mathbf{y}) d\mathbf{y} d\mathbf{x} = \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} (\mathbf{x}+\mathbf{y})^{\boldsymbol{\alpha}} \Phi^r(\mathbf{x}) d\mathbf{x} \Phi^r(\mathbf{y}) d\mathbf{y} \\ &= \sum_{\boldsymbol{\beta} \le \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int\limits_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\beta}} \Phi^r(\mathbf{x}) d\mathbf{x} \int\limits_{\mathbb{R}^d} \mathbf{y}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \Phi^r(\mathbf{y}) d\mathbf{y},\end{aligned} \tag{3.8}$$

where $\boldsymbol{\beta} \le \boldsymbol{\alpha}$ means that $\beta_k \le \alpha_k$ for all $1 \le k \le d$. Hence, Φ satisfies (1.2) if Φ^r satisfies (1.2). If, however, Φ satisfies (1.2) then we can use induction to show that Φ^r also satisfies (1.2). To see this, we start with $\boldsymbol{\alpha} = \mathbf{e}_j$ and find

$$\begin{aligned}0 &= \int\limits_{\mathbb{R}^d} x_j \Phi(\mathbf{x}) d\mathbf{x} = \int\limits_{\mathbb{R}^d} x_j \int\limits_{\mathbb{R}^d} \Phi^r(\mathbf{y}) \Phi^r(\mathbf{x}-\mathbf{y}) d\mathbf{y} d\mathbf{x} \\ &= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} (x_j + y_j) \Phi^r(\mathbf{x}) \Phi^r(\mathbf{y}) d\mathbf{y} d\mathbf{x} = 2 \int\limits_{\mathbb{R}^d} \Phi^r(\mathbf{y}) d\mathbf{y} \int\limits_{\mathbb{R}^d} x_j \Phi^r(\mathbf{x}) d\mathbf{x} \\ &= 2 \int\limits_{\mathbb{R}^d} x_j \Phi^r(\mathbf{x}) d\mathbf{x},\end{aligned}$$

which means that Φ^r satisfies (1.2) for $|\boldsymbol{\alpha}| = 1$. Next, if Φ^r satisfies (1.2) for an $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ with $|\boldsymbol{\alpha}| < m - 1$, we have

$$\begin{aligned}0 &= \int\limits_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}+\mathbf{e}_j} \Phi(\mathbf{x}) d\mathbf{x} = \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} (\mathbf{x}+\mathbf{y})^{\boldsymbol{\alpha}+\mathbf{e}_j} \Phi^r(\mathbf{y}) \Phi^r(\mathbf{x}) d\mathbf{y} d\mathbf{x} \\ &= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} (x_j + y_j) \sum_{\boldsymbol{\beta} \le \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \mathbf{y}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \Phi^r(\mathbf{y}) \Phi^r(\mathbf{x}) d\mathbf{y} d\mathbf{x} \\ &= 2 \sum_{\boldsymbol{\beta} \le \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int\limits_{\mathbb{R}^d} x^{\boldsymbol{\beta}} \Phi^r(\mathbf{x}) d\mathbf{x} \int\limits_{\mathbb{R}^d} y^{\boldsymbol{\alpha}-\boldsymbol{\beta}+\mathbf{e}_j} \Phi^r(\mathbf{y}) d\mathbf{y} \\ &= 2 \int\limits_{\mathbb{R}^d} \mathbf{y}^{\boldsymbol{\alpha}+\mathbf{e}_j} \Phi^r(\mathbf{y}) d\mathbf{y},\end{aligned}$$

where we have again used that for all $\boldsymbol{\beta} \le \boldsymbol{\alpha}$ the first integral in the penultimate line vanishes except for $\boldsymbol{\beta} = \mathbf{0}$.
It remains to show that $\int \|\mathbf{x}\|_2^m |\Phi(\mathbf{x})| d\mathbf{x} < \infty$ if and only if $\int \|\mathbf{x}\|_2^m |\Phi^r(\mathbf{x})| d\mathbf{x} < \infty$. Let $|\boldsymbol{\alpha}| = m$. Now assume that Φ^r satisfies (1.3). With $\Phi^r \in L^1(\mathbb{R}^d)$ and as it satisfies (1.2) it is easy to see that we also have $\int \|\mathbf{x}\|_2^j |\Phi^r(\mathbf{x})| d\mathbf{x} < \infty$ for all $0 \le j \le m$. Using the triangle inequality, this shows

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \|\mathbf{x}\|_2^m |\Phi(\mathbf{x})| d\mathbf{x} &\le \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \|\mathbf{x}+\mathbf{y}\|_2^m |\Phi^r(\mathbf{x}) \Phi^r(\mathbf{y})| d\mathbf{x} d\mathbf{y} \\ &\le \sum_{j=0}^{m} \binom{m}{j} \int\limits_{\mathbb{R}^d} \|\mathbf{x}\|_2^j |\Phi^r(\mathbf{x})| d\mathbf{x} \int\limits_{\mathbb{R}^d} \|\mathbf{y}\|_2^{m-j} |\Phi^r(\mathbf{y})| d\mathbf{y} < \infty,\end{aligned}$$

i.e Φ satisfies (1.3). Finally, assuming that Φ satisfies (1.3) we note that $\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 = |x_1| + \ldots + |x_d|$ implies

$$\int\limits_{\mathbb{R}^d} \|\mathbf{x}\|_2^m |\Phi^r(\mathbf{x})| d\mathbf{x} \le \sum_{|\boldsymbol{\alpha}|=m} \frac{m!}{\boldsymbol{\alpha}!} \int\limits_{\mathbb{R}^d} |\mathbf{x}^{\boldsymbol{\alpha}}| |\Phi^r(\mathbf{x})| d\mathbf{x} < \infty$$

provided the integrals on the right-hand side exist and are finite. To see this, we note for $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ with $|\boldsymbol{\alpha}| = m$ that in the last line of equation (3.8), all terms vanish except for $\boldsymbol{\beta} = \mathbf{0}$ and $\boldsymbol{\beta} = \boldsymbol{\alpha}$. This means

$$\int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} \Phi(\mathbf{x}) d\mathbf{x} = 2 \int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} \Phi^r(\mathbf{x}) d\mathbf{x} =: 2C_{\boldsymbol{\alpha}}.$$

Using this, we make the same calculation as in (3.8) and arrive at

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathbf{x}^{\boldsymbol{\alpha}} \Phi(\mathbf{x})| \, d\mathbf{x} &= \int_{\mathbb{R}^d} \left| \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \Phi^r(\mathbf{x}) \int_{\mathbb{R}^d} y^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \Phi^r(\mathbf{y}) d\mathbf{y} \right| d\mathbf{x} \\
&= \int_{\mathbb{R}^d} \left| \mathbf{x}^{\boldsymbol{\alpha}} \Phi^r(\mathbf{x}) \int_{\mathbb{R}^d} \Phi^r(\mathbf{y}) d\mathbf{y} + \Phi^r(\mathbf{x}) \int_{\mathbb{R}^d} y^{\boldsymbol{\alpha}} \Phi^r(\mathbf{y}) d\mathbf{y} \right| d\mathbf{x} \\
&= \int_{\mathbb{R}^d} |\mathbf{x}^{\boldsymbol{\alpha}} \Phi^r(\mathbf{x}) + C_{\boldsymbol{\alpha}} \Phi^r(\mathbf{x})| \, d\mathbf{x}
\end{aligned}$$

If now $\int_{\mathbb{R}^d} \|\mathbf{x}\|_2^m |\Phi(\mathbf{x})| d\mathbf{x} < \infty$ then we easily see that

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathbf{x}^{\boldsymbol{\alpha}} \Phi^r(\mathbf{x})| \, d\mathbf{x} &\leq \int_{\mathbb{R}^d} |\mathbf{x}^{\boldsymbol{\alpha}} \Phi^r(\mathbf{x}) + C_{\boldsymbol{\alpha}} \Phi^r(x)| \, d\mathbf{x} + \int_{\mathbb{R}^d} |C_{\boldsymbol{\alpha}} \Phi^r(\mathbf{x})| \, d\mathbf{x} \\
&= \int_{\mathbb{R}^d} |\mathbf{x}^{\boldsymbol{\alpha}} \Phi(\mathbf{x})| \, d\mathbf{x} + |C_{\boldsymbol{\alpha}}| \|\Phi^r\|_{L^1(\mathbb{R}^d)} < \infty
\end{aligned}$$

for $|\boldsymbol{\alpha}| = m$. □

The third and last condition that we impose on our kernel function is actually a condition on the root kernel. In particular, we want the convolution root to satisfy a so-called approximation condition.

Definition 3.5
For $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ let $p_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The kernel $\Phi^r \in W^{1,2}(\mathbb{R}^d)$ satisfies the approximation condition of order $L \in \mathbb{N}$ if there exists a constant $C > 0$ such that

$$|(p_{\boldsymbol{\alpha}} \partial_j \Phi^r)^\wedge(\boldsymbol{\omega})| \leq C |\widehat{\Phi^r}(\boldsymbol{\omega})|, \quad 1 \leq |\boldsymbol{\alpha}| \leq L, \quad 1 \leq j \leq d, \tag{3.9}$$

holds almost everywhere and such that

$$p_{\boldsymbol{\alpha}} \partial_j \Phi^r \in L^2(\mathbb{R}^d), \quad |\boldsymbol{\alpha}| = L + 1, \quad 1 \leq j \leq d. \tag{3.10}$$

This condition was first introduced by Oelschläger in [Oel90] and [Oel91] in a slightly different form. While this property is a very important part in the proof of convergence, there is a lack of understanding the meaning of this condition in a descriptive way. For the proof following in the next section, we will need to rephrase this condition.

Proposition 3.6
Let $\Phi^r \in W^{1,2}(\mathbb{R}^d)$. Then, Φ^r satisfies the approximation condition of order L if and only if it satisfies (3.10) *and*

$$|D^{\boldsymbol{\alpha}}[\omega_j \widehat{\Phi^r}(\boldsymbol{\omega})]| \leq C |\widehat{\Phi^r}(\boldsymbol{\omega})|, \quad 1 \leq |\boldsymbol{\alpha}| \leq L, \quad 1 \leq j \leq d, \tag{3.11}$$

almost everywhere. For $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ with $|\boldsymbol{\alpha}| \geq 1$, $1 \leq j \leq d$ and $\varepsilon > 0$ let $p_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}}$ and

$$W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{x}) := \frac{(-1)^{|\boldsymbol{\alpha}|+1}}{\boldsymbol{\alpha}!} p_{\boldsymbol{\alpha}}(\mathbf{x}) \partial_j \Phi^r_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \tag{3.12}$$

If Φ^r is in addition even, then we have $W^j_{\varepsilon,\boldsymbol{\alpha}}(-\mathbf{x}) = (-1)^{|\boldsymbol{\alpha}|+1} W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{x})$ almost everywhere and for $1 \leq j \leq d$ conditions (3.9) *and* (3.10) *imply*

$$|\widehat{W^j_{\varepsilon,\boldsymbol{\alpha}}}(\boldsymbol{\omega})| \leq C\varepsilon^{|\boldsymbol{\alpha}|-1} |\widehat{\Phi^r_\varepsilon}(\boldsymbol{\omega})|, \qquad 1 \leq |\boldsymbol{\alpha}| \leq L, \quad 1 \leq j \leq d, \tag{3.13}$$

$$\|W^j_{\varepsilon,\boldsymbol{\alpha}}\|_{L^2(\mathbb{R}^d)} = \varepsilon^{L-d/2} \|p_{\boldsymbol{\alpha}} \partial_j \Phi^r\|_{L^2(\mathbb{R}^d)}, \quad |\boldsymbol{\alpha}| = L+1, \tag{3.14}$$

where (3.13) *holds almost everywhere.*

Proof. With Lemma 1.2, we see that (3.11) is equivalent to (3.9) for a Φ^r in $L^1(\mathbb{R}^d)$. Since $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, this property carries over to $L^2(\mathbb{R}^d)$.
Next, we see that if Φ^r is even, its first partial derivation is odd and we have

$$W^j_{\varepsilon,\boldsymbol{\alpha}}(-\mathbf{x}) = \frac{(-1)^{|\boldsymbol{\alpha}|+1}}{\boldsymbol{\alpha}!} (-1)^{|\boldsymbol{\alpha}|} p_{\boldsymbol{\alpha}}(\mathbf{x}) \left(-\partial_j \Phi^r_\varepsilon(\mathbf{x})\right) = (-1)^{|\boldsymbol{\alpha}|+1} W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{x})$$

almost everywhere. Now let $1 \leq |\boldsymbol{\alpha}| \leq L+1$. Then we note that

$$\begin{aligned}
\widehat{W^j_{\varepsilon,\boldsymbol{\alpha}}}(\boldsymbol{\omega}) &= \frac{(-1)^{|\boldsymbol{\alpha}|+1}}{\boldsymbol{\alpha}!(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\mathbf{x}\cdot\boldsymbol{\omega}} p_{\boldsymbol{\alpha}}(\mathbf{x}) \partial_j \Phi^r_\varepsilon(\mathbf{x}) d\mathbf{x} \\
&= \frac{(-1)^{|\boldsymbol{\alpha}|+1}}{\boldsymbol{\alpha}!(2\pi)^{d/2}} \varepsilon^{-d-1} \int_{\mathbb{R}^d} e^{-i\mathbf{x}\cdot\boldsymbol{\omega}} p_{\boldsymbol{\alpha}}(\mathbf{x}) \partial_j \Phi^r(\mathbf{x}/\varepsilon) d\mathbf{x} \\
&= \frac{(-1)^{|\boldsymbol{\alpha}|+1}}{\boldsymbol{\alpha}!(2\pi)^{d/2}} \varepsilon^{|\boldsymbol{\alpha}|-1} \int_{\mathbb{R}^d} e^{-i\varepsilon\mathbf{y}\cdot\boldsymbol{\omega}} p_{\boldsymbol{\alpha}}(\mathbf{y}) \partial_j \Phi^r(\mathbf{y}) d\mathbf{x} \\
&= \varepsilon^{|\boldsymbol{\alpha}|-1} \frac{(-1)^{|\boldsymbol{\alpha}|+1}}{\boldsymbol{\alpha}!} \widehat{p_{\boldsymbol{\alpha}} \partial_j \Phi^r}(\varepsilon\boldsymbol{\omega})
\end{aligned}$$

for functions $W^j_{\varepsilon,\boldsymbol{\alpha}} \in L^1(\mathbb{R}^d)$. Therefore, the identity holds in $L^2(\mathbb{R}^d)$, too. Hence, equation (3.9) implies (3.13) almost everywhere. Finally, for a $|\alpha| = L+1$, Plancherel's identity shows

$$\|W^j_{\varepsilon,\boldsymbol{\alpha}}\|_{L^2(\mathbb{R}^d)} = \|\widehat{W^j_{\varepsilon,\boldsymbol{\alpha}}}\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^{|\boldsymbol{\alpha}|-1} \|\widehat{p_{\boldsymbol{\alpha}} \partial_j \Phi^r}(\varepsilon\cdot)\|_{L^2(\mathbb{R}^d)},$$

where we note that

$$\|\widehat{p_{\boldsymbol{\alpha}} \partial_j \Phi^r}(\varepsilon\cdot)\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\widehat{p_{\boldsymbol{\alpha}} \partial_j \Phi^r}(\varepsilon\boldsymbol{\omega})|^2 d\boldsymbol{\omega} = \varepsilon^{-d} \int_{\mathbb{R}^d} |\widehat{p_{\boldsymbol{\alpha}} \partial_j \Phi^r}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} = \varepsilon^{-d} \|\widehat{p_{\boldsymbol{\alpha}} \partial_j \Phi^r}\|^2_{L^2(\mathbb{R}^d)}.$$

Applying Plancharel's identity once again finishes the proof. □

Note that the constant factor $(-1)^{|\boldsymbol{\alpha}|+1}/\boldsymbol{\alpha}!$ in the definition of $W^j_{\varepsilon,\boldsymbol{\alpha}}$ is not important for deriving the result in the next section but will simplify some calculations.

3.3 Convergence of the Energy Error Term

In this section, we want to give a first convergence result for the SPH method. For this, we need to define an error term which we will derive from the energy of the system of Euler's equations and the energy of the SPH system. First, we assume that we have a kernel function $\Phi : \mathbb{R}^d \to \mathbb{R}$

with an even convolution root Φ^r. Using the definition, we note that

$$\begin{aligned}\Phi(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)-\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)) &= \int_{\mathbb{R}^d} \Phi^r(\mathbf{y})\Phi^r(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)-\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)-\mathbf{y})d\mathbf{y} \\ &= \int_{\mathbb{R}^d} \Phi^r(\mathbf{x}-\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t))\Phi^r(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)-\mathbf{x})d\mathbf{x} \\ &= \int_{\mathbb{R}^d} \Phi^r(\mathbf{x}-\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t))\Phi^r(\mathbf{x}-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t))d\mathbf{x}\end{aligned}$$

for two indices $\mathbf{j},\mathbf{k}\in\mathbb{Z}^d$ with associated particle position $\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)\in\mathbb{R}^d$. With this identity and the definition of the approximated density (2.15), we can rewrite the potential energy of the SPH method from (3.7) as

$$\begin{aligned}\mathcal{E}_{\mathrm{SPH,pot}}(t) &= \frac{1}{2}h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\rho_{\mathbf{j}}^{\varepsilon,h}(t) \\ &= \frac{1}{2}h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\Phi_\varepsilon(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)-\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)) \\ &= \frac{1}{2}h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\int_{\mathbb{R}^d}\Phi_\varepsilon^r(\mathbf{x}-\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t))\Phi_\varepsilon^r(\mathbf{x}-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t))d\mathbf{x} \\ &= \frac{1}{2}\int_{\mathbb{R}^d}\left(h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\Phi_\varepsilon^r(\mathbf{x}-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t))\right)^2 d\mathbf{x},\end{aligned} \tag{3.15}$$

which looks closer to the integral in the potential energy term of Euler's equations. For the sake of simplicity, we will write

$$\rho^{\varepsilon,h,r}(\mathbf{x},t) := h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\Phi_\varepsilon^r(\mathbf{x}-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)) \tag{3.16}$$

for $(\mathbf{x},t)\in\mathbb{R}^d\times[0,T]$ in the following, which can be interpreted as a alternative approximation of the density ρ. Now, we are able to define an error term for the energy.

Definition 3.7
Let $(\mathbf{u},\rho)$ *be the solution of the weakly compressible Euler equations* (3.1) - (3.3) *up to a time* $T>0$, *and let* $(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h},\mathbf{u}_{\mathbf{k}}^{\varepsilon,h})_{\mathbf{k}\in\mathbb{Z}^d}$ *be a solution of the corresponding SPH equations* (3.4) - (3.6). *The energy error* $Q:[0,T]\to\mathbb{R}$ *is defined by*

$$Q(t) := h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\|\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)-\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\|_2^2+\int_{\mathbb{R}^d}\left(\rho^{\varepsilon,h,r}(\mathbf{x},t)-\rho(\mathbf{x},t)\right)^2d\mathbf{x}.$$

We identify the sum as the kinetic part of the energy error and the integral as the potential part of the energy error. Note that we omitted the constant factor $1/2$ in front of both parts due to simplicity. Our goal is to give a bound of the energy error Q with respect to both parameters h and ε by using the lemma of Gronwall. Note that such a bound will not give us instantly the convergence of the particle trajectories. However, it is a first step to derive such a convergence result. Before we investigate the behavior of Q in time, we will give a bound for the initial error $Q(0)$.

Theorem 3.8
Assume $\Phi^r\in W^{s,1}(\mathbb{R}^d)\cap C(\mathbb{R}^d)$, *satisfies the moment condition of order* $m\geq 1$. *If* $\rho_0\in W^{\max\{m,s\},2}(\mathbb{R}^d)\cap C(\mathbb{R}^d)$, *then there exists a constant* $C>0$ *such that the energy at time* $t=0$

can be bounded by

$$Q(0) \leq C\left(\frac{h^{2s}}{\varepsilon^{2s}}\|\rho_0\|^2_{W^{s,2}(\mathbb{R}^d)} + \varepsilon^{2m}|\rho_0|^2_{W^{m,2}(\mathbb{R}^d)}\right).$$

Proof. At time $t = 0$, we have $\mathbf{x}_\mathbf{j}^{\varepsilon,h}(0) = h\mathbf{j} = \mathbf{x}_\mathbf{j}(0)$ and $\mathbf{u}_\mathbf{j}^{\varepsilon,h}(0) = \mathbf{u}_0(jh) = \mathbf{u}_\mathbf{j}(0)$, such that the kinetic part of the error vanishes. Hence, $Q(0)$ reduces to its potential part, i.e.

$$Q(0) = \int\limits_{\mathbb{R}^d}\left|h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\Phi^r_\varepsilon(\mathbf{x}-h\mathbf{j}) - \rho_0(\mathbf{x})\right|^2 d\mathbf{x} = \left\|h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\Phi^r_\varepsilon(\cdot-h\mathbf{j}) - \rho_0\right\|^2_{L^2(\mathbb{R}^d)}$$

Applying Corollary 1.35 with $p = r = 2$ and $q = 1$ we easily arrive at our bound. □

We will see that, interestingly, the error bound of $Q(0)$ is the only place, where the spatial discretisation parameter $h > 0$ enters the equation. Note that the initial placement of the particles on a cartesian grid might not be optimal. It should be possible to derive similar error bounds for other regular distributions of the initial placement of the particles, for example the densest sphere packing or a randomized placement with a given separation distance. In this cases a different quadrature formula has to be used.
We are now in the position to formulate and proof the main theorem of this chapter. Note that, according to Corollary 2.19, the solution of the SPH approximated system exists globally in time. Hence, the following convergence theorem is limited in time by the existence of the solution of the Euler equations.

Theorem 3.9
Let $\Phi^r \in W^{1,2}(\mathbb{R}^d)$ be an even root kernel, which satisfies the moment condition of order $m \geq 1$ and the approximation condition of order $L > d/2$. Let $\Phi = \Phi^r * \Phi^r$ be the corresponding convolution kernel. Assume finite discrete mass and that the solution $(\mathbf{u}, \rho)$ of Euler's equations (3.1) - (3.3) *satisfy*

$$\begin{aligned} u_j &\in L^\infty(0,T;W^{\eta,2}(\mathbb{R}^d)), \qquad 1 \leq j \leq d,\\ \rho &\in L^\infty(0,T;L^1(\mathbb{R}^d)) \cap L^\infty(0,T;W^{\sigma,2}(\mathbb{R}^d)), \end{aligned} \tag{3.17}$$

for some time $T > 0$ with $\eta > \max\{L, m\} + \frac{d}{2} + 1$ and $\sigma > m + \frac{d}{2} + 1$. Let $(\mathbf{x}_\mathbf{k}^{\varepsilon,h}, \mathbf{u}_\mathbf{k}^{\varepsilon,h})_{\mathbf{k}\in\mathbb{Z}^d}$ be a solution of the corresponding SPH equations (3.4) - (3.6)*. Then, there exists a constant $C > 0$ such that the energy can be bounded by*

$$Q(t) \leq Q(0) + C\varepsilon^{\min\{m,2L-d\}}, \qquad t \in [0,T].$$

Proof. We start the proof by rewriting the energy error as

$$\begin{aligned} Q(t) =& h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\|\mathbf{u}_\mathbf{k}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h}(t),t)\|_2^2 + \int\limits_{\mathbb{R}^d}\left(\rho^{\varepsilon,h,r}(\mathbf{x},t) - \rho(\mathbf{x},t)\right)^2 d\mathbf{x}\\ =& h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\|\mathbf{u}_\mathbf{k}^{\varepsilon,h}(t)\|_2^2 - 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_\mathbf{k}^{\varepsilon,h}(t)\cdot\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h}(t),t)\\ &+ h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\|\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h}(t),t)\|_2^2 + \int\limits_{\mathbb{R}^d}\rho^{\varepsilon,h,r}(\mathbf{x},t)^2 d\mathbf{x}\\ &- 2\int\limits_{\mathbb{R}^d}\rho^{\varepsilon,h,r}(\mathbf{x},t)\rho(\mathbf{x},t)d\mathbf{x} + \int\limits_{\mathbb{R}^d}\rho(\mathbf{x},t)^2 d\mathbf{x}. \end{aligned}$$

We will now differentiate $Q(t)$ with respect to t. First, we note that combining the first and fourth term of $Q(t)$ and taking (3.15) and (3.16) into account results in the energy of the SPH system.

Applying Lemma 2.15, we have

$$\frac{d}{dt}\left(h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\|\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\|_2^2+\int\limits_{\mathbb{R}^d}\rho^{\varepsilon,h,r}(\mathbf{x},t)^2d\mathbf{x}\right)=0.$$

The other four terms will be differentiated separately. Before doing this, using equation (3.1), we note that

$$\begin{aligned}\frac{d}{dt}\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)&=\left(\left(\frac{d}{dt}\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)\right)\cdot\nabla\right)\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)+\partial_t\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&=\left(\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\cdot\nabla\right)\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&\quad-\left(\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\cdot\nabla\right)\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)-\nabla\rho(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&=\left(\left(\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)-\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\right)\cdot\nabla\right)\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)-\nabla\rho(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t).\end{aligned}\tag{3.18}$$

Hence, we have for the derivative of the second term of $Q(t)$

$$\begin{aligned}&-2\frac{d}{dt}h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\cdot\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&\qquad=-2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\left(\frac{d}{dt}\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\right)\cdot\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&\qquad\quad-2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\cdot\frac{d}{dt}\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&\qquad=2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\nabla\rho^{\varepsilon,h}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\cdot\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&\qquad\quad-2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\cdot\left(\left(\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)-\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\right)\cdot\nabla\right)\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&\qquad\quad+2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\cdot\nabla\rho(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&\qquad=:S_1+S_2+S_3.\end{aligned}$$

Using equation (3.18) again, the third term of $Q(t)$ has the derivative

$$\begin{aligned}\frac{d}{dt}h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\|\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\|^2&=2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\cdot\frac{d}{dt}\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&=2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\cdot\left(\left(\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)-\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\right)\cdot\nabla\right)\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&\quad-2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\cdot\nabla\rho(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\\&=:S_4+S_5.\end{aligned}$$

For the penultimate term of $Q(t)$, using the definition of $\rho^{\varepsilon,h,r}$ and equation (3.2), we have

$$\begin{aligned}-2\frac{d}{dt}\int\limits_{\mathbb{R}^d}\rho^{\varepsilon,h,r}(\mathbf{x},t)\rho(\mathbf{x},t)d\mathbf{x}&=2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\int\limits_{\mathbb{R}^d}\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t)\cdot\nabla\Phi_\varepsilon^r(\mathbf{x}-\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t))\rho(\mathbf{x},t)d\mathbf{x}\\&\quad+2\int\limits_{\mathbb{R}^d}\rho^{\varepsilon,h,r}(\mathbf{x},t)\nabla\cdot(\rho\mathbf{u})(\mathbf{x},t)d\mathbf{x}\\&=:S_6+S_7.\end{aligned}$$

Using equation (3.2), the last term has the derivative

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho(\mathbf{x},t)^2 d\mathbf{x} = -2\int_{\mathbb{R}^d} \rho(\mathbf{x},t)\nabla\cdot(\rho\mathbf{u})\,(\mathbf{x},t)d\mathbf{x} =: S_8.$$

Summarized, we can express the time derivative of $Q(t)$ as

$$\frac{d}{dt}Q(t) = (S_1+S_5+S_7+S_8)+(S_2+S_4)+(S_3+S_6) =: A_1+A_2+A_3, \tag{3.19}$$

which we now have to bound. From now on, we will suppress the time variable due to readability, i.e. we will, for example, write $\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h})$ instead of $\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h}(t),t)$.
Combining the two parts in the term A_2 leads to

$$\begin{aligned} A_2 &= -2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_\mathbf{k}^{\varepsilon,h}\cdot\left(\left(\mathbf{u}_\mathbf{k}^{\varepsilon,h}-\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h})\right)\cdot\nabla\right)\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) \\ &\quad + 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h})\cdot\left(\left(\mathbf{u}_\mathbf{k}^{\varepsilon,h}-\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h})\right)\cdot\nabla\right)\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) \\ &= 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\left(\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h})-\mathbf{u}_\mathbf{k}^{\varepsilon,h}\right)\cdot\left(\left(\mathbf{u}_\mathbf{k}^{\varepsilon,h}-\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h})\right)\cdot\nabla\right)\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h}). \end{aligned}$$

To find an estimate for the inner dot product, we note that for a $\mathbf{v}\in\mathbb{R}^d$ we have

$$\left|\mathbf{v}\cdot((\mathbf{v}\cdot\nabla)\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h}))\right| = \left|\sum_{j,k=1}^d v_j v_k \partial_k u_j(\mathbf{x}_\mathbf{k}^{\varepsilon,h})\right| \le \|\mathbf{u}\|_{L^\infty(W^{1,\infty})}\sum_{j,k=1}^d |v_j v_k|,$$

where Lemma 2.17 gives

$$\sum_{j,k=1}^d |v_j v_k| = \left(\sum_{j=1}^d |v_j|\right)^2 \le d\sum_{j=1}^d v_j^2 = d\|\mathbf{v}\|_2^2.$$

Hence, A_2 can be bounded by

$$|A_2| \le 2d\|\mathbf{u}\|_{L^\infty(W^{1,\infty})}h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\|\mathbf{u}_\mathbf{k}^{\varepsilon,h}-\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h})\|_2^2 \le C(u)Q(t).$$

With integration by parts and the fact that Φ^r is even we have for the term A_3

$$\begin{aligned} A_3 &= 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_\mathbf{k}^{\varepsilon,h}\cdot\nabla\rho(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) + 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_\mathbf{k}^{\varepsilon,h}\cdot\int_{\mathbb{R}^d}\nabla\Phi_\varepsilon^r(\mathbf{x}-\mathbf{x}_\mathbf{k}^{\varepsilon,h})\rho(\mathbf{x})d\mathbf{x} \\ &= 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_\mathbf{k}^{\varepsilon,h}\cdot\left(\nabla\rho(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) - \int_{\mathbb{R}^d}\Phi_\varepsilon^r(\mathbf{x}-\mathbf{x}_\mathbf{k}^{\varepsilon,h})\nabla\rho(\mathbf{x})d\mathbf{x}\right) \\ &= 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}_\mathbf{k}^{\varepsilon,h}\cdot\left(\nabla\rho(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) - (\nabla\rho)*\Phi_\varepsilon^r(\mathbf{x}_\mathbf{k}^{\varepsilon,h})\right), \end{aligned}$$

which can be bounded by

$$\begin{aligned} |A_3| &\le 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\|\mathbf{u}_\mathbf{k}^{\varepsilon,h}\|_2\|\nabla\rho(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) - (\nabla\rho)*\Phi_\varepsilon^r(\mathbf{x}_\mathbf{k}^{\varepsilon,h})\|_2 \\ &\le 2\|\nabla\rho - (\nabla\rho)*\Phi_\varepsilon^r\|_{L^\infty(\mathbb{R}^d)}h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\|\mathbf{u}_\mathbf{k}^{\varepsilon,h}\|_2. \end{aligned}$$

Noting that the kernel Φ^r satisfies the moment condition of order m, the first norm can be bounded with Theorem 1.28 by

$$\|\nabla\rho - (\nabla\rho) * \Phi^r_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq c\varepsilon^m |\nabla\rho|_{W^{m,\infty}(\mathbb{R}^d)} \leq C\varepsilon^m \|\rho\|_{L^\infty(W^{m+1,\infty})}.$$

The remaining sum can be estimated by

$$\begin{aligned} h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \|\mathbf{u}^{\varepsilon,h}_\mathbf{k}\|_2 &\leq h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left(\|\mathbf{u}^{\varepsilon,h}_\mathbf{k} - \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k})\|_2 + \|\mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k})\|_2 \right) \\ &\leq \left(h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \right)^{1/2} \left(h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \|\mathbf{u}^{\varepsilon,h}_\mathbf{k} - \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k})\|_2^2 \right)^{1/2} \\ &\quad + h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \|\mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k})\|_2 \\ &\leq \frac{M}{2} + \frac{1}{2} h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \|\mathbf{u}^{\varepsilon,h}_\mathbf{k} - \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k})\|_2^2 + M\|\mathbf{u}\|_{L^\infty(L^\infty)} \\ &\leq C(\mathbf{u}, M)(1 + Q(t)), \end{aligned}$$

where we used the finite discrete mass of the particles $h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \leq M$. Altogether, A_3 can be bounded by

$$|A_3| \leq C(u, \rho, M)\varepsilon^m (1 + Q(t)). \tag{3.20}$$

For the term A_1, we have to work a little harder. First of all, we rewrite A_1 to arrive at

$$\begin{aligned} A_1 &= 2h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \nabla\rho^{\varepsilon,h}(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) \cdot \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) - 2h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) \cdot \nabla\rho(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) \\ &\quad + 2 \int_{\mathbb{R}^d} \rho^{\varepsilon,h,r}(\mathbf{x}) \nabla\cdot(\rho\mathbf{u})(\mathbf{x}) d\mathbf{x} - 2 \int_{\mathbb{R}^d} \rho(\mathbf{x}) \nabla\cdot(\rho\mathbf{u})(\mathbf{x}) d\mathbf{x} \\ &= 2h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) \cdot \left(\nabla\rho^{\varepsilon,h}(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) - \nabla\rho(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) \right) + 2 \int_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r}(\mathbf{x}) - \rho(\mathbf{x}) \right) \nabla\cdot(\rho\mathbf{u})(\mathbf{x}) d\mathbf{x} \\ &= 2h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) \cdot \nabla \left(\rho^{\varepsilon,h} - \rho \right) (\mathbf{x}^{\varepsilon,h}_\mathbf{k}) - 2 \int_{\mathbb{R}^d} (\rho\mathbf{u})(\mathbf{x}) \cdot \nabla \left(\rho^{\varepsilon,h,r} - \rho \right) (\mathbf{x}) d\mathbf{x} \\ &= 2h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) \cdot \nabla \left(\rho^{\varepsilon,h} - \rho * \Phi_\varepsilon \right) (\mathbf{x}^{\varepsilon,h}_\mathbf{k}) - 2 \int_{\mathbb{R}^d} (\rho\mathbf{u})(\mathbf{x}) \cdot \nabla \left(\rho^{\varepsilon,h} - \rho * \Phi_\varepsilon \right) (\mathbf{x}) d\mathbf{x} + R \end{aligned}$$

with the remainder R given by

$$\begin{aligned} R &= 2h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) \cdot \nabla \left(\rho * \Phi_\varepsilon - \rho \right) (\mathbf{x}^{\varepsilon,h}_\mathbf{k}) - 2 \int_{\mathbb{R}^d} (\rho\mathbf{u})(\mathbf{x}) \cdot \nabla \left(\rho^{\varepsilon,h,r} - \rho^{\varepsilon,h} \right) (\mathbf{x}) d\mathbf{x} \\ &\quad - 2 \int_{\mathbb{R}^d} (\rho\mathbf{u})(\mathbf{x}) \cdot \nabla \left(\rho * \Phi_\varepsilon - \rho \right) (\mathbf{x}) d\mathbf{x} \\ &= 2h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \mathbf{u}(\mathbf{x}^{\varepsilon,h}_\mathbf{k}) \cdot \left((\nabla\rho) * \Phi_\varepsilon - \nabla\rho \right) (\mathbf{x}^{\varepsilon,h}_\mathbf{k}) + 2 \int_{\mathbb{R}^d} \nabla\cdot(\rho\mathbf{u})(\mathbf{x}) \left(\rho^{\varepsilon,h,r} - \rho^{\varepsilon,h} \right) (\mathbf{x}) d\mathbf{x} \\ &\quad + 2 \int_{\mathbb{R}^d} \nabla\cdot(\rho\mathbf{u})(\mathbf{x}) \left(\rho * \Phi_\varepsilon - \rho \right) (\mathbf{x}) d\mathbf{x} \\ &=: R_1 + R_2 + R_3. \end{aligned}$$

The first part of the remainder can be bounded similarly to the term A_3. Using Theorem 1.28 we have

$$|R_1| \leq 2M\|\mathbf{u}\|_{L^\infty(L^\infty)}\|(\nabla\rho)*\Phi_\varepsilon - \nabla\rho\|_{L^\infty(\mathbb{R}^d)} \leq C(\mathbf{u},M)\varepsilon^m\|\rho\|_{L^\infty(W^{m+1,\infty})}.$$

With the same arguments we can bound the third term of the remainder by

$$|R_3| \leq 2\|\nabla\cdot(\rho\mathbf{u})\|_{L^\infty(L^\infty)}\|\rho*\Phi_\varepsilon - \rho\|_{L^1(\mathbb{R}^d)} \leq C(\rho,\mathbf{u})\varepsilon^m\|\rho\|_{L^\infty(W^{m,1})}.$$

For the second part R_2 we insert the definition of $\rho^{\varepsilon,h}$ and $\rho^{\varepsilon,h,r}$ and derive

$$\begin{aligned}
R_2 &= 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\left(\int_{\mathbb{R}^d}\nabla\cdot(\rho\mathbf{u})(\mathbf{x})\left(\Phi_\varepsilon^r(\mathbf{x}-\mathbf{x}_\mathbf{k}^{\varepsilon,h}) - \Phi_\varepsilon(\mathbf{x}-\mathbf{x}_\mathbf{k}^{\varepsilon,h})\right)d\mathbf{x}\right)\\
&= 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\left((\nabla\cdot(\rho\mathbf{u}))*\Phi_\varepsilon^r(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x}_\mathbf{k}^{\varepsilon,h}))\right)\\
&\quad + 2h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\left(\nabla\cdot(\rho\mathbf{u})(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) - (\nabla\cdot(\rho\mathbf{u}))*\Phi_\varepsilon(\mathbf{x}_\mathbf{k}^{\varepsilon,h}))\right),
\end{aligned}$$

where we just have two convolution errors so that we have the bound

$$|R_2| \leq C(\mathbf{u},\rho,M)\varepsilon^m.$$

Altogether, the remainder R can be bounded by

$$|R| \leq C(\mathbf{u},\rho,M)\varepsilon^m.$$

It remains to show that the first part of A_1 can be bounded. First of all, we define

$$\widetilde{A}_1 := h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})\mathbf{u}(\mathbf{x}_\mathbf{k}^{\varepsilon,h})\cdot\nabla\left(\rho^{\varepsilon,h}-\rho*\Phi_\varepsilon\right)(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) - \int_{\mathbb{R}^d}(\rho\mathbf{u})(\mathbf{x})\cdot\nabla\left(\rho^{\varepsilon,h}-\rho*\Phi_\varepsilon\right)(\mathbf{x})d\mathbf{x}.$$

We will derive a bound for each component $u_j\partial_j$ of $\mathbf{u}\cdot\nabla$. Using $\Phi_\varepsilon = \Phi_\varepsilon^r * \Phi_\varepsilon^r$, we can write

$$\partial_j(\rho^{\varepsilon,h}-\rho*\Phi_\varepsilon) = \partial_j(\rho^{\varepsilon,h,r}*\Phi_\varepsilon^r - \rho*\Phi_\varepsilon^r*\Phi_\varepsilon^r) =: (\partial_j\Phi_\varepsilon^r)*f^{\varepsilon,h}$$

with $f^{\varepsilon,h}$ given by

$$f^{\varepsilon,h} = \rho^{\varepsilon,h,r} - \rho*\Phi_\varepsilon^r.$$

With this, we can write the j-th part of $\widetilde{A}_1$ by

$$\widetilde{A}_{1,j} := h^d\sum_{\mathbf{k}\in\mathbb{Z}^d}\rho_0(h\mathbf{k})u_j(\mathbf{x}_\mathbf{k}^{\varepsilon,h})(\partial_j\Phi_\varepsilon^r)*f^{\varepsilon,h}(\mathbf{x}_\mathbf{k}^{\varepsilon,h}) - \int_{\mathbb{R}^d}(\rho u_j)(\mathbf{x})(\partial_j\Phi_\varepsilon^r)*f^{\varepsilon,h}(\mathbf{x})d\mathbf{x},$$

where we have the equality $\widetilde{A}_1 = \sum_{j=1}^d \widetilde{A}_{1,j}$.
Since $u_j \in C^{L+1}(\mathbb{R}^d)$ for all $1\leq j\leq d$ by the Sobolev embedding theorem and our assumption on the smoothness of $\mathbf{u}$, we may use a Taylor expansion of $u_j(\mathbf{x})$ about $\mathbf{y}\in\mathbb{R}^d$ given by

$$u_j(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|\leq L}\frac{D^{\boldsymbol{\alpha}}u_j(\mathbf{y})}{\boldsymbol{\alpha}!}(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}} + (L+1)\sum_{|\boldsymbol{\alpha}|=L+1}\frac{(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\int_0^1(1-s)^L D^{\boldsymbol{\alpha}}u_j(\mathbf{x}-s(\mathbf{x}-\mathbf{y}))ds.$$

Inserting the Taylor expansion of $u_j(\mathbf{x})$ and using the function $W_{\varepsilon,\boldsymbol{\alpha}}^j$ from Proposition 3.6 gives

$$\begin{aligned}
u_j(\mathbf{x})(\partial_j\Phi_\varepsilon^r)*f^{\varepsilon,h}(\mathbf{x}) &= \int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y})u_j(\mathbf{x})\partial_j\Phi_\varepsilon^r(\mathbf{x}-\mathbf{y})d\mathbf{y}\\
&= \int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y})\sum_{|\boldsymbol{\alpha}|\leq L}\frac{D^{\boldsymbol{\alpha}}u_j(\mathbf{y})}{\boldsymbol{\alpha}!}(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}}\partial_j\Phi_\varepsilon^r(\mathbf{x}-\mathbf{y})d\mathbf{y} + R_L(\mathbf{x})\\
&= \sum_{|\boldsymbol{\alpha}|\leq L}\int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y})D^{\boldsymbol{\alpha}}u_j(\mathbf{y})W_{\varepsilon,\boldsymbol{\alpha}}^j(\mathbf{y}-\mathbf{x})d\mathbf{y} + R_L(\mathbf{x}),
\end{aligned}$$

where the remainder is given by

$$R_L(\mathbf{x}) = (L+1) \sum_{|\boldsymbol{\alpha}|=L+1} \int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x}) \int_0^1 (1-s)^L D^{\boldsymbol{\alpha}} u_j(\mathbf{x}-s(\mathbf{x}-\mathbf{y})) ds d\mathbf{y}.$$

Inserting this into $\widetilde{A}_{1,j}$ gives

$$\begin{aligned}
\widetilde{A}_{1,j} &= h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \left[\sum_{|\boldsymbol{\alpha}|\le L} \int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y}) D^{\boldsymbol{\alpha}} u_j(\mathbf{y}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) d\mathbf{y} + R_L(\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) \right] \\
&\quad - \int_{\mathbb{R}^d} \rho(\mathbf{x}) \left[\sum_{|\boldsymbol{\alpha}|\le L} \int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y}) D^{\boldsymbol{\alpha}} u_j(\mathbf{y}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x}) d\mathbf{y} + R_L(\mathbf{x}) \right] d\mathbf{x} \\
&= \sum_{|\boldsymbol{\alpha}|\le L} \int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y}) D^{\boldsymbol{\alpha}} u_j(\mathbf{y}) \left[h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) - \int_{\mathbb{R}^d} \rho(\mathbf{x}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x}) d\mathbf{x} \right] d\mathbf{y} \\
&\quad + h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) R_L(\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) - \int_{\mathbb{R}^d} \rho(\mathbf{x}) R_L(\mathbf{x}) d\mathbf{x} \\
&=: \sum_{|\boldsymbol{\alpha}|\le L} \widetilde{R}_{\boldsymbol{\alpha}} + \widetilde{R}_L.
\end{aligned}$$

We will bound these terms separately. For $\boldsymbol{\alpha} = \mathbf{0}$ we have $W^j_{\varepsilon,\mathbf{0}} = -\partial_j \Phi^r_\varepsilon$. With the definition of $f^{\varepsilon,h}$ we find

$$\begin{aligned}
|\widetilde{R}_{\mathbf{0}}| &= \left| \int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y}) u_j(\mathbf{y}) \left[h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \partial_j \Phi^r_\varepsilon(\mathbf{y}-\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) - \int_{\mathbb{R}^d} \rho(\mathbf{x}) \partial_j \Phi^r_\varepsilon(\mathbf{y}-\mathbf{x}) d\mathbf{x} \right] d\mathbf{y} \right| \\
&= \left| \int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y}) u_j(\mathbf{y}) \partial_j f^{\varepsilon,h}(\mathbf{y}) d\mathbf{y} \right| = \frac{1}{2} \left| \int_{\mathbb{R}^d} u_j(\mathbf{y}) \partial_j (f^{\varepsilon,h}(\mathbf{y}))^2 d\mathbf{y} \right| \\
&= \frac{1}{2} \left| \int_{\mathbb{R}^d} \partial_j u_j(\mathbf{y}) (f^{\varepsilon,h}(\mathbf{y}))^2 d\mathbf{y} \right| \le \frac{1}{2} \|\partial_j u_j\|_{L^\infty(\mathbb{R}^d)} \|f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

The L^2-norm of $f^{\varepsilon,h}$ will be bounded later. For $1 \le |\boldsymbol{\alpha}| \le L$, $\widetilde{R}_{\boldsymbol{\alpha}}$ can be bounded with the Cauchy-Schwarz inequality by

$$\begin{aligned}
|\widetilde{R}_{\boldsymbol{\alpha}}| &\le \|D^{\boldsymbol{\alpha}} u_j\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |f^{\varepsilon,h}(\mathbf{y})| \left| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) - \int_{\mathbb{R}^d} \rho(\mathbf{x}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x}) d\mathbf{x} \right| d\mathbf{y} \\
&\le \|D^{\boldsymbol{\alpha}} u_j\|_{L^\infty(\mathbb{R}^d)} \|f^{\varepsilon,h}\|_{L^2(\mathbb{R}^d)} \left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\cdot - \mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) - \rho * W^j_{\varepsilon,\boldsymbol{\alpha}} \right\|_{L^2(\mathbb{R}^d)} \\
&=: \|D^{\boldsymbol{\alpha}} u_j\|_{L^\infty(\mathbb{R}^d)} \|f^{\varepsilon,h}\|_{L^2(\mathbb{R}^d)} \left\| F^j_{\boldsymbol{\alpha}} \right\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where we defined, analogously to $f^{\varepsilon,h}$,

$$F^j_{\boldsymbol{\alpha}} := h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\cdot - \mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) - \rho * W^j_{\varepsilon,\boldsymbol{\alpha}}. \tag{3.21}$$

With Plancherel's identity, Theorem 1.3, we can bound the $L^2(\mathbb{R}^d)$-norm of $F^j_{\boldsymbol{\alpha}}$ by

$$\begin{aligned}
\|F^j_{\boldsymbol{\alpha}}\|^2_{L^2(\mathbb{R}^d)} &= \left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\cdot - \mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) - \rho * W^j_{\varepsilon,\boldsymbol{\alpha}} \right\|^2_{L^2(\mathbb{R}^d)} \\
&= \left\| \left(h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) e^{-i\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}\cdot} \right) \widehat{W^j_{\varepsilon,\boldsymbol{\alpha}}} - (2\pi)^{d/2} \widehat{\rho}\, \widehat{W^j_{\varepsilon,\boldsymbol{\alpha}}} \right\|^2_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} |\widehat{W^j_{\varepsilon,\boldsymbol{\alpha}}}(\boldsymbol{\omega})|^2 \left| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) e^{-i\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}\boldsymbol{\omega}} - (2\pi)^{d/2}\widehat{\rho}(\boldsymbol{\omega}) \right|^2 d\boldsymbol{\omega}.
\end{aligned}$$

Since Φ^r_ε satisfies the approximation condition of order L, we have $|\widehat{W^j_{\varepsilon,\boldsymbol{\alpha}}}| \le C\varepsilon^{|\boldsymbol{\alpha}|-1}|\widehat{\Phi^r_\varepsilon}|$ by (3.13) almost everywhere. Hence,

$$\begin{aligned}
\|F^j_{\boldsymbol{\alpha}}\|^2_{L^2(\mathbb{R}^d)} &\le C\varepsilon^{2|\boldsymbol{\alpha}|-2} \int_{\mathbb{R}^d} |\widehat{\Phi^r_\varepsilon}(\boldsymbol{\omega})|^2 \left| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) e^{-i\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}\boldsymbol{\omega}} - (2\pi)^{d/2}\widehat{\rho}(\boldsymbol{\omega}) \right|^2 d\boldsymbol{\omega} \\
&= C\varepsilon^{2|\boldsymbol{\alpha}|-2} \left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \Phi^r_\varepsilon(\cdot - \mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) - \rho * \Phi^r_\varepsilon \right\|^2_{L^2(\mathbb{R}^d)} \\
&= C\varepsilon^{2|\boldsymbol{\alpha}|-2} \|f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)},
\end{aligned} \tag{3.22}$$

so that we have

$$|\widetilde{R}_{\boldsymbol{\alpha}}| \le C\varepsilon^{|\boldsymbol{\alpha}|-1} \|D^{\boldsymbol{\alpha}} u_j\|_{L^\infty(\mathbb{R}^d)} \|f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)}.$$

It remains to bound $\widetilde{R}_L$. We have

$$\begin{aligned}
|\widetilde{R}_L| &= \left| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) R_L(\mathbf{x}^{\varepsilon,h}_{\mathbf{k}}) - \int_{\mathbb{R}^d} \rho(\mathbf{x}) R_L(\mathbf{x}) d\mathbf{x} \right| \\
&\le \|R_L\|_{L^\infty(\mathbb{R}^d)} \left(M + \|\rho\|_{L^1(\mathbb{R}^d)} \right).
\end{aligned}$$

For R_L we have

$$\begin{aligned}
|R_L(\mathbf{x})| &= \left| (L+1) \sum_{|\boldsymbol{\alpha}|=L+1} \int_{\mathbb{R}^d} f^{\varepsilon,h}(\mathbf{y}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x}) \int_0^1 (1-s)^L D^{\boldsymbol{\alpha}} u_j(\mathbf{x} - s(\mathbf{x}-\mathbf{y})) ds d\mathbf{y} \right| \\
&\le (L+1) \int_0^1 (1-s)^L ds \sum_{|\boldsymbol{\alpha}|=L+1} \|D^{\boldsymbol{\alpha}} u_j\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |f^{\varepsilon,h}(\mathbf{y}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x})| d\mathbf{y},
\end{aligned}$$

where we first note that

$$(L+1)\int_0^1 (1-s)^L ds = 1.$$

For the remaining integral, we are using the Cauchy-Schwarz inequality and (3.14) to conclude

$$\int_{\mathbb{R}^d} |f^{\varepsilon,h}(\mathbf{y}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{x})| d\mathbf{y} \le \|f^{\varepsilon,h}\|_{L^2(\mathbb{R}^d)} \|W^j_{\varepsilon,\boldsymbol{\alpha}}\|_{L^2(\mathbb{R}^d)} \le \varepsilon^{L-d/2} \|f^{\varepsilon,h}\|_{L^2(\mathbb{R}^d)}.$$

Overall, we have

$$
\begin{aligned}
|\widetilde{R}_L| &\leq \varepsilon^{L-d/2}\left(M + \|\rho\|_{L^1(\mathbb{R}^d)}\right)\|u\|_{L^\infty(W^{L+1,\infty})}\|f^{\varepsilon,h}\|_{L^2(\mathbb{R}^d)} \\
&\leq C(\mathbf{u},\rho,M)\left(\varepsilon^{2L-d} + \|f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)}\right).
\end{aligned}
$$

In conclusion, we have for $\widetilde{A}_1$, and hence A_1, the estimates

$$
\begin{aligned}
|\widetilde{A}_1| &\leq C(\mathbf{u},\rho,M)\left(\varepsilon^{2L-d} + \|f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)}\right), \\
|A_1| &\leq C(\mathbf{u},\rho,M)\left(\varepsilon^{m} + \varepsilon^{2L-d} + \|f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)}\right).
\end{aligned}
$$

For $f^{\varepsilon,h}$ we derive the bound

$$
\|f^{\varepsilon,h}\|_{L^2(\mathbb{R}^d)} \leq \|\rho^{\varepsilon,h,r} - \rho\|_{L^2(\mathbb{R}^d)} + \|\rho - \rho * \Phi^r_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq Q^{1/2}(t) + C\varepsilon^m\|\rho\|_{L^\infty(W^{m,2})},
$$

which gives us

$$
|A_1| \leq C(\mathbf{u},\rho,M)\left(\varepsilon^m + \varepsilon^{2L-d} + Q(t)\right). \tag{3.23}
$$

Together with the estimates of A_2 and A_3, we finally arrive at

$$
\frac{d}{dt}Q(t) \leq C\left(\varepsilon^{\min\{m,2L-d\}} + Q(t)\right),
$$

so that applying Gronwall's inequality yields

$$
Q(t) \leq Q(0) + C\varepsilon^{\min\{m,2L-d\}}
$$

for all $t \in [0,T]$. □

Before we combine Theorem 3.8 and Theorem 3.9, we will take a look at the conditions (3.17) we required of the solution of Euler's equations. The given conditions are very restrictive, Theorem 3.9 only holds for smooth solutions of the Euler equations. However, using the theory for nonlinear hyperbolic equations, we know for which initial data $\mathbf{u}_0$ and ρ_0 conditions (3.17) are satisfied.

