THE RELATIVISTIC VLASOV–MAXWELL SYSTEM WITH EXTERNAL ELECTROMAGNETIC FIELDS

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ABSTRACT

The time evolution of a collisionless plasma is modeled by the relativistic Vlasov–Maxwell system which couples the Vlasov equation (the transport equation) with the Maxwell equations of electrodynamics. We consider the case that the plasma consists of several particle species, the particles are located in a container $\Omega \subset \mathbb{R}^3$, and are subject to boundary conditions on $\partial\Omega$.

In the first two parts of this work, we deal with the situation that there are external currents, typically in the exterior of the container, that may serve as a control of the plasma if adjusted suitably. In order to allow interaction between the exterior and the interior of the container, we do not impose perfect conductor boundary conditions for the electromagnetic fields—in contrast to other papers dealing with a similar setting, but without external currents—but consider the fields as functions on whole space $\mathbb{R}^3$ and model objects that are placed in space via given matrix-valued functions ε (the permittivity) and μ (the permeability). Firstly, a weak solution concept is introduced and existence of global-in-time solutions is proved, as well as the redundancy of the divergence part of the Maxwell equations in this weak solution concept. Secondly, since a typical aim in fusion plasma physics is to keep the amount of particles hitting $\partial\Omega$ as small as possible (since they damage the reactor wall), while the control costs should not be too exhaustive (to ensure efficiency), we consider a suitable minimization problem with the Vlasov–Maxwell system as a constraint. This problem is analyzed in detail. In particular, we prove existence of minimizers and establish an approach to derive first order optimality conditions.

In the third part of this work, we consider the case that the plasma is located in an infinitely long cylinder and is influenced by an external magnetic field. We prove existence of stationary solutions (extending in the third space direction infinitely) and give conditions on the external magnetic field under which the plasma is confined inside the cylinder, that is, it stays away from the boundary of the cylinder.

ZUSAMMENFASSUNG

Die zeitliche Entwicklung eines kollisionsfreien Plasmas wird durch das relativistische Vlasov-Maxwell-System modelliert, das die Vlasov-Gleichung (die Transportgleichung) mit den Maxwell-Gleichungen der Elektrodynamik koppelt. Es wird der Fall betrachtet, dass das Plasma aus mehreren Teilchenspezies besteht, die Teilchen sich in einem Behälter $\Omega \subset \mathbb{R}^3$ befinden und auf $\partial\Omega$ Randbedingungen genügen.

In den ersten beiden Teilen dieser Arbeit wird die Situation behandelt, dass externe Ströme vorhanden sind, typischerweise außerhalb des Behälters, die bei entsprechender Justierung als Steuerung des Plasmas dienen können. Um eine Interaktion zwischen dem Äußeren und dem Inneren des Behälters zu ermöglichen, werden keine Randbedingungen eines perfekten Leiters für die elektromagnetischen Felder verlangt – im Gegensatz zu anderen Arbeiten, die ein ähnliches Setting, jedoch ohne externe Ströme, behandeln –, sondern die Felder als Funktionen auf den gesamten Raum $\mathbb{R}^3$ betrachtet und Objekte, die im Raum platziert sind, mittels gegebener, matrixwertiger Funktionen ε (die Permittivität) und μ (die Permeabilität) modelliert. Zuerst werden ein schwaches Lösungskonzept eingeführt und die Existenz von globalen Lösungen sowie die Redundanz des Divergenzteils der Maxwell-Gleichungen in diesem schwachen Lösungskonzept nachgewiesen. Da ein typisches Ziel in der Fusionsplasmaphysik darin besteht, die Menge der Teilchen, die $\partial\Omega$ treffen, so klein wie möglich zu halten (da solche die Reaktorwand beschädigen), während die Kontrollkosten nicht allzu hoch sein sollten (um Effizienz zu gewährleisten), wird danach ein geeignetes Minimierungsproblem mit dem Vlasov-Maxwell-System als Nebenbedingung betrachtet. Dieses Problem wird detailliert analysiert. Insbesondere werden die Existenz von Minimierern nachgewiesen und eine Vorgehensweise zur Herleitung von Optimalitätsbedingungen erster Ordnung etabliert.

Im dritten Teil dieser Arbeit wird der Fall betrachtet, dass sich das Plasma in einem unendlich langen Zylinder befindet und durch ein äußeres Magnetfeld beeinflusst wird. Die Existenz von stationären Lösungen (die sich in die dritte Raumrichtung unendlich weit erstrecken) wird bewiesen und Bedingungen an das äußere Magnetfeld werden hergeleitet, unter denen das Plasma im Inneren des Zylinders eingeschlossen ist, also vom Zylinderrand entfernt bleibt.

CONTENTS

CHAPTER 0

Introduction

0.1 The PDE system

The time evolution of a collisionless plasma is modeled by the relativistic Vlasov–Maxwell system. Collisions among the plasma particles can be neglected if the plasma is sufficiently rarefied or hot. The particles only interact through electromagnetic fields created collectively. We consider the following setting: There are N species of particles, all of which are located in a container $\Omega \subset \mathbb{R}^3$, which is a bounded domain, for example, a fusion reactor. Thus, boundary conditions on $\partial\Omega$ have to be imposed.

In the exterior of Ω, there are external currents, for example, in electric coils, that may serve as a control of the plasma if adjusted suitably. In order to model objects that are placed somewhere in space, for example, the reactor wall, electric coils, and (almost perfect) superconductors, we consider the permittivity ε and permeability μ, which are functions of the space coordinate, take values in the set of symmetric, positive definite matrices of dimension three, and do not depend on time, as given. With this assumption we can model linear, possibly anisotropic materials that stay fixed in time. We should mention that in reality ε and μ will on the one hand additionally depend on the particle density inside Ω and on the other hand additionally locally on the electromagnetic fields, typically via their frequencies (maybe even nonlocally because of hysteresis). However, this would cause further nonlinearities which we avoid in this work.

The unknowns are on the one hand the particle densities $f^\alpha = f^\alpha(t, x, v)$, $\alpha = 1, \ldots, N$, which are functions of time $t \geq 0$, the space coordinate $x \in \Omega$, and the momentum coordinate $v \in \mathbb{R}^3$. Roughly speaking, $f^\alpha(t, x, v)$ indicates how many particles of the α-th species are at time t at position x with momentum v. On the other hand there are the electromagnetic fields $E = E(t, x)$, $H = H(t, x)$, which depend on time t and space coordinate $x \in \mathbb{R}^3$. The D- and B-fields are computed from E and H by the linear constitutive equations $D = \varepsilon E$ and $B = \mu H$. We will only view E and H as unknowns in the following.

The Vlasov part, which is to hold for each α, reads as follows:

$$\partial_t f^\alpha + \widehat{v}_\alpha \cdot \partial_x f^\alpha + q_\alpha (E + \widehat{v}_\alpha \times H) \cdot \partial_v f^\alpha = 0, \tag{0.1a}$$

$$f^\alpha_- = \mathcal{K}_\alpha f^\alpha_+ + g^\alpha, \tag{0.1b}$$

$$f^\alpha(0) = \mathring{f}^\alpha. \tag{0.1c}$$

Here, (0.1a) is the Vlasov equation equipped with the boundary condition (0.1b) on $\partial\Omega$ and the initial condition (0.1c) for $t = 0$. In (0.1c), $f^\alpha(0)$ denotes the evaluation of f^α at time $t = 0$, that is to say, the function $f^\alpha(0, \cdot, \cdot)$. We will use this notation often, also similarly for the electromagnetic fields and other functions.

Note that throughout this work we use modified Gaussian units such that the speed of light (in vacuum) is normalized to unity and all rest masses m_α of a particle of the respective species are at least 1. In (0.1a), q_α is the charge of the α-th particle species and $\widehat{v}_\alpha$ the velocity, which is computed from the momentum v via

$$\widehat{v}_\alpha = \frac{v}{\sqrt{m_\alpha^2 + |v|^2}}$$

according to special relativity. Clearly, $|\widehat{v}_\alpha| < 1$, that is, the velocities are bounded by the speed of light. Moreover, we assume that $\varepsilon = \mu = \mathrm{Id}$ on Ω, Id denoting the 3×3-identity matrix. Thus, the speed of light is constant in Ω and $B = H$ on Ω.

To derive a precise statement of the boundary condition (0.1b) and a definition of $f^\alpha_\pm$, the operator $\mathcal{K}_\alpha$, and where (0.1b) has to hold, we have a look at typical examples at first. Most commonly, the operator $\mathcal{K}_\alpha$ describes a specular boundary condition. For this, we assume that Ω has a (at least piecewise) C^1-boundary that is a submanifold of $\mathbb{R}^3$, and denote the outer unit normal of $\partial\Omega$ at some $x \in \partial\Omega$ by $n(x)$. Now consider a particle moving inside Ω and then hitting the surface $\partial\Omega$ at some time t at $x \in \partial\Omega$. Its momentum v (shortly) after the reflection satisfies $v \cdot n(x) < 0$ and its momentum (shortly) before the hit is thus given by $v - 2(v \cdot n(x))n(x)$. In other words, this means that the components of the momentum which are tangential to $n(x)$ stay the same, and that the component which is normal to $n(x)$ changes the sign. On the level of a particle density f^α, this consideration yields the condition

$$f^\alpha(t, x, v) = f^\alpha(t, x, v - 2(v \cdot n(x))n(x)) =: \big(Kf^\alpha\big)(t, x, v) \tag{0.2}$$

for $x \in \partial\Omega$ and $v \cdot n(x) < 0$.

More generally, we can consider the case that only a portion of the particles that hit the boundary are reflected and the rest is absorbed and, additionally, more particles are added from outside. Thus, we may demand

$$f^\alpha(t, x, v) = a^\alpha(t, x, v)\big(Kf^\alpha\big)(t, x, v) + g^\alpha(t, x, v) \tag{0.3}$$

for $x \in \partial\Omega$ and $v \cdot n(x) < 0$. Here, $0 \le a^\alpha(t, x, v) \le 1$ is a coefficient; that is to say, $a^\alpha(t, x, v)$-times the amount of the particles hitting the boundary at time t at $x \in \partial\Omega$ with momentum v are reflected and the rest is absorbed. Furthermore, $g^\alpha(t, x, v) \ge 0$ is the source term describing how many particles are added from outside.

Since the boundary condition is to hold only if $v \cdot n(x) < 0$, it is natural to decompose $[0, \infty[\times \partial\Omega \times \mathbb{R}^3$ into three parts:

$$\gamma^+ := \big\{(t, x, v) \in [0, \infty[\times \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) > 0\big\},$$

$$\gamma^- := \left\{(t,x,v) \in [0,\infty[\times \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) < 0\right\},$$
$$\gamma^0 := \left\{(t,x,v) \in [0,\infty[\times \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) = 0\right\}.$$

Therefore, (0.3) is to hold for $(t,x,v) \in \gamma^-$. Moreover, K can be seen as an operator mapping functions on γ^+ to functions on γ^-. In accordance with (0.1b), we define $f^\alpha_\pm$ to be the restriction of f^α to $\gamma^\pm$. Of course, this only makes sense if we have some regularity of f^α, for example, continuity on $[0,\infty[\times \overline{\Omega} \times \mathbb{R}^3$. But even if a solution f^α (of a Vlasov equation) is only an L^p-function, it is possible to define a trace $f^\alpha_\pm$ of f^α on $\gamma^\pm$; see Definition 1.2.7.(ii). Note that $\mathcal{K}_\alpha = a^\alpha K$ in (0.1b) yields (0.3). Since the time variable in the sets above is somewhat unnecessary, we abbreviate

$$\gamma^+_T := \left\{(t,x,v) \in [0,T[\times \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) > 0\right\},$$
$$\gamma^-_T := \left\{(t,x,v) \in [0,T[\times \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) < 0\right\},$$
$$\gamma^0_T := \left\{(t,x,v) \in [0,T[\times \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) = 0\right\},$$
$$\tilde{\gamma}^+ := \left\{(x,v) \in \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) > 0\right\},$$
$$\tilde{\gamma}^- := \left\{(x,v) \in \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) < 0\right\},$$
$$\tilde{\gamma}^0 := \left\{(x,v) \in \partial\Omega \times \mathbb{R}^3 \mid v \cdot n(x) = 0\right\}$$

for $0 < T \leq \infty$. For ease of notation it will be convenient to introduce a surface measure on $[0,\infty[\times \partial\Omega \times \mathbb{R}^3$, namely,

$$d\gamma_\alpha = |\widehat{v}_\alpha \cdot n(x)| \, dv dS_x dt.$$

Furthermore, the Vlasov part is coupled with Maxwell's equations, which describe the time evolution of the electromagnetic fields:

$$\varepsilon \partial_t E - \operatorname{curl}_x H = -4\pi j, \tag{0.4a}$$
$$\mu \partial_t H + \operatorname{curl}_x E = 0, \tag{0.4b}$$
$$(E,H)(0) = \left(\mathring{E}, \mathring{H}\right). \tag{0.4c}$$

Here, the current $j = j^{\text{int}} + u$ is typically the sum of the internal currents

$$j^{\text{int}} := \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv$$

and some external current u, that is supported in some open set $\Gamma \subset \mathbb{R}^3$. We will always extend j^{int} (u) by zero outside Ω (Γ). Concerning set-ups with boundary conditions on the plasma, the papers we are aware of deal with perfect conductor boundary conditions for the electromagnetic fields; see, for example, [Guo93]. Such a set-up can model no interaction between the interior and the exterior. However, considering fusion reactors, there are external currents in the exterior, for example, in field coils. These external currents induce electromagnetic fields and thus influence the behavior of the internal plasma. Even more important, the main aim of fusion plasma research is to adjust these external currents "suitably". Thus, we impose Maxwell's equations globally in space.

Actually, Maxwell's equations additionally include conditions on the divergence of $D = \varepsilon E$ and $B = \mu H$, namely,

$$\operatorname{div}_x(\varepsilon E) = 4\pi\rho, \tag{0.5a}$$

$$\operatorname{div}_x\left(\mu H\right) = 0, \tag{0.5b}$$

where ρ denotes the charge density. Usually, these equations are known to be redundant if all functions are smooth enough, local conservation of charge is satisfied, i.e.,

$$\partial_t \rho + \operatorname{div}_x j = 0,$$

and (0.5) holds initially, which we then view as a constraint on the initial data. Therefore, in Chapters 1 and 2 we largely ignore (0.5) and discuss in Section 1.5 in what sense (0.5) is satisfied in the context of a weak solution concept.

We thus arrive at the following Vlasov–Maxwell system, which is (0.1) and (0.4) combined, on a time interval with given final time $0 < T_\bullet \leq \infty$:

$$\begin{aligned}
\partial_t f^\alpha + \widehat{v}_\alpha \cdot \partial_x f^\alpha + q_\alpha(E + \widehat{v}_\alpha \times H) \cdot \partial_v f^\alpha &= 0 && \text{on } I_{T_\bullet} \times \Omega \times \mathbb{R}^3, && \text{(VM.1)}\\
f_-^\alpha &= \mathcal{K}_\alpha f_+^\alpha + g^\alpha && \text{on } \gamma_{T_\bullet}^-, && \text{(VM.2)}\\
f^\alpha(0) &= \mathring{f}^\alpha && \text{on } \Omega \times \mathbb{R}^3, && \text{(VM.3)}\\
\varepsilon \partial_t E - \operatorname{curl}_x H &= -4\pi j && \text{on } I_{T_\bullet} \times \mathbb{R}^3, && \text{(VM.4)}\\
\mu \partial_t H + \operatorname{curl}_x E &= 0 && \text{on } I_{T_\bullet} \times \mathbb{R}^3, && \text{(VM.5)}\\
(E, H)(0) &= \left(\mathring{E}, \mathring{H}\right) && \text{on } \mathbb{R}^3, && \text{(VM.6)}
\end{aligned}$$

where (VM.1) to (VM.3) have to hold for all $\alpha = 1, \ldots, N$ and $I_{T_\bullet}$ denotes the given time interval. Here and in the following, $I_T := [0, T]$ for $0 \leq T < \infty$ and $I_\infty := [0, \infty[$.

0.2 Outline

This work splits into three parts. In Chapter 1 we prove existence of weak solutions of (VM) for given (and suitable) $\mathring{f}$, $\mathring{E}$, $\mathring{H}$, $\mathcal{K}_\alpha$, g^α, and u. To this end, we firstly define in Section 1.1 what we call weak solutions of (VM). The strategy to construct a weak solution follows the strategy of Guo [Guo93], who considered (VM) with $\varepsilon = \mu = \mathrm{Id}$, $u = 0$, and (VM.4) and (VM.5) only imposed on Ω and equipped with perfect conductor boundary conditions on $\partial\Omega$. Firstly, we consider the Vlasov part in Section 1.2 and state some important results of Beals and Protopopescu [BP87], who dealt with transport equations with Lipschitz continuous vector field subject to boundary conditions; here, we also refer to the book of Greenberg, Mee, and Protopopescu [GMP87]. Going to the level of characteristics and exploiting that the characteristic flow is measure preserving (which follows from the fact that the Lorentz force of electrodynamics has no sources and sinks with respect to momentum), L^p-bounds on f^α and f_+^α are derived. After shortly discussing the Maxwell part in Section 1.3, we proceed with the construction of a weak solution in Section 1.4. Additionally to L^p-bounds on f^α and f_+^α, we make use of an energy consideration. For classical solutions of (VM) one can easily derive the energy balance

$$\frac{d}{dt}\left(\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3}\sqrt{m_\alpha^2+|v|^2}f^\alpha\,dvdx+\frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\varepsilon E\cdot E+\mu H\cdot H\right)dx\right)\leq C-\int_{\mathbb{R}^3}E\cdot u\,dx,$$

if $\mathcal{K}_\alpha$ takes the form $\mathcal{K}_\alpha = a^\alpha K$ with $0 \leq a^\alpha \leq 1$, and where C is some expression in the g^α; if $a^\alpha = 1$ for all α, equality holds above. In order to apply a quadratic Gronwall argument and to conclude that the left bracket is bounded for each time, the map

$$(E,H)\mapsto\left(\int_{\mathbb{R}^3}\left(\varepsilon E\cdot E+\mu H\cdot H\right)dx\right)^{\frac{1}{2}}$$

should be a norm on $L^2(\mathbb{R}^3;\mathbb{R}^6)$ which is equivalent to the standard L^2-norm. Thus, assumptions about uniform positive definiteness of ε and μ will be made. Then, it is natural to search for a weak solution in those spaces for whose norms the above a priori bounds have been established. It turns out that firstly a cut-off system has to be investigated in Section 1.4.2. Afterwards, the cut-off is removed in Section 1.4.3 and the main result is proved in Theorem 1.4.4.

As already mentioned, in Section 1.5 we turn to the redundancy of the divergence part of Maxwell's equations. Guo [Guo93] proved that the divergence equations are redundant if one imposes them on Ω. However, in our set-up the Maxwell equations are imposed on whole space. Thus, things are more complicated since we have to "cross over" $\partial\Omega$. Whereas (0.5b) is easy to handle, the consideration of (0.5a) is much more difficult and requires the property of local conservation of charge and the correct definition of the charge density ρ. The idea is to show that the weak form—(1.1.2), in particular—also holds for test functions that do not depend on v and thus to have a weak form of conservation of internal charge at hand. Therefore, we have to perform some technical approximations under a smoothness assumption about $\partial\Omega$. It turns out that a part of ρ is a distribution which is supported on $\partial\Omega$ and arises due to the boundary conditions. The main result is stated in Theorem 1.5.6.

In Chapter 2 we analyze an optimal control problem. A typical aim in fusion plasma physics is to keep the amount of particles hitting $\partial\Omega$ as small as possible (since they damage the reactor wall), while the control costs should not be too exhaustive (to ensure efficiency). This leads to a minimization problem where a certain objective function shall be driven to a minimum over a certain set of functions satisfying (VM) in a weak sense. More precisely, the objective function is

$$\frac{1}{q}\sum_{\alpha=1}^{N}w_\alpha\left\|f_+^\alpha\right\|_{L^q\left(\gamma_{T,\bullet}^+,d\gamma_\alpha\right)}^q+\frac{1}{r}\|u\|_{\mathcal{U}}^r.$$

Here, $1 < q < \infty$, $w_\alpha > 0$, and $\mathcal{U} = W^{1,r}\big(]0,T_\bullet[\times\Gamma;\mathbb{R}^3\big)$ with $\frac{4}{3} < r < \infty$. Thus, the objective function penalizes hits of the particles on $\partial\Omega$ and exhaustive control costs. In addition to (VM), it is necessary to impose two inequality constraints, namely, (2.1.1) and (2.1.2), which are natural in the sense that they come from formal a priori bounds. After discussing the minimization problem in detail in Section 2.1, we firstly prove existence of a minimizer in Section 2.2; see Theorem 2.2.1. Secondly, we establish an approach to derive first order optimality conditions for a minimizer under the assumption $q > 2$ in Sections 2.3 and 2.4. To this end, the one main idea is to write the weak form of (VM) equivalently as an identity

$$\mathcal{G}\big(\big(f^\alpha, f_+^\alpha\big)_\alpha, E, H, u\big) = 0 \text{ in } \Lambda^*,$$

where $\mathcal{G}$ is differentiable, Λ is a uniformly convex, reflexive test function space, and Λ^* is its topological dual space; see Section 2.3. The other main idea, which is motivated by approaches of Lions [Lio85], is to introduce an approximate minimization problem with a penalization parameter $s > 0$ which is driven to infinity later; see Section 2.4. In particular, we add the differentiable term

$$\frac{s}{2}\left\|\mathcal{G}\big(\big(f^\alpha, f_+^\alpha\big)_\alpha, E, H, u\big)\right\|_{\Lambda^*}^2$$

to the original objective function and abolish the constraint that (VM) be solved. For this approximate problem, we prove existence of a minimizer and establish first order optimality conditions; see Theorems 2.4.3 and 2.4.11. After that, we let $s \to \infty$ and prove that, along a suitable sequence, a minimizer of the original problem is obtained in the limit, and the convergence of the controls u is even strong; see Theorem 2.4.13. Lastly, we briefly discuss in Section 2.5 how these results can also be verified in case of similar set-ups or different objective functions. We should point out that the main problem we have to deal with is that existence of global-in-time solutions to (VM) is only known in a weak solution concept. In fact, one cannot expect C^1-solutions in general as a result of the boundary conditions for the plasma particles; this was observed by Guo [Guo95] even in a one-dimensional setting. It is an open problem whether or not such weak solutions are unique for given u. Thus, standard approaches to derive first order optimality conditions via introducing a (preferably differentiable) control-to-state operator, as is, for example, done in the books of Hinze et al. [Hin+09] and Tröltzsch [Trö10], cannot be applied.

In Chapter 3 we consider the case that only an external magnetic field influences the internal system. The aim then is to answer the following two questions: Firstly, for a given time-independent external magnetic field, is there a corresponding stationary solution? Secondly, are there stationary solutions that are confined in Ω, i.e., the particles stay away from the boundary of their container, if the external magnetic field is adjusted suitably? Results are obtained in the case that Ω is an infinitely long cylinder (hence no longer bounded) and that the electromagnetic fields are subject to perfect conductor boundary conditions on $\partial\Omega$. In particular, proceeding similarly to Degond [Deg90], Batt and Fabian [BF93], Knopf [Kno19], and Skubachevskii [Sku14], we state some basic assumptions on the symmetry of the appearing functions and state the corresponding invariant quantities $\mathcal{E}^\alpha$, $\mathcal{F}^\alpha$, and $\mathcal{G}^\alpha$ in Section 3.2, which lead to the natural ansatz

$$f^\alpha = \eta^\alpha(\mathcal{E}^\alpha, \mathcal{F}^\alpha, \mathcal{G}^\alpha).$$

This ansatz, together with a basic definition and some useful preliminary lemmas and tools, is the content of Section 3.3. In Sections 3.4 and 3.5 we answer the above-mentioned questions. In particular, we firstly prove existence of a steady state for a given external magnetic field in Theorem 3.4.4; see also Theorems 3.4.6 and 3.4.9 for further properties. Here, the main idea is to formulate the problem equivalently as a fixed point problem

$$\big(\phi, A_\varphi, A_3\big) = \mathcal{M}\big(\phi, A_\varphi, A_3\big)$$

for (some components of) the electromagnetic four-potential, which is then handled by Schaefer's fixed point theorem. Secondly, we give conditions on the external magnetic potential under which the steady state is confined; see Theorem 3.5.1.

0.3 Further literature

Vlasov–Maxwell systems have been studied extensively. In case of no reactor wall, i.e., the Vlasov equation is imposed globally in space (as well as Maxwell's equations), global well-posedness of the Cauchy problem is a famous open problem. Global existence and uniqueness of classical solutions has been proved in lower dimensional settings; see Glassey and Schaeffer [GS90; GS97; GS98a; GS98b]. In the full three-dimensional setting, a continuation criterion was given by Glassey and Strauss [GS86]. Furthermore, global existence of weak solutions was proved by Di Perna and Lions [DL89]. Their momentum-averaging lemma is fundamental for proving existence of weak solutions in any setting (with or without boundary, with or without perfect conductor boundary conditions and so on), since it handles the nonlinearity in the Vlasov equation. However, uniqueness of these weak solutions is not known. For a more detailed overview we refer to Rein [Rei04] and to the book of Glassey [Gla96], which also deals with other PDE systems in kinetic theory.

Controllability of the relativistic Vlasov–Maxwell system in two dimensions was studied by Glass and Han-Kwan [GH15]. Knopf [Kno18] and later Knopf and the author [KW18] analyzed optimal control problems for the Vlasov–Poisson system, where Maxwell's equations are replaced by the electrostatic Poisson equation. Here, an external magnetic field was considered as a control. Studying control problems with the Vlasov–Poisson system as the governing PDE system enjoys the advantage of having existence and uniqueness of global-in-time classical solutions at hand, due to the results of Pfaffelmoser [Pfa92] and Schaeffer [Sch91]. Also, an optimal control problem for the two-dimensional Vlasov–Maxwell system was considered in [Web18].

Stationary solutions have already been obtained in similar set-ups; see, for example, Poupaud [Pou92] and Rein [Rei92]. Approaches for confinement of Vlasov plasmas can be found in a series of works of Caprino, Cavallaro, and Marchioro [CCM12;

CCM14; CCM15; CCM16], who dealt with Vlasov–Poisson plasmas, and in Han-Kwan [Han10] and Nguyen, Nguyen, and Strauss [NNS15] in the case of a Vlasov–Maxwell plasma. Stability of stationary solutions was discussed in Nguyen and Strauss [NS14], Zhang [Zha19], and (for a two-fluid model) in Zhelyazov, Han-Kwan, and Rademacher [ZHR15].

0.4 Some notation

Throughout this work, C^k-spaces ($k \in \mathbb{N} \cup \{\infty\}$) on the closure of some open set U are defined to be the space of C^k-functions h on U such that all derivatives of h of order less or equal k can be continuously extended to $\overline{U}$. Moreover, the index 'b' in C^k_b indicates that all derivatives of order less or equal k of such functions shall be bounded, and the index 'c' in C^k_c indicates that such functions shall be compactly supported. As usual, $C^{k,s}$ ($k \in \mathbb{N}_0$, $0 < s \leq 1$) denotes Hölder spaces.

Furthermore, we denote by χ_M the characteristic function of some set M and by χ_T the characteristic function of $[0, T]$. For $1 \leq p < \infty$ we define

$$L^p_{\alpha\mathrm{kin}}(A, da) := \left\{ u \in L^p(A, da) \mid \int_A v^0_\alpha |u|^p \, da < \infty \right\},$$

equipped with the corresponding weighted norm. Here, $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ or $A \subset \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ is some Borel set equipped with a measure a and the weight v^0_α is given by

$$v^0_\alpha := \sqrt{m^2_\alpha + |v|^2}.$$

By $m_\alpha \geq 1$ we have $v^0_\alpha \geq 1$. Moreover, we write

$$L^p_{\mathrm{lt}}(A, da) := \{u \colon A \to \mathbb{R} \mid \chi_T u \in L^p(A, da) \text{ for all } T > 0\}$$

for $1 \leq p \leq \infty$. If a is the Lebesgue measure we write $L^p_{\alpha\mathrm{kin}}(A)$ and $L^p_{\mathrm{lt}}(A)$, respectively. A combination $L^p_{\alpha\mathrm{kin,lt}}(A, da)$ is defined accordingly. Furthermore, we abbreviate

$$G_{\mathrm{lt}}(I; X) := \{u \colon I \to X \mid u \in G([0, T]; X) \text{ for all } T \in I\}$$

where $0 \in I \subset [0, \infty[$ is some interval, G is some C^k or L^p, and X is a normed, separable vector space. Also, the somewhat sloppy notation

$$L^\infty(I; L^\infty(A)) := L^\infty(I \times A)$$

and

$$G(I; X \cap Y) := G(I; X) \cap G(I; Y)$$

(and likewise with index 'lt', respectively) occur.

Since ε is already used for the permittivity, the letter ι, and not ε, will always denote a small positive number.

For a matrix $A \in \mathbb{R}^{n \times n}$ ($n \in \mathbb{N}$) and a positive number $\sigma > 0$, we write $A \geq \sigma$ ($A \leq \sigma$) if $Ax \cdot x \geq \sigma |x|^2$ ($Ax \cdot x \leq \sigma |x|^2$) for all $x \in \mathbb{R}^n$. For a measurable $A \colon \mathbb{R}^n \to \mathbb{R}^{n \times n}$ and $\sigma > 0$, we write $A \geq \sigma$ ($A \leq \sigma$) if $A(x) \geq \sigma$ ($A(x) \leq \sigma$) for almost all $x \in \mathbb{R}^n$.

For $x, y \in \mathbb{R}^n$ ($n \in \mathbb{N}$), $[x, y]$ denotes the closed line segment connecting x and y; similar notations are used for segments not including one or two of the endpoints.

Finally, for a normed space X, we write X^* for the topological dual space. For some $x \in X$ and $r > 0$, $B_r(x)$ denotes the open ball in X with center x and radius r. Furthermore, we abbreviate $B_r := B_r(0)$.

CHAPTER 1

Existence of weak solutions

1.1 Preliminaries

In this chapter we consider the case that some particle species, say, for $\alpha = 1, \dots, N'$ with $N' \in \{0, \dots, N\}$, are subject to partially absorbing boundary conditions with possibly a source term g^α, and the other particle species, for $\alpha = N'+1, \dots, N$, are subject to (partially) purely reflecting boundary conditions with no source term g^α. To be more precise, for $\alpha = 1, \dots, N'$ assume $a^\alpha \in L^\infty\left(\gamma_{T_\bullet}^-\right)$, $a_0^\alpha := \|a^\alpha\|_{L^\infty\left(\gamma_{T_\bullet}^-\right)} < 1$, and $g^\alpha \in \left(L^1_{\alpha\text{kin,lt}} \cap L^\infty_{\text{lt}}\right)\left(\gamma_{T_\bullet}^-, d\gamma_\alpha\right)$. For $\alpha = N'+1, \dots, N$, however, assume $a^\alpha \in L^\infty\left(\gamma_{T_\bullet}^-\right)$, $\|a^\alpha\|_{L^\infty\left(\gamma_{T_\bullet}^-\right)} = 1$, and $g^\alpha = 0$. For all α we define $\mathcal{K}_\alpha := a^\alpha K$.

The space of test functions for (VM.1) to (VM.3) is $\Psi_{T_\bullet}$, where

$$\Psi_T := \Big\{\psi \in C^\infty\left(I_T \times \overline{\Omega} \times \mathbb{R}^3\right) \mid \operatorname{supp}\psi \subset [0,T[\, \times \overline{\Omega} \times \mathbb{R}^3 \text{ compact}, \\ \operatorname{dist}\left(\operatorname{supp}\psi, \gamma_T^0\right) > 0,\ \operatorname{dist}\left(\operatorname{supp}\psi, \{0\} \times \partial\Omega \times \mathbb{R}^3\right) > 0\Big\} \quad (1.1.1)$$

for $0 < T \leq \infty$. The restriction that $\operatorname{supp}\psi$ be away from certain sets will be important later; see Definition 1.2.2 and Lemma 1.2.5. On the other hand, $\Theta_{T_\bullet}$ is the space of test functions for (VM.4) to (VM.6), where

$$\Theta_T := \left\{\vartheta \in C^\infty\left(I_T \times \mathbb{R}^3; \mathbb{R}^3\right) \mid \operatorname{supp}\vartheta \subset [0,T[\, \times \mathbb{R}^3 \text{ compact}\right\}$$

for $0 < T \leq \infty$.

We start with the definition of what we call weak solutions of (VM).

Definition 1.1.1. Let $0 < T_\bullet \leq \infty$, $u \in L^1_{\text{loc}}\left(\mathbb{R}^3; \mathbb{R}^3\right)$. We call a tuple $\left(\left(f^\alpha, f_+^\alpha\right)_\alpha, E, H, j\right)$ a weak solution of (VM) on the time interval $I_{T_\bullet}$ with external current u if (for all α):

(i) $f^\alpha \in L^1_{\text{loc}}\left(I_{T_\bullet} \times \overline{\Omega} \times \mathbb{R}^3\right)$, $f_+^\alpha \in L^1_{\text{loc}}\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)$, $E, H, j \in L^1_{\text{loc}}\left(I_{T_\bullet} \times \mathbb{R}^3; \mathbb{R}^3\right)$.

(ii) For all $\psi \in \Psi_{T_\bullet}$ it holds that

$$0 = -\int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} \big(\partial_t \psi + \widehat{v}_\alpha \cdot \partial_x \psi + q_\alpha (E + \widehat{v}_\alpha \times H) \cdot \partial_v \psi\big) f^\alpha \, dv dx dt + \int_{\gamma_{T_\bullet}^+} f_+^\alpha \psi \, d\gamma_\alpha - \int_{\gamma_{T_\bullet}^-} \big(\mathcal{K}_\alpha f_+^\alpha + g^\alpha\big) \psi \, d\gamma_\alpha - \int_\Omega \int_{\mathbb{R}^3} \mathring{f}^\alpha \psi(0) \, dv dx \tag{1.1.2}$$

(in particular, especially the integral of $(E + \widehat{v}_\alpha \times H) f^\alpha \cdot \partial_v \psi$ is supposed to exist).

(iii) For all $\vartheta \in \Theta_{T_\bullet}$ it holds that

$$0 = \int_0^{T_\bullet} \int_{\mathbb{R}^3} \big(\varepsilon E \cdot \partial_t \vartheta - H \cdot \operatorname{curl}_x \vartheta - 4\pi j \cdot \vartheta\big) \, dx dt + \int_{\mathbb{R}^3} \varepsilon \mathring{E} \cdot \vartheta(0) \, dx, \tag{1.1.3a}$$

$$0 = \int_0^{T_\bullet} \int_{\mathbb{R}^3} \big(\mu H \cdot \partial_t \vartheta + E \cdot \operatorname{curl}_x \vartheta\big) \, dx dt + \int_{\mathbb{R}^3} \mu \mathring{H} \cdot \vartheta(0) \, dx. \tag{1.1.3b}$$

(iv) The current j is the sum of the internal and the external currents, i.e.,

$$j = j^{\text{int}} + u := \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv + u.$$

Whereas the weak formulation of the Maxwell equations is standard, the weak formulation of the Vlasov part will be explained in Section 1.2.1.

To obtain certain energy estimates we will need the following quadratic version of Gronwall's lemma, which is a slight improvement of [Dra03, Theorem 5].

Lemma 1.1.2. *Let $a, b \in \mathbb{R}$, $a < b$, $y, h \colon [a,b] \to [0, \infty[$ and $g \colon [a,b] \to \mathbb{R}$ be continuous, and $\overline{y} \colon [a,b] \to \mathbb{R}$. Assume that the following inequality holds for all $t \in [a,b]$:*

$$\frac{1}{2}\overline{y}(t)^2 + \frac{1}{2}y(t)^2 \le \frac{1}{2}g(t)^2 + \int_a^t h(s) y(s) \, ds.$$

Then we have

$$\sqrt{\overline{y}(t)^2 + y(t)^2} \le \big|g(t)\big| + \int_a^t h(s) \, ds$$

for all $t \in [a,b]$.

Proof. Let $\iota > 0$ and choose $G_\iota \in C^1([a,b])$ such that $G_\iota \ge 0$ and $\big|G_\iota - g^2\big| < \iota$ on $[a,b]$. Now consider

$$y_\iota \colon [a,b] \to \,]0, \infty[, \quad y_\iota(t) = \frac{1}{2}(G_\iota(t) + \iota) + \int_a^t h(s) y(s) \, ds.$$

By assumption we have $y(t) \leq \sqrt{\overline{y}(t)^2 + y(t)^2} \leq \sqrt{2y_\iota(t)}$. Furthermore, $\sqrt{2y_\iota}$ is differentiable with

$$\frac{d}{dt}\sqrt{2y_\iota(t)} = \frac{\frac{1}{2}G_\iota'(t) + h(t)y(t)}{\sqrt{2y_\iota(t)}} \leq \frac{G_\iota'(t)}{2\sqrt{G_\iota(t)+\iota}} + h(t).$$

Integrating this estimate from a to t yields

$$\begin{aligned}
\sqrt{\overline{y}(t)^2 + y(t)^2} \leq \sqrt{2y_\iota(t)} \leq \sqrt{2y_\iota(a)} &+ \int_a^t \frac{G_\iota'(s)}{2\sqrt{G_\iota(s)+\iota}}\,ds + \int_a^t h(s)\,ds \\
&= \sqrt{G_\iota(a)+\iota} + \sqrt{G_\iota(t)+\iota} - \sqrt{G_\iota(a)+\iota} + \int_a^t h(s)\,ds \\
&\leq \sqrt{g(t)^2 + 2\iota} + \int_a^t h(s)\,ds \leq |g(t)| + \sqrt{2\iota} + \int_a^t h(s)\,ds.
\end{aligned}$$

Since $\iota > 0$ is arbitrary, the proof is finished. □

Following a general strategy, existence of weak solutions to (VM) is proved by constructing a sequence of solutions to approximating PDE systems and then extracting a weakly converging subsequence whose limit is a candidate for a solution of the original PDE system. Since (VM) as a whole is nonlinear, it is natural to decouple the Vlasov part from the Maxwell part by taking the already known fields from the previous iteration step to construct the new particle densities out of the Vlasov equation. Vice versa, one then proceeds with the Maxwell part to construct the new fields out of an already known current. Thus, it is useful to dissociate the Vlasov part from the Maxwell part and consider the force field in the Vlasov part and the current in the Maxwell part, respectively, as given for the time being.

1.2 The Vlasov part

Throughout this section, $\alpha \in \{1, \ldots, N\}$ is fixed.