Lemma 3.10
Suppose that $\mathbf{u}_0 \in H^\sigma(\mathbb{R}^d)^d$ *and* $\rho_0^{1/2} \in H^\sigma(\mathbb{R}^d)$ *for a* $\sigma > \max\{L,m\}+1+d/2$. *Then, there exists a time* $T > 0$ *such that the unique solution* $\mathbf{u} : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ *and* $\rho : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ *of Euler's equations* (3.1) - (3.3) *satisfy the conditions* (3.17).

Proof. From Theorem 2.9 we know that u_j, $1 \leq j \leq d$, and $\rho^{1/2}$ are in $C([0,T], W^{\sigma,2}(\mathbb{R}^d))$.
For the density, we note that $\rho^{1/2} \in C([0,T], W^{\sigma,2}(\mathbb{R}^d))$ and Theorem 1.18 imply that $\rho \in C([0,T], W^{\sigma,2}(\mathbb{R}^d))$. Since u_j, $1 \leq j \leq d$, and ρ are continuous in time, Lemma 1.25 states that $u_j, \rho \in L^\infty(0,T; W^{\sigma,2}(\mathbb{R}^d))$.
Finally, knowing that the root $\rho^{1/2}$ of the density ρ is in $C([0,T]; L^2(\mathbb{R}^d))$ implies that the density ρ itself is in $C([0,T]; L^1(\mathbb{R}^d))$. Thus, Lemma 1.25 again yields $\rho \in L^\infty(0,T; L^1(\mathbb{R}^d))$. □

Combining the last three results will finally give us a convergence result for the SPH method, which depends only on the kernel and the initial conditions of the Euler equations. Note that if $\rho_0^{1/2} \in H^\sigma(\mathbb{R}^d)$ for a $\sigma > \max\{L,m\} + 1 + d/2$, Theorem 1.18 yields that we have $\rho_0 \in H^\sigma(\mathbb{R}^d)$. Hence ρ_0 is continuous according to the sobolev embedding theorem.

Corollary 3.11
Let $\Phi^r \in W^{s,1}(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, *be an even root kernel, which satisfies the moment condition of order* $m \geq 1$ *and the approximation condition of order* $L > d/2$. *Let* $\Phi = \Phi^r * \Phi^r$ *be the corresponding convolution kernel. Let* $\varepsilon > 0$ *and* $h > 0$.
Assume initial data $\mathbf{u}_0 \in W^{\sigma,2}(\mathbb{R}^d)^d$ *and finite discrete mass with density* ρ_0 *satisfying* $\rho_0^{1/2} \in$

$W^{\max\{\sigma,s\},2}(\mathbb{R}^d)$ for a $\sigma > \max\{L,m\} + 1 + d/2$ and that the solution $(\mathbf{u},\rho)$ of Euler's equations (3.1) - (3.3) *exists up to a time $T > 0$. Let $(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}, \mathbf{u}_{\mathbf{k}}^{\varepsilon,h})_{\mathbf{k}\in\mathbb{Z}^d}$ be a solution of the corresponding SPH equations* (3.4) - (3.6).
Then, there exists a constant $C > 0$ such that the energy error can be bounded by

$$Q(t) \leq C\left(\frac{h^{2s}}{\varepsilon^{2s}} + \varepsilon^{\min\{m,2L-d\}}\right)$$

for all $t \in [0,T]$.

We will finish the section with a few remarks about the given convergence result.

Remark 3.12

i) Theorem 3.9 is only valid for smooth solutions of the Euler equations. Unfortunately, it is a well-known fact that hyperbolic equations like the Euler equations can form discontinuities after a period of time even with smooth initial values, see, for example, [NS04] or [Sid85]. Thus, the convergence result is not applicable to shocks or other irregular solutions and is only valid as long as the smooth solution exists.

ii) The error bound is only valid for the given equation of state. A generalization of the equation of state to the form $p = c\rho^\gamma$ for a $\gamma \geq 1$ has a significant effect to the right-hand side of the SPH equations, see equations (2.12) - (2.14), and also to the associated energy. This results in a much more complex energy error $Q(t)$ and the current proof of convergence does not hold.

iii) The given theory is only valid in the absence of boundary conditions. Boundary conditions, especially the treatment of walls, play an important role in the application of the SPH method. Even though there exist some ideas on how to treat boundaries, see for example [Vio12, ch. 6], there is so far no mathematical investigation of these boundary treatments.

3.4 Pointwise Convergence

In this section we want to extend the given convergence result in Corollary 3.11 to a pointwise convergence of the particle trajectories, i.e. we want to show that $\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t)$ converges to $\mathbf{X}(h\mathbf{j},t)$ for every $\mathbf{j} \in \mathbb{Z}^d$ and $t \in [0,T]$, as ε and h tends to zero in a specific way. Unfortunately, the result of Corollary 3.11 will not suffice to have a uniform error bound for all particles.
Nevertheless, we are able to extend the result to pointwise convergence on compact sets where the initial density is sufficiently far away from zero. For particles with too small initial mass, the error cannot be bounded. However, these particles are of minor interest since they carry less information.
We will provide two ways of how we can achieve a form of compact convergence.

Theorem 3.13
Let the assumptions of Corollary 3.11 hold. This means in particular, that the kernel satisfies the moment condition of order $m \geq 1$, the approximation condition of order $L > d/2$ and has smoothness $s \geq 1$. Then, the following holds:

i) If $\rho_0(\mathbf{x}) > 0$ for all $\|\mathbf{x}\|_2 < 1$ then, for each $\mathbf{j} \in \mathbb{Z}^d$, there exists a constant $C > 0$ such that we have for sufficiently small h,

$$\|\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{X}(h\mathbf{j},t)\|_2 + \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{X}(h\mathbf{j},t),t)\|_2 \leq C\left(\frac{h^{s-d/2}}{\varepsilon^s} + \frac{\varepsilon^{\min\{m/2,L-d/2\}}}{h^{d/2}}\right)$$

for all $t \in [0,T]$.

ii) Let $K \subset \mathbb{R}^d$ be compact with $\inf_{\mathbf{x}\in K} \rho_0(\mathbf{x}) > 0$. Then, there exists a constant $C_K > 0$ such that we have

$$\|\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{X}(h\mathbf{j},t)\|_2 + \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{X}(h\mathbf{j},t),t)\|_2 \leq C_K \left(\frac{h^{s-d/2}}{\varepsilon^s} + \frac{\varepsilon^{\min\{m/2, L-d/2\}}}{h^{d/2}} \right)$$

for each $\mathbf{j} \in \mathbb{Z}^d$ with $h\mathbf{j} \in K$ and all $t \in [0,T]$.

Proof. First we note that we have

$$\frac{d}{dt}\|\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{X}(h\mathbf{j},t)\|_2 \leq \left\| \frac{d}{dt}\left(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{X}(h\mathbf{j},t)\right)\right\|_2 \leq \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{X}(h\mathbf{j},t),t)\|_2$$

which, together with $\|\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(0) - \mathbf{X}(h\mathbf{j},0)\|_2 = 0$, instantly implies that

$$\|\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{X}(h\mathbf{j},t)\|_2 \leq T \sup_{\tau\in[0,T]} \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(\tau) - \mathbf{u}(\mathbf{X}(h\mathbf{j},\tau),\tau)\|_2$$

for each $\mathbf{j} \in \mathbb{Z}^d$. We also note that ρ_0 is continuous since $\rho_0^{1/2} \in W^{s,2}(\mathbb{R}^d)$. Now we have to distinguish both cases.
Let $\mathbf{j} \in \mathbb{Z}^d$ be fix with $\rho_0(h\mathbf{j}) > 0$. As ρ_0 is positive on the closed unit ball, it attains its positive minimum $\eta := \min\{\rho_0(\mathbf{x}) \mid \|\mathbf{x}\|_2 \leq 1\}$. For sufficiently small h we thus have $\rho_0(h\mathbf{j}) \geq \eta$. Hence, we have

$$\begin{aligned}
\|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t),t)\|_2^2 &\leq \frac{\rho_0(h\mathbf{j})}{\eta}\|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t),t)\|_2^2 \\
&\leq \eta^{-1}h^{-d}\left(h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k})\|\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t),t)\|^2\right) \\
&= \eta^{-1}h^{-d}Q(t) \\
&\leq Ch^{-d}\left(\frac{h^{2s}}{\varepsilon^{2s}} + \varepsilon^{\max\{m,2L-d\}}\right),
\end{aligned}$$

for all $t \in [0,T]$, where we used Corollary 3.11. The triangle inequality then yields

$$\begin{aligned}
\|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{X}(h\mathbf{j},t),t)\|_2 &\leq \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t),t)\|_2 + \|\mathbf{u}(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t),t) - \mathbf{u}(\mathbf{X}(h\mathbf{j},t),t)\|_2 \\
&\leq \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t),t)\|_2 + \|\mathbf{u}\|_{L^\infty(W^{1,\infty})}\|\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{X}(h\mathbf{j},t)\|_2 \\
&\leq \left(1 + T\|\mathbf{u}\|_{L^\infty(W^{1,\infty})}\right) \sup_{\tau\in[0,T]} \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(\tau) - \mathbf{u}(\mathbf{X}(h\mathbf{j},\tau),\tau)\|_2 \\
&\leq C(T,\mathbf{u})\left(\frac{h^{s-d/2}}{\varepsilon^s} + \frac{\varepsilon^{\max\{m/2,L-d/2\}}}{h^{d/2}}\right),
\end{aligned}$$

which gives us the case i).
For ii), ρ_0 is positive again on K, and thus attains its positive minimum $\eta_K := \min\{\rho_0(\mathbf{x}) \mid \mathbf{x} \in K\}$. Hence, for all $\mathbf{j} \in \mathbb{Z}^d$ with $h\mathbf{j} \in K$ we have

$$\|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t),t)\|_2^2 \leq C_K h^{-d}\left(\frac{h^{2s}}{\varepsilon^{2s}} + \varepsilon^{\max\{m,2L-d\}}\right),$$

from which it follows that

$$\|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{X}(h\mathbf{j},t),t)\|_2 \leq C_K(T,u)\left(\frac{h^{s-d/2}}{\varepsilon^s} + \frac{\varepsilon^{\max\{m/2,L-d/2\}}}{h^{d/2}}\right)$$

with the same calculations and arguments as in the first case. □

Both representations of the pointwise convergence in Theorem 3.13 have their advantages and drawbacks. The first case gives us convergence for a fixed $\mathbf{j} \in \mathbb{Z}^d$ for a sufficiently small h, which means that we trace the same $\mathbf{j}$ as we sent ε and h to zero. Unfortunately, tracing the same $\mathbf{j} \in \mathbb{Z}^d$ means that the particle starting at $h\mathbf{j}$ represents different positions as h tends to zero, which means that we trace different particle trajectories. As we set h sufficiently small such that $\rho_0(h\mathbf{j}) \geq \min\{\rho_0(\mathbf{x}) \,|\, \|\mathbf{x}\|_2 \leq 1\}$ means in general that $\|h\mathbf{j}\|_2 < 1$. Hence, this theoretically gives us only convergence for particles starting in the closed unit ball.
The second case states convergence for an arbitrary compact set $K \in \mathbb{R}^d$. However, if h tends to zero, the set of indices, such that $h\mathbf{j}$ is in K, changes and we do not get convergence of a specific index $\mathbf{j}$. Hence, both cases do not state uniform convergence on the whole space $\mathbb{R}^d$.

3.5 Outlook: The Periodic Case

At the end of this chapter we will briefly review the SPH method for the barotropic Euler's equations on the periodic d-dimensional Torus $\mathbb{T}^d = [0,1[^d$. Even if the proof of the SPH method on the whole space $\mathbb{R}^d$ is interesting from a theoretical point of view, it has its limitations in applications. Since we have an infinite number of particles, the only case where we can verify the result is for an initial density with compact support.
However, in the case of periodic boundary conditions, we only have a finite number of particles, which allows us an easy computation and verification of the given results. This is why, in this section, we will give the underlying equations and the expected results. However, since a complete proof of the convergence result in the periodic case would mean to repeat the whole, given theory we will restrict ourselves to a short overview. Note that in [MO13], a large part of the theory we would need can be found.
For the rest of this section, we will call a function $f : \mathbb{R}^d \to \mathbb{R}$ periodic, if $f(\mathbf{x}+\mathbf{j}) = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{j} \in \mathbb{Z}^d$.
The underlying Euler's equations remain as in (3.1) - (3.3), with the small modification that the given initial velocity $\mathbf{u}_0 : \mathbb{R}^d \to \mathbb{R}^d$ and initial density $\rho_0 : \mathbb{R}^d \to \mathbb{R}$ are supposed to be periodic, and we now seek a solution $(\mathbf{u}, \rho)$ which is periodic in space as well. This is equivalent to solving Euler's equations on $\mathbb{T}^d$ with periodic boundary conditions.
For the associated SPH system, we also have to make some modifications. Let $N \in \mathbb{N}$ and let $h = 1/N$ be the spatial discretization parameter, with which we will divide the Torus $\mathbb{T}^d$ into N^d cubes with edge length h and midpoint $h\mathbf{j}$ for $\mathbf{j} \in \mathbb{G}_N^d$, where the set of indices $\mathbb{G}_N^d$ is defined by

$$\mathbb{G}_N^d := \{\mathbf{j} \in \mathbb{N}_0^d;\ j_k < N \text{ for all } 1 \leq k \leq d\}.$$

Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be an at least one time differentiable, even and periodic function, and let $\Phi_\varepsilon := \varepsilon^{-d}\Phi(\cdot/\varepsilon)$ be its scaled version for a smoothing parameter $\varepsilon > 0$. The SPH system for the barotropic Euler equations on the Torus $\mathbb{T}^d$ is given by

$$\frac{d}{dt}\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) = \mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t), \tag{3.24}$$

$$\frac{d}{dt}\mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(t) = -\nabla\rho^{\varepsilon,h}(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t),t) = h^d \sum_{\mathbf{k}\in\mathbb{G}_N^d} \rho_0(h\mathbf{k})\nabla\Phi_\varepsilon(\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(t) - \mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t)), \tag{3.25}$$

for all $\mathbf{j} \in \mathbb{G}_N^d$ and all $t \in]0,T]$ and

$$\mathbf{x}_{\mathbf{j}}^{\varepsilon,h}(0) = h\mathbf{j}, \quad \mathbf{u}_{\mathbf{j}}^{\varepsilon,h}(0) = \mathbf{u}_0(h\mathbf{j}), \tag{3.26}$$

for all $\mathbf{j} \in \mathbb{G}_N^d$.
To achieve similar results like those in Theorem 3.8, Theorem 3.9 and Corollary 3.11 for the periodic case, we would need the theory, including the definitions for the moment and approximation conditions in the periodic case. We will neglect the technical details here because it would go

beyond the scope of this work at this point, but refer to [MO13]. Instead, we want to give a short overview of the expected results.
The most significant difference will be that the density ρ_0 does not need to vanish at infinity. Since we are on the periodic but bounded domain $\mathbb{T}^d$, it suffices to assume $\rho_0 \in L^\infty(\mathbb{T}^d)$ to ensure that $\rho_0 \in L^p(\mathbb{T}^d)$ for every $1 \leq p \leq \infty$. This means that we can assume finite discrete mass even if $\inf_{\mathbf{x}\in\mathbb{T}^d} \rho_0(\mathbf{x}) > 0$, which makes a simplification for the whole theory.
The convergence of the energy error term, which is now given by

$$Q(t) := h^d \sum_{\mathbf{k}\in\mathbb{G}_N^d} \rho_0(h\mathbf{k}) \|\mathbf{u}_{\mathbf{k}}^{\varepsilon,h}(t) - \mathbf{u}(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}(t), t)\|_2^2 + \int\limits_{\mathbb{T}^d} \left(\rho^{\varepsilon,h,r}(\mathbf{x}, t) - \rho(\mathbf{x}, t)\right)^2 d\mathbf{x}, \tag{3.27}$$

remains as in Theorem 3.8 and Theorem 3.9, since the proofs of both theorems will not change.
An improvement of the convergence result can be expected in the pointwise convergence result, Theorem 3.13. Since we investigate Euler's equations on the bounded domain $\mathbb{T}^d$, we have finite mass even if we have the case that $\inf_{\mathbf{x}\in\mathbb{T}^d} \rho_0(\mathbf{x}) > 0$. Hence, we expect that the pointwise convergence result will hold uniformly on $\mathbb{T}^d$.

Chapter 4

Kernel Construction

In this chapter we want to derive a class of kernel functions Φ for the SPH system in (3.4) - (3.6). As we have seen, the conditions we need for the kernel are originally conditions on the root kernel Φ^r. More precisely, the root kernel has to satisfy the moment and approximation conditions. However, for the calculation of the SPH system (3.4) - (3.6) we do not need the root kernel, but only the kernel function Φ. Hence, we have two possibilities how to construct the kernel function Φ.

The first possibility is to derive a root kernel Φ^r which satisfies the conditions given in Corollary 3.11. The kernel Φ can then be calculated via $\Phi = \Phi^r * \Phi^r$. Some properties like radiality could be transfered from the root kernel to Φ. Nevertheless, the actual computation of Φ^r is often not explicitly possible and the numerical calculation would be very expensive.

The other possibility, which we are pursuing now, is to derive conditions on the kernel Φ such that it possesses a root kernel Φ^r with the required properties, and then to verify these conditions directly for Φ. This has the advantage that we do not have to calculate Φ^r explicitly, since its existence will suffice.

This chapter will be divided in three parts. In the first part, we want to derive some important tools which we need later on. In the second part we want to derive the conditions for Φ. After that we will discuss a class of functions which satisfy these conditions in the third part.

4.1 Required Tools

In this section we will give some tools we require for the construction of the kernel Φ. We will discuss a multivariate version of the theorem of Faà di Bruno and give a short introduction to radial functions.

4.1.1 The Formula of Faà di Bruno and Applications

The formula of Faà die Bruno is a generalization of the univariate chain rule to higher derivatives. For our purpose, we will need a generalization of this formula to a multivariate setting, i.e. to the multivariate version of the Faà di Bruno formula. For further details, see [CS96].

Theorem 4.1 (Multivariate Formula of Faà di Bruno)
Let $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^m$ for $d, m \in \mathbb{N}$. Let $k \in \mathbb{N}$, $\mathbf{f} : U \to V$ and $g : V \to \mathbb{R}$ with $\mathbf{f} \in C^k(U)^m$ and $g \in C^k(V)$. Then

$$D^{\boldsymbol{\alpha}}(g(\mathbf{f}))(\mathbf{x}) = \sum_{1 \le |\boldsymbol{\nu}| \le |\boldsymbol{\alpha}|} (D^{\boldsymbol{\nu}} g)(\mathbf{f}(\mathbf{x})) \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{(\mathbf{n}, \boldsymbol{\beta}) \in p_s(\boldsymbol{\alpha}, \boldsymbol{\nu})} \boldsymbol{\alpha}! \prod_{j=1}^{s} \frac{(D^{\boldsymbol{\beta}_j} \mathbf{f}(\mathbf{x}))^{\mathbf{n}_j}}{\mathbf{n}_j! \boldsymbol{\beta}_j!}, \quad \mathbf{x} \in U,$$

for all $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ *with* $|\boldsymbol{\alpha}| \leq k$, *where* $\boldsymbol{\nu} \in \mathbb{N}_0^m$,

$$p_s(\boldsymbol{\alpha}, \boldsymbol{\nu}) = \Big\{(\mathbf{n}_1, \ldots, \mathbf{n}_s, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_s) \in (\mathbb{N}_0^m)^s \times (\mathbb{N}_0^d)^s, |\mathbf{n}_i| > 0,$$
$$0 \prec \boldsymbol{\beta}_1 \prec \ldots \prec \boldsymbol{\beta}_s, \sum_{i=1}^{s} \mathbf{n}_i = \boldsymbol{\nu} \text{ and } \sum_{i=1}^{s} |\mathbf{n}_i| \boldsymbol{\beta}_i = \boldsymbol{\alpha}\Big\}$$

and $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ *means, that either* $|\boldsymbol{\alpha}| < |\boldsymbol{\beta}|$ *or* $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ *with* $\alpha_1 = \beta_1, \ldots, \alpha_{j-1} = \beta_{j-1}$ *and* $\alpha_j < \beta_j$ *for a* $1 \leq j \leq d$.

We will omit the proof and refer to [CS96]. In the following, we will need Theorem 4.1 only for a specific case. We will start by setting the dimension of V to $m = 1$.

Corollary 4.2
Let the conditions be given as in Theorem 4.1 with $m = 1$. *Then*

$$D^{\boldsymbol{\alpha}}(g(f))(\mathbf{x}) = \sum_{\ell=1}^{|\boldsymbol{\alpha}|} g^{(\ell)}(f(\mathbf{x})) \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{(\mathbf{n}, \boldsymbol{\beta}) \in p_s(\boldsymbol{\alpha}, \ell)} \boldsymbol{\alpha}! \prod_{j=1}^{s} \frac{(D^{\boldsymbol{\beta}_j} f(\mathbf{x}))^{n_j}}{n_j! \boldsymbol{\beta}_j!}, \quad \mathbf{x} \in U,$$

for $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ *with* $|\boldsymbol{\alpha}| \leq k$ *and*

$$p_s(\boldsymbol{\alpha}, \ell) = \Big\{(n_1, \ldots, n_s, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_s) \in (\mathbb{N}_0)^s \times (\mathbb{N}_0^d)^s,$$
$$0 \prec \boldsymbol{\beta}_1 \prec \ldots \prec \boldsymbol{\beta}_s, \sum_{i=1}^{s} n_i = \ell \text{ and } \sum_{i=1}^{s} n_i \boldsymbol{\beta}_i = \boldsymbol{\alpha}\Big\}.$$

In particular, we will need Corollary 4.2 in the case that $g := \sqrt{\cdot}$. The higher derivatives of the square root function are given by the following lemma.

Lemma 4.3
Let $g := \sqrt{\cdot}$. *Then*

$$g^{(k)}(r) = \frac{c_k}{g(r)^{2k-1}} = c_k g(r)^{1-2k}$$

for all $r > 0$, $k \in \mathbb{N}$ *and* $s > 0$ *with* $c_k = -(-2)^{-k} \prod_{\ell=2}^{k} (2\ell - 3)$.

Proof. The proof is a simple induction over k. For $k = 1$ we just have the derivative of the root function with $c_1 = 1/2$. For the induction step we have

$$g^{(k+1)}(r) = c_k \left(g(r)^{1-2k}\right)' = (-2)^{-1}(2k-1) c_k g(r)^{1-2(k+1)}.$$

The constant c_k yields

$$(-2)^{-1}(2k-1)c_k = (-2)^{-(k+1)}(2k-1) \prod_{\ell=2}^{k} (2\ell - 3) = c_{k+1},$$

which completes the proof. □

In the next step we will combine Corollary 4.2 and Lemma 4.3.

Lemma 4.4
Let the conditions be given as in Theorem 4.1 with $U = \mathbb{R}^d$, $V = \mathbb{R}^+$, $g := \sqrt{\cdot}$ *and* $f(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \mathbb{R}^d$. *Then*

$$D^{\boldsymbol{\alpha}} \sqrt{f(\mathbf{x})} = \sum_{\ell=1}^{|\boldsymbol{\alpha}|} \frac{c_\ell}{\sqrt{f(\mathbf{x})}^{2\ell-1}} \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{(\mathbf{n}, \boldsymbol{\beta}) \in p_s(\boldsymbol{\alpha}, \ell)} \boldsymbol{\alpha}! \prod_{j=1}^{s} \frac{(D^{\boldsymbol{\beta}_j} f(\mathbf{x}))^{n_j}}{n_j! \boldsymbol{\beta}_j!}, \quad \mathbf{x} \in \mathbb{R}^d,$$

for $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ with $|\boldsymbol{\alpha}| \leq k$.
Furthermore, assume that there exists a constant $C > 0$ such that

$$|D^{\boldsymbol{\alpha}} f(\mathbf{x})| \leq C f(\mathbf{x}) \tag{4.1}$$

for all $\mathbf{x} \in \mathbb{R}^d$ and $1 \leq |\boldsymbol{\alpha}| \leq k$. Then, there exists a constant $\widetilde{C}$, depending on C, such that

$$|D^{\boldsymbol{\alpha}} \sqrt{f(\mathbf{x})}| \leq \widetilde{C} \sqrt{f(\mathbf{x})}$$

for all $\mathbf{x} \in \mathbb{R}^d$ and $1 \leq |\boldsymbol{\alpha}| \leq k$.

Proof. The first part of the lemma follows from Corollary 4.2 and Lemma 4.3. For the second part, we note that

$$|D^{\boldsymbol{\alpha}} \sqrt{f(\mathbf{x})}| \leq c(\boldsymbol{\alpha}) \sum_{\ell=1}^{|\boldsymbol{\alpha}|} \frac{1}{\sqrt{f(\mathbf{x})}^{2\ell-1}} \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{(\mathbf{n},\boldsymbol{\beta}) \in p_s(\boldsymbol{\alpha},\ell)} \prod_{j=1}^{s} |D^{\boldsymbol{\beta}_j} f(\mathbf{x})|^{n_j}.$$

If $(\mathbf{n}, \boldsymbol{\beta}) \in p_s(\boldsymbol{\alpha}, \ell)$, we have $\sum_{i=1}^{s} n_i = \ell$. Using (4.1), the product yields

$$\prod_{j=1}^{s} |D^{\boldsymbol{\beta}_j} f(\mathbf{x})|^{n_j} \leq C \prod_{j=1}^{s} f(\mathbf{x})^{n_j} = C f(\mathbf{x})^{\ell}.$$

Finally, since all occurring sums are finite, we have

$$|D^{\boldsymbol{\alpha}} \sqrt{f(\mathbf{x})}| \leq c(\boldsymbol{\alpha}, C) \sum_{\ell=1}^{|\boldsymbol{\alpha}|} \frac{f(\mathbf{x})^{\ell}}{\sqrt{f(\mathbf{x})}^{2\ell-1}} \leq \widetilde{C}(\boldsymbol{\alpha}) \sqrt{f(\mathbf{x})},$$

which completes the prove. □

The first part of this lemma is the specific Formula of Faà di Bruno we wanted to derive. The interesting part is the second part of this lemma, which is a direct conclusion of the of Faà di Bruno formula and gives us a kind of relation of a function with its root. In the following, we will investigate and modify this relation further.

Lemma 4.5
Let $k \in \mathbb{N}$ and $f \in C^k(\mathbb{R}^d)$ with $f(0) \neq 0$. If there is a $C > 0$ with

$$|D^{\boldsymbol{\alpha}} x_j f(\mathbf{x})| \leq C |f(\mathbf{x})|$$

for all $\mathbf{x} \in \mathbb{R}^d$, $1 \leq j \leq d$ and $1 \leq |\boldsymbol{\alpha}| \leq k$, then there exists a $\widetilde{C} > 0$, such that

$$|D^{\boldsymbol{\alpha}} f(\mathbf{x})| \leq \widetilde{C} |f(\mathbf{x})|$$

for all $\mathbf{x} \in \mathbb{R}^d$ and $1 \leq |\boldsymbol{\alpha}| \leq k$.

Proof. Let $1 \leq |\boldsymbol{\alpha}| \leq k$. Because f is continuous with $f(0) \neq 0$, there exists an $r > 0$ with $f(\mathbf{x}) \neq 0$ for all $\|\mathbf{x}\|_2 \leq r$. Set $\widetilde{C} = \max_{\|\mathbf{x}\|_2 \leq r} |D^{\boldsymbol{\alpha}} f(\mathbf{x})| / |f(\mathbf{x})|$. Then $|D^{\boldsymbol{\alpha}} f(\mathbf{x})| \leq \widetilde{C} |f(\mathbf{x})|$ for all $\|\mathbf{x}\|_2 \leq r$.
Now consider $\|\mathbf{x}\|_2 > r$ and $1 \leq j \leq d$. If $\alpha_j \geq 1$, we obtain by differentiation

$$D^{\boldsymbol{\alpha}} x_j f(\mathbf{x}) = x_j D^{\boldsymbol{\alpha}} f(\mathbf{x}) + \alpha_j D^{\boldsymbol{\alpha} - \mathbf{e}_j} f(\mathbf{x}) \tag{4.2}$$

for all $\mathbf{x} \in \mathbb{R}^d$ and $1 \leq |\boldsymbol{\alpha}| \leq k$. With this we obtain the estimate

$$|D^{\boldsymbol{\alpha}} f(\mathbf{x})| \leq r^{-1} |x_j D^{\boldsymbol{\alpha}} f(\mathbf{x})| \leq r^{-1} |D^{\boldsymbol{\alpha}} x_j f(\mathbf{x})| + r^{-1} |\alpha_j D^{\boldsymbol{\alpha} - \mathbf{e}_j} f(\mathbf{x})|,$$

If $\alpha_j = 0$, the second term on the right hand side of (4.2) vanishes, so that we have the estimate

$$|D^{\boldsymbol{\alpha}} f(\mathbf{x})| \leq r^{-1} |x_j D^{\boldsymbol{\alpha}} f(\mathbf{x})| \leq r^{-1} |D^{\boldsymbol{\alpha}} x_j f(\mathbf{x})|$$

in this case. The proof will be completed by induction on the length of $\boldsymbol{\alpha}$. □

With this result and the second part of Lemma 4.4 we have the following corollary.

Corollary 4.6
Let $k \in \mathbb{N}$ and $f \in C^k(\mathbb{R}^d)$ be nonnegative with $f(0) \neq 0$. If there exists a $C > 0$ with

$$|D^{\boldsymbol{\alpha}} x_j f(\mathbf{x})| \leq C f(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^d$, $1 \leq j \leq d$ and $1 \leq |\boldsymbol{\alpha}| \leq k$, then there exists a $\widetilde{C} > 0$, such that

$$|D^{\boldsymbol{\alpha}} \sqrt{f(\mathbf{x})}| \leq \widetilde{C} \sqrt{f(\mathbf{x})}$$

for all $\mathbf{x} \in \mathbb{R}^d$ and $1 \leq |\boldsymbol{\alpha}| \leq k$.

In a last step, we will modify this result to a slightly different version with a multiplicative monomial.

Theorem 4.7
Let $k \in \mathbb{N}$ and $f \in C^k(\mathbb{R}^d)$ be nonnegative with $f(0) \neq 0$. If there is a $C > 0$ with

$$|D^{\boldsymbol{\alpha}} x_j f(\mathbf{x})| \leq C f(\mathbf{x}) \tag{4.3}$$

for all $\mathbf{x} \in \mathbb{R}^d$, $1 \leq j \leq d$ and $1 \leq |\boldsymbol{\alpha}| \leq k$, then there exists a $\widetilde{C} > 0$, such that

$$|D^{\boldsymbol{\alpha}} x_j \sqrt{f(\mathbf{x})}| \leq \widetilde{C} \sqrt{f(\mathbf{x})}$$

for all $\mathbf{x} \in \mathbb{R}^d$, $1 \leq j \leq d$ and $1 \leq |\boldsymbol{\alpha}| \leq k$.

***Proof*.** Let $1 \leq j \leq d$. With (4.2), we have

$$|D^{\boldsymbol{\alpha}} x_j \sqrt{f(\mathbf{x})}| \leq |x_j D^{\boldsymbol{\alpha}} \sqrt{f(\mathbf{x})}| + \alpha_j |D^{\boldsymbol{\alpha} - \mathbf{e}_j} \sqrt{f(\mathbf{x})}|,$$

where, with respect to Corollary 4.6, the second summand fullfills the proposed inequality.
The estimate of the first summand is clear for $|x_j| \leq 1$. So let $|x_j| > 1$.
The formula of Faà di Bruno, Lemma 4.4, implies

$$\begin{aligned}
|x_j D^{\boldsymbol{\alpha}} \sqrt{f(\mathbf{x})}| &= \left| x_j \sum_{l=1}^{|\boldsymbol{\alpha}|} \frac{c_l}{\sqrt{f(\mathbf{x})}^{2l-1}} \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{p_s(\boldsymbol{\alpha}, l)} \boldsymbol{\alpha}! \prod_{i=1}^{s} \frac{(D^{\boldsymbol{\beta}_i} f(\mathbf{x}))^{k_i}}{k_i! \boldsymbol{\beta}_i!} \right| \\
&\leq \sum_{l=1}^{|\boldsymbol{\alpha}|} \frac{c_l}{\sqrt{f(\mathbf{x})}^{2l-1}} \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{p_s(\boldsymbol{\alpha}, l)} \boldsymbol{\alpha}! |x_j| \prod_{i=1}^{s} \left| \frac{(D^{\boldsymbol{\beta}_i} f(\mathbf{x}))^{k_i}}{k_i! \boldsymbol{\beta}_i!} \right|.
\end{aligned}$$

With $|x_j| \leq |x_j^l|$ for all $1 \leq l \leq |\boldsymbol{\alpha}|$ and $\sum\limits_{i=1}^{s} k_i = l$ we have

$$|x_j D^{\boldsymbol{\alpha}} \sqrt{f(\mathbf{x})}| \leq \sum_{l=1}^{|\boldsymbol{\alpha}|} \frac{c_l}{\sqrt{f(\mathbf{x})}^{2l-1}} \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{p_s(\boldsymbol{\alpha}, l)} \boldsymbol{\alpha}! \prod_{i=1}^{s} \left| \frac{(x_j D^{\boldsymbol{\beta}_i} f(\mathbf{x}))^{k_i}}{k_i! \boldsymbol{\beta}_i!} \right|.$$

Now equation (4.2), the condition from equation (4.3) and Lemma 4.5 yield

$$\begin{aligned}
|x_j D^{\boldsymbol{\beta}_i} f(\mathbf{x})| =& |D^{\boldsymbol{\beta}_i} x_j f(\mathbf{x}) - (\boldsymbol{\beta}_i)_j D^{\boldsymbol{\beta}_i - \mathbf{e}_j} f(\mathbf{x})| \\
\leq& |D^{\boldsymbol{\beta}_i} x_j f(\mathbf{x})| + |(\boldsymbol{\beta}_i)_j D^{\boldsymbol{\beta}_i - \mathbf{e}_j} f(\mathbf{x})| \\
\leq& c |f(\mathbf{x})|.
\end{aligned}$$

Hence, with $f(\mathbf{x}) > 0$, we have

$$
\begin{aligned}
|x_j D^{\boldsymbol{\alpha}} \sqrt{f(\mathbf{x})}| \leq & c \sum_{l=1}^{|\boldsymbol{\alpha}|} \frac{c_l}{\sqrt{f(\mathbf{x})}^{2l-1}} \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{p_s(\boldsymbol{\alpha},l)} \boldsymbol{\alpha}! \prod_{i=1}^{s} \frac{|f(\mathbf{x})|^{k_i}}{k_i! \boldsymbol{\beta}_i!} \\
\leq & c \sum_{l=1}^{|\boldsymbol{\alpha}|} \frac{f(\mathbf{x})^l}{\sqrt{f(\mathbf{x})}^{2l-1}} \\
\leq & c \sqrt{f(\mathbf{x})},
\end{aligned}
$$

which completes the proof. □

4.1.2 Radial Functions

In this section we will give an outline of radial functions, i.e. functions that only depend on the distance of its argument to the origin. For further details see, for example, [Wen04]. First of all we will give a formal definition of radial functions.

Definition 4.8
A function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is said to be radial if there exists a function $g : [0, \infty[\mapsto \mathbb{R}$ such that $f(\mathbf{x}) = g(\|\mathbf{x}\|_2)$ for all $\mathbf{x} \in \mathbb{R}^d$.

A important property of the following kernel construction theory is the positive definiteness of the kernel functions. For that we have to define what it means for a radial function to be positive definite.

Definition 4.9
A univariate function $g : [0, \infty[\mapsto \mathbb{R}$ is said to be positive definite on $\mathbb{R}^d$ if the corresponding multivariate function $f = g(\| \cdot \|_2) : \mathbb{R}^d \to \mathbb{R}$ is positive definite.

As we saw in Theorem 1.9, the Fourier transform can be an important tool to show if a function is positive definite. One can show that the Fourier transform of a radial function will be radial as well. Moreover, it is possible to derive a formula of the Fourier transform of radial functions by their radial part. The proof of this formula can be found in [Wen04].

Theorem 4.10
Suppose that $g \in C([0, \infty[)$ satisfies $r \mapsto r^{d-1} g(r) \in L^1([0, \infty[)$. Let

$$
\mathcal{F}_d g(s) := s^{-\frac{d-2}{2}} \int_0^\infty g(t) t^{\frac{d}{2}} J_{\frac{d-2}{2}}(st) dt
$$

for $s \in [0, \infty[$, where J_k denotes the Bessel function of order k, see [GR00, 8.402]. Let $f : \mathbb{R}^d \to \mathbb{R}$ be defined by $\mathbf{x} \to g(\|\mathbf{x}\|_2)$. Then

$$
\widehat{f}(\boldsymbol{\omega}) = \mathcal{F}_d g(\|\boldsymbol{\omega}\|_2)
$$

for all $\boldsymbol{\omega} \in \mathbb{R}^d$.

With this specification of the Fourier transformation of radial functions by their radial part, we can conclude the inverse Fourier transform by

$$
\mathcal{F}_d^{-1} f(t) := t^{-\frac{d-2}{2}} \int_0^\infty f(s) s^{\frac{d}{2}} J_{\frac{d-2}{2}}(ts) ds = \mathcal{F}_d f(t)
$$

for $t \in [0, \infty[$. Later in this chapter we are particularly interested in the Fourier transformation of radial functions for $d = 1$. In this case, the radial Fourier transformation is equal to the classical

Fourier transformation for even functions. Hence, the one dimensional radial Fourier transform is given by

$$\mathcal{F}_1 g(s) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(t)\cos(st)dt = \sqrt{2}\pi^{-1/2} \int_{0}^{\infty} g(t)\cos(st)dt. \tag{4.4}$$

Another useful property, especially for the moment condition from Definition 1.26, are the vanishing odd moments of radial functions.

Lemma 4.11
Let $\boldsymbol{\alpha} \in \mathbb{N}^d$ with $|\boldsymbol{\alpha}|$ odd. Suppose that $g : [0, \infty[\to \mathbb{R}$ satisfies $r \mapsto r^{d-1+|\alpha|} g(r) \in L^1[0, \infty[$. Then

$$\int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} g(\|\mathbf{x}\|_2) d\mathbf{x} = 0.$$

Proof. Since $|\boldsymbol{\alpha}|$ is odd, there is at least one $i \in \{1, \ldots, d\}$ such that α_i is odd. We may assume $i = 1$ without loss of generality. By splitting up the integral, we have

$$\int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} g(\|\mathbf{x}\|_2) d\mathbf{x} = \int_{x_1 \geq 0} \mathbf{x}^{\boldsymbol{\alpha}} g(\|\mathbf{x}\|_2) d\mathbf{x} + \int_{x_1 \leq 0} \mathbf{x}^{\boldsymbol{\alpha}} g(\|\mathbf{x}\|_2) d\mathbf{x},$$

where $\int_{x_1 \geq 0}$ denotes the integral over the set $\{\mathbf{x} \in \mathbb{R}^d;\ x_1 \geq 0\}$, and $\int_{x_1 \leq 0}$ the integral over the set $\{\mathbf{x} \in \mathbb{R}^d;\ x_1 \leq 0\}$. Now we note that we can transform the second integral to

$$\int_{x_1 \leq 0} \mathbf{x}^{\boldsymbol{\alpha}} g(\|\mathbf{x}\|_2) d\mathbf{x} = (-1)^{\alpha_1} \int_{x_1 \geq 0} \mathbf{x}^{\boldsymbol{\alpha}} g(\|\mathbf{x}\|_2) d\mathbf{x}.$$

Since α_i is odd, $(-1)^{\alpha_1} = -1$ and we conclude

$$\int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} g(\|\mathbf{x}\|_2) d\mathbf{x} = \int_{x_1 \geq 0} \mathbf{x}^{\boldsymbol{\alpha}} g(\|\mathbf{x}\|_2) d\mathbf{x} - \int_{x_1 \geq 0} \mathbf{x}^{\boldsymbol{\alpha}} g(\|\mathbf{x}\|_2) d\mathbf{x} = 0,$$

which completes the proof. □

The last lemma of this section will give us a formula for the derivatives of radial functions by using the results of Section 4.1.1. The formula is not optimal in the sense that it might contain zero coefficients. However, we are interest in the highest possible degree of the occuring monomials.

Lemma 4.12
Suppose that $g \in C^k([0, \infty[)$ and $f : \mathbb{R}^d \mapsto \mathbb{R}$ defined by $f(\mathbf{x}) = g(\|\mathbf{x}\|_2)$ for $\mathbf{x} \in \mathbb{R}^d$. Then the $\boldsymbol{\alpha}$-th derivative of f is given by

$$D^{\boldsymbol{\alpha}} f(\mathbf{x}) = \sum_{k=1}^{|\boldsymbol{\alpha}|} g^{(k)}(\|\mathbf{x}\|_2) \left(\sum_{l=1}^{2|\boldsymbol{\alpha}|-1} \frac{\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}|-k)} c_{\boldsymbol{\nu}} \mathbf{x}^{\boldsymbol{\nu}}}{\|\mathbf{x}\|_2^l} \right) \tag{4.5}$$

for all $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and $\boldsymbol{\alpha} \in \mathbb{N}^d$ with $|\boldsymbol{\alpha}| \leq k$, where $c_{\boldsymbol{\nu}} \in \mathbb{N}$.

Proof. We will prove the formula via induction on the length of $\boldsymbol{\alpha}$. For $|\boldsymbol{\alpha}| = 1$ we start without restriction with $\boldsymbol{\alpha} = \mathbf{e}_1$, where we have

$$D^{\mathbf{e}_1} g(\|\mathbf{x}\|_2) = g'(\|\mathbf{x}\|_2) \frac{x_1}{\|\mathbf{x}\|_2},$$

which is of the form (4.5). For the induction step, it suffices to consider multiindices of the form $\boldsymbol{\alpha}+\mathbf{e}_1$ with $|\boldsymbol{\alpha}|<k$. The induction hypothesis then yields

$$\begin{aligned}
D^{\boldsymbol{\alpha}+\mathbf{e}_1}f(\mathbf{x}) =& D^{\mathbf{e}_1}\sum_{k=1}^{|\boldsymbol{\alpha}|} g^{(k)}(\|\mathbf{x}\|_2)\left(\sum_{l=1}^{2|\boldsymbol{\alpha}|-1}\frac{\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}|-k)} c_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}}}{\|\mathbf{x}\|_2^l}\right)\\
=&\sum_{k=1}^{|\boldsymbol{\alpha}|}\frac{x_1}{\|\mathbf{x}\|_2}g^{(k+1)}(\|\mathbf{x}\|_2)\left(\sum_{l=1}^{2|\boldsymbol{\alpha}|-1}\frac{\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}|-k)} c_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}}}{\|\mathbf{x}\|_2^l}\right)\\
&-\sum_{k=1}^{|\boldsymbol{\alpha}|} g^{(k)}(\|\mathbf{x}\|_2)\left(\sum_{l=1}^{2|\boldsymbol{\alpha}|-1} l\frac{\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}|-k)} c_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}+\mathbf{e}_1}}{\|\mathbf{x}\|_2^{l+2}}\right)\\
&+\sum_{k=1}^{|\boldsymbol{\alpha}|} g^{(k)}(\|\mathbf{x}\|_2)\left(\sum_{l=1}^{2|\boldsymbol{\alpha}|-1}\frac{\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}|-k)} \nu_1 c_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}-\mathbf{e}_1}}{\|\mathbf{x}\|_2^l}\right).
\end{aligned}$$

It remains to show that each of these sums has the form (4.5). We will show this only for the first sum, which we will call S_1 for short. The other two sums will be transformed analogously. For S_1, we will need three shifts in the indices. First, by a shift of the summation from k to $k-1$ in the first sum, we have

$$\begin{aligned}
S_1 &= \sum_{k=1}^{|\boldsymbol{\alpha}|} g^{(k+1)}(\|\mathbf{x}\|_2)\left(\sum_{l=1}^{2|\boldsymbol{\alpha}|-1}\frac{\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}|-k)} c_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}+\mathbf{e}_1}}{\|\mathbf{x}\|_2^{l+1}}\right)\\
&= \sum_{k=2}^{|\boldsymbol{\alpha}+\mathbf{e}_1|} g^{(k)}(\|\mathbf{x}\|_2)\left(\sum_{l=1}^{2|\boldsymbol{\alpha}|-1}\frac{\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}+\mathbf{e}_1|-k)} c_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}+\mathbf{e}_1}}{\|\mathbf{x}\|_2^{l+1}}\right).
\end{aligned}$$

where we used that $|\boldsymbol{\alpha}+\mathbf{e}_1|=|\boldsymbol{\alpha}|+1$. Then, by shifting from $\boldsymbol{\nu}$ to $\boldsymbol{\nu}-\mathbf{e}_1$, we will sum over all $|\boldsymbol{\nu}|=l+1-(|\boldsymbol{\alpha}|-k+1)$ with $\nu_1\geq 1$ in the inner sum. We will write $\widetilde{c}_{\boldsymbol{\nu}}=c_{\boldsymbol{\nu}-\mathbf{e}_1}$, where we set $\widetilde{c}_{\boldsymbol{\nu}}=0$ for all $\boldsymbol{\nu}$ with $\nu_1=0$. Thus, we can sum over all $|\boldsymbol{\nu}|=l+1-(|\boldsymbol{\alpha}|-k+1)$ and have

$$\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}+\mathbf{e}_1|-k)} c_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}+\mathbf{e}_1} = \sum_{|\boldsymbol{\nu}|=l+1-(|\boldsymbol{\alpha}+\mathbf{e}_1|-k)} \widetilde{c}_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}}$$

Finally, shifting from l to $l-1$ yields

$$\begin{aligned}
S_1 &= \sum_{k=2}^{|\boldsymbol{\alpha}+\mathbf{e}_1|} g^{(k)}(\|\mathbf{x}\|_2)\left(\sum_{l=1}^{2|\boldsymbol{\alpha}|-1}\frac{\sum_{|\boldsymbol{\nu}|=l+1-(|\boldsymbol{\alpha}+\mathbf{e}_1|-k)} \widetilde{c}_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}}}{\|\mathbf{x}\|_2^{l+1}}\right)\\
&= \sum_{k=2}^{|\boldsymbol{\alpha}+\mathbf{e}_1|} g^{(k)}(\|\mathbf{x}\|_2)\left(\sum_{l=2}^{2|\boldsymbol{\alpha}|}\frac{\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}+\mathbf{e}_1|-k)} \widetilde{c}_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}}}{\|\mathbf{x}\|_2^{l}}\right)\\
&= \sum_{k=2}^{|\boldsymbol{\alpha}+\mathbf{e}_1|} g^{(k)}(\|\mathbf{x}\|_2)\left(\sum_{l=2}^{2|\boldsymbol{\alpha}+\mathbf{e}_1|-2}\frac{\sum_{|\boldsymbol{\nu}|=l-(|\boldsymbol{\alpha}+\mathbf{e}_1|-k)} \widetilde{c}_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}}}{\|\mathbf{x}\|_2^{l}}\right).
\end{aligned}$$

For the form of equation (4.5), we have to increase the limit of the sum over l. But this is no problem since we set $\widetilde{c}_{\boldsymbol{\nu}}=0$ for all $|\boldsymbol{\nu}|=|\boldsymbol{\alpha}+\mathbf{e}_1|+k-1$. □

One has to be careful with the derivatives of radial functions in the point $\mathbf{x} = \mathbf{0}$. Whether the derivatives of $f = g(\|\cdot\|_2)$ are continuous in $\mathbf{x} = \mathbf{0}$ does not only depend on the smoothness of the univariate function $g : [0,\infty[\to \mathbb{R}$, but on its even extension $g(|\cdot|) : \mathbb{R} \to \mathbb{R}$. If $g(|\cdot|)$ is k-times continuously differentiable in zero, so is g. Whenever we will say that g is continuously differentiable in 0, we mean its even extension $g(|\cdot|)$.

4.2 Construction Method

The main goal of this section is to establish easy-to-check conditions for the kernel $\Phi : \mathbb{R}^d \to \mathbb{R}$, so that the conditions in Theorem 3.9 and Corollary 3.11 are satisfied for a convolution root Φ^r of Φ. We will investigate radial functions with compact support for this purpose, but initially we want to start as general as possible. We begin with investigating the existence and the regularity of convolution roots.

4.2.1 Existence of Convolution Roots

To recall Definition 3.1, a convolution root of a function Φ is a function Φ^r with $\Phi = \Phi^r * \Phi^r$. At first, it is not clear if such a function even exists, and if it exists which regularity this function has. In the case of compactly supported functions, Boas and Kac, [BJK45], found an existence result in one dimension. Based on this result, Ehm, Gneiting and Richards, [EGR04], published a version of this existence theorem in arbitrary dimensions, including conditions for some characteristics like radiality. Unfortunately, the conditions they stated are too complicated to verify.
We will begin with a first result on the convolution of an $L^2(\mathbb{R}^d)$ function.

Lemma 4.13
Let $f \in L^2(\mathbb{R}^d)$. Then

$$f * f = (2\pi)^{d/2}([\widehat{f}]^2)^\vee$$

is in $L^\infty(\mathbb{R}^d)$.

Proof. Using Young's inequality, it is clear that $f*f$ is in $L^\infty(\mathbb{R}^d)$. Moreover, since f is in $L^2(\mathbb{R}^d)$, so is $\widehat{f}$, which means that $[\widehat{f}]^2$ is in $L^1(\mathbb{R}^d)$. Thus, $([\widehat{f}]^2)^\vee$ is also in $L^\infty(\mathbb{R}^d)$.
It remains to show equality. Since both sides are only in $L^\infty(\mathbb{R}^d)$, we cannot simply use Fourier transformation. However, for γ from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, see, for example, [Wen04], we have

$$\gamma * \gamma = ([\gamma * \gamma]^\wedge)^\vee = (2\pi)^{d/2}([\widehat{\gamma}]^2)^\vee.$$

As $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, there exists a sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ which converges to f in $L^2(\mathbb{R}^d)$. Then,

$$\begin{aligned}\|f * f - (2\pi)^{d/2}([\widehat{f}]^2)^\vee\|_{L^\infty(\mathbb{R}^d)} \leq &\|f * f - \gamma_n * \gamma_n\|_{L^\infty(\mathbb{R}^d)} \\ &+ \|\gamma_n * \gamma_n - (2\pi)^{d/2}([\widehat{f}]^2)^\vee\|_{L^\infty(\mathbb{R}^d)}.\end{aligned} \tag{4.6}$$

The first term on the right-hand side can be bounded, using again Young's inequality, by

$$\begin{aligned}\|f * f - \gamma_n * \gamma_n\|_{L^\infty(\mathbb{R}^d)} \leq &\|f * (f - \gamma_n)\|_{L^\infty(\mathbb{R}^d)} + \|(f - \gamma_n) * \gamma_n\|_{L^\infty(\mathbb{R}^d)} \\ \leq &\|f\|_{L^2(\mathbb{R}^d)}\|f - \gamma_n\|_{L^2(\mathbb{R}^d)} + \|f - \gamma_n\|_{L^2(\mathbb{R}^d)}\|\gamma_n\|_{L^2(\mathbb{R}^d)} \\ \leq &2\|f\|_{L^2(\mathbb{R}^d)}\|f - \gamma_n\|_{L^2(\mathbb{R}^d)} + \|f - \gamma_n\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

which tends to zero with $n \to \infty$, where we used $\|\gamma_n\|_{L^2(\mathbb{R}^d)} \leq \|f - \gamma_n\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)}$ in the last step.
For the second term on the right-hand side of (4.6) we have

$$\begin{aligned}\|\gamma_n * \gamma_n - (2\pi)^{d/2}([\widehat{f}]^2)^\vee\|_{L^\infty(\mathbb{R}^d)} &= (2\pi)^{d/2}\|([\widehat{\gamma_n}]^2)^\vee - ([\widehat{f}]^2)^\vee\|_{L^\infty(\mathbb{R}^d)} \\ &\leq (2\pi)^{d/2}\|[\widehat{\gamma_n}]^2 - [\widehat{f}]^2\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

For the remaining term we can use the binomial formula and the Cauchy-Schwarz inequality to derive

$$\begin{aligned}
\|[\widehat{\gamma_n}]^2 - [\widehat{f}]^2\|_{L^1(\mathbb{R}^d)} &= \| \left(\widehat{\gamma_n} + \widehat{f}\right) \left(\widehat{\gamma_n} - \widehat{f}\right) \|_{L^1(\mathbb{R}^d)} \\
&\leq \|\widehat{\gamma_n} + \widehat{f}\|_{L^2(\mathbb{R}^d)} \|\widehat{\gamma_n} - \widehat{f}\|_{L^2(\mathbb{R}^d)} \\
&= \|\gamma_n + f\|_{L^2(\mathbb{R}^d)} \|\gamma_n - f\|_{L^2(\mathbb{R}^d)} \\
&\leq \left(2\|f\|_{L^2(\mathbb{R}^d)} + \|\gamma_n - f\|_{L^2(\mathbb{R}^d)}\right) \|\gamma_n - f\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

which also tends to zero with $n \to \infty$. □

With this result, we are now able to give a first existence result for root kernels.