1.2.1 Weak formulation

Let $F\colon I_{T_\bullet} \times \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}^3$ be an already known force field; consider this to be the Lorentz force induced by some electromagnetic fields. In order to have local conservation of charge, it is natural to assume that F is divergence free with respect to v, at least in the sense of distributions. Of course, the Lorentz force in our situation satisfies this assumption.

We want to solve the following system:

$$\partial_t f^\alpha + \widehat{v}_\alpha \cdot \partial_x f^\alpha + F \cdot \partial_v f^\alpha = 0 \qquad \text{on } I_{T_\bullet} \times \Omega \times \mathbb{R}^3, \tag{1.2.1a}$$

$$f_-^\alpha = \mathcal{K}_\alpha f_+^\alpha + g^\alpha \qquad \text{on } \gamma_{T_\bullet}^-, \tag{1.2.1b}$$

$$f^\alpha(0) = \mathring{f}^\alpha \qquad \text{on } \Omega \times \mathbb{R}^3. \tag{1.2.1c}$$

The first step is to derive a weak formulation of (1.2.1). To this end, assume that $f^\alpha \in C^1\left(I_{T_\bullet} \times \overline{\Omega} \times \mathbb{R}^3\right)$ and that F is locally integrable, and continuously differentiable and divergence free (both) with respect to v. Taking a test function $\psi \in \Psi_{T_\bullet}$, multiplying (1.2.1a) with ψ, and then integrating over $I_{T_\bullet} \times \Omega \times \mathbb{R}^3$ leads to

$$\begin{aligned} 0 &= \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} \left(\partial_t f^\alpha + \widehat{v}_\alpha \cdot \partial_x f^\alpha + F \cdot \partial_v f^\alpha\right) \psi \, dv dx dt \\ &= -\int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} \left(\partial_t \psi + \widehat{v}_\alpha \cdot \partial_x \psi + F \cdot \partial_v \psi\right) f^\alpha \, dv dx dt \\ &\quad - \int_\Omega \int_{\mathbb{R}^3} f^\alpha(0)\psi(0) \, dv dx + \int_0^{T_\bullet} \int_{\partial\Omega} \int_{\mathbb{R}^3} f^\alpha \psi \widehat{v}_\alpha \cdot n \, dv dS_x dt. \end{aligned} \tag{1.2.2}$$

Here, the assumption that F is divergence free with respect to v enters. The only term we have to take care about further is the third one. We decompose the domain of integration and write $f_\pm^\alpha$ for the restriction of f^α to $\gamma_{T_\bullet}^\pm$ to get

$$\begin{aligned} &\int_0^{T_\bullet} \int_{\partial\Omega} \int_{\mathbb{R}^3} f^\alpha \psi \widehat{v}_\alpha \cdot n \, dv dS_x dt \\ &= \iiint\limits_{\gamma_{T_\bullet}^+} f^\alpha \psi \widehat{v}_\alpha \cdot n \, dv dS_x dt + \iiint\limits_{\gamma_{T_\bullet}^-} f^\alpha \psi \widehat{v}_\alpha \cdot n \, dv dS_x dt + \iiint\limits_{\gamma_{T_\bullet}^0} f^\alpha \psi \widehat{v}_\alpha \cdot n \, dv dS_x dt \\ &= \iiint\limits_{\gamma_{T_\bullet}^+} f_+^\alpha \psi \widehat{v}_\alpha \cdot n \, dv dS_x dt + \iiint\limits_{\gamma_{T_\bullet}^-} f_-^\alpha \psi \widehat{v}_\alpha \cdot n \, dv dS_x dt \end{aligned} \tag{1.2.3}$$

because of $\widehat{v}_\alpha \cdot n = 0$ on γ^0. If we demand (1.2.1b) the very last term has to equal

$$\iiint\limits_{\gamma_{T_\bullet}^-} \left(\mathcal{K}_\alpha f_+^\alpha + g^\alpha\right) \psi \widehat{v}_\alpha \cdot n \, dv dS_x dt. \tag{1.2.4}$$

For ease of notation we use the abbreviation

$$d\gamma_\alpha = |\widehat{v}_\alpha \cdot n(x)| \, dv dS_x dt,$$

that was already introduced earlier. Note that

$$d\gamma_\alpha = \pm \widehat{v}_\alpha \cdot n(x) \, dv dS_x dt \quad \text{on } \gamma^\pm.$$

Combining this and (1.2.2) to (1.2.4) we conclude that (1.2.1) is equivalent to the property that

$$0 = -\int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} \left(\partial_t \psi + \widehat{v}_\alpha \cdot \partial_x \psi + F \cdot \partial_v \psi\right) f^\alpha \, dv dx dt$$

$$+\int_{\gamma_{T_\bullet}^+} f_+^\alpha \psi\, d\gamma_\alpha - \int_{\gamma_{T_\bullet}^-} (\mathcal{K}_\alpha f_+^\alpha + g^\alpha)\psi\, d\gamma_\alpha - \int_\Omega \int_{\mathbb{R}^3} \mathring{f}^\alpha \psi(0)\, dv dx$$

for all $\psi \in \Psi_{T_\bullet}$.

We explain in the following remark in what sense we can speak of traces f_+^α of f^α in a weak solution concept.

Remark 1.2.1. If Definition 1.1.1.(ii) is satisfied, f_+^α is the trace of f^α in the following sense:

- As we have just seen, f_+^α is the restriction of f^α to $\gamma_{T_\bullet}^+$ if $f^\alpha \in C^1\left(I_{T_\bullet} \times \overline{\Omega} \times \mathbb{R}^3\right)$.
- There is no other $\tilde{f}_+^\alpha \in L^1_{\mathrm{loc}}\left(\gamma_{T_\bullet}^+\right)$ such that Definition 1.1.1.(ii) is satisfied as well, since for such $\tilde{f}_+^\alpha$ we have

$$\int_{\gamma_{T_\bullet}^+} \left(f_+^\alpha - \tilde{f}_+^\alpha\right)\psi\, d\gamma_\alpha = 0$$

  for all $\psi \in C^\infty\left(I_{T_\bullet} \times \overline{\Omega} \times \mathbb{R}^3\right)$ with $\operatorname{supp}\psi \subset [0, T_\bullet[\times \overline{\Omega} \times \mathbb{R}^3$ compact that vanish on $\gamma_{T_\bullet}^- \cup \gamma_{T_\bullet}^0$. Consequently, $\tilde{f}_+^\alpha = f_+^\alpha$.

1.2.2 Solutions of the Vlasov part

We give a brief introduction to the techniques and statements of Beals and Protopopescu [BP87], who used an approach via characteristics to tackle linear transport problems with initial-boundary conditions in a very general setting. Since we do not need the full statements of [BP87], we formulate those results in the way we will need them in our situation.

Throughout this subsection, let $T > 0$, $\Omega \subset \mathbb{R}^3$ be an open, bounded set with $C^{1,\kappa}$-boundary for some $\kappa > 0$, and $\Sigma_T :=]0, T[\times \Omega \times \mathbb{R}^3$. Furthermore, let Y be a first order linear differential operator that is divergence free and whose coefficients are Lipschitz continuous on $\overline{\Sigma_T}$. In accordance to our situation, we choose

$$Y := \partial_t + \widehat{v}_\alpha \cdot \partial_x + F \cdot \partial_v.$$

Thus, the assumptions about Y here reduce to two conditions on F, namely, that F is Lipschitz continuous on $\overline{\Sigma_T}$ and divergence free with respect to v. We additionally assume that F is bounded on $\overline{\Sigma_T}$. By Lipschitz continuity of F, for each $(t, x, v) \in \Sigma_T$ there is a well-defined integral curve $s \mapsto (S, X, V)(s, t, x, v)$ satisfying

$$\frac{d}{ds}S = 1, \quad \frac{d}{ds}X = \widehat{V}_\alpha, \quad \frac{d}{ds}V = F(s, X, V), \quad (S, X, V)(t, t, x, v) = (t, x, v).$$

This curve is defined as long as it remains in $\overline{\Sigma_T}$ and there is a corresponding maximal time interval $I \subset \mathbb{R}$ for which it is defined. We define the length of this curve to be the length of the maximal time interval for which the curve remains in Σ_T, that is to say, the length equals $s^+ - s^-$ where

$$s^+ := \sup\{s \in I \mid (S, X, V)(s, t, x, v) \in \Sigma_T\},$$
$$s^- := \inf\{s \in I \mid (S, X, V)(s, t, x, v) \in \Sigma_T\}.$$

The next assumption is that there is a finite upper bound to all lengths of such integral curves. This condition is trivially satisfied in our case $\Sigma_T =]0, T[\times \Omega \times \mathbb{R}^3$ since T is an upper bound. The last assumption is that each integral curve has a left and right limit point, i.e.,

$$\lim_{s\to s^-, s>s^-} (S, X, V)(s, t, x, v), \lim_{s\to s^+, s<s^+} (S, X, V)(s, t, x, v) \in \overline{\Sigma_T}.$$

These limits, if they exist, have to be elements of $\partial\Sigma_T$. For their existence it is sufficient that F is bounded by some constant $C > 0$ since then both $\dot{X}$ and $\dot{V}$ are bounded because of

$$\left|\dot{X}\right| = \left|\widehat{V}_\alpha\right| \le 1, \quad \left|\dot{V}\right| = |F(s, X, V)| \le C.$$

Accordingly, we define D_T^- (D_T^+) to be the subset of $\partial\Sigma_T$ consisting of all such left (right) limits, often referred to as incoming (outgoing) sets. These sets are Borel sets since D_T^- (D_T^+) is the image of the open set Σ_T under the continuous function that maps a point in Σ_T to the left (right) limit point of the integral curve passing through this point. Note that possibly $D_T^\pm$ are not disjoint and/or do not exhaust $\partial\Sigma_T$ but both $D_T^+ \cap D_T^-$ and $\partial\Sigma_T \setminus \left(D_T^+ \cup D_T^-\right)$ are negligible in the sense that the union of all associated integral curves in Σ_T has Lebesgue measure zero.

We proceed with the definition of the test function space corresponding to Y.

Definition 1.2.2. Let Φ_T^Y be the space of all measurable functions $\phi\colon \Sigma_T \to \mathbb{R}$ with the following three properties:

(i) ϕ is continuously differentiable along each integral curve.
(ii) ϕ and $Y\phi$ are bounded functions.
(iii) The support of ϕ is bounded and there is a positive lower bound to the lengths of the integral curves which meet the support of ϕ.

Remark 1.2.3. • Here and in the following, the term Yh, where $h \in L^1_{\mathrm{loc}}(\Sigma_T)$, is in general to be understood as a distribution, i.e.,

$$(Yh)(\varphi) = -\int_{\Sigma_T} \left(\partial_t\varphi + \widehat{v}_\alpha \cdot \partial_x\varphi + F \cdot \partial_v\varphi\right) h \, d(t, x, v), \quad \varphi \in C_c^\infty(\Sigma_T).$$

In Definition 1.2.2.(ii) or later in Definition 1.2.7.(i), this distribution is assumed to be given by a function on Σ_T.

• Because of Definition 1.2.2.(ii) and 1.2.2.(iii) we have $\phi, Y\phi \in L^q(\Sigma_T)$ for any $\phi \in \Phi_T^Y$ and $1 \le q \le \infty$.

- Note that a function $\phi \in \Phi_T^Y$ only has to be continuously differentiable along each integral curve but may be discontinuous in other directions. Because of Definition 1.2.2.(i) and 1.2.2.(ii) every $\phi \in \Phi_T^Y$ can be extended to be continuous at the endpoints of each integral curve.

Since Φ_T^Y depends on F, it cannot be suitable for the whole nonlinear system (VM), where F is unknown. Thus, an important (technical) statement is that our test function space $\Psi_{T_\bullet}$, which is independent of F, belongs to Φ_T^Y after a cut-off in the time variable (if $T \leq T_\bullet$). This is verified in the following two lemmas, where we follow the proof of [Guo93, Lemma 2.1.].

Lemma 1.2.4. *(i) For any $\iota > 0$ there is a $\delta = \delta(\iota) > 0$ such that for all $(x, v) \in \tilde{\gamma}^-$ satisfying* $\operatorname{dist}\big((x,v), \tilde{\gamma}^0\big) > \iota$ *we have $\widehat{v}_\alpha \cdot n(x) \leq -\delta$.*

(ii) For any $\iota > 0$ there is an $\eta = \eta(\iota) > 0$ such that for any $x \in \partial\Omega$, $y \in \mathbb{R}^3$ we have $y \in \Omega$ if $\big|y - x\big| < \eta$ and $\big(y - x\big) \cdot n(x) \leq -\iota\big|y - x\big| < 0$.

Proof. As for part 1.2.4.(i), suppose the contrary. Then we can find a $\iota > 0$ and a sequence $(x_k, v_k) \subset \tilde{\gamma}^-$ with $\operatorname{dist}\big((x_k, v_k), \tilde{\gamma}^0\big) > \iota$ for $k \in \mathbb{N}$ and $\widehat{v}_{k,\alpha} \cdot n(x_k) \longrightarrow 0$ for $k \longrightarrow \infty$. Without loss of generality we can assume that (v_k) is bounded: If $|v_k| \geq 1$ let $w_k := \frac{v_k}{|v_k|}$. Then,

$$\begin{aligned} 0 > \widehat{w}_{k,\alpha} \cdot n(x_k) &= |\widehat{w}_{k,\alpha}| \cos(\measuredangle(\widehat{w}_{k,\alpha}, n(x_k))) \geq |\widehat{v}_{k,\alpha}| \cos(\measuredangle(\widehat{v}_{k,\alpha}, n(x_k))) \\ &= \widehat{v}_{k,\alpha} \cdot n(x_k) \longrightarrow 0 \end{aligned}$$

for $k \longrightarrow \infty$ because of $|\widehat{w}_{k,\alpha}| \leq |\widehat{v}_{k,\alpha}|$.

Therefore, $(x_k, v_k) \subset \partial\Omega \times \mathbb{R}^3$ converges, after extracting a suitable subsequence, to some $(x, v) \in \partial\Omega \times \mathbb{R}^3$. On the one hand, we have $\operatorname{dist}\big((x,v), \tilde{\gamma}^0\big) \geq \iota$, and on the other hand $\widehat{v}_\alpha \cdot n(x) = 0$ which is a contradiction.

The proof of part 1.2.4.(ii) exploits that $\partial\Omega$ is of class $C^{1,\kappa}$. Suppose that the assertion does not hold, i.e., we can find a $\iota > 0$ and sequences $(x_k) \subset \partial\Omega$, $\big(y_k\big) \subset \mathbb{R}^3$ with $\big|y_k - x_k\big| < \frac{1}{k}$ and $\big(y_k - x_k\big) \cdot n(x_k) \leq -\iota\big|y_k - x_k\big| < 0$ but $y_k \notin \Omega$. We may assume that both sequences converge because of $(x_k) \subset \partial\Omega$ and $\big(y_k\big) \subset \partial\Omega + B_1$. The limits of both sequences have to be the same; we call the limit $x \in \partial\Omega$. Since $x_k + t\big(y_k - x_k\big) \in \Omega$ for $t > 0$ small enough and $y_k \notin \Omega$, there has to be a $\tilde{x}_k \in \big]x_k, y_k\big] \cap \partial\Omega$. Obviously we have $|\tilde{x}_k - x_k| < \frac{1}{k}$ and

$$(\tilde{x}_k - x_k) \cdot n(x_k) = \big(y_k - x_k\big) \cdot n(x_k) \frac{|\tilde{x}_k - x_k|}{\big|y_k - x_k\big|} \leq -\iota|\tilde{x}_k - x_k| < 0. \tag{1.2.5}$$

Since $\partial\Omega$ is compact and $\tilde{x}_k \longrightarrow x$, $x_k \longrightarrow x$ for $k \longrightarrow \infty$, $\tilde{x}_k$, x_k, and x lie in the image of the same $C^{1,\kappa}$-chart $\varphi\colon \mathbb{R}^2 \supset W \longrightarrow \partial\Omega$ if k is large enough. Let $\tilde{p}_k := \varphi^{-1}(\tilde{x}_k)$, $p_k := \varphi^{-1}(x_k)$, and $p := \varphi^{-1}(x)$. By continuity of φ^{-1}, both $\big(\tilde{p}_k\big)$ and $\big(p_k\big)$ converge to p. Thus, we may assume that $\tilde{p}_k, p_k \in K_p := \overline{B_r\big(p\big)} \subset W$ for suitable $r > 0$ and large k. We expand the left-hand side of (1.2.5) to get the estimate

$$|(\tilde{x}_k - x_k) \cdot n(x_k)|$$

$$
\begin{aligned}
&= \left|\varphi'(p_k)(\tilde{p}_k - p_k)\cdot n(x_k) + \left(\varphi(\tilde{p}_k) - \varphi(p_k) - \varphi'(p_k)(\tilde{p}_k - p_k)\right)\cdot n(x_k)\right| \\
&= \left|\left(\varphi(\tilde{p}_k) - \varphi(p_k) - \varphi'(p_k)(\tilde{p}_k - p_k)\right)\cdot n(x_k)\right| \\
&\le \sup_{\xi\in[p_k,\tilde{p}_k]} \left|\left(\varphi'(\xi) - \varphi'(p_k)\right)(\tilde{p}_k - p_k)\right| \le \|\varphi\|_{C^{1,\kappa}(K_p;\mathbb{R}^3)}\left|\tilde{p}_k - p_k\right|^{1+\kappa}
\end{aligned}
$$

since $n(x_k)$ is perpendicular to the columns of $\varphi'(p_k)$. Together with (1.2.5) and the fact that φ^{-1} is Lipschitz continuous on $\varphi(K_p)$ with some Lipschitz constant $L_{\varphi,K_p} > 0$—see proof below—, this yields for large k

$$
0 < \frac{\iota}{L_{\varphi,K_p}}\left|\tilde{p}_k - p_k\right| \le \iota|\tilde{x}_k - x_k| \le |(\tilde{x}_k - x_k)\cdot n(x_k)| \le \|\varphi\|_{C^{1,\kappa}(K_p;\mathbb{R}^3)}\left|\tilde{p}_k - p_k\right|^{1+\kappa}.
$$

But this contradicts $\left|\tilde{p}_k - p_k\right| \to 0$ for $k\to\infty$.

So there remains to show the Lipschitz continuity of φ^{-1} on $\varphi(K_p)$. This relies on the fact that, since φ is a chart, the function

$$
G\colon K_p\times\partial B_1 \to \mathbb{R},\quad G(\tilde{p},\delta p) = \left|\varphi'(\tilde{p})\delta p\right|
$$

is continuous and positive so that it is bounded from below by some positive constant $c > 0$. For $\tilde{x},\overline{x}\in\varphi(K_p)$ with $\tilde{x}\neq\overline{x}$ and $\tilde{p} := \varphi^{-1}(\tilde{x})$, $\overline{p} := \varphi^{-1}(\overline{x}) \in K_p$, we thus have

$$
\begin{aligned}
\frac{|\tilde{x}-\overline{x}|}{\left|\tilde{p}-\overline{p}\right|} &= \left|\varphi'(\overline{p})\frac{\tilde{p}-\overline{p}}{\left|\tilde{p}-\overline{p}\right|} + \frac{\varphi(\tilde{p}) - \varphi(\overline{p}) - \varphi'(\overline{p})(\tilde{p}-\overline{p})}{\left|\tilde{p}-\overline{p}\right|}\right| \\
&\ge c - \frac{\sup_{\xi\in[\overline{p},\tilde{p}]}\left|\left(\varphi'(\xi) - \varphi'(\overline{p})\right)(\tilde{p}-\overline{p})\right|}{\left|\tilde{p}-\overline{p}\right|} \ge c - \|\varphi\|_{C^{1,\kappa}(K_p;\mathbb{R}^3)}\left|\tilde{p}-\overline{p}\right|^{\kappa}.
\end{aligned}
$$

If φ^{-1} were not Lipschitz continuous on $\varphi(K_p)$ we would find sequences $(\tilde{x}_k)$, $(\overline{x}_k)\subset\varphi(K_p)$ and $(\tilde{p}_k) = (\varphi^{-1}(\tilde{x}_k))$, $(\overline{p}_k) = (\varphi^{-1}(\overline{x}_k))\subset K_p$ such that

$$
\frac{1}{k} \ge \frac{|\tilde{x}_k - \overline{x}_k|}{\left|\tilde{p}_k - \overline{p}_k\right|} \ge c - \|\varphi\|_{C^{1,\kappa}(K_p;\mathbb{R}^3)}\left|\tilde{p}_k - \overline{p}_k\right|^{\kappa}. \tag{1.2.6}
$$

Due to compactness of $\varphi(K_p)$ we may assume that $(\tilde{x}_k)$ and $(\overline{x}_k)$ converge to the same limit (the same because of $|\tilde{x}_k - \overline{x}_k| \le \frac{1}{k}\operatorname{diam}K_p$, where $\operatorname{diam}K_p$ is the diameter of K_p) and that the corresponding $(\tilde{p}_k)$, $(\overline{p}_k)$ also converge to the same limit due to the continuity of φ^{-1}. But this contradicts (1.2.6). Hence, φ^{-1} is Lipschitz continuous on $\varphi(K_p)$ and the proof is finished. □

Lemma 1.2.5. *For each $\psi\in\Psi_T$ we have $\psi\big|_{\Sigma_T}\in\Phi_T^Y$.*

Proof. Let $\psi\in\Psi_T$ and define

$$
d_0 := \min\left\{\operatorname{dist}\left(\operatorname{supp}\psi,\gamma_T^0\right), \operatorname{dist}\left(\operatorname{supp}\psi, \{0\}\times\partial\Omega\times\mathbb{R}^3\right)\right\} > 0.
$$

Since $\psi\big|_{\Sigma_T}$ obviously satisfies Definition 1.2.2.(i) and 1.2.2.(ii), we only have to take care about Definition 1.2.2.(iii). First note that, since the support of ψ is compact in $[0,T[\times\overline{\Omega}\times\mathbb{R}^3$, there is a $0\le s_0<T$ such that $\psi(t,x,v)=0$ for $s_0\le t<T$, $x\in\overline{\Omega}$, $v\in\mathbb{R}^3$.

We consider an integral curve which meets $\operatorname{supp}\psi=\operatorname{supp}\psi\big|_{\Sigma_T}$. This curve can be written as $s\mapsto(S,X,V)(s,s^-,x,v)$ and remains in Σ_T for a maximal time interval $]s^-,s^+[\,\subset\,]0,T[$ so that $(s^-,x,v)\in D_T^-$. Obviously it holds that $s^-\le s_0$. We have to find a positive lower bound for s^+-s^- that does not depend on s^-, s^+, x, and v. In the following, let $s\in\,]s^-,s^+[$.

Case 1. If

$$\operatorname{dist}\big((s^-,x,v),\operatorname{supp}\psi\big)\ge\frac{d_0}{2}$$

we can find an s such that $(S,X,V)(s,s^-,x,v)\in\operatorname{supp}\psi$ since the curve meets the support of ψ. By $\big|\dot X\big|\le 1$ and $\big|\dot V\big|\le\sup_{\Sigma_T}|F|$ we have

$$\begin{aligned}\frac{d_0}{2}&\le\operatorname{dist}\big((s^-,x,v),\operatorname{supp}\psi\big)\le|(S,X,V)(s,s^-,x,v)-(s^-,x,v)|\\&\le\sqrt{2+\|F\|_\infty^2}(s-s^-)\end{aligned}$$

so that $\frac{d_0}{2\sqrt{2+\|F\|_\infty^2}}$ is such a desired lower bound in this case.

Case 2. The more complicated case is

$$\operatorname{dist}\big((s^-,x,v),\operatorname{supp}\psi\big)<\frac{d_0}{2}.$$

Since $\{T\}\times\overline{\Omega}\times\mathbb{R}^3$ and γ_T^+ do not intersect D_T^-, we have

$$D_T^-\subset\gamma_T^-\cup\gamma_T^0\cup\Big(\{0\}\times\overline{\Omega}\times\mathbb{R}^3\Big).$$

Clearly, it holds that $(s^-,x,v)\notin\gamma_T^0$ because of

$$\operatorname{dist}\big((s^-,x,v),\operatorname{supp}\psi\big)<\frac{d_0}{2}<d_0\le\operatorname{dist}\big(\operatorname{supp}\psi,\gamma_T^0\big).$$

If $(s^-,x,v)\in\{0\}\times\overline{\Omega}\times\mathbb{R}^3$ we have

$$\begin{aligned}\operatorname{dist}(x,\partial\Omega)&=\operatorname{dist}\big((s^-,x,v),\{0\}\times\partial\Omega\times\mathbb{R}^3\big)\\&\ge\operatorname{dist}\big(\operatorname{supp}\psi,\{0\}\times\partial\Omega\times\mathbb{R}^3\big)-\operatorname{dist}\big((s^-,x,v),\operatorname{supp}\psi\big)>d_0-\frac{d_0}{2}=\frac{d_0}{2}.\end{aligned}$$

Thus, $X(s,s^-,x,v)\in\Omega$ for $0\le s<\min\Big\{\frac{d_0}{2},T\Big\}$ again because of $\big|\dot X\big|\le 1$. Therefore, a positive lower bound to the length of the integral curve in this case is $\min\Big\{\frac{d_0}{2},T\Big\}$.

Finally, suppose $(s^-, x, v) \in \gamma_T^-$. First note that

$$\operatorname{dist}\big((x,v),\tilde{\gamma}^0\big) = \operatorname{dist}\big((s^-,x,v),\gamma_T^0\big) \geq \operatorname{dist}\big(\operatorname{supp}\psi,\gamma_T^0\big) - \operatorname{dist}\big((s^-,x,v),\operatorname{supp}\psi\big) > d_0 - \frac{d_0}{2} = \frac{d_0}{2}. \tag{1.2.7}$$

Let $\delta = \delta\left(\frac{d_0}{2}\right)$ and $\eta = \eta\left(\frac{\delta}{2}\right)$ according to Lemma 1.2.4. We claim that

$$m := \min\left\{T - s_0, \frac{\eta}{2}, \frac{\delta}{\frac{9}{2}\|F\|_{L^\infty(\Sigma_T;\mathbb{R}^3)} + 1}\right\}$$

is such a positive lower bound (to the length of the integral curve) we search for. Indeed, we firstly have $[s^-, s^- + m] \subset [0,T]$ due to $s^- \leq s_0$. Secondly, let

$$\overline{s} := \sup\{s \in \,]s^-, s^- + m] \mid X(\tilde{s}, s^-, x, v) \in \Omega \text{ for all } \tilde{s} \in \,]s^-, s[\}.$$

Because of

$$|X(\overline{s}, s^-, x, v) - x| \leq \overline{s} - s^- < \eta$$

and

$$\begin{aligned}
(X(\overline{s}, s^-, x, v) - x)\cdot n(x) &= \left(\int_{s^-}^{\overline{s}} \widehat{V}_\alpha(\tau, s^-, x, v)\,d\tau\right)\cdot n(x) \\
&= (\overline{s} - s^-)\widehat{v}_\alpha \cdot n(x) + \int_{s^-}^{\overline{s}}\int_{s^-}^{\tau}\left(\frac{d}{ds}\widehat{V}_\alpha\right)(l, s^-, x, v)\,dl\,d\tau \cdot n(x) \\
&\leq -\delta(\overline{s} - s^-) + \frac{9}{2}\|F\|_{L^\infty(\Sigma_T;\mathbb{R}^3)} \cdot \frac{1}{2}(\overline{s} - s^-)^2 \leq -\frac{\delta}{2}(\overline{s} - s^-) \leq -\frac{\delta}{2}|X(\overline{s}, s^-, x, v) - x|
\end{aligned}$$

(which also implies $X(\overline{s}, s^-, x, v) \neq x$ since $-\frac{\delta}{2}(\overline{s} - s^-) < 0$) by (1.2.7) and $\left|\frac{d\widehat{v}_{\alpha,i}}{dv_j}\right| \leq \frac{3}{2}$, $i, j = 1, 2, 3$, we have $X(\overline{s}, s^-, x, v) \in \Omega$ and thus $\overline{s} = s^- + m$. This completes the proof. □

We should remark that the three conditions on $\psi \in \Psi_T$ in (1.1.1) are really necessary: Let $\iota > 0$ be small and, for simplicity, take $F = 0$. Firstly, if we allow a test function ψ that does not vanish before time T and has support on γ_T^-, we can find an integral curve entering $\overline{\Sigma_T}$ on $\gamma_T^- \cap \operatorname{supp}\psi$ at time $s^- = T - \iota$. Secondly, if we allow a test function ψ with support on γ_T^0, then for some $(t, x, v) \in \gamma_T^0$—such that in a neighborhood of x there are no common points of Ω and the tangent space of $\partial\Omega$ at x—the curves $(S, X, V)(s) = (s, x - \iota n(x) + (s - t)\widehat{v}_\alpha, v)$, defined for all $s \in [0, T]$, will meet the support of ψ. Thirdly, if we allow a test function ψ with support on $\{0\} \times \partial\Omega \times \mathbb{R}^3$ we can find an integral curve meeting the support of ψ, (its X-coordinate) starting at time 0 near $\partial\Omega$ and leaving Ω at time ι. In all three cases, there will be no positive lower bound to the length of these curves.

Conversely, these restrictions cause no problems for later considerations. Firstly, we do not want to test a solution of (1.2.1) at time T since we are interested in an initial, and not final, value problem. Secondly, we only want to impose a boundary condition on γ^- and not on γ^0. Thirdly, proper initial data of the distribution function have to satisfy the boundary condition at time 0 a priori so that this property need not be tested, and $\{0\} \times \partial\Omega \times \mathbb{R}^3$ is even a null set with respect to $d\gamma_\alpha$.

We now proceed with some important results of [BP87]. There, the main idea is to use the "identifications"

$$\Sigma_T \approx \left\{(s,z) \mid z \in D_T^-, 0 < s < l(z)\right\}, \quad Y \approx \frac{d}{ds},$$

where $l(z)$ is the length of the integral curve corresponding to z. The first important result is the following property which is closely related to Green's identity; see [BP87, Proposition 7].

Proposition 1.2.6. *There are two unique Borel measures $\nu^\pm$ on $D_T^\pm$ such that*

$$\int_{\Sigma_T} Y\phi \, d(t,x,v) = \int_{D_T^+} \phi \, d\nu^+ - \int_{D_T^-} \phi \, d\nu^-$$

for all $\phi \in \Phi_T^Y$.

We have to define the space of functions in which we search for solutions of some initial-boundary problem.

Definition 1.2.7. For $1 \leq p < \infty$ let $E^p(\Sigma_T; Y)$ be the space of functions $f \in L^p(\Sigma_T)$ with the following two properties:

(i) $Yf \in L^p(\Sigma_T)$.
(ii) There is a trace of f on $D_T^\pm$, i.e., a pair of functions $f^\pm \in L^p\left(D_T^\pm, d\nu^\pm\right)$ satisfying the extended Green's identity

$$\int_{\Sigma_T} \left(\phi Y f + f Y \phi\right) d(t,x,v) = \int_{D_T^+} f^+ \phi \, d\nu^+ - \int_{D_T^-} f^- \phi \, d\nu^-$$

for all $\phi \in \Phi_T^Y$.

Note that a trace in the sense as stated above is unique and that all terms are well-defined according to Remark 1.2.3.

Lemma 1.2.8. *Let $1 \leq p < \infty$, $f \in E^p(\Sigma_T; Y)$ and $w \in C^\infty(\Sigma_T) \cap C_b^1\left(\overline{\Sigma_T}\right)$. Then, $wf \in E^p(\Sigma_T; Y)$ and $\left(wf\right)^\pm = wf^\pm$.*

Proof. Because of

$$\left(Y(wf)\right)(\varphi) = -\int_{\Sigma_T} \left(\partial_t \varphi + \widehat{v}_\alpha \cdot \partial_x \varphi + F \cdot \partial_v \varphi\right) wf \, d(t,x,v)$$

$$
\begin{aligned}
&= -\int_{\Sigma_T} \big(\partial_t\big(w\varphi\big) + \widehat{v}_\alpha \cdot \partial_x\big(w\varphi\big) + F\cdot\partial_v\big(w\varphi\big)\big) f\, d(t,x,v) \\
&\quad + \int_{\Sigma_T} (\partial_t w + \widehat{v}_\alpha \cdot \partial_x w + F\cdot \partial_v w) f\varphi\, d(t,x,v) \\
&= \int_{\Sigma_T} (Yf) w\varphi\, d(t,x,v) + \int_{\Sigma_T} (Yw) f\varphi\, d(t,x,v)
\end{aligned}
$$

for any $\varphi \in C_c^\infty(\Sigma_T)$, it holds that $Y\big(wf\big) = wYf + fYw \in L^p(\Sigma_T)$. Now let $\phi \in \Phi_T^Y$. We have $w\phi \in \Phi_T^Y$ since Definition 1.2.2.(i) and 1.2.2.(ii) are satisfied because of the regularity of w and Definition 1.2.2.(iii) is satisfied because of $\operatorname{supp}\big(w\phi\big) \subset \operatorname{supp}\phi$. Thus, it holds that

$$
\begin{aligned}
\int_{\Sigma_T} \big(\phi Y\big(wf\big) + wfY\phi\big) d(t,x,v) &= \int_{\Sigma_T} \big(w\phi Yf + fY\big(w\phi\big)\big) d(t,x,v) \\
&= \int_{D_T^+} f^+ w\phi\, d\nu^+ - \int_{D_T^-} f^- w\phi\, d\nu^-,
\end{aligned}
$$

which proves the assertion. □

In the following it is convenient to split $D_T^\pm$ as follows:

$$
\begin{aligned}
D_\pm^T &:= \big\{(t,x,v) \in D_T^\pm \mid 0 < t < T\big\}, \\
D_0 := \big\{(t,x,v) \in D_T^- \mid t = 0\big\}, &\quad D_T^T := \big\{(t,x,v) \in D_T^+ \mid t = T\big\},
\end{aligned}
$$

so that $D_T^- = D_-^T \cup D_0$ and $D_T^+ = D_+^T \cup D_T^T$. Note that D_0 does not depend on T (in the sense that any $0 < \tilde{T} < T$ yields the same set D_0). According to this decomposition we write $d\nu_- = d\nu^-\big|_{D_-^T}$, $d\nu_0 = d\nu^-\big|_{D_0}$, $d\nu_+ = d\nu^+\big|_{D_+^T}$, $d\nu_T = d\nu^+\big|_{D_T^T}$, $f_- = f^-\big|_{D_-^T}$, $f_0 = f^-\big|_{D_0}$, $f_+ = f^+\big|_{D_+^T}$, and $f_T = f^+\big|_{D_T^T}$. We have

$$
\begin{aligned}
\{0\}\times\Omega\times\mathbb{R}^3 \subset D_0 \subset \{0\}\times\overline{\Omega}\times\mathbb{R}^3, &\quad \{T\}\times\Omega\times\mathbb{R}^3 \subset D_T^T \subset \{T\}\times\overline{\Omega}\times\mathbb{R}^3, \\
\gamma_T^- \subset D_-^T \subset \gamma_T^- \cup \gamma_T^0, &\quad \gamma_T^+ \subset D_+^T \subset \gamma_T^+ \cup \gamma_T^0.
\end{aligned}
$$

Therefore, we can identify L^p-functions on D_0 (or D_T^T) with L^p-functions on $\Omega\times\mathbb{R}^3$ since $\left(\overline{\Omega}\times\mathbb{R}^3\right) \setminus \big(\Omega\times\mathbb{R}^3\big)$ has (x,v)-Lebesgue measure zero. Additionally, we may write $f(0)$ and $f(T)$ instead of f_0 and f_T pointing out that we may evaluate f at time 0 and T in some sense.

For each $\psi \in \Psi_T$ we have

$$
\begin{aligned}
\int_{\Sigma_T} Y\psi\, d(t,x,v) &= -\int_\Omega\int_{\mathbb{R}^3} \psi(0) dv dx + \int_0^T \int_{\partial\Omega}\int_{\mathbb{R}^3} \psi \widehat{v}_\alpha \cdot n\, dv dS_x dt \\
&= -\int_\Omega\int_{\mathbb{R}^3} \psi(0) dv dx + \int_{\gamma_T^+} \psi\, d\gamma_\alpha - \int_{\gamma_T^-} \psi\, d\gamma_\alpha.
\end{aligned}
$$

This shows that $d\nu_0 = d(x, v)$ on D_0 and $d\nu_\pm = d\gamma_\alpha$ on $\gamma_T^\pm$. With an analog reasoning (consider test functions $\tilde{\psi}(t, x, v) = \psi(T - t, x, v)$, $\psi \in \Psi_T$) we conclude that $d\nu_T = d(x, v)$ on D_T^T as well.

We proceed with a definition of some properties of operators.

Definition 1.2.9. Let O be an operator between two function spaces on subsets of some $\mathbb{R}^n$, whose first component we call time. O is called

(i) local in time if $O(uv) = uO(v)$ for all continuous functions u that only depend on time and all possible v;
(ii) nonnegative if $O(v) \geq 0$ for all $v \geq 0$.

Now we are ready to state the following result regarding the unique solvability of linear transport problems with initial-boundary conditions; see [BP87, Proposition 1, Theorems 1 and 2].

Proposition 1.2.10. *Let* $1 \leq p < \infty$, $h \in L^\infty(\Sigma_T)$, $F\colon \overline{\Sigma_T} \to \mathbb{R}$ *be Lipschitz continuous, differentiable with respect to* v*, and divergence free with respect to* v*, and* $Y = \partial_t + \widehat{v}_\alpha \cdot \partial_x + F \cdot \partial_v$.

(i) For all $f \in E^p(\Sigma_T; Y)$ *we have*

$$\int_{D_T^T} |f_T|^p \, d\nu_T + \int_{D_+^T} |f_+|^p \, d\nu_+ + p \int_{\Sigma_T} h|f|^p \, d(t, x, v)$$
$$= \int_{D_0} |f_0|^p \, d\nu_0 + \int_{D_-^T} |f_-|^p \, d\nu_- + p \int_{\Sigma_T} \operatorname{sign}(f)|f|^{p-1}(Y + h)f \, d(t, x, v). \quad (1.2.8)$$

(ii) Let moreover $\mathfrak{R}\colon L^p(D_+^T, d\nu_+) \to L^p(D_-^T, d\nu_-)$ *be a bounded linear operator, that is local in time and has operator norm less than* 1*, and* $g_0 \in L^p(D_0)$, $g_- \in L^p(D_-^T, d\nu_-)$*. Then the problem*

$$Yf = 0 \qquad \text{on } \Sigma_T, \qquad (1.2.9a)$$
$$f_0 = g_0 \qquad \text{on } D_0, \qquad (1.2.9b)$$
$$f_- = \mathfrak{R} f_+ + g_- \qquad \text{on } D_-^T \qquad (1.2.9c)$$

has a unique solution $f \in E^p(\Sigma_T; Y)$*. Here,* (1.2.9a) *holds pointwise almost everywhere (cf. Definition 1.2.7.(i) and Remark 1.2.3), and* (1.2.9b) *and* (1.2.9c) *hold pointwise almost everywhere (with respect to the corresponding measures) in the sense of trace (cf. Definition 1.2.7.(ii)). Moreover, the solution is nonnegative if* $\mathfrak{R}$, g_0*, and* g_- *are nonnegative.*

Here and in the following, for functions the property "nonnegative" usually means "nonnegative almost everywhere". We want to express, in some way, the theorem above in words that fit to our problem (1.2.1), that is to say, we should somehow replace D_0, D_-^T (and so on) by $\Omega \times \mathbb{R}^3$, γ_T^- (and so on). Moreover, we search for solutions of (1.2.1) on $I_{T_\bullet}$ instead of solutions on some time interval $[0, T]$. To this end, we first have to define what we call a strong solution of (1.2.1). From now on, the force term F shall satisfy the following condition.