Lemma 4.14
Let $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be positive definite. Then, $\Phi^r := (2\pi)^{-d/4}[\widehat{\Phi}^{1/2}]^\vee \in L^2(\mathbb{R}^d)$ is a real-valued root kernel of Φ. If Φ is in addition radial, so is the root kernel Φ^r.

Proof. As Φ is positive definite, we have $\widehat{\Phi}(\boldsymbol{\omega}) \geq 0$ for all $\boldsymbol{\omega} \in \mathbb{R}^d$ and we can define the root $\boldsymbol{\omega} \mapsto \widehat{\Phi}^{1/2}(\boldsymbol{\omega})$. Moreover, $\widehat{\Phi}$ is in $L^1(\mathbb{R}^d)$ due to Lemma 1.12 since $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is positive definite. Hence, $\widehat{\Phi}^{1/2}$ is in $L^2(\mathbb{R}^d)$, which has therefore an inverse Fourier transform $\Phi^r := (2\pi)^{-d/4}[\widehat{\Phi}^{1/2}]^\vee \in L^2(\mathbb{R}^d)$. Setting $f = \Phi^r$ in Lemma 4.13 finishes the proof. □

Our next goal is to have more regularity for the convolution root Φ^r.

Lemma 4.15
Let $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be positive definite and in addition let $\Phi \in W^{2\sigma,2}(\mathbb{R}^d)$ for a $\sigma > d/4$. Then, the convolution root from Lemma 4.14 satisfies $\Phi^r \in W^{\tau,2}(\mathbb{R}^d)$ for all $\tau < \sigma - d/4$.

Proof. With $\widehat{\Phi^r}^2 = (2\pi)^{-d/2}\widehat{\Phi}$ we see that $\Phi^r \in W^{\tau,2}(\mathbb{R}^d)$ if

$$\begin{aligned}
\int_{\mathbb{R}^d} (1 + \|\boldsymbol{\omega}\|_2^2)^\tau |\widehat{\Phi^r}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} =& (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} (1 + \|\boldsymbol{\omega}\|_2^2)^\tau \widehat{\Phi}(\boldsymbol{\omega}) d\boldsymbol{\omega} \\
\leq& (2\pi)^{-\frac{d}{2}} \left(\int_{\mathbb{R}^d} (1 + \|\boldsymbol{\omega}\|_2^2)^{2\sigma} \widehat{\Phi}(\boldsymbol{\omega})^2 d\boldsymbol{\omega} \right)^{1/2} \left(\int_{\mathbb{R}^d} (1 + \|\boldsymbol{\omega}\|_2^2)^{2(\tau-\sigma)} d\boldsymbol{\omega} \right)^{1/2} \\
\leq& c\|\Phi\|_{H^{2\sigma}(\mathbb{R}^d)},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Note that the second integral is bounded since $\tau < \sigma - d/4$. □

Next, we investigate the integrability of the convolution root from Lemma 4.14. From this result we already know that Φ^r is in $L^2(\mathbb{R}^d)$. If Φ^r would be continuous and would have compact support, Φ^r would be in $L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$. Unfortunately, Φ having compact support does not imply that Φ^r has compact support, so it is not useful to pursue this possibility. For this reason, we have to make a new condition on Φ^r, which, in a similar way, will later also appear in another condition.

Lemma 4.16
Let $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be positive definite and assume that there exist constants $C > 0$ and $L \in \mathbb{N}$ with $L \geq \lfloor d/2 \rfloor + 1$ such that

$$|D^{\boldsymbol{\alpha}} \widehat{\Phi^r}(\boldsymbol{\omega})| \leq C \widehat{\Phi^r}(\boldsymbol{\omega}), \qquad \boldsymbol{\omega} \in \mathbb{R}^d, \tag{4.7}$$

for all $1 \leq |\boldsymbol{\alpha}| \leq L$. Then, the convolution root from Lemma 4.14 satisfies $\Phi^r \in L^1(\mathbb{R}^d)$.

Proof. Since we already know from Lemma 4.14 that Φ^r is in $L^2(\mathbb{R}^d)$, we also know that its Fourier transform $\widehat{\Phi^r}$ is in $L^2(\mathbb{R}^d)$. Hence, using the Cauchy-Schwarz inequality, we can calculate

$$\int_{\mathbb{R}^d} |\Phi^r(\mathbf{x})| d\mathbf{x} \leq \left(\int_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)^{-L} d\mathbf{x} \right)^{1/2} \left(\int_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)^{L} |\Phi^r(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2},$$

where the first integral on the right hand side is bounded since $L \geq \lfloor d/2 \rfloor + 1 > d/2$. For the second integral on the right-hand side we have to distinguish if L is an even or an odd integer. If L is even, we have

$$\begin{aligned} \int_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)^{L} |\Phi^r(\mathbf{x})|^2 d\mathbf{x} &= \int_{\mathbb{R}^d} |(I - \Delta)^{L/2} \widehat{\Phi^r}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} \\ &= \sum_{|\boldsymbol{\alpha}|, |\boldsymbol{\beta}| \leq L} c_{\boldsymbol{\alpha}} c_{\boldsymbol{\beta}} \int_{\mathbb{R}^d} D^{\boldsymbol{\alpha}} \widehat{\Phi^r}(\boldsymbol{\omega}) D^{\boldsymbol{\beta}} \widehat{\Phi^r}(\boldsymbol{\omega}) d\boldsymbol{\omega} \\ &\leq C \int_{\mathbb{R}^d} |\widehat{\Phi^r}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}, \end{aligned} \tag{4.8}$$

where we needed that $L/2$ is an integer and used inequality (4.7) in the last step. If L is an odd integer, we have

$$\begin{aligned} \int_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)^{L} |\Phi^r(\mathbf{x})|^2 d\mathbf{x} &= \int_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)(1 + \|\mathbf{x}\|_2^2)^{L-1} |\Phi^r(\mathbf{x})|^2 d\mathbf{x} \\ &= \int_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)^{L-1} |\Phi^r(\mathbf{x})|^2 d\mathbf{x} + \sum_{j=1}^{d} \int_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)^{L-1} |x_j \Phi^r(\mathbf{x})|^2 d\mathbf{x}. \end{aligned}$$

Both integrals can be discussed as in (4.8) since $L-1$ is even, where we note for the second integral that $\left(p_{\mathbf{e}_j} \Phi^r\right)^\wedge = i\partial_j \widehat{\Phi^r}$ with $p_{\mathbf{e}_j}(\mathbf{x}) = x_j$ according to Lemma 1.2. $\square$

The newly occurred condition (4.7) seems to be very restrictive, but also similar to the approximation condition. As we will see later, this condition will not limit us in the choice of our kernel function Φ. Finally, we combine the last two lemmas.

Theorem 4.17
Let $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be positive definite and in addition let $\Phi \in W^{2\sigma,2}(\mathbb{R}^d)$ for a $\sigma > d/4$. Assume that there exists constants $C > 0$ and $L \in \mathbb{N}$ with $L \geq \lfloor d/2 \rfloor + 1$ such that

$$|D^{\boldsymbol{\alpha}} \widehat{\Phi^r}(\boldsymbol{\omega})| \leq C \widehat{\Phi^r}(\boldsymbol{\omega}), \qquad \boldsymbol{\omega} \in \mathbb{R}^d,$$

for all $1 \leq |\boldsymbol{\alpha}| \leq L$. Then, the convolution root from Lemma 4.14 satisfies $\Phi^r \in W^{\tau,1}(\mathbb{R}^d)$ for all integer $\tau < \sigma - d/4$.

Proof. From Lemma 4.15 we already know that $\Phi^r \in W^{\tau,2}(\mathbb{R}^d)$ for all $\tau < \sigma - d/4$. This yields

$$\|p_{\boldsymbol{\beta}} D^{\boldsymbol{\alpha}} \widehat{\Phi^r}\|_{L^2(\mathbb{R}^d)} \leq C \|p_{\boldsymbol{\beta}} \widehat{\Phi^r}\|_{L^2(\mathbb{R}^d)} = C \|D^{\boldsymbol{\beta}} \Phi^r\|_{L^2(\mathbb{R}^d)} \tag{4.9}$$

for all $|\boldsymbol{\alpha}| \leq L$ and $|\boldsymbol{\beta}| \leq \tau$, so that we have $p_{\boldsymbol{\beta}} D^{\boldsymbol{\alpha}} \widehat{\Phi^r} \in L^2(\mathbb{R}^d)$. This also means that $D^{\boldsymbol{\alpha}}[p_{\boldsymbol{\beta}} \widehat{\Phi^r}]$ is in $L^2(\mathbb{R}^d)$ for all $|\boldsymbol{\alpha}| \leq L$ and $|\boldsymbol{\beta}| \leq \tau$, since the latter is just a linear combination of the former. Now we have to distinguish whether L is an even or an odd integer.
If L is even, let $k = L/2$. With this we can conclude that $(I - \Delta)^k [p_{\boldsymbol{\beta}} \widehat{\Phi^r}] \in L^2(\mathbb{R}^d)$ for all $|\boldsymbol{\beta}| \leq \tau$, where I denotes the identity operator. Noting that a real-valued Φ^r with a nonnegative Fourier

transform $\widehat{\Phi^r}$ means $(\Phi^r)^\vee = \overline{\widehat{\Phi^r}} = \widehat{\Phi^r}$ we can conclude that $(1+\|\cdot\|_2^2)^k[D^{\boldsymbol{\beta}}\Phi^r] \in L^2(\mathbb{R}^d)$ for all $|\boldsymbol{\beta}| \le \tau$ using Plancherel's identity. Hence, we arrive at

$$\int_{\mathbb{R}^d} |D^{\boldsymbol{\beta}}\Phi^r(\mathbf{x})|d\mathbf{x} \le \left(\int_{\mathbb{R}^d}(1+\|\mathbf{x}\|_2^2)^{2k}|D^{\boldsymbol{\beta}}\Phi^r(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2}\left(\int_{\mathbb{R}^d}(1+\|\mathbf{x}\|_2^2)^{-2k}d\mathbf{x}\right)^{1/2}$$
$$\le C\|(1+\|\cdot\|_2^2)^k D^{\boldsymbol{\beta}}\Phi^r\|_{L^2(\mathbb{R}^d)},$$

where the second integral in the product in the first line is finite because of $k > d/4$.
If L is an odd integer, we set $k = (L-1)/2$. With this we can conclude that $(I-\Delta)^k\partial_j[p_{\boldsymbol{\beta}}\widehat{\Phi^r}] \in L^2(\mathbb{R}^d)$ for all $|\boldsymbol{\beta}| \le \tau$ and for all $1 \le j \le d$. Similar to the first case, we can conclude that $(1+\|\cdot\|_2^2)^k[p_{\mathbf{e}_j}D^{\boldsymbol{\beta}}\Phi^r] \in L^2(\mathbb{R}^d)$ for all $|\boldsymbol{\beta}| \le \tau$ and for all $1 \le j \le d$. Hence, we arrive at

$$\int_{\mathbb{R}^d} |D^{\boldsymbol{\beta}}\Phi^r(\mathbf{x})|d\mathbf{x} \le \left(\int_{\mathbb{R}^d}(1+\|\mathbf{x}\|_2^2)^{2k+1}|D^{\boldsymbol{\beta}}\Phi^r(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2}\left(\int_{\mathbb{R}^d}(1+\|\mathbf{x}\|_2^2)^{-2k-1}d\mathbf{x}\right)^{1/2},$$

where the second integral in the product on the first line is finite because of $k+1/2 > d/4$. Finally, for the first integral have

$$\int_{\mathbb{R}^d}(1+\|\mathbf{x}\|_2^2)^{2k+1}|D^{\boldsymbol{\beta}}\Phi^r(\mathbf{x})|^2 d\mathbf{x} \le \int_{\mathbb{R}^d}(1+\|\mathbf{x}\|_2^2)^{2k}|D^{\boldsymbol{\beta}}\Phi^r(\mathbf{x})|^2 d\mathbf{x}$$
$$+\sum_{j=1}^d \int_{\mathbb{R}^d}(1+\|\mathbf{x}\|_2^2)^{2k}|x_j D^{\boldsymbol{\beta}}\Phi^r(\mathbf{x})|^2 d\mathbf{x},$$

where both integrals can be bounded since $(1+\|\cdot\|_2^2)^k[D^{\boldsymbol{\beta}}\Phi^r] \in L^2(\mathbb{R}^d)$ and $(1+\|\cdot\|_2^2)^k[p_{\mathbf{e}_j}D^{\boldsymbol{\beta}}\Phi^r] \in L^2(\mathbb{R}^d)$. □

4.2.2 Conditions on the Convolution Kernel

After we investigated the existence of a convolution root of the kernel, we want to transfer the two conditions we have attached to the root kernel to the convolution kernel. First, for the approximation condition of the convolution root Φ^r we derive the following result.

Lemma 4.18
Let $\Phi \in C(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ be positive definite. Assume that Φ satisfies the approximation condition of order $L \ge 2$. Then, the convolution root Φ^r from Lemma 4.14 satisfies the approximation condition of order $L-1$.

Proof. From Proposition 3.6 we know that Φ satisfying the approximation condition of order L implies that there exists a $C > 0$ such that

$$|D^{\boldsymbol{\alpha}}[\omega_j\widehat{\Phi}(\boldsymbol{\omega})]| \le C\widehat{\Phi}(\boldsymbol{\omega}), \quad \boldsymbol{\omega}\in\mathbb{R}^d, \quad 1 \le |\boldsymbol{\alpha}| \le L.$$

From Theorem 4.7 and $\widehat{\Phi^r} = (2\pi)^{-d/4}\widehat{\Phi}^{1/2}$ it then follows that

$$|D^{\boldsymbol{\alpha}}[\omega_j\widehat{\Phi^r}(\boldsymbol{\omega})]| \le C\widehat{\Phi^r}(\boldsymbol{\omega}), \quad \boldsymbol{\omega}\in\mathbb{R}^d, \quad 1 \le |\boldsymbol{\alpha}| \le L.$$

which means in particular that $p_{\boldsymbol{\alpha}}\partial_j\Phi^r \in L^2(\mathbb{R}^d)$ for $|\boldsymbol{\alpha}| = L$, so that Φ^r satisfies the approximation condition of order $L-1$. □

For the momentum condition, we already know from Proposition 3.4 that Φ^r satisfies the moment condition of order m if and only if Φ satisfies the moment condition of order m as long as $\Phi^r \in L^1(\mathbb{R}^d)$. Using Lemma 4.16, Φ^r is in $L^1(\mathbb{R}^d)$ if condition (4.7) holds. Hence, using Corollary 4.6, we can derive the following result.

Lemma 4.19
Let $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be positive definite. Assume that Φ satisfies the moment condition of order m and assume that there exists a constant $C > 0$ and an $L \geq \lfloor d/2 \rfloor + 1$ such that

$$|D^{\boldsymbol{\alpha}} \omega_j \widehat{\Phi}(\boldsymbol{\omega})| \leq C \widehat{\Phi}(\boldsymbol{\omega}), \qquad \boldsymbol{\omega} \in \mathbb{R}^d, \tag{4.10}$$

for all $|\boldsymbol{\alpha}| \leq L$ and $1 \leq j \leq d$. Then, the convolution root Φ^r from Lemma 4.14 is in $L^1(\mathbb{R}^d)$ and satisfies the moment condition of order m.

Proof. Using (4.10), Corollary 4.6 implies that there exists a constant $\widetilde{C} > 0$ such that

$$|D^{\boldsymbol{\alpha}} \widehat{\Phi^r}(\boldsymbol{\omega})| \leq \widetilde{C} |\widehat{\Phi^r}(\boldsymbol{\omega})|, \qquad \boldsymbol{\omega} \in \mathbb{R}^d$$

for all $1 \leq |\boldsymbol{\alpha}| \leq L$. Since $L \geq \lfloor d/2 \rfloor + 1$, using Lemma 4.16 gives that $\Phi^r \in L^1(\mathbb{R}^d)$ and hence satisfies the moment condition of order m according to Proposition 3.4. □

Note that condition (4.10) is nothing else but the first part of the approximation condition. If we combine the last two results we come to one of our main results in this section.

Theorem 4.20
Let $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be positive definite. Assume that Φ satisfies the moment condition of order m and the approximation condition of order $L \geq \lfloor d/2 \rfloor + 1$. Then, the convolution root Φ^r from Lemma 4.14 is in $L^1(\mathbb{R}^d)$, satisfies the moment condition of order m and the approximation condition of order $L - 1$.
If Φ is in addition in $W^{2\sigma,2}(\mathbb{R}^d)$ for a $\sigma > d/4$, then $\Phi^r \in W^{\tau,1}(\mathbb{R}^d)$ for all integer $\tau < \sigma - d/4$.

Proof. It follows from Lemma 4.18 and Lemma 4.19 that the root kernel Φ^r satisfies the moment condition of order m, the approximation conditions of order $L - 1$ and lies in $L^1(\mathbb{R}^d)$. Moreover, since Φ satisfies the approximation condition of order L, (4.10) is satisfied for all $1 \leq |\boldsymbol{\alpha}| \leq L$. Hence, Corollary 4.6 implies that (4.7) is satisfied for all $1 \leq |\boldsymbol{\alpha}| \leq L$ and using Theorem 4.17 completes the proof. □

Theorem 4.20 is a very important result to construct the kernel because we managed to formulate conditions on the convolution kernel Φ so that the conditions of Theorem 3.9 on the root kernel Φ^r are satisfied. Since Φ^r is only needed in the proof of the convergence but not in the numerical SPH scheme (3.4) - (3.6), we can neglect Φ^r now and can focus on the construction of an easy to implement and efficient to calculate kernel Φ that will give us automatically the existence and the required properties of a root kernel.

4.2.3 Radial Kernels with Compact Support

From now on we can concentrate on the construction of the kernel function Φ. Due to simplicity, we restrict our choice to radial functions with compact support. This is a good choice since those functions are easy to implement and efficient to calculate in the SPH formalism. Moreover, radial functions simplify the required conditions we need. Note that with the choice of a radial kernel $\Phi = \phi(\|\cdot\|_2)$, its Fourier transform $\widehat{\Phi} = \mathcal{F}_d \phi(\|\cdot\|_2)$ as well as its convolution root Φ^r will be radial, too. Unfortunately, as mentioned before, this is not true for the compact support any more.
We first have to check how the conditions look like in the radial version. To do this, we want to rewrite the moment and approximation conditions for radial functions. We will start with the approximation condition.

Lemma 4.21
Let $\phi \in C^1([0, \infty[)$ be positive definite on $\mathbb{R}^d$ with compact support and with $\int_{\mathbb{R}^d} \phi(\|\mathbf{x}\|_2) d\mathbf{x} = 1$. Suppose there exists a constant $C > 0$ such that

$$\left| \frac{d^k}{ds^k} s \mathcal{F}_d \phi(s) \right| \leq C \mathcal{F}_d \phi(s) \tag{4.11}$$

holds for all $s \in [0, \infty[$ and $1 \leq k \leq L$. Then, $\Phi = \phi(\|\cdot\|_2)$ satisfies the approximation condition of order L.

Proof. We have to prove that Φ satisfies conditions (3.10) and (3.11). For the first condition, we note that since $\phi \in C^1([0,\infty[)$ has compact support, $p_{\boldsymbol{\alpha}}\partial_j\Phi$ is continuous and has compact support for all $|\boldsymbol{\alpha}| = L+1$ and $1 \le j \le d$. Thus, $p_{\boldsymbol{\alpha}}\partial_j\Phi \in L^2(\mathbb{R}^d)$.
To show condition (3.11), we suppose $1 \le |\boldsymbol{\alpha}| \le L$, and $1 \le j \le d$. We have to show that there exists a constant $C > 0$ such that $|D^{\boldsymbol{\alpha}} x_j \widehat{\Phi}(\mathbf{x})| \le C\widehat{\Phi}(\mathbf{x})$ holds for all $\mathbf{x} \in \mathbb{R}^d$.
We will first take a look at small $\mathbf{x} \in \mathbb{R}^d$. Since $\Phi = \phi(\|\cdot\|_2)$ has compact support, $\widehat{\Phi}$ lies in $C^\infty(\mathbb{R}^d)$. Moreover, Φ is normalized, i.e. $\int_{\mathbb{R}^d} \Phi(\mathbf{x})d\mathbf{x} = 1$, and thus $\widehat{\Phi}(\mathbf{0}) > 0$. Due to the continuity of $\widehat{\Phi}$ there exists a radius $r > 0$ with $\widehat{\Phi}(\mathbf{x}) > 0$ for all $\|\mathbf{x}\|_2 \le r$. Therefore, we can choose

$$C_1 = \max_{\|\mathbf{x}\|_2 \le r} \frac{|D^{\boldsymbol{\alpha}} x_j \widehat{\Phi}(\mathbf{x})|}{\widehat{\Phi}(\mathbf{x})}$$

such that $|D^{\boldsymbol{\alpha}} x_j \widehat{\Phi}(\mathbf{x})| \le C_1\widehat{\Phi}(\mathbf{x})$ for all $\|\mathbf{x}\|_2 \le r$.
Now let $\|\mathbf{x}\|_2 > r$ and suppose that $\alpha_j \ge 1$ in the first instance. With (4.2), we note that

$$|D^{\boldsymbol{\alpha}} x_j \widehat{\Phi}(\mathbf{x})| \le \alpha_j |D^{\boldsymbol{\alpha}-\mathbf{e}_j}\widehat{\Phi}(\mathbf{x})| + |x_j D^{\boldsymbol{\alpha}}\widehat{\Phi}(\mathbf{x})|. \tag{4.12}$$

Applying Lemma 4.12 to the first absolute value we have

$$\begin{aligned} |D^{\boldsymbol{\alpha}-\mathbf{e}_j}\widehat{\Phi}(\mathbf{x})| &\le \sum_{k=1}^{|\boldsymbol{\alpha}|-1} \left|(\mathcal{F}_d\phi)^{(k)}(\|\mathbf{x}\|_2)\right| \sum_{\ell=1}^{2|\boldsymbol{\alpha}|-3} \frac{\sum_{|\boldsymbol{\nu}|=\ell-(|\boldsymbol{\alpha}|-1-k)} c_{\boldsymbol{\nu}}\|\mathbf{x}\|_2^{|\boldsymbol{\nu}|}}{\|\mathbf{x}\|_2^\ell} \\ &\le c\sum_{k=1}^{|\boldsymbol{\alpha}|-1} \|\mathbf{x}\|_2^{-(|\boldsymbol{\alpha}|-1-k)} \left|(\mathcal{F}_d\phi)^{(k)}(\|\mathbf{x}\|_2)\right| \\ &\le c\max\{1, r^{-|\boldsymbol{\alpha}|+2}\} \sum_{k=1}^{|\boldsymbol{\alpha}|-1} \left|(\mathcal{F}_d\phi)^{(k)}(\|\mathbf{x}\|_2)\right|, \end{aligned} \tag{4.13}$$

where we used $\|\mathbf{x}\|_2 > r$. Assumption (4.11) and Lemma 4.5 in the univariate case imply that there exists a $C > 0$ such that $\left|(\mathcal{F}_d\phi)^{(k)}(\|\mathbf{x}\|_2)\right| \le C\mathcal{F}_d\phi(\|\mathbf{x}\|_2)$ for every $1 \le k \le |\boldsymbol{\alpha}|-1$. Hence,

$$|D^{\boldsymbol{\alpha}-\mathbf{e}_j}\widehat{\Phi}(\mathbf{x})| \le c\mathcal{F}_d\phi(\|\mathbf{x}\|_2) = c\widehat{\Phi}(\mathbf{x}).$$

For the second part we apply Lemma 4.12 as we did it in (4.13) to have

$$\begin{aligned} |x_j D^{\boldsymbol{\alpha}}\widehat{\Phi}(\mathbf{x})| \le \|\mathbf{x}\|_2 |D^{\boldsymbol{\alpha}}\widehat{\Phi}(\mathbf{x})| &\le c\|\mathbf{x}\|_2 \sum_{k=1}^{|\boldsymbol{\alpha}|} \|\mathbf{x}\|_2^{-(|\boldsymbol{\alpha}|-k)} \left|(\mathcal{F}_d\phi)^{(k)}(\|\mathbf{x}\|_2)\right| \\ &= c\left(\sum_{k=1}^{|\boldsymbol{\alpha}|-1} \|\mathbf{x}\|_2^{-(|\boldsymbol{\alpha}|-k-1)} \left|(\mathcal{F}_d\phi^{(k)}(\|\mathbf{x}\|_2)\right| + \|\mathbf{x}\|_2 \left|(\mathcal{F}_d\phi^{(|\boldsymbol{\alpha}|)}(\|\mathbf{x}\|_2)\right|\right), \end{aligned}$$

where we split up the sum in the last line in the sum over the first $|\boldsymbol{\alpha}|-1$ terms and the term for $k = |\boldsymbol{\alpha}|$. The occurring sum is exactly the sum from (4.13) and can be estimated as above. For the remaining term, we expand the absolute value to apply a reverse chain rule. With $s = \|\mathbf{x}\|_2$ and using the triangle inequality, we conclude

$$\begin{aligned} \left|s\left(\mathcal{F}_d\phi^r\right)^{(|\boldsymbol{\alpha}|)}(s)\right| &\le \left|s\left(\mathcal{F}_d\phi^r\right)^{(|\boldsymbol{\alpha}|)}(s) + |\boldsymbol{\alpha}|\left(\mathcal{F}_d\phi^r\right)^{(|\boldsymbol{\alpha}|-1)}(s)\right| + |\boldsymbol{\alpha}|\left|\left(\mathcal{F}_d\phi^r\right)^{(|\boldsymbol{\alpha}|-1)}(s)\right| \\ &= \left|\frac{d^{|\boldsymbol{\alpha}|}}{ds^{|\boldsymbol{\alpha}|}}\left(s\mathcal{F}_d\phi^r(s)\right)\right| + |\boldsymbol{\alpha}|\left|\left(\mathcal{F}_d\phi^r\right)^{(|\boldsymbol{\alpha}|-1)}(s)\right|. \end{aligned}$$

The first term can be bounded by assumption (4.11). For the second term, assumption (4.11) and Lemma 4.5 in the univariate case imply again that $\left|(\mathcal{F}_d\phi)^{(|\alpha-1|)}(s)\right| \le C\mathcal{F}_d\phi(s)$. Finally, we have

$$\|\mathbf{x}\|_2 \left|(\mathcal{F}_d\phi^{(|\boldsymbol{\alpha}|)}(\|\mathbf{x}\|_2)\right| \le C\mathcal{F}_d\phi(\|\mathbf{x}\|_2).$$

If $\alpha_j = 0$, the first term on the right hand side of (4.12) vanishes, so that the proof can be adapted. □

For the moment condition we recall Lemma 4.11 which gives us that odd moments vanish if the kernel is radial. A first result is the following one.

Lemma 4.22
A compactly supported, radial and normalized kernel $\Phi = \phi(\|\cdot\|_2)$ always satisfies the moment condition of an even order, in particular, Φ has at least order 2. A non-negative compactly supported and normalized kernel satisfies the moment condition with order at most 2. Hence, a non-negative, radial and normalized kernel always satisfies the moment condition of order 2.

Hence, we will concentrate on moment conditions of even order since odd-order conditions are always satisfied. We can rewrite the moment conditions as follows.

Lemma 4.23
Let $\phi \in C[0,\infty[$ be positive definite on $\mathbb{R}^d$ with compact support and let $m \in \mathbb{N}$. Suppose ϕ satisfies

$$\int_0^\infty s^{d-1}\phi(s)ds = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}, \tag{4.14}$$

$$\int_0^\infty s^{2k+d-1}\phi(s)ds = 0, \qquad 1 \le k \le m-1, \tag{4.15}$$

$$\int_0^\infty s^{2m+d-1}\phi(s)ds < \infty, \tag{4.16}$$

where Γ denotes the Gamma function, see [GR00, 8.310]. Then, $\Phi = \phi(\|\cdot\|_2)$ satisfies the moment condition of order $2m$.

***Proof*.** Applying the transformation theorem with spherical coordinates for an arbitrary $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ gives

$$\int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}}\Phi(\mathbf{x})d\mathbf{x} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty s^{|\boldsymbol{\alpha}|+d-1}\phi(s)ds$$

For $|\boldsymbol{\alpha}| = 0$ and $|\boldsymbol{\alpha}| = 2m$ we can simply apply assumption (4.14) and (4.16).
For $1 \le |\boldsymbol{\alpha}| < 2m$, we first notice that with Lemma 4.11 the integral is zero if $|\boldsymbol{\alpha}|$ is odd since Φ is a radial function. For an even $|\boldsymbol{\alpha}|$ we have $|\boldsymbol{\alpha}| = 2k$ for a $1 \le k \le m-1$. Hence,

$$\int_0^\infty s^{|\boldsymbol{\alpha}|+d-1}\phi(s)ds = 0,$$

where we used assumption (4.15). □

Note that condition (4.16) is automatically satisfied if our kernel ϕ has compact support.

4.2.4 A Kernel Construction Scheme

With Lemma 4.23 in mind, we now want to give a simple way to construct radial kernel functions satisfying the moment condition of arbitrary order. For that, we follow the ideas of [BM85], see also [MB02] and [RW16] by constructing a linear combination of scaled kernel functions. Suppose that $m \in \mathbb{N}$. Consider the kernel as a linear combination of scaled functions, i.e. consider

$$\phi(s) = \sum_{j=1}^m \lambda_j \psi\left(\frac{s}{a_j}\right), \quad s \in [0,\infty[, \tag{4.17}$$

with $\lambda_j \in \mathbb{R}$ for $1 \leq j \leq m$, $0 < a_1 < a_2 < \ldots < a_m \in \mathbb{R}$ and $\psi : [0, \infty[\to \mathbb{R}$ be a function with compact support. With this kind of kernel, we can change the conditions on the kernel from Lemma 4.23 to conditions on the parameters $\boldsymbol{\lambda}$ and $\mathbf{a}$. Inserting (4.17) into condition (4.14), we find the new condition

$$\sum_{j=1}^{m} \lambda_j a_j^d = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}} \left(\int_0^\infty s^{d-1} \psi(s) ds \right)^{-1} =: c_0.$$

Furthermore, condition (4.15) will be satisfied if

$$\sum_{j=1}^{m} \lambda_j a_j^{2k+d} = 0$$

for all $1 \leq k \leq m-1$. Since we have m equations for our conditions, we receive the following linear system

$$\begin{pmatrix} a_1^d & a_2^d & a_3^d & \cdots & a_m^d \\ a_1^{2+d} & a_2^{2+d} & a_3^{2+d} & \cdots & a_m^{2+d} \\ a_1^{4+d} & a_2^{4+d} & a_3^{4+d} & \cdots & a_m^{4+d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{2m-2+d} & a_2^{2m-2+d} & a_3^{2m-2+d} & \cdots & a_m^{2m-2+d} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_m \end{pmatrix} = \begin{pmatrix} c_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.18}$$

To simplify this system we will transfer the a_j^d from the matrix to the vector and denote $\lambda_j a_j^d$ by $\widetilde{\lambda}_j$. Then, we have the system

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_m^2 \\ a_1^4 & a_2^4 & a_3^4 & \cdots & a_m^4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{2m-2} & a_2^{2m-2} & a_3^{2m-2} & \cdots & a_m^{2m-2} \end{pmatrix} \begin{pmatrix} \widetilde{\lambda}_1 \\ \widetilde{\lambda}_2 \\ \widetilde{\lambda}_3 \\ \vdots \\ \widetilde{\lambda}_m \end{pmatrix} = \begin{pmatrix} c_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.19}$$

The matrix we received is nothing else than a transposed Vandermonde matrix for the distinct numbers $a_1^2, a_2^2, \ldots, a_m^2$, which means that unique solvability is given.

Proposition 4.24
Let $m \in \mathbb{N}$, $a_j > 0$ be pairwise distinct for $1 \leq j \leq m$ and $c_0 \in \mathbb{R}$. Then, the system given in (4.18) *has exactly one solution. Furthermore, the coefficients λ_j are given by*

$$\begin{aligned} \lambda_j &= \frac{c_0}{a_j^d} \prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{a_k^2}{a_k^2 - a_j^2} \\ &= (-1)^{j-1} \frac{c_0}{a_j^d} \frac{a_1^2 \cdots a_{j-1}^2 a_{j+1}^2 \cdots a_m^2}{(a_j^2 - a_1^2) \cdots (a_j^2 - a_{j-1}^2)(a_{j+1}^2 - a_j^2) \cdots (a_m^2 - a_j^2)} \end{aligned}$$

for $1 \leq j \leq m$.

Proof. Let $\widetilde{\lambda}_j = a_j^d \lambda_j$ and $\widetilde{a}_j = a_j^2$ for $1 \leq j \leq m$ and denote the matrix given in equation (4.19) by A. Since the $\widetilde{a}_j$ are pairwise distinct for $1 \leq j \leq m$, the determinant of the transposed Vandermonde-matrix A, given by $\det(A) = \prod_{1 \leq j < k \leq m} (\widetilde{a}_k - \widetilde{a}_j)$, is different from zero and the solution of system (4.19) can be calculated by Cramers rule via

$$\widetilde{\lambda}_j = \frac{\det(A_j)}{\det(A)},$$

where the matrices A_j result from A by replacing the j-th column with the right-hand side of equation (4.18). For the determinant of A_j we conclude

$$\begin{aligned}\det(A_j) &= \begin{vmatrix} 1 & \cdots & 1 & c_0 & 1 & \cdots & 1 \\ \widetilde{a}_1 & \cdots & \widetilde{a}_{j-1} & 0 & \widetilde{a}_{j+1} & \cdots & \widetilde{a}_m \\ \widetilde{a}_1^2 & \cdots & \widetilde{a}_{j-1}^2 & 0 & \widetilde{a}_{j+1}^2 & \cdots & \widetilde{a}_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_1^{m-1} & \cdots & \widetilde{a}_{j-1}^{m-1} & 0 & \widetilde{a}_{j+1}^{m-1} & \cdots & \widetilde{a}_m^{m-1} \end{vmatrix} \\ &= c_0(-1)^j \begin{vmatrix} \widetilde{a}_1 & \cdots & \widetilde{a}_{j-1} & \widetilde{a}_{j+1} & \cdots & \widetilde{a}_m \\ \widetilde{a}_1^2 & \cdots & \widetilde{a}_{j-1}^2 & \widetilde{a}_{j+1}^2 & \cdots & \widetilde{a}_m^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_1^{m-1} & \cdots & \widetilde{a}_{j-1}^{m-1} & \widetilde{a}_{j+1}^{m-1} & \cdots & \widetilde{a}_m^{m-1} \end{vmatrix} \\ &= c_0(-1)^j \left(\prod_{\substack{1\le k\le m \\ k\neq j}} \widetilde{a}_k \right) \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \widetilde{a}_1 & \cdots & \widetilde{a}_{j-1} & \widetilde{a}_{j+1} & \cdots & \widetilde{a}_m \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_1^{m-2} & \cdots & \widetilde{a}_{j-1}^{m-2} & \widetilde{a}_{j+1}^{m-2} & \cdots & \widetilde{a}_m^{m-2} \end{vmatrix} \\ &= c_0(-1)^j \left(\prod_{\substack{1\le k\le m \\ k\neq j}} \widetilde{a}_k \right) \left(\prod_{\substack{1\le i<k\le m \\ i,k\neq j}} (\widetilde{a}_k - \widetilde{a}_i) \right), \end{aligned}$$

which completes the proof. □

This construction method of the kernel gives us a very simple way to satisfy the moment condition of arbitrary order. Moreover, this construction is, except for the constant c_0, independent of the choice for the function ψ. However, a few problems arise along this construction. We have to be very careful not to violate other properties of the kernel Φ. In particular, Φ has to remain positive definite and has to have a convolution root.
Unfortunately, to keep these two properties, new conditions on the function ψ, the parameters a_j and m will appear. To keep Φ positive definite, the d-dimensional Fourier transform of ψ has to be decreasing and $(\widetilde{\lambda}_j)_{j=1}^m = (\lambda_j a_j^d)_{j=1}^m$ have to be a monotonically decreasing sequence in the absolute value, i.e. $|\widetilde{\lambda}_j| > |\widetilde{\lambda}_{j+1}|$ for $1 \le j \le m-1$. This leads us to the following result.

Lemma 4.25
Let $\psi \in C([0,\infty[)$, so that $s \mapsto s^{d-1}\psi(s) \in L^1(\mathbb{R}^d)$, be positive definite on $\mathbb{R}^d$ with a decreasing d-dimensional Fourier transform. Let $m \in \mathbb{N}$ and $0 < a_1 < a_2 \ldots < a_m$ so that $(|\widetilde{\lambda}_j|)_{j=1}^m = (|\lambda_j|a_j^d)_{j=1}^m$ is a monotonically decreasing sequence and let $\widetilde{\lambda}_j > 0$ if j is odd. Then, ϕ, defined in (4.17) *is positive definite on $\mathbb{R}^d$.*
Furthermore, if $\mathcal{F}_d\psi$ is positive in $[0,\infty[$, then $\mathcal{F}_d\phi$ is also positive in $[0,\infty[$.

Proof. First of all, let m be even. The d-dimensional Fourier transformation of ϕ can be written as

$$\mathcal{F}_d\phi(s) = \sum_{j=1}^m \widetilde{\lambda}_j \mathcal{F}_d\psi(a_j s) = c \sum_{j=1}^{m/2} \left(\widetilde{\lambda}_{2j-1}\mathcal{F}_d\psi(a_{2j-1}s) + \widetilde{\lambda}_{2j}\mathcal{F}_d\psi(a_{2j}s) \right),$$

where we split up the sum in terms with odd and even indices, respectively. Using that $\widetilde{\lambda}_j$ is positive if j is odd, we have $\widetilde{\lambda}_{2j-1} > |\widetilde{\lambda}_{2j}|$. Moreover, since $\mathcal{F}_d\psi$ is decreasing, meaning in particular $\mathcal{F}_d\psi(a_{2j-1}s) \ge \mathcal{F}_d\psi(a_{2j}s)$ for all $s > 0$ and $1 \le j \le m/2$, we have

$$\widetilde{\lambda}_{2j-1}\mathcal{F}_d\psi(a_{2j-1}s) > |\widetilde{\lambda}_{2j}|\mathcal{F}_d\psi(a_{2j}s),$$

and hence $\mathcal{F}_d\phi(s) > 0$. Now, let m be odd. Then, we have

$$\mathcal{F}_d\phi(s) = c\sum_{j=1}^{m-1} \widetilde{\lambda}_j \mathcal{F}_d\psi\left(a_j s\right) + \widetilde{\lambda}_m \mathcal{F}_d\psi\left(a_m s\right),$$

where the sum is positive according to the case where m was even. The second term is also positive, since λ_m is positive if m is odd. □

For the sake of simplicity, we now choose $\mathbf{a} \in \mathbb{R}^m$ to be equidistant, more precisely $a_j = bj$ for a fixed $b > 0$ and for $1 \leq j \leq m$. Then, we can calculate $\widetilde{\lambda}_j$ via

$$\begin{aligned}
\widetilde{\lambda}_j &= c_0 \prod_{\substack{k=1\\k\neq j}}^{m} \frac{(bk)^2}{(bk)^2-(bj)^2} = c_0 \prod_{\substack{k=1\\k\neq j}}^{m} \frac{k^2}{k^2-j^2} = c_0 \prod_{\substack{k=1\\k\neq j}}^{m} \frac{k}{k-j}\frac{k}{k+j} \\
&= c_0 \frac{(m!)^2}{j^2} \left(\prod_{k=1}^{j-1} \frac{1}{k-j}\right) \left(\prod_{k=j+1}^{m} \frac{1}{k-j}\right) \left(2j \prod_{k=1}^{m} \frac{1}{k+j}\right) \\
&= (-1)^{j-1} 2c_0 \frac{(m!)^2}{j} \left(\prod_{k=1}^{j-1} \frac{1}{k}\right) \left(\prod_{k=1}^{m-j} \frac{1}{k}\right) \left(\prod_{k=1+j}^{m+j} \frac{1}{k}\right) \\
&= (-1)^{j-1} 2c_0 \frac{m!}{(m+j)!}\frac{m!}{(m-j)!} \\
&= (-1)^{j-1} 2c_0 \prod_{k=1}^{j} \frac{m+1-k}{m+k}.
\end{aligned}$$

With this representation for $\widetilde{\lambda}_j$, which is independent of the choice of b, we can easily verify that $\widetilde{\lambda}_j = -\frac{m+1-j}{m+j}\widetilde{\lambda}_{j-1}$. Since $\frac{m+1-j}{m+j} < 1$, $(|\widetilde{\lambda}_j|)_{j=1}^m$ is a monotonically decreasing sequence. Moreover, since $\widetilde{\lambda}_1$ is positive, $\widetilde{\lambda}_1$ is positive if j is odd. Hence, the sequence $(|\lambda_j|)_{j=1}^m$ from Proposition 4.24 can be used to construct ϕ.
Finally, it remains to derive a condition on ψ so that Φ, resulting from the construction in (4.17), satisfies the approximation condition. This is the case if ψ itself satisfies a kind of approximation condition.

Lemma 4.26
Let $\psi \in C^1([0,\infty[)$ be positive definite on $\mathbb{R}^d$ with compact support and a decreasing d-dimensional Fourier transform. Let $a_j = bj$ for a given $b > 0$, $1 \leq j \leq m$ and λ_j from Proposition 4.24. Assume there exists a constant $C > 0$ such that

$$\left|\frac{d^k}{ds^k} s\mathcal{F}_d\psi(s)\right| \leq C\mathcal{F}_d\psi(s), \quad s \in [0,\infty[, \tag{4.20}$$

for all $1 \leq k \leq L$. Then, $\Phi = \phi(\|\cdot\|_2)$ with ϕ defined in (4.17) satisfies the approximation condition of order L.

***Proof*.** First we note that $\phi \in C^1([0,\infty[)$ has compact support. Moreover, as we have shown before Lemma 4.26, $(|\widetilde{\lambda}_j|)_{j=1}^m$ is a monotonically decreasing sequence and λ_j is positive if j is odd. Since ψ is positive definite with decreasing d-dimensional Fourier transform, Lemma 4.25 states that ϕ is positive definite.
To apply Lemma 4.21 it remains to show that

$$\left|\frac{d^k}{ds^k} s\mathcal{F}_d\phi(s)\right| \leq C\mathcal{F}_d\phi(s), \quad s \in [0,\infty[,$$

for all $1 \le k \le L$. Inserting the definition of ϕ, we achieve

$$\begin{aligned}
\left|\frac{d^k}{ds^k} s\mathcal{F}_d\phi(s)\right| &= \left|\frac{d^k}{ds^k} s\sum_{j=1}^m \lambda_j a_j^d \mathcal{F}_d\psi(sa_j)\right| \\
&\le \sum_{j=1}^m |\widetilde{\lambda}_j| \left|\frac{d^k}{ds^k} s\mathcal{F}_d\psi(sa_j)\right| \\
&\le \sum_{j=1}^m |\widetilde{\lambda}_j||a_j^{k-1}| \left|\frac{d^k}{dr^k} r\mathcal{F}_d\psi(r)\Big|_{r=sa_j}\right| \\
&\le C|a_m^{k-1}| \sum_{j=1}^m |\widetilde{\lambda}_j| \mathcal{F}_d\psi(sa_j)
\end{aligned}$$

for all $s \ge 0$ and $1 \le k \le L$, where we used the assumption (4.20). To complete the proof, we have to show that there exists a $\widetilde{C} > 0$ with $\sum_{j=1}^m |\widetilde{\lambda}_j|\mathcal{F}_d\psi(sa_j) \le \widetilde{C}\sum_{j=1}^m \widetilde{\lambda}_j\mathcal{F}_d\psi(sa_j) = \widetilde{C}\mathcal{F}_d\phi(s)$, or

$$\sum_{j=1}^m \left(\widetilde{C}\widetilde{\lambda}_j - |\widetilde{\lambda}_j|\right)\mathcal{F}_d\psi(sa_j) \ge 0, \qquad s \ge 0. \tag{4.21}$$

According to Lemma 4.25, inequality (4.21) holds if $(|\widetilde{C}\widetilde{\lambda}_j - |\widetilde{\lambda}_j||)_{j=1}^m$ is a decreasing sequence. We recall the properties of $\widetilde{\lambda}_j$. As we have shown before Lemma 4.26, $\widetilde{\lambda}_j = -\frac{m+1-j}{m+j}\widetilde{\lambda}_{j-1}$ and λ_j is positive if and only if j is odd, otherwise λ_j is negative. With this, assuming that $\widetilde{C} \ge 1$, we have

$$\widetilde{C}\widetilde{\lambda}_j - |\widetilde{\lambda}_j| = \begin{cases} (\widetilde{C}+1)\widetilde{\lambda}_j, & j \text{ is even,} \\ (\widetilde{C}-1)\widetilde{\lambda}_j, & j \text{ is odd,} \end{cases}$$

which shows that $\widetilde{C}\widetilde{\lambda}_j - |\widetilde{\lambda}_j|$ is positive if j is odd. Moreover, if j is odd, we have

$$|\widetilde{C}\widetilde{\lambda}_j - |\widetilde{\lambda}_j|| = (\widetilde{C}-1)|\widetilde{\lambda}_j| = (\widetilde{C}-1)\frac{m+1-j}{m+j}|\widetilde{\lambda}_{j-1}| < (\widetilde{C}+1)|\widetilde{\lambda}_{j-1}|,$$

which is true for every $\widetilde{C} > 1$ since $(m+1-j)/(m+j) < 1$. If, however, j is even we have

$$|\widetilde{C}\widetilde{\lambda}_j - |\widetilde{\lambda}_j|| = (\widetilde{C}+1)|\widetilde{\lambda}_j| = (\widetilde{C}+1)\frac{m+1-j}{m+j}|\widetilde{\lambda}_{j-1}| < (\widetilde{C}-1)|\widetilde{\lambda}_{j-1}|,$$

which is true for every $\widetilde{C}$ satisfying $\widetilde{C} > (2m+1)/(2j-1)$ for $1 \le j \le m$, so that $\widetilde{C} > 2m+1$ suffices to ensure that inequality (4.21) holds. Applying Lemma 4.21 completes the proof. □

We will finish this section with our final result which unites all conditions on ψ, so that our corresponding radial kernel Φ satisfies the condition from Theorem 4.20.

Theorem 4.27
Let $\psi \in C([0,\infty[)$ be positive definite on $\mathbb{R}^d$ with compact support and a decreasing d-dimensional Fourier transform. Let $m \in \mathbb{N}$ and $L \ge \lfloor d/2 \rfloor + 1$. Assume there exists a constant $C > 0$ such that

$$\left|\frac{d^k}{ds^k} s\mathcal{F}_d\psi(s)\right| \le C\mathcal{F}_d\psi(s), \qquad s \in [0,\infty[, \tag{4.22}$$

for all $1 \le k \le L$. Let Φ be defined as in (4.17) with $a_j = bj$ for a given $b > 0$, $1 \le j \le m$ and let λ_j be determined as in Proposition 4.24. Then, the convolution root Φ^r from Lemma 4.14 is in $L^1(\mathbb{R}^d)$, satisfies the moment condition of order $2m$ and the approximation condition of order $L-1$.
If $\psi(\|\cdot\|)$ is in addition in $W^{2\sigma,2}(\mathbb{R}^d)$ for $\sigma > d/4$, then $\Phi^r \in W^{\tau,1}(\mathbb{R}^d)$ for all $\tau < \sigma - d/4$.

Proof. With the given properties we know that Φ defined as in (4.17) is a continuous, positive definite function with compact support that satisfies the moment condition of order $2m$ according to Lemma 4.23 and the approximation condition of order L according to Lemma 4.26. Theorem 4.20 gives us the existence of Φ^r with the required properties. □

4.3 The Wendland Radial Basis Functions

In the last section, we derived a construction scheme for the convolution kernel Φ as a function of an univariate kernel ψ. To complete this construction, we still need a class of nonnegative, positive definite functions $\psi \in C([0,\infty[)$ with compact support and decreasing Fourier transforms, satisfying the conditions of Theorem 4.27. Moreover, these functions should be easy to implement and efficient to calculate.
The original Wendland functions, first derived in [Wen95], will be a good starting point to find those kernels. It is a well-known fact that these functions are nonnegative, positive definite with compact support. Furthermore, they can be constructed having any given smoothness. However, it is not clear whether they have a decreasing Fourier transform and whether they satisfy the approximation condition. For both conditions, we need to know the Fourier transform of the Wendland functions. Unfortunately, the form of the Fourier transform of the Wendland functions differs in even and odd spatial dimensions, whereby especially the case for odd space dimension becomes technical. Nevertheless, we will provide an extension of the original Wendland functions such that they have a decreasing Fourier transform and satisfy the approximation condition of any order in even and in odd space dimension.
This theory can easily be transfered to the missing Wendland functions, see [Sch11], by a simple dimension step argument. Hence, we provide an extension for the missing Wendland functions such they have a decreasing Fourier transform and satisfy the approximation condition of any order.
First of all, we will give a generalized theory of the Wendland functions.

4.3.1 The generalized Wendland functions

We will start with the basic construction of the generalized Wendland functions. For that, we will need a special integral operator. For the following theory, see also [Sch11], [Che13] and [Hub12].

Definition 4.28
Let $\alpha \geq 0$ and f be given such that $s \mapsto sf(s)(s^2-r^2)^{\alpha-1}$ is in $L^1([0,\infty[)$ for all $r \geq 0$. Then, the operator I^α is defined by

$$I^\alpha f(r) = \begin{cases} \frac{1}{2^{\alpha-1}\Gamma(\alpha)} \int\limits_r^\infty sf(s)(s^2-r^2)^{\alpha-1}ds, & \alpha \neq 0, \\ f(r), & \alpha = 0, \end{cases}$$

for all $r \geq 0$.

This operator has some important properties we will derive. To show these properties, we will need the following two equations which can be found in [GR00, 3.196.3] and [GR00, 6.567.1].

Lemma 4.29
i) Let $b > a$ and $\mu, \nu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 0$ and $\mathrm{Re}(\nu) > 0$. Then, the following equation holds

$$\int\limits_a^b (s-a)^{\mu-1}(b-s)^{\nu-1}ds = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}(b-a)^{\mu+\nu-1}.$$

ii) Let $b > 0$ and $\mu, \nu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > -1$ and $\mathrm{Re}(\nu) > -1$. Then, the following equation holds

$$\int_0^1 t^{\nu+1} J_\nu(bt)(1-t^2)^\mu dt = 2^\mu \Gamma(\mu+1) b^{-(\mu+1)} J_{\nu+\mu+1}(b).$$

The first important property of the operator defined in Definition 4.28 is additivity in the parameter α. The linearity in f is clear since the integral is a linear operator itself.

Lemma 4.30
Let $\alpha, \beta \geq 0$ and f such that $s \mapsto sf(s)(s^2-r^2)^{\gamma-1}$ is in $L^1([0,\infty[)$ for $\gamma \in \{\alpha, \beta, \alpha+\beta\}$. Then

$$I^\alpha I^\beta f(r) = I^{\alpha+\beta} f(r)$$

for all $r \geq 0$.

***Proof*.** For $\alpha = 0$ or $\beta = 0$ there is nothing to show. Hence, we suppose that $\alpha, \beta > 0$. First of all, we insert the definition of I^α and I^β. Thus, we have

$$\begin{aligned} I^\alpha I^\beta f(r) &= \frac{1}{2^{\alpha-1}\Gamma(\alpha)} \int_r^\infty s I^\beta f(s)(s^2-r^2)^{\alpha-1} ds \\ &= \frac{1}{2^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta)} \int_r^\infty s \int_s^\infty tf(t)(t^2-s^2)^{\beta-1} dt (s^2-r^2)^{\alpha-1} ds. \end{aligned}$$

We will use $C_{\alpha,\beta} := (2^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta))^{-1}$ as an abbreviation. By reordering the integral, we conclude

$$\begin{aligned} I^\alpha I^\beta f(r) &= C_{\alpha,\beta} \int_r^\infty \int_s^\infty s(s^2-r^2)^{\alpha-1}(t^2-s^2)^{\beta-1} tf(t) dt ds \\ &= C_{\alpha,\beta} \int_r^\infty tf(t) \int_r^t s(s^2-r^2)^{\alpha-1}(t^2-s^2)^{\beta-1} ds dt, \end{aligned}$$

where we changed the order of integration using Fubini's theorem. Note, that we had to adjust the integration boundaries. Substituting s by $\sqrt{s}$ implies

$$I^\alpha I^\beta f(r) = C_{\alpha,\beta} \int_r^\infty tf(t) \frac{1}{2} \int_{r^2}^{t^2} (s-r^2)^{\alpha-1}(t^2-s)^{\beta-1} ds dt.$$

From Lemma 4.29 i) we know the latter integral. Inserting this gives us

$$\begin{aligned} I^\alpha I^\beta f(r) &= \frac{C_{\alpha,\beta}}{2} \int_r^\infty tf(t) \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} (t^2-r^2)^{\alpha+\beta-1} dt \\ &= \frac{1}{2^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \int_r^\infty tf(t)(t^2-r^2)^{\alpha+\beta-1} dt \\ &= I^{\alpha+\beta} f(r), \end{aligned}$$

which completes the proof. □

The operator I^α has a very useful property in combination with the Fourier transformation $\mathcal{F}_d$ of radial functions given in Theorem 4.10.

Lemma 4.31
Let $0 \leq \alpha \leq (d-1)/2$ *and* f *such that* $s \mapsto f(s)s^{d/2}J_{(d-2)/2}(rs)$ *is in* $L^1([0,\infty[)$ *for all* $r \geq 0$. *Then,*

$$\mathcal{F}_{d-2\alpha} I^\alpha f(r) = \mathcal{F}_d f(r)$$

for all $r \geq 0$.

Proof. For $\alpha = 0$ there is nothing to show. Hence, let $\alpha > 0$. We will start by inserting the definition of the radial Fourier operator and the definition of the operator I^α. Hence,

$$\begin{aligned}
\mathcal{F}_{d-2\alpha} I^\alpha f(r) &= r^{-\frac{d-2\alpha-2}{2}} \int_0^\infty I^\alpha f(t) t^{\frac{d-2\alpha}{2}} J_{\frac{d-2\alpha-2}{2}}(rt)dt \\
&= r^{-\frac{d-2\alpha-2}{2}} \int_0^\infty \frac{1}{2^{\alpha-1}\Gamma(\alpha)} \int_t^\infty sf(s)(s^2-t^2)^{\alpha-1} ds\, t^{\frac{d-2\alpha}{2}} J_{\frac{d-2\alpha-2}{2}}(rt)dt \\
&= \frac{r^{-\frac{d-2\alpha-2}{2}}}{2^{\alpha-1}\Gamma(\alpha)} \int_0^\infty \int_t^\infty sf(s)(s^2-t^2)^{\alpha-1} t^{\frac{d-2\alpha}{2}} J_{\frac{d-2\alpha-2}{2}}(rt)dsdt.
\end{aligned}$$

For the sake of simplicity, we will write $\nu = (d-2\alpha-2)/2$ and $\mu = \alpha - 1$. By changing the order of integration with Fubini's theorem, where we have to adjust the integration boundaries, we conclude

$$\begin{aligned}
\mathcal{F}_{d-2\alpha} I^\alpha f(r) &= \frac{r^{-\nu}}{2^\mu \Gamma(\alpha)} \int_0^\infty \int_t^\infty sf(s)(s^2-t^2)^\mu t^{\nu+1} J_\nu(rt)dsdt \\
&= \frac{r^{-\nu}}{2^\mu \Gamma(\alpha)} \int_0^\infty sf(s) \int_0^s t^{\nu+1} J_\nu(rt)(s^2-t^2)^\mu dtds \\
&= \frac{r^{-\nu}}{2^\mu \Gamma(\alpha)} \int_0^\infty f(s)s^{\nu+2\mu+3} \int_0^1 t^{\nu+1} J_\nu(rst)(1-t^2)^\mu dtds,
\end{aligned}$$

where we substituted t by t/s in the last step. At this point we can use Lemma 4.29 ii) with $b = rs$ to arrive at

$$\begin{aligned}
\mathcal{F}_{d-2\alpha} I^\alpha f(r) &= \frac{r^{-\nu}}{2^\mu \Gamma(\alpha)} \int_0^\infty f(s)s^{\nu+2\mu+3} 2^\mu \Gamma(\alpha)(rs)^{-(\mu+1)} J_{\nu+\mu+1}(rs)ds \\
&= r^{-\nu-\mu-1} \int_0^\infty f(s)s^{\nu+\mu+2} J_{\nu+\mu+1}(rs)ds \\
&= r^{-\frac{d-2}{2}} \int_0^\infty f(s)s^{\frac{d}{2}} J_{\frac{d-2}{2}}(rs)ds.
\end{aligned}$$

This completes the proof. □

We will shortly review the last result. The operator I^α enables a form of dimension walk, which means, for example, that we can calculate the d-dimensional Fourier transform of f by calculating

the 1-dimensional Fourier transform of $I^{(d-1)/2}f$. In particular, we will also make use of one-dimensional steps where we have $\alpha = 1/2$.
With this property in mind, we now give the definition of the generalized Wendland function. To this end, we introduce the cutoff function $(\cdot)_+$, which is defined by

$$(x)_+ = \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

and recall the notation $\lfloor x \rfloor$, which denotes the largest integer less than or equal to $x \in \mathbb{R}$.

Definition 4.32
Let $\alpha \geq 0$, $\mu > -1$ and f_μ defined by $f_\mu(s) := (1-s)_+^\mu$. The generalized Wendland functions are given by

$$\phi_{\mu,\alpha}(s) :\doteq I^\alpha(f_\mu)(s) \tag{4.23}$$

for $s \geq 0$.

Note that $:\doteq$ means that the generalized Wendland functions are defined up to a multiplicative constant. This will be important later when we demand that $\phi_{\mu,\alpha}$ is normalized in some way. For the greater part of the following theory, however, we will neglect this constant.
It is a well-known fact that these functions are positive definite on $\mathbb{R}^d$ for $\alpha \in \mathbb{N}_0/2$ and $\mu \in \mathbb{N}_0$ if $\mu \geq \lfloor d/2 + \alpha \rfloor + 1$, see [Sch11]. Moreover, Chernih [Che13] showed that these functions are positive definite on $\mathbb{R}^d$ for $\alpha > 0$ if and only if $\mu \geq (d+1)/2 + \alpha$.
A very useful property of the generalized Wendland function is a combination of Lemma 4.30 and Lemma 4.31, which enables the dimension walk introduced in Lemma 4.31.

Theorem 4.33
Let $\alpha \geq 0$, $\mu > -1$ and $0 \leq \beta \leq (d-1)/2$. Then $\mathcal{F}_d\phi_{\mu,\alpha}(r) = \mathcal{F}_{d-2\beta}\phi_{\mu,\alpha+\beta}(r)$ for all $r \geq 0$.

Proof. By Definition 4.32, f_μ has compact support, so that $s \mapsto s f_\mu(s)(s^2 - r^2)^\gamma$ is in $L^1([0,\infty[)$ for all $r > 0$. Hence, using Lemma 4.30, $\phi_{\mu,\alpha+\beta} = I^\beta \phi_{\mu,\alpha}$ holds. Moreover, using that $\phi_{\mu,\alpha}$ has compact support, the map $s \mapsto \phi_{\mu,\alpha} s^{d/2} J_{(d-2)/2}(rs)$ is in $L^1([0,\infty[)$ for all $r > 0$. Hence, applying Lemma 4.31 completes the proof. □

Theorem 4.33 allows to calculate the d-dimensional Fourier transform of the generalized Wendland function $\phi_{\mu,\alpha}$ by calculating the one-dimensional Fourier transform of $\phi_{\mu,\alpha+(d-1)/2}$. This dimension walk is the key idea to determine the Fourier transform of the Wendland functions later on.
Another easy-to-show property is the monotonicity of the Fourier transform for specific μ. To prove this, we will need the following lemma.

Lemma 4.34
Let $f : [0,\infty[\to \mathbb{R}$ with $r \mapsto r^{d+a} f(r) \in L^1(\mathbb{R}^d)$ for $a \in \{-1, 0, +1\}$. Then,

$$-\frac{1}{r}(\mathcal{F}_d f)'(r) = \mathcal{F}_{d+2} f(r)$$

for all $r \geq 0$.

Proof. Using the dominated convergence theorem we may change the order of differentiation and integration, to conclude

$$\begin{aligned} -\frac{1}{r}(\mathcal{F}_d f(r))' &= -\frac{1}{r}\left(r^{-\frac{d-2}{2}} \int_0^\infty f(s) s^{\frac{d}{2}} J_{\frac{d-2}{2}}(rs)\,ds \right)' \\ &= -\frac{1}{r} \int_0^\infty f(s) s^d \left(t^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(t) \right)' \Big|_{t=rs} ds. \end{aligned}$$

With the equation $(t^{-\nu}J_\nu(t))' = -t^{-\nu}J_{\nu+1}(t)$, which can, for example, be found in [Wen04], we arrive at

$$\begin{aligned}-\frac{1}{r}(\mathcal{F}_d f(r))' &= \frac{1}{r}\int_0^\infty f(s)s^d(rs)^{-\frac{d-2}{2}}J_{\frac{d}{2}}(rs)ds\\ &= r^{-\frac{d}{2}}\int_0^\infty f(s)s^{\frac{d+2}{2}}J_{\frac{d}{2}}(rs)ds\\ &= \mathcal{F}_{d+2}f(r),\end{aligned}$$

which completes the proof. □

The monotonicity of the Fourier transform of Wendland's functions was firstly showen by Chernih in [Che13] with a more complicated proof. It can be shown that the generalized Wendland functions are monotonically decreasing if μ is chosen large enough. This is a crucial property in the sense of Theorem 4.27.

Theorem 4.35
Let $\alpha \geq 0$. Then, the d-dimensional Fourier transform of the generalized Wendland function $\phi_{\mu,\alpha}$ is monotonically decreasing if $\mu \geq \lfloor d/2+\alpha\rfloor + 2$.

Proof. We have to show that the derivative of $\mathcal{F}_d\phi_{\mu,\alpha}$ is negative. With Lemma 4.34 we know that $(\mathcal{F}_d\phi_{\mu,\alpha})'(s) = -s\mathcal{F}_{d+2}\phi_{\mu,\alpha}(s)$. For the $(d+2)$-dimensional Fourier transform we have the dimension step from Theorem 4.33 to go back to the d-dimensional Fourier transform. Hence, we have $\mathcal{F}_{d+2}\phi_{\mu,\alpha} = \mathcal{F}_d\phi_{\mu,\alpha+1}$. Finally we have

$$(\mathcal{F}_d\phi_{\mu,\alpha})'(r) = -r\mathcal{F}_d\phi_{\mu,\alpha+1}(r) < 0$$

for every $r > 0$, since $\phi_{\mu,\alpha+1}$ is a positive definite function for all $\mu \geq \lfloor d/2+\alpha+1\rfloor + 1$. □

We defined the Wendland functions and derived some basic properties we will need. From now on, we will consider two different cases, one in which α is assumed to be an integer and the other in which $\alpha + 1/2$ is assumed to be an integer.
The final goal is to find functions ψ that satisfy the conditions in Theorem 4.27, that means, among other things, that ψ is positive definite, has a decreasing d-dimensional Fourier transform and satisfies a kind of approximation condition (4.22). While the first two conditions are satisfied by the general Wendland functions, the approximation condition (4.22) is a little bit harder to achieve. For this, we will calculate an explicit formula of the Fourier transform, and will show that (4.22) holds. Moreover, we will differ between the original Wendland functions ($\alpha = k$ and $\mu = \lfloor d/2\rfloor + k+1$) and the missing Wendland functions ($\alpha = k+1/2$ and $\mu = \lfloor (d+1)/2\rfloor + k+1$).

4.3.2 The original Wendland functions

The original Wendland functions were first proposed in 1995 by Wendland in [Wen95]. These functions are given by equation (4.23) with $\alpha = k$ for a $k \in \mathbb{N}$ and $\mu = \lfloor d/2\rfloor + k + 1$, i.e.

$$\psi_{d,k} := \phi_{\lfloor\frac{d}{2}\rfloor+k+1,k} = I^k f_{\lfloor\frac{d}{2}\rfloor+k+1}.$$

In our case, we need an extended version of the original Wendland functions. Let $l \geq 0$. We define the extended original Wendland functions by

$$\psi_{d,k,l} := \phi_{\lfloor\frac{d}{2}\rfloor+k+l+1,k} = I^k f_{\lfloor\frac{d}{2}\rfloor+k+l+1}.$$

We will write $\mu = \lfloor d/2\rfloor + k + 1$ since it is a frequently used parameter in the context of Wendland functions. If $l = 0$, then $\psi_{d,k,0} = \psi_{d,k}$, which means that the original Wendland functions are

a special case of the extended original Wendland functions. Note, that the new parameter l can also be seen as an increment of the spatial dimension of the Wendland functions, since we have $\psi_{d,k,l} = \psi_{d+2l,k}$ for $l \in \mathbb{N}$. Thus, all we do is considering original Wendland functions of higher spatial dimension in lower dimensional spaces. Note, that $\psi_{d,k,l}$ possess $2k$ continuous derivatives around 0, like the original ones, but $\lfloor d/2 \rfloor + 2k + l = \mu + k + l - 1$ continuous derivatives around 1, l more than the original Wendland functions.
We will start with arbitrary spatial dimension. Later on, when we have to calculate the Fourier transform of the extended original Wendland functions, we have to differ between even and odd spatial dimension.
To calculate the Fourier transform analytically, we need a closed form representation of the Wendland functions. Such a closed representation was first given by Hubbert in [Hub12]. We will extend this representation to include the new parameter l as follows.

Lemma 4.36
Let $l, k \in \mathbb{N}_0$ and $\mu = \lfloor d/2 \rfloor + k + 1$. Then, the extended Wendland function can be written as

$$\psi_{d,k,l}(r) = \phi_{\mu+l,k}(r) = \frac{1}{2^k k!}(1-r)_+^{\mu+l+k} \sum_{j=0}^{k} \frac{\binom{k}{j}}{\binom{\mu+l+k+j}{k+j}} 2^{k-j} r^{k-j} (1-r)^j$$

for $r \geq 0$.

Proof. Hubbert proved the formula for $l = 0$ in [Hub12]. Since the parameter l can be interpreted as an increase of the spatial dimension, i.e. $\psi_{d,k,l} = \psi_{d+2l,k}$, the formula is also valid for arbitrary $l \in \mathbb{N}$. □

Lemma 4.36 gives us a representation of the extended Wendland kernels of the form

$$\psi_{d,k,l} = (1-r)_+^{\lfloor d/2 \rfloor + 2k + l + 1} p(r),$$

where p is a polynomial of degree k. With this formula, it is obvious that the first $\lfloor d/2 \rfloor + 2k + l$ derivatives will vanish at $r = 1$.

Corollary 4.37
Let $l, k \in \mathbb{N}_0$. Then

$$\psi_{d,k,l}^{(n)}(1) = 0$$

for all $0 \leq n \leq \lfloor d/2 \rfloor + 2k + l$.

Chernih [Che13] simplified the representation from Lemma 4.36 to the following, general result.

Theorem 4.38
Let $l, k \in \mathbb{N}_0$ and $\nu \in \mathbb{N}$ with $\nu \geq (d + 2k + 1)/2$. Then, the Wendland functions are given by

$$\phi_{\nu,k}(r) = \frac{2^k k! \nu!}{(\nu + 2k)!} \sum_{j=0}^{\nu+2k} (-1)^{k+j} \binom{\nu + 2k}{j} \binom{\frac{j-1}{2}}{k} r^j$$

for all $0 \leq r \leq 1$, where we used the generalized binomial coefficient

$$\binom{a}{k} := \begin{cases} \frac{a(a-1)\cdots(a-(k-1))}{k!}, & k > 0, \\ 1, & k = 0, \\ 0, & k < 0, \end{cases}$$

for $a \in \mathbb{C}$.

Proof. For the proof of this formula, see [Che13]. □

This formula holds in particular for the extended original Wendland functions $\psi_{d,k,l} = \phi_{\mu+l,k}$ with $\nu = \mu + l \geq (d + 2k + 1)/2$ for all $l \in \mathbb{N}_0$. By combining the constants of the previous formula, we can finally write the extended original Wendland functions as follows.

Corollary 4.39
Let $l, k \in \mathbb{N}_0$ and $\mu = \lfloor d/2 \rfloor + k + 1$. Then, the extended Wendland functions are given by

$$\psi_{d,k,l}(r) = C_{d,k,l} \sum_{j=0}^{\mu+2k+l} a_{d,k,l,j} r^j$$

for all $0 \leq r \leq 1$, with

$$a_{d,k,l,j} = (-1)^{k+j} \binom{\mu + 2k + l}{j} \binom{\frac{j-1}{2}}{k}$$

and

$$C_{d,k,l} = \frac{2^k k! (\mu + l)!}{(\mu + 2k + l)!}.$$

Note that $a_{d,k,l,j} = 0$ if $(j - 1)/2 - k$ is a negative integer because of the appearing generalized binomial coefficient.
Furthermore, from Corollary 4.37 we know that the first $\lfloor d/2 \rfloor + 2k + l$ derivatives vanish at $r = 1$. This gives us the following relations for the parameters $a_{d,k,l,j}$.

Corollary 4.40
Let $l, k \in \mathbb{N}_0$ and $\mu = \lfloor d/2 \rfloor + k + 1$. Then

$$\sum_{j=n}^{\mu+2k+l} a_{d,k,l,j} \frac{j!}{(j-n)!} = 0.$$

for all $0 \leq n \leq \lfloor d/2 \rfloor + 2k + l = \mu + k + l - 1$.

The formula in Corollary 4.39 will serve us as a starting point for the calculation of the Fourier transform of the extended original Wendland functions. From this point, we have to differ between even and odd space dimensions.

The original Wendland function in odd space dimension

Next, we will restrict ourselves to odd space dimension. Note that in this case we have $\lfloor d/2 \rfloor = (d - 1)/2$. As we will see, we will need the following integrals.

Lemma 4.41
Let $j \in \mathbb{N}_0$. Then, the following equation holds

$$\begin{aligned} \int_0^1 s^j \cos(sr) ds = &\sin(r) \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^n \frac{j!}{(j-2n)!} r^{-(2n+1)} \\ &+ \cos(r) \sum_{n=0}^{\lfloor \frac{j-1}{2} \rfloor} (-1)^n \frac{j!}{(j-2n-1)!} r^{-(2n+2)} \\ &+ \left(j - 2 \left\lfloor \frac{j}{2} \right\rfloor\right) (-1)^{\frac{j+1}{2}} j! r^{-(j+1)} \end{aligned} \tag{4.24}$$

for all $r > 0$.

Remark
Note that in the term $j - 2\left\lfloor \frac{j}{2} \right\rfloor$ *is zero if* j *is even, so that the last term of* (4.24) *vanishes in this case. Hence,* $(-1)^{(j+1)/2}$ *has not to be defined for even* j*.*

Proof. We will denote the integrals with

$$C_j(r) := \int_0^1 s^j \cos(sr) ds.$$

We will give the proof by induction, where we will first show that the formula above is correct for $j = 0$ and $j = 1$, and then do an induction step from $j - 2$ to j. For $j = 0$, we can simply calculate the integral by

$$C_0(r) = \frac{1}{r} \sin(r).$$

For $j = 1$, we use partial integration to arrive at

$$C_1(r) = \frac{1}{r} \sin(r) + \frac{1}{r^2} \cos(r) - \frac{1}{r^2},$$

where we see that formula (4.24) is correct. Now suppose that the formula is correct for C_{j-2}. For C_j, using partial integration twice, we derive

$$\begin{aligned} C_j(r) &= \frac{1}{r} \sin(r) - \frac{j}{r} \int_0^1 s^{j-1} \sin(rs) ds \\ &= \frac{1}{r} \sin(r) + \frac{j}{r^2} \cos(r) - \frac{j(j-1)}{r^2} C_{j-2}(r). \end{aligned}$$

Now, we want to use the induction hypothesis and insert C_{j-2}. To differ between the terms occurring in C_{j-2}, we will split up the term to $C_{j-2}(r) = C_{j-2,\sin}(r) + C_{j-2,\cos}(r) + C_{j-2,0}(r)$. Hence, for the term that contains the sinus terms, we have

$$\begin{aligned} \frac{1}{r} \sin(r) - \frac{j(j-1)}{r^2} C_{j-2,\sin}(r) &= \frac{1}{r} \sin(r) - \frac{j(j-1)}{r^2} \sin(r) \sum_{n=0}^{\lfloor \frac{j-2}{2} \rfloor} (-1)^n \frac{(j-2)!}{(j-2-2n)!} r^{-(2n+1)} \\ &= \sin(r) \left(\frac{1}{r} + \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor - 1} (-1)^{n+1} \frac{j!}{(j-2-2n)!} r^{-(2n+3)} \right) \\ &= \sin(r) \left(\frac{1}{r} + \sum_{n=1}^{\lfloor \frac{j}{2} \rfloor} (-1)^n \frac{j!}{(j-2n)!} r^{-(2n+1)} \right) \\ &= \sin(r) \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^n \frac{j!}{(j-2n)!} r^{-(2n+1)}. \end{aligned}$$

In the same way we can combine the parts containing the cosinus terms. For the remaining third term $C_{j-2,0}$, we note that $(j-2) - 2\lfloor (j-2)/2 \rfloor = j - 2\lfloor j/2 \rfloor$, which completes the proof. □

With the calculated integrals we can give a closed form representation of the Fourier transform of the extended original Wendland functions in odd space dimensions.

Theorem 4.42
Let $d \in \mathbb{N}$ be odd, $k, l \in \mathbb{N}_0$ and $\mu = (d-1)/2 + k + 1$. Then, the Fourier transform of the extended Wendland function is given by

$$\mathcal{F}_d\psi_{d,k,l}(r) = \sqrt{\frac{2}{\pi}} C_{d,k+\frac{d-1}{2},l-\frac{d-1}{2}} \Bigg(\sin(r) \sum_{n=0}^{\lfloor \frac{\mu+l}{2} \rfloor - \lfloor \frac{l}{2} \rfloor - 1} b_{1,d,k,l,n} r^{-(2\lfloor \frac{l}{2} \rfloor + 2\mu + 2n + 1)} + \cos(r) \sum_{n=0}^{\lfloor \frac{\mu+l-1}{2} \rfloor - \lfloor \frac{l-1}{2} \rfloor - 1} b_{2,d,k,l,n} r^{-(2\lfloor \frac{l-1}{2} \rfloor + 2\mu + 2n + 2)} + \sum_{n=0}^{\lfloor \frac{\mu+l-1}{2} \rfloor} b_{3,d,k,l,n} r^{-(2n+2\mu)} \Bigg),$$

where the coefficients are given by

$$b_{1,d,k,l,n} = (-1)^{\lfloor \frac{l}{2} \rfloor + \mu + n} \sum_{j=2\lfloor \frac{l}{2} \rfloor + 2\mu + 2n}^{3\mu+l-2} a_{d,k+\frac{d-1}{2},l-\frac{d-1}{2},j} \frac{j!}{(j - 2\lfloor \frac{l}{2} \rfloor - 2\mu - 2n)!},$$

$$b_{2,d,k,l,n} = (-1)^{\lfloor \frac{l-1}{2} \rfloor + \mu + n} \sum_{j=2\lfloor \frac{l-1}{2} \rfloor + 2\mu + 2n + 1}^{3\mu+l-2} a_{d,k+\frac{d-1}{2},l-\frac{d-1}{2},j} \frac{j!}{(j - 2\lfloor \frac{l-1}{2} \rfloor - 2\mu - 2n - 1)!},$$

$$b_{3,d,k,l,n} = (-1)^{\mu+n} a_{d,k+\frac{d-1}{2},l-\frac{d-1}{2},2n+2\mu-1} (2n + 2\mu - 1)!.$$

Proof. Taking the d-dimensional Fourier transform of $\psi_{d,k,l}$, we can step through the dimension via Theorem 4.33 and arrive at

$$\mathcal{F}_d\psi_{d,k,l}(r) = \mathcal{F}_d\phi_{\mu+l,k}(r) = \mathcal{F}_1\phi_{\mu+l,k+\frac{d-1}{2}}(r) = \mathcal{F}_1\psi_{d,k+\frac{d-1}{2},l-\frac{d-1}{2}}(r),$$

where we just have to calculate the one-dimensional Fourier transform of $\psi_{d,k+\frac{d-1}{2},l-\frac{d-1}{2}}$. Now we denote $k' = k + \frac{d-1}{2}$ and $l' = l - \frac{d-1}{2}$ and $\mu' = \frac{d-1}{2} + k' + 1$. Recalling Corollary 4.39, the extended Wendland functions can be written as

$$\psi_{d,k',l'}(r) = \phi_{\mu'+l',k'}(r) = C_{d,k',l'} \sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} r^j, \quad r \in [0, 1].$$

Note that k' and l' are integer since d is odd, so that using Corollary 4.39 is allowed. Using the one dimensional radial Fourier transform (4.4) then yields

$$\mathcal{F}_d\psi_{d,k,l}(r) = \sqrt{\frac{2}{\pi}} C_{d,k',l'} \sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} \int_0^1 s^j \cos(sr) ds.$$

By inserting the integrals from Lemma 4.41, the sum that occurs in the Fourier transform becomes

$$\sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} \int_0^1 s^j \cos(rs) ds = \sin(r)A(r) + \cos(r)B(r) + C(r),$$

where the functions A, B and C are given by

$$A(r) = \sum_{j=0}^{2k'+\mu'+l'} \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^n a_{d,k',l',j} \frac{j!}{(j-2n)!} r^{-(2n+1)},$$

$$B(r) = \sum_{j=1}^{2k'+\mu'+l'} \sum_{n=0}^{\lfloor \frac{j-1}{2} \rfloor} (-1)^n a_{d,k',l',j} \frac{j!}{(j-2n-1)!} r^{-(2n+2)}$$

and

$$C(r) = \sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} \left(j - 2 \left\lfloor \frac{j}{2} \right\rfloor \right) (-1)^{\frac{j+1}{2}} j! r^{-(j+1)}.$$

Note, that we start the summation over j in B at 1 since for $j = 0$ there are no terms. By changing the order of summation we obtain for A

$$\begin{aligned} A(r) &= \sum_{n=0}^{\lfloor \frac{2k'+\mu'+l'}{2} \rfloor} \sum_{j=2n}^{2k'+\mu'+l'} (-1)^n a_{d,k',l',j} \frac{j!}{(j-2n)!} r^{-(2n+1)} \\ &= \sum_{n=0}^{\lfloor \frac{2k'+\mu'+l'}{2} \rfloor} (-1)^n r^{-(2n+1)} \sum_{j=2n}^{2k'+\mu'+l'} a_{d,k',l',j} \frac{j!}{(j-2n)!}. \end{aligned}$$

We know from Corollary 4.40 that the second sum vanishes for $0 \leq 2n \leq \mu' + k' + l' - 1$. So the first inner sum, which does not vanish is for $n = (\mu' + k' + l')/2$, if the numerator is even, and $n = (\mu' + k' + l' + 1)/2$, if the numerator is odd. Hence, we have

$$A(r) = \sum_{n=\lfloor \frac{\mu'+k'+l'+1}{2} \rfloor}^{\lfloor \frac{2k'+\mu'+l'}{2} \rfloor} (-1)^n r^{-(2n+1)} \sum_{j=2n}^{2k'+\mu'+l'} a_{d,k',l',j} \frac{j!}{(j-2n)!}.$$

Now we want to change the parameters back to k and l. Using $\mu' + l' = \mu + l$ and $k' = \mu - 1$ yields $\lfloor \frac{2k'+\mu'+l'}{2} \rfloor = \lfloor \frac{3\mu+l-2}{2} \rfloor = \lfloor \frac{\mu+l}{2} \rfloor + \mu - 1$ and $\lfloor \frac{\mu'+k'+l'+1}{2} \rfloor = \lfloor \frac{2\mu+l}{2} \rfloor = \lfloor \frac{l}{2} \rfloor + \mu$. Inserting the change of parameters, we derive

$$\begin{aligned} A(r) &= \sum_{n=\lfloor \frac{l}{2} \rfloor + \mu}^{\lfloor \frac{\mu+l}{2} \rfloor + \mu - 1} (-1)^n r^{-(2n+1)} \sum_{j=2n}^{3\mu+l-2} a_{d,k',l',j} \frac{j!}{(j-2n)!} \\ &= \sum_{n=0}^{\lfloor \frac{\mu+l}{2} \rfloor - \lfloor \frac{l}{2} \rfloor - 1} b_{1,d,k,l,n} r^{-(2\lfloor \frac{l}{2} \rfloor + 2\mu + 2n + 1)}, \end{aligned}$$

where we shifted the index n and used the coefficients given by

$$b_{1,d,k,l,n} = (-1)^{\lfloor \frac{l}{2} \rfloor + \mu + n} \sum_{j=2\lfloor \frac{l}{2} \rfloor + 2\mu + 2n}^{3\mu+l-2} a_{d,k',l',j} \frac{j!}{(j - 2\lfloor \frac{l}{2} \rfloor - 2\mu - 2n)!}.$$

We will apply the same procedure to B. Again, by changing the order of summation we have

$$\begin{aligned} B(r) &= \sum_{n=0}^{\lfloor \frac{\mu'+2k'+l'-1}{2} \rfloor} \sum_{j=2n+1}^{\mu'+2k'+l'} (-1)^n a_{d,k',l',j} \frac{j!}{(j-2n-1)!} r^{-(2n+2)} \\ &= \sum_{n=0}^{\lfloor \frac{\mu'+2k'+l'-1}{2} \rfloor} (-1)^n r^{-(2n+2)} \sum_{j=2n+1}^{\mu'+2k'+l'} a_{d,k',l',j} \frac{j!}{(j-2n-1)!}. \end{aligned}$$

The second sum vanishes again for $0 \leq 2n+1 \leq \mu' + k' + l' - 1$ according to Corollary 4.40. Hence,

$$B(r) = \sum_{n=\lfloor \frac{\mu'+k'+l'}{2} \rfloor}^{\lfloor \frac{\mu'+2k'+l'-1}{2} \rfloor} (-1)^n r^{-(2n+2)} \sum_{j=2n+1}^{\mu'+2k'+l'} a_{d,k',l',j} \frac{j!}{(j-2n-1)!}.$$

Now, we change back the parameters to the original ones. For the borders of the sum we have $\lfloor \frac{\mu'+2k'+l'-1}{2} \rfloor = \lfloor \frac{3\mu+l-3}{2} \rfloor = \lfloor \frac{\mu+l-1}{2} \rfloor + \mu - 1$ and $\lfloor \frac{\mu'+k'+l'}{2} \rfloor = \lfloor \frac{2\mu+l-1}{2} \rfloor = \lfloor \frac{l-1}{2} \rfloor + \mu$. We arrive at

$$\begin{aligned} B(r) &= \sum_{n=\lfloor \frac{l-1}{2} \rfloor + \mu}^{\lfloor \frac{\mu+l-1}{2} \rfloor + \mu - 1} (-1)^n r^{-(2n+2)} \sum_{j=2n+1}^{3\mu+l-2} a_{d,k',l',j} \frac{j!}{(j-2n-1)!} \\ &= \sum_{n=0}^{\lfloor \frac{\mu+l-1}{2} \rfloor - \lfloor \frac{l-1}{2} \rfloor - 1} b_{2,d,k,l,n} r^{-(2\lfloor \frac{l-1}{2} \rfloor + 2\mu + 2n + 2)}, \end{aligned}$$

where we used the coefficient

$$b_{2,d,k,l,n} = (-1)^{\lfloor \frac{l-1}{2} \rfloor + \mu + n} \sum_{j=2\lfloor \frac{l-1}{2} \rfloor + 2\mu + 2n + 1}^{3\mu+l-2} a_{d,k',l',j} \frac{j!}{(j - 2\lfloor \frac{l-1}{2} \rfloor - 2\mu - 2n - 1)!}$$

Finally, we have to rewrite the function C. Since $j - 2\left\lfloor \frac{j}{2} \right\rfloor$ is 1 if and only if j is odd and 0 if j is even, we conclude

$$C(r) = \sum_{j=0}^{\lfloor \frac{\mu'+2k'+l'-1}{2} \rfloor} a_{d,k',l',2j+1} (-1)^{j+1} (2j+1)! r^{-(2j+2)}.$$

We note that $a_{d,k',l',2j+1} = 0$ if $j < \mu - 1$ since $\binom{j}{\mu-1} = 0$. Hence, we arrive at

$$\begin{aligned} C(r) &= \sum_{j=\mu-1}^{\lfloor \frac{\mu+l-1}{2} \rfloor + \mu - 1} a_{d,k',l',2j+1} (-1)^{j+1} (2j+1)! r^{-(2j+2)} \\ &= \sum_{j=0}^{\lfloor \frac{\mu+l-1}{2} \rfloor} a_{d,k',l',2j+2\mu-1} (-1)^{\mu+j} (2j+2\mu-1)! r^{-(2j+2\mu)}, \end{aligned}$$

where we also returned to the original parameters. Setting the coefficient

$$b_{3,d,k,l,n} = (-1)^{\mu+n} a_{d,k+\frac{d-1}{2},l-\frac{d-1}{2},2n+2\mu-1} (2n+2\mu-1)!$$

finishes the proof. □

We derived a closed form representation of the extended Wendland functions in odd space dimensions. Now our interest lies in the general form of the Fourier transform. As we can see from Theorem 4.42, this form depends on whether the parameter l is odd or even.

Proposition 4.43
Let $d \in \mathbb{N}$ be odd, $k, l \in \mathbb{N}_0$ and $\mu = (d-1)/2 + k + 1$. Then

$$\mathcal{F}_d \psi_{d,k,l}(r) = r^{-(3\mu+l-1)} \left(p_{1,d,k,l}(r) \sin(r) + p_{2,d,k,l}(r) \cos(r) + q_{d,k,l}(r) \right), \quad r \geq 0,$$

where $q_{d,k,l}$ is a polynomial of degree $\mu + l - 1$ and $p_{1,d,k,l}, p_{2,d,k,l}$ are polynomials of degree at most $\mu - 1$.
To be more precise, either $p_{1,d,k,l} \in \Pi_{\mu-2}$ and $p_{2,d,k,l} \in \Pi_{\mu-1}$ if l is even or $p_{1,d,k,l} \in \Pi_{\mu-1}$ and $p_{2,d,k,l} \in \Pi_{\mu-2}$ if l is odd.

Note carefully that the coefficient of the highest monomial part in q, given by $b_{3,k,l,0}$, does not vanish according to the definition of the coefficients $a_{d,k,l,j}$ given in Corollary 4.39. Hence, a notable fact of Proposition 4.43 is that the polynomial $q_{d,k,l}$ has a degree higher by at least l than $p_{1,d,k,l}$

and $p_{2,d,k,l}$. This fact will be important to prove that the extended original Wendland functions satisfy the approximation condition.
As a direct consequence of Proposition 4.43, we know that the asymptotically behavior of $\mathcal{F}_d\psi_{d,k,l}$ has to be like $r^{-2\mu}$, in particular that $\mathcal{F}_d\psi_{d,k,l} = \Theta(r^{-2\mu})$. Since $\mathcal{F}_d\psi_{d,k,l}$ is positive, there exists a $C > 0$ such that

$$\mathcal{F}_d\psi_{d,k,l}(r) \geq Cr^{-2\mu}, \quad r > 1. \tag{4.25}$$

Using this representation, we can conclude two different properties of the extended original Wendland functions. The first one is about the regularity.

Lemma 4.44
Let $d \in \mathbb{N}$ be odd, $k, l \in \mathbb{N}_0$ and $\mu = (d-1)/2 + k + 1$. Then, $\psi_{d,k,l}(\|\cdot\|_2)$ belongs to $W^{\sigma,2}(\mathbb{R}^d)$ for all $\sigma < 2\mu - d/2$.

Proof. From Proposition 4.43 we know that

$$\mathcal{F}_d\psi_{d,k,l} = \Theta(r^{-2\mu})$$

for $r \to \infty$. Since $\mathcal{F}_d\psi_{d,k,l}$ is continuous, there exists a $C > 0$ such that

$$\mathcal{F}_d\psi_{d,k,l}(r) \leq C\frac{1}{(1+r^2)^\mu}$$

for all $r \geq 0$. Using this bound, we arrive at

$$\int\limits_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)^\sigma |\mathcal{F}_d\psi_{d,k,l}(\|\mathbf{x}\|_2)|^2 d\mathbf{x} \leq \int\limits_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)^{\sigma - 2\mu} d\mathbf{x} < \infty$$

which holds if $\sigma - 2\mu < -d/2$. □

The second property, which follows from inequality (4.25), is that the extended original Wendland functions satisfy the approximation condition. Here, the previously mentioned difference in the degree of the polynomial $q_{d,k,l}$ and the polynomials $p_{1,d,k,l}, p_{2,d,k,l}$ becomes important.

Theorem 4.45
Let $d \in \mathbb{N}$ be odd, $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$. Then, for every $L \in \mathbb{N}$ there exists a constant $C_L > 0$ such that

$$\left|\frac{d^n}{dr^n} r\mathcal{F}_d\psi_{d,k,l}(r)\right| \leq C_L \mathcal{F}_d\psi_{d,k,l}(r)$$

for all $1 \leq n \leq L$ and $r \geq 0$.

Proof. In the following we will neglect the dependence of d, k and l and write e.g. p_1 instead of $p_{1,d,k,l}$. We will assume $L \geq l$ without loss of generality.
Since $\mathcal{F}_d\psi_{d,k,l} \in C^\infty([0,\infty[)$ is positive and monotonically decreasing according to Theorem 4.35, such a constant C can easily be found for $r \leq 1$ and it suffices to show that the inequality holds for all $r > 1$. According to Proposition 4.43, $\mathcal{F}_d\psi_{d,k,l}$ has the form

$$\mathcal{F}_d\psi_{d,k,l}(r) = r^{-(3\mu+l-1)}\left(p(r) + q(r)\right), \quad r \geq 0$$

with $\mu = (d-1)/2 + k + 1$, where we denoted $p(r) = p_1(r)\sin(r) + p_2(r)\cos(r)$ for all $r \geq 0$ with $p_1, p_2 \in \Pi_{\leq\mu-1}$ and $q \in \Pi_{\mu+l-1}$, where the set $\Pi_{\leq\mu-1}$ denotes all polynomials with degree at least $\mu - 1$.
Hence, for an arbitrary derivative of order $j \in \mathbb{N}$ we have that $q^{(j)} \in \Pi_{\mu+l-1-j}$ and thus that there exists a constant $c_q > 0$ with $|q^{(j)}(r)| \leq c_q r^{\mu+l-1-j}$ for all $r > 1$ if $j \leq \mu + l - 1$ or $q^{(j)} \equiv 0$ if $j > \mu + l - 1$.

The function p, however, does not change its form when we differentiate it. For the first derivative, we have

$$p'(r) = (p_1'(r) - p_2(r))\sin(r) + (p_2'(r) + p_1(r))\cos(r)$$

where $p_1' - p_2, p_2' - p_1 \in \Pi_{\leq \mu-1}$. Hence, for an arbitrary derivative of order $j \leq \mu - 1$, we have that $p^{(j)}(r) = \widetilde{p}_1(r)\sin(r) + \widetilde{p}_2(r)\cos(r)$, where $\widetilde{p}_1$ and $\widetilde{p}_2$ have the same degree as p_1 and p_2, i.e. $\widetilde{p}_1, \widetilde{p}_2 \in \Pi_{\leq \mu-1}$. Thus, there exists a constant $c_p > 0$ such that $|p^{(j)}(r)| \leq c_p r^{\mu-1}$ for all $r > 1$.
Hence, for an $1 \leq n \leq L$, we can calculate the n-th derivative by

$$\begin{aligned}\frac{d^n}{dr^n} r\mathcal{F}_d\psi_{d,k,l}(r) &= \sum_{j=0}^{n} \binom{n}{j} \frac{d^{n-j}}{dr^{n-j}} r^{-(3\mu+l-2)} \frac{d^j}{dr^j}\left(p(r) + q(r)\right) \\ &= \sum_{j=0}^{n} c_j r^{-(3\mu+l-2)-n+j}\left(p^{(j)}(r) + q^{(j)}(r)\right)\end{aligned}$$

with certain nonvanishing constants c_j. The considerations above show that for each $0 \leq j \leq n$ we have $|p^{(j)}(r)| \leq c_p r^{\mu-1}$. Moreover, if $0 \leq j \leq l$, then we have

$$|q^{(j)}(r)| \leq cr^{\mu+l-1-j}, \quad |p^{(j)}(r)| \leq cr^{\mu-1} \leq cr^{\mu+l-1-j}.$$

For $j > l$ we have have on the one hand

$$|q^{(j)}(r)| \leq cr^{\mu-1},$$

if $j \leq \mu + l - 1$ and, on the other hand, $|q^{(j)}(r)| = 0 \leq cr^{\mu-1}$ if $j > \mu + l - 1$. Hence, we have

$$\left|\frac{d^n}{dr^n} r\mathcal{F}_d\psi_{d,k,l}(r)\right| \leq c\sum_{j=0}^{l} r^{-(3\mu+l-2)-n+j} r^{\mu+l-1-j} + \sum_{j=l+1}^{n} r^{-(3\mu+l-2)-n+j} r^{\mu-1}$$

for all $r > 1$. The first sum simplifies to $\sum_{j=0}^{l} r^{-2\mu-n+1} \leq cr^{-2\mu}$ as $n \geq 1$, $r > 1$. The second sum becomes

$$\sum_{j=l+1}^{n} r^{-2\mu-n+j-l+1} = \sum_{j=0}^{n-l-1} r^{-2\mu-n+j+2} \leq cr^{-2\mu-l+1} \leq cr^{-2\mu}$$

since $l \geq 1$. Using (4.25) finishes the proof. □

Theorem 4.45 states that the extended original Wendland functions satisfy the approximation condition of any order if the parameter $l \geq 1$. Hence, it will be possible to eliminate the dependency of the order of the approximation condition in Theorem 3.9.
Unfortunately, the proof does not hold for the original Wendland functions, where $l = 0$. Using the notation from the proof of Theorem 4.45 and setting $a = 3\mu - 1$, the derivative of $r\mathcal{F}_d\psi_{d,k,0}(r)$ is given by

$$\begin{aligned}\frac{d}{dr}(r\mathcal{F}_d\psi_{d,k,0}(r)) = r^{-a}\Big(&\big((-a+1)p_1(r) + rp_1'(r) - rp_2(r)\big)\sin(r) \\ &+ \big((-a+1)p_2(r) + rp_1(r) + r(p_2'(r)\big)\cos(r) + (-a+1)q(r) + rq'(r)\Big).\end{aligned}$$

In the case of $l = 0$, $rp_1(r)$ is a polynomial of degree $\mu - 1$ and $rp_2(r)$ is a polynomial of degree μ. Hence, $\frac{d}{dr}(r\mathcal{F}_d\psi_{3,1,0}(r))$ decays like $r^{-a+\mu}{=}r^{-2\mu+1}$, while $\mathcal{F}_d\psi_{d,k,0}$ itself decays like $r^{-2\mu}$. Hence, a constant C_L as given in Theorem 4.45 cannot be found for any $L \geq 1$.

Theorem 4.46
Let $d \in \mathbb{N}$ be odd and $k \in \mathbb{N}_0$. The original Wendland functions $\psi_{d,k,0}$ do not satisfy condition (4.22) *of any order.*

Finally, we come to our main theorems of this chapter. The first result gives us the properties of the extended original Wendland functions themselves without using the construction from equation (4.17) for a higher moment condition. Nevertheless, since $\psi_{d,k,l}(\|\cdot\|_2)$ is a radial and therefore an even function, it satisfies automatically the moment condition of order 2. Note that this also requires that $\psi_{d,k,l}(\|\cdot\|_2)$ is normalized, i.e. that $\int_{\mathbb{R}^d} \psi_{d,k,l}(\|\mathbf{x}\|_2)d\mathbf{x} = 1$. Since we defined the generalized Wendland functions up to a multiplicative constant, we will choose this constant such that the normalization is satisfied.

Theorem 4.47
Let $d \in \mathbb{N}$ be odd and $k, l \in \mathbb{N}$. Then, the root kernel of the normalized extended original Wendland function $\psi_{d,k,l}(\|\cdot\|_2)$ exists, belongs to $W^{k,1}(\mathbb{R}^d) \cap W^{k,2}(\mathbb{R}^d)$ and satisfies the moment condition of order 2 and the approximation condition of any order $L \in \mathbb{N}$.

Proof. We know that $\psi_{d,k,l}$ has compact support and since $k > 0$, $\psi_{d,k,l}$ is in $C^1([0,\infty[)$. Hence, using Theorem 4.45, Lemma 4.21 states that $\psi_{d,k,l}(\|\cdot\|_2)$ satisfies the approximation condition of any order $L \in \mathbb{N}$.
Furthermore, from Lemma 4.44 we know that $\psi_{d,k,l}(\|\cdot\|_2) \in W^{2\sigma,2}(\mathbb{R}^d)$ for all $\sigma < \mu - d/4$. Lemma 4.15 then yields that the convolution root of $\psi_{d,k,l}(\|\cdot\|_2)$ belongs to $W^{\tau,2}(\mathbb{R}^d)$ for all $\tau < \sigma - d/4 < \mu - d/2 = k + 1/2$, which includes $\tau = k$. Since $\psi_{d,k,l}$, and hence its convolution root satisfy an approximation condition of any order, Theorem 4.17 yields that the root kernel belongs also to $W^{k,1}(\mathbb{R}^d)$. □

The next result takes advantage of the construction from equation (4.17) to satisfy the moment condition of an arbitrary order. Here, the requirement $l \geq 1$ to ensure that the Fourier transform of the original extended Wendland functions is monotonically decreasing matches with the requirement for the approximation condition.

Theorem 4.48
*Let $d \in \mathbb{N}$ be odd and $k, l \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $\Phi = \phi(\|\cdot\|_2)$ with ϕ be defined as in (4.17) with $\psi = \psi_{d,k,l}$, $a_j = bj$ for a fixed $b > 0$, $1 \leq j \leq m$ and λ_j from Proposition 4.24.
Then, the convolution root Φ^r from Lemma 4.14 is in $W^{k,1}(\mathbb{R}^d) \cap W^{k,2}(\mathbb{R}^d)$, satisfies the moment condition of order $2m$ and the approximation condition of any order $L \in \mathbb{N}$.*

Proof. We have to check the requirements of Theorem 4.27. First of all, $\psi_{d,k,l} \in W^{2\sigma,2}(\mathbb{R}^d)$, with $d/4 < \sigma < \mu - d/4$, has compact support, is positive definite and monotonically decreasing according to Theorem 4.35. Moreover it satisfies inequality (4.22) for all $L \in \mathbb{N}$. Applying Theorem 4.27 finishes the proof. □

Note that if $k > d/2$, the Sobolev embedding theorem yields that Φ^r is also continuous. Hence, using Theorem 4.48, we can finally complete the convergence result for the SPH method in odd space dimensions. Note that we can eliminate the dependency of the order of the approximation condition in the estimate. Hence, the convergence order of the SPH method only depends on the smoothness of the kernel function and the order of the moment condition.

Corollary 4.49
*Let $d \in \mathbb{N}$ be odd and $k, l \in \mathbb{N}$ with $k > d/2$. Let $m \in \mathbb{N}$ and $\Phi = \phi(\|\cdot\|_2)$ with ϕ be defined as in (4.17) with $\psi = \psi_{d,k,l}$, $a_j = bj$ for a fixed $b > 0$, $1 \leq j \leq m$ and λ_j from Proposition 4.24.
Assume initial data $\mathbf{u}_0 \in W^{\sigma,2}(\mathbb{R}^d)^d$ and finite discrete mass with density ρ_0 satisfying $\rho_0^{1/2} \in W^{\max\{\sigma,k\},2}(\mathbb{R}^d)$ for a $\sigma > m + 1 + d/2$ and that the solution $(\mathbf{u}, \rho)$ of the Euler equations (3.1) - (3.3) exists up to a time $T > 0$. Let $(\mathbf{x}_\mathbf{k}^{\varepsilon,h}, \mathbf{u}_\mathbf{k}^{\varepsilon,h})_{\mathbf{k}\in\mathbb{Z}^d}$ be the solution of the weakly compressible SPH equations (3.4) - (3.6).
Then, there exists a constant $C > 0$ such that the energy can be bounded by*

$$Q(t) \leq C \left(\frac{h^{2k}}{\varepsilon^{2k}} + \varepsilon^{2m} \right).$$

for all $t \in [0, T]$.

Of course, these results can also be transferred to pointwise convergence. Since nothing changes at the error bound except for the omission of the parameter L, we just refer to Theorem 3.13 at this point.
Thus, we derived a class of kernel functions that leads to convergence of the SPH method in odd space dimensions. We would like to mention once again that these kernels consist of simple polynomials, making them easy to implement and efficient to calculate. A few examples of such kernel functions from Corollary 4.49, including the most useful ones, can be found in Table 4.1 for $m = 1$ and Table 4.2.

k	l	Extended original Wendland functions $\psi_{3,k,l}$
1	1	$\psi_{3,1,1}(r) \doteq (1-r)_+^5(5r+1)$
	2	$\psi_{3,1,2}(r) \doteq (1-r)_+^6(6r+1)$
	3	$\psi_{3,1,3}(r) \doteq (1-r)_+^7(7r+1)$
2	1	$\psi_{3,2,1}(r) \doteq (1-r)_+^7(16r^2+7r+1)$
	2	$\psi_{3,2,2}(r) \doteq (1-r)_+^8(21r^2+8r+1)$
	3	$\psi_{3,2,3}(r) \doteq (1-r)_+^9(80r^2+27r+3)$
3	1	$\psi_{3,3,1}(r) \doteq (1-r)_+^9(231r^3+159r^2+45r+5)$
	2	$\psi_{3,3,2}(r) \doteq (1-r)_+^{10}(320r^3+197r^2+50r+5)$
	3	$\psi_{3,3,3}(r) \doteq (1-r)_+^{11}(429r^3+239r^2+55r+5)$

Table 4.1: Extended original Wendland functions in $d = 3$ for various parameters of k and l. These functions satisfy the moment condition of order $m = 2$ and the approximation condition of any order $L \in \mathbb{N}$. These functions can be used in the construction method from (4.17).

$m=4$	k	$\Phi(r) \doteq 32\psi_{3,k,1}(2r) - \psi_{3,k,1}(r)$
	1	$\Phi(r) \doteq 32(1-2r)_+^5(10r+1) - (1-r)_+^5(5r+1)$
	2	$\Phi(r) \doteq 32(1-2r)_+^7(64r^2+14r+1) - (1-r)_+^7(16r^2+7r+1)$
	3	$\Phi(r) \doteq 32(1-2r)_+^9(1848r^3+636r^2+90r+5)$
		$-(1-r)_+^9(231r^3+159r^2+45r+5)$
$m=6$	k	$\Phi(r) \doteq 1620\psi_{3,k,1}(3r) - 81\psi_{3,k,1}(2r) + 4\psi_{3,k,1}(r)$
	1	$\Phi(r) \doteq 1620(1-3r)_+^5(15r+1) - 81(1-2r)_+^5(10r+1)$
		$+4(1-r)_+^5(5r+1)$
	2	$\Phi(r) \doteq 1620(1-3r)_+^7(144r^2+21r+1) - 81(1-2r)_+^7(64r^2+14r+1)$
		$+4(1-r)_+^7(16r^2+7r+1)$
	3	$\Phi(r) \doteq 1620(1-3r)_+^9(6237r^3+1431r^2+135r+5)$
		$-81(1-2r)_+^9(1848r^3+636r^2+90r+5)$
		$+4(1-r)_+^9(231r^3+159r^2+45r+5)$

Table 4.2: Extended original Wendland functions in $d = 3$ for $l = 1$ and various parameters of k. These functions satisfy the moment condition of order $m = 4$ or $m = 6$, respectively, and the approximation condition of any order $L \in \mathbb{N}$.

The original Wendland function in even space dimension

It this section, we want to show that the extended original Wendland functions satisfy condition (4.22), and hence the approximation condition, also in even space dimensions. Note that we now have $\lfloor d/2 \rfloor = d/2$. Before we start to derive a closed form representation for the Fourier transform in even space dimensions, we will need the following auxiliary results which can be found in [Rui96].

Lemma 4.50
Let $n \in \mathbb{N}$.

i) For all $x \in \mathbb{R}$ we have

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(x-j)^n = n!.$$

ii) For all $0 \le m < n$ we have

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^m = 0.$$

Proof. Let $f_n(x) := \sum_{j=0}^{n}(-1)^j\binom{n}{j}(x-j)^n$. We will proof the first equality by induction on n. For $n = 0$, we easily see that

$$f_0(x) = 1 = 0!.$$

Now let the assumption be true for $n \in \mathbb{N}$. Then, by deriving f_{n+1} and extracting the terms for $j = 0$ and $j = n+1$, we have

$$\begin{aligned} f'_{n+1}(x) &= (n+1)\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(x-j)^n \\ &= (n+1)\left(x^n + \sum_{j=1}^{n}(-1)^j\left(\binom{n}{j}+\binom{n}{j-1}\right)(x-j)^n + (-1)^{n+1}(x-n-1)^n\right). \end{aligned}$$

On the one hand, we have that

$$x^n + \sum_{j=1}^{n}(-1)^j\binom{n}{j}(x-j)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}(x-j)^n = f_n(x).$$

On the other hand, shifting the summation index, we have

$$\begin{aligned} &\sum_{j=1}^{n}(-1)^j\binom{n}{j-1}(x-j)^n + (-1)^{n+1}(x-n-1)^n \\ &\qquad = -\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}(x-j-1)^n + (-1)^{n+1}(x-n-1)^n \\ &\qquad = -\sum_{j=0}^{n}(-1)^j\binom{n}{j}(x-j-1)^n \\ &\qquad = -f_n(x-1). \end{aligned}$$

Hence, we see that f_{n+1} is constant since its derivative vanishes. Using that $\binom{n+1}{j}(n+1-j) = (n+1)\binom{n}{j}$ we conclude

$$\begin{aligned} f_{n+1}(x) = f_{n+1}(n+1) &= \sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(n+1-j)^{n+1} \\ &= (n+1)\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n+1-j)^n \\ &= (n+1)f_n(n+1) = (n+1)!, \end{aligned}$$

which finishes the first part of the proof. For the second part of this lemma, we note that

$$0 = f_n^{(m)}(x) = \frac{n!}{(n-m)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x-j)^{n-m}$$

for every $0 < m \leq n$. Inserting $x = 0$ finishes the proof. □

Next we derive an extension of Corollary 4.40 regarding the coefficients $a_{d,k,l,j}$ of the polynomial version of the original Wendland functions from Corollary 4.39.