Condition 1.2.11. $F\colon I_{T_\bullet} \times \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}^3$ is Lipschitz continuous and bounded on $\overline{\Sigma_T}$ for any $T \in I_{T_\bullet}$, and moreover differentiable and divergence free (both) with respect to v.

Definition 1.2.12. Assume that $1 \le p < \infty$, $\mathcal{K}\colon L^p_{\mathrm{lt}}\Big(\gamma^+_{T_\bullet}, d\gamma_\alpha\Big) \to L^p_{\mathrm{lt}}\Big(\gamma^-_{T_\bullet}, d\gamma_\alpha\Big)$, $\mathring{f} \in L^p\big(\Omega \times \mathbb{R}^3\big)$, and $g \in L^p_{\mathrm{lt}}\Big(\gamma^-_{T_\bullet}, d\gamma_\alpha\Big)$. We call a function $f\colon I_{T_\bullet} \times \Omega \times \mathbb{R}^3 \to \mathbb{R}$ a strong solution of (1.2.1) if:

(i) $\chi_T f \in E^p(\Sigma_T; Y)$ for all $0 < T \in I_{T_\bullet}$.
(ii) For all $\psi \in \Psi_{T_\bullet}$ it holds that

$$\begin{aligned}
&\int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3}\big(\partial_t\psi + \widehat{v}_\alpha\cdot\partial_x\psi + F\cdot\partial_v\psi\big)f\,dv\,dx\,dt\\
&= \int_{\gamma^+_{T_\bullet}} f_+\psi\,d\gamma_\alpha - \int_{\gamma^-_{T_\bullet}}(\mathcal{K}f_+ + g)\psi\,d\gamma_\alpha - \int_\Omega\int_{\mathbb{R}^3}\mathring{f}\psi(0)\,dv\,dx.
\end{aligned}$$

Note that, for each $0 < T \in I_{T_\bullet}$, at first only a trace of $\chi_T f$ is defined. By uniqueness, for another $I_{T_\bullet} \ni T' \ge T$, the traces of $\chi_{T'} f$ and $\chi_T f$ coincide on the common time interval $[0,T]$. Thus, we may write $f_\pm$, which is defined on all of $I_{T_\bullet}$, and may drop the dependence on some T.

Proposition 1.2.13. *Let $1 \le p < \infty$, F satisfy Condition 1.2.11, and $\mathcal{K}\colon L^p_{\mathrm{lt}}\Big(\gamma^+_{T_\bullet}, d\gamma_\alpha\Big) \to L^p_{\mathrm{lt}}\Big(\gamma^-_{T_\bullet}, d\gamma_\alpha\Big)$ be a linear operator, that is local in time and such that there is a $0 \le k_0 < 1$ satisfying*

$$\|\mathcal{K}c\|_{L^p\left(\gamma^-_T, d\gamma_\alpha\right)} \le k_0\|c\|_{L^p\left(\gamma^+_T, d\gamma_\alpha\right)}$$

for all $c \in L^p_{\mathrm{lt}}\Big(\gamma^+_{T_\bullet}, d\gamma_\alpha\Big)$, $0 < T \in I_{T_\bullet}$. Furthermore, let $\mathring{f} \in L^p\big(\Omega \times \mathbb{R}^3\big)$ and $g \in L^p_{\mathrm{lt}}\Big(\gamma^-_{T_\bullet}, d\gamma_\alpha\Big)$. Then:

(i) There is exactly one strong solution of (1.2.1) *in the sense of Definition 1.2.12.*
(ii) This solution is nonnegative if $\mathcal{K}$, $\mathring{f}$, and g are nonnegative.

Proof. Let $0 < T \in I_{T_\bullet}$ and define

$$\begin{aligned}
&g^T_-\colon D^T_- \to \mathbb{R}, \quad g^T_-(t,x,v) = \begin{cases} g(t,x,v), & (t,x,v) \in \gamma^-_T, \\ 0, & \text{otherwise;}\end{cases}\\
&g_0\colon D_0 \to \mathbb{R}, \quad g_0(0,x,v) = \mathring{f}(x,v).
\end{aligned}$$

Note that the latter definition makes sense since, as mentioned above, D_0 coincides with $\{0\} \times \Omega \times \mathbb{R}^3$ up to a negligible set. We have

$$\big\|g^T_-\big\|_{L^p\left(D^T_-, d\nu_-\right)} = \big\|\chi_T g\big\|_{L^p(\gamma^-, d\gamma_\alpha)}, \quad \|g_0\|_{L^p(D_0)} = \Big\|\mathring{f}\Big\|_{L^p(\Omega\times\mathbb{R}^3)}$$

so that $g_-^T \in L^p(D_-^T, d\nu^\pm)$ and $g_0 \in L^p(D_0)$. Furthermore, for $h \in L^p(D_+^T, d\nu_+)$ let

$$\overline{h}\colon \gamma_{T_\bullet}^+ \to \mathbb{R}, \quad \overline{h}(t,x,v) = \begin{cases} h(t,x,v), & (t,x,v) \in \gamma_T^+, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathfrak{K}_T h\colon D_-^T \to \mathbb{R}, \quad (\mathfrak{K}_T h)(t,x,v) = \begin{cases} \left(\mathcal{K}\overline{h}\right)(t,x,v), & (t,x,v) \in \gamma_T^-, \\ 0, & \text{otherwise.} \end{cases}$$

Because of

$$\|\mathfrak{K}_T h\|_{L^p(D_-^T, d\nu_-)} = \left\|\chi_T \mathcal{K}\overline{h}\right\|_{L^p(\gamma_T^-, d\gamma_\alpha)} \le k_0 \left\|\chi_T \overline{h}\right\|_{L^p(\gamma_T^+, d\gamma_\alpha)} = k_0 \|h\|_{L^p(D_+^T, d\nu_+)}$$

we conclude that $\mathfrak{K}_T$ maps $L^p(D_+^T, d\nu_+)$ to $L^p(D_-^T, d\nu_-)$ and has operator norm less than 1. Moreover, $\mathfrak{K}_T$ is local in time. Thus, by Proposition 1.2.10.(ii) there is a solution of (1.2.9) (with $\mathfrak{K}_T$, g_0, g_-^T given). By uniqueness and $D_-^T \subset D_-^{T'}$ for $T \le T'$, such a solution (and its trace) does not depend on T, whence there is a function f such that $\chi_T f \in E^p(\Sigma_T; Y)$ is the unique solution of (1.2.9) for any given T. Now take $\psi \in \Psi_{T_\bullet}$ and $0 < T < T_\bullet$ such that $\psi(t,x,v) = 0$ if $t \ge T$. By Lemma 1.2.5 we have $\psi\big|_{\Sigma_T} \in \Phi_T^Y$. Applying the definition of trace and using the properties of ψ, this leads to

$$\begin{aligned}
\int_{\Sigma_{T_\bullet}} fY\psi \, d(t,x,v) &= \int_{\Sigma_T} fY\psi \, d(t,x,v) \\
&= -\int_{D_0} f\psi \, d\nu_0 + \int_{D_+^T} f^+\psi \, d\nu_+ - \int_{D_-^T} (\mathfrak{K}_T f^+ + g_-^T)\psi \, d\nu_- \\
&= -\int_\Omega \int_{\mathbb{R}^3} \mathring{f}\psi(0) \, dv dx + \int_{\gamma_T^+} f^+\psi \, d\gamma_\alpha - \int_{\gamma_T^-} (\mathfrak{K}_T f^+ + g_-^T)\psi \, d\gamma_\alpha \\
&= -\int_\Omega \int_{\mathbb{R}^3} \mathring{f}\psi(0) \, dv dx + \int_{\gamma_{T_\bullet}^+} f^+\psi \, d\gamma_\alpha - \int_{\gamma_{T_\bullet}^-} \left(\mathcal{K} f^+ + g\right)\psi \, d\gamma_\alpha
\end{aligned}$$

and the proof of part 1.2.13.(i) is complete.

Part 1.2.13.(ii) follows from the fact that $\mathfrak{K}_T$, g_0, and g_-^T are nonnegative if $\mathcal{K}$, $\mathring{f}$, and g are nonnegative, and Proposition 1.2.10.(ii). □

We now turn to the special situation that $\mathcal{K} = aK$, where K is the reflection operator. According to (0.2), Kf is defined for any function f (that is defined on a subset of $\mathbb{R} \times \partial\Omega \times \mathbb{R}^3$) and is self inverse, i.e., $K^2 = \mathrm{id}$. Its restriction to $L^p_{\mathrm{lt}}\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)$ yields an operator $K\colon L^p_{\mathrm{lt}}\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right) \to L^p_{\mathrm{lt}}\left(\gamma_{T_\bullet}^-, d\gamma_\alpha\right)$ for any $1 \le p \le \infty$. Its inverse is $K\colon L^p_{\mathrm{lt}}\left(\gamma_{T_\bullet}^-, d\gamma_\alpha\right) \to L^p_{\mathrm{lt}}\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)$.

Lemma 1.2.14. *For any $T > 0$, $1 \le p < \infty$, and any $a, b \in L^\infty(\gamma_T^-)$, $c \in L^p(\gamma_T^+, d\gamma_\alpha)$, and $h \in L^p(\gamma_T^-, d\gamma_\alpha)$ we have*

$$\int_{\gamma_T^-} b|aKc + h|^p \, d\gamma_\alpha = \int_{\gamma_T^+} Kb|cKa + Kh|^p \, d\gamma_\alpha.$$

If additionally $a_0 := \|a\|_{L^\infty(\gamma_T^-)} < 1$ and a, b, c, h are nonnegative, the estimate

$$\int_{\gamma_T^-} b|aKc + h|^p \, d\gamma_\alpha \le a_0 \int_{\gamma_T^+} (Kb)c^p \, d\gamma_\alpha + (1 - a_0)^{1-p} \int_{\gamma_T^-} bh^p \, d\gamma_\alpha$$

holds.

Proof. We compute

$$\begin{aligned}
&\int_{\gamma_T^-} b|aKc + h|^p \, d\gamma_\alpha \\
&= \iiint_{\gamma_T^-} b(t,x,v)|a(t,x,v)c(t,x,v - 2(v\cdot n(x))n(x)) + h(t,x,v)|^p |\widehat{v}_\alpha \cdot n(x)| \, dv dS_x dt \\
&= \iiint_{\gamma_T^+} b(t,x,v - 2(v\cdot n(x))n(x))|a(t,x,v - 2(v\cdot n(x))n(x))c(t,x,v) \\
&\qquad\qquad + h(t,x,v - 2(v\cdot n(x))n(x))|^p |-\widehat{v}_\alpha \cdot n(x)| \, dv dS_x dt \\
&= \int_{\gamma_T^+} Kb|cKa + Kh|^p \, d\gamma_\alpha.
\end{aligned}$$

using the change of variables $v \mapsto v - 2(v \cdot n(x))n(x)$. Note that the determinant of the corresponding Jacobian equals -1 since the map is a reflection. As for the second statement, we estimate

$$\begin{aligned}
\int_{\gamma_T^-} b|aKc + h|^p d\gamma_\alpha &= \int_{\gamma_T^+} Kb|cKa + Kh|^p d\gamma_\alpha \\
&\le \int_{\gamma_T^+} Kb\left|a_0 c + (1 - a_0)(1 - a_0)^{-1}Kh\right|^p \, d\gamma_\alpha \\
&\le a_0 \int_{\gamma_T^+} (Kb)c^p d\gamma_\alpha + (1 - a_0) \int_{\gamma_T^+} Kb\left|(1 - a_0)^{-1}Kh\right|^p \, d\gamma_\alpha \\
&= a_0 \int_{\gamma_T^+} (Kb)c^p d\gamma_\alpha + (1 - a_0)^{1-p} \int_{\gamma_T^-} bh^p \, d\gamma_\alpha
\end{aligned}$$

using the convexity of the p-th power and the first statement. □

Proposition 1.2.15. *Consider $\mathcal{K} = aK$, where $0 \leq a \in L^\infty\left(\gamma_{T_\bullet}^-\right)$ with $a_0 := \|a\|_{L^\infty\left(\gamma_{T_\bullet}^-\right)} < 1$. Let F satisfy Condition 1.2.11, $0 \leq \mathring{f} \in \left(L^1 \cap L^\infty\right)(\Omega \times \mathbb{R}^3)$, $0 \leq g \in \left(L^1_{\mathrm{lt}} \cap L^\infty_{\mathrm{lt}}\right)\left(\gamma_{T_\bullet}^-, d\gamma_\alpha\right)$. Then there is a unique, nonnegative strong solution $f \in L^\infty_{\mathrm{lt}}\left(I_{T_\bullet}; \left(L^1 \cap L^\infty\right)(\Omega \times \mathbb{R}^3)\right)$ of* (1.2.1) *with nonnegative trace $f_\pm \in \left(L^1_{\mathrm{lt}} \cap L^\infty_{\mathrm{lt}}\right)\left(\gamma_{T_\bullet}^\pm, d\gamma_\alpha\right)$ in the sense that the conditions of Definition 1.2.12 are satisfied for all $1 \leq p < \infty$. Moreover, we have the estimates*

$$(1-a_0)^{\frac{1}{p}}\|f_+\|_{L^p\left(\gamma_T^+, d\gamma_\alpha\right)}, \|f(T)\|_{L^p(\Omega\times\mathbb{R}^3)} \leq \left\|\mathring{f}\right\|_{L^p(\Omega\times\mathbb{R}^3)} + (1-a_0)^{\frac{1}{p}-1}\|g\|_{L^p\left(\gamma_T^-, d\gamma_\alpha\right)} \tag{1.2.10}$$

and

$$\begin{aligned}(1-a_0)\int_{\gamma_T^+\cap\{|v|<R\}} \theta f_+ \, d\gamma_\alpha + \int_\Omega\int_{B_R} \theta f(T)\, dvdx \\ \leq \int_\Omega\int_{\mathbb{R}^3} \theta \mathring{f}\, dvdx + \int_{\gamma_T^-} \theta g\, d\gamma_\alpha + \int_0^T\int_\Omega\int_{B_R} F\cdot f\nabla\theta\, dvdxdt\end{aligned} \tag{1.2.11}$$

for any $0 < T \in I_{T_\bullet}$, $1 \leq p \leq \infty$ (where $\frac{1}{\infty} = 0$), $0 < R \leq \infty$ (where $B_\infty = \mathbb{R}^3$), and any nonnegative C^1-function $\theta = \theta(v)$ on $\mathbb{R}^3$ that only depends on $|v|$, is monotonically increasing in $|v|$ (i.e., $\theta(v) = \tilde{\theta}(|v|)$ for some monotonically increasing $\tilde{\theta} \in C^1(\mathbb{R}_{\geq 0})$ with $\tilde{\theta}'(0) = 0$), and has the property that $\nabla\theta \in L^q\left(\mathbb{R}^3; \mathbb{R}^3\right)$ for some $1 \leq q \leq \infty$. In particular,

$$\begin{aligned}(1-a_0)\int_{\gamma_T^+\cap\{|v|<R\}} v_\alpha^0 f_+ \, d\gamma_\alpha + \int_\Omega\int_{B_R} v_\alpha^0 f(T)\, dvdx \\ \leq \int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}\, dvdx + \int_{\gamma_T^-} v_\alpha^0 g\, d\gamma_\alpha + \int_0^T\int_\Omega\int_{B_R} F\cdot \widehat{v}_\alpha f\, dvdxdt\end{aligned} \tag{1.2.12}$$

*and hence $f \in L^\infty_{\mathrm{lt}}\left(I_{T_\bullet}; \left(L^1_{\alpha\mathrm{kin}} \cap L^\infty\right)(\Omega\times\mathbb{R}^3)\right)$ and $f_\pm \in \left(L^1_{\alpha\mathrm{kin,lt}} \cap L^\infty_{\mathrm{lt}}\right)\left(\gamma_{T_\bullet}^\pm, d\gamma_\alpha\right)$ if additionally $\mathring{f} \in L^1_{\alpha\mathrm{kin}}(\Omega\times\mathbb{R}^3)$ and $g \in L^1_{\alpha\mathrm{kin,lt}}\left(\gamma_{T_\bullet}^-, d\gamma_\alpha\right)$.
Furthermore,*

$$\begin{aligned}&\left\|\int_{B_R} f(T,\cdot,v)\, dv\right\|_{L^{\frac{4}{3}}(\Omega)} \\ &\leq \left(\frac{4\pi}{3}\left\|\mathring{f}\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} + \frac{4\pi}{3}(1-a_0)^{-1}\|g\|_{L^\infty(\gamma_T^-)} + 1\right)\left(\int_\Omega\int_{B_R} v_\alpha^0 f(T)\, dvdx\right)^{\frac{3}{4}}\end{aligned} \tag{1.2.13}$$

for any $0 < T \in I_{T_\bullet}$ and $0 < R \leq \infty$.

Proof. Let $1 \leq p < \infty$ and consider $\mathcal{K} = aK\colon L^p_{\mathrm{lt}}\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right) \to L^p_{\mathrm{lt}}\left(\gamma_{T_\bullet}^-, d\gamma_\alpha\right)$, which is linear and local in time. We have

$$\|\mathcal{K}c\|_{L^p\left(\gamma_T^-, d\gamma_\alpha\right)} = \|cKa\|_{L^p\left(\gamma_T^+, d\gamma_\alpha\right)} \leq a_0\|c\|_{L^p\left(\gamma_T^+, d\gamma_\alpha\right)}$$

for all $c \in L^p_{\mathrm{lt}}\Big(\gamma^+_{T_\bullet}, d\gamma_\alpha\Big)$ and $0 < T \in I_{T_\bullet}$ by Lemma 1.2.14. Thus, by Proposition 1.2.13 there is a unique solution for this p in the sense of Definition 1.2.12. Since p was arbitrary, it follows that, for all $1 \le p < \infty$, $f \in L^p_{\mathrm{lt}}(\Sigma_{T_\bullet})$ and $f_\pm \in L^p_{\mathrm{lt}}\Big(\gamma^\pm_{T_\bullet}, d\gamma_\alpha\Big)$, and all conditions of Definition 1.2.12 are satisfied.

Let $0 < T \in I_{T_\bullet}$ and recall that in the proof of Proposition 1.2.13 the solution on $[0,T]$ was given by a solution of (1.2.9) with $\mathfrak{K}_T$, g_0, g^T_-. Thus, $f_- = \mathfrak{K}_T f_+ + g^T_- = 0$ on $D^T_- \setminus \gamma^-_T$. Applying Proposition 1.2.10.(i) (with $h = 0$), dropping negligible terms in (1.2.8), and using Lemma 1.2.14 we arrive at

$$\begin{aligned}\int_{\gamma^+_T} f_+^p \, d\gamma_\alpha + \int_\Omega\int_{\mathbb{R}^3} f(T)^p \, dvdx &= \int_\Omega\int_{\mathbb{R}^3} \mathring{f}^p \, dvdx + \int_{\gamma^-_T} \big(aKf_+ + g\big)^p \, d\gamma_\alpha \\ &\le \int_\Omega\int_{\mathbb{R}^3} \mathring{f}^p \, dvdx + a_0 \int_{\gamma^+_T} f_+^p \, d\gamma_\alpha + (1-a_0)^{1-p} \int_{\gamma^-_T} g^p \, d\gamma_\alpha.\end{aligned}$$

This yields

$$(1-a_0)\int_{\gamma^+_T} f_+^p \, d\gamma_\alpha + \int_\Omega\int_{\mathbb{R}^3} f(T)^p \, dvdx \le \int_\Omega\int_{\mathbb{R}^3} \mathring{f}^p \, dvdx + (1-a_0)^{1-p}\int_{\gamma^-_T} g^p \, d\gamma_\alpha$$

and therefore

$$\begin{aligned}(1-a_0)^{\frac{1}{p}} \|f_+\|_{L^p(\gamma^+_T, d\gamma_\alpha)}, \|f(T)\|_{L^p(\Omega\times\mathbb{R}^3)} &\le \left(\int_\Omega\int_{\mathbb{R}^3} \mathring{f}^p \, dvdx + (1-a_0)^{1-p}\int_{\gamma^-_T} g^p \, d\gamma_\alpha\right)^{\frac{1}{p}} \\ &\le \big\|\mathring{f}\big\|_{L^p(\Omega\times\mathbb{R}^3)} + (1-a_0)^{\frac{1}{p}-1}\|g\|_{L^p(\gamma^-_T, d\gamma_\alpha)}\end{aligned}$$

by $r^p + s^p \le (r+s)^p$ for $r, s \ge 0$. Letting $p \to \infty$ we deduce (1.2.10) also for $p = \infty$. This also shows that $f \in L^\infty_{\mathrm{lt}}(\Sigma_{T_\bullet})$ (thus $f \in L^\infty_{\mathrm{lt}}\big(I_{T_\bullet}; \big(L^1 \cap L^\infty\big)\big(\Omega \times \mathbb{R}^3\big)\big)$ altogether) and $f_+ \in L^\infty_{\mathrm{lt}}\Big(\gamma^+_{T_\bullet}, d\gamma_\alpha\Big)$ (thus $f_+ \in \Big(L^1_{\mathrm{lt}} \cap L^\infty_{\mathrm{lt}}\Big)\Big(\gamma^+_{T_\bullet}, d\gamma_\alpha\Big)$ altogether and therefore $f_- \in \Big(L^1_{\mathrm{lt}} \cap L^\infty_{\mathrm{lt}}\Big)\Big(\gamma^-_{T_\bullet}, d\gamma_\alpha\Big)$).

To prove (1.2.11), let first $0 < R < \infty$ and

$$\beta\colon \mathbb{R}^3 \to \mathbb{R}, \quad \beta(v) = \begin{cases} \theta(v), & |v| < R, \\ \theta(R), & |v| \ge R. \end{cases}$$

Clearly we have $\beta - \theta(R) \in W^{1,1}\big(\mathbb{R}^3\big)$. Now let $\iota > 0$ be arbitrary and choose $\beta_\iota \in C^1_b\big(\mathbb{R}^3\big) \cap C^\infty\big(\mathbb{R}^3\big)$ with $\|\beta_\iota - \beta\|_{W^{1,1}(\mathbb{R}^3)} < \iota$. This β_ι can be chosen in a such a way that it is nonnegative and only depends on $|v|$ since β is nonnegative and only depends on $|v|$. Proceeding similarly as before, we define $\tilde{f}(t,x,v) := \beta_\iota(v) f(t,x,v) \ge 0$, noticing that $Y\tilde{f} = F \cdot f\nabla\beta\iota$ and $K\beta_\iota = \beta_\iota$, and apply Lemmas 1.2.8 and 1.2.14 and Proposition 1.2.10.(i) to $\tilde{f}$ for $p = 1$:

$$\int_{\gamma^+_T} \beta_\iota f_+ \, d\gamma_\alpha + \int_\Omega\int_{\mathbb{R}^3} \beta_\iota f(T) \, dvdx$$

$$
\begin{aligned}
&= \int_\Omega \int_{\mathbb{R}^3} \beta_\iota \mathring{f}\, dv dx + \int_{\gamma_T^-} \beta_\iota \big(aKf_+ + g\big)\, d\gamma_\alpha + \int_{\Sigma_T} F \cdot f \nabla \beta\iota \, d(t,x,v) \\
&\leq \int_\Omega \int_{\mathbb{R}^3} \beta_\iota \mathring{f}\, dv dx + a_0 \int_{\gamma_T^+} \beta_\iota f_+ \, d\gamma_\alpha + \int_{\gamma_T^-} \beta_\iota g\, d\gamma_\alpha + \int_{\Sigma_T} F \cdot f \nabla \beta\iota \, d(t,x,v),
\end{aligned}
$$

so that

$$
\begin{aligned}
&(1-a_0) \int_{\gamma_T^+} \beta_\iota f_+ \, d\gamma_\alpha + \int_\Omega \int_{\mathbb{R}^3} \beta_\iota f(T)\, dv dx \\
&\leq \int_\Omega \int_{\mathbb{R}^3} \beta_\iota \mathring{f}\, dv dx + \int_{\gamma_T^-} \beta_\iota g\, d\gamma_\alpha + \int_{\Sigma_T} F \cdot f \nabla \beta\iota \, d(t,x,v). \qquad (1.2.14)
\end{aligned}
$$

Taking the limit $\iota \to 0$ does not cause any problem because we have $\left\|\beta_\iota - \beta\right\|_{W^{1,1}(\mathbb{R}^3)} \to 0$ for $\iota \to 0$, $F \in L^\infty\big(\Sigma_T;\mathbb{R}^3\big)$, $f \in L^\infty(\Sigma_T)$, $f_+ \in L^\infty\big(\gamma_T^+, d\gamma_\alpha\big)$, $\mathring{f}, f(T) \in L^\infty\big(\Omega \times \mathbb{R}^3\big)$, $g \in L^\infty\big(\gamma_T^-, d\gamma_\alpha\big)$, and the fact that the (surface) measures of Ω, $\partial\Omega$, and $[0,T]$ are all finite. Hence, (1.2.14) holds with ι removed. Next we insert the definition of β and drop the terms where $|v| \geq R$ on the left-hand side to get

$$
\begin{aligned}
&(1-a_0) \int_{\gamma_T^+ \cap \{|v|<R\}} \theta f_+ \, d\gamma_\alpha + \int_\Omega \int_{B_R} \theta f(T)\, dv dx \\
&\leq \int_\Omega \int_{B_R} \theta \mathring{f}\, dv dx + \theta(R) \int_\Omega \int_{|v|\geq R} \mathring{f}\, dv dx + \int_{\gamma_T^- \cap \{|v|<R\}} \theta g\, d\gamma_\alpha \\
&\quad + \theta(R) \int_{\gamma_T^- \cap \{|v|\geq R\}} g\, d\gamma_\alpha + \int_0^T \int_\Omega \int_{B_R} F \cdot f \nabla \theta \, dv dx dt \\
&\leq \int_\Omega \int_{\mathbb{R}^3} \theta \mathring{f}\, dv dx + \int_{\gamma_T^-} \theta g\, d\gamma_\alpha + \int_0^T \int_\Omega \int_{B_R} F \cdot f \nabla \theta \, dv dx dt
\end{aligned}
$$

since θ is monotonically increasing in $|v|$. Note that it is important that $\nabla\beta$ vanishes for $|v| > R$. This proves (1.2.11) for $0 < R < \infty$. Because of $\nabla\theta \in L^q\big(\mathbb{R}^3;\mathbb{R}^3\big)$, $F \in L^\infty\big(\Sigma_T;\mathbb{R}^3\big)$, $f \in L^{q'}(\Sigma_T)$ (where $\frac{1}{q} + \frac{1}{q'} = 1$), and the fact that the measures of Ω and $[0,T]$ are finite, we know that $F \cdot f \nabla\theta \in L^1(\Sigma_T)$. By letting $R \to \infty$ we thus obtain (1.2.11) for $R = \infty$. In particular, we get (1.2.12) for $\theta(v) = v_\alpha^0$ noticing that $\nabla\theta(v) = \widehat{v}_\alpha$ is a bounded function.

As for (1.2.13), let $0 < R \leq \infty$ and first derive the following key estimate:

$$
\begin{aligned}
&\int_{B_R} f(T,x,v)\, dv \leq \int_{B_r} f(T,x,v)\, dv + \int_{r\leq|v|<R} f(T,x,v)\, dv \\
&\leq \frac{4\pi}{3} r^3 \left\|f(T)\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} + \frac{1}{r} \int_{B_R} v_\alpha^0 f(T,x,v)\, dv dx \\
&\leq \left(\int_{B_R} v_\alpha^0 f(T,x,v)\, dv\right)^{\frac{3}{4}} \left(\frac{4\pi}{3} \left\|\mathring{f}\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} + \frac{4\pi}{3} (1-a_0)^{-1} \left\|g\right\|_{L^\infty(\gamma_T^-, d\gamma_\alpha)} + 1\right) \qquad (1.2.15)
\end{aligned}
$$

where we set $r := \left(\int_{B_R} v_\alpha^0 f(T,x,v)\,dv\right)^{\frac{1}{4}} \in [0,\infty[$ and used (1.2.10); if $r = 0$, the second step makes no sense but clearly both the left-hand side and the right-hand side of (1.2.15) are zero in this case. Note that the integral on the left-hand side exists for almost all $(T,x) \in I_{T_\bullet} \times \Omega$ by Fubini's theorem and that the first estimate above holds trivially if $r > R$ by $f \geq 0$ and is an equality if $r \leq R$. Taking both sides of (1.2.15) to the power $\frac{4}{3}$ and then integrating over Ω yields (1.2.13). □

Remark 1.2.16. The L^∞-spaces on $\gamma_T^\pm$ with respect to $d\gamma_\alpha$ and the standard surface measure are the same and the respective norms coincide since null sets with respect to $d\gamma_\alpha$ are null sets with respect to the standard surface measure and vice versa by $\widehat{v}_\alpha \cdot n(x) > 0$ (< 0) on γ_T^+ (γ_T^-). Consequently, from now on we will (mostly) not point out the measure in the denotation of such L^∞-spaces and simply write $L^\infty\left(\gamma_T^\pm\right)$.

1.3 The Maxwell part

We proceed with the Maxwell part. For a given current density j, permittivity ε, permeability μ, and initial data $\mathring{E}$, $\mathring{H}$ we want to solve the following system:

$$\varepsilon\partial_t E - \operatorname{curl}_x H = -4\pi j \qquad \text{on } I_{T_\bullet} \times \mathbb{R}^3, \tag{1.3.1a}$$

$$\mu\partial_t H + \operatorname{curl}_x E = 0 \qquad \text{on } I_{T_\bullet} \times \mathbb{R}^3, \tag{1.3.1b}$$

$$(E,H)(0) = \left(\mathring{E}, \mathring{H}\right) \qquad \text{on } \mathbb{R}^3. \tag{1.3.1c}$$

This system is a linear symmetric hyperbolic system. To tackle this problem, we state (a shortened version of) a theorem of Kato [Kat75, Theorem I].

Proposition 1.3.1. *Let $T > 0$ and consider the problem*

$$a_0\partial_t w + \sum_{i=1}^{3} a_i \partial_{x_i} w = h \qquad \text{on } [0,T] \times \mathbb{R}^3, \tag{1.3.2a}$$

$$w(0) = \mathring{w} \qquad \text{on } \mathbb{R}^3 \tag{1.3.2b}$$

with $h\colon [0,T] \times \mathbb{R}^3 \to \mathbb{R}^6$, $a_i\colon [0,T] \times \mathbb{R}^3 \to \mathbb{R}^{6\times 6}$, $0 \leq i \leq 3$, and $\mathring{w}\colon \mathbb{R}^3 \to \mathbb{R}^6$ given. Let $s, s' \in \mathbb{N}$ with $s \geq 3$, $1 \leq s' \leq s$, and let the following assumptions hold for some $M, L, \sigma > 0$ for all $0 \leq t, t' \leq T$, $x \in \mathbb{R}^3$, and $0 \leq i \leq 3$:

(i) $a_i \in C\left([0,T]; L^2_{\mathrm{ul}}\left(\mathbb{R}^3; \mathbb{R}^{6\times 6}\right)\right)$,

(ii) $\|a_i(t)\|_{H^s_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{6\times 6})} \leq M$,

(iii) $\|a_0(t) - a_0(t')\|_{H^{s-1}_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{6\times 6})} \leq L|t - t'|$,

(iv) $a_i(t,x)$ *is symmetric,*

(v) $a_0(t, x) \geq \sigma$,

(vi) $h \in L^1\big([0,T]; H^{s'}(\mathbb{R}^3;\mathbb{R}^6)\big) \cap C\big([0,T]; H^{s'-1}(\mathbb{R}^3;\mathbb{R}^6)\big)$,

(vii) $\mathring{w} \in H^{s'}(\mathbb{R}^3;\mathbb{R}^6)$.

Then, (1.3.2) *has a solution* $w \in C\big([0,T]; H^{s'}(\mathbb{R}^3;\mathbb{R}^6)\big) \cap C^1\big([0,T]; H^{s'-1}(\mathbb{R}^3;\mathbb{R}^6)\big)$ *which is unique in the larger class* $C\big([0,T]; H^1(\mathbb{R}^3;\mathbb{R}^6)\big) \cap C^1\big([0,T]; L^2(\mathbb{R}^3;\mathbb{R}^6)\big)$.

Here, for $m \in \mathbb{N}$ the space $L^2_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{m\times m})$ is the set of all measurable functions $z\colon \mathbb{R}^3 \to \mathbb{R}^{m\times m}$ such that

$$\|z\|_{L^2_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{m\times m})} := \sup_{x\in\mathbb{R}^3} \|z\|_{L^2(B_1(x);\mathbb{R}^{m\times m})} < \infty,$$

and the so-called "uniform local Sobolev space" $H^k_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{m\times m})$, $k \in \mathbb{N}_0$, is the set of all $z \in L^2_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{m\times m})$ such that all distribution derivatives of order less or equal k are elements of $L^2_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{m\times m})$. The space $H^k_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{m\times m})$ is equipped with the norm

$$\|z\|_{H^k_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{m\times m})} := \sup_{|\beta|\leq k} \left\|D^\beta z\right\|_{L^2_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{m\times m})}.$$

Due to Kato we have the continuous embedding $H^3_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{3\times 3}) \subset C_b(\mathbb{R}^3;\mathbb{R}^{3\times 3})$, so that ε and μ are bounded functions in the following theorem.

Proposition 1.3.2. *Let* $\varepsilon, \mu \in H^3_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ *have the following properties:* $\varepsilon(x)$, $\mu(x)$ *are symmetric for each* $x \in \mathbb{R}^3$ *and there is a* $\sigma > 0$ *such that* $\varepsilon(x), \mu(x) \geq \sigma$ *for all* $x \in \mathbb{R}^3$. *Moreover, let* $j \in L^1_{\mathrm{lt}}\big(I_{T_\bullet}; H^3(\mathbb{R}^3;\mathbb{R}^3)\big) \cap C_{\mathrm{lt}}\big(I_{T_\bullet}; H^2(\mathbb{R}^3;\mathbb{R}^3)\big)$ *and* $\mathring{E}, \mathring{H} \in H^3(\mathbb{R}^3;\mathbb{R}^3)$. *Then there is a unique solution* $(E, H) \in C_{\mathrm{lt}}\big(I_{T_\bullet}; H^3(\mathbb{R}^3;\mathbb{R}^6)\big) \cap C^1_{\mathrm{lt}}\big(I_{T_\bullet}; H^2(\mathbb{R}^3;\mathbb{R}^6)\big)$ *of* (1.3.1). *Furthermore, we have*

$$\frac{1}{8\pi}\int_{\mathbb{R}^3} \big(\varepsilon E\cdot E + \mu H\cdot H\big)(T)\,dx = \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(\varepsilon \mathring{E}\cdot\mathring{E} + \mu\mathring{H}\cdot\mathring{H}\Big)\,dx - \int_0^T\int_{\mathbb{R}^3} E\cdot j\,dxdt \tag{1.3.3}$$

and

$$\begin{aligned}\|(E,H)(T)\|_{L^2(\mathbb{R}^3;\mathbb{R}^6)} &:= \Big(\|E(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|H(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\Big)^{\frac{1}{2}} \\ &\leq \sigma^{-\frac{1}{2}}\left(\int_{\mathbb{R}^3}\Big(\varepsilon\mathring{E}\cdot\mathring{E} + \mu\mathring{H}\cdot\mathring{H}\Big)\,dx\right)^{\frac{1}{2}} + 4\pi\sigma^{-1}\|j\|_{L^1([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))}\end{aligned} \tag{1.3.4}$$

for any $0 < T \in I_{T_\bullet}$.

Proof. Let $0 < T \in I_{T_\bullet}$ and define

$$a_0, a_1, a_2, a_3 \colon [0,T] \times \mathbb{R}^3 \longrightarrow \mathbb{R}^{6\times 6}, \quad a_0(t,x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix},$$

$$a_1(t,x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad a_2(t,x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$a_3(t,x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$w := \begin{pmatrix} E \\ H \end{pmatrix}, \quad h := \begin{pmatrix} -4\pi j \\ 0_{\mathbb{R}^3} \end{pmatrix}, \quad \mathring{w} := \begin{pmatrix} \mathring{E} \\ \mathring{H} \end{pmatrix}.$$

With this, it is easy to see that (1.3.2) is equivalent to (1.3.1) on $[0,T]$. The given conditions on $\varepsilon, \mu, j, \mathring{E}$, and $\mathring{H}$ imply the conditions on a_0, h, and $\mathring{w}$ needed in Proposition 1.3.1 (with $s = s' = 3$). Applying this proposition, we find a solution $(E,H) \in C([0,T]; H^3(\mathbb{R}^3;\mathbb{R}^6)) \cap C^1([0,T]; H^2(\mathbb{R}^3;\mathbb{R}^6))$ of (1.3.1) on the time interval $[0,T]$. Because of the uniqueness in Proposition 1.3.1 the solutions on $[0,T]$ and $[0,T']$ coincide on the common time interval $[0,T]$ if $0 < T \leq T' \in I_{T_\bullet}$. Hence, there is a unique global-in-time solution $(E,H) \in C_{\mathrm{lt}}(I_{T_\bullet}; H^3(\mathbb{R}^3;\mathbb{R}^6)) \cap C^1_{\mathrm{lt}}(I_{T_\bullet}; H^2(\mathbb{R}^3;\mathbb{R}^6))$ of (1.3.1).