Lemma 4.51
Let $k, l \in \mathbb{N}_0$ and $\mu = \lfloor d/2 \rfloor + k + 1$. Then

$$\sum_{j=0}^{\mu+2k+l} a_{d,k,l,j} j^n = 0.$$

for all $0 \leq n \leq \mu + k + l - 1$.

Proof. From Corollary 4.40 we know that

$$0 = \sum_{j=n}^{\mu+2k+l} a_{d,k,l,j} \frac{j!}{(j-n)!} = \sum_{j=n}^{\mu+2k+l} a_{d,k,l,j} \prod_{m=0}^{n-1} (j-m).$$

We note that the product is zero if $0 \leq j \leq n-1$. Hence, we can extend the sum over j to conclude

$$\sum_{j=n}^{\mu+2k+l} a_{d,k,l,j} \prod_{m=0}^{n-1} (j-m) = \sum_{j=0}^{\mu+2k+l} a_{d,k,l,j} \prod_{m=0}^{n-1} (j-m) = 0$$

for all $0 \leq n \leq \mu + k + l - 1$. Now, the proof can be finished by induction on n. In the case $n = 0$ there is nothing to show since the product degenerates to one. For $n - 1 \to n$, we can rewrite the product as an polynomial in j of degree $n + 1$ to see that

$$0 = \sum_{j=0}^{\mu+2k+l} a_{d,k,l,j} \prod_{m=0}^{n} (j-m) = \sum_{m=0}^{n} c_m \sum_{j=0}^{\mu+2k+l} a_{d,k,l,j} j^m$$

The incuction hypothesis yields that the sum is zero for all $0 \leq m \leq n-1$. Hence, only the sum for $m = n$ remains, which has to vanish. □

Now we want to calculate the Fourier transform of the original Wendland functions in even dimensions. Therefore, we need to define the following coefficients

$$d_{j,n} := \begin{cases} \prod_{m=0}^{n-1} (j-2m), & n \geq 1, \\ 1, & \text{else.} \end{cases} \tag{4.26}$$

Note that we can interpret $d_{j,n}$ as a polynomial in j of degree n with simple zeros at $0, 2, \ldots 2n-2$. A conclusion from this interpretation is the following auxiliary result.

Lemma 4.52
Let $d \in \mathbb{N}$ be even, $k, l \in \mathbb{N}_0$ and $\mu = d/2 + k + 1$. Then, we have

$$\sum_{j=2\mu-3}^{3\mu+l-4} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j} d_{j,\mu-1} j^n = 0$$

for all $0 \leq n \leq \mu + l - 2$

Proof. Let $k' = k + (d-2)/2$, $l' = l - (d-2)/2$ and $\mu' = d/2 + k' + 1$, so that $\mu = k' + 2$. Using that $d_{j,k'+1}$ is a polynomial in j of degree $k'+1$ and that $n + k' + 1 \leq \mu' + k' + l' - 1$ if $n \leq \mu + l - 2$, Lemma 4.51 states that the sum can be written as

$$\sum_{j=2\mu-3}^{3\mu+l-4} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j} d_{j,\mu-1} j^n = \sum_{j=2k'+1}^{2k'+\mu'+l'} a_{d,k',l',j} d_{j,k'+1} j^n = -\sum_{j=0}^{2k'} a_{d,k',l',j} d_{j,k'+1} j^n.$$

On the one hand we note that $d_{j,k'+1} = 0$ if $0 \leq j \leq 2k'$ is even. On the other hand, using the definition of $a_{d,k',l',j}$ from Corollary 4.39, we note that $a_{d,k',l',j} = 0$ if $1 \leq j \leq 2k' - 1$ is odd since the appearing binomial coefficient vanishes. Hence, the sum is zero, which completes the proof. □

Using the polynomial representation of $d_{j,n}$, we can conclude the following two identities from Lemma 4.52.

Corollary 4.53
Let $d \in \mathbb{N}$ be even, $k, l \in \mathbb{N}_0$ and $\mu = d/2 + k + 1$. Then, we have

$$\sum_{j=2\mu-3}^{3\mu+l-4} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j} d_{j,\mu-1}^2 j^n = 0$$

for all $0 \leq n \leq l - 1$ and

$$\sum_{j=2\mu-3}^{3\mu+l-4} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j} d_{j,\mu-1} d_{j,\mu-2} j^n = 0$$

for all $0 \leq n \leq l$.

We will need a special case for $l = 0$ where the first sum in Corollary 4.53 does not vanish.

Lemma 4.54
Let $d \in \mathbb{N}$ be even, $k \in \mathbb{N}_0$ and $\mu = d/2 + k + 1$. Then, we have

$$\sum_{j=2\mu-3}^{3\mu-4} a_{d,k+\frac{d-2}{2},-\frac{d-2}{2},j} d_{j,\mu-1}^2 = \frac{(3\mu-4)!}{2^{\mu-2}(\mu-2)!}.$$

Proof. Let $k' = k + (d-2)/2$, $l' = -(d-2)/2$ and $\mu' = d/2 + k' + 1$, so that $\mu = k' + 2$. First, we note that for $2k' + 1 \leq j \leq 2k' + \mu' + l'$ we have

$$\binom{\frac{j-1}{2}}{k'} = \frac{1}{2^{k'} k'!} \prod_{m=0}^{k'-1} (j - 1 - 2m) = \frac{d_{j-1,k'}}{2^{k'} k'!}.$$

Inserting this and the definition of $a_{d,k',l',j}$ in the sum we conclude that

$$\sum_{j=2\mu-3}^{3\mu-4} a_{d,k+\frac{d-2}{2},-\frac{d-2}{2},j} d_{j,\mu-1}^2 = \frac{(-1)^{k'}}{2^{k'} k'!} \sum_{j=2k'+1}^{2k'+\mu'+l'} (-1)^j \binom{2k'+\mu'+l'}{j} d_{j-1,k'} d_{j,k'+1}^2. \tag{4.27}$$

We note that for $0 \leq j \leq 2k'$ the coefficients $d_{j-1,k'} = 0$ if j odd and $d_{j,k'+1} = 0$ is j is even, so that we can start the summation at $j = 0$. Moreover we note that $d_{j-1,k'} d_{j,k'+1}^2$ is a polynomial in j of degree $3k' + 2$. Hence, we have the representation

$$d_{j-1,k'} d_{j,k'+1}^2 = \sum_{n=0}^{3k'+2} c_k j^k$$

with coefficients $c_k \in \mathbb{R}$, where $c_{3k'+2} = c_{3\mu-4} = 1$. Using this representation, we can write the remaining sum from (4.27) as

$$\begin{aligned}\sum_{j=0}^{2k'+\mu'+l'} (-1)^j \binom{2k'+\mu'+l'}{j} d_{j-1,k'} d_{j,k'+1}^2 &= \sum_{n=0}^{3k'+2} c_k \sum_{j=0}^{2k'+\mu'+l'} (-1)^j \binom{2k'+\mu'+l'}{j} j^n \\ &= \sum_{n=0}^{3\mu-4} c_k \sum_{j=0}^{3\mu-4} (-1)^j \binom{3\mu-4}{j} j^n,\end{aligned}$$

According to Lemma 4.50 the sum over j vanishes for all $0 \leq n < 3\mu - 4$, while for $n = 3\mu - 4$ the sum is equal to $(-1)^{3\mu-4}(3\mu-4)!$. Thus, we have

$$\sum_{j=0}^{2k'+\mu'+l'} (-1)^j \binom{2k'+\mu'+l'}{j} d_{j-1,k'} d_{j,k'+1}^2 = c_{3\mu-4}(-1)^{3\mu-4}(3\mu-4)!.$$

Inserting this result into the sum from the beginning completes the proof. □

Finally, to calculate the Fourier transform we will also need the following integrals.

Lemma 4.55

Let $j \in \mathbb{N}$. Then, the following equation holds

$$\begin{aligned}\int_0^1 s^{j+1} J_0(sr)ds =& J_0(r) \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor - 1} (-1)^n d_{j,n+1} d_{j,n} r^{-(2n+2)} \\ &+ J_1(r) \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^n d_{j,n}^2 r^{-(2n+1)} \\ &+ \left(J_0(r)H_1(r) - J_1(r)H_0(r)\right)(-1)^{\frac{j-1}{2}} d_{j,\lfloor \frac{j+2}{2} \rfloor}^2 \frac{\pi}{2} r^{-(j+1)},\end{aligned} \tag{4.28}$$

where J_k denotes the Bessel function of order k, see [GR00, 8.402], and H_k denotes the Struve function of order k, see [GR00, 8.550].

Proof. We will denote the integrals by

$$C_j(r) := \int_0^1 s^{j+1} J_0(sr)ds.$$

We will give the proof by induction, where we will first show that the formula above is correct for $j = 0$ and $j = 1$, and then do an induction step from $j - 2$ to j. For $j = 0$, we have from [GR00, 6.561.5]

$$C_0(r) = \frac{1}{r} J_1(r),$$

and for $j = 1$, partial integration, [GR00, 8.472.3] and [GR00, 6.561.1] yield

$$C_1(r) = \frac{1}{r} J_1(r) + \frac{1}{r^2} \frac{\pi}{2} \left(J_0(r)H_1(r) - J_1(r)H_0(r)\right),$$

which shows that formula (4.28) is correct for $j = 0$ and $j = 1$. Now suppose the formula is correct for $j - 2$ with $j \geq 2$. For C_j, substituting $t = rs$ and using partial integration, we derive

$$\begin{aligned}C_j(r) = r^{-(j+2)} \int_0^r J_0(t) t^{j+1} dt &= r^{-(j+2)} \left(J_1(t) t^{j+1} \Big|_{t=0}^{r} - j \int_0^r J_1(t) t^j dt \right) \\ &= \frac{1}{r} J_1(r) - \frac{j}{r^{j+2}} \int_0^r J_1(t) t^j dt,\end{aligned}$$

where we used that $\frac{d}{dt}t^\nu J_\nu(t) = t^\nu J_{\nu-1}(t)$. For the remaining integral, we use $J_{-1}(t) = -J_1(t)$ and partial integration again, such that we have

$$\int_0^r J_1(t)t^j dt = -\int_0^r J_{-1}(t)t^j dt = -J_0(r)r^j + j\int_0^r J_0(t)t^{j-1}dt.$$

Substituting $s = t/r$ back and yields

$$\begin{aligned}
C_j(r) =& \frac{1}{r}J_1(r) + \frac{j}{r^2}J_0(r) - \frac{j^2}{r^2}C_{j-2}(r)\\
=& J_0(r)\left(\frac{j}{r^2} - \frac{j^2}{r^2}\sum_{n=0}^{\lfloor\frac{j-2}{2}\rfloor-1}(-1)^n d_{j-2,n+1}d_{j-2,n}r^{-(2n+2)}\right)\\
&+ J_1(r)\left(\frac{1}{r} - \frac{j^2}{r^2}\sum_{n=0}^{\lfloor\frac{j-2}{2}\rfloor}(-1)^n d_{j-2,n}^2 r^{-(2n+1)}\right)\\
&- \frac{j^2}{r^2}\left(J_0(r)H_1(r) - J_1(r)H_0(r)\right)(-1)^{\frac{j-3}{2}}d^2_{j-2,\lfloor\frac{j}{2}\rfloor}\frac{\pi}{2}r^{-(j-1)},
\end{aligned}$$

where we inserted C_{j-2} in the last equality. Using that $jd_{j-2,m} = d_{j,m+1}$ for every $m \in \mathbb{N}$ and shifting the summation index in the two appearing sums from n to $n+1$ gives the required expression. □

With the integrals above, we are now able to calculate the Fourier transform.

Theorem 4.56
Let $d \in \mathbb{N}$ be even, $k, l \in \mathbb{N}_0$ and $\mu = d/2 + k + 1$. Then, the Fourier transform of the extended original Wendland functions is given by

$$\begin{aligned}
\mathcal{F}_d\psi_{d,k,l}(r) = C_{d,k+\frac{d-2}{2},l-\frac{d-2}{2}}\Bigg(& J_0(r)\sum_{n=0}^{\lfloor\frac{\mu+l}{2}\rfloor-1}c_{1,d,k,l,n}r^{-(2\mu+2n-2)}\\
&+ J_1(r)\sum_{n=0}^{\lfloor\frac{\mu+l}{2}\rfloor-1}c_{2,d,k,l,n}r^{-(2\mu+2n-1)}\\
&+ \left(J_0(r)H_1(r) - J_1(r)H_0(r)\right)\sum_{n=0}^{\lfloor\frac{\mu+l-1}{2}\rfloor}c_{3,d,k,l,n}r^{-(2\mu+2n-2)}\Bigg),
\end{aligned}$$

where the coefficients are given by

$$\begin{aligned}
c_{1,d,k,l,n} &= (-1)^{\mu+n}\sum_{j=2\mu+2n-2}^{3\mu+l-4}a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j}d_{j,\mu+n-1}d_{j,\mu+n-2},\\
c_{2,d,k,l,n} &= (-1)^{\mu+n-1}\sum_{j=2\mu+2n-2}^{3\mu+l-4}a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j}d^2_{j,\mu+n-1},\\
c_{3,d,k,l,n} &= (-1)^{\mu+n-2}\frac{\pi}{2}a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},2\mu+2n-3}d^2_{2\mu+2n-3,\mu+n-1}
\end{aligned}$$

and where $C_{d,k,l}$ and $a_{d,k,l,j}$ are from Corollary 4.39 while $d_{j,n}$ is defined in (4.26).

Proof. Recalling Corollary 4.39, we can write the extended Wendland functions as

$$\psi_{d,k,l}(r) = \phi_{\mu+l,k}(r) = C_{d,k,l}\sum_{j=0}^{2k+\mu+l}a_{d,k,l,j}r^j, \quad r \in [0,1]. \tag{4.29}$$

Calculating the Fourier transform, we can step through the dimensions via Theorem 4.33 and arrive at

$$\mathcal{F}_d\psi_{d,k,l}(r) = \mathcal{F}_d\phi_{\mu+l,k}(r) = \mathcal{F}_2\phi_{\mu+l,k+\frac{d-2}{2}}(r) = \mathcal{F}_2\psi_{d,k+\frac{d-2}{2},l-\frac{d-2}{2}}(r).$$

Now we denote $k' = k + \frac{d-2}{2}$, $l' = l - \frac{d-2}{2}$ and $\mu' = \frac{d}{2} + k' + 1$. Using the 2-dimensional radial Fourier transformation according to Theorem 4.10, we have

$$\mathcal{F}_d\psi_{d,k,l}(r) = C_{d,k',l'} \sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} \int_0^1 s^{j+1} J_0(rs) ds.$$

Note that k' and l' are integers since d is even, so that we are able to apply this formula. Hence, with Lemma 4.55 we see that the Fourier transform $\mathcal{F}_d\psi_{d,k,l}$ becomes, up to the $C_{d,k',l'}$ factor,

$$\sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} \int_0^1 s^{j+1} J_0(rs) ds = J_0(r)A(r) + J_1(r)B(r) + \left(J_0(r)H_1(r) - J_1(r)H_0(r)\right)C(r),$$

where the functions A, B and C are given by

$$A(r) = \sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor - 1} (-1)^n d_{j,n+1} d_{j,n} r^{-(2n+2)},$$

$$B(r) = \sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^n d_{j,n}^2 r^{-(2n+1)}$$

and

$$C(r) = \frac{\pi}{2} \sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} (-1)^{\frac{j-1}{2}} \frac{\pi}{2} d^2_{j,\lfloor \frac{j+2}{2} \rfloor} r^{-(j+1)}.$$

We will now rewrite the three functions beginning with A. We can start the summation over j in A at 2 since for $j = 0$ and $j = 1$ there are no terms in the inner sum. By changing the order of summation we obtain

$$\begin{aligned} A(r) &= \sum_{j=2}^{2k'+\mu'+l'} a_{d,k',l',j} \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor - 1} (-1)^n d_{j,n+1} d_{j,n} r^{-(2n+2)} \\ &= \sum_{n=0}^{k'+\lfloor \frac{\mu'+l'}{2} \rfloor - 1} r^{-(2n+2)} (-1)^n \sum_{j=2n+2}^{2k'+\mu'+l'} a_{d,k',l',j} d_{j,n+1} d_{j,n}. \end{aligned}$$

We now have to take a closer look at the sum over j. On the one hand, using the definition, we see that $d_{j,n+1} = 0$ for all $j \in \{0, 2, \ldots, 2n\}$. On the other hand, we have that $a_{d,k',l',j} = 0$ for all $j \in \{1, 3, \ldots 2k' - 1\}$. Thus, we know that $a_{d,k',l',j} = 0$ for all $j \in \{1, 3, \ldots 2n + 1\}$ if $n \leq k' - 1$. Hence,

$$\sum_{j=2n+2}^{2k'+\mu'+l'} a_{d,k',l',j} d_{j,n+1} d_{j,n} = \sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} d_{n+1,j} d_{n,j}$$

for all $n \leq k' - 1$. Now we interpret $d_{j,n+1} d_{j,n}$ as a polynomial in j of degree $2n + 1$, e.g. $d_{n+1,j} d_{n,j} = \sum_{m=0}^{2n+1} c_{n,m} j^m$ with some real-valued coefficients $c_{n,m}$. Inserting this polynomial expression, and by changing the order of summation, we have

$$\sum_{j=2n+2}^{2k'+\mu'+l'} a_{d,k',l',j} d_{n+1,j} d_{n,j} = \sum_{m=0}^{2n+1} c_{n,m} \sum_{j=0}^{2k'+\mu'+l'} a_{d,k',l',j} j^m.$$

Using Lemma 4.51, the sum over j vanishes for $m \in \{0, 1, \ldots, \mu'+k'+l'-1\}$. The summation index m goes up to $2n+1$, so the two sums vanish if $n \le (\mu'+k'+l'-1)/2$. Since $k'-1 \le (\mu'+k'+l'-1)/2$, this is particularly the case for $n \le k'-1$, so that we can start the sum over n in A at k', and have

$$\begin{aligned} A(r) &= \sum_{n=k'}^{k'+\lfloor\frac{\mu'+l'}{2}\rfloor-1} r^{-(2n+2)}(-1)^n \sum_{j=2n+2}^{2k'+\mu'+l'} a_{d,k',l',j} d_{j,n+1} d_{j,n} \\ &= \sum_{n=0}^{\lfloor\frac{\mu+l}{2}\rfloor-1} c_{1,d,k,l,n} r^{-(2\mu+2n-2)}, \end{aligned}$$

where we shifted the summation index n and used that $\mu' + l' = \mu + l$ and $k' = \mu - 2$. The coefficients are given by

$$c_{1,d,k,l,n} = (-1)^{\mu+n} \sum_{j=2\mu+2n-2}^{3\mu+l-4} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j} d_{j,\mu+n-1} d_{j,\mu+n-2}.$$

We will apply the same procedure to B. Again, by changing the order of summation, we have

$$B(r) = \sum_{n=0}^{k'+\lfloor\frac{\mu'+l'}{2}\rfloor} (-1)^n r^{-(2n+1)} \sum_{j=2n}^{\mu'+2k'+l'} a_{d,k',l',j} d_{j,n}^2.$$

With the same arguments as above, we can start the sum over j at 0 instead of $2n$ if $n \le k'$. Moreover, $d_{j,n}^2$ can be written as a polynomial in j of degree $2n$, i.e. $d_{j,n}^2 = \sum_{m=0}^{2n} c_{n,m} j^m$ with some real valued coefficients $c_{n,m}$. Hence, by changing the order of summation, we have

$$\sum_{j=2n}^{\mu'+2k'+l'} a_{d,k',l',j} d_{j,n}^2 = \sum_{m=0}^{2n} c_{n,m} \sum_{j=2n}^{\mu'+2k'+l'} a_{d,k',l',j} j^m.$$

Using Lemma 4.51, the sum over j vanishes for $m \in \{0, 1, \ldots \mu' + k' + l' - 1\}$, such that the whole expression is zero for $n \in \{0, 1, \ldots, \lfloor(\mu' + k' + l' - 1)/2\rfloor\}$ if $n \le k'$. Hence, we can write

$$\begin{aligned} B(r) &= \sum_{n=k'+1}^{k'+\lfloor\frac{\mu'+l'}{2}\rfloor} (-1)^n r^{-(2n+1)} \sum_{j=2n}^{\mu'+2k'+l'} a_{d,k',l',j} d_{j,n}^2 \\ &= \sum_{n=0}^{\lfloor\frac{\mu+l}{2}\rfloor-1} c_{2,d,k,l,n} r^{-(2\mu+2n-1)}, \end{aligned}$$

where we shifted the summation index n and used that $\mu' + l' = \mu + l$ and $k' = \mu - 2$. The coefficients are given by

$$c_{2,d,k,l,n} = (-1)^{\mu+n-1} \sum_{j=2\mu+2n-2}^{3\mu+l-4} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j} d_{j,\mu+n-1}^2.$$

Finally, we have to rewrite the function C. Since $d_{j,\lfloor\frac{j+2}{2}\rfloor}$ is 0 if j is even, we will transform the index by $j = 2n + 1$ and arrive at

$$C(r) = \frac{\pi}{2} \sum_{n=0}^{k'+\lfloor\frac{\mu'+l'-1}{2}\rfloor} a_{d,k',l',2n+1} (-1)^n d_{2n+1,n+1}^2 r^{-(2n+2)}.$$

Now we note that $a_{d,k',l',2n+1} = 0$ if $n < k'$ according to Corollary 4.39. Hence, by shifting the summation index n, we have

$$C(r) = \frac{\pi}{2} \sum_{n=0}^{\lfloor \frac{\mu'+l'-1}{2} \rfloor} a_{d,k',l',2k'+2n+1} (-1)^{k'+n} d^2_{2k'+2n+1,k'+n+1} r^{-(2k'+2n+2)}$$
$$= \sum_{n=0}^{\lfloor \frac{\mu+l-1}{2} \rfloor} c_{3,d,k,l,n} r^{-(2\mu+2n-2)}$$

where we also returned to the original parameters. Setting the coefficient

$$c_{3,d,k,l,n} = (-1)^{\mu+n-2} \frac{\pi}{2} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},2\mu+2n-3} d^2_{2\mu+2n-3,\mu+n-1}$$

finishes the proof. □

We derived a closed form representation of the Fourier transform of the extended original Wendland functions in even space dimensions. This representation can be summarized as follows.

Proposition 4.57
Let $d \in \mathbb{N}$ be even, $k, l \in \mathbb{N}_0$ and $\mu = d/2 + k + 1$. Then

$$\mathcal{F}_d \psi_{d,k,l}(r) = r^{-(3\mu+l-3)} \Big(p_{1,d,k,l}(r) J_0(r) + p_{2,d,k,l}(r) J_1(r) + q_{d,k,l}(r) \left(J_0(r) H_1(r) - J_1(r) H_0(r) \right) \Big), \quad r \geq 0,$$

with $p_{1,d,k,l} \in \Pi_{\mu+l-1}$, $p_{2,d,k,l} \in \Pi_{\mu+l-2}$ and $q_{d,k,l} \in \Pi_{\mu+l-1}$ given by

$$p_{1,d,k,l}(r) := C_{d,k+\frac{d-2}{2},l-\frac{d-2}{2}} \sum_{n=0}^{\lfloor \frac{\mu+l}{2} \rfloor - 1} c_{1,d,k,l,n} r^{\mu+l-1-2n},$$
$$p_{2,d,k,l}(r) := C_{d,k+\frac{d-2}{2},l-\frac{d-2}{2}} \sum_{n=0}^{\lfloor \frac{\mu+l}{2} \rfloor - 1} c_{2,d,k,l,n} r^{\mu+l-2-2n}$$

and

$$q_{d,k,l}(r) := C_{d,k+\frac{d-2}{2},l-\frac{d-2}{2}} \sum_{n=0}^{\lfloor \frac{\mu+l-1}{2} \rfloor} c_{3,d,k,l,n} r^{\mu+l-1-2n},$$

where $C_{d,k+\frac{d-2}{2},l-\frac{d-2}{2}}$ is given in Corollary 4.39 and $c_{1,d,k,l,n}$, $c_{2,d,k,l,n}$ and $c_{3,d,k,l,n}$ are given in Theorem 4.56.

We are interested in the asymptotic behavior of $\mathcal{F}_d \psi_{d,k,l}$. We will just write p_1, p_2 and q instead of $p_{1,d,k,l}$, $p_{2,d,k,l}$ and $q_{d,k,l}$, respectively, for the sake of readability. From [GR00, 8.554], we have that the asymptotic representation of the Struve functions is given by

$$H_0(r) = Y_0(r) + \frac{2}{\pi} r^{-1} + h_0(r), \quad H_1(r) = Y_1(r) + \frac{2}{\pi} + h_1(r), \tag{4.30}$$

where Y_n is the Bessel function of the second kind of order $n \in \mathbb{Z}$, see [GR00, 8.403], and where $h_0 = \Theta(r^{-3})$ and $h_1 = \Theta(r^{-2})$. Hence, an asymptotic representation of $\mathcal{F}_d \psi_{d,k,l}$ is given by

$$\mathcal{F}_d \psi_{d,k,l} = r^{-(3\mu+l-3)} \left(J_0(r) \left(p_1(r) + \frac{2}{\pi} q(r) + q(r) h_1(r) \right) + J_1(r) \left(p_2(r) - \frac{2}{\pi} \frac{q(r)}{r} - q(r) h_0(r) \right) + q(r) \left(J_0(r) Y_1(r) - J_1(r) Y_0(r) \right) \right).$$

According to [GR00, 8.477.1], the sum of the Bessel functions yields $J_0(r)Y_1(r) - J_1(r)Y_0(r) = -2(r\pi)^{-1}$, so that we have

$$J_0(r)H_1(r) - J_1(r)H_0(r) = -\frac{2}{\pi}r^{-1} + J_0(r)\left(\frac{2}{\pi} + h_1(r)\right) - J_1(r)\left(\frac{2}{\pi}r^{-1} + h_0(r)\right), \qquad (4.31)$$

and hence

$$\begin{aligned}\mathcal{F}_d\psi_{d,k,l} = r^{-(3\mu+l-3)}\Bigg(&J_0(r)\left(p_1(r) + \frac{2}{\pi}q(r) + q(r)h_1(r)\right)\\ &+ J_1(r)\left(p_2(r) - \frac{2}{\pi}\frac{q(r)}{r} - q(r)h_0(r)\right) - \frac{2}{\pi}\frac{q(r)}{r}\Bigg).\end{aligned}$$

The term $p_2(r) - 2(\pi r)^{-1}q(r)$ is a polynomial of degree $\mu + l - 2$ or less. The coefficient of the highest degree of p_1 is given by $c_{1,d,k,l,0}$, the coefficient of $2\pi^{-1}q(r)$ is given by $2\pi^{-1}c_{3,d,k,l,0}$. Using that $d_{2\mu-3,\mu-1} = d_{2\mu-3,\mu-2}$, the coefficient of the highest degree of $p_1 + 2\pi^{-1}q$ is given by

$$-c_{1,d,k,l,0} - 2\pi^{-1}c_{3,d,k,l,0} = (-1)^{\mu+1}\sum_{j=2\mu-3}^{3\mu+l-4} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j}\,d_{j,\mu-1}d_{j,\mu-2} = 0, \qquad (4.32)$$

according to Corollary 4.53. Hence, $p_1(r) + 2\pi^{-1}q(r)$ is also a polynomial of degree $\mu + l - 2$ or less. Using that $J_0, J_1 = \Theta(r^{-1/2})$ according to [GR00, 8.451.1], we can conclude that

$$\begin{aligned} r &\mapsto J_0(r)\left(p_1(r) + \frac{2}{\pi}q(r) + q(r)h_1(r)\right) \in \Theta(r^{\mu+l-5/2}),\\ r &\mapsto J_1(r)\left(p_2(r) - \frac{2}{\pi}\frac{q(r)}{r} - q(r)h_0(r)\right) \in \Theta(r^{\mu+l-5/2}).\end{aligned}$$

Since $2(\pi r)^{-1}q(r)$ is a polynomial of degree $\mu + l - 2$, we have that

$$r \mapsto 2(\pi r)^{-1}q(r) \in \Theta(r^{\mu+l-2})$$

and we can conclude that $\mathcal{F}_d\psi_{d,k,l} = \Theta(r^{-2\mu+1})$. In particular, this means that there exists a constant $C > 0$ such that

$$\mathcal{F}_d\psi_{d,k,l}(r) \geq Cr^{-2\mu+1} \qquad (4.33)$$

for $r \geq 1$. Using this asymptotic behaviour, we can conclude two different properties of the extended original Wendland functions. The first one is about the regularity.

Lemma 4.58
Let $d \in \mathbb{N}$ be even, $k, l \in \mathbb{N}_0$ and $\mu = d/2 + k + 1$. Then, $\psi_{d,k,l}(\|\cdot\|_2)$ belongs to $W^{\sigma,2}(\mathbb{R}^d)$ for all $\sigma < 2\mu - d/2 - 1$.

We will omit the proof since it is literally the same as the proof of Lemma 4.44. The second property, which follows from inequality (4.33), is that the extended Wendland functions satisfy the approximation condition in even space dimensions of any order. The key point for this result is that the highest order monomial which occurs in the derivative of the $\mathcal{F}_d\psi_{d,k,l}$ vanishes.

Theorem 4.59
Let $d \in \mathbb{N}$ be even, $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$. Then, for every $L \in \mathbb{N}$ there exists a constant $C_L > 0$ such that

$$\left|\frac{d^n}{dr^n}r\mathcal{F}_d\psi_{d,k,l}(r)\right| \leq C_L\mathcal{F}_d\psi_{d,k,l}(r)$$

for all $1 \leq n \leq L$ and $r \geq 0$.

Proof. We set $a = 3\mu + l - 3$ with $\mu = d/2 + k + 1$. We will just write q instead of $q_{d,k,l}$ for the sake of readability, such that we have

$$\mathcal{F}_d \psi_{d,k,l}(r) = r^{-a}\Big(p_1(r)J_0(r) + p_2(r)J_1(r) \\ + q(r)\left(J_0(r)H_1(r) - J_1(r)H_0(r)\right)\Big), \quad r \geq 0.$$

Since $\mathcal{F}_d \psi_{d,k,l} \in C^\infty([0,\infty[)$ is positive on $[0,\infty[$, such a constant C_L can easily be found for $r \leq 1$ and it suffices to show that the inequality holds for all $r > 1$.
Referring to [GR00] equations 8.472.1, 8.473.4, 8.553.2 and 8.553.3, we note that J_0, J_1, H_0 and H_1 yield the derivatives

$$\frac{d}{dr}J_0(r) = -J_1(r), \qquad \frac{d}{dr}J_1(r) = J_0(r) - \frac{1}{r}J_1(r),$$
$$\frac{d}{dr}H_0(r) = \frac{2}{\pi} - H_1(r), \quad \frac{d}{dr}H_1(r) = H_0(r) - \frac{1}{r}H_1(r),$$

so that we conclude that

$$\frac{d}{dr}(J_0(r)H_1(r) - J_1(r)H_0(r)) = -\frac{1}{r}\left(J_0(r)H_1(r) - J_1(r)H_0(r)\right) - \frac{2}{\pi}J_1(r).$$

Hence, the first derivative yields

$$\frac{d}{dr}\left(r\mathcal{F}_d\psi_{d,k,l}(r)\right) = r^{-a}\Big(\widetilde{p}_{1,1}(r)J_0(r) + \widetilde{p}_{2,1}(r)J_1(r) + \widetilde{q}_1(r)\left(J_0(r)H_1(r) - J_1(r)H_0(r)\right)\Big),$$

where the new polynomials are given by

$$\widetilde{p}_{1,1}(r) = rp'_1(r) + rp_2(r) - (a-1)p_1(r)$$
$$\widetilde{p}_{2,1}(r) = rp'_2(r) - rp_1(r) - \frac{2r}{\pi}q(r) - ap_2(r)$$

and

$$\widetilde{q}_1(r) = rq'(r) - aq(r).$$

Using (4.31), we can conclude that the derivative has the asymptotic representation

$$\frac{d}{dr}\left(r\mathcal{F}_d\psi_{d,k,l}(r)\right) = r^{-a}\Bigg(J_0(r)\left(\widetilde{p}_{1,1}(r) + \frac{2}{\pi}\widetilde{q}_1(r) + \widetilde{q}_1(r)h_1(r)\right) \\ + J_1(r)\left(\widetilde{p}_{2,1}(r) - \frac{2}{\pi}\frac{\widetilde{q}_1(r)}{r} - \widetilde{q}_1(r)h_0(r)\right) - \frac{2}{\pi}\frac{\widetilde{q}_1(r)}{r}\Bigg). \tag{4.34}$$

We have to show that the single polynomial terms in the brackets are at most of degree $\mu + l - 2$. First, $\widetilde{p}_{1,1}$ and $\widetilde{q}_1$ are polynomials of order at most $\mu + l - 1$. The coefficient of the highest degree of $\widetilde{p}_{1,1}$ is the sum of the coefficients of the highest degrees of the single polynomials, which is given by

$$(\mu + l - 1)c_{1,d,k,l,0} + c_{2,d,k,l,0} - (a-1)c_{1,d,k,l,0} = (2 - 2\mu)c_{1,d,k,l,0} + (c_{1,d,k,l,0} + c_{2,d,k,l,0}).$$

Analogously, the coefficient of the highest degree of $2\pi^{-1}\widetilde{q}_1$ is given by $2\pi^{-1}(2-2\mu)c_{3,d,k,l,0}$. We firstly note that if $l \geq 1$, Corollary 4.53 states that

$$c_{2,d,k,l,0} = (-1)^{\mu-1}\sum_{j=2\mu-2}^{3\mu+l-4} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j}\, d^2_{j,\mu-1} \\ = (-1)^{\mu} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},2\mu-3}\, d^2_{2\mu-3,\mu-1}. \tag{4.35}$$

Noting that $d_{2\mu-3,\mu-1} = d_{2\mu-3,\mu-2}$, Corollary 4.53 yields

$$c_{1,d,k,l,0} + c_{2,d,k,l,0} = (-1)^{\mu} \sum_{j=2\mu-3}^{3\mu+l-4} a_{d,k+\frac{d-2}{2},l-\frac{d-2}{2},j} d_{j,\mu-1} d_{j,\mu-2} = 0. \tag{4.36}$$

For the remaining parts of the coefficient of the highest degree of $\widetilde{p}_{1,1}(r) + 2\pi^{-1}\widetilde{q}_1(r)$ we have

$$(2-2\mu)c_{1,d,k,l,0} + 2\pi^{-1}(2-2\mu)c_{3,d,k,l,0} = (2-2\mu)\left(c_{1,d,k,l,0} + 2\pi^{-1}c_{3,d,k,l,0}\right) = 0$$

according to (4.32). Hence, $\widetilde{p}_{1,1} + 2\pi^{-1}\widetilde{q}_1$ is a polynomial of degree $\mu+l-2$ or less. As $h_1 \in \Theta(r^{-2})$ and hence $\widetilde{q}_1 h_1 = \Theta(r^{\mu+l-3})$, the whole coefficient function $\widetilde{p}_{1,1} + 2\pi^{-1}\widetilde{q}_1 + \widetilde{q}_1 h_1$ in front of J_0 grows at most like $r^{\mu+l-2}$.
Next, we look at the coefficient function in front of J_1. For $\widetilde{p}_{2,1}$, we note that the terms $rp'_2(r)$ and $ap_2(r)$ are polynomials of degree $\mu+l-2$, while $rp_1(r)$ and $2\pi^{-1}rq(r)$ are both polynomials of degree $\mu+l$. The coefficient of the highest degree of $rp_1(r)$ is given by $c_{1,d,k,l,0}$, the coefficient of $2\pi^{-1}rq(r)$ is given by $2\pi^{-1}c_{3,d,k,l,0}$. Hence, the coefficient of the highest degree of $\widetilde{p}_{2,1}$ is again given by (4.32), and hence vanishes. Since the polynomials $rp_1(r)$ and $2\pi^{-1}rq(r)$ do not have a monomial part of degree $\mu+l-1$ as we can see in Proposition 4.57, $\widetilde{p}_{2,1}$ is a polynomial of order $\mu+l-2$. Finally, $\widetilde{q}_1$ is a polynomial of degree $\mu+l-1$, so that $\widetilde{q}_1(r)/r$ behaves like $r^{\mu+l-2}$. As $\widetilde{q}_1 h_0 \in \Theta(r^{\mu+l-4})$ since $h_0 \in \Theta(r^{-2})$, the whole coefficient function $\widetilde{p}_{2,1}(r) - 2\pi^{-1}\widetilde{q}_1(r)/r - \widetilde{q}_1(r)h_0(r)$ in front of J_1 grows also at most like $r^{\mu+l-2}$.
Altogether, using that $J_0, J_1 \in \Theta(r^{-1/2})$, we have

$$r \mapsto \frac{d}{dr}\left(r\mathcal{F}_d\psi_{d,k,l}(r)\right) \in \Theta(r^{-2\mu+1})$$

so that there exist constants $c > 0$ and $C > 0$ such that

$$\left|\frac{d}{dr}(r\mathcal{F}_d\psi_{d,k,l}(r))\right| \le cr^{-2\mu+1} \le C\mathcal{F}_d\psi_{d,k,l}(r)$$

for all $r > 1$. For higher derivatives of order $n \ge 2$ we have

$$\frac{d^n}{dr^n}\left(r\mathcal{F}_d\psi_{d,k,l}(r)\right) = r^{-a}\left(\widetilde{p}_{1,n}(r)J_0(r) + \widetilde{p}_{2,n}(r)J_1(r) + \widetilde{q}_n(r)\left(J_0(r)H_1(r) - J_1(r)H_0(r)\right)\right)$$

where the new functions are given by

$$\widetilde{p}_{1,n}(r) = \widetilde{p}'_{1,n-1}(r) + \widetilde{p}_{2,n-1}(r) - a\frac{\widetilde{p}_{1,n-1}(r)}{r}$$
$$\widetilde{p}_{2,n}(r) = \widetilde{p}'_{2,n-1}(r) - \widetilde{p}_{1,n-1}(r) - \frac{2}{\pi}\widetilde{q}_{n-1}(r) - (a+1)\frac{\widetilde{p}_{2,n-1}(r)}{r}$$

and

$$\widetilde{q}_n(r) = \widetilde{q}'_{n-1}(r) - (a+1)\frac{\widetilde{q}_{n-1}(r)}{r}.$$

We first note that since $\widetilde{q}_1$ is a polynomial of degree $\mu+l-1$, there exists a constant $c_{q,n}$ with $|\widetilde{q}_n(r)| \le c_{q,n}r^{\mu+l-n}$ for all $n \in \mathbb{N}$ and $r \ge 1$. For $\widetilde{p}_{1,n}$ and $\widetilde{p}_{2,n}$, we first note that the highest degree occurring in $\widetilde{p}_{1,2}$ is $\mu+l-2$ since $\widetilde{p}_{1,1}$ is a polynomial of degree $\mu+l-1$ and $\widetilde{p}_{2,1}$ of degree $\mu+l-2$. For $\widetilde{p}_{2,2}$ we note that, using (4.35) and (4.36) as above, $\widetilde{p}_{1,1}(r)+2\pi^{-1}\widetilde{q}_{n-1}$ is a polynomial of degree $\mu+l-2$, so that the highest degree occurring in $\widetilde{p}_{2,2}$ is $\mu+l-2$.
Since the highest degree occurring in $\widetilde{p}_{1,2}$ and $\widetilde{p}_{2,2}$ is $\mu+l-2$, the highest degree that can occur in $\widetilde{p}_{1,n}$,$\widetilde{p}_{2,n}$ for $n \ge 3$ is also $\mu+l-2$. Hence, for all $n \ge 2$ we find a constant $c_{p,n} > 0$ with

$$|\widetilde{p}_{1,n}(r)| + |\widetilde{p}_{2,n}(r)| \le c_{p,n}r^{\mu+l-2}$$

for $r \geq 1$.
According to [GR00, 8.451.1],[GR00, 8.451.2] and (4.31), there exists a constant $c > 0$, so that

$$|J_0(r)| + |J_1(r)| + |J_0(r)H_1(r) - H_1(r)J_0(r)| \leq cr^{-1/2}.$$

Hence, we can conclude that for every $n \geq 2$ there exists a constant $C > 0$ such that

$$\begin{aligned}\left|\frac{d^n}{dr^n}(r\mathcal{F}_d\psi_{d,k,l}(r))\right| &\leq cr^{-a-1/2}\big(|\widetilde{p}_{1,n}(r)| + |\widetilde{p}_{2,n}(r)| + |\widetilde{q}_n(r)|\big)\\ &\leq cr^{-3\mu-l+3-1/2+\mu+l-2} = cr^{-2\mu+1/2}\\ &\leq cr^{-2\mu+1} \leq C\mathcal{F}_d\psi_{d,k,l}(r),\end{aligned}$$

according to inequality (4.33), which completes the proof. □

Hence, we can conclude that the original Wendland functions satisfy an approximation condition of any order in even space dimension if $l \geq 1$. Again, the proof does not hold if $l = 0$. Recalling the asymptotic representation of $\frac{d}{dr}(r\mathcal{F}_d\psi_{d,k,l}(r))$ from (4.34), we note that for $\widetilde{p}_{1,1} + 2\pi^{-1}\widetilde{q}_1$ the coefficient for the monomial of degree $\mu + l - 1$ vanishes if and only if $l \geq 1$, see Corollary 4.53 and Lemma 4.54. Hence, if $l = 0$, $\widetilde{p}_{1,1} + 2\pi^{-1}\widetilde{q}_1$ is a polynomial of degree $\mu + l - 1$ and $\frac{d}{dr}(r\mathcal{F}_d\psi_{d,k,l}(r))$ decays like $r^{-2\mu+3/2}$. However $\psi_{d,k,l}$ itself decays like $r^{-2\mu+1}$. Hence, a constant C_L as given in Theorem 4.59 cannot be found for any $L \geq 1$.

Theorem 4.60
Let $d \in \mathbb{N}$ be even and $k \in \mathbb{N}_0$. The original Wendland functions $\psi_{d,k,0}$ do not satisfy condition (4.22) *of any order.*

Moreover, we can conclude that the root kernel of $\psi_{d,k,l}$ is in $W^{k,1}(\mathbb{R}^d)$ according to Theorem 4.17.

Theorem 4.61
Let $d \in \mathbb{N}$ be even, $k, l \in \mathbb{N}$. Then, the root kernel of the normalized generalized Wendland function $\psi_{d,k,l}(\|\cdot\|_2)$ exists, belongs to $W^{k,1}(\mathbb{R}^d) \cap W^{k,2}(\mathbb{R}^d)$ and satisfies the moment condition of order 2 *and the approximation condition of any order $L \in \mathbb{N}$.*

The proof is exactly the proof of Theorem 4.47. The next result takes advantage of the construction from equation (4.17) to satisfy the moment condition of an arbitrary order. In this case, the condition $l \geq 1$ also ensures that the Fourier transform of the original Wendland functions is monotonically decreasing, see Theorem 4.35.

Theorem 4.62
Let $d \in \mathbb{N}$ be even and $k, l \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $\Phi = \phi(\|\cdot\|_2)$ with ϕ be defined as in (4.17) *with $\psi = \psi_{d,k,l}$, $a_j = bj$ for a fixed $b > 0$, $1 \leq j \leq m$ and λ_j from Proposition 4.24.*
Then, the convolution root Φ^r from Lemma 4.14 is in $W^{k,1}(\mathbb{R}^d) \cap W^{k,2}(\mathbb{R}^d)$, satisfies the moment condition of order $2m$ and the approximation condition of any order $L \in \mathbb{N}$.

Finally, we can derive a convergence result for the SPH method in even space dimension. As in the case of odd space dimension, we can eliminate the dependency on the parameter L in the estimate.

Corollary 4.63
Let $d \in \mathbb{N}$ be even and let $l, k \in \mathbb{N}$ with $k > d/2$. Let $m \in \mathbb{N}$ and $\Phi = \phi(\|\cdot\|_2)$ with ϕ be defined as in (4.17) *with $\psi = \psi_{d,k,l}$, $a_j = bj$ for a fixed $b > 0$, $1 \leq j \leq m$ and λ_j from Proposition 4.24.*
Assume initial data $\mathbf{u}_0 \in W^{\sigma,2}(\mathbb{R}^d)^d$ and finite discrete mass with density ρ_0 satisfying $\rho_0^{1/2} \in W^{\max\{\sigma,k\},2}(\mathbb{R}^d)$ for a $\sigma > m + 1 + d/2$ and that the solution $(\mathbf{u}, \rho)$ of the Euler equations (3.1) - (3.3) *exists up to a time $T > 0$. Let $(\mathbf{x}_\mathbf{k}^{\varepsilon,h}, \mathbf{u}_\mathbf{k}^{\varepsilon,h})_{\mathbf{k}\in\mathbb{Z}^d}$ be the solution of the weakly compressible SPH equations* (3.4) - (3.6)*.*
Then, there exists a constant $C > 0$ such that the energy can be bounded by

$$Q(t) \leq C\left(\frac{h^{2k}}{\varepsilon^{2k}} + \varepsilon^{2m}\right).$$

for all $t \in [0, T]$.

A few examples of the kernels in even space dimension, including the most useful ones, can be found in Table 4.3 and Table 4.4.

k	l	Extended original Wendland function $\psi_{2,k,l}$
1	1	$\psi_{2,1,1}(r) \doteq (1-r)_+^5(5r+1)$
	2	$\psi_{2,1,2}(r) \doteq (1-r)_+^6(6r+1)$
	3	$\psi_{2,1,3}(r) \doteq (1-r)_+^7(7r+1)$
2	1	$\psi_{2,2,1}(r) \doteq (1-r)_+^7(16r^2+7r+1)$
	2	$\psi_{2,2,2}(r) \doteq (1-r)_+^8(21r^2+8r+1)$
	3	$\psi_{2,2,3}(r) \doteq (1-r)_+^9(80r^2+27r+3)$
3	1	$\psi_{2,3,1}(r) \doteq (1-r)_+^9(231r^3+159r^2+45r+5)$
	2	$\psi_{2,3,2}(r) \doteq (1-r)_+^{10}(320r^3+197r^2+50r+5)$
	3	$\psi_{2,3,3}(r) \doteq (1-r)_+^{11}(429r^3+239r^2+55r+5)$

Table 4.3: Extended original Wendland functions in $d=2$ for various parameters of k and l. These functions satisfy the moment condition of order $m=2$ and the approximation condition of any order $L \in \mathbb{N}$. These functions can be used for the construction method from (4.17).

$m=4$	k	Kernel Construction $\Phi(r) \doteq 16\psi_{2,k,1}(2r) - \psi_{2,k,1}(r)$
	1	$\Phi(r) \doteq 16(1-2r)_+^5(10r+1) - (1-r)_+^5(5r+1)$
	2	$\Phi(r) \doteq 16(1-2r)_+^7(64r^2+14r+1)$
	3	$\Phi(r) \doteq 16(1-2r)_+^9(1848r^3+636r^2+90r+5)$ $-(1-r)_+^9(231r^3+159r^2+45r+5)$
$m=6$	k	Kernel Construction $\Phi(r) \doteq 270\psi_{2,k,1}(3r) - 27\psi_{2,k,1}(2r) + 2\psi_{2,k,1}(r)$
	1	$\Phi(r) \doteq 270(1-3r)_+^5(15r+1) - 27(1-2r)_+^5(10r+1)$ $+2(1-r)_+^5(5r+1)$
	2	$\Phi(r) \doteq 270(1-3r)_+^7(144r^2+21r+1) - 27(1-2r)_+^7(64r^2+14r+1)$ $+2(1-r)_+^7(16r^2+7r+1)$
	3	$\Phi(r) \doteq 270(1-3r)_+^9(6237r^3+1431r^2+135r+5)$ $-27(1-2r)_+^9(1848r^3+636r^2+90r+5)$ $+2(1-r)_+^9(231r^3+159r^2+45r+5)$

Table 4.4: Extended original Wendland functions in $d=2$ for $l=1$ and various parameters of k. These functions satisfy the moment condition of order $m=4$ or $m=6$, respectively, and the approximation condition of any order $L \in \mathbb{N}$.

4.3.3 The missing Wendland functions

In odd space dimensions, the original Wendland functions are reproducing kernels in Hilbert spaces which are norm–equivalent to the Sobolev spaces $H^\sigma(\mathbb{R}^d)$ of integer order σ. The missing Wendland functions were proposed in 2011 by Schaback in [Sch11] to cover also Sobolev spaces of integer order in even dimensions. Although the extended original Wendland functions already provide convergence for the SPH method, we want to consider the missing Wendland functions for theoretical purposes.
The missing Wendland functions are given by equation (4.23) with $\alpha = k + 1/2$ for $k \in \mathbb{N}$ and $\mu = \lfloor d/2 + \alpha \rfloor + 1 = \lfloor (d+1)/2 \rfloor + k + 1$, i.e.

$$\psi_{d,k+\frac{1}{2}} := \phi_{\lfloor \frac{d+1}{2} \rfloor + k + 1, k + \frac{1}{2}} = I^{k+\frac{1}{2}} f_{\lfloor \frac{d+1}{2} \rfloor + k + 1}.$$

For our purpose, we again need an extended version of the missing Wendland functions. Let $l \geq 0$. We define the extended missing Wendland functions by

$$\psi_{d,k+\frac{1}{2},l} := \phi_{\lfloor \frac{d+1}{2} \rfloor + k + l + 1, k + \frac{1}{2}} = I^{k+\frac{1}{2}} f_{\lfloor \frac{d+1}{2} \rfloor + k + l + 1}.$$

The most considerable difference to the original Wendland functions is that the missing Wendland functions are not purely polynomial but have a logarithmic and a square root part. This, however, make them less interesting for applications, since they are less efficient to calculate.
A closed form representation was given by Chernih in [Che13]. We will generalize this representation to our extended kernels. For the use of simplicity, we will use

$$S(r) := \sqrt{1-r^2}, \quad L(r) := \ln\left(\frac{1+\sqrt{1-r^2}}{r}\right)$$

for all $r \in]0,1]$ throughout this section. Before we begin, we will derive some integral equations we will need in what follows.

Lemma 4.64
Let $j,k \in \mathbb{N}$ and let $g_{j,k}(r) = \int_r^1 s^{j+1}\left(s^2-r^2\right)^{k-\frac{1}{2}}\,ds$. Then,

$$g_{j,k}(r) = S(r)\left(q_{1,j,k}(r) + q_{2,j,k}(r) + q_{3,j,k}(r)\right)$$

if j is even, and

$$g_{j,k}(r) = S(r)\left(q_{1,j,k}(r) + q_{2,j,k}(r)\right) + \frac{1}{2} q_{3,j,k}(r)\left(S(r) + r^2 L(r)\right)$$

if j is odd, with

$$q_{1,j,k}(r) := \sum_{n=0}^{k-1}\left(\sum_{i=0}^{k-1-n} (-1)^{i+n}\binom{k-1-i}{n}\frac{d_{2k-1,i}}{d_{j+2i+2,i+1}}\right) r^{2n}$$

$$q_{2,j,k}(r) := (-1)^k \frac{d_{2k-1,k}}{d_{2k+j,k}}\left(\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor + k - 1}\frac{d_{2k+j,i}}{d_{2k+j+1,i+1}} r^{2i}\right)$$

$$q_{3,j,k}(r) := (-1)^k \frac{d_{2k-1,k}}{d_{2k+j,k}}\frac{d_{2k+j,\lfloor \frac{j}{2}\rfloor + k}}{d_{2k+j+1,\lfloor \frac{j}{2}\rfloor + k}} r^{2\lfloor \frac{j}{2}\rfloor + 2k}$$

for all $0 < r < 1$, and where $d_{j,n}$ is given by (4.26).