To get (1.3.3), we use the following energy balance:

$$\begin{aligned} \frac{d}{dt}\frac{1}{8\pi}\int_{\mathbb{R}^3} \big(\varepsilon E \cdot E + \mu H \cdot H\big)\,dx &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \big(E \cdot \varepsilon \partial_t E + H \cdot \mu \partial_t H\big)\,dx \\ &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \big(E \cdot \operatorname{curl}_x H - H \cdot \operatorname{curl}_x E - 4\pi E \cdot j\big)\,dx = -\int_{\mathbb{R}^3} E \cdot j\,dx. \end{aligned}$$

Note that all integrals exist by the boundedness of ε, μ and by the known regularity of E and H. In the first line, the symmetry of ε and μ enters. For the last step, it is important that $(E,H)(t) \in H^1(\mathbb{R}^3;\mathbb{R}^6)$ so that the boundary term that occurs after integrating one of the curl-terms by parts vanishes at infinity. Integrating this identity from 0 to T yields (1.3.3). By positive definiteness of ε and μ we can further estimate

$$\frac{1}{8\pi}\int_{\mathbb{R}^3} \big(\varepsilon E \cdot E + \mu H \cdot H\big)(T)\,dx$$

$$
\begin{aligned}
&\le \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\varepsilon\mathring{E}\cdot\mathring{E}+\mu\mathring{H}\cdot\mathring{H}\right)dx+\int_0^T\left(\int_{\mathbb{R}^3}|E|^2\,dx\right)^{\frac{1}{2}}\|j(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,dt\\
&\le \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\varepsilon\mathring{E}\cdot\mathring{E}+\mu\mathring{H}\cdot\mathring{H}\right)dx+\sigma^{-\frac{1}{2}}\int_0^T\left(\int_{\mathbb{R}^3}\varepsilon E\cdot E\,dx\right)^{\frac{1}{2}}\|j(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,dt\\
&\le \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\varepsilon\mathring{E}\cdot\mathring{E}+\mu\mathring{H}\cdot\mathring{H}\right)dx\\
&\quad+\sigma^{-\frac{1}{2}}\int_0^T\left(\int_{\mathbb{R}^3}\left(\varepsilon E\cdot E+\mu H\cdot H\right)dx\right)^{\frac{1}{2}}\|j(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,dt,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^3}\left(\varepsilon E\cdot E+\mu H\cdot H\right)(T)\,dx\right)^{\frac{1}{2}}\\
&\le\left(\int_{\mathbb{R}^3}\left(\varepsilon\mathring{E}\cdot\mathring{E}+\mu\mathring{H}\cdot\mathring{H}\right)dx\right)^{\frac{1}{2}}+4\pi\sigma^{-\frac{1}{2}}\|j\|_{L^1([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))}
\end{aligned}
$$

by the quadratic version of Gronwall's lemma; see Lemma 1.1.2. Using the positive definiteness of ε, μ again, we arrive at (1.3.4). □

1.4 The iteration scheme

We shall now construct weak solutions by means of an iteration scheme.

1.4.1 Regularity of the data and approximations

Throughout this section we assume the following conditions on the data, source terms, and material parameters.

Condition 1.4.1.

- $0 \le \mathring{f}^\alpha \in \left(L^1_{\alpha\mathrm{kin}} \cap L^\infty\right)(\Omega\times\mathbb{R}^3)$ for all $\alpha = 1,\dots,N$;
- $0 \le a^\alpha \in L^\infty\left(\gamma^-_{T_\bullet}\right)$, $a^\alpha_0 := \|a^\alpha\|_{L^\infty\left(\gamma^-_{T_\bullet}\right)} < 1$, $0 \le g^\alpha \in \left(L^1_{\alpha\mathrm{kin,lt}} \cap L^\infty_{\mathrm{lt}}\right)\left(\gamma^-_{T_\bullet}\right)$ for $\alpha = 1,\dots,N'$;
- $0 \le a^\alpha \in L^\infty\left(\gamma^-_{T_\bullet}\right)$, $\|a^\alpha\|_{L^\infty\left(\gamma^-_{T_\bullet}\right)} = 1$, $g^\alpha = 0$ for $\alpha = N'+1,\dots,N$;
- $\mathring{E}, \mathring{H} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$;
- $\varepsilon,\mu \in L^\infty(\mathbb{R}^3;\mathbb{R}^{3\times3})$ such that there are $\sigma,\sigma' > 0$ satisfying $\sigma \le \varepsilon,\mu \le \sigma'$, and $\varepsilon = \mu = \mathrm{Id}$ on Ω;
- $u \in L^1_{\mathrm{lt}}\left(I_{T_\bullet}; L^2(\Gamma;\mathbb{R}^3)\right)$.

For $\alpha = 1, \dots, N'$ we already have $\|a^\alpha\|_{L^\infty\left(\gamma^-_{T_\bullet}\right)} < 1$ so that we can simply define $a^\alpha_k := a^\alpha$ in order to be able to apply the results of Section 1.2. Conversely, for $\alpha = N'+1, \dots, N$ we have to modify a^α. The easiest way is to scale a^α with a positive number depending on k and smaller than 1 that converges to 1 for $k \to \infty$ so that we somehow get back the original a^α in the limit $k \to \infty$. Hence, we define $a^\alpha_k := \frac{k}{k+1} a^\alpha$ satisfying $\left\|a^\alpha_k\right\|_{L^\infty\left(\gamma^-_{T_\bullet}\right)} = \frac{k}{k+1} < 1$.

Since in the results of Section 1.3 all coefficients and data have to be smooth enough, on the one hand we have to choose approximating sequences $\left(\mathring{E}_k\right), \left(\mathring{H}_k\right) \subset H^3(\mathbb{R}^3;\mathbb{R}^3)$ with $\mathring{E}_k \to \mathring{E}, \mathring{H}_k \to \mathring{H}$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ for $k \to \infty$. On the other hand, we have to smooth ε and μ. In the following, have in mind that for a symmetric, positive definite matrix $A \in \mathbb{R}^{3\times3}$ and some $C \geq 0$ we have the equivalence

$$A \leq C \Leftrightarrow \|A\|_{\mathbb{R}^{3\times3}} \leq C$$

where we use the norm

$$\|A\|_{\mathbb{R}^{3\times3}} = \sup_{|x|\leq 1}|Ax| = \max\left\{\lambda \in \mathbb{R} \mid \lambda \text{ eigenvalue of } A\right\},$$

where the last equality holds for symmetric, positive definite A. Thus, for some measurable $A\colon \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ such that $A(x)$ is symmetric and positive definite for almost all $x \in \mathbb{R}^3$, the property $A \leq C$ is equivalent to $\|A\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^{3\times3})} \leq C$.

We want to construct sequences $(\varepsilon_k), (\mu_k) \subset H^3_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{3\times3})$ with $\sigma \leq \varepsilon_k, \mu_k \leq \sigma'$ in order to be able to apply Proposition 1.3.2 later such that these sequences converge to ε, μ in a certain sense. We perform the construction of (ε_k); the one for (μ_k) works completely analogously. Let $\omega \in C^\infty_c(\mathbb{R}^3)$ with $\omega \geq 0$, $\operatorname{supp}\omega \subset \overline{B_1}$, $\int_{\mathbb{R}^3} \omega\, dx = 1$ be a Friedrich's mollifier and define $\omega_s := s^{-3}\omega\left(\frac{\cdot}{s}\right)$ for $s > 0$. Now let

$$\tilde{\varepsilon}_k(x) := \begin{cases} \varepsilon(x) - \sigma \mathrm{Id}, & x \in B_k, \\ 0, & x \notin B_k \end{cases}$$

for $k \in \mathbb{N}$. Clearly, $\tilde{\varepsilon}_k \in L^\infty(\mathbb{R}^3;\mathbb{R}^{3\times3})$ and $\tilde{\varepsilon}_k$ vanishes on $\mathbb{R}^3 \setminus B_k$. This implies $\omega_s * \tilde{\varepsilon}_k \in C^\infty_c(\mathbb{R}^3;\mathbb{R}^{3\times3})$ (the convolution understood componentwise) for any $s > 0$. By $\tilde{\varepsilon}_k \in L^2(B_k;\mathbb{R}^{3\times3})$ we have $\omega_s * \tilde{\varepsilon}_k \to \tilde{\varepsilon}_k$ in $L^2(B_k;\mathbb{R}^{3\times3})$ for $s \to 0$. Hence, we can choose $s_k > 0$ such that

$$\left\|\omega_{s_k} * \tilde{\varepsilon}_k - \tilde{\varepsilon}_k\right\|_{L^2(B_k;\mathbb{R}^{3\times3})} < \frac{1}{k}.$$

Finally, define $\varepsilon_k := \omega_{s_k} * \tilde{\varepsilon}_k + \sigma\mathrm{Id}$. It holds that $\varepsilon_k \in H^3_{\mathrm{ul}}(\mathbb{R}^3;\mathbb{R}^{3\times3})$ since it is of class C^∞ and constant for $|x|$ large. By construction $\varepsilon_k(x)$ is symmetric for all $x \in \mathbb{R}^3$ and

$$\|\varepsilon - \varepsilon_k\|_{L^2(B_k;\mathbb{R}^{3\times3})} < \frac{1}{k}. \tag{1.4.1}$$

Furthermore, for any $E, x \in \mathbb{R}^3$ it holds that

$$\varepsilon_k(x)E \cdot E = \int_{\mathbb{R}^3} \omega_{s_k}(x-y)\tilde{\varepsilon}_k(y)E \cdot E\,dy + \sigma|E|^2$$

$$= \int_{B_k} \omega_{s_k}(x-y)\varepsilon(y)E \cdot E\,dy - \sigma|E|^2 \int_{B_k} \omega_{s_k}(x-y)\,dy + \sigma|E|^2$$

$$\begin{cases} \geq \sigma|E|^2 \int_{B_k} \omega_{s_k}(x-y)\,dy - \sigma|E|^2 \int_{B_k} \omega_{s_k}(x-y)\,dy + \sigma|E|^2 = \sigma|E|^2, \\ \leq \sigma'|E|^2 \int_{B_k} \omega_{s_k}(x-y)\,dy - \sigma|E|^2 \int_{B_k} \omega_{s_k}(x-y)\,dy + \sigma|E|^2 \leq \sigma'|E|^2. \end{cases}$$

Note that for the last line we used the fact that the integral of ω_s over whole $\mathbb{R}^3$ equals 1 for any $s > 0$. Altogether, ε_k and the similarly defined μ_k satisfy all conditions needed in Proposition 1.3.2.

1.4.2 A cut-off problem

We now follow Guo [Guo93], who considered the problem with $\varepsilon = \mu = \mathrm{Id}$, $u = 0$, and perfect conductor boundary conditions for the electromagnetic fields on $\partial\Omega$. However, Lemma 2.5. therein, cf. Proposition 1.3.2 here, is incorrect. In order to construct a weak solution of (VM), we first turn to a cut-off problem where we consider bounded time and momentum domains. Whereas the cut-off in time is no real drawback, the cut-off in momentum space is on the one hand unpleasant but on the other hand necessary. To understand this necessity, we should recall (1.3.4). Consider there j to be the sum of some external current and the current j^{int} induced by the particle densities. In an iteration scheme we would like to have an estimate like (1.3.4) for the fields where the right-hand side is uniformly bounded along the iteration. Then we could extract some weakly converging subsequence. However, for this uniformity we would need that j^{int} is uniformly bounded in $L^1([0,T]; L^2(\mathbb{R}^3;\mathbb{R}^3))$ along the iteration. This would require a better estimate than (1.2.13) where we only were able to put our hands on the $L^{\frac{4}{3}}(\mathbb{R}^3;\mathbb{R}^3)$-norm of j^{int} (at each time). Moreover, in an energy balance along the iteration, the crucial terms describing the energy transfer within the internal system will not cancel out; this would only be the case if we solve (VM) simultaneously along an iteration.

Now if we consider a cut-off problem (the cut-off referring to momentum space) we can simply estimate the L^2-norm of j^{int} with respect to x by a linear combination of the L^2-norms of the f^α with respect to (x, v), cf. (1.4.4), and then use (1.2.10) for $p = 2$ so that we get the desired uniform boundedness along the iteration. Later, adding the limit versions of (1.2.12) and (1.3.3), we observe that the problematic terms on the right-hand side, that is to say, the terms $\pm E \cdot j^{\mathrm{int}}$, cancel out. Thus, now (after a Gronwall argument) having a full energy estimate with only expressions of the given functions on the right-hand side, we find that a posteriori the cut-off does not substantially enter this estimate so that we will be able to get a solution of the system without a cut-off by considering a sequence of solutions corresponding to larger and larger cut-off domains.

To make things more precise, let $0 < R < \infty$, define $R^* := \min\{R, T_\bullet\}$, and start the iteration with $E_0, H_0\colon [0,R^*]\times\mathbb{R}^3 \to \mathbb{R}^3$, $(E_0,H_0)(t,x,v) = \left(\mathring{E}_0, \mathring{H}_0\right)(x,v)$. We assume that we already have $E_k, H_k \in L^\infty\big([0,R^*]; L^2(\mathbb{R}^3;\mathbb{R}^3)\big) \cap C^{0,1}\big([0,R^*]\times\mathbb{R}^3;\mathbb{R}^3\big)$ of the k-th step. We first define f^α_{k+1} as the solution of the Vlasov part

$$\partial_t f^\alpha_{k+1} + \widehat{v}_\alpha\cdot\partial_x f^\alpha_{k+1} + F^\alpha_k\cdot\partial_v f^\alpha_{k+1} = 0 \qquad \text{on } [0,R^*]\times\Omega\times\mathbb{R}^3, \tag{1.4.2a}$$

$$f^\alpha_{k+1,-} = a^\alpha_{k+1} K f^\alpha_{k+1,+} + g^\alpha \qquad \text{on } \gamma^-_{R^*}, \tag{1.4.2b}$$

$$f^\alpha_{k+1}(0) = \mathring{f}^\alpha \qquad \text{on } \Omega\times\mathbb{R}^3 \tag{1.4.2c}$$

with given force field $F^\alpha_k := q_\alpha(E_k + \widehat{v}_\alpha\times H_k)$, which satisfies Condition 1.2.11 by the regularity of E_k and H_k. Indeed, we can solve (1.4.2) applying Proposition 1.2.15 (with final time R^* instead of $T_\bullet$) and noticing that a^α_{k+1} is bounded away from 1 on $\gamma^-_{R^*}$. Therefore, we have $0 \le f^\alpha_{k+1} \in L^\infty\Big([0,R^*]; \Big(L^1_{\alpha\mathrm{kin}}\cap L^\infty\Big)(\Omega\times\mathbb{R}^3)\Big)$ and $0 \le f^\alpha_{k+1,\pm} \in \Big(L^1_{\alpha\mathrm{kin}}\cap L^\infty\Big)\big(\gamma^\pm_{R^*}, d\gamma_\alpha\big)$.

Next we want to solve the Maxwell part. Now the cut-off appears: We define the current

$$j_{k+1} := j^{\mathrm{int}}_{k+1} + u := \sum_{\alpha=1}^N q_\alpha \int_{B_R} \widehat{v}_\alpha f^\alpha_{k+1}\, dv + u \tag{1.4.3}$$

where we integrate only over the cut-off domain B_R rather than over the whole momentum space. Note that j^{int}_{k+1} (u) is defined to be 0 outside Ω (Γ). By

$$\left(\int_\Omega \left|j^{\mathrm{int}}_{k+1}\right|^2 dx\right)^{\frac{1}{2}} \le \sqrt{\frac{4\pi}{3}R^3}\sum_{\alpha=1}^N |q_\alpha| \left(\int_\Omega\int_{\mathbb{R}^3} \left|f^\alpha_{k+1}\right|^2 dv dx\right)^{\frac{1}{2}} \tag{1.4.4}$$

and $f^\alpha_{k+1} \in L^\infty\big([0,R^*]; L^2(\Omega\times\mathbb{R}^3)\big)$ we have $j_{k+1} \in L^1\big([0,R^*]; L^2(\mathbb{R}^3;\mathbb{R}^3)\big)$. Unfortunately, this regularity is not enough to apply Proposition 1.3.2. Thus, we approximate j_{k+1} by a smooth function, that is to say, take $\bar{j}_{k+1} \in C^\infty_c\big(]0,R^*[\times\mathbb{R}^3;\mathbb{R}^3\big)$ such that

$$4\pi\left\|j_{k+1} - \bar{j}_{k+1}\right\|_{L^1([0,R^*];L^2(\mathbb{R}^3;\mathbb{R}^3))} < \frac{1}{k+1}. \tag{1.4.5}$$

With this smoothed current as the source term in the Maxwell system we solve

$$\varepsilon_{k+1}\partial_t E_{k+1} - \mathrm{curl}_x H_{k+1} = -4\pi\bar{j}_{k+1} \qquad \text{on } [0,R^*]\times\mathbb{R}^3, \tag{1.4.6a}$$

$$\mu_{k+1}\partial_t H_{k+1} + \mathrm{curl}_x E_{k+1} = 0 \qquad \text{on } [0,R^*]\times\mathbb{R}^3, \tag{1.4.6b}$$

$$(E_{k+1}, H_{k+1})(0) = \left(\mathring{E}_{k+1}, \mathring{H}_{k+1}\right) \qquad \text{on } \mathbb{R}^3. \tag{1.4.6c}$$

Indeed, applying Proposition 1.3.2, we see that there is a unique solution $(E_{k+1}, H_{k+1}) \in C\big([0,R^*]; H^3(\mathbb{R}^3;\mathbb{R}^6)\big) \cap C^1\big([0,R^*]; H^2(\mathbb{R}^3;\mathbb{R}^6)\big)$. By Sobolev's embedding theorem it holds that $E_{k+1}, H_{k+1} \in C^{0,1}\big([0,R^*]\times\mathbb{R}^3;\mathbb{R}^3\big)$. Altogether, the induction hypothesis is satisfied so that we can proceed with the next iteration step.

In order to extract some weakly converging subsequence, we have to establish suitable estimates. To this end, consider (1.2.10) and (1.3.4) applied to (1.4.2) and (1.4.6):

$$\begin{aligned}&\left(1-\left\|a^\alpha_{k+1}\right\|_{L^\infty\left(\gamma^-_{R^*}\right)}\right)^{\frac{1}{p}}\left\|f^\alpha_{k+1,+}\right\|_{L^p\left(\gamma^+_T,d\gamma_\alpha\right)}, \left\|f^\alpha_{k+1}(T)\right\|_{L^p(\Omega\times\mathbb{R}^3)}\\ &\leq \left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)}+\left(1-\left\|a^\alpha_{k+1}\right\|_{L^\infty\left(\gamma^-_{R^*}\right)}\right)^{\frac{1}{p}-1}\left\|g^\alpha\right\|_{L^p\left(\gamma^-_T,d\gamma_\alpha\right)}\end{aligned}\tag{1.4.7}$$

and

$$\begin{aligned}&\left\|(E_{k+1},H_{k+1})(T)\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\\ &\leq \sigma^{-\frac{1}{2}}\left(\int_{\mathbb{R}^3}\left(\varepsilon_{k+1}\mathring{E}_{k+1}\cdot\mathring{E}_{k+1}+\mu_{k+1}\mathring{H}_{k+1}\cdot\mathring{H}_{k+1}\right)dx\right)^{\frac{1}{2}}+4\pi\sigma^{-1}\left\|\bar{j}_{k+1}\right\|_{L^1([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))}\end{aligned}\tag{1.4.8}$$

for $0 < T \leq R^*$, $1 \leq p \leq \infty$. Note that we need $\varepsilon_k(x), \mu_k(x) \geq \sigma$ uniformly in x and k to get (1.4.8).

For $\alpha = 1, \dots, N'$, (1.4.7) reduces to

$$\begin{aligned}&\left(1-a^\alpha_0\right)^{\frac{1}{p}}\left\|f^\alpha_{k+1,+}\right\|_{L^p\left(\gamma^+_T,d\gamma_\alpha\right)}, \left\|f^\alpha_{k+1}(T)\right\|_{L^p(\Omega\times\mathbb{R}^3)}\\ &\leq \left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)}+\left(1-a^\alpha_0\right)^{\frac{1}{p}-1}\left\|g^\alpha\right\|_{L^p\left(\gamma^-_T,d\gamma_\alpha\right)}\end{aligned}\tag{1.4.9}$$

and to

$$(k+2)^{-\frac{1}{p}}\left\|f^\alpha_{k+1,+}\right\|_{L^p\left(\gamma^+_T,d\gamma_\alpha\right)}, \left\|f^\alpha_{k+1}(T)\right\|_{L^p(\Omega\times\mathbb{R}^3)} \leq \left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)}\tag{1.4.10}$$

for $\alpha = N'+1, \dots, N$. Thus, we conclude that any sequence $\left(f^\alpha_k\right)$ is bounded in any $L^p\big([0,R^*]\times\Omega\times\mathbb{R}^3\big)$, $1 \leq p \leq \infty$, so that we may extract subsequences (also denoted by $\left(f^\alpha_k\right)$) that converge weakly in $L^p\big([0,R^*]\times\Omega\times\mathbb{R}^3\big)$ for $1 < p < \infty$ and weak-* in $L^\infty\big([0,R^*]\times\Omega\times\mathbb{R}^3\big)$ to some nonnegative f^α_R. As in (1.4.3) we define

$$j_R := j^{\text{int}}_R + u := \sum_{\alpha=1}^N q_\alpha \int_{B_R} \widehat{v}_\alpha f^\alpha_R \, dv + u.$$

As for the boundary values, we have to distinct absorbing and reflecting boundary conditions. For $\alpha = 1, \dots, N'$, (1.4.9) yields the boundedness of $\left(f^\alpha_{k,+}\right)$ in any $L^p\big(\gamma^+_{R^*}, d\gamma_\alpha\big)$, $1 \leq p \leq \infty$, so we may extract a subsequence that converges weakly in $L^p\big(\gamma^+_{R^*}, d\gamma_\alpha\big)$ for $1 < p < \infty$ and weak-* in $L^\infty\big(\gamma^+_{R^*}\big)$ to some nonnegative $f^\alpha_{R,+}$. For

$\alpha = N' + 1, \dots, N$, (1.4.10) yields a uniform estimate only for $p = \infty$, so here we may extract a subsequence that only converges weak-* to some nonnegative $f^\alpha_{R,+}$ in $L^\infty\left(\gamma^+_{R^*}\right)$.

Note that we do not claim that the $f^\alpha_{R,+}$ are traces of the f^α_R in the sense of Section 1.2—because we cannot assume the force term in the limit Vlasov equations to be Lipschitz continuous, as we see below, and therefore an approach via characteristics as in Section 1.2.2 is not applicable—but f^α_R and $f^\alpha_{R,+}$ are rather related to each other in the sense of Remark 1.2.1; note that Definition 1.1.1.(ii) is satisfied (for f^α_R, $f^\alpha_{R,+}$, E_R, H_R), as is shown below. This clarification also applies to the f^α and f^α_+ constructed later in Section 1.4.3.

Next we have a look at L^p-estimates for f^α_R and $f^\alpha_{R,+}$ and let $T \in]0, R^*]$. Clearly, we have

$$\begin{aligned} \left\|f^\alpha_R\right\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^3)} &\le \liminf_{k\to\infty}\left\|f^\alpha_k\right\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^3)} \\ &\le \left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} + \begin{cases} \left(1-a^\alpha_0\right)^{-1}\left\|g^\alpha\right\|_{L^\infty(\gamma^-_T)}, & \alpha = 1,\dots,N' \\ 0, & \alpha = N'+1,\dots,N \end{cases} \end{aligned} \tag{1.4.11}$$

and

$$\begin{aligned} \left\|f^\alpha_{R,+}\right\|_{L^\infty\left(\gamma^+_T\right)} &\le \liminf_{k\to\infty}\left\|f^\alpha_{k,+}\right\|_{L^\infty\left(\gamma^+_T\right)} \\ &\le \left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} + \begin{cases} \left(1-a^\alpha_0\right)^{-1}\left\|g^\alpha\right\|_{L^\infty(\gamma^-_T)}, & \alpha = 1,\dots,N' \\ 0, & \alpha = N'+1,\dots,N \end{cases} \end{aligned}$$

by weak-* convergence and (1.4.9) and (1.4.10), respectively. As for the other L^p-norms, let $A \subset [0, T]$ be measurable with Lebesgue measure $\lambda(A)$, and $r > 0$. For $1 < p < \infty$ it holds that

$$\begin{aligned} &\int_A\int_\Omega\int_{\mathbb{R}^3}\left|f^\alpha_R\right|^p dvdxdt \le \liminf_{k\to\infty}\int_A\int_\Omega\int_{\mathbb{R}^3}\left|f^\alpha_k\right|^p dvdxdt \\ &\le \lambda(A)\left(\left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)} + \begin{cases} \left(1-a^\alpha_0\right)^{\frac{1}{p}-1}\left\|g^\alpha\right\|_{L^p\left(\gamma^-_T,d\gamma_\alpha\right)}, & \alpha = 1,\dots,N' \\ 0, & \alpha = N'+1,\dots,N \end{cases}\right)^p \end{aligned}$$

by weak convergence and (1.4.9) and (1.4.10), respectively. Therefore, we have $f^\alpha_R \in L^\infty\left([0,R^*];L^p\left(\Omega\times\mathbb{R}^3\right)\right)$ with

$$\left\|f^\alpha_R\right\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))} \le \left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)} + \begin{cases} \left(1-a^\alpha_0\right)^{\frac{1}{p}-1}\left\|g^\alpha\right\|_{L^p\left(\gamma^-_T,d\gamma_\alpha\right)}, & \alpha = 1,\dots,N' \\ 0, & \alpha = N'+1,\dots,N \end{cases}$$

for all $T \in]0, R^*]$. For $\alpha = 1, \dots, N'$, it additionally holds that

$$\left\|f^\alpha_{R,+}\right\|_{L^p\left(\gamma^+_T,d\gamma_\alpha\right)} \le \liminf_{k\to\infty}\left\|f^\alpha_{k,+}\right\|_{L^p\left(\gamma^+_T,d\gamma_\alpha\right)}$$

$$\leq \left(1 - a_0^\alpha\right)^{-\frac{1}{p}} \left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)} + \left(1 - a_0^\alpha\right)^{-1} \left\|g^\alpha\right\|_{L^p\left(\gamma_T^-, d\gamma_\alpha\right)}$$

by weak convergence and (1.4.9).

Finally, we turn to $p = 1$. On the one hand, for any measurable $A \subset [0, T]$ and $r > 0$ it holds that

$$\begin{aligned} \int_A \int_\Omega \int_{B_r} f_R^\alpha \, dv dx dt &= \lim_{k\to\infty} \int_A \int_\Omega \int_{B_r} f_k^\alpha \, dv dx dt \\ &\leq \lambda(A) \left(\left\|\mathring{f}^\alpha\right\|_{L^1(\Omega\times\mathbb{R}^3)} + \begin{cases} \left\|g^\alpha\right\|_{L^1\left(\gamma_T^-, d\gamma_\alpha\right)}, & \alpha = 1, \ldots, N' \\ 0, & \alpha = N'+1, \ldots, N \end{cases} \right) \end{aligned}$$

by weak convergence (in L^2, for example) and (1.4.9) and (1.4.10), respectively. This estimate implies that $f_R^\alpha \in L^\infty\left([0, R^*]; L^1\left(\Omega \times \mathbb{R}^3\right)\right)$ with

$$\left\|f_R^\alpha\right\|_{L^\infty([0,T];L^1(\Omega\times\mathbb{R}^3))} \leq \left\|\mathring{f}^\alpha\right\|_{L^1(\Omega\times\mathbb{R}^3)} + \begin{cases} \left\|g^\alpha\right\|_{L^1\left(\gamma_T^-, d\gamma_\alpha\right)}, & \alpha = 1, \ldots, N' \\ 0, & \alpha = N'+1, \ldots, N \end{cases}$$

for all $T \in \,]0, R^*]$. On the other hand, we have

$$\begin{aligned} \int_{\gamma_T^+ \cap \{|v|<r\}} f_{R,+}^\alpha \, d\gamma_\alpha &= \lim_{k\to\infty} \int_{\gamma_T^+ \cap \{|v|<r\}} f_{k,+}^\alpha \, d\gamma_\alpha \\ &\leq \left(1 - a_0^\alpha\right)^{-1} \left(\left\|\mathring{f}^\alpha\right\|_{L^1(\Omega\times\mathbb{R}^3)} + \left\|g^\alpha\right\|_{L^1\left(\gamma_T^-, d\gamma_\alpha\right)} \right) \end{aligned}$$

for $\alpha = 1, \ldots, N'$ by weak convergence and (1.4.9). Therefore, $f_{R,+}^\alpha \in L^1\left(\gamma_{R^*}^+, d\gamma_\alpha\right)$ and

$$\left\|f_{R,+}^\alpha\right\|_{L^1\left(\gamma_T^+, d\gamma_\alpha\right)} \leq \left(1 - a_0^\alpha\right)^{-1} \left(\left\|\mathring{f}^\alpha\right\|_{L^1(\Omega\times\mathbb{R}^3)} + \left\|g^\alpha\right\|_{L^1\left(\gamma_T^-, d\gamma_\alpha\right)} \right)$$

for all $T \in \,]0, R^*]$.

Next, we turn to an estimate on the electromagnetic fields. To examine (1.4.8) further, we first note that

$$\begin{aligned} \left\|\bar{j}_{k+1}\right\|_{L^1([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))} &\leq \frac{1}{4\pi(k+1)} + \left\|j_{k+1}\right\|_{L^1([0,R^*];L^2(\mathbb{R}^3;\mathbb{R}^3))} \\ &\leq 1 + \sqrt{\frac{4\pi}{3} R^3} \sum_{\alpha=1}^N |q_\alpha| \left\|f_{k+1}^\alpha\right\|_{L^1([0,R^*];L^2(\Omega\times\mathbb{R}^3))} + \|u\|_{L^1([0,R^*];L^2(\Gamma;\mathbb{R}^3))} \end{aligned}$$

for $0 < T \leq R^*$ using (1.4.4). Hence, the right-hand side of (1.4.8) is bounded uniformly in k because we already have a uniform bound on $\left(f_k^\alpha\right)$ in $L^\infty\left([0, R^*]; L^2\left(\Omega \times \mathbb{R}^3\right)\right)$ and because of $\varepsilon_k, \mu_k \leq \sigma'$ and the L^2-convergence of the approximating initial data. Thus, we may assume without loss of generality that (E_k, H_k) converges weakly in $L^2\left([0, R^*] \times \mathbb{R}^3; \mathbb{R}^6\right)$ to some (E_R, H_R).

We now show that $\left(\left(f_R^\alpha, f_{R,+}^\alpha\right)_\alpha, E_R, H_R, j_R\right)$ satisfies Definition 1.1.1.(i) to 1.1.1.(iii) with final time R^*. Clearly, all functions are of class L^1_{loc}. The main task is to show that we may pass to the limit in (1.1.2) and (1.1.3) applied to the iterates: We have for all $\psi \in \Psi_{R^*}$, $\vartheta \in \Theta_{R^*}$, and $k \geq 1$

$$\begin{aligned} 0 = &-\int_0^{R^*}\int_\Omega\int_{\mathbb{R}^3}\left(\partial_t\psi + \widehat{v}_\alpha\cdot\partial_x\psi + q_\alpha(E_k + \widehat{v}_\alpha\times H_k)\cdot\partial_v\psi\right)f_{k+1}^\alpha\,dv\,dx\,dt \\ &+\int_{\gamma_{R^*}^+} f_{k+1,+}^\alpha\psi\,d\gamma_\alpha - \int_{\gamma_{R^*}^-}\left(a_{k+1}^\alpha K f_{k+1,+}^\alpha + g^\alpha\right)\psi\,d\gamma_\alpha - \int_\Omega\int_{\mathbb{R}^3}\mathring{f}^\alpha\psi(0)\,dv\,dx, \end{aligned} \tag{1.4.12}$$

$$0 = \int_0^{R^*}\int_{\mathbb{R}^3}\left(\varepsilon_k E_k\cdot\partial_t\vartheta - H_k\cdot\operatorname{curl}_x\vartheta - 4\pi\bar{j}_k\cdot\vartheta\right)dx\,dt + \int_{\mathbb{R}^3}\varepsilon_k\mathring{E}_k\cdot\vartheta(0)\,dx, \tag{1.4.13}$$

$$0 = \int_0^{R^*}\int_{\mathbb{R}^3}\left(\mu_k H_k\cdot\partial_t\vartheta + E_k\cdot\operatorname{curl}_x\vartheta\right)dx\,dt + \int_{\mathbb{R}^3}\mu_k\mathring{H}_k\cdot\vartheta(0)\,dx. \tag{1.4.14}$$

We can pass to the limit in (1.4.13) and (1.4.14): Whereas the terms including the curl are easy to handle by weak convergence of E_k, H_k, we have to take more care about the terms including ε_k, μ_k, and $\bar{j}_k$. For the first ones, let $L \in \mathbb{N}$ such that ϑ vanishes for $|x| \geq L$ so that we in fact only integrate over B_L. For $k \geq L$ we have

$$\|\varepsilon - \varepsilon_k\|_{L^2(B_L;\mathbb{R}^{3\times3})} \leq \|\varepsilon - \varepsilon_k\|_{L^2(B_k;\mathbb{R}^{3\times3})} < \frac{1}{k}$$

by (1.4.1) so that $\varepsilon_k \to \varepsilon$ in $L^2(B_L;\mathbb{R}^{3\times3})$. This is enough for passing to the limit in the terms including ε_k since we additionally have $E_k \rightharpoonup E_R$ in $L^2([0,R^*]\times\mathbb{R}^3;\mathbb{R}^3)$, even strong convergence of the approximating initial data, and the boundedness of the time interval $[0,R^*]$. Similarly, we argue for the terms with μ_k. So there only remains the term including $\bar{j}_k$. To tackle this one, we estimate

$$\begin{aligned} &\left|\int_0^{R^*}\int_{\mathbb{R}^3}\left(\bar{j}_k - j_R\right)\cdot\vartheta\,dx\,dt\right| \leq \left|\int_0^{R^*}\int_{\mathbb{R}^3}\left(\bar{j}_k - j_k\right)\cdot\vartheta\,dx\,dt\right| + \left|\int_0^{R^*}\int_{\mathbb{R}^3}\left(j_k - j_R\right)\cdot\vartheta\,dx\,dt\right| \\ &\leq \left\|\bar{j}_k - j_k\right\|_{L^1([0,R^*];L^2(\mathbb{R}^3;\mathbb{R}^3))}\|\vartheta\|_{L^\infty([0,R^*];L^2(\mathbb{R}^3;\mathbb{R}^3))} \\ &\quad + \sum_{\alpha=1}^N|q_\alpha|\left|\int_0^{R^*}\int_{\mathbb{R}^3}\int_{B_R}\widehat{v}_\alpha\left(f_k^\alpha - f_R^\alpha\right)dv\cdot\vartheta\,dx\,dt\right|, \end{aligned}$$

where the first term on the right-hand side converges to 0 for $k \to \infty$ by construction of $\bar{j}_k$, cf. (1.4.5), and each summand of the second term by weak convergence of the f_k^α; note that $\widehat{v}_\alpha\cdot\vartheta\chi_{\{|v|\leq R\}} \in L^2([0,R^*]\times\mathbb{R}^3\times\mathbb{R}^3)$.

Passing to the limit in (1.4.12) is more complicated, especially because of the nonlinear product term including E_k, H_k, and f_k^α. The other terms are easy to handle due to weak convergence of f_k^α and weak (or weak-*) convergence of $f_{k,+}^\alpha$; for this note that

$$\lim_{k\to\infty}\int_{\gamma_{R^*}^-} a_{k+1}^\alpha\left(K f_{k+1,+}^\alpha\right)\psi\,d\gamma_\alpha$$

$$= \lim_{k\to\infty} \left(\int_{\gamma^-_{R^*}} a^\alpha \left(K f^\alpha_{k+1,+} \right) \psi \, d\gamma_\alpha \cdot \begin{cases} 1, & \alpha = 1, \dots, N' \\ \frac{k+1}{k+2}, & \alpha = N'+1, \dots, N \end{cases} \right)$$
$$= \lim_{k\to\infty} \left(\int_{\gamma^+_{R^*}} (Ka^\alpha) f^\alpha_{k+1,+} K\psi \, d\gamma_\alpha \cdot \begin{cases} 1, & \alpha = 1, \dots, N' \\ \frac{k+1}{k+2}, & \alpha = N'+1, \dots, N \end{cases} \right) = \int_{\gamma^+_{R^*}} (Ka^\alpha) f^\alpha_{R,+} K\psi \, d\gamma_\alpha$$
$$= \int_{\gamma^-_{R^*}} a^\alpha \left(K f^\alpha_{R,+} \right) \psi \, d\gamma_\alpha \tag{1.4.15}$$

using Lemma 1.2.14—the second time applied to $c := f^\alpha_{R,+} \chi_{\{|v|\le r\}}$ where $r > 0$ is chosen such that ψ vanishes for $|v| > r$, as $f^\alpha_{R,+}$ is not necessarily of class $L^1\left(\gamma^+_{R^*}, d\gamma_\alpha\right)$.

So there only remains the crucial product term. In order to be able to pass to the limit, we need some compactness. To this end, the key is the following momentum-averaging lemma; see [DL89], or [Rei04] for a shortened proof.

Lemma 1.4.2. *Let $r > 0$ and $\zeta \in C^\infty_c(B_r)$. There exists a constant $C > 0$ such that for any functions $h, g_0 \in L^2\left(\mathbb{R} \times \mathbb{R}^3 \times B_r\right)$, $g_1 \in L^2\left(\mathbb{R} \times \mathbb{R}^3 \times B_r; \mathbb{R}^3\right)$ which satisfy the inhomogeneous transport equation*

$$\partial_t h + \widehat{v}_\alpha \cdot \partial_x h = g_0 + \mathrm{div}_v \, g_1$$

in the sense of distributions we have

$$\int_{B_r} \zeta(v) h(\cdot, \cdot, v) \, dv \in H^{\frac{1}{4}}\left(\mathbb{R} \times \mathbb{R}^3\right)$$

with

$$\left\| \int_{B_r} \zeta(v) h(\cdot, \cdot, v) \, dv \right\|_{H^{\frac{1}{4}}(\mathbb{R}\times\mathbb{R}^3)} \\ \le C\left(\|h\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times B_r)} + \|g_0\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times B_r)} + \|g_1\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times B_r;\mathbb{R}^3)} \right).$$

Note that, in the references above, this lemma was proved for $\widehat{v} = \frac{v}{\sqrt{1+|v|^2}}$ instead of $\widehat{v}_\alpha$, i.e., for $m_\alpha = 1$, but this slight difference plays no role for the analysis.

Let $\zeta \in C^\infty_c\left(\mathbb{R}^3\right)$ and $r > 0$ such that ζ vanishes for $|v| > r-1$. Our goal is to show that $\int_{\mathbb{R}^3} \zeta f^\alpha_k \, dv$ converges strongly (and not only weakly) to $\int_{\mathbb{R}^3} \zeta f^\alpha_R \, dv$ in $L^2([0, R^*] \times \Omega)$. To this end, let $\eta \in C^\infty_c(]0, R^*[\times \Omega \times B_r)$. We have

$$\begin{aligned} &\partial_t \left(\eta f^\alpha_{k+1}\right) + \widehat{v}_\alpha \cdot \partial_x \left(\eta f^\alpha_{k+1}\right) \\ &= -\mathrm{div}_v \left(q_\alpha (E_k + \widehat{v}_\alpha \times H_k) \left(\eta f^\alpha_{k+1}\right) \right) + f^\alpha_{k+1} \partial_t \eta + f^\alpha_{k+1} \widehat{v}_\alpha \cdot \partial_x \eta \\ &\quad + q_\alpha f^\alpha_{k+1} (E_k + \widehat{v}_\alpha \times H_k) \cdot \partial_v \eta \\ &=: \mathrm{div}_v \, g^k_1 + g^k_0 \end{aligned} \tag{1.4.16}$$

on $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ in the sense of distributions. This holds if we first extend all functions by 0 so that they are defined on $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$, then take an arbitrary test function

$\xi \in C_c^\infty\big(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\big)$ and notice that $\eta\xi\big|_{[0,R^*]\times\overline{\Omega}\times\mathbb{R}^3} \in \Psi_{R^*}$ since the support of η is away from $\partial\Omega$. Thus, $\eta\xi$, which vanishes on $\partial\Omega$ and for $t = 0$, is a proper test function for system (1.4.2). But testing this system with this test function is nothing else than testing (1.4.16) with ξ.

Clearly, the L^2-norms of g_0^k and g_1^k on $\mathbb{R}\times\mathbb{R}^3\times B_r$ are uniformly bounded in k because of $\eta \in C_c^\infty(]0,R^*[\times\Omega\times B_r)$ and the already known uniform boundedness of f_{k+1}^α in $L^2\big([0,R^*]\times\Omega\times\mathbb{R}^3\big)$ and $L^\infty\big([0,R^*]\times\Omega\times\mathbb{R}^3\big)$ and E_k, H_k in $L^2\big([0,R^*]\times\mathbb{R}^3;\mathbb{R}^3\big)$. Thus, applying Lemma 1.4.2 yields the uniform boundedness of

$$\left\|\int_{B_r}\zeta(v)\big(\eta f_k^\alpha\big)(\cdot,\cdot,v)\,dv\right\|_{H^{\frac14}(\mathbb{R}\times\mathbb{R}^3)} = \left\|\int_{B_r}\zeta(v)\big(\eta f_k^\alpha\big)(\cdot,\cdot,v)\,dv\right\|_{H^{\frac14}(]0,R^*[\times\Omega)}.$$

By boundedness of $]0,R^*[\times\Omega$ we have the compact embedding $H^{\frac14}(]0,R^*[\times\Omega) \subset L^2(]0,R^*[\times\Omega)$ so that the sequence $\left(\int_{B_r}\zeta(v)\Big(\eta f_k^\alpha\Big)(\cdot,\cdot,v)\,dv\right)$ converges, after extracting a suitable subsequence, strongly to $\int_{B_r}\zeta(v)\big(\eta f_R^\alpha\big)(\cdot,\cdot,v)\,dv$ in $L^2(]0,R^*[\times\Omega)$.

Again by the uniform boundedness of f_k^α in $L^\infty\big([0,R^*]\times\Omega\times\mathbb{R}^3\big)$ it holds that

$$\begin{aligned}&\left\|\int_{\mathbb{R}^3}\zeta(v)\big((1-\eta)\big(f_k^\alpha - f_R^\alpha\big)\big)(\cdot,\cdot,v)\,dv\right\|_{L^2([0,R^*]\times\Omega)}\\ &= \left\|\int_{B_r}\zeta(v)\big((1-\eta)\big(f_k^\alpha - f_R^\alpha\big)\big)(\cdot,\cdot,v)\,dv\right\|_{L^2([0,R^*]\times\Omega)} \le C\|1-\eta\|_{L^2([0,R^*]\times\Omega\times B_r)}\end{aligned} \tag{1.4.17}$$

with a constant $C \geq 0$ that does not depend on k. Now let $l \in \mathbb{N}$ be arbitrary and choose $\eta = \eta_l \in C_c^\infty(]0,R^*[\times\Omega\times B_r)$ such that the right-hand side of (1.4.17) is smaller than $\frac{1}{l}$; note that $1 \in L^2([0,R^*]\times\Omega\times B_r)$. We iteratively choose subsequences $\left(f_{l,k}^\alpha\right)_{k\in\mathbb{N}}$ of $\left(f_k^\alpha\right)$ such that $\left(f_{l+1,k}^\alpha\right)_{k\in\mathbb{N}}$ is a subsequence $\left(f_{l,k}^\alpha\right)_{k\in\mathbb{N}}$ and such that

$$\lim_{k\to\infty}\int_{B_r}\zeta(v)\Big(\eta_l f_{l,k}^\alpha\Big)(\cdot,\cdot,v)\,dv = \int_{B_r}\zeta(v)\big(\eta_l f_R^\alpha\big)(\cdot,\cdot,v)\,dv$$

in $L^2(]0,R^*[\times\Omega)$ for all $l \in \mathbb{N}$. Considering the diagonal sequence, now again denoted by $\left(f_k^\alpha\right)$, these considerations imply

$$\int_{\mathbb{R}^3}\zeta(v)f_k^\alpha(\cdot,\cdot,v)\,dv \to \int_{\mathbb{R}^3}\zeta(v)f_R^\alpha(\cdot,\cdot,v)\,dv \text{ strongly in } L^2([0,R^*]\times\Omega) \text{ for } k\to\infty \tag{1.4.18}$$

because of

$$\left\|\int_{\mathbb{R}^3}\zeta(v)f_k^\alpha(\cdot,\cdot,v)\,dv - \int_{\mathbb{R}^3}\zeta(v)f_R^\alpha(\cdot,\cdot,v)\,dv\right\|_{L^2([0,R^*]\times\Omega)}$$

$$\leq \frac{1}{k} + \left\| \int_{B_r} \zeta(v)\big(\eta_k f_k^\alpha\big)(\cdot,\cdot,v)\,dv - \int_{B_r} \zeta(v)\big(\eta_k f_R^\alpha\big)(\cdot,\cdot,v)\,dv \right\|_{L^2([0,R^*]\times\Omega)}.$$

Finally, take $\psi \in \Psi_{R^*}$ and consider the limit of the crucial product term in (1.4.12). By a density argument—in particular, the approximation theorem of Weierstraß, cf. [Wal02, Section 7.24]—we may assume that ψ factorizes, i.e.,

$$\psi(t,x,v) = \psi_1(t,x)\psi_2(v).$$

On the one hand, we have

$$\begin{aligned}
&\lim_{k\to\infty} \int_0^{R^*} \int_\Omega \int_{\mathbb{R}^3} q_\alpha E_k \cdot \big(\partial_v \psi\big) f_{k+1}^\alpha \,dvdxdt \\
&= \lim_{k\to\infty} \int_0^{R^*} \int_\Omega q_\alpha E_k \psi_1 \cdot \left(\int_{\mathbb{R}^3} f_{k+1}^\alpha \nabla\psi_2\,dv \right) dxdt \\
&= \int_0^{R^*} \int_\Omega q_\alpha E_R \psi_1 \cdot \left(\int_{\mathbb{R}^3} f_R^\alpha \nabla\psi_2\,dv \right) dxdt = \int_0^{R^*} \int_\Omega \int_{\mathbb{R}^3} q_\alpha E_R \cdot \big(\partial_v\psi\big) f_R^\alpha\,dvdxdt
\end{aligned}$$

by $\psi_1 \in L^\infty([0,R^*]\times\Omega)$, $E_k \rightharpoonup E_R$ weakly in $L^2\big([0,R^*]\times\Omega;\mathbb{R}^3\big)$, and (1.4.18) defining $\zeta := \big(\nabla\psi_2\big)_i$, $i = 1,2,3$. On the other hand, it holds that

$$\begin{aligned}
&\lim_{k\to\infty} \int_0^{R^*} \int_\Omega \int_{\mathbb{R}^3} q_\alpha(\widehat{v}_\alpha \times H_k) \cdot \big(\partial_v\psi\big) f_{k+1}^\alpha\,dvdxdt \\
&= \lim_{k\to\infty} \int_0^{R^*} \int_\Omega q_\alpha H_k \psi_1 \cdot \left(\int_{\mathbb{R}^3} \big(\nabla\psi_2 \times \widehat{v}_\alpha\big) f_{k+1}^\alpha\,dv \right) dxdt \\
&= \int_0^{R^*} \int_\Omega q_\alpha H_R \psi_1 \cdot \left(\int_{\mathbb{R}^3} \big(\nabla\psi_2 \times \widehat{v}_\alpha\big) f_R^\alpha\,dv \right) dxdt \\
&= \int_0^{R^*} \int_\Omega \int_{\mathbb{R}^3} q_\alpha(\widehat{v}_\alpha \times H_R) \cdot \big(\partial_v\psi\big) f_R^\alpha\,dvdxdt
\end{aligned}$$

by $\psi_1 \in L^\infty([0,R^*]\times\Omega)$, $H_k \rightharpoonup H_R$ weakly in $L^2\big([0,R^*]\times\Omega;\mathbb{R}^3\big)$, and (1.4.18) defining $\zeta(v) := \big(\nabla\psi_2(v) \times \widehat{v}_\alpha\big)_i$, $i = 1,2,3$.

Altogether, $\left(\left(f_R^\alpha, f_{R,+}^\alpha\right)_\alpha, E_R, H_R, j_R\right)$ satisfies Definition 1.1.1.(i) to 1.1.1.(iii) with final time R^*.

In order to have good estimates for $R \to \infty$, the right-hand side of an energy inequality should not depend on R. To this end, consider (1.2.12) and (1.3.3) applied to the k-iterated functions. Note that the estimate on the term on the left-hand side of (1.2.12) including the boundary values is only worth anything for $k \to \infty$ for $\alpha = 1,\dots,N'$. Therefore, it is convenient to introduce

$$b_k^\alpha(T) := \begin{cases} \big(1 - a_0^\alpha\big) \displaystyle\int_{\gamma_T^+ \cap \{|v|<R\}} v_\alpha^0 f_{k,+}^\alpha\,d\gamma_\alpha, & \alpha = 1,\dots,N' \\ 0, & \alpha = N'+1,\dots,N \end{cases}$$

and similarly $b_R^\alpha(T)$ where k is replaced by R. Now we have

$$\begin{aligned}
&b_k^\alpha(T) + \int_\Omega \int_{B_R} v_\alpha^0 f_k^\alpha(T)\, dvdx \\
&\leq \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\, dvdx + \int_{\gamma_T^-} v_\alpha^0 g^\alpha\, d\gamma_\alpha \\
&\quad + \int_0^T \int_\Omega \int_{B_R} q_\alpha (E_{k-1} + \widehat{v}_\alpha \times H_{k-1}) \cdot \widehat{v}_\alpha f_k^\alpha\, dvdxdt \\
&= \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\, dvdx + \int_{\gamma_T^-} v_\alpha^0 g^\alpha\, d\gamma_\alpha + \int_0^T \int_\Omega E_{k-1} \cdot \int_{B_R} q_\alpha \widehat{v}_\alpha f_k^\alpha\, dvdxdt \qquad (1.4.19)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{8\pi} \int_{\mathbb{R}^3} \big(\varepsilon_k E_k \cdot E_k + \mu_k H_k \cdot H_k\big)(T)\, dx \\
&= \frac{1}{8\pi} \int_{\mathbb{R}^3} \Big(\varepsilon_k \mathring{E}_k \cdot \mathring{E}_k + \mu_k \mathring{H}_k \cdot \mathring{H}_k\Big)\, dx - \int_0^T \int_{\mathbb{R}^3} E_k \cdot \overline{j}_k\, dxdt \qquad (1.4.20)
\end{aligned}$$

for any $k \geq 1$ and $T \in \,]0, R^*]$. We consider the right-hand sides of (1.4.19) and (1.4.20) further. The term including the initial data of the electromagnetic fields is bounded uniformly in k due to

$$\int_{\mathbb{R}^3} \Big(\varepsilon_k \mathring{E}_k \cdot \mathring{E}_k + \mu_k \mathring{H}_k \cdot \mathring{H}_k\Big)\, dx \leq \sigma' \int_{\mathbb{R}^3} \Big(\Big|\mathring{E}_k\Big|^2 + \Big|\mathring{H}_k\Big|^2\Big)\, dx \overset{k\to\infty}{\to} \sigma' \int_{\mathbb{R}^3} \Big(\Big|\mathring{E}\Big|^2 + \Big|\mathring{H}\Big|^2\Big)\, dx.$$

Next we show that, up to a subsequence,

$$\lim_{k\to\infty} \int_0^T \int_\Omega E_{k-1} \cdot \int_{B_R} q_\alpha \widehat{v}_\alpha f_k^\alpha\, dvdxdt = \int_0^T \int_\Omega E_R \cdot \int_{B_R} q_\alpha \widehat{v}_\alpha f_R^\alpha\, dvdxdt. \qquad (1.4.21)$$

To this end, let $l \in \mathbb{N}$ and $\zeta_l \in C_c^\infty\big(B_R; \mathbb{R}^3\big)$ with $\big\|q_\alpha \widehat{\cdot}_\alpha - \zeta_l\big\|_{L^2(B_R;\mathbb{R}^3)} < \frac{1}{l}$. By the uniform boundedness of E_k in $L^2\big([0, R^*] \times \mathbb{R}^3; \mathbb{R}^3\big)$ and f_k^α in $L^\infty\big([0, R^*] \times \Omega \times \mathbb{R}^3\big)$ (and by the limit functions being elements of these spaces) and by the finiteness of the measures of the time interval, Ω, and B_R, it holds that

$$\left|\int_0^T \int_\Omega E_{k-1} \cdot \int_{B_R} \big(q_\alpha \widehat{v}_\alpha - \zeta_l\big) f_k^\alpha\, dvdxdt\right|, \left|\int_0^T \int_\Omega E_R \cdot \int_{B_R} \big(q_\alpha \widehat{v}_\alpha - \zeta_l\big) f_R^\alpha\, dvdxdt\right| \leq \frac{C}{l}$$

where $0 \leq C < \infty$ does not depend on k. Similarly as before, after again exploiting the compactness result of Lemma 1.4.2, we deduce

$$\lim_{k\to\infty} \int_0^T \int_\Omega E_{k-1} \cdot \int_{B_R} \zeta_l f_k^\alpha\, dvdxdt = \int_0^T \int_\Omega E_R \cdot \int_{B_R} \zeta_l f_R^\alpha\, dvdxdt,$$

possibly after extracting a suitable subsequence depending on l. Via a similar diagonal sequence argument as before, we get (1.4.21) up to a subsequence. Summing (1.4.21) over α yields

$$\lim_{k\to\infty}\int_0^T\int_\Omega E_{k-1}\cdot j_k^{\mathrm{int}}\,dxdt = \int_0^T\int_\Omega E_R\cdot j_R^{\mathrm{int}}\,dxdt.$$

Similarly,

$$\lim_{k\to\infty}\int_0^T\int_\Omega E_k\cdot j_k^{\mathrm{int}}\,dxdt = \int_0^T\int_\Omega E_R\cdot j_R^{\mathrm{int}}\,dxdt,$$

whence we have

$$\lim_{k\to\infty}\int_0^T\int_\Omega\left(E_{k-1}\cdot j_k^{\mathrm{int}} - E_k\cdot j_k^{\mathrm{int}}\right)dxdt = 0. \tag{1.4.22}$$

However, this is not enough since we in fact have to consider $E_{k-1}\cdot j_k^{\mathrm{int}} - E_k\cdot\bar{j}_k$. To get hands on this term, we choose $\varphi_k^1, \varphi_k^2 \in C_c^\infty(]0,R^*[\times\Omega)$ with

$$\left\|E_{k-1}\cdot j_k^{\mathrm{int}} - \varphi_k^1\right\|_{L^1(]0,R^*[\times\Omega)}, \left\|E_k\cdot j_k^{\mathrm{int}} - \varphi_k^2\right\|_{L^1(]0,R^*[\times\Omega)} < \frac{1}{k} \tag{1.4.23}$$

and choose $u_k \in C_c^\infty\left(]0,R^*[\times\Gamma;\mathbb{R}^3\right)$ such that

$$\|u-u_k\|_{L^1([0,R^*];L^2(\Gamma;\mathbb{R}^3))} < \frac{1}{k}. \tag{1.4.24}$$

Using these approximations, (1.4.3), and (1.4.5) we estimate

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^3}\left(E_{k-1}\cdot j_k^{\mathrm{int}} - E_k\cdot\bar{j}_k\right)dxdt\right| \\
&\le \left|\int_0^T\int_\Omega\left(E_{k-1}\cdot j_k^{\mathrm{int}} - \varphi_k^1\right)dxdt\right| + \left|\int_0^T\int_\Omega\left(\varphi_k^1-\varphi_k^2\right)dxdt\right| \\
&\quad + \left|\int_0^T\int_\Omega\left(\varphi_k^2 - E_k\cdot j_k^{\mathrm{int}}\right)dxdt\right| + \left|\int_0^T\int_{\mathbb{R}^3}E_k\cdot\left(j_k-\bar{j}_k\right)dxdt\right| \\
&\quad + \left|\int_0^T\int_\Gamma E_k\cdot(u-u_k)\,dxdt\right| + \left|\int_0^T\int_\Gamma E_k\cdot u_k\,dxdt\right| \\
&\le \int_0^T\|E_k(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\|u_k(t)\|_{L^2(\Gamma;\mathbb{R}^3)}\,dt + \left|\int_0^T\int_\Omega\left(\varphi_k^1-\varphi_k^2\right)dxdt\right| + \frac{C}{k} \\
&=: \int_0^T\|E_k(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\|u_k(t)\|_{L^2(\Gamma;\mathbb{R}^3)}\,dt + h_k(T),
\end{aligned} \tag{1.4.25}$$

where $C > 0$ does not depend on k since we have a uniform bound on the E_k in $L^\infty\left([0,R^*];L^2\left(\mathbb{R}^3;\mathbb{R}^3\right)\right)$. Furthermore, h_k is continuous with respect to T and

$$h_k(T)\to 0 \text{ for } k\to\infty \text{ for each } T\in[0,R^*] \tag{1.4.26}$$

by (1.4.22) and (1.4.23). Moreover, we have

$$0 \le h_k(T) \le \frac{C+2}{k} + \left\|E_{k-1} \cdot j_k^{\mathrm{int}}\right\|_{L^1(]0,R^*[\times\Omega)} + \left\|E_k \cdot j_k^{\mathrm{int}}\right\|_{L^1(]0,R^*[\times\Omega)}$$
$$\le \frac{C+2}{k} + \Big(\|E_{k-1}\|_{L^\infty([0,R^*];L^2(\mathbb{R}^3;\mathbb{R}^3))} + \|E_k\|_{L^\infty([0,R^*];L^2(\mathbb{R}^3;\mathbb{R}^3))}\Big)\left\|j_k^{\mathrm{int}}\right\|_{L^1([0,R^*];L^2(\Omega;\mathbb{R}^3))} \le \tilde{C}, \tag{1.4.27}$$

where $\tilde{C} > 0$ does not depend on k (and T) by the uniform boundedness of the E_k in $L^\infty\big([0,R^*];L^2\big(\mathbb{R}^3;\mathbb{R}^3\big)\big)$ and (1.4.4) (combined with (1.4.9) and (1.4.10), respectively).

Now let $0 < T \le T' \le R^*$. Exploiting $\sigma \le \varepsilon_k, \mu_k \le \sigma'$, summing (1.4.19) over α, adding (1.4.20), and then using (1.4.25) yields

$$\begin{aligned}
&\sum_{\alpha=1}^N b_k^\alpha(T) + \sum_{\alpha=1}^N \int_\Omega \int_{B_R} v_\alpha^0 f_k^\alpha(T)\,dv\,dx + \frac{\sigma}{8\pi}\|(E_k,H_k)(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)} \\
&\le \sum_{\alpha=1}^N b_k^\alpha(T) + \sum_{\alpha=1}^N \int_\Omega \int_{B_R} v_\alpha^0 f_k^\alpha(T)\,dv\,dx + \frac{1}{8\pi}\int_{\mathbb{R}^3} \big(\varepsilon_k E_k \cdot E_k + \mu_k H_k \cdot H_k\big)(T)\,dx \\
&\le \sum_{\alpha=1}^N \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\,dv\,dx + \sum_{\alpha=1}^{N'} \int_{\gamma_T^-} v_\alpha^0 g^\alpha\,d\gamma_\alpha + \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(\varepsilon_k \mathring{E}_k \cdot \mathring{E}_k + \mu_k \mathring{H}_k \cdot \mathring{H}_k\Big)\,dx \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \Big(E_{k-1} \cdot j_k^{\mathrm{int}} - E_k \cdot \overline{j}_k\Big)\,dx\,dt \\
&\le \sum_{\alpha=1}^N \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\,dv\,dx + \sum_{\alpha=1}^{N'} \int_{\gamma_{T'}^-} v_\alpha^0 g^\alpha\,d\gamma_\alpha + \frac{\sigma'}{8\pi}\Big\|\Big(\mathring{E}_k, \mathring{H}_k\Big)\Big\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)} \\
&\quad + \int_0^T \|E_k(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\|u_k(t)\|_{L^2(\Gamma;\mathbb{R}^3)}\,dt + h_k(T) \\
&\le \sum_{\alpha=1}^N \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\,dv\,dx + \sum_{\alpha=1}^{N'} \int_{\gamma_{T'}^-} v_\alpha^0 g^\alpha\,d\gamma_\alpha + \frac{\sigma'}{8\pi}\Big\|\Big(\mathring{E}_k, \mathring{H}_k\Big)\Big\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)} \\
&\quad + \sqrt{4\pi}\sigma^{-\frac{1}{2}} \int_0^T \frac{\sqrt{\sigma}}{\sqrt{4\pi}}\|(E_k,H_k)(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\|u_k(t)\|_{L^2(\Gamma;\mathbb{R}^3)}\,dt + h_k(T).
\end{aligned}$$

By $E_k, H_k \in C\big([0,R^*];L^2\big(\mathbb{R}^3;\mathbb{R}^3\big)\big)$, $u_k \in C\big([0,R^*];L^2\big(\Gamma;\mathbb{R}^3\big)\big)$, and by continuity of h_k we can apply Lemma 1.1.2 and thus obtain

$$\begin{aligned}
&\left(2\sum_{\alpha=1}^N b_k^\alpha(T) + 2\sum_{\alpha=1}^N \int_\Omega \int_{B_R} v_\alpha^0 f_k^\alpha(T)\,dv\,dx + \frac{\sigma}{4\pi}\|(E_k,H_k)(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right)^{\frac{1}{2}} \\
&\le \sqrt{2}\left(\sum_{\alpha=1}^N \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\,dv\,dx + \sum_{\alpha=1}^{N'} \int_{\gamma_{T'}^-} v_\alpha^0 g^\alpha\,d\gamma_\alpha + \frac{\sigma'}{8\pi}\Big\|\Big(\mathring{E}_k, \mathring{H}_k\Big)\Big\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)} + h_k(T)\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{4\pi}\sigma^{-\frac{1}{2}} \|u_k\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))} \\
&\le \sqrt{2}\left(\sum_{\alpha=1}^{N} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha \, dv dx + \sum_{\alpha=1}^{N'} \int_{\gamma_{T'}^-} v_\alpha^0 g^\alpha \, d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E}_k, \mathring{H}_k\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)} + h_k(T)\right)^{\frac{1}{2}} \\
&+ \sqrt{4\pi}\sigma^{-\frac{1}{2}} \|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))} + \frac{\sqrt{4\pi}\sigma^{-\frac{1}{2}}}{k}
\end{aligned}
$$

by (1.4.24) so that

$$
\begin{aligned}
&\sum_{\alpha=1}^{N} b_k^\alpha(T) + \sum_{\alpha=1}^{N} \int_\Omega \int_{B_R} v_\alpha^0 f_k^\alpha(T) \, dv dx + \frac{\sigma}{8\pi} \|(E_k, H_k)(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)} \\
&\le \left(\left(\sum_{\alpha=1}^{N} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha \, dv dx + \sum_{\alpha=1}^{N'} \int_{\gamma_{T'}^-} v_\alpha^0 g^\alpha \, d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E}_k, \mathring{H}_k\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)} + h_k(T)\right)^{\frac{1}{2}}\right. \\
&\qquad\left. + \sqrt{2\pi}\sigma^{-\frac{1}{2}} \|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))} + \frac{\sqrt{2\pi}\sigma^{-\frac{1}{2}}}{k}\right)^2 \qquad (1.4.28)
\end{aligned}
$$

altogether. To be able to let $k \to \infty$, we have to integrate (1.4.28) in time since the evaluation of the limit functions f_R^α, E_R, H_R at a certain time makes no sense here (which is contrary to the time evaluation $b_k^\alpha(T)$ because there a full (t, x, v)-integral is included). Now let $A \subset [0, T']$ be measurable with Lebesgue measure $\lambda(A)$. As for $\sum_{\alpha=1}^N b_k^\alpha(T)$, we note that $\sum_{\alpha=1}^N b_R^\alpha(T)$ is the pointwise limit of $\sum_{\alpha=1}^N b_k^\alpha(T)$ by weak convergence and that we have a pointwise bound uniformly in T and k in view of (1.4.28). Additionally exploiting weak convergence and weak lower semicontinuity, respectively, the strong convergence of the initial electromagnetic fields, (1.4.26), and (1.4.27) we conclude

$$
\begin{aligned}
&\int_A \left(\sum_{\alpha=1}^{N} b_R^\alpha(T) + \sum_{\alpha=1}^{N} \int_\Omega \int_{B_R} v_\alpha^0 f_R^\alpha(T) \, dv dx + \frac{\sigma}{8\pi} \|(E_R, H_R)(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right) dT \\
&\le \liminf_{k\to\infty} \int_A \left(\sum_{\alpha=1}^{N} b_k^\alpha(T) + \sum_{\alpha=1}^{N} \int_\Omega \int_{B_R} v_\alpha^0 f_k^\alpha(T) \, dv dx + \frac{\sigma}{8\pi} \|(E_k, H_k)(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right) dT \\
&\le \lim_{k\to\infty} \int_A \left(\left(\sum_{\alpha=1}^{N} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha \, dv dx + \sum_{\alpha=1}^{N'} \int_{\gamma_{T'}^-} v_\alpha^0 g^\alpha \, d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E}_k, \mathring{H}_k\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right.\right. \\
&\qquad\left.\left. + h_k(T)\right)^{\frac{1}{2}} + \sqrt{2\pi}\sigma^{-\frac{1}{2}} \|u\|_{L^1([0,T'];L^2(\Gamma;\mathbb{R}^3))} + \frac{\sqrt{2\pi}\sigma^{-\frac{1}{2}}}{k}\right)^2 dT
\end{aligned}
$$

$$= \lambda(A)\left(\left(\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\,dv dx + \sum_{\alpha=1}^{N'}\int_{\gamma_{T'}^-} v_\alpha^0 g^\alpha\,d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right)^{\frac{1}{2}} + \sqrt{2\pi}\sigma^{-\frac{1}{2}}\|u\|_{L^1([0,T'];L^2(\Gamma;\mathbb{R}^3))}\right)^2.$$

Therefore, we have $(E_R, H_R) \in L^\infty\big([0,R^*];L^2\big(\mathbb{R}^3;\mathbb{R}^6\big)\big)$ and (after taking $T = T'$)

$$\left(\sum_{\alpha=1}^{N'}\left(1-a_0^\alpha\right)\int_{\gamma_T^+\cap\{|v|<R\}} v_\alpha^0 f_{R,+}^\alpha\,d\gamma_\alpha + \left\|\sum_{\alpha=1}^{N}\int_\Omega\int_{B_R} v_\alpha^0 f_R^\alpha(\cdot)\,dv dx + \frac{\sigma}{8\pi}\|(E_R,H_R)(\cdot)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right\|_{L^\infty([0,T])}\right)^{\frac{1}{2}}$$
$$\le \left(\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\,dv dx + \sum_{\alpha=1}^{N'}\int_{\gamma_T^-} v_\alpha^0 g^\alpha\,d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right)^{\frac{1}{2}} + \sqrt{2\pi}\sigma^{-\frac{1}{2}}\|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))} \tag{1.4.29}$$

for all $T \in\,]0,R^*]$. This is exactly the kind of energy estimate we wanted to derive since R does no longer appear on the right-hand side.

Lastly, we show that, up to a subsequence,

$$j_k^{\text{int}} \rightharpoonup j_R^{\text{int}} \text{ in } L^{\frac{4}{3}}\left([0,R^*]\times\Omega;\mathbb{R}^3\right) \tag{1.4.30}$$

for $k \to \infty$ and derive an $L^\infty\Big([0,R^*];L^{\frac{4}{3}}\left(\Omega;\mathbb{R}^3\right)\Big)$-bound for j_R^{int}. To this end, applying (1.2.13) yields

$$\left\|j_k^{\text{int}}(T)\right\|_{L^{\frac{4}{3}}(\Omega;\mathbb{R}^3)} \le \sum_{\alpha=1}^{N}|q_\alpha|\left\|\int_{B_R} f_k^\alpha(T,\cdot,v)\,dv\right\|_{L^{\frac{4}{3}}(\Omega)}$$
$$\le \sum_{\alpha=1}^{N}\left(\frac{4\pi}{3}\left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} + 1 + \begin{cases}\frac{4\pi}{3}\left(1-a_0^\alpha\right)^{-1}\|g^\alpha\|_{L^\infty(\gamma_T^-)}, & \alpha = 1,\dots,N'\\ 0, & \alpha = N'+1,\dots,N\end{cases}\right) \cdot|q_\alpha|\left(\int_\Omega\int_{B_R} v_\alpha^0 f_k^\alpha(T)\,dv dx\right)^{\frac{3}{4}}$$

for $0 < T \le R^*$ and the right-hand side is bounded in $L^{\frac{4}{3}}([0,R^*])$ uniformly in k by virtue of (1.4.28), where all terms on the right-hand side are uniformly bounded in k. Therefore, we may assume without loss of generality that j_k^{int} converges weakly in

$L^{\frac{4}{3}}\left([0,R^*]\times\Omega;\mathbb{R}^3\right)$ to some $\widetilde{j}_R^{\mathrm{int}}$. Indeed, $\widetilde{j}_R^{\mathrm{int}} = j_R^{\mathrm{int}}$ almost everywhere since

$$\int_0^{R^*}\int_\Omega\left(j_R^{\mathrm{int}} - \widetilde{j}_R^{\mathrm{int}}\right)\cdot\vartheta\,dxdt = \int_0^{R^*}\int_\Omega\left(\sum_{\alpha=1}^N q_\alpha\int_{B_R}\widehat{v}_\alpha f_R^\alpha\,dv - \widetilde{j}_R^{\mathrm{int}}\right)\cdot\vartheta\,dxdt$$
$$= \lim_{k\to\infty}\int_0^{R^*}\int_\Omega\left(\sum_{\alpha=1}^N q_\alpha\int_{B_R}\widehat{v}_\alpha f_k^\alpha\,dv - j_k^{\mathrm{int}}\right)\cdot\vartheta\,dxdt = 0$$

for any $\vartheta\in C_c^\infty\left(]0,R^*[\times\Omega;\mathbb{R}^3\right)$ by (for example, L^2-) weak convergence of the f_k^α. Altogether, we have shown (1.4.30). As for the desired bound, we proceed similarly to (1.2.13) and (1.2.15), respectively. Let $0<T\le R^*$ and $A\subset[0,T]$ measurable. For almost all $(t,x)\in A\times\Omega$ we have

$$\begin{aligned}\int_{B_R} f_R^\alpha(t,x,v)\,dv &\le \int_{B_r} f_R^\alpha(t,x,v)\,dv + \int_{r\le|v|\le R} f_R^\alpha(t,x,v)\,dv\\ &\le \frac{4\pi}{3}r^3\left\|f_R^\alpha\right\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^3)} + \frac{1}{r}\int_{B_R} v_\alpha^0 f_R^\alpha(t,x,v)\,dv\\ &= \left(\int_{B_R} v_\alpha^0 f_R^\alpha(t,x,v)\,dv\right)^{\frac{3}{4}}\left(\frac{4\pi}{3}\left\|f_R^\alpha\right\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^3)}+1\right) \qquad (1.4.31)\end{aligned}$$

where we set $r := \left(\int_{B_R} v_\alpha^0 f_R^\alpha(t,x,v)\,dv\right)^{\frac{1}{4}}\in[0,\infty[$. Summing over α, taking the $L^{\frac{4}{3}}(\Omega)$-norm, then integrating over A yields

$$\int_A\left(\int_\Omega\left|j_R^{\mathrm{int}}\right|^{\frac{4}{3}}dx\right)^{\frac{3}{4}}dt \le \sum_{\alpha=1}^N|q_\alpha|\int_A\left(\int_\Omega\left|\int_{B_R} f_R^\alpha\,dv\right|^{\frac{4}{3}}dx\right)^{\frac{3}{4}}dt$$
$$\le \int_A\sum_{\alpha=1}^N|q_\alpha|\left(\frac{4\pi}{3}\left\|f_R^\alpha\right\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^3)}+1\right)\left(\int_\Omega\int_{B_R} v_\alpha^0 f_R^\alpha\,dvdx\right)^{\frac{3}{4}}dt$$
$$\le \left(\sum_{\alpha=1}^N|q_\alpha|^4\left(\frac{4\pi}{3}\left\|f_R^\alpha\right\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^3)}+1\right)^4\right)^{\frac{1}{4}}\int_A\left(\sum_{\alpha=1}^N\int_\Omega\int_{B_R} v_\alpha^0 f_R^\alpha\,dvdx\right)^{\frac{3}{4}}dt \qquad (1.4.32)$$

by the triangle inequality in $L^{\frac{4}{3}}$ and Hölder's inequality for the sum. Inserting (1.4.11) and (1.4.29), respectively, we conclude $j_R^{\mathrm{int}}\in L^\infty\left([0,R^*];L^{\frac{4}{3}}\left(\Omega;\mathbb{R}^3\right)\right)$ with

$$\left\|j_R^{\mathrm{int}}\right\|_{L^\infty\left([0,T];L^{\frac{4}{3}}(\Omega;\mathbb{R}^3)\right)}$$
$$\le \left(\sum_{\alpha=1}^N|q_\alpha|^4\left(\frac{4\pi}{3}\left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)}+1+\begin{cases}\frac{4\pi}{3\left(1-a_0^\alpha\right)}\left\|g^\alpha\right\|_{L^\infty(\gamma_T^-)}, & \alpha\le N'\\ 0, & \alpha>N'\end{cases}\right)^4\right)^{\frac{1}{4}}$$

$$\cdot\left(\left(\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\,dvdx + \sum_{\alpha=1}^{N'}\int_{\gamma_T^-} v_\alpha^0 g^\alpha\,d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right)^{\frac{1}{2}} + \sqrt{2\pi}\sigma^{-\frac{1}{2}}\|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))}\right)^{\frac{3}{2}}$$

for any $0 < T \leq R^*$.

We summarize the results of this section in the following lemma.