Proof. We will first eliminate the second parameter k of $g_{j,k}$, before we will derive a recursive formula with respect to the parameter j. Partial integration yields

$$\begin{aligned} g_{j,k}(r) &= \int_r^1 s^{j+1}\left(s^2-r^2\right)^{k-\frac{1}{2}}\,ds \\ &= \frac{s^{j+2}}{j+2}\left(s^2-r^2\right)^{k-\frac{1}{2}}\Big|_r^1 - \frac{2k-1}{j+2}\int_r^1 s^{j+3}\left(s^2-r^2\right)^{k-\frac{3}{2}}\,ds \\ &= \frac{(1-r^2)^{k-\frac{1}{2}}}{j+2} - \frac{2k-1}{j+2} g_{j+2,k-1}(r). \end{aligned}$$

By induction, we can show that applying partial integration k times leads to

$$
\begin{aligned}
g_{j,k}(r) =& \sqrt{1-r^2}\sum_{i=0}^{k-1}(-1)^i \frac{\prod_{m=0}^{i-1}(2k-1-2m)}{\prod_{m=0}^{i}(j+2+2m)}(1-r^2)^{k-1-i} \\
&+ (-1)^k \prod_{m=0}^{k-1}\left(\frac{2k-1-2m}{j+2+2m}\right) g_{j+2k,0}(r) \\
=& \sqrt{1-r^2}\sum_{i=0}^{k-1}(-1)^i \frac{d_{2k-1,i}}{d_{j+2i+2,i+1}} \sum_{n=0}^{k-1-i}\binom{k-1-i}{n}(-1)^n r^{2n} \\
&+ (-1)^k \frac{d_{2k-1.k}}{d_{2k+j,k}} g_{j+2k,0}(r),
\end{aligned}
$$

where we also used the binomial formula. We have to change the order of summation in the first term in order to obtain a closed polynomial form. Note that we have to adjust the indices of the summation. Hence, we arrive at the representation

$$
\begin{aligned}
g_{j,k}(r) =& \sqrt{1-r^2}\sum_{n=0}^{k-1}\left(\sum_{i=0}^{k-1-n}(-1)^{i+n}\binom{k-1-i}{n}\frac{d_{2k-1,i}}{d_{j+2i+2,i+1}}\right) r^{2n} \\
&+ (-1)^k \frac{d_{2k-1,k}}{d_{2k+j,k}} g_{j+2k,0}(r).
\end{aligned}
$$

So far, we have eliminated the second parameter. To derive a recursive formula for the first parameter, applying partial integration once again yields

$$
\begin{aligned}
g_{j,0}(r) &= \int_r^1 s^j s\left(s^2-r^2\right)^{-\frac{1}{2}} ds \\
&= s^j\left(s^2-r^2\right)^{\frac{1}{2}}\Big|_r^1 - j\int_r^1 s^{j-1}\left(s^2-r^2\right)^{\frac{1}{2}} ds \\
&= \sqrt{1-r^2} - j\int_r^1 s^{j-1}\left(s^2-r^2\right)\left(s^2-r^2\right)^{-\frac{1}{2}} ds \\
&= \sqrt{1-r^2} - j g_{j,0}(r) + j r^2 g_{j-2,0}(r).
\end{aligned}
$$

Solving this equation for $g_{j,0}(r)$, we reach at the recursive formula

$$
g_{j,0}(r) = \frac{\sqrt{1-r^2}}{j+1} + \frac{j}{j+1} r^2 g_{j-2,0}(r).
$$

Applying this recursive formula $\lfloor \frac{j}{2} \rfloor$-times leads to

$$
g_{j,0}(r) = \sqrt{1-r^2}\left(\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor -1}\frac{d_{j,i}}{d_{j+1,i+1}} r^{2i}\right) + \frac{d_{j,\lfloor \frac{j}{2}\rfloor}}{d_{j+1,\lfloor \frac{j}{2}\rfloor}} r^{2\lfloor \frac{j}{2}\rfloor} g_{j-2\lfloor \frac{j}{2}\rfloor,0}(r).
$$

Finally, simple calculations show that for the starting point at $j=0$ we have

$$
g_{0,0}(r) = \sqrt{1-r^2}
$$

and for $j = 1$

$$g_{1,0}(r) = \frac{1}{2}\left(\sqrt{1-r^2} + r^2 \ln\left(\frac{1+\sqrt{1-r^2}}{r}\right)\right),$$

which completes the proof. □

Using this integral, we can now provide a closed form representation for the extended missing Wendland functions.

Theorem 4.65
Let $k, l \in \mathbb{N}_0$ and $\mu = \lfloor \frac{d+1}{2} \rfloor + k + 1$. Then,

$$\psi_{d,k+\frac{1}{2},l}(r) = S(r)P_{d,l,k}(r) + L(r)Q_{d,l,k}(r),$$

for all $r \in]0,1]$, where the functions $P_{d,l,k}$ and $Q_{d,l,k}$ are given by

$$P_{d,l,k}(r) = \frac{1}{2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2})}\left(\sum_{j=0}^{\mu+l}(-1)^j \binom{\mu+l}{j}\left(q_{1,j,k}(r) + q_{2,j,k}(r) + \frac{1}{2}q_{3,j,k}(r)\right) + \frac{1}{2}\sum_{j=0}^{\lfloor\frac{\mu+l}{2}\rfloor}\binom{\mu+l}{2j} q_{3,2j,k}(r)\right)$$

and

$$Q_{d,l,k}(r) = -\frac{r^2}{2^{k+\frac{1}{2}}\Gamma(k+\frac{1}{2})}\sum_{j=0}^{\lfloor\frac{\mu+l-1}{2}\rfloor}\binom{\mu+l}{2j+1} q_{3,2j+1,k}(r).$$

Proof. With Lemma 4.64 we have

$$\begin{aligned}
\psi_{d,k+\frac{1}{2},l}(r) =& \frac{1}{2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2})}\int_r^1 s(1-s)^{\mu+l}(s^2-r^2)^{k-\frac{1}{2}}ds \\
=& \frac{1}{2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2})}\sum_{j=0}^{\mu+l}(-1)^j\binom{\mu+l}{j}\int_r^1 s^{j+1}(s^2-r^2)^{k-\frac{1}{2}}ds \\
=& \frac{S(r)}{2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2})}\left(\sum_{j=0}^{\mu+l}(-1)^j\binom{\mu+l}{j}\left(q_{1,j,k}(r) + q_{2,j,k}(r) + \frac{1}{2}q_{3,j,k}(r)\right) + \frac{1}{2}\sum_{j=0}^{\lfloor\frac{\mu+l}{2}\rfloor}\binom{\mu+l}{2j}q_{3,2j,k}(r)\right) \\
& - \frac{r^2L(r)}{2^{k+\frac{1}{2}}\Gamma(k+\frac{1}{2})}\sum_{j=0}^{\lfloor\frac{\mu+l-1}{2}\rfloor}\binom{\mu+l}{2j+1}q_{3,2j+1,k}(r),
\end{aligned}$$

where we splitted up a sum in its even and its odd part. □

The missing Wendland function in even space dimension

Now we consider the missing Wendland function in even space dimension. Besides the closed form representation, we additionally require a formula for the Fourier transform of the missing Wendland functions. Fortunately, we can achieve the formula for the Fourier transform easily by stepping through the dimensions and use the Fourier transformation formula of the original Wendland functions.

Theorem 4.66
Let $d \in \mathbb{N}$ be even, $k, l \in \mathbb{N}_0$ and $\mu = d/2 + k + 1$. Then,

$$\mathcal{F}_d \psi_{d,k+\frac{1}{2},l}(r) = \sqrt{\frac{2}{\pi}} C_{d+1,k+\frac{d}{2},l-\frac{d}{2}} \left(\sin(r) \sum_{n=0}^{\lfloor \frac{\mu+l}{2} \rfloor - \lfloor \frac{l}{2} \rfloor - 1} b_{1,d+1,k,l,n} r^{-(2\lfloor \frac{l}{2} \rfloor + 2\mu + 2n + 1)} + \cos(r) \sum_{n=0}^{\lfloor \frac{\mu+l-1}{2} \rfloor - \lfloor \frac{l-1}{2} \rfloor - 1} b_{2,d+1,k,l,n} r^{-(2\lfloor \frac{l-1}{2} \rfloor + 2\mu + 2n + 2)} + \sum_{n=0}^{\lfloor \frac{\mu+l-1}{2} \rfloor} b_{3,d+1,k,l,n} r^{-(2n+2\mu)} \right),$$

where the coefficients $b_{1,d,k,l,n}$, $b_{2,d,k,l,n}$ and $b_{3,d,k,l,n}$ are defined in Theorem 4.42.

***Proof*.** With Theorem 4.33 we have

$$\mathcal{F}_d \psi_{d,k+\frac{1}{2},l}(r) = \mathcal{F}_d \phi_{\mu+l,k+\frac{1}{2}}(r) = \mathcal{F}_{d+1} \phi_{\mu+l,k}(r).$$

Setting $d' := d + 1$, and $\mu' := (d' - 1)/2 + k + 1 = \mu$ we conclude

$$\mathcal{F}_d \psi_{d,k+\frac{1}{2},l}(r) = \mathcal{F}_{d+1} \phi_{\mu+l,k}(r) = \mathcal{F}_{d'} \phi_{\mu'+l,k}(r) = \mathcal{F}_{d'} \psi_{d',k,l}(r).$$

Applying Theorem 4.42 leads us to our representation. □

With this reference to the Fourier transform of the original Wendland functions, we can easily adopt the rest of the results. Firstly, we will shorten the expression from Theorem 4.66.

Corollary 4.67
Let $d \in \mathbb{N}$ be even, $k, l \in \mathbb{N}_0$ and $\mu = d/2 + k + 1$. Then,

$$\mathcal{F}_d \psi_{d,k+\frac{1}{2},l}(r) = r^{-(3\mu+l-1)} \left(p_{1,d,k,l}(r) \sin(r) + p_{2,d,k,l}(r) \cos(r) + q_{d,k,l}(r) \right), \quad r \geq 0,$$

where $q_{d,k,l}$ is a polynomial of degree $\mu + l - 1$ and $p_{1,d,k,l}, p_{2,d,k,l}$ are polynomials of degree at most $\mu - 1$.
To be more precise, either $p_{1,d,k,l} \in \Pi_{\mu-2}$ and $p_{2,d,k,l} \in \Pi_{\mu-1}$ if l is even or $p_{1,d,k,l} \in \Pi_{\mu-1}$ and $p_{2,d,k,l} \in \Pi_{\mu-2}$ if l is odd.

The next lemma will give us the regularity off the extended missing Wendland functions in even dimensions.

Lemma 4.68
Let $d \in \mathbb{N}$ be even, $k, l \in \mathbb{N}_0$ and $\mu = d/2 + k + 1$. Then, $\psi_{d,k+\frac{1}{2},l}(\|\cdot\|_2)$ belongs to $W^{\sigma,2}(\mathbb{R}^d)$ for all $\sigma < 2\mu - d/2$.

Using Corollary 4.67, the proof is literary the same as the one of Lemma 4.44, which is why we omit it here. With the given connection to the Fourier transform of the original Wendland functions it is easy to verify that the missing Wendland functions also satisfy the approximation condition.

Theorem 4.69
Let $d \in \mathbb{N}$ be even, $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$. Then, for every $L \in \mathbb{N}$, there exists a $C_L > 0$ such that

$$\left| \frac{d^n}{dr^n} r \mathcal{F}_d \psi_{d,k+\frac{1}{2},l}(r) \right| \leq C_L \mathcal{F}_d \psi_{d,k+\frac{1}{2},l}(r)$$

for all $1 \leq n \leq L$ and all $r \geq 0$.

Proof. Since $\mathcal{F}_d\psi_{d,k+\frac{1}{2},l} = \mathcal{F}_{d+1}\psi_{d+1,k,l}$, Theorem 4.45 yields

$$\left|\frac{d^n}{dr^n} r\mathcal{F}_d\psi_{d,k+\frac{1}{2},l}(r)\right| = \left|\frac{d^n}{dr^n} r\mathcal{F}_{d+1}\psi_{d+1,k,l}(r)\right| \leq C_L\mathcal{F}_{d+1}\psi_{d+1,k,l}(r) = C_L\mathcal{F}_d\psi_{d,k+\frac{1}{2},l}(r)$$

for all $1 \leq n \leq L$ and all $r \geq 0$. □

Finally, we can prove our main result for the extended missing Wendland functions. Its proof is literally the same as the proof of Theorem 4.47 and will hence be omitted

Theorem 4.70
Let $d \in \mathbb{N}$ be even and $k, l \in \mathbb{N}$. Then, the root kernel of the extended missing Wendland function $\psi_{d,k+\frac{1}{2},l}(\|\cdot\|_2)$ exists, belongs to $W^{k,1}(\mathbb{R}^d) \cap W^{k,2}(\mathbb{R}^d)$ and satisfies the moment condition of order 2 and the approximation condition of any order $L \in \mathbb{N}$.

The next result takes advantage of the construction from equation (4.17) to satisfy the moment condition of arbitrary order. Again, the proof for the missing Wendland functions is the same as the proof of Theorem 4.48 which contains the results for the original Wendland functions.

Theorem 4.71
Let $d \in \mathbb{N}$ be even and $k, l \in \mathbb{N}$. Let $m \in \mathbb{N}$ and Φ be defined like in (4.17) with $\psi = \psi_{d,k+\frac{1}{2},l}$, $a_j > bj$ for a $b > 0$, $1 \leq j \leq m$ and λ_j from Proposition 4.24.
Then, the convolution root Φ^r from Lemma 4.14 is in $W^{k,1}(\mathbb{R}^d) \cap W^{k,2}(\mathbb{R}^d)$, satisfies an moment condition of order $2m$ a approximation property of any order $L \in \mathbb{N}$.

Finally, we can state the convergence result of the SPH method for the extended missing Wendland functions in even space dimension.

Corollary 4.72
Let $d \in \mathbb{N}$ be even and $k, l \in \mathbb{N}$ with $k > d/2$. Let $m \in \mathbb{N}$ and $\Phi = \phi(\|\cdot\|_2)$ with ϕ be defined as in (4.17) with $\psi = \psi_{d,k+\frac{1}{2},l}$, $a_j = bj$ for a fixed $b > 0$, $1 \leq j \leq m$ and λ_j from Proposition 4.24.
Assume initial data $\mathbf{u}_0 \in W^{\sigma,2}(\mathbb{R}^d)^d$ and finite discrete mass with density ρ_0 satisfying $\rho_0^{1/2} \in W^{\max\{\sigma,k\},2}(\mathbb{R}^d)$ for a $\sigma > m + 1 + d/2$ and that the solution $(\mathbf{u}, \rho)$ of the Euler equations (3.1) - (3.3) exists up to a time $T > 0$. Let $(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}, \mathbf{u}_{\mathbf{k}}^{\varepsilon,h})_{\mathbf{k}\in\mathbb{Z}^d}$ be the solution of the weakly compressible SPH equations (3.4) - (3.6).
Then, there exists a constant $C > 0$ such that the energy can be bounded by

$$Q(t) \leq C\left(\frac{h^{2k}}{\varepsilon^{2k}} + \varepsilon^{2m}\right).$$

for all $t \in [0, T]$.

A few examples of the kernel functions which fits in Corollary 4.71, including the most useful ones, can be found in Table 4.5.

The missing Wendland function in odd space dimension

Now we finally consider the missing Wendland function in odd space dimension. As in the case of even spatial dimensions, we can achieve the formula for the Fourier transform easily by stepping through the dimensions and use the Fourier transformation formula of the original Wendland functions.

k	l	Extended missing Wendland function $\psi_{2,k,l}(r)$
1	1	$(105r^6 + 210r^4)L(r)$
		$-(32r^6 + 247r^4 + 40r^2 - 4)S(r)$
	2	$(105r^8 + 1680r^6 + 1680r^4)L(r)$
		$-(919r^6 + 2346r^4 + 216r^2 - 16)S(r)$
2	1	$-(945r^{10} + 18900r^8 + 25200r^6)L(r)$
		$+(9295r^8 + 31670r^6 + 4704r^4 - 6888r^2 + 64)S(r)$
	2	$-(10395r^{10} + 69300r^8 + 55440r^6)L(r)$
		$+(2048r^{10} + 46949r^8 + 79418r^6 + 7504r^4 - 848r^2 + 64)S(r)$

Table 4.5: Extended missing Wendland functions in $d = 2$ for various parameters of k and l. These functions satisfy a moment condition of order $m = 2$ and an approximation condition of any order $L \in \mathbb{N}$. These functions can be used for the construction method from (4.17).

Theorem 4.73
Let $d \in \mathbb{N}$ be odd, $k, l \in \mathbb{N}_0$ and $\mu = (d+1)/2 + k + 1$. Then, the Fourier transform of the extended missing Wendland functions is given by

$$\begin{aligned}\mathcal{F}_d\psi_{d,k+\frac{1}{2},l}(r) = C_{d+1,k+\frac{d-1}{2},l-\frac{d-1}{2}} \Bigg(& J_0(r) \sum_{n=0}^{\lfloor \frac{\mu+l}{2} \rfloor -1} c_{1,d+1,k,l,n} r^{-(2\mu+2n-2)} \\ & + J_1(r) \sum_{n=0}^{\lfloor \frac{\mu+l}{2} \rfloor -1} c_{2,d+1,k,l,n} r^{-(2\mu+2n-1)} \\ & + (J_0(r)H_1(r) - J_1(r)H_0(r)) \sum_{n=0}^{\lfloor \frac{\mu+l-1}{2} \rfloor} c_{3,d+1,k,l,n} r^{-(2\mu+2n-2)} \Bigg),\end{aligned}$$

where the coefficients $c_{1,d,k,l,n}$, $c_{2,d,k,l,n}$ and $c_{3,d,k,l,n}$ are defined in Theorem 4.56.

The proof is exactly like the proof of Theorem 4.66 and will be omitted here. With this reference to the Fourier transform of the original Wendland functions, we can summarize the expression to the following form.

Proposition 4.74
Let $d \in \mathbb{N}$ be odd, $k, l \in \mathbb{N}_0$ and $\mu = (d+1)/2 + k + 1$. Then

$$\begin{aligned}\mathcal{F}_d\psi_{d,k+\frac{1}{2},l}(r) = r^{-(3\mu+l-3)} \Big(& p_{1,d,k,l}(r) J_0(r) + p_{2,d,k,l}(r) J_r(r) \\ & + q_{d,k,l}(r) \left(J_0(r)H_1(r) - J_1(r)H_0(r)\right) \Big), \quad r \geq 0,\end{aligned}$$

with $q_{d,k,l} \in \Pi_{\mu+l-1}$, $p_{1,d,k,l} \in \Pi_{\mu+l-1}$ and $p_{2,d,k,l} \in \Pi_{\mu+l-2}$.

Hence, we can conclude the regularity of the missing Wendland functions from the previous result.

Lemma 4.75
Let $d \in \mathbb{N}$ be odd, $l, k \in \mathbb{N}_0$ and $\mu = (d+1)/2 + k + 1$. Then, $\psi_{d,k+\frac{1}{2},l}(\| \cdot \|_2)$ belongs to $W^{\sigma,2}(\mathbb{R}^d)$ for all $\sigma < 2\mu - d/2 - 1$.

Once more, the proof can be taken from Lemma 4.44. By stepping through the dimension, we can easily show that the missing Wendland function satisfy the approximation condition of arbitrary order in odd space dimension. Therefore we need the following result.

k	l	Extended missing Wendland function $\psi_{3,k,l}(r)$
1	1	$(105r^8 + 1680r^6 + 1680r^4)L(r)$
		$-(919r^6 + 2346r^4 + 216r^2 - 16)S(r)$
	2	$(4725r^8 + 25200r^6 + 15120r^4)L(r)$
		$-(1024r^8 + 18827r^6 + 23874r^4 + 1400r^2 - 80)S(r)$
	3	$(315r^{10} + 9450r^8 + 25200r^6 + 10080r^4)L(r)$
		$-(3781r^8 + 23108r^6 + 17484r^4 + 704r^2 - 32)S(r)$
2	1	$-(10395r^{10} + 69300r^8 + 55440r^6)L(r)$
		$+(2048r^{10} + 46949r^8 + 79418r^6 + 7504r^4 - 848r^2 + 64)S(r)$
	2	$-3465(r^{12} + 36r^{10} + 120r^8 + 64r^6)L(r)$
		$+(45687r^{10} + 348086r^8 + 351160r^6 + 22752r^4 - 2048r^2 + 128)S(r)$
	3	$-45045(r^{12} + 84r^{10} + 168r^8 + 64r^6)L(r)$
		$+(49152r^{12} + 1945677r^{10} + 7392890r^8 + 4946104r^6 + 231840r^4 - 17024r^2 + 896)S(r)$

Table 4.6: Extended missing Wendland functions in $d = 3$ for various parameters of k and l. These functions satisfy a moment condition of order $m = 2$ and an approximation condition of any order $L \in \mathbb{N}$. These functions can be used for the construction method from (4.17).

Theorem 4.76
Let $d \in \mathbb{N}$ be odd, $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$. Then, for every $L \in \mathbb{N}$ there exists a constant $C_L > 0$ such that

$$\left| \frac{d^n}{dr^n} r\mathcal{F}_d\psi_{d,k,l}(r) \right| \leq C_L \mathcal{F}_d\psi_{d,k,l}(r)$$

for all $1 \leq n \leq L$ and $r \geq 0$.

Again, the proof is nearly the same as the one of Theorem 4.69 by using the result of Theorem 4.59. Using the last results, we have the following result for the root kernel of the extended missing Wendland function.

Theorem 4.77
Let $d \in \mathbb{N}$ be odd, $k, l \in \mathbb{N}$. Then, the root kernel of the normalized missing Wendland function $\psi_{d,k+\frac{1}{2},l}(\|\cdot\|_2)$ exists, belongs to $W^{k,1}(\mathbb{R}^d) \cap W^{k,2}(\mathbb{R}^d)$ and satisfies the moment condition of order 2 and the approximation condition of any order $L \in \mathbb{N}$.

The proof is exactly the proof of Theorem 4.47. The next result takes advantage of the construction from equation (4.17) to satisfy the moment condition of an arbitrary order.

Theorem 4.78
Let $d \in \mathbb{N}$ be odd and let $k, l \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $\Phi = \phi(\|\cdot\|_2)$ with ϕ be defined as in (4.17) with $\psi = \psi_{d,k+\frac{1}{2},l}$, $a_j = bj$ for a fixed $b > 0$, $1 \leq j \leq m$ and λ_j from Proposition 4.24.
Then, the convolution root Φ^r from Lemma 4.14 is in $W^{k,1}(\mathbb{R}^d) \cap W^{k,2}(\mathbb{R}^d)$, satisfies the moment condition of order $2m$ and the approximation condition of any order $L \in \mathbb{N}$.

Finally, we can derive a convergence result for the SPH method in even space dimension for the extended missing Wendland functions. As in the case of odd space dimension, we can eliminate the dependency on the parameter L in the error estimate.

Corollary 4.79
Let $d \in \mathbb{N}$ be odd and let $k, l \in \mathbb{N}$ with $k > d/2$. Let $m \in \mathbb{N}$ and $\Phi = \phi(\|\cdot\|_2)$ with ϕ be defined as in (4.17) with $\psi = \psi_{d,k+\frac{1}{2},l}$, $a_j = bj$ for a fixed $b > 0$, $1 \leq j \leq m$ and λ_j from Proposition 4.24.
Assume initial data $\mathbf{u}_0 \in W^{\sigma,2}(\mathbb{R}^d)^d$ and finite discrete mass with density ρ_0 satisfying $\rho_0^{1/2} \in W^{\max\{\sigma,k\},2}(\mathbb{R}^d)$ for a $\sigma > m + 1 + d/2$ and that the solution $(\mathbf{u}, \rho)$ of the Euler equations (3.1)

- (3.3) *exists up to a time* $T > 0$. *Let* $(\mathbf{x}_{\mathbf{k}}^{\varepsilon,h}, \mathbf{u}_{\mathbf{k}}^{\varepsilon,h})_{\mathbf{k}\in\mathbb{Z}^d}$ *be the solution of the weakly compressible SPH equations* (3.4) - (3.6).
Then, there exists a constant $C > 0$ *such that the energy can be bounded by*

$$Q(t) \leq C\left(\frac{h^{2k}}{\varepsilon^{2k}} + \varepsilon^{2m}\right).$$

for all $t \in [0, T]$.

A few examples of the kernel functions which fits in Corollary 4.79, including the most useful ones, can be found in Table 4.6.

Chapter 5

Time Discretization

In Chapter 3, we discretized the Euler equations in space by the SPH approximation and derived a convergence result for this approximation. However, in applications, we will also need the Euler equations to be discretized in time. Hence, the goal of this chapter is to give a convergence result for a fully discretized SPH system for the Euler equations.
For the sake of simplicity, we will concentrate on a simple explicit and implicit Euler time discretization to demonstrate the possibility of the convergence of a fully discretized SPH system. Unfortunately, the right-hand side of the SPH approximated Euler equations (3.5) satisfy only a Lipschitz condition depending on ε^{-1}, such that the classical results for numerical methods for ordinary differential equations do not lead to a convergence result for the fully discretized system.

5.1 Auxiliary Tools

We will start this chapter by giving some auxiliary results, which we require for the subsequent analysis. In the first lemma, we will derive an explicit and an implicit discretized version of Gronwall's inequality.

Lemma 5.1
Let $N \in \mathbb{N}$, $\tau = 1/N$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda > 0$. Let $(a_n)_{n=0}^N \subset \mathbb{R}$ be a sequence of positive numbers.

i) If the sequence $(a_n)_{n=0}^N$ satisfies

$$a_{n+1} \leq a_n (1 + \lambda\tau) + \mu\tau, \qquad 0 \leq n \leq N-1,$$

then, the following inequality holds

$$a_n \leq (a_0 + \mu) e^{\lambda}, \qquad 0 \leq n \leq N.$$

ii) If $\lambda\tau < 1$ and the sequence $(a_n)_{n=0}^N$ satisfies

$$a_{n+1} \leq a_n + \lambda\tau a_{n+1} + \mu\tau, \qquad 0 \leq n \leq N-1,$$

then, the following inequality holds

$$a_n \leq (a_0 + \mu) e^{\lambda/(1-\lambda\tau)}, \qquad 0 \leq n \leq N.$$

Proof. We start with the first inequality. Per induction, one can show that

$$a_n \leq a_0 (1 + \lambda\tau)^n + \mu\tau \sum_{k=0}^{n-1} (1 + \lambda\tau)^k \tag{5.1}$$

for all $0 \leq n \leq N$. Since $\lambda > 0$, $(1+\lambda/k)^k$ is monotonically increasing in k and tends to $\exp(\lambda)$ if k tends to infinity. Using that $N\tau = 1$, we have that

$$(1+\lambda\tau)^k \leq \left(1+\frac{\lambda\tau N}{k}\right)^k \leq e^{\lambda}$$

for all k with $1 \leq k \leq N$. Inserting this inequality in (5.1) yields

$$a_n \leq a_0 e^{\lambda} + \mu\tau n e^{\lambda} \leq (a_0+\mu)e^{\lambda}, \quad 1 \leq n \leq N,$$

which finishes the proof for the first inequality. The second inequality can be solved for a_{n+1}, which gives

$$a_{n+1} \leq a_n (1-\lambda\tau)^{-1} + \mu\tau (1-\lambda\tau)^{-1}$$

for all $0 \leq n \leq N$. Again, per induction, one can show that

$$a_n \leq a_0 (1-\lambda\tau)^{-n} + \mu\tau \sum_{k=1}^{n} (1-\lambda\tau)^{-k}.$$

For the first term we have

$$(1-\lambda\tau)^{-k} = \left(1+\frac{\lambda\tau}{1-\lambda\tau}\right)^k \leq \left(1+\frac{\lambda\tau}{1-\lambda\tau}\frac{N}{k}\right)^k \leq e^{\lambda/(1-\lambda\tau)}$$

for all k with $1 \leq k \leq N$, which, inserted in the last inequality, yields the required bounding. □

In the upcoming analysis we will also need the following version of the Gagliardo–Nirenberg interpolation inequality for the derivatives of a $W^{k,2}(\mathbb{R}^d)$ function.

Lemma 5.2
Let $f \in W^{k,2}(\mathbb{R}^d)$ and $\tau > 0$. Then

$$\|\partial_j f\|^2_{L^2(\mathbb{R}^d)} \leq \frac{l-1}{l\tau}\|f\|^2_{L^2(\mathbb{R}^d)} + \frac{\tau^{l-1}}{l}\|\partial_j^l f\|^2_{L^2(\mathbb{R}^d)}$$

for all $1 \leq j \leq d$ and $1 \leq l \leq k$.

Proof. Let $1 \leq j \leq d$. Using partial integration and the inequality of Cauchy Schwarz, we first note that

$$\begin{aligned}\|\partial_j^n f\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \partial_j^n f(\mathbf{x})\partial_j^n f(\mathbf{x})d\mathbf{x} = -\int_{\mathbb{R}^d} \partial_j^{n-1} f(\mathbf{x})\partial_j^{n+1} f(\mathbf{x})d\mathbf{x} \\ &\leq \|\partial_j^{n-1} f\|_{L^2(\mathbb{R}^d)}\|\partial_j^{n+1} f\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{2\tau}\|\partial_j^{n-1} f\|^2_{L^2(\mathbb{R}^d)} + \frac{\tau}{2}\|\partial_j^{n+1} f\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

for all $1 \leq n \leq k-1$, where we used Young's inequality for products in the last line. We will now set $a_n = \|\partial_j^n f\|^2_{L^2(\mathbb{R}^d)}$ to simplify the notation, so that the inequality above becomes

$$a_n \leq \frac{1}{2\tau}a_{n-1} + \frac{\tau}{2}a_{n+1}. \tag{5.2}$$

Now we will prove that

$$a_{l-1} \leq \frac{1}{l\tau^{l-1}}a_0 + \frac{(l-1)\tau}{l}a_l. \tag{5.3}$$

for all $1 \leq l \leq k$ per induction. The case $l = 1$ is obvious. For $l-1 \to l$, using inequality (5.2) yields

$$a_l \leq \frac{1}{2\tau}a_{l-1} + \frac{\tau}{2}a_{l+1} \leq \frac{1}{2l\tau^l}a_0 + \frac{l-1}{2l}a_l + \frac{\tau}{2}a_{l+1}.$$

Solving this inequality for a_l and multiplying with $2l/(l+1)$ leads to (5.3). Now we are able to prove the stated lemma using induction. We have to show that

$$a_1 \leq \frac{l-1}{l\tau} a_0 + \frac{\tau^{l-1}}{l} a_l.$$

The case $l = 1$ is obvious again. For $l-1 \to l$, using inequality (5.3) yields for $l \geq 2$

$$\begin{aligned} a_1 &\leq \frac{l-2}{(l-1)\tau} a_0 + \frac{\tau^{l-2}}{l-1} a_{l-1} \leq \frac{l-2}{(l-1)\tau} a_0 + \frac{1}{l(l-1)\tau} a_0 + \frac{\tau^{l-1}}{l} a_l \\ &= \frac{l-1}{l\tau} a_0 + \frac{\tau^{l-1}}{l} a_l, \end{aligned}$$

which finishes the proof. □

5.2 Explicit Discretization in Time

In the following, we will assume that the analytical solution of the Euler equations will hold up to the time $T > 0$. The maximum number of time steps will be denoted by $N \in \mathbb{N}$, hence the size of a time step is given by $\tau = T/N$. The position of the particle $\mathbf{j} \in \mathbb{Z}^d$ at the time step n is given by $\mathbf{x}_\mathbf{j}^{\varepsilon,h,n} \in \mathbb{R}^d$, and its associated velocity by $\mathbf{u}_\mathbf{j}^{\varepsilon,h,n} \in \mathbb{R}^d$. Due to readability, we will neglect the dependence on ε and h and write $\mathbf{x}_\mathbf{j}^n$ for $\mathbf{x}_\mathbf{j}^{\varepsilon,h,n}$ and $\mathbf{u}_\mathbf{j}^n$ for $\mathbf{u}_\mathbf{j}^{\varepsilon,h,n}$, even if $\mathbf{x}_\mathbf{j}^n$ and $\mathbf{u}_\mathbf{j}^n$ still depend on ε and h.
Using an explicit Euler time step algorithm, the fully discretized SPH system for the Euler equations (3.1) - (3.3) is given by

$$\mathbf{x}_\mathbf{j}^{n+1} = \mathbf{x}_\mathbf{j}^n + \tau \mathbf{u}_\mathbf{j}^n, \tag{5.4}$$

$$\mathbf{u}_\mathbf{j}^{n+1} = \mathbf{u}_\mathbf{j}^n - \tau h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \nabla\Phi_\varepsilon(\mathbf{x}_\mathbf{j}^n - \mathbf{x}_\mathbf{k}^n), \tag{5.5}$$

for every $\mathbf{j} \in \mathbb{Z}^d$ and $n \leq N-1$ and

$$\mathbf{x}_\mathbf{j}^0 = h\mathbf{j}, \quad \mathbf{u}_\mathbf{j}^0 = \mathbf{u}_0(h\mathbf{j}) \tag{5.6}$$

for every $\mathbf{j} \in \mathbb{Z}^d$. The approximated density is given by

$$\rho^{\varepsilon,h,n}(\mathbf{x}) := h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \Phi_\varepsilon(\mathbf{x} - \mathbf{x}_\mathbf{k}^n).$$

As in the case of the semi-discretized system in Chapter 3, we will need an error term to measure the error between the solution of the Euler equations (3.1) - (3.3) and the solution of the fully discretized SPH system (5.4) - (5.6). As mentioned before, the classical convergence theory for one step methods does not hold since the Lipschitz constant of the right-hand side of the SPH equations behaves proportionally to ε^{-d-2}, which would lead to an estimate of the form $\sim \exp\left(c\,\varepsilon^{-2-d}\right)$. Hence, this would not lead to convergence for $\varepsilon \to 0$.
For this reason, we will again investigate the energy error term from Definition 3.7 in a time discretized version, which is given by

$$Q^n := h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2^2 + \int_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x}, t_n)\right)^2 d\mathbf{x},$$

where $t_n = n\tau$ and

$$\rho^{\varepsilon,h,r,n}(\mathbf{x}) := h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_\mathbf{k}^n), \qquad \mathbf{x} \in \mathbb{R}^d.$$

We will now give a bound on Q^n by a discrete Gronwall argument and by using the result from Theorem 3.9. As in the semi-discretized case, the error bound will depend on the initial error Q^0, the energy error in the zeroth time step. Note that Q^0 is equal to $Q(0)$, the energy error at the time $t = 0$ for the semi-discretized system. Therefore, Q^0 can be bounded exactly as in Theorem 3.8.

Theorem 5.3
Let $s \geq 2$ and $\Phi^r \in W^{s,2}(\mathbb{R}^d)$ be an even root kernel, which satisfies the moment condition of order $m \geq 1$ and the approximation condition of order $L > d/2$. Let $\Phi = \Phi^r * \Phi^r$ be the corresponding convolution kernel. Assume finite discrete mass and that the solution $(\mathbf{u}, \rho)$ of the Euler equations (3.1) - (3.3) *satisfies*

$$u_j \in L^\infty(0,T; W^{\eta,2}(\mathbb{R}^d)), \qquad 1 \leq j \leq d,$$
$$\rho \in L^\infty(0,T; L^1(\mathbb{R}^d)) \cap L^\infty(0,T; W^{\sigma,2}(\mathbb{R}^d)),$$

for some time $T > 0$ with $\eta > \max\{L, m\} + \frac{d}{2} + 1$ and $\sigma > m + \frac{d}{2} + 1$ with $\sigma \geq s$. Let $\tau > 0$ with $\tau \leq \varepsilon^2$ and let $(\mathbf{x}_\mathbf{k}^n, \mathbf{u}_\mathbf{k}^n)_{\mathbf{k}\in\mathbb{Z}^d}$, $1 \leq n \leq N$, be a solution of the corresponding, fully discretized SPH equations (5.4) - (5.6).
Then, there exist two constants $C_1, C_2 > 0$ such that the energy can be bounded by

$$Q^N \leq C_1 \left(Q^0 + \varepsilon^{\min\{m,2L-d\}} + \frac{\tau^2}{\varepsilon^{d+4}} \right) \exp\left(C_2 \frac{\tau}{\varepsilon^{d+2}} \right).$$

Proof. We will prove this theorem with a discrete Gronwall argument. Hence, we have to bound the energy Q^{n+1} by the energy of the previous timestep Q^n. We will begin by rewriting the kinetic and the potential energy part before we will give bounds for each occurring term.
We start with the kinetic energy term. Since $\mathbf{u}$ is continuously differentiable in both arguments, using (5.4) and the mean value theorem yield

$$\begin{aligned}\mathbf{u}(\mathbf{x}_\mathbf{j}^{n+1}, t_{n+1}) =& \mathbf{u}(\mathbf{x}_\mathbf{j}^n + \tau \mathbf{u}_\mathbf{j}^n, n\tau + \tau) \\ =& \mathbf{u}(\mathbf{x}_\mathbf{j}^n, n\tau) + \frac{d}{ds} \mathbf{u}(\mathbf{x}_\mathbf{j}^n + s\tau \mathbf{u}_\mathbf{j}^n, n\tau + s\tau)\big|_{s=\zeta_\mathbf{u}}\end{aligned}$$

with a $\zeta_\mathbf{u} \in [0,1]$. For the sake of readability, we will shortly write $\mathbf{x}_\mathbf{j}^{n+s} = \mathbf{x}_\mathbf{j}^n + s\tau \mathbf{u}_\mathbf{j}^n$ and $t_{n+s} = t_n + s\tau$ for $s \in [0,1]$. Using the Euler equation (3.1), we have

$$\frac{d}{ds} \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+s}, t_{n+s})\big|_{s=\zeta_\mathbf{u}} = -\tau \nabla \rho(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}}) + \tau \left(\left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}}) \right) \cdot \nabla \right) \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}}).$$

Hence, using (5.4) and (5.5), the difference of the solution and the approximated velocity yields

$$\begin{aligned}\mathbf{u}_\mathbf{j}^{n+1} - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+1}, t_{n+1}) =& \mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n) - \tau \left(\nabla \rho^{\varepsilon,h,n}(\mathbf{x}_\mathbf{j}^n) - \nabla \rho(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}}) \right) \\ & - \tau \left(\left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}}) \right) \cdot \nabla \right) \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}}).\end{aligned}$$

Inserting this equation, the kinetic energy error term can be written as

$$\begin{aligned}Q_{kin}^{n+1} =& h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^{n+1} - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+1}, t_{n+1})\|_2^2 \\ =& Q_{kin}^n + \tau \left(S_{1,1}^n + S_{1,2}^n \right) + \tau^2 \left(S_{2,1}^n + S_{2,2}^n + S_{2,3}^n \right),\end{aligned}$$

where the single terms are given by

$$S_{1,1}^n := -2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\right) \cdot \left(\nabla\rho^{\varepsilon,h,n}(\mathbf{x}_\mathbf{j}^n) - \nabla\rho(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\right),$$

$$S_{1,2}^n := -2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\right) \cdot \left(\left(\left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\right) \cdot \nabla\right) \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\right),$$

$$S_{2,1}^n := h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\nabla\rho^{\varepsilon,h,n}(\mathbf{x}_\mathbf{j}^n) - \nabla\rho(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\|_2^2,$$

$$S_{2,2}^n := 2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\nabla\rho^{\varepsilon,h,n}(\mathbf{x}_\mathbf{j}^n) - \nabla\rho(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\right) \cdot \left(\left(\left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\right) \cdot \nabla\right) \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\right)$$

and

$$S_{2,3}^n := h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left\| \left(\left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\right) \cdot \nabla\right) \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}}) \right\|_2^2.$$

For the potential energy error term, we use the same procedure by using the mean value theorem. For the density ρ we have

$$\rho(\mathbf{x}, t_{n+1}) = \rho(\mathbf{x}, t_n + \tau) = \rho(\mathbf{x}, t_n) + \frac{d}{ds}\rho(\mathbf{x}, t_n + s\tau)\big|_{s=\zeta_\rho},$$

with a $\zeta_\rho \in [0, 1]$, where the Euler equation (3.2) yields

$$\frac{d}{ds}\rho(\mathbf{x}, t_n + s\tau)\big|_{s=\zeta_\rho} = \tau\partial_t\rho(\mathbf{x}, t_{n+\zeta_\rho}) = -\tau\nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t_{n+\zeta_\rho}).$$

For the approximated density, (5.4) yields

$$\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_\mathbf{j}^{n+1}) = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_\mathbf{j}^n - \tau\mathbf{u}_\mathbf{j}^n).$$

Since the kernel Φ_ε^r is continuously differentiable, we can again use the mean value theorem to derive

$$\Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_\mathbf{j}^n - \tau\mathbf{u}_\mathbf{j}^n) = \Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_\mathbf{j}^n) - \tau\mathbf{u}_\mathbf{j}^n \cdot \nabla\Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_\mathbf{j}^n - \zeta_\Phi\tau\mathbf{u}_\mathbf{j}^n)$$

for a $\zeta_\Phi \in [0, 1]$. Inserting this equation into the approximated density, we have

$$\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) = \rho^{\varepsilon,h,r,n}(\mathbf{x}) - \tau h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}_\mathbf{j}^n \cdot \nabla\Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_\mathbf{j}^{n+\zeta_\Phi}).$$

Hence, the potential energy term can be written as

$$\begin{aligned} Q_{pot}^{n+1} &= \int_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) - \rho(\mathbf{x}, t_{n+1})\right)^2 d\mathbf{x} \\ &= Q_{pot}^n + \tau S_{1,3}^n + \tau^2 S_{2,4}^n, \end{aligned}$$

where the single terms are given by

$$S_{1,3}^n := -2\int_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x}, t_n)\right) \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}_\mathbf{j}^n \cdot \nabla\Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_\mathbf{j}^{n+\zeta_\Phi}) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t_{n+\zeta_\rho})\right) d\mathbf{x},$$

$$S_{2,4}^n := \int_{\mathbb{R}^d} \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}_\mathbf{j}^n \cdot \nabla\Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_\mathbf{j}^{n+\zeta_\Phi}) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t_{n+\zeta_\rho})\right)^2 d\mathbf{x}.$$

Combining the kinetic and the potential error term, we can write the time discretized energy error at the $(n+1)$-th time step as

$$Q^{n+1} = Q^{n+1}_{kin} + Q^{n+1}_{pot} = Q^n + \tau\left(S^n_{1,1} + S^n_{1,2} + S^n_{1,3}\right) + \tau^2\left(S^n_{2,1} + S^n_{2,2} + S^n_{2,3} + S^n_{2,4}\right).$$

We will now split up the occurring terms separately, starting with $S^n_{1,1}$, $S^n_{1,2}$ and $S^n_{1,3}$. The first one can be written as $S^n_{1,1} = S^n_{1,1,1} + S^n_{1,1,2}$ with

$$\begin{aligned} S^n_{1,1,1} :=& -2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\left(\mathbf{u}^n_{\mathbf{j}} - \mathbf{u}(\mathbf{x}^n_{\mathbf{j}}, t_n)\right)\cdot\left(\nabla\rho^{\varepsilon,h,n}(\mathbf{x}^n_{\mathbf{j}}, t_n) - \nabla\rho(\mathbf{x}^n_{\mathbf{j}}, t_n)\right),\\ S^n_{1,1,2} :=& -2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\left(\mathbf{u}^n_{\mathbf{j}} - \mathbf{u}(\mathbf{x}^n_{\mathbf{j}}, t_n)\right)\cdot\left(\nabla\rho(\mathbf{x}^n_{\mathbf{j}}, t_n) - \nabla\rho(\mathbf{x}^{n+\zeta_{\mathbf{u}}}_{\mathbf{j}}, t_{n+\zeta_{\mathbf{u}}})\right). \end{aligned}$$

The second one can be written as $S^n_{1,2} = S^n_{1,2,1} + S^n_{1,2,2}$ with

$$\begin{aligned} S^n_{1,2,1} :=& -2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\left(\mathbf{u}^n_{\mathbf{j}} - \mathbf{u}(\mathbf{x}^n_{\mathbf{j}}, t_n)\right)\cdot\left(\left(\left(\mathbf{u}^n_{\mathbf{j}} - \mathbf{u}(\mathbf{x}^n_{\mathbf{j}}, t_n)\right)\cdot\nabla\right)\mathbf{u}(\mathbf{x}^{n+\zeta_{\mathbf{u}}}_{\mathbf{j}}, t_{n+\zeta_{\mathbf{u}}})\right),\\ S^n_{1,2,2} :=& -2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\left(\mathbf{u}^n_{\mathbf{j}} - \mathbf{u}(\mathbf{x}^n_{\mathbf{j}}, t_n)\right)\cdot\left(\left(\left(\mathbf{u}(\mathbf{x}^n_{\mathbf{j}}, t_n) - \mathbf{u}(\mathbf{x}^{n+\zeta_{\mathbf{u}}}_{\mathbf{j}}, t_{n+\zeta_{\mathbf{u}}})\right)\cdot\nabla\right)\mathbf{u}(\mathbf{x}^{n+\zeta_{\mathbf{u}}}_{\mathbf{j}}, t_{n+\zeta_{\mathbf{u}}})\right). \end{aligned}$$

The third one can be written as $S^n_{1,3} = S^n_{1,3,1} + S^n_{1,3,2} + S^n_{1,3,3}$ with

$$\begin{aligned} S^n_{1,3,1} :=& -2\int_{\mathbb{R}^d}\left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x}, t_n)\right)\left(h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\mathbf{u}^n_{\mathbf{j}}\cdot\nabla\Phi^r_\varepsilon(\mathbf{x} - \mathbf{x}^n_{\mathbf{j}}) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t_n)\right)d\mathbf{x},\\ S^n_{1,3,2} :=& -2\int_{\mathbb{R}^d}\left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x}, t_n)\right)\left(h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\mathbf{u}^n_{\mathbf{j}}\cdot\left(\nabla\Phi^r_\varepsilon(\mathbf{x} - \mathbf{x}^{n+\zeta_\Phi}_{\mathbf{j}}) - \nabla\Phi^r_\varepsilon(\mathbf{x} - \mathbf{x}^n_{\mathbf{j}})\right)\right)d\mathbf{x} \end{aligned}$$

and

$$S^n_{1,3,3} := -2\int_{\mathbb{R}^d}\left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x}, t_n)\right)\left(\nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t_n) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t_{n+\zeta_\rho})\right)d\mathbf{x}.$$

$S^n_{1,1,1}$, $S^n_{1,2,1}$ and $S^n_{1,3,1}$ can be bounded by identifying these terms with the corresponding terms occurring in the proof of Theorem 3.8. First of all, $S^n_{1,2,1}$ can be identified with A_2 from equation (3.19). Note that the arguments of the derivatives of $\mathbf{u}$ in $S^n_{1,2,1}$ differ from the ones in A_2, which can be neglected since we bound $\mathbf{u}$ in the $L^\infty(W^{1,\infty})$ norm. Hence, we have

$$S^n_{1,2,1} \le C(\mathbf{u})Q^n.$$

Separating the terms in $S^n_{1,1,1}$ and $S^n_{1,3,1}$ yields

$$\begin{aligned} S^n_{1,1,1} =& -2h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\mathbf{u}^n_{\mathbf{j}}\cdot\nabla\rho^{\varepsilon,h,n}(\mathbf{x}^n_{\mathbf{j}}, t_n)\\ &+ 2h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\mathbf{u}^n_{\mathbf{j}}\cdot\nabla\rho(\mathbf{x}^n_{\mathbf{j}}, t_n)\\ &+ 2h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho_0(h\mathbf{j})\mathbf{u}(\mathbf{x}^n_{\mathbf{j}}, t_n)\cdot\left(\nabla\rho^{\varepsilon,h,n}(\mathbf{x}^n_{\mathbf{j}}, t_n) - \nabla\rho(\mathbf{x}^n_{\mathbf{j}}, t_n)\right)\\ =& S^n_{1,1,1,1} + S^n_{1,1,1,2} + S^n_{1,1,1,3} \end{aligned}$$

and

$$\begin{aligned}
S^n_{1,3,1} =& -2\int\limits_{\mathbb{R}^d} \rho^{\varepsilon,h,r,n}(\mathbf{x}) h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}^n_{\mathbf{j}} \cdot \nabla\Phi^r_\varepsilon(\mathbf{x}-\mathbf{x}^n_{\mathbf{j}}) d\mathbf{x} \\
&+ 2\int\limits_{\mathbb{R}^d} \rho(\mathbf{x},t_n) h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}^n_{\mathbf{j}} \cdot \nabla\Phi^r_\varepsilon(\mathbf{x}-\mathbf{x}^n_{\mathbf{j}}) d\mathbf{x} \\
&+ 2\int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x},t_n)\right) \nabla\cdot(\rho\mathbf{u})(\mathbf{x},t_n) d\mathbf{x} \\
=& S^n_{1,3,1,1} + S^n_{1,3,1,2} + S^n_{1,3,1,3}.
\end{aligned}$$

Using that $\nabla\Phi^r_\varepsilon(-\mathbf{x}) = -\nabla\Phi^r_\varepsilon(\mathbf{x})$ and that $\Phi^r * \Phi^r = \Phi$ yields

$$S^n_{1,3,1,1} = 2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}^n_{\mathbf{j}} \cdot \int\limits_{\mathbb{R}^d} \rho^{\varepsilon,h,r,n}(\mathbf{x}) \nabla\Phi^r_\varepsilon(\mathbf{x}^n_{\mathbf{j}}-\mathbf{x}) d\mathbf{x} = 2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}^n_{\mathbf{j}} \cdot \nabla\rho^{\varepsilon,h,n}(\mathbf{x}),$$

which cancels out with $S^n_{1,1,1,1}$. Moreover, the sum $S^n_{1,1,1,2} + S^n_{1,3,1,2}$ and the sum $S^n_{1,1,1,3} + S^n_{1,3,1,3}$ can be identified with A_3 and A_1 from equation (3.19) from the proof of Theorem 3.9, respectively. Hence, using (3.20) and (3.23) we have

$$S^n_{1,1,1} + S^n_{1,3,1} \leq C(\mathbf{u},\rho,M)\left(\varepsilon^m + \varepsilon^{2L-d} + Q^n\right).$$

The remaining terms have to be bounded separately. Using the Cauchy-Schwarz inequality, $S^n_{1,1,2}$ yields

$$\begin{aligned}
S^n_{1,1,2} &\leq h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}^n_{\mathbf{j}} - \mathbf{u}(\mathbf{x}^n_{\mathbf{j}},t_n)\|_2^2 + h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\nabla\rho(\mathbf{x}^n_{\mathbf{j}},t_n) - \nabla\rho(\mathbf{x}^{n+\zeta_{\mathbf{u}}}_{\mathbf{j}},t_{n+\zeta_{\mathbf{u}}})\|_2^2 \\
&= Q^n_{kin} + R^n_{1,1,2},
\end{aligned}$$

where the remainder $R^n_{1,1,2}$ of $S^n_{1,1,2}$ is given by

$$R^n_{1,1,2} := h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\nabla\rho(\mathbf{x}^n_{\mathbf{j}},t_n) - \nabla\rho(\mathbf{x}^{n+\zeta_{\mathbf{u}}}_{\mathbf{j}},t_{n+\zeta_{\mathbf{u}}})\|_2^2.$$

Now, we want to apply the mean value theorem once again. This can be done since the Sobolev embedding theorem states that $\mathbf{u}$ and ρ are at least $m+1$ times continuously differentiable. Hence, the difference of the densities can be written as

$$\begin{aligned}
\nabla\rho(\mathbf{x}^n_{\mathbf{j}},t_n) - \nabla\rho(\mathbf{x}^{n+\zeta_{\mathbf{u}}}_{\mathbf{j}},t_{n+\zeta_{\mathbf{u}}}) =& -\zeta_{\mathbf{u}} \frac{d}{ds}\nabla\rho(\mathbf{x}^n_{\mathbf{j}} + s\tau\mathbf{u}^n_{\mathbf{j}}, t_n + s\tau)\big|_{s=\widetilde{\zeta_{\mathbf{u}}}} \\
=& -\zeta_{\mathbf{u}}\tau\left(\mathbf{u}^n_{\mathbf{j}}\cdot\nabla\right)(\nabla\rho)(\mathbf{x}^n_{\mathbf{j}} + \widetilde{\zeta_{\mathbf{u}}}\tau\mathbf{u}^n_{\mathbf{j}}, t_n + \widetilde{\zeta_{\mathbf{u}}}\tau) \\
&+ \zeta_{\mathbf{u}}\tau\nabla\left(\nabla\cdot(\rho\mathbf{u})\right)(\mathbf{x}^n_{\mathbf{j}} + \widetilde{\zeta_{\mathbf{u}}}\tau\mathbf{u}^n_{\mathbf{j}}, t_n + \widetilde{\zeta_{\mathbf{u}}}\tau)
\end{aligned}$$

for a $\widetilde{\zeta_{\mathbf{u}}} \in [0,\zeta_{\mathbf{u}}]$, where we used the continuity equation (3.2) in the last line. Hence, with $|\zeta_{\mathbf{u}}| \leq 1$, the remaining sum yields

$$\begin{aligned}
R^n_{1,1,2} \leq& \tau^2 h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\left(\mathbf{u}^n_{\mathbf{j}}\cdot\nabla\right)(\nabla\rho)(\mathbf{x}^n_{\mathbf{j}} + \widetilde{\zeta_{\mathbf{u}}}\tau\mathbf{u}^n_{\mathbf{j}}, t_n + \widetilde{\zeta_{\mathbf{u}}}\tau)\|_2^2 \\
&+ \tau^2 h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\nabla\left(\nabla\cdot(\rho\mathbf{u})\right)(\mathbf{x}^n_{\mathbf{j}} + \widetilde{\zeta_{\mathbf{u}}}\tau\mathbf{u}^n_{\mathbf{j}}, t_n + \widetilde{\zeta_{\mathbf{u}}}\tau)\|_2^2 \\
\leq& \tau^2 \left(\|\rho\|^2_{L^\infty(W^{2,\infty})} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}^n_{\mathbf{j}}\|_2^2 + \|\rho\mathbf{u}\|^2_{L^\infty(W^{2,\infty})} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \right).
\end{aligned}$$