Lemma 1.4.3. *Let $R > 0$ and $R^* = \min\{R, T_\bullet\}$. There exist functions*

- $f_R^\alpha \in L^\infty\left([0,R^*];\left(L^1\cap L^\infty\right)(\Omega\times\mathbb{R}^3)\right)\cap L^\infty\left([0,R^*];L^1_{\alpha\mathrm{kin}}(\Omega\times B_R)\right)$, *$\alpha = 1,\ldots,N$, all nonnegative,*
- $f_{R,+}^\alpha \in \left(L^1\cap L^\infty\right)\left(\gamma_{R^*}^+, d\gamma_\alpha\right)\cap L^1_{\alpha\mathrm{kin}}\left(\gamma_{R^*}^+\cap\{|v|<R\}, d\gamma_\alpha\right)$, *$\alpha = 1,\ldots,N'$, and* $f_{R,+}^\alpha \in L^\infty\left(\gamma_{R^*}^+\right)$, *$\alpha = N'+1,\ldots,N$, all nonnegative,*
- $(E_R, H_R) \in L^\infty\left([0,R^*];L^2\left(\mathbb{R}^3;\mathbb{R}^6\right)\right)$

such that $\left(\left(f_R^\alpha, f_{R,+}^\alpha\right)_\alpha, E_R, H_R, j_R\right)$ *satisfies Definition 1.1.1.(i) to 1.1.1.(iii) with final time R^*, where*

$$j_R = j_R^{\mathrm{int}} + u = \sum_{\alpha=1}^{N} q_\alpha\int_{B_R}\widehat{v}_\alpha f_R^\alpha\,dv + u,\quad j_R^{\mathrm{int}} \in L^\infty\left([0,R^*];\left(L^1\cap L^{\frac{4}{3}}\right)(\Omega;\mathbb{R}^3)\right).$$

Furthermore, we have the following estimates for any $1 \leq p \leq \infty$ and $T \in]0,R^*]$:
Estimates on $f_R^\alpha, f_{R,+}^\alpha$:

$$\left\|f_R^\alpha\right\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))} \leq \left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)} + \left(1-a_0^\alpha\right)^{\frac{1}{p}-1}\left\|g^\alpha\right\|_{L^p\left(\gamma_T^-,d\gamma_\alpha\right)}, \tag{1.4.33}$$

$$\left\|f_{R,+}^\alpha\right\|_{L^p\left(\gamma_T^+,d\gamma_\alpha\right)} \leq \left(1-a_0^\alpha\right)^{-\frac{1}{p}}\left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)} + \left(1-a_0^\alpha\right)^{-1}\left\|g^\alpha\right\|_{L^p\left(\gamma_T^-,d\gamma_\alpha\right)} \tag{1.4.34}$$

for $\alpha = 1,\ldots,N'$ and

$$\left\|f_R^\alpha\right\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))} \leq \left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)}, \tag{1.4.35}$$

$$\left\|f_{R,+}^\alpha\right\|_{L^\infty\left(\gamma_T^+\right)} \leq \left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} \tag{1.4.36}$$

for $\alpha = N'+1,\ldots,N$.
Energy-like estimate:

$$\left(\sum_{\alpha=1}^{N'}\left(1-a_0^\alpha\right)\int_{\gamma_T^+\cap\{|v|<R\}} v_\alpha^0 f_{R,+}^\alpha\,d\gamma_\alpha\right.$$

$$+\left\|\sum_{\alpha=1}^{N}\int_\Omega\int_{B_R} v_\alpha^0 f_R^\alpha(\cdot)\,dvdx + \frac{\sigma}{8\pi}\|(E_R,H_R)(\cdot)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right\|_{L^\infty([0,T])}\Bigg)^{\frac{1}{2}}$$

$$\le \left(\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\,dvdx + \sum_{\alpha=1}^{N'}\int_{\gamma_T^-} v_\alpha^0 g^\alpha\,d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right)^{\frac{1}{2}}$$

$$+ \sqrt{2\pi}\sigma^{-\frac{1}{2}}\|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))}. \tag{1.4.37}$$

Estimate on j_R^{int}*:*

$$\left\|j_R^{\text{int}}\right\|_{L^\infty\left([0,T];L^{\frac{4}{3}}(\Omega;\mathbb{R}^3)\right)}$$

$$\le \left(\sum_{\alpha=1}^{N}\left|q_\alpha\right|^4\left(\frac{4\pi}{3}\left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} + 1 + \begin{cases}\frac{4\pi}{3\left(1-a_0^\alpha\right)}\left\|g^\alpha\right\|_{L^\infty(\gamma_T^-)}, & \alpha\le N'\\ 0, & \alpha > N'\end{cases}\right)^4\right)^{\frac{1}{4}}$$

$$\cdot\left(\left(\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\,dvdx + \sum_{\alpha=1}^{N'}\int_{\gamma_T^-} v_\alpha^0 g^\alpha\,d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right)^{\frac{1}{2}}\right.$$

$$\left.+ \sqrt{2\pi}\sigma^{-\frac{1}{2}}\|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))}\right)^{\frac{3}{2}}. \tag{1.4.38}$$

1.4.3 Removing the cut-off

Finally, we obtain a weak solution of (VM) on the time Interval $I_{T_\bullet}$ by letting $R\to\infty$. To this end, it is crucial that the right-hand sides of the estimates (1.4.33) to (1.4.37) do not depend on R. We choose a sequence $(R_m)\subset\,]0,\infty[$ converging to ∞ and denote $f_m^\alpha = f_{R_m}^\alpha$ and so on, and $m^* = \min\{R_m, T_\bullet\}$. Now take $L\in\mathbb{N}$ and define $L^* = \min\{L, T_\bullet\}$. By (1.4.33) to (1.4.38) we may extract subsequences $\left(f_{m_l,L}^\alpha\right)_l$, $\left(f_{m_l,+,L}^\alpha\right)_l$, $\left(\left(E_{m_l,L}, H_{m_l,L}\right)\right)_l$, $\left(j_{m_l,L}^{\text{int}}\right)_l$ such that for any $1<p<\infty$, $\left(f_{m_l,L}^\alpha\right)_l$ converges weakly in $L^p\left([0,L^*]\times\Omega\times\mathbb{R}^3\right)$ and weak-* in $L^\infty\left([0,L^*]\times\Omega\times\mathbb{R}^3\right)$, $\left(f_{m_l,+,L}^\alpha\right)_l$ converges weak-* in $L^\infty(\gamma_{L^*})$, for $\alpha = 1,\dots,N'$ additionally weakly in $L^p(\gamma_{L^*}, d\gamma_\alpha)$, and moreover $\left(\left(E_{m_l,L}, H_{m_l,L}\right)\right)_l$ converges weakly in $L^2\left([0,L^*]\times\mathbb{R}^3;\mathbb{R}^6\right)$ and $\left(j_{m_l,L}^{\text{int}}\right)_l$ weakly in $L^{\frac{4}{3}}\left([0,L^*]\times\Omega;\mathbb{R}^3\right)$, and such that these subsequences are subsequences of the previous ones with index $L-1$ (if $L\ge 2$). By considering the respective diagonal sequences with indices m_L, L and L running we have found subsequences (now again denoted by index m) and limit functions

$f^\alpha, f_+^\alpha, E, H, \widetilde{j}^{\text{int}}$ such that

$$\begin{aligned} f_m^\alpha &\rightharpoonup f^\alpha && \text{in } L^p\big([0,M^*]\times\Omega\times\mathbb{R}^3\big), \\ f_m^\alpha &\overset{*}{\rightharpoonup} f^\alpha && \text{in } L^\infty\big([0,M^*]\times\Omega\times\mathbb{R}^3\big), \\ f_{m,+}^\alpha &\overset{*}{\rightharpoonup} f_+^\alpha && \text{in } L^\infty(\gamma_{M^*}), \\ f_{m,+}^\alpha &\rightharpoonup f_+^\alpha && \text{in } L^p(\gamma_{M^*}, d\gamma_\alpha) \quad (\text{only for } \alpha = 1,\dots,N'), \\ (E_m, H_m) &\rightharpoonup (E,H) && \text{in } L^2\big([0,M^*]\times\mathbb{R}^3;\mathbb{R}^6\big), \\ j_m^{\text{int}} &\rightharpoonup \widetilde{j}^{\text{int}} && \text{in } L^{\frac{4}{3}}\big([0,M^*]\times\Omega;\mathbb{R}^3\big) \end{aligned}$$

for $m\to\infty$ for all $1<p<\infty$ and $M>0$ (where $M^*=\min\{M,T_\bullet\}$ as usual). We will show later that indeed $\widetilde{j}^{\text{int}} = j^{\text{int}} := \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha\, dv$. Clearly, all f^α and f_+^α are nonnegative.

Applying the same techniques that were used to derive (1.4.33) to (1.4.36), we conclude $f^\alpha \in L^\infty_{\text{lt}}\big(I_{T_\bullet}; (L^1\cap L^\infty)(\Omega\times\mathbb{R}^3)\big)$, $f_+^\alpha \in \Big(L^1_{\text{lt}}\cap L^\infty_{\text{lt}}\Big)\Big(\gamma_{T_\bullet}^+, d\gamma_\alpha\Big)$ for $\alpha = 1,\dots,N'$, and $f_+^\alpha \in L^\infty\Big(\gamma_{T_\bullet}^+\Big)$ for $\alpha = N'+1,\dots,N$, satisfying

$$\begin{aligned} \|f^\alpha\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))} &\le \Big\|\mathring{f}^\alpha\Big\|_{L^p(\Omega\times\mathbb{R}^3)} + \big(1-a_0^\alpha\big)^{\frac{1}{p}-1}\|g^\alpha\|_{L^p(\gamma_T^-, d\gamma_\alpha)}, \\ \|f_+^\alpha\|_{L^p(\gamma_T^+, d\gamma_\alpha)} &\le \big(1-a_0^\alpha\big)^{-\frac{1}{p}}\Big\|\mathring{f}^\alpha\Big\|_{L^p(\Omega\times\mathbb{R}^3)} + \big(1-a_0^\alpha\big)^{-1}\|g^\alpha\|_{L^p(\gamma_T^-, d\gamma_\alpha)} \end{aligned}$$

for $\alpha = 1,\dots,N'$ and

$$\|f^\alpha\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))} \le \Big\|\mathring{f}^\alpha\Big\|_{L^p(\Omega\times\mathbb{R}^3)}, \quad \|f_+^\alpha\|_{L^\infty(\gamma_T^+)} \le \Big\|\mathring{f}^\alpha\Big\|_{L^\infty(\Omega\times\mathbb{R}^3)}$$

for $\alpha = N'+1,\dots,N$, respectively, for any $0<T\in I_{T_\bullet}$ and $1\le p\le\infty$. As for the energy estimate (1.4.37), we also consider $m\to\infty$. Similarly as in the previous section, take $0<T\in I_{T_\bullet}$, $A\subset[0,T]$ measurable, and $r>0$. We have

$$\begin{aligned} &\int_A \left(\sum_{\alpha=1}^N \int_\Omega\int_{B_r} v_\alpha^0 f^\alpha(T)\, dv dx + \frac{\sigma}{8\pi}\|(E,H)(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right) dT \\ &\le \liminf_{m\to\infty} \int_A \left(\sum_{\alpha=1}^N \int_\Omega\int_{B_r} v_\alpha^0 f_m^\alpha(T)\, dv dx + \frac{\sigma}{8\pi}\|(E_m,H_m)(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right) dT \\ &\le \liminf_{m\to\infty} \int_A \left(\sum_{\alpha=1}^N \int_\Omega\int_{B_{R_m}} v_\alpha^0 f_m^\alpha(T)\, dv dx + \frac{\sigma}{8\pi}\|(E_m,H_m)(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right) dT \end{aligned}$$

by weak convergence and $R_m\to\infty$. Similarly,

$$\sum_{\alpha=1}^{N'}\big(1-a_0^\alpha\big)\int_{\gamma_T^+\cap\{|v|<r\}} v_\alpha^0 f_+^\alpha\, d\gamma_\alpha = \lim_{m\to\infty}\sum_{\alpha=1}^{N'}\big(1-a_0^\alpha\big)\int_{\gamma_T^+\cap\{|v|<r\}} v_\alpha^0 f_{m,+}^\alpha\, d\gamma_\alpha$$

$$\leq \liminf_{m\to\infty} \sum_{\alpha=1}^{N'} \left(1 - a_0^\alpha\right) \int_{\gamma_T^+ \cap \{|v|<R_m\}} v_\alpha^0 f_{m,+}^\alpha \, d\gamma_\alpha .$$

Combining these estimates with (1.4.37) and using their uniformity in r, we deduce $f^\alpha \in L^\infty_{\mathrm{lt}}\left(I_{T_\bullet}; L^1_{\alpha\mathrm{kin}}(\Omega \times \mathbb{R}^3)\right)$, $(E,H) \in L^\infty_{\mathrm{lt}}\left(I_{T_\bullet}; L^2(\mathbb{R}^3;\mathbb{R}^6)\right)$, and $f_+^\alpha \in L^1_{\alpha\mathrm{kin,lt}}\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)$ (only for $\alpha = 1, \ldots, N'$), and we have for any $0 < T \in I_{T_\bullet}$

$$\begin{aligned}
&\left(\sum_{\alpha=1}^{N'} \left(1 - a_0^\alpha\right) \int_{\gamma_T^+} v_\alpha^0 f_+^\alpha \, d\gamma_\alpha \right.\\
&\qquad \left. + \left\| \sum_{\alpha=1}^{N} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 f^\alpha(\cdot) \, dv dx + \frac{\sigma}{8\pi} \|(E,H)(\cdot)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)} \right\|_{L^\infty([0,T])} \right)^{\frac{1}{2}}\\
&\leq \left(\sum_{\alpha=1}^{N} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha \, dv dx + \sum_{\alpha=1}^{N'} \int_{\gamma_T^-} v_\alpha^0 g^\alpha \, d\gamma_\alpha + \frac{\sigma'}{8\pi} \left\| \left(\mathring{E}, \mathring{H}\right) \right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)} \right)^{\frac{1}{2}}\\
&\quad + \sqrt{2\pi}\sigma^{-\frac{1}{2}} \|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))}
\end{aligned} \tag{1.4.39}$$

by a reasoning similar to the one in the previous section.

Next we consider the internal current j^{int}. To show that indeed $j^{\mathrm{int}} = \tilde{j}^{\mathrm{int}}$ we take $\vartheta \in C_c^\infty(]0, T_\bullet[\times \Omega; \mathbb{R}^3)$ and $r > 0$. Using weak convergence of j_m^{int} and f_m^α, respectively, we deduce

$$\begin{aligned}
&\left| \int_0^{T_\bullet} \int_\Omega \left(j^{\mathrm{int}} - \tilde{j}^{\mathrm{int}}\right) \cdot \vartheta \, dx dt \right| = \left| \lim_{m\to\infty} \iint_{\operatorname{supp}\vartheta} \left(j^{\mathrm{int}} - j_m^{\mathrm{int}}\right) \cdot \vartheta \, dx dt \right|\\
&= \left| \lim_{m\to\infty} \iint_{\operatorname{supp}\vartheta} \left(\sum_{\alpha=1}^{N} q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv - \sum_{\alpha=1}^{N} q_\alpha \int_{B_{R_m}} \widehat{v}_\alpha f_m^\alpha \, dv \right) \cdot \vartheta \, dx dt \right|\\
&\leq \limsup_{m\to\infty} \left| \sum_{\alpha=1}^{N} q_\alpha \iint_{\operatorname{supp}\vartheta} \int_{B_r} \widehat{v}_\alpha \left(f^\alpha - f_m^\alpha\right) \cdot \vartheta \, dv dx dt \right|\\
&\quad + \limsup_{m\to\infty} \left| \sum_{\alpha=1}^{N} q_\alpha \iint_{\operatorname{supp}\vartheta} \left(\int_{|v|\geq r} \widehat{v}_\alpha f^\alpha \, dv - \int_{r\leq|v|\leq R_m} \widehat{v}_\alpha f_m^\alpha \, dv \right) \cdot \vartheta \, dx dt \right|\\
&\leq 0 + \limsup_{m\to\infty} \frac{1}{r} \|\vartheta\|_{L^\infty(]0,T_\bullet[\times\Omega;\mathbb{R}^3)} \sum_{\alpha=1}^{N} |q_\alpha| \iint_{\operatorname{supp}\vartheta} \left(\int_{\mathbb{R}^3} v_\alpha^0 f^\alpha \, dv + \int_{B_{R_m}} v_\alpha^0 f_m^\alpha \, dv \right) dx dt\\
&\leq \frac{C}{r}
\end{aligned}$$

where C is finite by virtue of (1.4.37) and (1.4.39), respectively, and does not depend on r. Since $r > 0$ and $\vartheta \in C_c^\infty(]0, T_\bullet[\times \Omega; \mathbb{R}^3)$ was arbitrary, we conclude $j^{\mathrm{int}} = \tilde{j}^{\mathrm{int}}$ almost everywhere.

Clearly, $j^{\mathrm{int}} \in L^\infty_{\mathrm{lt}}(I_{T_\bullet}; L^1(\Omega; \mathbb{R}^3))$ by $f^\alpha \in L^\infty_{\mathrm{lt}}(I_{T_\bullet}; L^1(\Omega \times \mathbb{R}^3))$. But we even have $j^{\mathrm{int}} \in L^\infty_{\mathrm{lt}}\left(I_{T_\bullet}; L^{\frac{4}{3}}(\Omega; \mathbb{R}^3)\right)$ because of

$$\int_A \int_\Omega \left|j^{\mathrm{int}}\right|^{\frac{4}{3}} dx dt \le \liminf_{m\to\infty} \int_A \int_\Omega \left|j^{\mathrm{int}}_m\right|^{\frac{4}{3}} dx dt$$

for any measurable $A \subset I_{T_\bullet}$ by weak lower semicontinuity and because of (1.4.38) along the (m-)sequence. Thus, we conclude that j^{int} satisfies

$$\begin{aligned}
&\left\|j^{\mathrm{int}}\right\|_{L^\infty\left([0,T]; L^{\frac{4}{3}}(\Omega;\mathbb{R}^3)\right)} \\
&\le \left(\sum_{\alpha=1}^N |q_\alpha|^4 \left(\frac{4\pi}{3}\left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} + 1 + \begin{cases} \frac{4\pi}{3(1-a_0^\alpha)}\left\|g^\alpha\right\|_{L^\infty(\gamma_T^-)}, & \alpha \le N' \\ 0, & \alpha > N' \end{cases}\right)^4\right)^{\frac{1}{4}} \\
&\quad \cdot \left(\left(\sum_{\alpha=1}^N \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha \, dv dx + \sum_{\alpha=1}^{N'} \int_{\gamma_T^-} v_\alpha^0 g^\alpha \, d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E}, \mathring{H}\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right)^{\frac{1}{2}}\right. \\
&\qquad \left. + \sqrt{2\pi}\sigma^{-\frac{1}{2}} \|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))}\right)^{\frac{3}{2}}
\end{aligned}$$

for any $0 < T \in I_{T_\bullet}$.

Finally, there remains to check that (VM) also holds in the limit, now on the time interval $I_{T_\bullet}$. As for the Maxwell equations, it is even easier to pass to the limit than before since ε and μ remain constant along the m-sequence. For some $\vartheta \in \Theta_{T_\bullet}$, (1.1.3) holds in the limit by weak convergence of E_m, H_m, and j^{int}_m. Note that for this only the weak convergence on a bounded time interval matters by ϑ being compactly supported with respect to time.

As for passing to the limit in (1.1.2), let $\psi \in \Psi_{T_\bullet}$. All terms but the nonlinear product term are again easy to handle by the known weak convergences. Note that again only a bounded time interval matters according to $\psi \in \Psi_{T_\bullet}$. As for the integral over $\gamma^-_{T_\bullet}$, we calculate

$$\begin{aligned}
&\lim_{m\to\infty} \int_{\gamma^-_{T_\bullet}} \mathcal{K}_\alpha f^\alpha_{m,+} \psi \, d\gamma_\alpha = \lim_{m\to\infty} \int_{\gamma^-_{T_\bullet} \cap \operatorname{supp} \psi} a^\alpha \left(K f^\alpha_{m,+}\right) \psi \, d\gamma_\alpha \\
&= \lim_{m\to\infty} \int_{\gamma^+_{T_\bullet} \cap \operatorname{supp} \psi} (K a^\alpha) f^\alpha_{m,+} K\psi \, d\gamma_\alpha = \int_{\gamma^+_{T_\bullet} \cap \operatorname{supp} \psi} (K a^\alpha) f^\alpha_+ K\psi \, d\gamma_\alpha
\end{aligned}$$

$$= \int_{\gamma_{T_\bullet}^- \cap \operatorname{supp}\psi} a^\alpha \big(K f_+^\alpha\big) \psi \, d\gamma_\alpha = \int_{\gamma_{T_\bullet}^-} \mathcal{K}_\alpha f_+^\alpha \psi \, d\gamma_\alpha$$

as in (1.4.15). To tackle the crucial product term, we proceed similarly to Section 1.4.2. We again may assume that ψ factorizes, i.e., $\psi(t,x,v) = \psi_1(t,x)\psi_2(v)$. For some $\zeta \in C_c^\infty(\mathbb{R}^3)$ and $r > 0$ such that ζ vanishes for $|v| > r-1$, and for given $l \in \mathbb{N}$, $0 < s \in I_{T_\bullet}$ we first choose $\eta_l \in C_c^\infty(]0,s[\times\Omega\times B_r)$ such that

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^3} \zeta(v)\big((1-\eta_l)\big(f_m^\alpha - f^\alpha\big)\big)(\cdot,\cdot,v)\, dv \right\|_{L^2([0,s]\times\Omega)} \\
&= \left\| \int_{B_r} \zeta(v)\big((1-\eta_l)\big(f_m^\alpha - f^\alpha\big)\big)(\cdot,\cdot,v)\, dv \right\|_{L^2([0,s]\times\Omega)} \le C\big\|1-\eta_l\big\|_{L^2([0,s]\times\Omega\times B_r)} < \frac{1}{l}
\end{aligned}$$

for m large, i.e., $R_m \geq s$. This is possible due to the uniform boundedness of the f_m^α in $L^2\big([0,s]\times\Omega\times\mathbb{R}^3\big)$. Arguing in the same way as in Section 1.4.2 only replacing R^* by s and noting that

$$\begin{aligned}
&\partial_t\big(\eta_l f_m^\alpha\big) + \widehat{v}_\alpha \cdot \partial_x\big(\eta_l f_m^\alpha\big) \\
&= -\operatorname{div}_v\big(q_\alpha(E_m + \widehat{v}_\alpha \times H_m)\big(\eta_l f_m^\alpha\big)\big) + f_m^\alpha \partial_t \eta_l + f_m^\alpha \widehat{v}_\alpha \cdot \partial_x \eta_l \\
&\quad + q_\alpha f_m^\alpha (E_m + \widehat{v}_\alpha \times H_m) \cdot \partial_v \eta_l
\end{aligned}$$

on $\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$ in the sense of distributions (if $R_m \geq s$), we conclude that, after extracting a suitable subsequence, $\int_{\mathbb{R}^3} \zeta(v) f_m^\alpha(\cdot,\cdot,v)\, dv \to \int_{\mathbb{R}^3} \zeta(v) f^\alpha(\cdot,\cdot,v)\, dv$ strongly in $L^2([0,s]\times\Omega)$ for $m \to \infty$. This is enough to pass to the limit in (1.1.2) for fixed $\psi \in \Psi_{T_\bullet}$ that factorizes by choosing s such that ψ vanishes for $t \geq s$.

We summarize our results in the following theorem.

Theorem 1.4.4. *Let $T_\bullet \in\,]0,\infty]$, $\Omega \subset \mathbb{R}^3$ be a bounded domain such that $\partial\Omega$ is a $C^{1,\kappa}$-submanifold of $\mathbb{R}^3$ for some $0 < \kappa \leq 1$, and let Condition 1.4.1 hold. Then there exist functions*

- $f^\alpha \in L^\infty_{\mathrm{lt}}\Big(I_{T_\bullet}; \Big(L^1_{\alpha\mathrm{kin}} \cap L^\infty\Big)\big(\Omega\times\mathbb{R}^3\big)\Big)$ *and* $f_+^\alpha \in \Big(L^1_{\alpha\mathrm{kin,lt}} \cap L^\infty_{\mathrm{lt}}\Big)\Big(\gamma_{T_\bullet}^+, d\gamma_\alpha\Big)$ *for* $\alpha = 1,\ldots,N'$, *all nonnegative,*
- $f^\alpha \in L^\infty\big(I_{T_\bullet}\times\Omega\times\mathbb{R}^3\big) \cap L^\infty_{\mathrm{lt}}\Big(I_{T_\bullet}; L^1_{\alpha\mathrm{kin}}\big(\Omega\times\mathbb{R}^3\big)\Big)$ *and* $f_+^\alpha \in L^\infty\Big(\gamma_{T_\bullet}^+\Big)$ *for* $\alpha = N'+1, \ldots, N$, *all nonnegative,*
- $(E,H) \in L^\infty_{\mathrm{lt}}\big(I_{T_\bullet}; L^2\big(\mathbb{R}^3;\mathbb{R}^6\big)\big)$

such that $\big(\big(f^\alpha, f_+^\alpha\big)_\alpha, E, H, j\big)$ *is a weak solution of* (VM) *on the time interval* $I_{T_\bullet}$ *with external current u in the sense of Definition 1.1.1, where*

$$j = j^{\mathrm{int}} + u = \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv + u, \quad j^{\mathrm{int}} \in L^\infty_{\mathrm{lt}}\Big(I_{T_\bullet}; \Big(L^1 \cap L^{\frac{4}{3}}\Big)\big(\Omega;\mathbb{R}^3\big)\Big).$$

Furthermore, we have the following estimates for any $1 \le p \le \infty$ *and* $0 < T \in I_{T_\bullet}$*:*
Estimates on f^α, f_+^α*:*

$$\left\|f^\alpha\right\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))} \le \left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)} + \left(1-a_0^\alpha\right)^{\frac{1}{p}-1}\left\|g^\alpha\right\|_{L^p\left(\gamma_T^-,d\gamma_\alpha\right)}, \tag{1.4.40}$$

$$\left\|f_+^\alpha\right\|_{L^p\left(\gamma_T^+,d\gamma_\alpha\right)} \le \left(1-a_0^\alpha\right)^{-\frac{1}{p}}\left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)} + \left(1-a_0^\alpha\right)^{-1}\left\|g^\alpha\right\|_{L^p\left(\gamma_T^-,d\gamma_\alpha\right)} \tag{1.4.41}$$

for $\alpha = 1, \dots, N'$ *and*

$$\left\|f^\alpha\right\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))} \le \left\|\mathring{f}^\alpha\right\|_{L^p(\Omega\times\mathbb{R}^3)}, \tag{1.4.42}$$

$$\left\|f_+^\alpha\right\|_{L^\infty\left(\gamma_T^+\right)} \le \left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} \tag{1.4.43}$$

for $\alpha = N'+1, \dots, N$.
Energy-like estimate:

$$\begin{aligned}
&\Bigg(\sum_{\alpha=1}^{N'}\left(1-a_0^\alpha\right)\int_{\gamma_T^+} v_\alpha^0 f_+^\alpha\, d\gamma_\alpha \\
&\qquad + \left\|\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 f^\alpha(\cdot)\, dv dx + \frac{\sigma}{8\pi}\|(E,H)(\cdot)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right\|_{L^\infty([0,T])}\Bigg)^{\frac{1}{2}} \\
&\le \left(\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\, dv dx + \sum_{\alpha=1}^{N'}\int_{\gamma_T^-} v_\alpha^0 g^\alpha\, d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right)^{\frac{1}{2}} \\
&\quad + \sqrt{2\pi}\sigma^{-\frac{1}{2}}\|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))}.
\end{aligned} \tag{1.4.44}$$

Estimate on j^{int}*:*

$$\begin{aligned}
&\left\|j^{\mathrm{int}}\right\|_{L^\infty\left([0,T];L^{\frac{4}{3}}(\Omega;\mathbb{R}^3)\right)} \\
&\le \left(\sum_{\alpha=1}^{N}\left|q_\alpha\right|^4\left(\frac{4\pi}{3}\left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} + 1 + \begin{cases}\frac{4\pi}{3\left(1-a_0^\alpha\right)}\left\|g^\alpha\right\|_{L^\infty\left(\gamma_T^-\right)}, & \alpha \le N' \\ 0, & \alpha > N'\end{cases}\right)^4\right)^{\frac{1}{4}} \\
&\quad \cdot \Bigg(\left(\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha\, dv dx + \sum_{\alpha=1}^{N'}\int_{\gamma_T^-} v_\alpha^0 g^\alpha\, d\gamma_\alpha + \frac{\sigma'}{8\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}\right)^{\frac{1}{2}} \\
&\qquad\qquad + \sqrt{2\pi}\sigma^{-\frac{1}{2}}\|u\|_{L^1([0,T];L^2(\Gamma;\mathbb{R}^3))}\Bigg)^{\frac{3}{2}}.
\end{aligned} \tag{1.4.45}$$

1.5 The redundant divergence equations and the charge balance

In this section we shall discuss in what sense the divergence equations (0.5) hold for a weak solution of (VM) in the sense of Definition 1.1.1. The weak formulation of (0.5) is

$$0 = \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\varepsilon E \cdot \partial_x \varphi + 4\pi \rho \varphi\right) dx dt, \tag{1.5.1a}$$

$$0 = \int_0^{T_\bullet} \int_{\mathbb{R}^3} \mu H \cdot \partial_x \varphi \, dx dt \tag{1.5.1b}$$

for all $\varphi \in C_c^\infty\left(]0, T_\bullet[\times \mathbb{R}^3\right)$. Obviously, this is equivalent to (0.5) on $]0, T_\bullet[\times \mathbb{R}^3$ in the sense of distributions.

For (0.5) should propagate in time, we have to demand that (0.5) holds initially as a constraint on the initial data, that is to say,

$$\operatorname{div}\left(\varepsilon \mathring{E}\right) = 4\pi \mathring{\rho}, \quad \operatorname{div}\left(\mu \mathring{H}\right) = 0$$

on $\mathbb{R}^3$ in the sense of distributions, or, equivalently,

$$0 = \int_{\mathbb{R}^3} \left(\varepsilon \mathring{E} \cdot \partial_x \xi + 4\pi \mathring{\rho} \xi\right) dx, \tag{1.5.2a}$$

$$0 = \int_{\mathbb{R}^3} \mu \mathring{H} \cdot \partial_x \xi \, dx \tag{1.5.2b}$$

for all $\xi \in C_c^\infty\left(\mathbb{R}^3\right)$.

Now let $\left(\left(f^\alpha, f_+^\alpha\right)_\alpha, E, H, j\right)$ be a weak solution of (VM) on the time interval $I_{T_\bullet}$ with external current u. It is easy to see that (1.5.1b) holds: Define

$$\vartheta \colon I_{T_\bullet} \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \vartheta(t, x) = -\int_t^{T_\bullet} \partial_x \varphi(s, x) \, ds.$$

Clearly, $\vartheta \in \Theta_{T_\bullet}$. Hence, (1.1.3b) and $\xi = \int_0^{T_\bullet} \varphi(s, \cdot) \, ds \in C_c^\infty\left(\mathbb{R}^3\right)$ in (1.5.2b) yields

$$\begin{aligned}
0 &= \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\mu H \cdot \partial_t \vartheta + E \cdot \operatorname{curl}_x \vartheta\right) dx dt + \int_{\mathbb{R}^3} \mu \mathring{H} \cdot \vartheta(0) \, dx \\
&= \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\mu H \cdot \partial_x \varphi - E \cdot \int_t^{T_\bullet} \operatorname{curl}_x \partial_x \varphi(s, x) \, ds\right) dx dt - \int_{\mathbb{R}^3} \mu \mathring{H} \cdot \partial_x \xi \, dx \\
&= \int_0^{T_\bullet} \int_{\mathbb{R}^3} \mu H \cdot \partial_x \varphi \, dx dt
\end{aligned}$$

and we are done.

As for (1.5.1a), we have to exploit local conservation of charge and have to determine what ρ is. Therefore, we have to make use of (1.1.2) in order to put the internal charge density into play. However, the test functions there have to satisfy $\psi \in \Psi_{T_\bullet}$ but a test function of (1.5.1a) does not depend on v. Consequently, we, on the one hand, have to consider a cut-off in momentum space, and, on the other hand, have to show that (1.1.2) also holds if the support of ψ is not away from $\gamma_{T_\bullet}^0$ or $\{0\}\times\partial\Omega\times\mathbb{R}^3$. To this end, the following technical lemma is useful. There and throughout the rest of this section, we assume that $\Omega\subset\mathbb{R}^3$ is a bounded domain such that $\partial\Omega$ is a $C^1\cap W^{2,\infty}$-submanifold of $\mathbb{R}^3$. Here, $\partial\Omega$ being of class $C^1\cap W^{2,\infty}$ means that it is of class C^1 and all local flattenings are locally of class $W^{2,\infty}$.

Lemma 1.5.1. *Let $1\le p<2$ and $\psi\in C^1\left(I_{T_\bullet}\times\mathbb{R}^3\times\mathbb{R}^3\right)$ with* $\operatorname{supp}\psi\subset[0,T_\bullet[\times\mathbb{R}^3\times\mathbb{R}^3$ *compact. Then there is a sequence $\left(\psi_k\right)\subset\Psi_{T_\bullet}$ such that*

$$\left\|\psi_k-\psi\right\|_{W^{1,p_t2_x1_v}(]0,T_\bullet[\times\Omega\times\mathbb{R}^3)}\to 0 \tag{1.5.3}$$

for $k\to\infty$ and there is $0<r<\infty$ such that ψ and all ψ_k vanish for $t\ge r$. Here,

$$\|h\|_{W^{1,p_t2_x1_v}(]0,T_\bullet[\times\Omega\times\mathbb{R}^3)}:=\left(\int_0^{T_\bullet}\left(\int_\Omega\left(\int_{\mathbb{R}^3}(|h|+|\partial_t h|+|\partial_x h|+|\partial_v h|)\,dv\right)^2dx\right)^{\frac{p}{2}}dt\right)^{\frac{1}{p}}.$$

Proof. First, we extend ψ to a C^1-function on $\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$ such that $\operatorname{supp}\psi\subset\,]-T_\bullet,T_\bullet[\,\times\mathbb{R}^3\times\mathbb{R}^3$ is compact (which can be achieved since the hyperplane where $t=0$ is smooth).

By assumption about $\partial\Omega$, for each $x\in\partial\Omega$ there exist open sets $\tilde U_x,\tilde U_x'\subset\mathbb{R}^3$ with $x\in\tilde U_x$ and a C^1-diffeomorphism $F^x\colon\tilde U_x\to\tilde U_x'$, that has the property $F^x\in W^{2,\infty}_{\mathrm{loc}}\left(\tilde U_x;\tilde U_x'\right)$, such that $F^x\left(\tilde U_x\cap\partial\Omega\right)=\tilde U_x'\cap\left(\mathbb{R}^2\times\{0\}\right)$. For any $x\in\partial\Omega$ we choose an open set $U_x\subset\mathbb{R}^3$ such that $x\in U_x$ and $U_x\subset\subset\tilde U_x$ (here and in the following, $A\subset\subset B$ is shorthand for "A bounded and $\overline{A}\subset B$"). Then, $\partial\Omega\subset\bigcup_{x\in\partial\Omega}U_x$, whence there are a finite number of points, say, $x_i\in\partial\Omega$, $i=1,\dots m$, such that $\partial\Omega\subset\bigcup_{i=1}^m U_i$, since $\partial\Omega$ is compact. Here and in the following, we write $U_i:=U_{x_i}$, $\tilde U_i:=\tilde U_{x_i}$, and $F^i:=F^{x_i}$. Since it holds that $\overline{\Omega}\setminus\bigcup_{i=1}^m U_i\subset\subset\Omega$, there is an open set $U_0\subset\mathbb{R}^3$ satisfying $\overline{\Omega}\setminus\bigcup_{i=1}^m U_i\subset\subset U_0\subset\subset\Omega$. Therefore, we have $\overline{\Omega}\subset\bigcup_{i=0}^m U_i$. Finally, we choose an open set $M\subset\mathbb{R}^3$ such that $\overline{\Omega}\subset M\subset\subset\bigcup_{i=0}^m U_i$.

Now let ζ_i, $i=0,\dots,m$, be a partition of unity on M subordinate to U_i, $i=0,\dots,m$, i.e., $\zeta_i\in C_c^\infty\left(\mathbb{R}^3\right)$, $0\le\zeta_i\le 1$, $\operatorname{supp}\zeta_i\subset U_i$, and $\sum_{i=0}^m\zeta_i=1$ on M (and hence on $\overline{\Omega}$, in particular). Furthermore, let $\eta\in C^\infty(\mathbb{R})$ such that $0\le\eta\le 1$, $\eta(y)=0$ for $|y|\le\frac{1}{2}$, and $\eta(y)=1$ for $|y|\ge 1$.

Next, for $i=1,\dots,m$ define $G^i\colon U_i\times\mathbb{R}^3\to\mathbb{R}^6$, $G^i(x,v)=\left(F^i(x),A^i(x)v\right)$, where the rows $A^i_j(x)$, $j=1,2,3$, of $A^i(x)$ are given by

$$A_1^i(x)=\frac{\nabla F_1^i(x)\times\nabla F_3^i(x)}{\left|\nabla F_1^i(x)\times\nabla F_3^i(x)\right|},\;A_2^i(x)=\frac{\nabla F_3^i(x)\times\left(\nabla F_1^i(x)\times\nabla F_3^i(x)\right)}{\left|\nabla F_3^i(x)\times\left(\nabla F_1^i(x)\times\nabla F_3^i(x)\right)\right|},\;A_3^i(x)=\frac{\nabla F_3^i(x)}{\left|\nabla F_3^i(x)\right|}.$$

Note that the rows are orthogonal and have length one, and that A^i is of class $C \cap W^{1,\infty}$ on U_i since F^i is of class $C^1 \cap W^{2,\infty}$ on U_i, $\det DF^i \neq 0$ on $\tilde{U}_i$, and hence the denominators in $A^i(x)$ are bounded away from zero on U_i because of $U_i \subset\subset \tilde{U}_i$. Therefore, G^i is of class $C \cap W^{1,\infty}$ on $U_i \times B_R$ for any $R > 0$.

The key idea is that, for any $(x, v) \in U_i \times \mathbb{R}^3$, $x \in \partial\Omega$ is equivalent to $G_3^i(x, v) = 0$ and, moreover, $(x, v) \in \tilde{\gamma}^0$ is equivalent to $G_3^i(x, v) = G_6^i(x, v) = 0$, since $n(x)$ and $\nabla F_3^i(x)$ are parallel (and both nonzero). Thus, since the supports of the approximating functions ψ_k shall be away from $\gamma^0_{T_\bullet}$ and $\{0\} \times \partial\Omega \times \mathbb{R}^3$, it is natural to consider the following C^∞-function in the variables (t, G), that cuts off a region near the two sets where $G_3 = G_6 = 0$ and where $t = G_3 = 0$:

$$\eta_k \colon \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}, \quad \eta_k(t, G) = \eta\big(k^2\big(G_3^2 + G_6^2\big)\big)\eta\big(k^2\big(t^2 + G_3^2\big)\big).$$

For $k \in \mathbb{N}$ we then define

$$\tilde{\psi}_k \colon \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \quad \tilde{\psi}_k(t, x, v) = \zeta_0(x)\psi(t, x, v) + \sum_{i=1}^{m} \zeta_i(x)\psi(t, x, v)\eta_k^{G^i}(t, x, v)$$

where

$$\eta_k^{G^i} \colon \mathbb{R} \times U_i \times \mathbb{R}^3 \to \mathbb{R}, \quad \eta_k^{G^i}(t, x, v) = \eta_k\big(t, G^i(x, v)\big).$$

We should mention that, according to $\operatorname{supp} \zeta_i \subset U_i$, $i = 0, \dots, m$, the i-th summand is (by definition) zero if $x \notin U_i$. Note that we can apply the chain rule for $\eta_k^{G^i}$ since η_k is smooth and $G^i \in W^{1,1}\big(U_i \times B_R; \mathbb{R}^6\big)$ for any $R > 0$. Therefore, $\tilde{\psi}_k$ is of class $C \cap W^{1,\infty}$.

First we show that (1.5.3) holds for $\tilde{\psi}_k$ (instead of ψ_k). By $\sum_{i=0}^m \zeta_i = 1$ on $\overline{\Omega}$ we have

$$\begin{aligned} \big\|\tilde{\psi}_k - \psi\big\|_{W^{1,p_t 2_x 1_v}(]0,T_\bullet[\times\Omega\times\mathbb{R}^3)} &\le \sum_{i=1}^{m} \Big\|\zeta_i \psi\Big(\eta_k^{G^i} - 1\Big)\Big\|_{W^{1,p_t 2_x 1_v}(]0,R[\times U_i \times B_R)} \\ &\le C \sum_{i=1}^{m} \Big\|\eta_k^{G^i} - 1\Big\|_{W^{1,p_t 2_x 1_v}(]0,R[\times U_i \times B_R)}, \end{aligned} \tag{1.5.4}$$

where $C > 0$ depends on the (finite) C_b^1-norms of ψ (and ζ_i) and where $R > 0$ is chosen such that ψ vanishes if $t \geq R$ or $|v| \geq R$. For fixed $i \in \{1, \dots, m\}$ and $(t, x, v) \in \mathbb{R} \times U_i \times \mathbb{R}^3$ the implications

$$\begin{aligned} \eta_k^{G^i}(t, x, v) \neq 1 &\Rightarrow k^2\Big(G_3^i(x, v)^2 + G_6^i(x, v)^2\Big) \le 1 \vee k^2\Big(t^2 + G_3^i(x, v)^2\Big) \le 1 \\ &\Rightarrow \big|F_3^i(x)\big| \le k^{-1} \wedge \big(\big|G_6^i(x, v)\big| \le k^{-1} \vee |t| \le k^{-1}\big) \end{aligned}$$

hold. Therefore, we have, recalling that $0 \le \eta \le 1$,

$$\left(\int_0^R \left(\int_{U_i} \left(\int_{B_R} \Big|\eta_k^{G^i} - 1\Big| \, dv\right)^2 dx\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^R \left(\int_{\left\{x\in U_i \,\middle|\, \left|F_3^i(x)\right| \leq k^{-1}\right\}} \left(\int_{\left\{v\in B_R \,\middle|\, \left|G_6^i(x,v)\right| \leq k^{-1}\right\}} dv \right)^2 dx \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}}$$
$$+ \left(\int_0^{k^{-1}} \left(\int_{\left\{x\in U_i \,\middle|\, \left|F_3^i(x)\right| \leq k^{-1}\right\}} \left(\frac{4\pi}{3} R^3 \right)^2 dx \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}}$$
$$=: I_1^k + I_2^k.$$

In the following, we will heavily make use of the facts that $A^i(x)$ is orthonormal for any $x \in U_i$, $\left|\det DF^i\right|$ is bounded away from zero on U_i, and $F^i(U_i)$ is bounded. Thus,

$$I_1^k \leq C \left(\int_0^R \left(\int_{\left\{y\in F^i(U_i) \,\middle|\, \left|y_3\right| \leq k^{-1}\right\}} \left(\int_{\{w\in B_R \mid |w_3| \leq k^{-1}\}} dw \right)^2 dy \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} \leq Ck^{-\frac{3}{2}} \longrightarrow 0$$

for $k \longrightarrow \infty$. Here and in the following, C denotes a positive, finite constant that may depend on p, R, and F^i, and that may change in each step. Similarly,

$$I_2^k \leq C \left(\int_0^{k^{-1}} \left(\int_{\left\{y\in F^i(U_i) \,\middle|\, \left|y_3\right| \leq k^{-1}\right\}} dy \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} \leq Ck^{-\frac{1}{2}-\frac{1}{p}} \longrightarrow 0$$

for $k \longrightarrow \infty$. Next we turn to the derivatives and start with the t-derivative. By

$$\partial_t \eta_k^{G^i}(t,x,v) = 2k^2 t \eta \left(k^2 \left(G_3^i(x,v)^2 + G_6^i(x,v)^2 \right) \right) \eta' \left(k^2 \left(t^2 + G_3^i(x,v)^2 \right) \right)$$

we have

$$\left| \partial_t \eta_k^{G^i}(t,x,v) \right| \leq Ck^2 t$$

and

$$\partial_t \eta_k^{G^i}(t,x,v) \neq 0 \Rightarrow k^2 \left(t^2 + G_3^i(x,v)^2 \right) \leq 1 \Rightarrow |t| \leq k^{-1} \wedge \left| F_3^i(x) \right| \leq k^{-1}.$$

Hence,

$$\left(\int_0^R \left(\int_{U_i} \left(\int_{B_R} \left| \partial_t \eta_k^{G^i} \right| dv \right)^2 dx \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}}$$
$$\leq Ck^2 \left(\int_0^{k^{-1}} \left(\int_{\left\{x\in U_i \,\middle|\, \left|F_3^i(x)\right| \leq k^{-1}\right\}} \left(\int_{B_R} t\, dv \right)^2 dx \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}}$$

$$\leq Ck^2\left(\int_0^{k^{-1}}\left(\int_{\{y\in F^i(U_i)|\,|y_3|\leq k^{-1}\}} t^2\,dy\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}} \leq Ck^{\frac{3}{2}}\left(\int_0^{k^{-1}} t^p\,dt\right)^{\frac{1}{p}} = Ck^{\frac{1}{2}-\frac{1}{p}} \longrightarrow 0$$

for $k \longrightarrow \infty$ by $p < 2$. As for the x-derivatives, we compute

$$\begin{aligned}
&\partial_{x_j}\eta_k^{G^i}(t,x,v)\\
&= k^2\eta'\Big(k^2\Big(G_3^i(x,v)^2 + G_6^i(x,v)^2\Big)\Big)\eta\Big(k^2\Big(t^2 + G_3^i(x,v)^2\Big)\Big)\partial_{x_j}\Big(G_3^i(x,v)^2 + G_6^i(x,v)^2\Big)\\
&\quad + k^2\eta\Big(k^2\Big(G_3^i(x,v)^2 + G_6^i(x,v)^2\Big)\Big)\eta'\Big(k^2\Big(t^2 + G_3^i(x,v)^2\Big)\Big)\partial_{x_j}\Big(G_3^i(x,v)^2\Big) \qquad (1.5.5)
\end{aligned}$$

for $j = 1,2,3$. If $k^2\Big(G_3^i(x,v)^2 + G_6^i(x,v)^2\Big) \geq 1$, the first summand vanishes and (1.5.5), on the one hand, implies

$$\left|\partial_{x_j}\eta_k^{G^i}(t,x,v)\right| \leq Ck^2\left|G_3^i(x,v)\right| = Ck^2\left|F_3^i(x)\right|$$

for $(t,x,v) \in \mathbb{R}\times U_i \times B_R$ since G^i is of class $W^{1,\infty}$ on $U_i \times B_R$, and, on the other hand,

$$\partial_{x_j}\eta_k^{G^i}(t,x,v) \neq 0 \Rightarrow k^2\Big(t^2 + G_3^i(x,v)^2\Big) \leq 1 \Rightarrow \left|F_3^i(x)\right| \leq k^{-1} \wedge |t| \leq k^{-1}.$$

If $k^2\Big(G_3^i(x,v)^2 + G_6^i(x,v)^2\Big) < 1$, we have, on the one hand,

$$\left|F_3^i(x)\right| \leq k^{-1} \wedge \left|G_6^i(x,v)\right| \leq k^{-1}$$

and (1.5.5), on the other hand, implies

$$\left|\partial_{x_j}\eta_k^{G^i}(t,x,v)\right| \leq Ck^2\left(\left|G_3^i(x,v)\right| + \left|G_6^i(x,v)\right|\right) = Ck^2\left(\left|F_3^i(x)\right| + \left|G_6^i(x,v)\right|\right).$$

Combing these two cases we conclude

$$\begin{aligned}
&\left(\int_0^R\left(\int_{U_i}\left(\int_{B_R}\left|\partial_{x_j}\eta_k^{G^i}\right|dv\right)^2 dx\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}}\\
&\leq Ck^2\left(\int_0^{k^{-1}}\left(\int_{\{x\in U_i|\,|F_3^i(x)|\leq k^{-1}\}}\left(\int_{B_R}\left|F_3^i(x)\right|dv\right)^2 dx\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}}\\
&\quad + Ck^2\left(\int_0^R\left(\int_{\{x\in U_i|\,|F_3^i(x)|\leq k^{-1}\}}\left(\int_{\{v\in B_R|\,|G_6^i(x,v)|\leq k^{-1}\}}\left|F_3^i(x)\right|dv\right)^2 dx\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&+ Ck^2\left(\int_0^R\left(\int_{\left\{x\in U_i\middle|\left|F_3^i(x)\right|\le k^{-1}\right\}}\left(\int_{\left\{v\in B_R\middle|\left|G_6^i(x,v)\right|\le k^{-1}\right\}}\left|G_6^i(x,v)\right|dv\right)^2 dx\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}}\\
&\le Ck^{2-\frac{1}{p}}\left(\int_{\left\{y\in F^i(U_i)\middle|\left|y_3\right|\le k^{-1}\right\}}\left|y_3\right|^2 dy\right)^{\frac{1}{2}}\\
&\quad + Ck^2\left(\int_{\left\{y\in F^i(U_i)\middle|\left|y_3\right|\le k^{-1}\right\}}\left|y_3\right|^2 dy\left(\int_{\{w\in B_R||w_3|\le k^{-1}\}} dw\right)^2\right)^{\frac{1}{2}}\\
&\quad + Ck^2\left(\int_{\left\{y\in F^i(U_i)\middle|\left|y_3\right|\le k^{-1}\right\}} dy\left(\int_{\{w\in B_R||w_3|\le k^{-1}\}} |w_3|\, dw\right)^2\right)^{\frac{1}{2}}\\
&\le Ck^{\frac{1}{2}-\frac{1}{p}} + Ck^{-\frac{1}{2}} \longrightarrow 0
\end{aligned}
$$

for $k \longrightarrow \infty$ again by $p < 2$. Finally, consider the v-derivatives and compute

$$
\partial_{v_j}\eta_k^{G^i}(t,x,v) = k^2\eta'\left(k^2\left(G_3^i(x,v)^2 + G_6^i(x,v)^2\right)\right)\eta\left(k^2\left(t^2 + G_3^i(x,v)^2\right)\right)\partial_{v_j}\left(G_6^i(x,v)^2\right)
$$

for $j = 1, 2, 3$, which implies

$$
\left|\partial_{v_j}\eta_k^{G^i}(t,x,v)\right| \le Ck^2\left|G_6^i(x,v)\right|
$$

and

$$
\begin{aligned}
\partial_{v_j}\eta_k^{G^i}(t,x,v) \ne 0 &\Rightarrow k^2\left(G_3^i(x,v)^2 + G_6^i(x,v)^2\right) \le 1\\
&\Rightarrow \left|F_3^i(x)\right| \le k^{-1} \wedge \left|G_6^i(x,v)\right| \le k^{-1}
\end{aligned}
$$

for $(t,x,v) \in \mathbb{R}\times U_i \times B_R$. Therefore, we have

$$
\begin{aligned}
&\left(\int_0^R\left(\int_{U_i}\left(\int_{B_R}\left|\partial_{v_j}\eta_k^{G^i}\right| dv\right)^2 dx\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}}\\
&\le Ck^2\left(\int_0^R\left(\int_{\left\{x\in U_i\middle|\left|F_3^i(x)\right|\le k^{-1}\right\}}\left(\int_{\left\{v\in B_R\middle|\left|G_6^i(x,v)\right|\le k^{-1}\right\}}\left|G_6^i(x,v)\right|dv\right)^2 dx\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}}\\
&\le Ck^{-\frac{1}{2}} \longrightarrow 0
\end{aligned}
$$

for $k \longrightarrow \infty$ as before. Altogether, we have shown that

$$
\lim_{k\to\infty}\left\|\eta_k^{G^i} - 1\right\|_{W^{1,p_t 2_x 1_v}(]0,R[\times U_i\times B_R)} = 0
$$

for any $i = 1, \dots, m$ and thus

$$\lim_{k\to\infty} \left\| \tilde{\psi}_k - \psi \right\|_{W^{1,p_t 2_x 1_v}(]0,T_\bullet[\times\Omega\times\mathbb{R}^3)} = 0 \tag{1.5.6}$$

by (1.5.4).

The next step is to show that, for each $k \in \mathbb{N}$, the support of $\tilde{\psi}_k$ is away from $\gamma_{T_\bullet}^0$ and $\{0\} \times \partial\Omega \times \mathbb{R}^3$. As for $\gamma_{T_\bullet}^0$, assume the contrary, i.e., $\operatorname{dist}\left(\operatorname{supp} \tilde{\psi}_k, \gamma_{T_\bullet}^0\right) = 0$. Then we find sequences $\left((\tilde{t}_l, \tilde{x}_l, \tilde{v}_l)\right) \subset \gamma_{T_\bullet}^0$ and $((t_l, x_l, v_l)) \subset \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ such that $\tilde{\psi}_k(t_l, x_l, v_l) \neq 0$ for all $l \in \mathbb{N}$ and

$$\lim_{l\to\infty} \left| (\tilde{t}_l, \tilde{x}_l, \tilde{v}_l) - (t_l, x_l, v_l) \right| = 0.$$

By compactness of $\operatorname{supp} \tilde{\psi}_k \subset \operatorname{supp} \psi$, both sequences are bounded, whence we may assume without loss of generality that both sequences converge to the same limit, say, $(t, x, v) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$. Since $\tilde{\gamma}^0$ is closed and $\tilde{t}_l \geq 0$ for $l \in \mathbb{N}$, we have $(x, v) \in \tilde{\gamma}^0$ and $t \geq 0$. By $\operatorname{dist}(x, U_0) > 0$ and since $\bigcup_{i=1}^m U_i$ is an open cover of $\partial\Omega$, we may also assume that

$$x_l \in \bigcup_{i \in I \cup J} U_i \setminus \bigcup_{i \in \{0,\dots,m\} \setminus (I \cup J)} U_i \tag{1.5.7}$$

where $I := \{i \in \{1, \dots, m\} \mid x \in U_i\}$, $J := \{i \in \{1, \dots, m\} \mid x \in \partial U_i\}$ (for l large, at least). Clearly, $\zeta_i(x_l) = 0$ for any $i \in J$ and large l. Now take $i \in I$. Since G^i is continuous and since $G_3^i(x, v) = G_6^i(x, v) = 0$ by $(x, v) \in \tilde{\gamma}^0$, we have

$$\lim_{l\to\infty} G_3^i(x_l, v_l) = \lim_{l\to\infty} G_6^i(x_l, v_l) = 0$$

and then

$$k^2 \left(G_3^i(x_l, v_l)^2 + G_6^i(x_l, v_l)^2 \right) \leq \frac{1}{2}$$

for l large. But then $\eta_k^{G^i}(t_l, x_l, v_l) = 0$ and therefore by (1.5.7)

$$\begin{aligned} 0 &\neq \tilde{\psi}_k(t_l, x_l, v_l) \\ &= \sum_{i \in I} \zeta_i(x_l)\psi(t_l, x_l, v_l)\eta_k^{G^i}(t_l, x_l, v_l) + \sum_{i \in J} \zeta_i(x_l)\psi(t_l, x_l, v_l)\eta_k^{G^i}(t_l, x_l, v_l) = 0, \end{aligned}$$

which is a contradiction. As for $\{0\} \times \partial\Omega \times \mathbb{R}^3$, the proof works completely analogously: If we assume $\operatorname{dist}\left(\operatorname{supp} \tilde{\psi}_k, \{0\} \times \partial\Omega \times \mathbb{R}^3\right) = 0$, we find sequences $\left((\tilde{t}_l, \tilde{x}_l, \tilde{v}_l)\right) \subset \{0\} \times \partial\Omega \times \mathbb{R}^3$ and $((t_l, x_l, v_l)) \subset \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ such that $\tilde{\psi}_k(t_l, x_l, v_l) \neq 0$ for all $l \in \mathbb{N}$ and

$$\lim_{l\to\infty} \left| (\tilde{t}_l, \tilde{x}_l, \tilde{v}_l) - (t_l, x_l, v_l) \right| = 0.$$

As before, we may assume that both sequences converge to the same limit, say, $(t, x, v) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$. Since $\{0\} \times \partial\Omega \times \mathbb{R}^3$ is closed, we have $(t, x, v) \in \{0\} \times \partial\Omega \times \mathbb{R}^3$. Again we may assume (1.5.7). Now take $i \in I$. Since G^i is continuous and since $t = G_3^i(x, v) = 0$ by $x \in \partial\Omega$, we have

$$\lim_{l\to\infty} t_l = \lim_{l\to\infty} G_3^i(x_l, v_l) = 0$$

and then

$$k^2\Big(t_l^2 + G_3^i(x_l, v_l)^2\Big) \le \frac{1}{2}$$

for l large. But then $\eta_k^{G^i}(t_l, x_l, v_l) = 0$ and the contradiction follows as before.

There only remains one problem: The approximating functions are only of class $C \cap W^{1,\infty}$ with compact support and not necessarily of class C^∞ as desired (which corresponds to the fact that $\partial\Omega$ is only of class $C^1 \cap W^{2,\infty}$ and not necessarily smooth). To this end, take a Friedrich's mollifier $\omega \in C_c^\infty(\mathbb{R}^7)$ with $\operatorname{supp}\omega \subset B_1$, $\int_{\mathbb{R}^7} \omega \, d(t, x, v) = 1$, and denote $\omega_\delta := \delta^{-7}\omega\big(\frac{\cdot}{\delta}\big)$ for $\delta > 0$. By $\tilde{\psi}_k \in H^1(\mathbb{R}^7)$, we know that $\omega_\delta * \tilde{\psi}_k$ converges to $\tilde{\psi}_k$ for $\delta \to 0$ in $H^1(\mathbb{R}^7)$. Moreover, since $\operatorname{supp}\tilde{\psi}_k \subset \,]-T_\bullet, T_\bullet[\, \times \mathbb{R}^3 \times \mathbb{R}^3$, $\operatorname{dist}\Big(\operatorname{supp}\tilde{\psi}_k, \gamma_{T_\bullet}^0\Big), \operatorname{dist}\Big(\operatorname{supp}\tilde{\psi}_k, \{0\} \times \partial\Omega \times \mathbb{R}^3\Big) > 0$, these properties also hold for $\omega_\delta * \tilde{\psi}_k$ instead of $\tilde{\psi}_k$ if δ is small enough. Choose $0 < \delta_k \le 1$ so small and such that

$$\big\|\omega_{\delta_k} * \tilde{\psi}_k - \tilde{\psi}_k\big\|_{H^1(\mathbb{R}^7)} \le \frac{1}{k}.$$

By $p < 2$, this implies

$$\big\|\omega_{\delta_k} * \tilde{\psi}_k - \tilde{\psi}_k\big\|_{W^{1,p_t 2_x 1_v}(]0,R+1[\times\Omega\times B_{R+1})} \le \frac{C}{k}$$

where $C > 0$ depends on p, Ω, and R. After combining this with (1.5.6), noting that $\tilde{\psi}_k$ and ψ vanish if $t \ge R$ or $|v| \ge R$ and $\omega_{\delta_k} * \tilde{\psi}_k$ if $t \ge R + 1$ (which implies the existence of r as asserted) or $|v| \ge R + 1$, and setting

$$\psi_k := \omega_{\delta_k} * \tilde{\psi}_k\big|_{I_{T_\bullet}\times\overline{\Omega}\times\mathbb{R}^3} \in \Psi_{T_\bullet},$$

we are finally done. □

With this lemma, we can extend (1.1.2) to test functions ψ whose supports do not necessarily have to be away from $\gamma_{T_\bullet}^0$ and $\{0\} \times \partial\Omega \times \mathbb{R}^3$ under a condition on the integrability of the solution.

Lemma 1.5.2. *Let* $\alpha \in \{1, \ldots, N\}$, $f^\alpha \in L_{\mathrm{lt}}^\infty(I_{T_\bullet} \times \Omega \times \mathbb{R}^3)$, $f_+^\alpha \in L_{\mathrm{lt}}^\infty\Big(\gamma_{T_\bullet}^+\Big)$, $(E, H) \in L_{\mathrm{lt}}^q\big(I_{T_\bullet}; L^2(\mathbb{R}^3; \mathbb{R}^6)\big)$ *for some* $q > 2$, $\mathcal{K}_\alpha \colon L_{\mathrm{lt}}^\infty\Big(\gamma_{T_\bullet}^+\Big) \to L_{\mathrm{lt}}^\infty\Big(\gamma_{T_\bullet}^-\Big)$, $g^\alpha \in L_{\mathrm{lt}}^\infty\Big(\gamma_{T_\bullet}^-\Big)$, *and* $\mathring{f}^\alpha \in L^\infty(\Omega \times \mathbb{R}^3)$ *such that Definition 1.1.1.(ii) is satisfied. Moreover, let* $\psi \in C^1(I_{T_\bullet} \times \mathbb{R}^3 \times \mathbb{R}^3)$ *with* $\operatorname{supp}\psi \subset [0, T_\bullet[\, \times \mathbb{R}^3 \times \mathbb{R}^3$ *compact. Then,* (1.1.2) *still holds for* ψ.

Proof. Let $1 \leq p < 2$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. In accordance with Lemma 1.5.1, let $(\psi_k) \subset \Psi_{T_\bullet}$ approximate ψ with respect to the $W^{1,p_t 2_x 1_v}$-norm, $0 < r < \infty$ such that ψ and all ψ_k vanish for $t \geq r$, and define $R := \min\{r, T_\bullet\}$. By assumption, (1.1.2) holds for ψ_k for all $k \in \mathbb{N}$. Hence, there remains to show that we can pass to the limit $k \to \infty$ in (1.1.2). First, we have

$$\left| \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} (\partial_t \psi_k - \partial_t \psi) f^\alpha \, dv dx dt \right| \leq \left\| \psi_k - \psi \right\|_{W^{1,1}(]0,R[\times\Omega\times\mathbb{R}^3)} \left\| f^\alpha \right\|_{L^\infty([0,R]\times\Omega\times\mathbb{R}^3)}$$
$$\leq C(R, \Omega, p, f^\alpha) \left\| \psi_k - \psi \right\|_{W^{1,p_t 2_x 1_v}(]0,R[\times\Omega\times\mathbb{R}^3)} \to 0$$

for $k \to \infty$, since R is finite and Ω is bounded. Similarly,

$$\lim_{k\to\infty} \left| \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} (\widehat{v}_\alpha \cdot \partial_x \psi_k - \widehat{v}_\alpha \cdot \partial_x \psi) f^\alpha \, dv dx dt \right| = 0$$

by $|\widehat{v}_\alpha| \leq 1$. Next,

$$\left| \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} (E + \widehat{v}_\alpha \times H) \cdot (\partial_v \psi_k - \partial_v \psi) f^\alpha \, dv dx dt \right|$$
$$\leq \left\| f^\alpha \right\|_{L^\infty([0,R]\times\Omega\times\mathbb{R}^3)} \int_0^R \int_\Omega (|E| + |H|) \int_{\mathbb{R}^3} \left| \partial_v \psi_k - \partial_v \psi \right| dv dx dt$$
$$\leq C(f^\alpha) \int_0^R \left(\int_\Omega \left(|E|^2 + |H|^2 \right) dx \right)^{\frac{1}{2}} \left(\int_\Omega \left(\int_{\mathbb{R}^3} \left| \partial_v \psi_k - \partial_v \psi \right| dv \right)^2 dx \right)^{\frac{1}{2}} dt$$
$$\leq C(f^\alpha) \|(E,H)\|_{L^q([0,R];L^2(\mathbb{R}^3;\mathbb{R}^6))} \left(\int_0^R \left(\int_\Omega \left(\int_{\mathbb{R}^3} \left| \partial_v \psi_k - \partial_v \psi \right| dv \right)^2 dx \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} \to 0$$

for $k \to \infty$. Note that this was the crucial estimate, for which we essentially needed the convergence of ψ_k to ψ in the $W^{1,p_t 2_x 1_v}$-norm. As for the integrals over $\gamma_{T_\bullet}^\pm$, we first have

$$\int_{\partial\Omega} \left| \psi_k - \psi \right| (t,x,v) \, dS_x \leq C(\Omega) \int_\Omega \left(\left| \psi_k - \psi \right| + \left| \partial_x \psi_k - \partial_x \psi \right| \right) (t,x,v) \, dx$$

for any $t \in I_{T_\bullet}$, $v \in \mathbb{R}^3$, since Ω is bounded and $\partial\Omega$ of class C^1. Therefore, by $|n(x) \cdot \widehat{v}_\alpha| \leq 1$ it holds that

$$\left| \int_{\gamma_{T_\bullet}^+} (\psi_k - \psi) f_+^\alpha \, d\gamma_\alpha \right| \leq C(\Omega) \left\| \psi_k - \psi \right\|_{W^{1,1}(]0,R[\times\Omega\times\mathbb{R}^3)} \left\| f_+^\alpha \right\|_{L^\infty(\gamma_R^+)} \to 0$$

for $k \to \infty$. Similarly,

$$\left| \int_{\gamma_{T_\bullet}^-} (\psi_k - \psi) (\mathcal{K}_\alpha f_+^\alpha + g^\alpha) \, d\gamma_\alpha \right|$$

$$\leq C(\Omega)\left\|\psi_k - \psi\right\|_{W^{1,1}(]0,R[\times\Omega\times\mathbb{R}^3)}\left(\left\|\mathcal{K}_\alpha f_+^\alpha\right\|_{L^\infty\left(\gamma_R^-\right)} + \left\|g^\alpha\right\|_{L^\infty\left(\gamma_R^-\right)}\right) \to 0$$

for $k \to \infty$. Lastly, by

$$\begin{aligned} 0 &= \psi_k(R,x,v) - \psi(R,x,v) \\ &= \psi_k(0,x,v) - \psi(0,x,v) + \int_0^R \left(\partial_t \psi_k(t,x,v) - \partial_t \psi(t,x,v)\right) dt \end{aligned}$$

for any $x \in \Omega$, $v \in \mathbb{R}^3$, we have

$$\left|\int_\Omega \int_{\mathbb{R}^3} \left(\psi_k(0) - \psi(0)\right) \mathring{f}^\alpha \, dv dx\right| \leq \left\|\psi_k - \psi\right\|_{W^{1,1}(]0,R[\times\Omega\times\mathbb{R}^3)} \left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} \to 0$$

for $k \to \infty$ and the proof is complete. □

The next step is to show that (1.1.2) still holds if ψ does not depend on v. This is done via a cut-off procedure in v. Note that in the following lemma it is essential that f^α is of class $L^1 \cap L^2_{\alpha\mathrm{kin}}$ locally in time.

Lemma 1.5.3. *For $\alpha \in \{1, \ldots, N\}$ let $f^\alpha \in \left(L^1_{\mathrm{lt}} \cap L^2_{\alpha\mathrm{kin},\mathrm{lt}} \cap L^\infty_{\mathrm{lt}}\right)\left(I_{T_\bullet} \times \Omega \times \mathbb{R}^3\right)$, $f_+^\alpha \in L^\infty_{\mathrm{lt}}\left(\gamma_{T_\bullet}^+\right)$, $(E,H) \in L^q_{\mathrm{lt}}\left(I_{T_\bullet}; L^2(\mathbb{R}^3;\mathbb{R}^6)\right)$ for some $q > 2$, $\mathcal{K}_\alpha \colon L^\infty_{\mathrm{lt}}\left(\gamma_{T_\bullet}^+\right) \to L^\infty_{\mathrm{lt}}\left(\gamma_{T_\bullet}^-\right)$, $g^\alpha \in L^\infty_{\mathrm{lt}}\left(\gamma_{T_\bullet}^-\right)$, and $\mathring{f}^\alpha \in \left(L^1 \cap L^\infty\right)(\Omega \times \mathbb{R}^3)$ such that Definition 1.1.1.(ii) is satisfied. Furthermore, let $\psi \in C^1\left(I_{T_\bullet} \times \mathbb{R}^3\right)$ with $\operatorname{supp}\psi \subset [0, T_\bullet[\times \mathbb{R}^3$ compact.*

(i) If $\operatorname{supp}\psi \subset [0, T_\bullet[\times \left(\mathbb{R}^3 \setminus \partial\Omega\right)$, we have

$$0 = \int_0^{T_\bullet} \int_\Omega \left(\partial_t \psi \int_{\mathbb{R}^3} f^\alpha \, dv + \partial_x \psi \cdot \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv\right) dx dt + \int_\Omega \psi(0) \int_{\mathbb{R}^3} \mathring{f}^\alpha \, dv dx. \tag{1.5.8}$$

(ii) If, additionally to the given assumptions, $f_+^\alpha \in L^1_{\mathrm{lt}}\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)$, $g^\alpha \in L^1_{\mathrm{lt}}\left(\gamma_{T_\bullet}^-, d\gamma_\alpha\right)$, and $\mathcal{K}_\alpha \colon \left(L^1_{\mathrm{lt}} \cap L^\infty_{\mathrm{lt}}\right)\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right) \to \left(L^1_{\mathrm{lt}} \cap L^\infty_{\mathrm{lt}}\right)\left(\gamma_{T_\bullet}^-, d\gamma_\alpha\right)$, but ψ need not vanish on $\partial\Omega$, then (1.1.2) is still satisfied for ψ, i.e.,

$$\begin{aligned} 0 = &- \int_0^{T_\bullet} \int_\Omega \left(\partial_t \psi \int_{\mathbb{R}^3} f^\alpha \, dv + \partial_x \psi \cdot \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv\right) dx dt + \int_{\gamma_{T_\bullet}^+} f_+^\alpha \psi \, d\gamma_\alpha \\ &- \int_{\gamma_{T_\bullet}^-} \left(\mathcal{K}_\alpha f_+^\alpha + g^\alpha\right) \psi \, d\gamma_\alpha - \int_\Omega \psi(0) \int_{\mathbb{R}^3} \mathring{f}^\alpha \, dv dx. \end{aligned} \tag{1.5.9}$$

Proof. The proof works similarly to the proof of [Guo93, Lemma 4.2.]. First, consider a test function ψ that may have support on $\partial\Omega$. Take $\eta \in C_c^\infty\left(\mathbb{R}^3\right)$, $0 \leq \eta \leq 1$, $\eta = 1$ on B_1, $\operatorname{supp}\eta \subset B_2$, and let $\eta_m := \eta\left(\frac{\cdot}{m}\right)$ for $m \in \mathbb{N}$. Then, $\psi_m \in C^1\left(I_{T_\bullet} \times \mathbb{R}^3 \times \mathbb{R}^3\right)$ with

$\operatorname{supp}\psi_m \subset [0,T_\bullet[\times\mathbb{R}^3\times\mathbb{R}^3$ compact, where $\psi_m(t,x,v) := \psi(t,x)\eta_m(v)$. Therefore, (1.1.2) holds for ψ_m by Lemma 1.5.2. We can pass to the limit $m\to\infty$ in all terms of (1.1.2) but the terms including integrals over $\gamma^\pm_{T_\bullet}$: Let $R>0$ such that ψ vanishes for $t\ge R$. First,

$$\left|\int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3} f^\alpha\partial_t\psi_m\,dvdxdt - \int_0^{T_\bullet}\int_\Omega\partial_t\psi\int_{\mathbb{R}^3} f^\alpha\,dvdxdt\right|$$
$$\le \left\|\partial_t\psi\right\|_{L^\infty(I_{T_\bullet}\times\mathbb{R}^3)}\int_0^R\int_\Omega\int_{\mathbb{R}^3}\left|\eta_m-1\right|\left|f^\alpha\right|dvdxdt\to 0$$

for $m\to\infty$ by dominated convergence since $\eta_m\to 1$ pointwise for $m\to\infty$ and $\left|\eta_m-1\right|\left|f^\alpha\right|\le\left|f^\alpha\right|\in L^1\big([0,R]\times\Omega\times\mathbb{R}^3\big)$. Similarly by $|\widehat{v}_\alpha|\le 1$,

$$\lim_{m\to\infty}\int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3}\partial_x\psi_m\cdot\widehat{v}_\alpha f^\alpha\,dvdxdt = \int_0^{T_\bullet}\int_\Omega\partial_x\psi\cdot\int_{\mathbb{R}^3}\widehat{v}_\alpha f^\alpha\,dvdxdt.$$

Because of

$$\partial_v\psi_m(t,x,v) = \frac{1}{m}\psi(t,x)\nabla\eta\Big(\frac{v}{m}\Big)$$

and

$$\partial_v\psi_m(t,x,v)\ne 0 \Longrightarrow m\le|v|\le 2m$$

for $(t,x,v)\in I_{T_\bullet}\times\Omega\times\mathbb{R}^3$, we get the following estimate, which is again the crucial one:

$$\begin{aligned}
&\left|\int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3}(E+\widehat{v}_\alpha\times H)f^\alpha\cdot\partial_v\psi_m\,dvdxdt\right|\\
&\le\left\|\psi\right\|_{L^\infty(I_{T_\bullet}\times\Omega)}\left\|\nabla\eta\right\|_{L^\infty(B_2;\mathbb{R}^3)}\int_0^R\int_\Omega(|E|+|H|)\int_{\{v\in\mathbb{R}^3|m\le|v|\le 2m\}}\frac{1}{m}\left|f^\alpha\right|dvdxdt\\
&\le C\big(\psi,\eta\big)\|(E,H)\|_{L^2([0,R]\times\Omega;\mathbb{R}^6)}\left(\int_0^R\int_\Omega\left(\int_{\{v\in\mathbb{R}^3|m\le|v|\le 2m\}}\frac{1}{m}\left|f^\alpha\right|dv\right)^2dxdt\right)^{\frac12}\\
&\le C\big(\psi,\eta,E,H\big)\left(\int_0^R\int_\Omega\int_{\{v\in\mathbb{R}^3|m\le|v|\le 2m\}}\frac{\frac{4\pi}{3}\big(8m^3-m^3\big)}{m^2}\left|f^\alpha\right|^2dv\,dxdt\right)^{\frac12}\\
&\le C\big(\psi,\eta,E,H\big)\left(\int_0^R\int_\Omega\int_{\{v\in\mathbb{R}^3|m\le|v|\le 2m\}}v^0_\alpha\left|f^\alpha\right|^2dv\,dxdt\right)^{\frac12}\to 0
\end{aligned}$$

for $m\to\infty$, since the last integral converges to 0 thanks to $f^\alpha\in L^2_{\alpha\mathrm{kin}}\big([0,R]\times\Omega\times\mathbb{R}^3\big)$. As for the term including the initial data, we see that

$$\left|\int_\Omega\int_{\mathbb{R}^3}\psi_m(0)\mathring{f}^\alpha\,dvdx - \int_\Omega\psi(0)\int_{\mathbb{R}^3}\mathring{f}^\alpha\,dvdx\right|$$

$$\leq \left\|\psi(0)\right\|_{L^\infty(\Omega)} \int_\Omega \int_{\mathbb{R}^3} \left|\eta_m - 1\right| \left|\mathring{f}^\alpha\right| dv dx \to 0$$

for $m \to \infty$ as well by dominated convergence and $\mathring{f}^\alpha \in L^1(\Omega \times \mathbb{R}^3)$.

Now if $\operatorname{supp} \psi \subset [0, T_\bullet[\times (\mathbb{R}^3 \setminus \partial\Omega)$, then ψ_m vanishes on $\partial\Omega$, too, and for ψ_m there vanish the integrals over $\gamma^\pm_{T_\bullet}$ appearing in (1.1.2). Hence, (1.5.8) is satisfied.

If the additional assumptions of part 1.5.3.(ii) hold but ψ need not vanish on $\partial\Omega$, we consider the integrals over $\gamma^\pm_{T_\bullet}$:

$$\left| \int_{\gamma^+_{T_\bullet}} f^\alpha_+ \psi_m \, d\gamma_\alpha - \int_{\gamma^+_{T_\bullet}} f^\alpha_+ \psi \, d\gamma_\alpha \right| \leq \left\|\psi\right\|_{L^\infty(I_{T_\bullet} \times \mathbb{R}^3)} \int_{\gamma^+_R} \left|\eta_m - 1\right| \left|f^\alpha_+\right| d\gamma_\alpha \to 0$$

and similarly

$$\begin{aligned} &\left| \int_{\gamma^-_{T_\bullet}} \left(\mathcal{K}_\alpha f^\alpha_+ + g^\alpha\right) \psi_m \, d\gamma_\alpha - \int_{\gamma^-_{T_\bullet}} \left(\mathcal{K}_\alpha f^\alpha_+ + g^\alpha\right) \psi \, d\gamma_\alpha \right| \\ &\leq \left\|\psi\right\|_{L^\infty(I_{T_\bullet} \times \mathbb{R}^3)} \int_{\gamma^-_R} \left|\eta_m - 1\right| \left(\left|\mathcal{K}_\alpha f^\alpha_+\right| + \left|g^\alpha\right|\right) d\gamma_\alpha \to 0 \end{aligned}$$

for $m \to \infty$ by dominated convergence and $f^\alpha_+ \in L^1(\gamma^+_R, d\gamma_\alpha)$, $\mathcal{K}_\alpha f^\alpha_+, g^\alpha \in L^1(\gamma^-_R, d\gamma_\alpha)$. Therefore, we obtain (1.5.9). □

In the following, we denote

$$\rho^{\text{int}} := \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} f^\alpha \, dv, \quad j^{\text{int}} := \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv$$

and extend these functions by zero outside Ω.

Equations (1.5.8) and (1.5.9) reflect the principle of local conservation of internal charge and imply a global charge balance after an integration.

Corollary 1.5.4. *Let the assumptions of Lemma 1.5.3 hold for all $\alpha \in \{1, \dots, N\}$.*

(i) We have

$$\partial_t \rho^{\text{int}} + \operatorname{div}_x j^{\text{int}} = 0$$

on $]0, T_\bullet[\times \Omega$ in the sense of distributions.