The second sum can be bound by the finite mass of the SPH system $h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \leq M$, the first sum can be bound by

$$\begin{aligned} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n\|_2^2 \leq& 2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2^2 + 2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2^2 \\ \leq& \left(2Q_{kin}^n + 2M\|\mathbf{u}\|_{L^\infty(L^\infty)}^2\right) \end{aligned} \tag{5.7}$$

so that we finally have

$$S_{1,1,2}^n \leq C(\mathbf{u}, \rho) \left(Q_{kin}^n + \tau^2 \left(1 + Q_{kin}^n\right)\right).$$

For $S_{1,2,2}^n$, the Cauchy-Schwarz inequality and Young's inequality yield

$$\begin{aligned} S_{1,2,2}^n \leq& 2\|\mathbf{u}\|_{L^\infty(W^{1,\infty})} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2 \|\mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n) - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\|_2 \\ \leq& \|\mathbf{u}\|_{L^\infty(W^{1,\infty})} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2^2 \\ &+ \|\mathbf{u}\|_{L^\infty(W^{1,\infty})} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n) - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\|_2^2 \\ =& C(\mathbf{u}) \left(Q_{kin}^n + R_{1,2,2}^n(\zeta_\mathbf{u})\right) \end{aligned}$$

where the remainder $R_{1,2,2}^n$ is given by

$$R_{1,2,2}^n(\zeta_\mathbf{u}) := h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n) - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}})\|_2^2. \tag{5.8}$$

Again, we can use the mean value theorem to find a $\widetilde{\zeta_\mathbf{u}} \in [0, \zeta_\mathbf{u}]$ such that

$$\begin{aligned} \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n) - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\zeta_\mathbf{u}}, t_{n+\zeta_\mathbf{u}}) =& -\zeta_\mathbf{u} \frac{d}{ds} \mathbf{u}(\mathbf{x}_\mathbf{j}^n + s\tau\mathbf{u}_\mathbf{j}^n, t_n + s\tau)\big|_{s=\widetilde{\zeta_\mathbf{u}}} \\ =& \zeta_\mathbf{u} \tau \nabla\rho(\mathbf{x}_\mathbf{j}^{n+\widetilde{\zeta_\mathbf{u}}}, t_{n+\widetilde{\zeta_\mathbf{u}}}) \\ &- \zeta_\mathbf{u} \tau \left(\left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\widetilde{\zeta_\mathbf{u}}}, t_{n+\widetilde{\zeta_\mathbf{u}}})\right) \cdot \nabla\right) \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\widetilde{\zeta_\mathbf{u}}}, t_{n+\widetilde{\zeta_\mathbf{u}}}), \end{aligned}$$

where we used the Euler equation (3.1). Inserting this into the remainder yields

$$\begin{aligned} R_{1,2,2}^n(\zeta_\mathbf{u}) \leq& 2\tau^2 h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\nabla\rho(\mathbf{x}_\mathbf{j}^{n+\widetilde{\zeta_\mathbf{u}}}, t_{n+\widetilde{\zeta_\mathbf{u}}})\|_2^2 \\ &+ 2\tau^2 h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left\| \left(\left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\widetilde{\zeta_\mathbf{u}}}, t_{n+\widetilde{\zeta_\mathbf{u}}})\right) \cdot \nabla\right) \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\widetilde{\zeta_\mathbf{u}}}, t_{n+\widetilde{\zeta_\mathbf{u}}}) \right\|_2^2 \\ \leq& 2\tau^2 M \|\rho\|_{L^\infty(W^{1,\infty})}^2 + 2\tau^2 \|\mathbf{u}\|_{L^\infty(W^{1,\infty})}^2 h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\widetilde{\zeta_\mathbf{u}}}, t_{n+\widetilde{\zeta_\mathbf{u}}}))\|_2^2. \end{aligned}$$

The remaining sum can be bounded by

$$\begin{aligned} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\widetilde{\zeta_\mathbf{u}}}, t_{n+\widetilde{\zeta_\mathbf{u}}}))\|_2^2 \leq& 2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n))\|_2^2 \\ &+ 2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n) - \mathbf{u}(\mathbf{x}_\mathbf{j}^{n+\widetilde{\zeta_\mathbf{u}}}, t_{n+\widetilde{\zeta_\mathbf{u}}}))\|_2^2 \\ =& 2Q_{kin}^n + 2R_{1,2,2}^n(\widetilde{\zeta_\mathbf{u}}) \end{aligned}$$

where we now use that $R^n_{1,2,2}(\widetilde{\zeta_{\mathbf{u}}}) \leq 8M\|u\|^2_{L^\infty(L^\infty)}$. Altogether, $S^n_{1,2,2}$ yields

$$S^n_{1,2,2} \leq C(\mathbf{u},\rho)\left(Q^n_{kin} + \tau^2\left(1+Q^n_{kin}\right)\right).$$

For the Term $S^n_{1,3,2}$ we first note that with $\mathbf{x}_{\mathbf{j}}^{n+\zeta_\Phi} = \mathbf{x}^n_{\mathbf{j}} + \tau\zeta_\Phi\mathbf{u}^n_{\mathbf{j}}$, there exists a $\widetilde{\zeta_\Phi} \in [0,\zeta_\Phi]$ such that

$$\nabla\Phi^r_\varepsilon(\mathbf{x}-\mathbf{x}_{\mathbf{j}}^{n+\zeta_\Phi}) - \nabla\Phi^r_\varepsilon(\mathbf{x}-\mathbf{x}^n_{\mathbf{j}}) = -\zeta_\Phi\tau\left(\mathbf{u}^n_{\mathbf{j}}\cdot\nabla\right)\nabla\Phi^r_\varepsilon(\mathbf{x}-\mathbf{x}^n_{\mathbf{j}} - \tau\widetilde{\zeta_\Phi}\mathbf{u}^n_{\mathbf{j}}). \tag{5.9}$$

Hence,

$$\begin{aligned} S^n_{1,3,2} &= 2\zeta_\Phi\tau h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x},t_n)\right)\left(\mathbf{u}^n_{\mathbf{j}}\cdot\left(\mathbf{u}^n_{\mathbf{j}}\cdot\nabla\right)\nabla\Phi^r_\varepsilon(\mathbf{x}-\mathbf{x}^n_{\mathbf{j}} - \tau\widetilde{\zeta_\Phi}\mathbf{u}^n_{\mathbf{j}})\right)d\mathbf{x} \\ &= 2\zeta_\Phi\tau \sum_{k,l=1}^d h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})(\mathbf{u}^n_{\mathbf{j}})_k(\mathbf{u}^n_{\mathbf{j}})_l \int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x},t_n)\right)\partial_k\partial_l\Phi^r_\varepsilon(\mathbf{x}-\mathbf{x}^n_{\mathbf{j}} - \tau\widetilde{\zeta_\Phi}\mathbf{u}^n_{\mathbf{j}})d\mathbf{x}. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have $2(\mathbf{u}^n_{\mathbf{j}})_k(\mathbf{u}^n_{\mathbf{j}})_l \leq (\mathbf{u}^n_{\mathbf{j}})^2_k + (\mathbf{u}^n_{\mathbf{j}})^2_l \leq 2\|\mathbf{u}^n_{\mathbf{j}}\|^2_2$ so that, using $|\zeta_\Phi| < 1$, we can bound $S^n_{1,3,2}$ by

$$S^n_{1,3,2} \leq 2\tau h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\|\mathbf{u}^n_{\mathbf{j}}\|^2_2 R^n_{1,3,2,\mathbf{j}},$$

where the remainder $R^n_{1,3,2,\mathbf{j}}$ is given by

$$R^n_{1,3,2,\mathbf{j}} := \int\limits_{\mathbb{R}^d} \left|\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x},t_n)\right| \sum_{k,l=1}^d \left|\partial_k\partial_l\Phi^r_\varepsilon(\mathbf{x}-\mathbf{x}^n_{\mathbf{j}} + \tau\widetilde{\zeta_\Phi}\mathbf{u}^n_{\mathbf{j}})\right| d\mathbf{x}$$

for every $\mathbf{j}\in\mathbb{Z}^d$. Using inequality (5.7) for the occurring sum, we have

$$S^n_{1,3,2} \leq 4\tau Q^n_{kin} \max_{\mathbf{j}\in\mathbb{Z}^d} R^n_{1,3,2,\mathbf{j}} + 4\tau M\|\mathbf{u}\|^2_{L^\infty(L^\infty)} \max_{\mathbf{j}\in\mathbb{Z}^d} R^n_{1,3,2,\mathbf{j}}.$$

We now have two possibilities to give a bound for the remainder $R^n_{1,3,2,\mathbf{j}}$. For the first term, we use that $|\partial_k\partial_l\Phi^r_\varepsilon(\mathbf{x}-\mathbf{x}^n_{\mathbf{j}} + \tau\widetilde{\zeta_\Phi}\mathbf{u}^n_{\mathbf{j}})| \leq \varepsilon^{-d-2}\|\Phi^r\|_{W^{2,\infty}}$ to bound the remainder by

$$R^n_{1,3,2,\mathbf{j}} \leq c\frac{1}{\varepsilon^{d+2}}\left(\|\rho^{\varepsilon,h,r,n}\|_{L^1(\mathbb{R}^d)} + \|\rho\|_{L^1(\mathbb{R}^d)}\right)$$

for every $\mathbf{j}\in\mathbb{Z}^d$, where $\|\rho^{\varepsilon,h,r,n}\|_{L^1}$ can be bounded by $cM\|\Phi^r\|_{L^1}$. For the second occurrence, we use Young's inequality to derive

$$\tau R^n_{1,3,2,\mathbf{j}} \leq \int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x},t_n)\right)^2 d\mathbf{x} + d^4\tau^2\|\Phi^r_\varepsilon\|^2_{W^{2,2}(\mathbb{R}^d)} \leq c\left(Q^n_{pot} + \frac{\tau^2}{\varepsilon^{d+4}}\right),$$

where the norm of the kernel was bounded by

$$\|\Phi^r_\varepsilon\|_{W^{2,2}(\mathbb{R}^d)} \leq \varepsilon^{-d/2-2}\|\Phi^r\|_{W^{2,2}(\mathbb{R}^d)}.$$

Taking these estimates together, we have

$$S^n_{1,3,2} \leq C(\mathbf{u},\rho,M)\left(Q^n_{pot} + Q^n_{kin}\frac{\tau}{\varepsilon^{d+2}} + \frac{\tau^2}{\varepsilon^{d+4}}\right).$$

For $S^n_{1,3,3}$, we easily see that

$$\begin{aligned}S^n_{1,3,3} &\leq \int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x},t_n)\right)^2 d\mathbf{x} + \int\limits_{\mathbb{R}^d} \left(\nabla\cdot(\rho\mathbf{u})(\mathbf{x},t_n) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x},t_{n+\zeta_\rho})\right)^2 d\mathbf{x}\\ &= Q^n_{pot} + R^n_{1,3,3},\end{aligned}$$

where the remainder is given by

$$R^n_{1,3,3} = \int\limits_{\mathbb{R}^d} \left(\nabla\cdot(\rho\mathbf{u})(\mathbf{x},t_n) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x},t_{n+\zeta_\rho})\right)^2 d\mathbf{x}.$$

Using the mean value theorem for every $1 \leq j \leq d$, we find a $\widetilde{\zeta}_{\rho,j} \in [0,\zeta_\rho]$, such that

$$\begin{aligned}\partial_j(\rho\mathbf{u})(\mathbf{x},t_n) - \partial_j(\rho\mathbf{u})(\mathbf{x},t_{n+\zeta_\rho}) &= -\zeta_\rho\tau\partial_t\partial_j(\rho\mathbf{u})(\mathbf{x},t_n+\widetilde{\zeta}_{\rho,j}\tau)\\ &= \zeta_\rho\tau\sum_{k=1}^d \partial_k\partial_j(\rho u_k\mathbf{u})(\mathbf{x},t_n+\widetilde{\zeta}_{\rho,j}\tau) + \zeta_\rho\tau\frac{1}{2}\partial_j\nabla\rho^2(\mathbf{x},t_n+\widetilde{\zeta}_{\rho,j}\tau),\end{aligned}$$

where we used that

$$\partial_t(\rho\mathbf{u}) = (\partial_t\rho)\mathbf{u} + \rho\partial_t\mathbf{u} = -(\rho\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\rho)\mathbf{u} - \rho(\nabla\rho + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\sum_{k=1}^d \partial_k(\rho u_k\mathbf{u}) - \frac{1}{2}\nabla\rho^2.$$

Inserting this equation into $R^n_{1,3,3}$, using Lemma 1.17 and that $|\zeta_\rho| < 1$ yields

$$\begin{aligned}R^n_{1,3,3} =& \zeta_\rho^2\tau^2 \int\limits_{\mathbb{R}^d} \left(\sum_{k,j=1}^d \partial_k\partial_j(\rho u_k u_j)(\mathbf{x},t_n+\widetilde{\zeta}_{\rho,j}\tau) + \frac{1}{2}\sum_{j=1}^d \partial_j\nabla\rho^2(\mathbf{x},t_n+\widetilde{\zeta}_{\rho,j}\tau)\right)^2 d\mathbf{x}\\ \leq& 2\zeta_\rho^2\tau^2 \int\limits_{\mathbb{R}^d} \left(\sum_{k,j=1}^d \partial_k\partial_j(\rho u_k u_j)(\mathbf{x},t_n+\widetilde{\zeta}_{\rho,j}\tau)\right)^2 d\mathbf{x} + \zeta_\rho^2 2\tau^2 \int\limits_{\mathbb{R}^d} \left(\frac{1}{2}\sum_{j=1}^d \partial_j\nabla\rho^2(\mathbf{x},t_n+\widetilde{\zeta}_{\rho,j}\tau)\right)^2 d\mathbf{x}\\ \leq& c\tau^2\left(\|\rho\|^2_{L^\infty(W^{2,2})}\|\mathbf{u}\|^4_{L^\infty(W^{2,2})} + \|\rho\|^2_{L^\infty(W^{1,2})}\right),\end{aligned}$$

where we also used the Cauchy-Schwarz inequality. Hence, we have

$$S^n_{1,3,3} \leq Q^n_{pot} + C(\mathbf{u},\rho)\tau^2.$$

It remains to find bounds for the terms $S^n_{2,1}$, $S^n_{2,2}$, $S^n_{2,3}$ and $S^n_{2,4}$. First of all, we note that using Young's inequality for products, we have

$$S^n_{2,2} \leq S^n_{2,1} + S^n_{2,3}.$$

The remaining three terms have to be bounded separately. $S^n_{2,1}$ can be estimated by

$$\begin{aligned}\frac{1}{2}S^n_{2,1} \leq& h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\|\nabla\rho^{\varepsilon,h,n}(\mathbf{x}^n_\mathbf{j}) - \nabla\rho(\mathbf{x}^n_\mathbf{j},t_n)\|_2^2\\ &+ h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\|\nabla\rho(\mathbf{x}^n_\mathbf{j},t_n) - \nabla\rho(\mathbf{x}^{n+\zeta_\mathbf{u}}_\mathbf{j},t_{n+\zeta_\mathbf{u}})\|_2^2\\ =&: S^n_{2,1,1} + S^n_{2,1,2}.\end{aligned}$$

We note that $S^n_{2,1,2} = R^n_{1,1,2}$, and so we have

$$S^n_{2,1,2} \leq C(\mathbf{u},M)\tau^2(1 + Q^n_{kin}).$$

For the term $S^n_{2,1,1}$, we note that $\rho^{\varepsilon,h,n}(\mathbf{x}^n_\mathbf{j}) = \rho^{\varepsilon,h,r,n}(\mathbf{x}^n_\mathbf{j}) * \Phi^r_\varepsilon$, so that using the triangle inequality and Lemma 1.17 yield

$$\begin{aligned} S^n_{2,1,1} \leq & 2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \| \left(\rho^{\varepsilon,h,r,n} - \rho(\cdot, t_n)\right) * \nabla\Phi^r_\varepsilon(\mathbf{x}^n_\mathbf{j}) \|_2^2 \\ & + 2h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\nabla\rho(\cdot, t_n) * \Phi^r_\varepsilon(\mathbf{x}^n_\mathbf{j}) - \nabla\rho(\mathbf{x}^n_\mathbf{j}, t_n)\|_2^2, \end{aligned} \tag{5.10}$$

where the second sum can be bounded using Theorem 1.28 by

$$h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\nabla\rho(\cdot, t_n) * \Phi^r_\varepsilon(\mathbf{x}^n_\mathbf{j}) - \nabla\rho(\mathbf{x}^n_\mathbf{j}, t_n)\|_2^2 \leq cM\varepsilon^m \|\rho\|_{L^\infty(W^{m+1,\infty})}.$$

In the first sum of inequality (5.10), using the $L^\infty(\mathbb{R}^d)$ norm, we can factorize the difference. Hence, using Young's inequality for convolutions, Theorem 1.5, this $L^\infty(\mathbb{R}^d)$ norm yields

$$\| \left(\rho^{\varepsilon,h,r,n} - \rho(\cdot, t_n)\right) * \nabla\Phi^r_\varepsilon\|^2_{L^\infty(\mathbb{R}^d)} \leq \|\rho^{\varepsilon,h,r,n} - \rho(\cdot, t_n)\|^2_{L^2(\mathbb{R}^d)} \|\nabla\Phi^r_\varepsilon\|^2_{L^2(\mathbb{R}^d)} \leq Q^n_{pot} \|\Phi^r_\varepsilon\|^2_{W^{1,2}(\mathbb{R}^d)}$$

with $\|\Phi^r_\varepsilon\|_{W^{1,2}(\mathbb{R}^d)} \leq \varepsilon^{-d/2-1} \|\Phi^r\|_{W^{1,2}(\mathbb{R}^d)}$. The remaining sum over the initial density can again be estimated by M. Combining the estimates above gives

$$S^n_{2,1} \leq S^n_{2,1,1} + S^n_{2,1,2} \leq C(\mathbf{u}, \rho, M) \left(\varepsilon^m + \frac{Q^n_{pot}}{\varepsilon^{d+2}} + \tau^2(1 + Q^n_{kin})\right).$$

For $S^n_{2,3}$, we first note for a $\mathbf{v} \in \mathbb{R}^d$ that we have

$$\| \left(\mathbf{v} \cdot \nabla\right) \mathbf{u}\|_2^2 = \sum_{k=1}^d \left| \sum_{j=1}^d v_j \partial_j u_k \right|^2 \leq d\|\mathbf{u}\|_{L^\infty(W^{1,\infty})} \left(\sum_{j=1}^d |v_j| \right)^2 \leq d^2 \|\mathbf{u}\|_{L^\infty(W^{1,\infty})} \|\mathbf{v}\|_2^2,$$

where we used Lemma 2.17. Applying this inequality to the terms in the sum of $S^n_{2,3}$ and using Lemma 1.17 once again, we have

$$\begin{aligned} S^n_{2,3} = & h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left\| \left(\left(\mathbf{u}^n_\mathbf{j} - \mathbf{u}(\mathbf{x}^{n+\zeta_\mathbf{u}}_\mathbf{j}, t_{n+\zeta_\mathbf{u}})\right) \cdot \nabla \right) \mathbf{u}(\mathbf{x}^{n+\zeta_\mathbf{u}}_\mathbf{j}, t_{n+\zeta_\mathbf{u}}) \right\|_2^2 \\ \leq & d^2 \|\mathbf{u}\|^2_{L^\infty(W^{1,\infty})} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}^n_\mathbf{j} - \mathbf{u}(\mathbf{x}^{n+\zeta_\mathbf{u}}_\mathbf{j}, t_{n+\zeta_\mathbf{u}})\|_2^2 \\ \leq & 2d^2 \|\mathbf{u}\|^2_{L^\infty(W^{1,\infty})} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}^n_\mathbf{j} - \mathbf{u}(\mathbf{x}^n_\mathbf{j}, t_n)\|_2^2 \\ & + 2d^2 \|\mathbf{u}\|^2_{L^\infty(W^{1,\infty})} h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}(\mathbf{x}^n_\mathbf{j}, t_n) - \mathbf{u}(\mathbf{x}^{n+\zeta_\mathbf{u}}_\mathbf{j}, t_{n+\zeta_\mathbf{u}})\|_2^2. \end{aligned}$$

The first occurring sum is identical to Q^n_{kin}, the second one can be identified with $R^n_{1,2,2}(\zeta_\mathbf{u})$, see (5.8). Hence, $S^n_{2,3}$ an be bound by

$$S^n_{2,3} \leq C(\mathbf{u}, M) \left(Q^n_{kin} + \tau^2(1 + Q^n_{kin})\right).$$

Finally we have to find an estimate for $S^n_{2,4}$. First, using Lemma 2.17 we split $S^n_{2,4}$ up into five parts by

$$\begin{aligned}
S_{2,4}^n \leq & 5 \left\| h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\right) \cdot \nabla \Phi_\varepsilon^r(\cdot - \mathbf{x}_\mathbf{j}^{n+\zeta_\Phi}) \right\|_{L^2(\mathbb{R}^d)}^2 \\
& + 5 \left\| h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n) \cdot \nabla \left(\Phi_\varepsilon^r(\cdot - \mathbf{x}_\mathbf{j}^{n+\zeta_\Phi}) - \Phi_\varepsilon^r(\cdot - \mathbf{x}_\mathbf{j}^n) \right) \right\|_{L^2(\mathbb{R}^d)}^2 \\
& + 5 \left\| h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n) \cdot \nabla \Phi_\varepsilon^r(\cdot - \mathbf{x}_\mathbf{j}^n) - \int_{\mathbb{R}^d} \rho(\mathbf{x}, t_n) \mathbf{u}(\mathbf{x}, t_n) \cdot \nabla \Phi_\varepsilon^r(\cdot - \mathbf{x}) d\mathbf{x} \right\|_{L^2(\mathbb{R}^d)}^2 \\
& + 5 \left\| (\nabla \cdot \rho \mathbf{u}(\cdot, t_n)) * \Phi_\varepsilon^r - \nabla \cdot (\rho \mathbf{u})(\cdot, t_n) \right\|_{L^2(\mathbb{R}^d)}^2 \\
& + 5 \left\| \nabla \cdot (\rho \mathbf{u})(\cdot, t_n) - \nabla \cdot (\rho \mathbf{u})(\cdot, t_{n+\zeta_\rho}) \right\|_{L^2(\mathbb{R}^d)}^2 \\
=: & 5 \left(S_{2,4,1}^n + S_{2,4,2}^n + S_{2,4,3}^n + S_{2,4,4}^n + S_{2,4,5}^n \right),
\end{aligned}$$

For the first norm we note that

$$S_{2,4,1}^n \leq h^{2d} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \rho_0(h\mathbf{k}) \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2 \|\mathbf{u}_\mathbf{k}^n - \mathbf{u}(\mathbf{x}_\mathbf{k}^n, t_n)\|_2 \|\Phi_\varepsilon^r\|_{W^{1,2}(\mathbb{R}^d)}^2,$$

where the norm of the kernel can be estimated by $\|\Phi_\varepsilon^r\|_{W^{1,2}(\mathbb{R}^d)}^2 \leq \varepsilon^{-d-2} \|\Phi^r\|_{W^{1,2}(\mathbb{R}^d)}^2$. Furthermore, using Young's inequality, we note that

$$2\|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2 \|\mathbf{u}_\mathbf{k}^n - \mathbf{u}(\mathbf{x}_\mathbf{k}^n, t_n)\|_2 \leq \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2^2 + \|\mathbf{u}_\mathbf{k}^n - \mathbf{u}(\mathbf{x}_\mathbf{k}^n, t_n)\|_2^2,$$

which yields, by switching the summation indices in one of the occurring double sums,

$$\begin{aligned}
S_{2,4,1}^n \leq & c \frac{1}{\varepsilon^{d+2}} h^{2d} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \rho_0(h\mathbf{j}) \left(\|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2^2 + \|\mathbf{u}_\mathbf{k}^n - \mathbf{u}(\mathbf{x}_\mathbf{k}^n, t_n)\|_2^2 \right) \\
= & 2c \frac{1}{\varepsilon^{d+2}} h^{2d} \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2^2 \\
\leq & C(M) \frac{Q_{kin}^n}{\varepsilon^{d+2}},
\end{aligned}$$

where we used the finite initial mass in the last line again. For $S_{2,4,2}^n$ we use equation (5.9) to find a $\zeta \in [0, \zeta_\Phi]$ such that

$$\begin{aligned}
S_{2,4,2}^n = & \| h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n) \cdot \left(\zeta_\Phi \tau \left(\mathbf{u}_\mathbf{j}^n \cdot \nabla \right) \Phi_\varepsilon^r(\cdot - \mathbf{x}_\mathbf{j}^n - \tau \zeta \mathbf{u}_\mathbf{j}^n) \right) \|_{L^2(\mathbb{R}^d)}^2 \\
\leq & c\tau^2 \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2 \|\mathbf{u}_\mathbf{j}^n\|_2 \right)^2 \|\Phi_\varepsilon^r\|_{W^{2,2}(\mathbb{R}^d)}^2.
\end{aligned}$$

The norm of the kernel function can be bounded by $\|\Phi_\varepsilon^r\|_{W^{2,2}(\mathbb{R}^d)}^2 \leq \varepsilon^{-d-4} \|\Phi^r\|_{W^{2,2}(\mathbb{R}^d)}^2$. For the remaining sum, we separate $\mathbf{u}$ to derive

$$\left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2 \|\mathbf{u}_\mathbf{j}^n\|_2 \right)^2 \leq c \|\mathbf{u}\|_{L^\infty(L^\infty)}^2 \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \|\mathbf{u}_\mathbf{j}^n\|_2 \right)^2.$$

The last term yields

$$\left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\|\mathbf{u}_\mathbf{j}^n\|_2\right)^2 \leq \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\left(\|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2 + \|\mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2\right)\right)^2$$
$$\leq \left(2\left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\|\mathbf{u}_\mathbf{j}^n - \mathbf{u}(\mathbf{x}_\mathbf{j}^n, t_n)\|_2\right)^2 + C(\mathbf{u})2M^2\right)$$
$$\leq C(\mathbf{u}, M)\left(1 + Q_{kin}^n\right),$$

where we used the same arguments in the last line as in the case of $S_{2,4,1}^n$ to get a bound for the remaining sum. Overall, $S_{2,4,2}^n$ can be bound by

$$S_{2,4,2}^n \leq C(\mathbf{u}, M)\frac{\tau^2}{\varepsilon^{d+4}}\left(1 + Q_{kin}^n\right).$$

For $S_{2,4,4}^n$ we apply Theorem 1.28 to achieve

$$S_{2,4,4}^n \leq \varepsilon^{2m}\|\rho\mathbf{u}\|_{L^\infty(W^{m+1,\infty})}^2.$$

For the fifth term we note that $S_{2,4,5}^n = R_{1,3,3}^n$, which yields

$$S_{2,4,5}^n \leq C(\mathbf{u}, \rho)\tau^2.$$

Finally, for $S_{2,4,3}^n$ we have to work a little harder. In the following we will omit the time dependence for the sake of readability. Let $1 \leq j \leq d$. We will derive a bound for the j-th part of the integral, which we denote by

$$S_{2,4,3,j}^n := \left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k})u_j(\mathbf{x}_\mathbf{k}^n)\partial_j\Phi_\varepsilon^r(\cdot - \mathbf{x}_\mathbf{k}^n) - \int_{\mathbb{R}^d} \rho(\mathbf{x})u_j(\mathbf{x})\partial_j\Phi_\varepsilon^r(\cdot - \mathbf{x})d\mathbf{x} \right\|_{L^2(\mathbb{R}^d)}^2. \tag{5.11}$$

By the Sobolev embedding theorem and our assumptions on the smoothness of $\mathbf{u}$, we may use a Taylor expansion of u_j about $\mathbf{y} \in \mathbb{R}^d$ given by

$$u_j(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|\leq L} \frac{D^{\boldsymbol{\alpha}}u_j(\mathbf{y})}{\boldsymbol{\alpha}!}(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}} + (L+1)\sum_{|\boldsymbol{\alpha}|=L+1} \frac{(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\int_0^1 (1-s)^L D^{\boldsymbol{\alpha}}u_j(\mathbf{x} - s(\mathbf{x}-\mathbf{y}))ds.$$

Together with the derivative of the kernel function, we can write

$$\begin{aligned} u_j(\mathbf{x})\partial_j\Phi_\varepsilon^r(\mathbf{y}-\mathbf{x}) &= \sum_{|\boldsymbol{\alpha}|\leq L} \frac{D^{\boldsymbol{\alpha}}u_j(\mathbf{y})}{\boldsymbol{\alpha}!}(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}}\partial_j\Phi_\varepsilon^r(\mathbf{y}-\mathbf{x}) \\ &\quad + \sum_{|\boldsymbol{\alpha}|=L+1} R_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y})\frac{(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\partial_j\Phi_\varepsilon^r(\mathbf{y}-\mathbf{x}) \\ &= \sum_{|\boldsymbol{\alpha}|\leq L} D^{\boldsymbol{\alpha}}u_j(\mathbf{y})W_{\varepsilon,\boldsymbol{\alpha}}^j(\mathbf{y}-\mathbf{x}) + \sum_{|\boldsymbol{\alpha}|=L+1} R_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y})W_{\varepsilon,\boldsymbol{\alpha}}^j(\mathbf{y}-\mathbf{x}), \end{aligned} \tag{5.12}$$

where $W_{\varepsilon,\boldsymbol{\alpha}}^j$ is defined in (3.12) by

$$W_{\varepsilon,\boldsymbol{\alpha}}^j(\mathbf{x}) := \frac{(-1)^{|\boldsymbol{\alpha}|+1}}{\boldsymbol{\alpha}!}p_{\boldsymbol{\alpha}}(\mathbf{x})\partial_j\Phi_\varepsilon^r(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d,$$

with $p_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}}$. For a $|\boldsymbol{\alpha}| = L+1$, the remainder $R_{\boldsymbol{\alpha}}$ is given by

$$R_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y}) := (L+1)\int_0^1 (1-s)^L D^{\boldsymbol{\alpha}} u_j(\mathbf{x} - s(\mathbf{x}-\mathbf{y}))ds.$$

Inserting (5.12) into equation (5.11) and separating the terms of the sums, using Lemma 2.17 yields the estimate

$$\begin{aligned}\sqrt{S^n_{2,4,3,j}} \leq & \sum_{|\boldsymbol{\alpha}|\leq L} \left\| D^{\boldsymbol{\alpha}} u_j \left(h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\cdot - \mathbf{x}^n_{\mathbf{k}}) - \int_{\mathbb{R}^d} \rho(\mathbf{x}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\cdot - \mathbf{x}) d\mathbf{x} \right) \right\|_{L^2(\mathbb{R}^d)} \\ & + \sum_{|\boldsymbol{\alpha}|=L+1} \left\| h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) R_{\boldsymbol{\alpha}}(\mathbf{x}^n_{\mathbf{k}},\cdot) W^j_{\varepsilon,\boldsymbol{\alpha}}(\cdot - \mathbf{x}^n_{\mathbf{k}}) \right\|_{L^2(\mathbb{R}^d)} \\ & + \sum_{|\boldsymbol{\alpha}|=L+1} \left\| \int_{\mathbb{R}^d} \rho(\mathbf{x}) R_{\boldsymbol{\alpha}}(\mathbf{x},\cdot) W^j_{\varepsilon,\boldsymbol{\alpha}}(\cdot - \mathbf{x}) d\mathbf{x} \right\|_{L^2(\mathbb{R}^d)} \\ =: & \sum_{|\boldsymbol{\alpha}|\leq L} \widetilde{R}_{\boldsymbol{\alpha}} + \sum_{|\boldsymbol{\alpha}|=L+1} \left(\widetilde{R}_{\boldsymbol{\alpha},1} + \widetilde{R}_{\boldsymbol{\alpha},2} \right).\end{aligned}$$

We will now give a bound for $\widetilde{R}_{\boldsymbol{\alpha}}$. First, for a $\boldsymbol{\alpha} = \mathbf{0}$ we note

$$\begin{aligned}\widetilde{R}_{\mathbf{0}} = & \left\| u_j \left(h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) \partial_j \Phi^r_\varepsilon(\cdot - \mathbf{x}^n_{\mathbf{k}}) - \int_{\mathbb{R}^d} \rho(\mathbf{x}) \partial_j \Phi^r_\varepsilon(\cdot - \mathbf{x}) d\mathbf{x} \right) \right\|_{L^2(\mathbb{R}^d)} \\ = & \left\| u_j \left(\partial_j \rho^{\varepsilon,h,r,n} - \partial_j(\rho * \Phi^r_\varepsilon) \right) \right\|_{L^2(\mathbb{R}^d)} \\ \leq & \|u_j\|_{L^\infty(L^\infty)} \|\partial_j f^{\varepsilon,h}\|_{L^2(\mathbb{R}^d)}\end{aligned}$$

where $f^{\varepsilon,h} := \rho^{\varepsilon,h,r,n} - \rho * \Phi^r_\varepsilon$. With $l = s$, Lemma 5.2 yields

$$\|\partial_j f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)} \leq \frac{s-1}{s\tau} \|f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)} + \frac{\tau^{s-1}}{s} \|\partial^s_j f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)}.$$

The s-th derivative of $f^{\varepsilon,h}$ can then be bound by

$$\begin{aligned}\|\partial^s_j f^{\varepsilon,h}\|_{L^2(\mathbb{R}^d)} & \leq \|\partial^s_j \rho^{\varepsilon,h,r,n}\|_{L^2(\mathbb{R}^d)} + \|\Phi^r_\varepsilon\|_{L^1(\mathbb{R}^d)} \|\partial^s_j \rho\|_{L^2(\mathbb{R}^d)} \\ & \leq M\varepsilon^{-d/2-s} + c\|\rho\|_{L^\infty(W^{s,2})},\end{aligned}$$

where we used Theorem 1.5 for the convolution of Φ^r_ε and $\partial^s_j \rho$. Hence, with

$$\|f^{\varepsilon,h}\|^2_{L^2(\mathbb{R}^d)} \leq 2\|\rho^{\varepsilon,h,r,n} - \rho * \Phi^r_\varepsilon\|^2_{L^2(\mathbb{R}^d)} + 2\|\rho - \rho\|^2_{L^2(\mathbb{R}^d)} \leq 2Q^n_{pot} + c\varepsilon^{2m}\|\rho\|^2_{L^\infty(W^{m,2})},$$

we achieve

$$\widetilde{R}_{\mathbf{0}} \leq c\left(\frac{Q^n_{pot}}{\tau} + \frac{\varepsilon^{2m}}{\tau} + \frac{\tau^{s-1}}{\varepsilon^{d+2s}} \right).$$

For $1 \leq |\boldsymbol{\alpha}| \leq L$ we note that

$$\begin{aligned}\widetilde{R}_{\boldsymbol{\alpha}} = & \int_{\mathbb{R}^d} \left(D^{\boldsymbol{\alpha}} u_j(\mathbf{y}) \left(h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y} - \mathbf{x}^n_{\mathbf{k}}) - \int_{\mathbb{R}^d} \rho(\mathbf{x}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y} - \mathbf{x}) d\mathbf{x} \right) \right)^2 d\mathbf{y} \\ \leq & \|u_j\|_{L^\infty(W^{L,\infty})} \int_{\mathbb{R}^d} \left(h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y} - \mathbf{x}^n_{\mathbf{k}}) - \rho * W^j_{\varepsilon,\boldsymbol{\alpha}}(\mathbf{y}) \right)^2 d\mathbf{y} \\ = & \|u_j\|_{L^\infty(W^{L,\infty})} \|F^j_{\boldsymbol{\alpha}}\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

where we defined, as in (3.21) in the proof of Theorem 3.9,

$$F_{\boldsymbol{\alpha}}^j := h^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \rho_0(h\mathbf{k}) W_{\varepsilon,\boldsymbol{\alpha}}^j(\cdot - \mathbf{x}_{\mathbf{k}}^{\varepsilon,h}) - \rho * W_{\varepsilon,\boldsymbol{\alpha}}^j.$$

Hence, using (3.22) gives

$$\begin{aligned}\|F_{\boldsymbol{\alpha}}^j\|_{L^2(\mathbb{R}^d)}^2 &\le C\varepsilon^{2|\boldsymbol{\alpha}|-2}\|f^{\varepsilon,h}\|_{L^2(\mathbb{R}^d)}^2 \le C\varepsilon^{2|\boldsymbol{\alpha}|-2}\left(\|\rho^{\varepsilon,h,r,n}-\rho\|_{L^2(\mathbb{R}^d)}^2 + \|\rho - \rho * \Phi_\varepsilon^r\|_{L^2(\mathbb{R}^d)}^2\right) \\ &\le C\varepsilon^{2|\boldsymbol{\alpha}|-2}\left(Q_{pot}^n + \varepsilon^{2m}\|\rho\|_{L^\infty(W^{m,2})}^2\right).\end{aligned}$$

The terms for $|\boldsymbol{\alpha}| = L+1$ can easily be bounded by

$$\widetilde{R}_{\boldsymbol{\alpha},1} \le M^2 \|R_{\boldsymbol{\alpha}}\|_{L^\infty\times L^\infty(\mathbb{R}^d)}^2 \|W_{\varepsilon,\boldsymbol{\alpha}}^j\|_{L^2(\mathbb{R}^d)}^2$$

and

$$\widetilde{R}_{\boldsymbol{\alpha},2} \le \|\rho\|_{L^1(\mathbb{R}^d)}^2 \|R_{\boldsymbol{\alpha}}\|_{L^\infty\times L^\infty(\mathbb{R}^d)}^2 \|W_{\varepsilon,\boldsymbol{\alpha}}^j\|_{L^2(\mathbb{R}^d)}^2,$$

where $\|R_{\boldsymbol{\alpha}}\|_{L^\infty\times L^\infty(\mathbb{R}^d)}^2 \le (L+1)\|u\|_{L^\infty(W^{L+1,\infty})}$ and Proposition 3.6 yields

$$\|W_{\varepsilon,\boldsymbol{\alpha}}^j\|_{L^2(\mathbb{R}^d)}^2 \le \varepsilon^{2L-d}\|p_{\boldsymbol{\alpha}}\partial_j\Phi^r\|_{L^2(\mathbb{R}^d)}^2 \le c\varepsilon^{2L-d}.$$

Combining these estimates, we arrive at

$$S_{2,4,3}^n \le C(\mathbf{u},\rho,M)\left(\frac{Q_{pot}^n}{\tau} + \frac{\varepsilon^{2m}}{\tau} + Q_{pot}^n + \varepsilon^{2m} + \varepsilon^{2L-d} + \frac{\tau^{s-1}}{\varepsilon^{d+2s}}\right),$$

such that we can bound $S_{2,4}^n$ by

$$S_{2,4}^n \le (\mathbf{u},\rho,M)\left(\frac{Q^n}{\tau} + \frac{\varepsilon^{2m}}{\tau} + \frac{Q^n}{\varepsilon^{d+2}} + \tau^2\left(1 + \frac{1+Q^n}{\varepsilon^{d+4}}\right) + \varepsilon^{2m} + \varepsilon^{2L-d} + \frac{\tau^{s-1}}{\varepsilon^{d+2s}}\right).$$

We recall all estimates we made. For the first part, $S_{1,1}^n$, $S_{1,2}^n$ and $S_{1,3}^n$, we have the seperate estimates

$$\begin{aligned} S_{1,1,1}^n + S_{1,2,1}^n + S_{1,3,1}^n &\le c\left(\varepsilon^m + \varepsilon^{2L-d} + Q^n\right), \\ S_{1,1,2}^n &\le c\left(Q^n + \tau^2\left(1+Q^n\right)\right), \\ S_{1,2,2}^n &\le c\left(Q^n + \tau^2\left(1+Q^n\right)\right), \\ S_{1,3,2}^n &\le c\left(Q^n + Q^n\frac{\tau}{\varepsilon^{d+2}} + \frac{\tau^2}{\varepsilon^{d+4}}\right), \\ S_{1,3,3}^n &\le Q^n + c\tau^2, \end{aligned}$$

so that we have the complete estimate

$$\begin{aligned}\tau\left(S_{1,1}^n + S_{1,2}^n + S_{1,3}^n\right) &\le c\tau\left[Q^n\left(1+\tau^2+\frac{\tau}{\varepsilon^{d+2}}\right) + \varepsilon^m + \varepsilon^{2L-d} + \tau^2 + \frac{\tau^2}{\varepsilon^{d+4}}\right] \\ &\le C\left[Q^n\tau\left(1+\frac{\tau}{\varepsilon^{d+2}}\right) + \tau\left(\varepsilon^{\min\{m,2L-d\}} + \frac{\tau^2}{\varepsilon^{d+4}}\right)\right],\end{aligned}$$

where we used that $\tau < 1$. For the second part, $S_{2,1}^n$, $S_{2,2}^n$, $S_{2,3}^n$ and $S_{2,4}^n$, we have the seperate estimates

$$\begin{aligned} S_{2,1}^n &\le c\left(\varepsilon^m + \frac{Q^n}{\varepsilon^{d+2}} + \tau^2(1+Q^n)\right), \\ S_{2,2}^n &\le S_{2,1}^n + S_{2,3}^n \\ S_{2,3}^n &\le c\left(Q^n + \tau^2(1+Q^n)\right), \\ S_{2,4}^n &\le c\left(\frac{Q^n}{\tau} + \frac{\varepsilon^{2m}}{\tau} + \frac{Q^n}{\varepsilon^{d+2}} + \tau^2\left(1+\frac{1+Q^n}{\varepsilon^{d+4}}\right) + \varepsilon^{2m} + \varepsilon^{2L-d} + \frac{\tau^{s-1}}{\varepsilon^{d+2s}}\right), \end{aligned}$$

so that we have the complete estimate

$$\begin{aligned}\tau^2\left(S^n_{2,1}+S^n_{2,2}+S^n_{2,3}+S^n_{2,4}\right) \leq c\tau^2\Bigg[&Q^n\left(1+\tau^2+\frac{1}{\tau}+\frac{1}{\varepsilon^{d+2}}+\frac{\tau^2}{\varepsilon^{d+4}}\right)\\ &+\tau^2+\varepsilon^m+\varepsilon^{2L-d}+\frac{\varepsilon^{2m}}{\tau}+\frac{\tau^2}{\varepsilon^{d+4}}+\frac{\tau^{s-1}}{\varepsilon^{d+2s}}\Bigg]\\ \leq C\Bigg[&Q^n\tau\left(1+\frac{\tau}{\varepsilon^{d+2}}\right)\\ &+\tau\left(\tau\varepsilon^{\min\{m,2L-d\}}+\varepsilon^{2m}+\frac{\tau^3}{\varepsilon^{d+4}}+\frac{\tau^s}{\varepsilon^{d+2s}}\right)\Bigg],\end{aligned}$$

where we used that $\tau \leq \varepsilon \leq 1$. Combining all estimates, we find constants $c_1 > 0$ and $c_2 > 0$ so that

$$\begin{aligned}Q^{n+1} =&Q^n+\tau\left(S^n_{1,1}+S^n_{1,2}+S^n_{1,3}\right)+\tau^2\left(S^n_{2,1}+S^n_{2,2}+S^n_{2,3}+S^n_{2,4}\right)\\ \leq&Q^n\left(1+\tau c_2\left(1+\frac{\tau}{\varepsilon^{d+2}}\right)\right)+\tau c_1\left(\varepsilon^{\min\{m,2L-d\}}+\frac{\tau^2}{\varepsilon^{d+4}}+\frac{\tau^s}{\varepsilon^{d+2s}}\right).\end{aligned}$$

Hence, using the discrete Gronwall's inequality, Lemma 5.1, we find

$$\begin{aligned}Q^n &\leq \left[Q^0+c_1\left(\varepsilon^{\min\{m,2L-d\}}+\frac{\tau^2}{\varepsilon^{d+4}}+\frac{\tau^s}{\varepsilon^{d+2s}}\right)\right]\exp\left(c_2\left(1+\frac{\tau}{\varepsilon^{d+2}}\right)\right)\\ &\leq C_1\left[Q^0+\left(\varepsilon^{\min\{m,2L-d\}}+\frac{\tau^2}{\varepsilon^{d+4}}\right)\right]\exp\left(C_2\frac{\tau}{\varepsilon^{d+2}}\right),\end{aligned}$$

where we used that $\tau \leq \varepsilon^2$ and $s \geq 2$, which finishes the proof. □

As we see, we have a strong dependence on the time discretization parameter τ. In order to obtain convergence, the parameter τ has to be at least $\sim \varepsilon^{d+2}$. Concerning the numerical efficiency of the SPH method, this would be bad. However, the error analysis above might not be optimal. Moreover, we prefer to use time stepping methods of higher order at this point, for example a high-order Runge Kutta method, even if the error analysis for time step algorithms of higher order has yet to be done.
Using Theorem 3.8 for the convergence of Q^0 and Lemma 3.10 to convert the conditions on the solution of the Euler equations to conditions on the initial data $(\mathbf{u}_0, \rho_0)$, we can conclude the following general result for the convergence of the energy error term.

Corollary 5.4
Let $s \geq 2$ and $\Phi^r \in W^{s,1}(\mathbb{R}^d)\cap W^{s,2}(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ be an even root kernel, which satisfies the moment condition of order $m \geq 1$ and the approximation condition of order $L > d/2$. Let $\Phi = \Phi^r * \Phi^r$ be the corresponding convolution kernel. Let $\varepsilon > 0$ and $h > 0$.
Assume initial data $\mathbf{u}_0 \in W^{\sigma,2}(\mathbb{R}^d)^d$ and finite discrete mass with ρ_0 so that $\rho_0^{1/2} \in W^{\max\{\sigma,s\},2}(\mathbb{R}^d)$ for a $\sigma > \max\{L, m\}+1+d/2$ and that the solution $(\mathbf{u}, \rho)$ of the Euler equations (3.1) - (3.3) *exists up to a time $T > 0$. Let $(\mathbf{x}^{\varepsilon,h,n}_{\mathbf{k}}, \mathbf{u}^{\varepsilon,h,n}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ for $0 \leq n \leq N$ be a solution of the corresponding, fully discretized SPH equations* (5.4) - (5.6).
Then, there exist constants $C_1, C_2 > 0$ such that the energy error can be bounded by

$$Q^n \leq C_1\left(\frac{h^{2s}}{\varepsilon^{2s}}+\varepsilon^{\min\{m,2L-d\}}+\frac{\tau^2}{\varepsilon^{d+4}}\right)\exp\left(C_2\frac{\tau}{\varepsilon^{d+2}}\right).$$

for all $0 \leq n \leq N$.

Since we now have a result for the convergence of the energy error term, we can also conclude pointwise convergence of the SPH method, analogously to Section 3.4.

Theorem 5.5
Let the assumptions of Corollary 5.4 hold. This means in particular that the kernel satisfies the moment condition of order $m \geq 1$, the approximation condition of order $L > d/2$ and has smoothness $s > d$. Then, the following holds:

i) If $\rho_0(\mathbf{x}) > 0$ for all $\|\mathbf{x}\|_2 < 1$ then, for each $\mathbf{j} \in \mathbb{Z}^d$, there exist constants $C_1, C_2 > 0$, such that we have for sufficiently small h,

$$\|\mathbf{x}_{\mathbf{j}}^{\varepsilon,h,n} - \mathbf{X}(h\mathbf{j}, n\tau)\|_2 + \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h,n}(t) - \mathbf{u}(\mathbf{X}(h\mathbf{j}, n\tau), n\tau)\|_2 \leq C_1 \left(\frac{h^{s-d/2}}{\varepsilon^s} + \frac{\varepsilon^{\max\{m/2, L-d/2\}}}{h^{d/2}} + \frac{\tau}{\varepsilon^{d/2+2} h^{d/2}} \right) \exp\left(C_2 \frac{\tau}{\varepsilon^{d+2}} \right)$$

for all $1 \leq n \leq N$.

ii) Let $K \subset \mathbb{R}^d$ be compact with $\inf_{\mathbf{x} \in K} \rho_0(\mathbf{x}) > 0$. Then, there exist constants $C_K, C_2 > 0$ and such that

$$\|\mathbf{x}_{\mathbf{j}}^{\varepsilon,h,n} - \mathbf{X}(h\mathbf{j}, n\tau)\|_2 + \|\mathbf{u}_{\mathbf{j}}^{\varepsilon,h,n}(t) - \mathbf{u}(\mathbf{X}(h\mathbf{j}, n\tau), n\tau)\|_2 \leq C_K \left(\frac{h^{s-d/2}}{\varepsilon^s} + \frac{\varepsilon^{\max\{m/2, L-d/2\}}}{h^{d/2}} + \frac{\tau}{\varepsilon^{d/2+2} h^{d/2}} \right) \exp\left(C_2 \frac{\tau}{\varepsilon^{d+2}} \right)$$

for each $j \in \mathbb{Z}^d$ with $h\mathbf{j} \in K$ and all $1 \leq n \leq N$.

5.3 Implicit Discretization in Time

In this section, we want to show that also an implicit time discretization scheme will lead to convergence. We will assume the same situation as in Section 5.2, including the definitions of the approximated density and the discretized energy error. The system for the implicit discretization in time is given by

$$\mathbf{x}_{\mathbf{j}}^{n+1} = \mathbf{x}_{\mathbf{j}}^{n} + \tau \mathbf{u}_{\mathbf{j}}^{n+1}, \tag{5.13}$$

$$\mathbf{u}_{\mathbf{j}}^{n+1} = \mathbf{u}_{\mathbf{j}}^{n} - \tau h^d \sum_{\mathbf{k} \in \mathbb{Z}^d} \rho_0(h\mathbf{k}) \nabla \Phi_\varepsilon(\mathbf{x}_{\mathbf{j}}^{n+1} - \mathbf{x}_{\mathbf{k}}^{n+1}) = \mathbf{u}_{\mathbf{j}}^{n} - \tau \nabla \rho^{\varepsilon,h,n+1}(\mathbf{x}_{\mathbf{j}}^{n+1}), \tag{5.14}$$

for every $\mathbf{j} \in \mathbb{Z}^d$ and $n \leq N-1$

$$\mathbf{x}_{\mathbf{j}}^{0} = h\mathbf{j}, \quad \mathbf{u}_{\mathbf{j}}^{0} = \mathbf{u}_0(h\mathbf{j}) \tag{5.15}$$

for every $\mathbf{j} \in \mathbb{Z}^d$. Analogously to Theorem 5.3, we can derive the following convergence result.

Theorem 5.6
Let $s \geq 2$ and $\Phi^r \in W^{s,2}(\mathbb{R}^d)$ be an even root kernel, which satisfies the moment condition of order $m \geq 1$ and the approximation condition of order $L > d/2$. Let $\Phi = \Phi^r * \Phi^r$ be the corresponding convolution kernel. Assume finite discrete mass and that the solution $(\mathbf{u}, \rho)$ of the Euler equations (3.1) - (3.3) *satisfies*

$$u_j \in L^\infty(0, T; W^{\eta,2}(\mathbb{R}^d)), \qquad 1 \leq j \leq d,$$
$$\rho \in L^\infty(0, T; L^1(\mathbb{R}^d)) \cap L^\infty(0, T; W^{\sigma,2}(\mathbb{R}^d)),$$

for some time $T > 0$ with $\eta > \max\{L, m\} + \frac{d}{2} + 1$ and $\sigma > m + \frac{d}{2} + 1$. Let $\tau > 0$ and let $(\mathbf{x}_{\mathbf{k}}^{n}, \mathbf{u}_{\mathbf{k}}^{n})_{\mathbf{k} \in \mathbb{Z}^d}$, $1 \leq n \leq N$, be a solution of the corresponding, fully discretized SPH equations (5.13) - (5.15)*. Then, for sufficiently small τ, there exist constants $C_1, C_2 > 0$ such that the energy can be bounded by*

$$Q^N \leq \left(Q^0 + C_1 \varepsilon^{\min\{m, 2L-d\}} + \frac{\tau^2}{\varepsilon^{d+4}} \right) \exp\left(\frac{C_2(\varepsilon^{d+2} + \tau)}{\varepsilon^{d+2}(1 - \tau C_2) - C_2 \tau^2} \right).$$

Proof. Since $\mathbf{u}$ is continuously differentiable in both arguments, using (5.13) and the mean value theorem yields

$$\begin{aligned}\mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1}, t_{n+1}) =& \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n} + \tau \mathbf{u}_{\mathbf{j}}^{n+1}, n\tau + \tau) \\ =& \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n}, n\tau) - \tau \nabla \rho(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}}, t_{n+\zeta_{\mathbf{u}}}) \\ &+ \tau \left(\left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}}, t_{n+\zeta_{\mathbf{u}}}) \right) \cdot \nabla \right) \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}}, t_{n+\zeta_{\mathbf{u}}})\end{aligned}$$

with a $\zeta_{\mathbf{u}} \in [0,1]$, where we shortly write $\mathbf{x}_{\mathbf{j}}^{n+s} = \mathbf{x}_{\mathbf{j}}^{n} + s\tau \mathbf{u}_{\mathbf{j}}^{n+1}$ and $t_{n+s} = t_n + s\tau$ for $s \in [0,1]$. Note that the definition of $\mathbf{x}_{\mathbf{j}}^{n+s}$ differs from the one we used in the proof of Theorem 5.3. Using (5.13) and (5.14), the difference of the solution and the approximated velocity yields

$$\begin{aligned}\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1}, t_{n+1}) =& \mathbf{u}_{\mathbf{j}}^{n} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n}, t_n) - \tau \left(\nabla \rho^{\varepsilon,h,n+1}(\mathbf{x}_{\mathbf{j}}^{n+1}) - \nabla \rho(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}}, t_{n+\zeta_{\mathbf{u}}}) \right) \\ &- \tau \left(\left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}}, t_{n+\zeta_{\mathbf{u}}}) \right) \cdot \nabla \right) \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}}, t_{n+\zeta_{\mathbf{u}}}).\end{aligned}$$

For the kinetic error term we have

$$\begin{aligned}Q_{kin}^{n+1} =& h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \rho_0(h\mathbf{j}) \| \mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1}, t_{n+1}) \|_2^2 \\ =& S_{0,1}^n + \tau (S_{1,1}^n + S_{1,2}^n),\end{aligned}$$

where the single terms are given by

$$\begin{aligned}S_{0,1}^n :=& h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1}, t_{n+1}) \right) \cdot \left(\mathbf{u}_{\mathbf{j}}^{n} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n}, t_n) \right), \\ S_{1,1}^n :=& - h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1}, t_{n+1}) \right) \cdot \left(\nabla \rho^{\varepsilon,h,n+1}(\mathbf{x}_{\mathbf{j}}^{n+1}) - \nabla \rho(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}}, t_{n+\zeta_{\mathbf{u}}}) \right)\end{aligned}$$

and

$$S_{1,2}^n := -h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1}, t_{n+1}) \right) \cdot \left(\left(\left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}}, t_{n+\zeta_{\mathbf{u}}}) \right) \cdot \nabla \right) \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}}, t_{n+\zeta_{\mathbf{u}}}) \right).$$

For the potential energy error term, we use the same procedure by using the mean value theorem. For the density ρ we have

$$\rho(\mathbf{x}, t_{n+1}) = \rho(\mathbf{x}, t_n + \tau) = \rho(\mathbf{x}, t_n) - \nabla \cdot (\rho \mathbf{u})(\mathbf{x}, t_{n+\zeta_\rho}),$$

with a $\zeta_\rho \in [0,1]$, where we used the Euler equation (3.1). For the approximated density, (5.13) and the mean value theorem yield

$$\begin{aligned}\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) =& h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \rho_0(h\mathbf{j}) \Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_{\mathbf{j}}^{n+1}) = h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \rho_0(h\mathbf{j}) \Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_{\mathbf{j}}^{n} + \tau \mathbf{u}_{\mathbf{j}}^{n+1}) \\ =& \rho^{\varepsilon,h,r,n}(\mathbf{x}) - \tau h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \rho_0(h\mathbf{j}) \mathbf{u}_{\mathbf{j}}^{n+1} \cdot \nabla \Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_{\mathbf{j}}^{n+\zeta_\Phi})\end{aligned}$$

with a $\zeta_\Phi \in [0,1]$. Hence, the potential energy term can be written as

$$\begin{aligned}Q_{pot}^{n+1} =& \int_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) - \rho(\mathbf{x}, t_{n+1}) \right)^2 d\mathbf{x} \\ =& S_{0,2}^n + \tau S_{1,3}^n,\end{aligned}$$

where the single terms are given by

$$S_{0,2}^n := \int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) - \rho(\mathbf{x},t_{n+1})\right) \left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x},t_n)\right) d\mathbf{x}$$

and

$$S_{1,3}^n := -\int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) - \rho(\mathbf{x},t_{n+1})\right) \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}_{\mathbf{j}}^{n+1} \cdot \nabla\Phi_\varepsilon^r(\mathbf{x} - \mathbf{x}_{\mathbf{j}}^{n+\zeta_\Phi}) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x},t_{n+\zeta_\rho})\right) d\mathbf{x}.$$

For the terms $S_{0,1}^n$ and $S_{0,2}^n$, we can apply that $ab \leq a^2/2 + b^2/2$ for $a,b\in\mathbb{R}$ to derive

$$S_{0,1}^n \leq h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\frac{1}{2}\|\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1},t_{n+1})\|_2^2 + \frac{1}{2}\|\mathbf{u}_{\mathbf{j}}^n - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^n,t_n)\|_2^2\right) = \frac{1}{2}\left(Q_{kin}^n + Q_{kin}^{n+1}\right)$$

and

$$S_{0,2}^n \leq \int\limits_{\mathbb{R}^d} \left(\frac{1}{2}\left(\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) - \rho(\mathbf{x},t_{n+1})\right)^2 + \frac{1}{2}\left(\rho^{\varepsilon,h,r,n}(\mathbf{x}) - \rho(\mathbf{x},t_n)\right)^2\right) d\mathbf{x} = \frac{1}{2}\left(Q_{pot}^n + Q_{pot}^{n+1}\right).$$

Adding both terms then yield

$$S_{0,1}^n + S_{0,2}^n \leq \frac{1}{2}\left(Q^n + Q^{n+1}\right).$$

Now we have to split up the three remaining terms. The first one can be written as $S_{1,1}^n = S_{1,1,1}^n + S_{1,1,2}^n$ with

$$S_{1,1,1}^n = -h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1},t_{n+1})\right) \cdot \left(\nabla\rho^{\varepsilon,h,n+1}(\mathbf{x}_{\mathbf{j}}^{n+1}) - \nabla\rho(\mathbf{x}_{\mathbf{j}}^{n+1},t_{n+1})\right)$$

and

$$S_{1,1,2}^n = -h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1},t_{n+1})\right) \cdot \left(\nabla\rho(\mathbf{x}_{\mathbf{j}}^{n+1},t_{n+1}) - \nabla\rho(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}},t_{n+\zeta_{\mathbf{u}}})\right).$$

The second one can be written as $S_{1,2}^n = S_{1,2,1}^n + S_{1,2,2}^n$ with

$$S_{1,2,1}^n = -h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1},t_{n+1})\right) \cdot \left(\left(\left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1},t_{n+1})\right)\cdot\nabla\right) \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}},t_{n+\zeta_{\mathbf{u}}})\right)$$

and

$$S_{1,2,2}^n = -h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j}) \left(\mathbf{u}_{\mathbf{j}}^{n+1} - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1},t_{n+1})\right) \cdot \left(\left(\left(\mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+1},t_{n+1}) - \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}},t_{n+\zeta_{\mathbf{u}}})\right)\cdot\nabla\right) \mathbf{u}(\mathbf{x}_{\mathbf{j}}^{n+\zeta_{\mathbf{u}}},t_{n+\zeta_{\mathbf{u}}})\right).$$

Finally, the third one can be written as $S^n_{1,3} = S^n_{1,3,1} + S^n_{1,3,2} + S^n_{1,3,3}$ with

$$S^n_{1,3,1} = -\int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) - \rho(\mathbf{x}, t_{n+1})\right) \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}^{n+1}_{\mathbf{j}} \cdot \nabla\Phi^r_\varepsilon(\mathbf{x} - \mathbf{x}^{n+1}_{\mathbf{j}}) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t_{n+1})\right) d\mathbf{x}$$

$$S^n_{1,3,2} = -\int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) - \rho(\mathbf{x}, t_{n+1})\right) \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \rho_0(h\mathbf{j})\mathbf{u}^{n+1}_{\mathbf{j}} \cdot \left(\nabla\Phi^r_\varepsilon(\mathbf{x} - \mathbf{x}^{n+\zeta_\Phi}_{\mathbf{j}}) - \nabla\Phi^r_\varepsilon(\mathbf{x} - \mathbf{x}^{n+1}_{\mathbf{j}})\right)\right) d\mathbf{x}$$

and

$$S^n_{1,3,3} = -\int\limits_{\mathbb{R}^d} \left(\rho^{\varepsilon,h,r,n+1}(\mathbf{x}) - \rho(\mathbf{x}, t_{n+1})\right) \left(\nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t_{n+\zeta_\rho}) - \nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t_{n+1})\right) d\mathbf{x}.$$

All these terms can be identified with terms that occurred in the proof of Theorem 5.3. To distinguish the terms of this proof and the terms occurring in the proof of Theorem 5.3, we will tag the terms of the proof of Theorem 5.3 with an ex for explicit euler method, for example $S^{n,\mathrm{ex}}_{1,1,1}$. First of all, we have $S^n_{1,1,1} = S^{n+1,\mathrm{ex}}_{1,1,1}/2$ and $S^n_{1,3,1} = S^{n+1,\mathrm{ex}}_{1,3,1}/2$, so that we have

$$S^n_{1,1,1} + S^n_{1,3,1} \leq C(\mathbf{u}, \rho, M)(\varepsilon^m + \varepsilon^{2L-d} + Q^{n+1}).$$

The five remaining terms $S^n_{1,1,2}$, $S^n_{1,2,1}$, $S^n_{1,2,2}$, $S^n_{1,3,2}$ and $S^n_{1,3,3}$ can be identified with $S^{n+1,\mathrm{ex}}_{1,1,2}$, $S^{n+1,\mathrm{ex}}_{1,2,1}$, $S^{n+1,\mathrm{ex}}_{1,2,2}$, $S^{n+1,\mathrm{ex}}_{1,3,2}$ and $S^{n+1,\mathrm{ex}}_{1,3,3}$, respectively. Note that these terms are not exactly the same, but differ only to the point that they satisfy the same error bounds under the same argumentation. Hence, we have

$$\begin{aligned}
S^n_{1,1,2} &\leq c\left(Q^{n+1} + \tau^2(1 + Q^{n+1}\right), \\
S^n_{1,2,1} &\leq cQ^{n+1}, \\
S^n_{1,2,2} &\leq c\left(Q^{n+1} + \tau^2(1 + Q^{n+1}\right), \\
S^n_{1,3,2} &\leq c\left(Q^{n+1} + Q^{n+1}\frac{\tau}{\varepsilon^{d+2}} + \frac{\tau^2}{\varepsilon^{d+4}}\right)
\end{aligned}$$

and

$$S^n_{1,3,3} \leq Q^{n+1} + c\tau^2,$$

so that we have the complete estimate

$$\begin{aligned}
\tau\left(S^n_{1,1} + S^n_{1,2} + S^n_{1,3}\right) &\leq c\tau\left[Q^{n+1}\left(1 + \tau^2 + \frac{\tau}{\varepsilon^{d+2}}\right) + \varepsilon^m + \varepsilon^{2L-d} + \tau^2 + \frac{\tau^2}{\varepsilon^{d+4}}\right] \\
&\leq C\left[Q^{n+1}\tau\left(1 + \frac{\tau}{\varepsilon^{d+2}}\right) + \tau\left(\varepsilon^{\min\{m,2L-d\}} + \frac{\tau^2}{\varepsilon^{d+4}}\right)\right],
\end{aligned}$$

where we used that $\tau \leq 1$. Hence, we have that

$$Q^{n+1} \leq \frac{1}{2}\left(Q^n + Q^{n+1}\right) + Q^{n+1}C\tau\left(1 + \frac{\tau}{\varepsilon^{d+2}}\right) + \tau C\left(\varepsilon^{\min\{m,2L-d\}} + \frac{\tau^2}{\varepsilon^{d+4}}\right),$$

which gives

$$Q^{n+1} \leq Q^n + Q^{n+1}\tau 2C\left(1+\frac{\tau}{\varepsilon^{d+2}}\right) + 2\tau C\left(\varepsilon^{\min\{m,2L-d\}} + \frac{\tau^2}{\varepsilon^{d+4}}\right).$$

Hence, using the discrete version of Gronwall's inequality Lemma 5.1, we find constants $C_1 > 0$ and $c_2 > 0$ so that

$$Q^n \leq \left[Q^0 + C_1\left(\varepsilon^{\min\{m,2L-d\}} + \frac{\tau^2}{\varepsilon^{d+4}}\right)\right]\exp\left(\frac{c_2\left(1+\frac{\tau}{\varepsilon^{d+2}}\right)}{1-\tau c_2\left(1+\frac{\tau}{\varepsilon^{d+2}}\right)}\right).$$

Setting $C_2 = 1/c_2$ finishes the proof. □

As we did for the explicit time discretization scheme, we can derive a general convergence result as in Corollary 5.4 and a pointwise convergence result as in Theorem 5.5 in the case of the implicit time discretization. Since nothing changes in both results, reference is made to Corollary 5.4 and Theorem 5.5 at this point.
Both time stepping methods, the explicit Euler method and the implicit Euler method, do not differ in their convergence rate. It is a well known fact that implicit methods are the better choice in terms of stability. For this reason, a fully stability analysis would have to be done. Unfortunately, standard methods fail in the analysis due to the non-linearity of the SPH method, so the analysis is a current part of research, see, for example, [DA12].
From a numerical point of view, however, explicit procedures are the better choice. In the implicit methods, a nonlinear system of equations must be solved at each time step, which would be inefficient.

Chapter 6

Numerical Experiments

In this chapter, we want to verify the convergence results from Chapter 3. Therefore, we compare the solution of the Euler equations (3.1) - (3.3) with the numerical solution of the SPH method in two different cases.
In the first case, we will set an initial density with compact support to have a test case which is suitable for the requirements of Corollary 3.11. For the second case, we will use periodic boundary conditions to compare the solutions in a non compactly-supported test case. Even though periodic boundary conditions do not satisfy the requirements of Corollary 3.11, we will expect the same convergence results according to Section 3.5.
First of all, we will need an non-trivial, analytical solution to the Euler equations. This is only insufficiently possible for our weakly compressible case, but in Section 6.1 we will show how to calculate a quasi-analytical solution by solving Burger's equation. This procedure has also been used in [Mol08] to find solutions of the shallow water equations. In the second section, we will then validate the results from Chapter 3 for both test cases.

6.1 Burgers' Equation and a Quasi-Analytical Solution of the Euler Equations

In this section, we will introduce Burgers' equation and will show a way to calculate quasi-analytical solutions to the one dimensional Euler equations once without boundary conditions and once with periodic boundary conditions. For the case of periodic boundary conditions, we will use the notation from Section 3.5. Since the following theory is the same on $\mathbb{R}$ and on $\mathbb{T}$, we will not distinguish both cases.
Let $\Omega = \mathbb{R}$ or $\mathbb{T}$. For given initial data $u_0 : \Omega \to \mathbb{R}$ and $\rho_0 : \Omega \to \mathbb{R}$, we seek the solution $u : \Omega \times [0, T[\to \mathbb{R}$ and $\rho : \Omega \times [0, T[\to \mathbb{R}$ of

$$\partial_t u + u\partial_x u = -\partial_x \rho \tag{6.1}$$

$$\partial_t \rho + u\partial_x \rho = -\rho\partial_x u \tag{6.2}$$

on $\Omega\times]0, T[$ and

$$u(\cdot, 0) = u_0, \quad \rho(\cdot, 0) = \rho_0 \tag{6.3}$$

on Ω.
Our main goal is to find a non-trivial analytical solution to this problem to which a numerical solution can be compared. For this, we will investigate the inviscid Burgers equation: For given initial data $f : \Omega \to \mathbb{R}$ we seek the solution $J : \Omega \times [0, T[\to \mathbb{R}$ of

$$\partial_t J + J\partial_x J = 0 \tag{6.4}$$

on $\Omega\times]0,\infty[$ and

$$J(\cdot,0) = f \tag{6.5}$$

on Ω. Before we show how to derive an analytical solution of the Euler equations from the solution of Burgers' equation, we have to investigate the solvability of the inviscid Burgers equation. Therefore, we will need the following Lemma.

Lemma 6.1
Let $f \in C^1(\Omega)$ be bounded, $x \in \Omega$ and $0 \le t < T$, where T is given by

$$T := \begin{cases} \infty & \text{if } \min_{x\in\Omega} f'(x) \ge 0, \\ \frac{-1}{\min_{x\in\Omega} f'(x)} & \text{if } \min_{x\in\Omega} f'(x) < 0. \end{cases} \tag{6.6}$$

Then, the nonlinear equation $x = y(x,t) + tf(y(x,t))$ has a unique solution $y(x,t) \in \mathbb{R}$. Moreover, $y : \Omega \times [0,T[\to \mathbb{R}$ is continuously differentiable in both variables.

Proof. We will prove this lemma first for $\Omega = \mathbb{R}$. Let $g_t : \mathbb{R} \to \mathbb{R}$, $y \mapsto y + tf(y)$ for $t > 0$. First, using that f is bounded, we have that $g_t(y)$ tends to $\pm\infty$ if y tends to $\pm\infty$. Since f is continuous, g_t is continuous, too. Hence, g_t is surjective.
Moreover, g_t is continuously differentiable on $\mathbb{R}$ since $f \in C^1(\mathbb{R})$, and the derivative yields $g_t' = 1 + tf'$. Hence, for all $y \in \mathbb{R}$ we have $g_t'(y) > 0$ either for all $t > 0$ if $\min_{y\in\mathbb{R}} f'(y) \ge 0$ or for all $t < -(\min_{y\in\mathbb{R}} f'(y))^{-1}$ if $\min_{y\in\mathbb{R}} f'(y) < 0$. This means that g_t is strictly monotonically increasing for all $0 \le t < T$ and hence it is injective.
Overall, g_t is bijective and there exists an inverse function $g_t^{-1} : \mathbb{R} \to \mathbb{R}$ of g_t. Setting $y(x,t) = g_t^{-1}(x)$ for all $x \in \mathbb{R}$ and $t < T$ finishes this part of the proof. Moreover, the implicit function theorem states that $y : \mathbb{R} \times [0,T[\to \mathbb{R}$ is continuously differentiable.
For $\Omega = \mathbb{T}$, we choose $g_t : \mathbb{T} \to [tf(0), 1+tf(1)[$, $y \mapsto y+tf(y)$. Hence, g_t is surjective and injective again, and the proof can be adopted from the proof of $\Omega = \mathbb{R}$. ☐

Using this result, we can prove the following local existence result for the Burgers equation.

Theorem 6.2
Let $f \in C^1(\Omega)$ be bounded. Let $J(x,t) = f(y(x,t))$, where $y(x,t)$ is the solution of the nonlinear equation $x = y(x,t) + tf(y(x,t))$ for $x \in \Omega$ and $0 \le t < T$. Then, J is a solution of the inviscid Burgers equation (6.4) *up to a maximum time $T > 0$, where T is given in* (6.6).

Proof. The derivation of the solution is based on the method of characteristics. In our case we can simply calculate the derivatives of J.
First of all, Lemma 6.1 states that the solution of the nonlinear equation exists up to the time T from (6.6). Hence, J exists for all $t < T$. Moreover, for $t = 0$ we have that $y(x,0) = x$ and hence that $J(x,0) = f(x)$ for all $x \in \Omega$.
For $0 < t < T$, we know that y is the solution of the nonlinear equation $x = y(x,t) + tf(y(x,t))$ for all $x \in \Omega$ and we can differentiate both sides with respect to x and t which yields

$$1 = \partial_x y(x,t) + tf'(y(x,t))\partial_x y(x,t) = \partial_x y(x,t)(1 + tf'(y(x,t)))$$

and

$$0 = \partial_t y(x,t) + tf'(y(x,t))\partial_t y(x,t) + f(y(x,t)) = \partial_t y(x,t)(1 + tf'(y(x,t))) + f(y(x,t)).$$

Multiplying the first equation with $f(y(x,t))$, subtracting $f(y(x,t))$ from the second equation and adding both results yields

$$\begin{aligned} 0 &= f(y(x,t))\partial_x y(x,t)\big(1 + tf'(y(x,t))\big) + \partial_t y(x,t)\big(1 + tf'(y(x,t))\big) \\ &= \big(1 + tf'(y(x,t))\big)\big(f(y(x,t))\partial_x y(x,t) + \partial_t y(x,t)\big), \end{aligned}$$

which means that $f(y(x,t))\partial_x y(x,t) + \partial_t y(x,t) = 0$ for all $0 \le t < T$ since $1 + tf'(y(x,t)) > 0$. Then, inserting this particular J into the left-hand side of Burgers' equation (6.4) yields

$$\begin{aligned}\partial_t J(x,t) + J(x,t)\partial_x J(x,t) &= \partial_t f(y(x,t)) + f(y(x,t))\partial_x f(y(x,t)) \\ &= f'(y(x,t))\partial_t y(x,t) + f(y(x,t))f'(y(x,t))\partial_x y(x,t) \\ &= f'(y(x,t))\big(\partial_t y(x,t) + f(y(x,t))\partial_x y(x,t)\big) = 0,\end{aligned}$$

which finishes the proof. □

Theorem 6.2 gives us the existence of the solution of the inviscid Burger's equation. Unfortunately, it also states this solution will only be local in time if the initial value f is not monotonically increasing, which is not the case if we choose f to be compactly supported.
Moreover, Theorem 6.2 does not offer us an analytical presentation of the solution. However, the nonlinear equation $x = y(x,t) + tf(y(x,t))$ can be solved sufficiently accurately using a numerical solver like Newton's method.
The following result allows us to construct a solution for the Euler equations from the solution of the inviscid Burgers equation.

Theorem 6.3
Let $f \in C^1(\Omega)$ be bounded and let $J : \Omega \times [0,T[\to \mathbb{R}$ be a solution of the inviscid Burgers equation (6.4) - (6.5) *up to the time $T > 0$ from* (6.6)*. Let $u_0 := \frac{2}{3}f + 1$ and $\rho_0 := (1 - \frac{1}{3}f)^2$ on Ω. Then, $u : \Omega \times [0,\infty[\to \mathbb{R}$ and $\rho : \Omega \times [0,\infty[\to \mathbb{R}$ defined by*

$$u = \frac{2}{3}J + 1, \quad \rho = \left(1 - \frac{1}{3}J\right)^2,$$

are a solution of the one dimensional Euler equation (6.1) - (6.3).

Proof. For the initial time $t = 0$, the initial conditions for u and ρ are obviously satisfied. Now let $0 < t \le T$. First note that according to equation (6.4) we have $\partial_t J = -J\partial_x J$. Hence, we have

$$\begin{aligned}\partial_t u + u\partial_x u &= \frac{2}{3}\partial_t J + \left(\frac{2}{3}J + 1\right)\frac{2}{3}\partial_x J = -\frac{2}{3}J\partial_x J + \left(\frac{2}{3}J + 1\right)\frac{2}{3}\partial_x J \\ &= \left(1 - \frac{1}{3}J\right)\frac{2}{3}\partial_x J = -\partial_x\left(1 - \frac{1}{3}J\right)^2 = -\partial_x \rho\end{aligned}$$

such that equation (6.1) is satisfied. For the continuum equation (6.2), a simple calculation shows that

$$\begin{aligned}\partial_t \rho + u\partial_x \rho &= \partial_t\left(1 - \frac{1}{3}J\right)^2 + \left(\frac{2}{3}J + 1\right)\partial_x\left(1 - \frac{1}{3}J\right)^2 \\ &= -\frac{2}{3}\left(1 - \frac{1}{3}J\right)\left(\partial_t J + \left(\frac{2}{3}J + 1\right)\partial_x J\right) \\ &= -\frac{2}{3}\left(1 - \frac{1}{3}J\right)\left(-J\partial_x J + \left(\frac{2}{3}J + 1\right)\partial_x J\right) \\ &= -\left(1 - \frac{1}{3}J\right)\left(1 - \frac{1}{3}J\right)\frac{2}{3}\partial_x J \\ &= -\left(1 - \frac{1}{3}J\right)^2\partial_x\left(\frac{2}{3}J + 1\right) = -\rho\partial_x u,\end{aligned}$$

which completes the proof. □

According to Theorem 6.3, a solution of the Euler equations can be calculated from the solution of the inviscid Burgers equation. As mentioned before, we are unable to derive an analytical solution to Burgers' equation but can calculate an arbitrarily accurate numerical solution by solving the nonlinear equation $x = y(x,t) + tf(y(x,t))$ with a numerical method, e.g. Newton's method. Since this approximated solution will be calculated with arbitrary accuracy, it will be sufficiently exact to be compared to the numerical solution of the SPH method. We will call this arbitrarily accurate numerical solution a quasi-analytical solution.

6.2 Results

In the following, we want to validate the theoretical results from Section 3.3 with the constructed kernel functions of Chapter 4. To do this, we implemented a C++ code which is based on the SPH scheme (3.4) - (3.6) coupled with a Runge-Kutta ODE solver.
As the kernel function we chose the extended original Wendland functions from Section 4.3.2. To compare the influence of the the smoothness k of the kernel, the moment condition order $2m$ and the parameter l, we took the parameters $k \in \{1,2,3,4\}$, $l \in \{0,1,2,3,4\}$ and $m \in \{1,2,3\}$. Note that for $l = 0$, the employed kernels do not fit in the derived theory and hence, convergence is theoretically not given. Nevertheless, it may be helpful to investigate this case as it may provide an indication of whether the theory derived here could be further improved.
To compare the error between the SPH algorithm and the quasi-analytical solution, we have to track the energy error term $Q = Q_{kin} + Q_{pot}$ from Definition 3.7. Unfortunately, it is not possible to calculate the potential energy error, since we neither have an explicit form of the root kernel Φ^r nor can we calculate the occurring integral exactly. Hence, we have to use an estimator for the potential energy error.
Both error terms we track are given by

$$Q_{kin}(t) = h^d \sum_{j \in \mathbb{J}} \rho_0(hj) \| u_j^{\varepsilon,h}(t) - \widetilde{u}(x_j^{\varepsilon,h}(t),t) \|_2^2$$

and

$$Q_{pot}(t) = h^d \sum_{j \in \mathbb{J}} \rho_0(hj) | \rho^{\varepsilon,h}(x_j^{\varepsilon,h}(t),t) - \widetilde{\rho}(x_j^{\varepsilon,h}(t),t) |,$$

where $\mathbb{J} = \mathbb{Z}$ if $\Omega = \mathbb{R}$ or $\mathbb{J} = \mathbb{G}_N$ if $\Omega = \mathbb{T}$, the pair $(\widetilde{u}, \widetilde{\rho})$ is the quasi-analytical solution calculated from the Euler equations (6.1) - (6.3) by solving the inviscid Burgers equation, and $(x_j, u_j)_{j \in \mathbb{J}}$ is the solution of the SPH system (3.4) - (3.6) or (3.24) - (3.26), respectively.
To solve the ODE to calculate the SPH solution, an explicit fourth order Runge-Kutta time-stepping algorithm was used. The time step width was chosen small enough so that the solution is sufficiently accurate to neglect the error occurring in the time discretization.
In the following, we want to calculate the convergence rate of the SPH method depending on the parameters h and ε in two different test cases. In the first test case, the Euler equations are solved on $\mathbb{R}$ with a compactly supported initial density. Even if this test case does not satisfy all of the conditions of Theorem 3.9 since u does not decay sufficiently fast, it will be sufficient to validate the numerical results.
In the second test case, the Euler equations are solved on the periodic domain $\mathbb{T}$ to validate our thoughts of Section 3.5 for extending our results to $\mathbb{T}$.

6.2.1 Test Case on $\mathbb{R}$

In the first case the one dimensional Euler equations (6.1) and (6.2) are solved on $\mathbb{R}$ with initial data $u_0 := \frac{2}{3}f + 1$ and $\rho_0 := (1 - \frac{1}{3}f)^2$, where the function f is given by

$$f(x) := \begin{cases} 3 - c_0 \exp(1/(4x^2 - 1)), & |x - \frac{1}{2}| < \frac{1}{2}, \\ 3, & \text{else}, \end{cases}$$

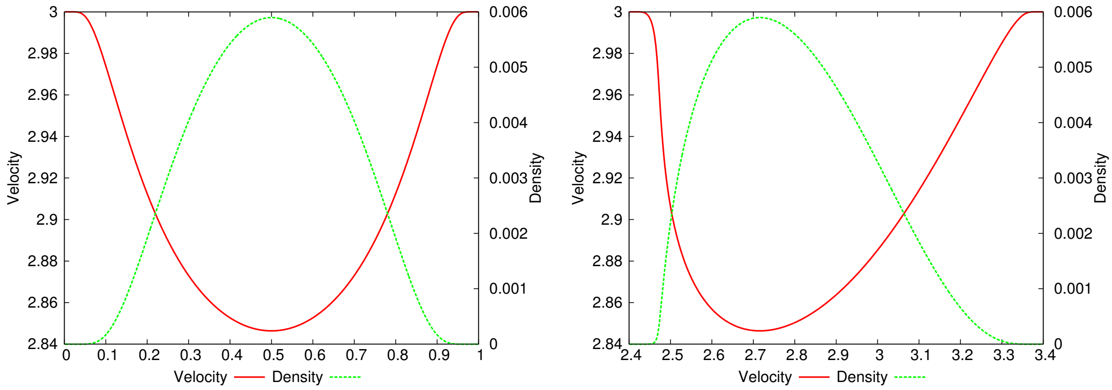

Figure 6.1: Velocity (red) and density (green) at time $t = 0$ (left) and $t = 0.8$ (right) for the solution of the Euler equations on $\mathbb{R}$. Note that the values at the x-axis differ since the initial wave is moving in time.

for $x \in \mathbb{R}$ with the constant

$$c_0 := \frac{\sqrt[4]{3}e^{\frac{3}{3-\sqrt{3}}}(4-\sqrt{3})}{12}.$$

The function f is in $C^\infty(\mathbb{R})$ and the constant c_0 was chosen so that $\min f' = -1$ on $\mathbb{R}$. According to Theorem 6.2, the solution of the Burgers equation exists up to $T = 1$. Hence, the analytical solution of the Euler equations exists up to $T = 1$, as well. To neglect any effect of the blow-up of the solution, the error between the quasi-analytical solution and the SPH solution was taken at time $t = 0.8$.
By the choice of the function f, the initial density ρ_0 has compact support, which results in a finite number of particles in the SPH scheme. However, the initial velocity is not integrable any more since it is constant 3 for all $|x - 1/2| > 1/2$. Hence, u_0 does not satisfy the requirements of Corollary 3.11 any more. However, our interest lies in a compact domain, such that we can assume the initial velocity to decay sufficiently fast at infinity, without changing the solution in our domain. The analytical solution of the Euler equations can be seen in Figure 6.1. In this first experiment, we set the parameters to $h = \varepsilon^2$. The single pairs of (h, ε) can be seen in Table 6.1.

ε	1.25e-1	1e-1	8.84e-2	6.25e-2	5e-2	2.5e-2	1e-2
h	1.5625e-2	1e-2	7.8125e-3	3.90625e-3	2.5e-3	6.25e-4	1e-4

Table 6.1: Values for ε and h with $h = \varepsilon^2$.

This choice was made to achieve a theoretical error which depends on the minimum of the kernel parameters k and m. According to Corollary 4.49 we expect that the energy error term Q behaves like

$$Q(t) \sim \varepsilon^{a_{\text{ana}}},$$

where the analytical constant is given by $a_{\text{ana}} = a_{\text{ana}}(k, m) = \min\{2k, 2m\}$. The goal is to determine the numerical constant $a_{\text{num}} = a_{\text{num}}(k, m, l)$ depending on all three kernel parameters. Note that we also investigate the dependency of the numerical constant on the parameter l, which is why a_{num} depends also on l. For this case, we set the time step to $\tau = 10^{-5}$ to neglect the error of the time discretization. Then, for each pair h, ε and each kernel function, the error was measured at time $t = 0.8$.
From these values, the constant a_{num} was calculated via the linear regression method for each kernel function, which means for each parameter constellation of k, l and m. However in some cases, especially for high values for the moment condition m, the error reached a kind of saturation, which

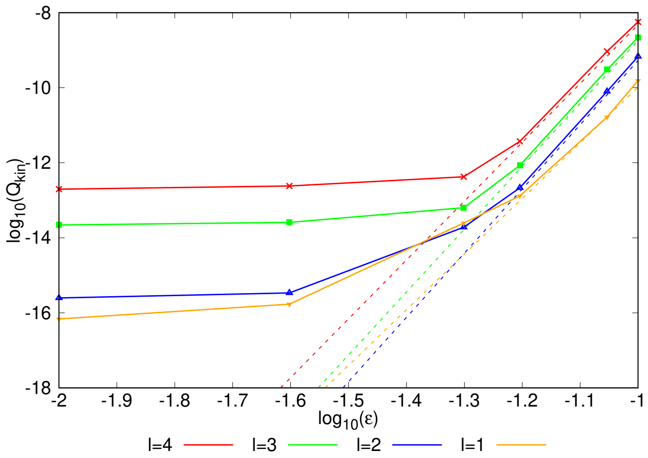

Figure 6.2: The error Q_{kin} at the $t = 0.8$ in the case of $\mathbb{R}$ as a function of ε with $h = \varepsilon^2$. The error is shown for kernel functions with $k = 4$, $m = 3$ and $l \in \{1, 2, 3, 4\}$. The dashed lines represent the asymptotic behavior of the error.

we discuss later, so that taking all (h, ε)-pair for the calculation of the constant a_{num} would distort the result, see Figure 6.2. For that reason, only the measured errors, for which the asymptotic behavior was recognizable, were taken for the calculation.
The resulting convergence rates can be found in Table 6.2. Note that the lower convergence rates of the potential error term can be explained since the potential error term is just an estimator.

		$k=1$			$k=2$			$k=3$			$k=4$		
l	m	$a_{\text{num}}^{\text{kin}}$	$a_{\text{num}}^{\text{pot}}$	a	$a_{\text{num}}^{\text{kin}}$	$a_{\text{num}}^{\text{pot}}$	a	$a_{\text{num}}^{\text{kin}}$	$a_{\text{num}}^{\text{pot}}$	a	$a_{\text{num}}^{\text{kin}}$	$a_{\text{num}}^{\text{pot}}$	a
0	1	3.30	1.91	-	3.39	1.93	-	3.45	1.94	-	3.50	1.94	-
	2	5.29	2.58	-	5.52	3.26	-	5.69	3.28	-	5.16^a	3.72^c	-
	3	5.18	2.68	-	7.32^a	5.02	-	8.46^b	7.12^a	-	13.24^c	9.13^c	-
1	1	3.36	1.92	2	3.43	1.93	2	3.48	1.94	2	3.52	1.95	2
	2	5.48	3.70	2	5.65	3.77	4	5.79	3.68	4	5.27^a	4.70^c	4
	3	7.53	3.99	2	8.57^a	5.82	4	11.39^b	7.53^a	4	15.73^c	9.19^c	6
2	1	3.42	1.92	2	3.47	1.94	2	3.51	1.95	2	3.55	1.95	2
	2	5.81	3.88	2	5.81	4.38	4	5.92	4.22	4	5.50^a	5.87^c	4
	3	7.87	3.97	2	10.10^a	5.85	4	13.49^b	7.53^a	6	16.56^c	8.78^c	6
3	1	3.47	1.90	2	3.51	1.94	2	3.54	1.95	2	3.57	1.95	2
	2	6.39	3.94	2	6.11	4.92	4	6.07	4.64	4	5.85^a	6.76^c	4
	3	7.88	3.94	2	11.03^a	5.80	4	14.07^b	7.34^a	6	15.71^c	8.35^c	6
4	1	3.51	1.88	2	3.54	1.94	2	3.57	1.95	2	3.59	1.96	2
	2	7.06	3.94	2	6.28	5.32	4	6.37^a	4.28	4	6.57^a	7.76^c	4
	3	7.71	3.89	2	11.07^a	5.72	4	13.86^b	7.06^a	6	14.46^c	7.81^c	6

Table 6.2: All values for $a_{\text{num}}^{\text{kin}}$ and $a_{\text{num}}^{\text{pot}}$ in $Q_{kin} \sim \varepsilon^{a_{\text{num}}^{\text{kin}}}$ and $Q_{pot} \sim \varepsilon^{a_{\text{num}}^{\text{pot}}}$ compared to its analytical value $a_{\text{ana}} = \min\{2k, 2m\}$, if $l \geq 0$, in the case of $\mathbb{R}$. Values indicated with an a are calculated without the error for $\varepsilon = 0.01$. Values indicated with a b are calculated only with the errors for $\varepsilon \in \{0.125, 0.1, 0.0884, 0.0625, 0.05\}$. Values indicated with a c are calculated only with the errors for $\varepsilon \in \{0.125, 0.1, 0.0884, 0.0625\}$.

As we see in Table 6.2, the convergence rate depends strongly on the kernel parameters, especially on the order $2m$ of the moment condition. For higher values of the moment condition parameter m we see that the convergence rates improve remarkably.
For $m = 1$ and $m = 2$, we see that the convergence rate is only slightly increasing if we increase the smoothness k of the kernel or the parameter l, as the changes only show up at the second digit. It seems that the convergence rate is limited by the moment condition in those cases. Interestingly,

we see comparable results for the cases $l = 0$ and $l \geq 1$.
For $m = 3$, we note a great improvement in most of the convergence rates. In this case, the convergence rate seems mostly to be limited by the smoothness k of the kernel, which can in especially be seen in the case $k = 1$. We also see small improvements of the convergence rate when we increase the parameter l as long as $l \leq 2$, while the convergence rate does not further improve for $l \geq 3$. Interestingly, the greatest difference of the convergence rate for the parameter l for Q_{kin} is between $l = 0$ and $l = 1$, which fits in our theory since for $l = 0$ the kernel functions do not satisfy condition (4.22). Nevertheless, the numerical convergence rates do not match the theoretical ones. However, the theoretical convergence rates are always lower than the numerical ones, so that the theoretical values could be seen as a lower bound to the numerical ones. Moreover, we see that the SPH algorithm is converging, even in the case of $l = 0$, where convergence where theoretically not proven. However, these observations may also be due to the fact that the chosen example is too good in some sense.

Saturation

In the left picture of Figure 6.3, we see that the errors for approximations using kernel functions with a higher parameter l are greater than those for a low value for l. In particular, for $m = 2$, the error for the kernel functions with $l \leq 2$ seem to have the same temporal progress while for $l = 3$ and $l = 4$ the error is greater up to a certain time depending on l. Especially, the initial error for $l \geq 3$ seems bigger than for $l \leq 2$. However, in the course of time, the error increases less strongly. As we see, there is a time t which depends on l, from which the error for $l = 3$ and $l = 4$ is smaller than for $l \leq 2$. From this time on both cases have the same further temporal progress.
For $m = 3$, we have the same effect more strongly. The error for $l \leq 1$ seem to have the same temporal progress while for $l \geq 2$ the error is significantly greater. However, there exists a time t, from which the error for the cases $l \geq 2$ is smaller than for $l \leq 1$.
Hence, we can separate the temporal progress of the error into two parts as shown in the right picture of Figure 6.3. In the first part the error is dominated by the initial error, which seems to increase if the value of the parameter l is increasing. However, in the first part the error only increases slowly in time. In the second part the error is increasing strongly in time, which seems that the error follows the expected temporal progress.

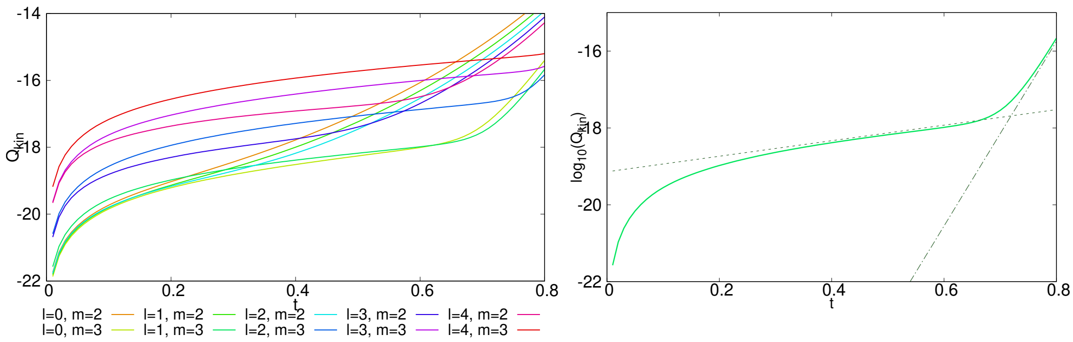

Figure 6.3: The error Q_{kin} as a function of time $0 \leq t \leq 0.8$ for the case $h = 0.000625$ and $\varepsilon = 0.025$. On the left, the error is shown for kernel functions with $k = 3$, $l \in \{0, 1, 2, 3, 4\}$ and $m \in \{2, 3\}$. On the right, the error is shown for the kernel with $k = 3$, $l = 1$ and $m = 3$, together with the asymptotes for small times (dotted line) and the asymptotes for larger times (dashed/dotted line).

This effect appears stronger if we use small values for ε, and can be strong enough that the time, where the error is in the first part, is higher than the time of the experiment $t = 0.8$. In this case, the error does not get smaller if we decrease ε any more and it seems that the error has reached a saturation. However, this is not true if we would look at longer time intervals. To neglect this effect, it would be possible to increase the maximum time T. However, this has not been possible since the analytical solution of the problem we solve does only exist up to $T = 1$.

6.2.2 Second Experiment

In the second experiment, the one dimensional periodic Euler's equations (6.1) and (6.2) were solved on $\mathbb{T}$ with initial data $u_0 := \frac{2}{3}f + 1$ and $\rho_0 := (1 - \frac{1}{3}f)^2$, where the function f is given by $f(x) = \sin(2\pi x)/(2\pi)$ for $x \in \mathbb{T}$. The function f is in $C^\infty(\mathbb{T})$ with $\min f' = -1$ on $\mathbb{T}$. Again, the solution of Burgers' equation, and hence of Euler's equations exists up to $T = 1$. The error was measured at $t = 0.8$ to avoid effects of the blow up of the solution at $T = 1$. The solution of Euler's equation can be seen in Figure 6.4.

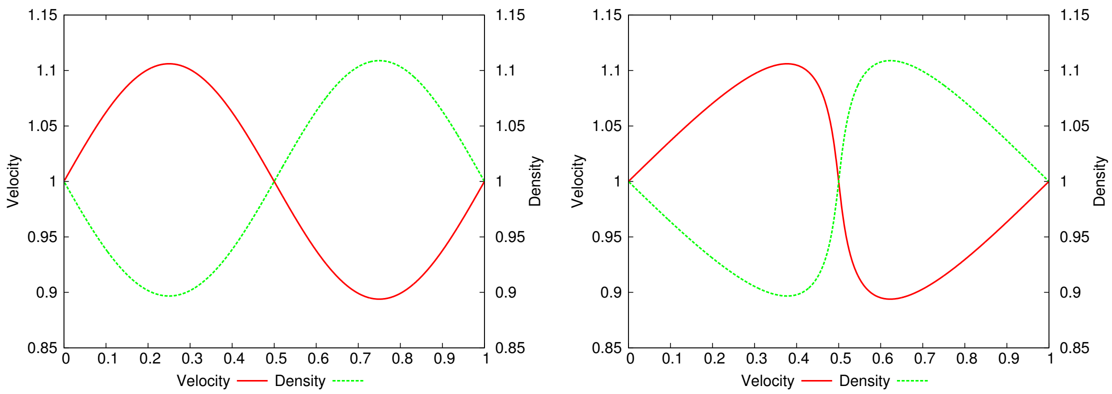

Figure 6.4: Velocity (red) and density (green) at the time $t = 0$ (left) and $t = 0.8$ (right) for the solution on $\mathbb{T}$.

In this second experiment, we set the parameters to $h = 0.1\varepsilon^2$. The factor 0.1 was chosen for stability reasons. The single pairs of (h, ε) can be seen in Table 6.3.
According to Corollary 4.49 we expect that the energy error term Q behaves like

$$Q(t) \sim 0.1^{2k}\varepsilon^{2k} + \varepsilon^{a_{\text{ana}}},$$

where the analytical constant is given by $a_{\text{ana}} = 2m$. The goal is to determine the numerical constant $a_{\text{num}} = a_{\text{num}}(k, l, m)$ depending on the kernel parameters. For this case, we set the time step to $\tau = 10^{-5}$ to neglect the error of the time discretization.

ε	2.5e-1	2e-1	1.25e-1	1e-1	6.25e-2	5e-2	2.5e-2
h	6.25e-3	4.0e-3	1.5625e-3	1.e-3	3.90625e-4	2.5e-4	6.25e-5

Table 6.3: Size of ε and h with $h = \varepsilon^2/10$.

As in the last case, for each pair h, ε and each kernel function, the error was measured at the time $t = 0.8$. The constant a_{num} was then calculated by the linear regression method. As in the first experiment, we have a kind of saturation, too. That is the reason we will calculate the constant a_{num} only using the data points where the asymptotic behavior is recognizable as we did in the first experiment.
As we see in Table 6.4, the convergence rates behave similar to those in the first experiment, but without any strong dependence on the smoothness k of the kernel function. This can be explained by the choice of $h = 0.1\varepsilon^2$, where the constant in front of the ε^{2k} part will be very small, so that the influence of the smoothness does not show up in the convergence rate.
We also see that the parameter l has hardly any effect on the convergence rate, so that the convergence rate is dominated by the order $2m$ of the moment condition. This fits in the theory for $l > 0$, since the kernels satisfy the approximation condition of arbitrary order. Interestingly,

the case $l = 0$ shows the same convergence rates, even if this case does not fit in the theory.
Again, the lower convergence rates of the potential error term can be explained since the potential error term is just an estimator.
Nevertheless, the numerical convergence rates do not match the theoretical ones. However, the theoretical convergence rates are always lower then the numerical ones, so that the theoretical values can be seen as a lower bound for the numerical ones. Moreover, we see that the SPH algorithm is converging, even in the case of $l = 0$, which is not covered by Corollary 3.11. However, these observations may also be due to the fact that the chosen example is too good in some sense.

		$k=1$			$k=2$			$k=3$			$k=4$		
l	m	$a_{\text{num}}^{\text{kin}}$	$a_{\text{num}}^{\text{pot}}$	a	$a_{\text{num}}^{\text{kin}}$	$a_{\text{num}}^{\text{pot}}$	a	$a_{\text{num}}^{\text{kin}}$	$a_{\text{num}}^{\text{pot}}$	a	$a_{\text{num}}^{\text{kin}}$	$a_{\text{num}}^{\text{pot}}$	a
0	1	2.51	1.56	-	2.61	1.60	-	2.69	1.64	-	2.75	1.66	-
	2	4.21	2.45	-	4.48	2.52	-	4.68	2.55	-	4.82	2.55	-
	3	5.26	2.76^a	-	5.24^a	2.91^a	-	5.52^a	2.96^a	-	5.74^a	2.98^b	-
1	1	2.59	1.59	2	2.67	1.63	2	2.73	1.66	2	2.79	1.68	2
	2	4.42	2.50	2	4.62	2.54	4	4.78	2.56	4	4.89	2.55^a	4
	3	5.24	2.88^a	2	5.44^a	2.95^a	4	5.68^a	2.97^a	6	5.84^a	2.90^b	6
2	1	2.66	1.63	2	2.72	1.65	2	2.78	1.67	2	2.83	1.69	2
	2	4.59	2.53	2	4.75	2.55	4	4.87	2.55	4	4.95	2.59^a	4
	3	5.19	2.93^a	2	5.62^a	2.96^a	4	5.80^a	2.96^b	6	5.84^a	2.94^b	6
3	1	2.73	1.65	2	2.77	1.67	2	2.82	1.69	2	2.86	1.71	2
	2	4.74	2.54	2	4.85	2.55	4	4.93	2.55	4	4.76	2.59^a	4
	3	5.11	2.96^a	2	5.78^a	2.95^a	4	5.85^a	2.97^b	6	4.99^b	2.57^c	6
4	1	2.79	1.68	2	2.82	1.69	2	2.86	1.70	2	2.90	1.72	2
	2	4.86	2.54	2	4.93	2.54	4	4.96	2.55^a	4	4.98	2.63^a	4
	3	5.03	2.98^a	2	5.88^a	2.91^a	4	5.74^a	2.86^b	6	4.70^c	1.93^c	6

Table 6.4: All values for $a_{\text{num}}^{\text{kin}}$ and $a_{\text{num}}^{\text{pot}}$ in $Q_{kin} \sim \varepsilon^{a_{\text{num}}^{\text{kin}}}$ and $Q_{pot} \sim \varepsilon^{a_{\text{num}}^{\text{pot}}}$ compared to their analytical values $a_{\text{ana}} = \min\{2k, 2m\}$ in the case of $\mathbb{T}$. Values indicated with an a are calculated without the error for $\varepsilon = 0.025$. Values indicated with a b are calculated only with the errors for $\varepsilon \in \{0.25, 0.2, 0.125, 0.1, 0.0625\}$. Values indicated with a c are calculated only with the errors for $\varepsilon \in \{0.25, 0.2\}$.

6.2.3 Discussion

As we have seen in both experiments, the SPH method is converging for all the employed kernel functions. Even with the kernel functions which do not match the requirements of Corollary 3.11, the SPH method seems to converge.
In both cases, the convergence rate depends strongly on the order $2m$ of the moment condition. However, the convergence rate can be limited by the smoothness of the kernel k, such that we need sufficiently high values for k to take advantage of a high moment condition.
There are a few problems we will discuss shortly.
The convergence rate does not match the theoretical predictions. A reason for this can be that our error analysis is far away from being optimal. An indication for this is the positive convergence rate of the kernel functions which do not satisfy the requirements of Corollary 3.11. Moreover, the constants, which we largely ignored in the error analysis, can significantly depend on the used kernel function. This can have an impact on the numerical error. However, another reason for this may also be due to the fact that the chosen example is too good in some sense. The initial error is greater for higher values of l, which may be an explanation for the saturation effect. It seems that a higher value for the parameter l may lead to greater constants in the initial error estimate Theorem 3.8. Fortunately, since the approximation condition is satisfied for arbitrary order if $l = 1$, this effect may not have a great impact in applications.

Kernel functions with higher order in k, l and especially m can cause numerical problems. Since these kernel functions are made of sums of high order polynomials with alternating signs, we have cancellation effects in the evaluation of the kernel functions. This can have an effect on the error since minor inaccuracies from the evaluation can disturb the theoretical properties of the kernel function. Also, this can be an explanation for the effect of the saturation for small times t.

Conclusion

We derived a convergence theory for an SPH discretization of the Euler equations for a specific barotropic flow. Based on the work of Oelschläger [Oel91], we carefully worked out the connection between the kernel size and the initial mesh width. Hence, we were able to derive an improved convergence result for the energy error term in Theorem 3.9 and in Corollary 3.11. The main result is then stated in Theorem 3.13, where we derived a first pointwise estimate for the SPH discretization, which can be seen as a first step on the way to a general convergence theory.

Nonetheless, there are a few restrictions to our theory, where further research is needed. The results are restricted to classical, smooth solutions on all of $\mathbb{R}^d$. The simulation of shocks or other discontinuities is not covered by the given theory. Moreover, an extension to bounded domains will require the treatment of boundary conditions and is the topic of current research.

The theory also depends very much on the assumed equation of state $p = \frac{1}{2}\rho^2$. A more general equation of state $p = c\rho^\gamma$ would be of greater interest for applications. We think that the generalization would be possible but requires substantial changes in the proof. Also, a generalization of the initial particle position seems to be possible. Since the initialization only contributes to the estimates in Theorem 3.8, but not Theorem 3.9, we believe that one can derive similar estimates with less regular initial distribution. It might even be possible to choose a better initialization process to improve the results in Theorem 3.8.

Also the convergence of a fully discretized scheme has been proven for an explicit and an implicit time discretization scheme in Theorem 5.3 and Theorem 5.6, respectively. These results show the possibility of the convergence of the fully discretized scheme. However, both results do not differ in their requirements, even if the implicit method is expected to be less restrictive. Hence, a full stability analysis would have do be made, which is part of current research. Moreover, further research has to be made to check which time discretization scheme would be an optimal choice for the SPH method.

To derive the error estimates, the employed kernel has to be a convolution kernel whose convolution root satisfies the moment condition from Definition 1.26 and the approximation condition from Definition 3.5. In our opinion, the best way to construct such a kernel function is to derive conditions on the convolution kernel such that the existence of a root kernel, that possess the required conditions, is given. It can also be possible to construct a kernel function by convolution. However, we see no advantage in such a construction since the root kernel is just needed in the convergence theory and not for the calculation of the SPH system. Moreover, the analytical calculation of the convolution may be very hard to achieve, and a numerical calculation seems to be inefficient.

The construction method derived in this work in Section 4.2 is specifically designed for radial functions with compact supports. This class of functions is very well suited for the SPH method since radial functions are easy to implement and the compact support ensures that the method is efficient. Since the construction method requires that the kernel function is positive definite, the Wendland functions seem to be the perfect choice since these functions are given by piecewise polynomials of minimal degree for a given smoothness. Moreover, they are frequently used in the SPH community. Unfortunately, the original Wendland functions do not fit in the kernel construction scheme, which is why we used an extension $\psi_{d,k,l}$ of the original Wendland functions $\psi_{d,k} = \psi_{d,k,0}$. In this extension we take the original Wendland functions of a higher spatial dimension. In appli-

cations, the only difference of using the extended Wendland functions instead of the original ones is that the polynomial degree is increased by one, which is only a minor disadvantage concerning the numerical efficiency. All other properties are conserved from the original functions.
We do not claim that the given construction method is optimal. It is possible that there exists a simpler construction method or that there are construction methods which require fever or weaker properties of the kernel function. Moreover, we do not claim that the Wendland function is the optimal choice as the kernel function, even if they are both suitable for the construction method and efficient to calculate. Other functions do also satisfy the required properties, as we showed for the missing Wendland functions. However, these functions are less efficient in applications because of their logarithmic and square-root part.
Finally, the conditions we are stating on the kernel function are only sufficient for deriving the convergence result. The numerical tests in Chapter 6 indicate that the approximation condition is not required to lead to convergence. Even the classical Wendland functions ($l = 0$), for which the theory does not hold, give comparable results. The numerical tests also show that the numerically observed order is often significantly better than the one predicted in this work. Hence, more research in this direction is required.

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