If moreover the additional assumptions of Lemma 1.5.3.(ii) are satisfied for all $\alpha \in \{1, \dots, N\}$, then:

(ii) It holds that

$$\partial_t \rho^{\text{int}} + T_{\partial\Omega} + \operatorname{div}_x j^{\text{int}} = 0 \tag{1.5.10}$$

on $]0,T_\bullet[\times\mathbb{R}^3$ *in the sense of distributions. Here, the distribution* $T_{\partial\Omega}$ *describes the boundary processes via*

$$T_{\partial\Omega}\psi=\sum_{\alpha=1}^N q_\alpha\left(\int_{\gamma^+_{T_\bullet}} f^\alpha_+\psi\,d\gamma_\alpha-\int_{\gamma^-_{T_\bullet}}\left(\mathcal{K}_\alpha f^\alpha_+ + g^\alpha\right)\psi\,d\gamma_\alpha\right).$$

(iii) For almost all $t\in I_{T_\bullet}$ *we have*

$$\int_\Omega \rho^{\mathrm{int}}(t,x)\,dx=\int_\Omega \mathring{\rho}^{\mathrm{int}}\,dx-\sum_{\alpha=1}^N q_\alpha\left(\int_{\gamma^+_t} f^\alpha_+\,d\gamma_\alpha-\int_{\gamma^-_t}\left(\mathcal{K}_\alpha f^\alpha_+ + g^\alpha\right)d\gamma_\alpha\right),$$

where

$$\mathring{\rho}^{\mathrm{int}}:=\sum_{\alpha=1}^N q_\alpha\int_{\mathbb{R}^3}\mathring{f}^\alpha\,dv.$$

Proof. As for parts 1.5.4.(i) and 1.5.4.(ii), simply multiply (1.5.8) and (1.5.9) with q_α and sum over α. As for part 1.5.4.(iii), take $\varphi\in C^\infty_c(]0,T_\bullet[)$ and let $\eta\in C^\infty_c(\mathbb{R}^3)$ with $\eta=1$ on $\overline{\Omega}$. We define

$$\psi\colon I_{T_\bullet}\times\mathbb{R}^3\to\mathbb{R},\quad \psi(t,x)=-\eta(x)\int_t^{T_\bullet}\varphi\,ds.$$

Then, $\psi\in C^\infty(I_{T_\bullet}\times\mathbb{R}^3)$ with $\operatorname{supp}\psi\subset[0,T_\bullet[\times\mathbb{R}^3$ compact. Thus, Lemma 1.5.3.(ii) yields, after summing over α,

$$\begin{aligned}
0&=\sum_{\alpha=1}^N q_\alpha\Bigg(-\int_0^{T_\bullet}\int_\Omega\left(\partial_t\psi\int_{\mathbb{R}^3} f^\alpha\,dv+\partial_x\psi\cdot\int_{\mathbb{R}^3}\widehat{v}_\alpha f^\alpha\,dv\right)dxdt+\int_{\gamma^+_{T_\bullet}} f^\alpha_+\psi\,d\gamma_\alpha\\
&\qquad-\int_{\gamma^-_{T_\bullet}}\left(\mathcal{K}_\alpha f^\alpha_+ + g^\alpha\right)\psi\,d\gamma_\alpha-\int_\Omega\psi(0)\int_{\mathbb{R}^3}\mathring{f}^\alpha\,dvdx\Bigg)\\
&=-\int_0^{T_\bullet}\varphi\int_\Omega\rho^{\mathrm{int}}\,dxdt+\int_0^{T_\bullet}\varphi\int_\Omega\mathring{\rho}^{\mathrm{int}}\,dxds\\
&\quad+\sum_{\alpha=1}^N q_\alpha\Bigg(-\int_0^{T_\bullet}\int_{\partial\Omega}\int_{\{v\in\mathbb{R}^3|n(x)\cdot v>0\}} f^\alpha_+(t,x,v)\int_t^{T_\bullet}\varphi(s)\,ds\,n(x)\cdot\widehat{v}_\alpha\,dvdS_xdt\\
&\quad-\int_0^{T_\bullet}\int_{\partial\Omega}\int_{\{v\in\mathbb{R}^3|n(x)\cdot v<0\}}\left(\mathcal{K}_\alpha f^\alpha_+ + g^\alpha\right)(t,x,v)\int_t^{T_\bullet}\varphi(s)\,ds\,n(x)\cdot\widehat{v}_\alpha\,dvdS_xdt\Bigg)\\
&=-\int_0^{T_\bullet}\varphi\left(\int_\Omega\rho^{\mathrm{int}}\,dx-\int_\Omega\mathring{\rho}^{\mathrm{int}}\,dx\right)dt\\
&\quad+\sum_{\alpha=1}^N q_\alpha\Bigg(-\int_0^{T_\bullet}\varphi(s)\int_0^s\int_{\partial\Omega}\int_{\{v\in\mathbb{R}^3|n(x)\cdot v>0\}} f^\alpha_+(t,x,v)n(x)\cdot\widehat{v}_\alpha\,dvdS_xdtds
\end{aligned}$$

$$- \int_0^{T_\bullet} \varphi(s) \int_0^s \int_{\partial\Omega} \int_{\{v \in \mathbb{R}^3 | n(x)\cdot v < 0\}} \left(\mathcal{K}_\alpha f_+^\alpha + g^\alpha\right)(t,x,v) n(x) \cdot \widehat{v}_\alpha \, dv dS_x dt ds \Bigg),$$

from which the assertion follows immediately. □

We can finally show the redundancy of the divergence equation $\mathrm{div}_x(\varepsilon E) = \rho$ with the help of Lemma 1.5.3; the redundancy of $\mathrm{div}_x(\mu H) = 0$ has already been proved. To this end, we have to introduce an external charge density such that the external charge is locally conserved, which is a natural assumption. Precisely, this means the following.

Condition 1.5.5. There are $\rho^u \in L^1_{\mathrm{loc}}(I_{T_\bullet} \times \Gamma)$ and $\mathring{\rho}^u \in L^1_{\mathrm{loc}}(\Gamma)$ such that $\partial_t \rho^u + \mathrm{div}_x\, u = 0$ on $]0, T_\bullet[\times \mathbb{R}^3$ and $\rho^u(0) = \mathring{\rho}^u$ on Γ, which is to be understood in the following weak sense:

$$0 = \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\rho^u \partial_t \psi + u \cdot \partial_x \psi\right) dx dt + \int_{\mathbb{R}^3} \mathring{\rho}^u \psi(0)\, dx$$

for any $\psi \in C^\infty(I_{T_\bullet} \times \mathbb{R}^3)$ with $\mathrm{supp}\, \psi \subset [0, T_\bullet[\times \mathbb{R}^3$ compact. Here, ρ^u and $\mathring{\rho}^u$ are extended by zero outside Γ.

Theorem 1.5.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that its boundary $\partial\Omega$ is a $C^1 \cap W^{2,\infty}$-submanifold of $\mathbb{R}^3$. Furthermore, we assume that, for all $\alpha \in \{1, \ldots, N\}$, $f^\alpha \in \left(L^1_{\mathrm{lt}} \cap L^2_{\alpha\mathrm{kin,lt}} \cap L^\infty_{\mathrm{lt}}\right)(I_{T_\bullet} \times \Omega \times \mathbb{R}^3)$, $f_+^\alpha \in L^\infty_{\mathrm{lt}}\left(\gamma^+_{T_\bullet}\right)$, $(E, H) \in L^q_{\mathrm{lt}}\left(I_{T_\bullet}; L^2(\mathbb{R}^3; \mathbb{R}^6)\right)$ for some $q > 2$, $\mathcal{K}_\alpha \colon L^\infty_{\mathrm{lt}}\left(\gamma^+_{T_\bullet}\right) \to L^\infty_{\mathrm{lt}}\left(\gamma^-_{T_\bullet}\right)$, $g^\alpha \in L^\infty_{\mathrm{lt}}\left(\gamma^-_{T_\bullet}\right)$, $\mathring{f}^\alpha \in \left(L^1 \cap L^\infty\right)(\Omega \times \mathbb{R}^3)$, $\left(\mathring{E}, \mathring{H}\right) \in L^2(\mathbb{R}^3; \mathbb{R}^6)$, $\varepsilon, \mu \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^{3\times3})$ with $\varepsilon = \mu = \mathrm{Id}$ on Ω, and $u \in L^1_{\mathrm{loc}}(I_{T_\bullet} \times \Gamma; \mathbb{R}^3)$ such that the tuple $\left(\left(f^\alpha, f_+^\alpha\right)_\alpha, E, H, j^{\mathrm{int}} + u\right)$ is a weak solution of* (VM) *on the time interval $I_{T_\bullet}$ with external current u in the sense of Definition 1.1.1. Furthermore, assume that Condition 1.5.5 holds and let initially*

$$\mathrm{div}_x\left(\varepsilon \mathring{E}\right) = 4\pi\left(\mathring{\rho}^{\mathrm{int}} + \mathring{\rho}^u\right)$$

on $\mathbb{R}^3$ be satisfied in the sense of distributions. Then:

(i) We have

$$\mathrm{div}_x(\varepsilon E) = 4\pi\left(\rho^{\mathrm{int}} + \rho^u\right)$$

on $]0, T_\bullet[\times \left(\mathbb{R}^3 \setminus \partial\Omega\right)$ in the sense of distributions, i.e.,

$$0 = \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\varepsilon E \cdot \partial_x \varphi + 4\pi\left(\rho^{\mathrm{int}} + \rho^u\right)\varphi\right) dx dt$$

for all $\varphi \in C_c^\infty\left(]0, T_\bullet[\times \left(\mathbb{R}^3 \setminus \partial\Omega\right)\right)$.

(ii) If, additionally to the given assumptions, $f_+^\alpha \in L^1_{\mathrm{lt}}\Big(\gamma^+_{T_\bullet}, d\gamma_\alpha\Big)$, $g^\alpha \in L^1_{\mathrm{lt}}\Big(\gamma^-_{T_\bullet}, d\gamma_\alpha\Big)$, *and* $\mathcal{K}_\alpha \colon \Big(L^1_{\mathrm{lt}} \cap L^\infty_{\mathrm{lt}}\Big)\Big(\gamma^+_{T_\bullet}, d\gamma_\alpha\Big) \to \Big(L^1_{\mathrm{lt}} \cap L^\infty_{\mathrm{lt}}\Big)\Big(\gamma^-_{T_\bullet}, d\gamma_\alpha\Big)$ *for all* $\alpha \in \{1, \dots, N\}$, *then*

$$\operatorname{div}_x(\varepsilon E) = 4\pi\big(\rho^{\mathrm{int}} + \rho^u + S_{\partial\Omega}\big) \tag{1.5.11}$$

on $]0, T_\bullet[\, \times \mathbb{R}^3$ *in the sense of distributions, i.e.,*

$$0 = \int_0^{T_\bullet} \int_{\mathbb{R}^3} \big(\varepsilon E \cdot \partial_x \varphi + 4\pi\big(\rho^{\mathrm{int}} + \rho^u\big)\varphi\big)\, dx dt + 4\pi S_{\partial\Omega}\varphi$$

for all $\varphi \in C_c^\infty\big(]0, T_\bullet[\, \times \mathbb{R}^3\big)$. *Here, the distribution* $S_{\partial\Omega}$*, whose support satisfies* $\operatorname{supp} S_{\partial\Omega} \subset I_{T_\bullet} \times \partial\Omega$*, is given by*

$$S_{\partial\Omega}\varphi = \int_0^{T_\bullet} \int_{\partial\Omega} \varphi(t,x) \int_0^t n(x) \cdot \left(\sum_{\alpha=1}^N q_\alpha \int_{\{v \in \mathbb{R}^3 | n(x)\cdot v > 0\}} \widehat{v}_\alpha f_+^\alpha(s,x,v)\, dv \right.$$
$$\left. + \sum_{\alpha=1}^N q_\alpha \int_{\{v \in \mathbb{R}^3 | n(x)\cdot v < 0\}} \widehat{v}_\alpha \big(\mathcal{K}_\alpha f_+^\alpha + g^\alpha\big)(s,x,v)\, dv \right) ds dS_x dt.$$

Proof. First take $\varphi \in C_c^\infty\big(]0, T_\bullet[\, \times \mathbb{R}^3\big)$ arbitrary and define

$$\psi \colon I_{T_\bullet} \times \mathbb{R}^3 \to \mathbb{R}, \qquad \psi(t,x) = -\int_t^{T_\bullet} \varphi(s,x)\, ds,$$
$$\vartheta \colon I_{T_\bullet} \times \mathbb{R}^3 \to \mathbb{R}^3, \qquad \vartheta(t,x) = -\int_t^{T_\bullet} \partial_x\varphi(s,x)\, ds,$$
$$\xi \colon \mathbb{R}^3 \to \mathbb{R}, \qquad \xi(x) = \int_0^{T_\bullet} \varphi(s,x)\, ds.$$

Clearly, $\psi \in C^\infty\big(I_{T_\bullet} \times \mathbb{R}^3\big)$ with $\operatorname{supp}\psi \subset [0, T_\bullet[\, \times \mathbb{R}^3$ compact, $\vartheta \in \Theta_{T_\bullet}$, and $\xi \in C_c^\infty\big(\mathbb{R}^3\big)$. Because of $\vartheta \in \Theta_{T_\bullet}$, (1.1.3a) holds, i.e.,

$$\begin{aligned}
0 &= \int_0^{T_\bullet} \int_{\mathbb{R}^3} \big(\varepsilon E \cdot \partial_t \vartheta - H \cdot \operatorname{curl}_x \vartheta - 4\pi\big(j^{\mathrm{int}} + u\big) \cdot \vartheta\big)\, dx dt + \int_{\mathbb{R}^3} \varepsilon \mathring{E} \cdot \vartheta(0)\, dx \\
&= \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\varepsilon E \cdot \partial_x \varphi + H \cdot \int_t^{T_\bullet} \operatorname{curl}_x \partial_x \varphi(s,x)\, ds - 4\pi\big(j^{\mathrm{int}} + u\big) \cdot \vartheta\right) dx dt \\
&\quad - \int_{\mathbb{R}^3} \varepsilon \mathring{E} \cdot \partial_x \xi\, dx \\
&= \int_0^{T_\bullet} \int_{\mathbb{R}^3} \big(\varepsilon E \cdot \partial_x \varphi - 4\pi\big(j^{\mathrm{int}} + u\big) \cdot \vartheta\big)\, dx dt - \int_{\mathbb{R}^3} \varepsilon \mathring{E} \cdot \partial_x \xi\, dx.
\end{aligned} \tag{1.5.12}$$

By Condition 1.5.5, we have

$$0 = \int_0^{T_\bullet} \int_{\mathbb{R}^3} \big(\rho^u \partial_t \psi + u \cdot \partial_x \psi\big)\, dx dt + \int_{\mathbb{R}^3} \mathring{\rho}^u \psi(0)\, dx$$

$$= \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\rho^u \varphi + u \cdot \vartheta\right) dx dt - \int_{\mathbb{R}^3} \mathring{\rho}^u \xi \, dx. \tag{1.5.13}$$

To prove part 1.5.6.(i), assume that $\varphi \in C_c^\infty\big(]0, T_\bullet[\times \left(\mathbb{R}^3 \setminus \partial\Omega\right)\big)$. Then we have $\psi \in C^\infty\left(I_{T_\bullet} \times \mathbb{R}^3\right)$ with $\operatorname{supp}\psi \subset [0, T_\bullet[\times \left(\mathbb{R}^3 \setminus \partial\Omega\right)$ compact and Lemma 1.5.3.(i) gives us, after multiplying with q_α and summing over α,

$$\begin{aligned} 0 &= \int_0^{T_\bullet} \int_\Omega \left(\rho^{\text{int}} \partial_t \psi + j^{\text{int}} \cdot \partial_x \psi\right) dx dt + \int_\Omega \mathring{\rho}^{\text{int}} \psi(0) \, dx \\ &= \int_0^{T_\bullet} \int_\Omega \left(\rho^{\text{int}} \varphi + j^{\text{int}} \cdot \vartheta\right) dx dt - \int_\Omega \mathring{\rho}^{\text{int}} \xi \, dx. \end{aligned} \tag{1.5.14}$$

Multiplying (1.5.13) and (1.5.14) with 4π and adding them to (1.5.12) yields

$$\int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\varepsilon E \cdot \partial_x \varphi + 4\pi\left(\rho^{\text{int}} + \rho^u\right)\varphi\right) dx = \int_{\mathbb{R}^3} \left(\varepsilon \mathring{E} \cdot \partial_x \xi + 4\pi\left(\mathring{\rho}^{\text{int}} + \mathring{\rho}^u\right)\xi\right) dx = 0$$

by $\operatorname{div}_x\left(\varepsilon \mathring{E}\right) = 4\pi\left(\mathring{\rho}^{\text{int}} + \mathring{\rho}^u\right)$ on $\mathbb{R}^3$ in the sense of distributions. Hence, $\operatorname{div}_x(\varepsilon E) = 4\pi\left(\rho^{\text{int}} + \rho^u\right)$ on $]0, T_\bullet[\times \left(\mathbb{R}^3 \setminus \partial\Omega\right)$ in the sense of distributions.

To prove part 1.5.6.(ii), let the additional assumptions stated there hold. Now the test function $\varphi \in C_c^\infty\left(]0, T_\bullet[\times \mathbb{R}^3\right)$ need not vanish on $\partial\Omega$. We have $\psi \in C^\infty\left(I_{T_\bullet} \times \mathbb{R}^3\right)$ with $\operatorname{supp}\psi \subset [0, T_\bullet[\times \mathbb{R}^3$ compact and Lemma 1.5.3.(ii) gives us, after multiplying with q_α and summing over α,

$$\begin{aligned} 0 &= \int_0^{T_\bullet} \int_\Omega \left(\rho^{\text{int}} \partial_t \psi + j^{\text{int}} \cdot \partial_x \psi\right) dx dt - T_{\partial\Omega}\psi + \int_\Omega \mathring{\rho}^{\text{int}} \psi(0) \, dx \\ &= \int_0^{T_\bullet} \int_\Omega \left(\rho^{\text{int}} \varphi + j^{\text{int}} \cdot \vartheta\right) dx dt - T_{\partial\Omega}\psi - \int_\Omega \mathring{\rho}^{\text{int}} \xi \, dx. \end{aligned} \tag{1.5.15}$$

We rewrite $T_{\partial\Omega}\psi$:

$$\begin{aligned} T_{\partial\Omega}\psi &= \sum_{\alpha=1}^N q_\alpha \left(\int_{\gamma_{T_\bullet}^+} f_+^\alpha \psi \, d\gamma_\alpha - \int_{\gamma_{T_\bullet}^-} \left(\mathcal{K}_\alpha f_+^\alpha + g^\alpha\right) \psi \, d\gamma_\alpha \right) \\ &= \sum_{\alpha=1}^N q_\alpha \Bigg(-\int_0^{T_\bullet} \int_{\partial\Omega} \int_{\{v \in \mathbb{R}^3 | n(x) \cdot v > 0\}} f_+^\alpha(t,x,v) \int_t^{T_\bullet} \varphi(s,x) \, ds \, n(x) \cdot \widehat{v}_\alpha \, dv dS_x dt \\ &\quad - \int_0^{T_\bullet} \int_{\partial\Omega} \int_{\{v \in \mathbb{R}^3 | n(x) \cdot v < 0\}} \left(\mathcal{K}_\alpha f_+^\alpha + g^\alpha\right)(t,x,v) \int_t^{T_\bullet} \varphi(s,x) \, ds \, n(x) \cdot \widehat{v}_\alpha \, dv dS_x dt \Bigg) \\ &= \sum_{\alpha=1}^N q_\alpha \Bigg(-\int_0^{T_\bullet} \int_{\partial\Omega} \varphi(s,x) \int_0^s \int_{\{v \in \mathbb{R}^3 | n(x) \cdot v > 0\}} f_+^\alpha(t,x,v) n(x) \cdot \widehat{v}_\alpha \, dv dt dS_x ds \\ &\quad - \int_0^{T_\bullet} \int_{\partial\Omega} \varphi(s,x) \int_0^s \int_{\{v \in \mathbb{R}^3 | n(x) \cdot v < 0\}} \left(\mathcal{K}_\alpha f_+^\alpha + g^\alpha\right)(t,x,v) n(x) \cdot \widehat{v}_\alpha \, dv dt dS_x ds \Bigg) \end{aligned}$$

$$= -S_{\partial\Omega}\varphi.$$

Similarly as before, multiplying (1.5.13) and (1.5.15) with 4π and adding them to (1.5.12) yields

$$\int_0^{T_\bullet}\int_{\mathbb{R}^3}\big(\varepsilon E\cdot\partial_x\varphi + 4\pi\big(\rho^{\text{int}}+\rho^u\big)\varphi\big)\,dx + 4\pi S_{\partial\Omega}\varphi$$
$$= \int_{\mathbb{R}^3}\Big(\varepsilon\mathring{E}\cdot\partial_x\xi + 4\pi\big(\mathring{\rho}^{\text{int}}+\mathring{\rho}^u\big)\xi\Big)\,dx = 0.$$

Hence, $\operatorname{div}_x(\varepsilon E) = 4\pi\big(\rho^{\text{int}}+\rho^u+S_{\partial\Omega}\big)$ on $]0,T_\bullet[\times\mathbb{R}^3$ in the sense of distributions. □

Remark 1.5.7. We discuss some assumptions and give some comments regarding Theorem 1.5.6 and Corollary 1.5.4:

- Clearly, we see by interpolation that $f^\alpha \in \Big(L^1_{\alpha\text{kin,lt}}\cap L^\infty_{\text{lt}}\Big)\big(I_{T_\bullet}\times\Omega\times\mathbb{R}^3\big)$ implies $f^\alpha \in \Big(L^1_{\text{lt}}\cap L^2_{\alpha\text{kin,lt}}\cap L^\infty_{\text{lt}}\Big)\big(I_{T_\bullet}\times\Omega\times\mathbb{R}^3\big)$ and that $(E,H)\in L^\infty_{\text{lt}}\big(I_{T_\bullet};L^2\big(\mathbb{R}^3;\mathbb{R}^6\big)\big)$ implies $(E,H)\in L^q_{\text{lt}}\big(I_{T_\bullet};L^2\big(\mathbb{R}^3;\mathbb{R}^6\big)\big)$ for any $q>2$. Hence, Theorem 1.5.6.(i) can be applied to solutions constructed as in Section 1.4; cf. Theorem 1.4.4. However, the boundary values f^α_+ constructed there only satisfy $f^\alpha_+\in L^1_{\text{lt}}\Big(\gamma^+_{T_\bullet},d\gamma_\alpha\Big)$ for $\alpha=1,\ldots,N'$, i.e., the particles are subject to partially absorbing boundary conditions, and not necessarily for $\alpha=N'+1,\ldots,N$, i.e., the particles are subject to (partially) purely reflecting boundary conditions. Therefore, whether the statement of Theorem 1.5.6.(ii) is true for solutions constructed as in Section 1.4, remains as an open problem, unless $N'=N$, i.e., all particles are subject to partially absorbing boundary conditions.

- Conversely, the assumption $f^\alpha_+\in L^1_{\text{lt}}\Big(\gamma^+_{T_\bullet},d\gamma_\alpha\Big)$ is necessary for Theorem 1.5.6.(ii) (and for Lemma 1.5.3.(ii)); otherwise, the integral $\int_{\gamma^+_{T_\bullet}} f^\alpha_+\psi\,d\gamma_\alpha$ will not exist in general since ψ need not vanish on $\partial\Omega$ and does not depend on v.

- The distribution $S_{\partial\Omega}$ can be interpreted as follows: The terms

  $$j^{\text{out}}_{\partial\Omega}(t,x) := \sum_{\alpha=1}^N q_\alpha\int_{\{v\in\mathbb{R}^3\mid n(x)\cdot v>0\}}\widehat{v}_\alpha f^\alpha_+(t,x,v)\,dv,$$
  $$j^{\text{in}}_{\partial\Omega}(t,x) := \sum_{\alpha=1}^N q_\alpha\int_{\{v\in\mathbb{R}^3\mid n(x)\cdot v<0\}}\widehat{v}_\alpha\big(\mathcal{K}_\alpha f^\alpha_+ + g^\alpha\big)(t,x,v)\,dv,$$

  where $(t,x)\in I_{T_\bullet}\times\partial\Omega$, can be interpreted as the outgoing and incoming boundary current density. Hence, $S_{\partial\Omega}$ can be rewritten as

  $$S_{\partial\Omega}\varphi = \int_0^{T_\bullet}\int_{\partial\Omega}\varphi(t,x)\int_0^t n(x)\cdot\Big(j^{\text{out}}_{\partial\Omega}(s,x)+j^{\text{in}}_{\partial\Omega}(s,x)\Big)\,ds\,dS_x\,dt.$$

Thus, $S_{\partial\Omega}$ makes up the balance of how many particles have left and entered Ω *up to* time t. On the other hand, the distribution $T_{\partial\Omega}$ makes up the balance of how many particles leave and enter Ω *at* time t via

$$T_{\partial\Omega}\psi = \int_0^{T_\bullet} \int_{\partial\Omega} \psi(t,x) n(x) \cdot \left(j_{\partial\Omega}^{\mathrm{out}}(t,x) + j_{\partial\Omega}^{\mathrm{in}}(t,x) \right) dS_x dt.$$

We easily see that $\partial_t S_{\partial\Omega} = T_{\partial\Omega}$ on $]0, T_\bullet[\times \mathbb{R}^3$ in the sense of distributions, which corresponds to the fact that $T_{\partial\Omega}$ appears as "a part of $\partial_t \rho$" in (1.5.10) and $S_{\partial\Omega}$ appears as "a part of ρ" in (1.5.11).

- The global charge balance, see Corollary 1.5.4.(iii), can similarly been written as follows:

$$\int_\Omega \rho^{\mathrm{int}}(t,x)\, dx = \int_\Omega \mathring{\rho}^{\mathrm{int}}\, dx - \int_0^t \int_{\partial\Omega} n \cdot \left(j_{\partial\Omega}^{\mathrm{out}} + j_{\partial\Omega}^{\mathrm{in}} \right) dS_x ds$$

for almost all $t \in I_{T_\bullet}$.

- As mentioned in the introduction, in a more realistic model ε and μ should depend on f^α, E, and H (maybe even nonlocally) and hence implicitly on time. In this situation, the weak formulation is the same as before, which is stated in Definition 1.1.1. If we assume $\varepsilon, \mu \in L^\infty_{\mathrm{loc}}\left(I_{T_\bullet} \times \mathbb{R}^3; \mathbb{R}^{3\times 3}\right)$ (and suitably introduce initial values for ε, μ), viewed as explicit functions of t and x, the proofs of Theorem 1.5.6 and the lemmas before are still valid, and Theorem 1.5.6 remains true, as well as the redundancy of $\mathrm{div}_x\left(\mu H\right) = 0$.

- Lastly, we emphasize that all results of this section hold, under the respective assumptions, for all weak solutions of (VM) in the sense of Definition 1.1.1 and not only for the solutions constructed as in Section 1.4.

CHAPTER 2

Optimal control problem

2.1 A prototype

In a fusion reactor, one of the main goals is to keep the particles away from the boundary of their container Ω since particles hitting the boundary damage the material there due to the usually very hot temperature of the plasma. Therefore, it is reasonable to penalize these hits, which, for example, can be achieved by taking some L^q-norms of the f_+^α as a part of the objective function that shall be minimized in an optimal control problem. Moreover, it is natural to consider the external current density u as a tool to reduce these hits on the reactor wall. For a prototype problem, we consider the case that all particles are subject to partially absorbing boundary conditions, i.e., $N = N'$, and assume $g^\alpha = 0$.

Apart from driving the amount of hits on the boundary to a minimum, one does not want too exhaustive control costs so that the fusion reactor may have a good efficiency. Thus, it is reasonable to add some norm of u to the objective function. Thereby, we also gain a mathematical advantage since then the objective function is coercive in u, which means that along a minimizing sequence this u-norm is bounded so that we can extract a weakly convergent subsequence whose weak limit is a candidate for an optimal control.

Conversely, as there are no terms including f^α, E, and H in the objective function, we do not have coercivity in these state variables because of the objective function itself. But there is still the PDE system (VM) as a constraint. Recalling (1.4.40) to (1.4.45) we see that these estimates yield uniform boundedness of f^α, E, H (and j^{int}) in various norms along a minimizing sequence. Unfortunately, we can only verify these estimates for solutions that are constructed as in Section 1.4. For general solutions of (VM) in the sense of Definition 1.1.1 these estimates may be violated as we do not know a way to prove these generally. Since in the classical context these estimates are easily heuristically established by exploiting an energy balance and the measure preserving nature of the characteristic flow of the Vlasov equation, it is reasonable to restrict ourselves to solutions that satisfy at least part of, maybe slightly weaker versions of (1.4.40) to (1.4.45).

To put our hands on the fields, only (1.4.44) is helpful. Considering this estimate along a minimizing, weakly converging sequence and trying to pass to the limit in this estimate, we see that the right-hand side, including some norm of u, has to be weakly continuous. But if we endow the control space with the norm that appears in (1.4.44), i.e., the $L^1\left([0,T_\bullet];L^2(\Gamma;\mathbb{R}^3)\right)$-norm, this weak continuity will not hold. Consequently, we consider a control space that is compactly embedded in $L^1\left([0,T_\bullet];L^2(\Gamma;\mathbb{R}^3)\right)$ so that the right-hand side of (1.4.44) converges even if the controls only converge weakly in this new smaller control space. This will be made clear in the proof of Theorem 2.2.1.

Altogether, we arrive at the following minimization problem:

$$\left.\begin{array}{ll}\min\limits_{y\in\mathcal{Y},u\in\mathcal{U}} & \mathcal{J}(y,u),\\ \text{s.t.} & \left(\left(f^\alpha,f_+^\alpha\right)_\alpha,E,H,j^{\text{int}}+u\right)\text{ solves (VM)},\\ & \text{(2.1.1) and (2.1.2) hold}\end{array}\right\}\tag{P}$$

where the objective function is

$$\mathcal{J}(y,u)=\frac{1}{q}\sum_{\alpha=1}^{N}w_\alpha\left\|f_+^\alpha\right\|^q_{L^q\left(\gamma_{T_\bullet}^+,d\gamma_\alpha\right)}+\frac{1}{r}\|u\|_{\mathcal{U}}^r$$

and the additional constraints are

$$0\le f^\alpha\le\left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)}\text{ a.e.},\quad \alpha=1,\dots,N,\tag{2.1.1}$$

$$\begin{aligned}&\sum_{\alpha=1}^{N}\int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3}v_\alpha^0f^\alpha\,dvdxdt+\frac{\sigma}{8\pi}\int_0^{T_\bullet}\int_{\mathbb{R}^3}\left(|E|^2+|H|^2\right)dxdt\\&\le 2T_\bullet\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3}v_\alpha^0\mathring{f}^\alpha\,dvdx+\frac{T_\bullet\sigma'}{4\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}+2\pi T_\bullet^2\sigma^{-1}\|u\|^2_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}\\&=:\mathcal{I}(u).\end{aligned}\tag{2.1.2}$$

Definition and Remark 2.1.1. We explain the formulation of the minimization problem in detail:

- We consider the optimal control problem on a finite time interval, i.e., $T_\bullet<\infty$.
- We assume that the given functions $\mathring{f}^\alpha$, a^α, $\mathring{E}$, $\mathring{H}$, ε, and μ satisfy the respective properties of Condition 1.4.1 with $N'=N$ and that $g^\alpha=0$, $\mathring{f}^\alpha\not\equiv 0$ for all $\alpha=1,\dots,N$.
- For ease of notation, we have abbreviated

$$y=\left(\left(f^\alpha,f_+^\alpha\right)_\alpha,E,H\right),$$

$$\mathcal{Y} = \left(\bigtimes_{\alpha=1}^{N} \left(\mathcal{Y}_{\text{pd}}^{\alpha} \times L^q \left(\gamma_{T_\bullet}^{+}, d\gamma_\alpha \right) \right) \right) \times L^2 \left([0, T_\bullet] \times \mathbb{R}^3; \mathbb{R}^3 \right)^2,$$

where $1 < q < \infty$ is fixed and

$$\begin{aligned} \mathcal{Y}_{\text{pd}}^{\alpha} := \big\{ f \in \left(L^1_{\alpha\text{kin}} \cap L^\infty \right) \left([0, T_\bullet] \times \Omega \times \mathbb{R}^3 \right) \mid \\ \forall \eta \in C_c^\infty \left( ]0, T_\bullet[\times \Omega \times \mathbb{R}^3 \right) : \partial_t (\eta f) + \widehat{v}_\alpha \cdot \partial_x (\eta f) \in L^2 \left([0, T_\bullet] \times \Omega; H^{-1} \left(\mathbb{R}^3 \right) \right), \\ \mathcal{N}_\alpha (f) < \infty \big\}. \end{aligned}$$

Here and in the following, for a distribution h on $]0, T_\bullet[\times \Omega \times \mathbb{R}^3$ the property $h \in L^2 \left([0, T_\bullet] \times \Omega; H^{-1} \left(\mathbb{R}^3 \right) \right)$ means that there exist functions $g_0 \in L^2 \left( ]0, T_\bullet[\times \Omega \times \mathbb{R}^3 \right)$ and $g_1 \in L^2 \left( ]0, T_\bullet[\times \Omega \times \mathbb{R}^3; \mathbb{R}^3 \right)$ such that

$$h = g_0 + \text{div}_v \, g_1 \text{ on }]0, T_\bullet[\times \Omega \times \mathbb{R}^3 \text{ in the sense of distributions.} \tag{2.1.3}$$

The space $L^2 \left([0, T_\bullet] \times \Omega; H^{-1} \left(\mathbb{R}^3 \right) \right)$ consisting of all such distributions is equipped with the norm

$$\begin{aligned} &\| h \|_{L^2([0,T_\bullet] \times \Omega; H^{-1}(\mathbb{R}^3))} \\ &= \min \left\{ \left(\| g_0 \|^2_{L^2(]0,T_\bullet[\times \Omega \times \mathbb{R}^3)} + \| g_1 \|^2_{L^2(]0,T_\bullet[\times \Omega \times \mathbb{R}^3; \mathbb{R}^3)} \right)^{\frac{1}{2}} \mid g_0, g_1 \text{ satisfy (2.1.3)} \right\}. \end{aligned}$$

Moreover, we denote

$$\mathcal{N}_\alpha (f) := \sup \left\| \partial_t (\eta f) + \widehat{v}_\alpha \cdot \partial_x (\eta f) \right\|_{L^2([0,T_\bullet] \times \Omega; H^{-1}(\mathbb{R}^3))}$$

where the supremum is taken over all $\eta \in C_c^\infty \left( ]0, T_\bullet[\times \Omega \times \mathbb{R}^3 \right)$ satisfying

$$\| \eta \|_{H^1(]0,T_\bullet[\times \Omega \times \mathbb{R}^3)} + \| \eta \|_{L^\infty([0,T_\bullet] \times \Omega; H^1(\mathbb{R}^3))} = 1. \tag{2.1.4}$$

The restriction in the definition of $\mathcal{Y}_{\text{pd}}^{\alpha}$ will not be important until Section 2.4 and is motivated by Lemma 2.1.2, which is stated below.

- The numbers $w_\alpha > 0$ are weights. For example, if we have two sorts of particles, say, ions and electrons, the weight corresponding to the ions should be larger than the one corresponding to the electrons since the heavy ions will cause more damage on the boundary of a fusion reactor if they hit it. Moreover, the weights also serve as an indicator of which of our two aims should rather be achieved, that is to say, no hits on the boundary and small control costs. More precisely, the w_α should be large if one rather wants no hits on the boundary, and should be small if one rather wants small control costs.

- The control space is

$$\mathcal{U} = W^{1,r} \left( ]0, T_\bullet[\times \Gamma; \mathbb{R}^3 \right)$$

where $\frac{4}{3} < r < \infty$ is fixed and $\Gamma \subset \mathbb{R}^3$ is open and bounded. By Sobolev's embedding theorem, $\mathcal{U}$ is compactly embedded in $L^2(]0, T_\bullet[\times \Gamma; \mathbb{R}^3)$. For this, the boundary of Γ has to satisfy some regularity condition, for example, the cone condition. From now on, we shall always assume that $\partial\Gamma$ is not "too bad", that is to say, this compact embedding holds. We endow $\mathcal{U}$ with the norm

$$\|u\|_{\mathcal{U}} := \left(\sum_{j=1}^{3} \int_0^{T_\bullet} \int_\Gamma \left(\left|u_j\right|^r + \kappa_1 \left|\partial_t u_j\right|^r + \kappa_2 \sum_{i=1}^{3} \left|\partial_{x_i} u_j\right|^r \right) dx dt \right)^{\frac{1}{r}},$$

which is equivalent to the standard $W^{1,r}(]0, T_\bullet[\times \Gamma; \mathbb{R}^3)$-norm. Here, $\kappa_1, \kappa_2 > 0$ are parameters chosen according to how much one wants to penalize u itself compared to its t-and x-derivatives.

- As usual,

$$j^{\text{int}} = \sum_{\alpha=1}^{N} q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv.$$

- The constraint that (VM) be solved is to be understood in the sense of Definition 1.1.1.
- The pointwise constraint (2.1.1) on f^α is on the one hand natural since any classical solution of (VM.1) with nonnegative initial datum satisfies this constraint, and, as we have seen in Theorem 1.4.4, also the weak solutions constructed in Section 1.4 do, and on the other hand necessary for a limit process when proving existence of a minimizer; see Section 2.2.
- The same applies *mutatis mutandis* for the energy constraint (2.1.2). Note that this inequality directly follows from the stronger inequality (1.4.44) (recall that we consider $g^\alpha = 0$) after an integration in time and Hölder's inequality:

$$\begin{aligned}
&\sum_{\alpha=1}^{N} \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 f^\alpha \, dv dx dt + \frac{\sigma}{8\pi} \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(|E|^2 + |H|^2 \right) dx dt \\
&\leq \int_0^{T_\bullet} \left\| \sum_{\alpha=1}^{N} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 f^\alpha(\cdot) \, dv dx + \frac{\sigma}{8\pi} \|(E,H)(\cdot)\|_{L^2(\mathbb{R}^3;\mathbb{R}^6)}^2 \right\|_{L^\infty([0,T])} dT \\
&\leq 2T_\bullet \left(\sum_{\alpha=1}^{N} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^\alpha \, dv dx + \frac{\sigma'}{8\pi} \left\| \left(\mathring{E}, \mathring{H} \right) \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^6)}^2 \right) \\
&\quad + 4\pi\sigma^{-1} \int_0^{T_\bullet} \left(\int_0^T \|u(t)\|_{L^2(\Gamma;\mathbb{R}^3)} \, dt \right)^2 dT
\end{aligned}$$

and

$$\int_0^{T_\bullet} \left(\int_0^T \|u(t)\|_{L^2(\Gamma;\mathbb{R}^3)} \, dt \right)^2 dT \leq \int_0^{T_\bullet} T \int_0^T \|u(t)\|_{L^2(\Gamma;\mathbb{R}^3)}^2 \, dt dT$$

$$\leq \frac{T_\bullet^2}{2} \|u\|^2_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}.$$

The main reason why we impose the weaker inequality (2.1.2) as a constraint is that no longer L^∞-terms or square roots appear, which would cause some trouble with respect to differentiability.

We proceed with the following lemma, that was already mentioned above.

Lemma 2.1.2. *Let $f^\alpha \in L^\infty([0,T_\bullet]\times\Omega\times\mathbb{R}^3)$, $f_+^\alpha \in L^1_{\mathrm{loc}}\big(\gamma^+_{T_\bullet}, d\gamma_\alpha\big)$ such that Definition 1.1.1.(ii) is satisfied with $E, H \in L^2([0,T_\bullet]\times\Omega;\mathbb{R}^3)$. Denote $F := q_\alpha(E + \widehat{v}_\alpha \times H)$. Then, for any $\eta \in C_c^\infty(]0,T_\bullet[\times\Omega\times\mathbb{R}^3)$ it holds that*

$$\partial_t(\eta f^\alpha) + \widehat{v}_\alpha \cdot \partial_x(\eta f^\alpha) = f^\alpha \partial_t \eta + f^\alpha \widehat{v}_\alpha \cdot \partial_x \eta + F f^\alpha \cdot \partial_v \eta - \mathrm{div}_v(\eta F f^\alpha) \tag{2.1.5}$$

on $]0,T_\bullet[\times\Omega\times\mathbb{R}^3$ in the sense of distributions and the left-hand side is an element of $L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))$. Furthermore,

$$\mathcal{N}_\alpha(f^\alpha) \leq 2\|f^\alpha\|_{L^\infty([0,T_\bullet]\times\Omega\times\mathbb{R}^3)}\Big(1 + \sqrt{2}|q_\alpha|\|(E,H)\|_{L^2([0,T_\bullet]\times\Omega;\mathbb{R}^6)}\Big). \tag{2.1.6}$$

Proof. It is easy to see that (2.1.5) holds. There remains to estimate the right-hand side:

$$\|f^\alpha \partial_t \eta\|_{L^2([0,T_\bullet]\times\Omega\times\mathbb{R}^3)}, \|f^\alpha \widehat{v}_\alpha \cdot \partial_x \eta\|_{L^2([0,T_\bullet]\times\Omega\times\mathbb{R}^3)} \leq \|f^\alpha\|_{L^\infty([0,T_\bullet]\times\Omega\times\mathbb{R}^3)}\|\eta\|_{H^1(]0,T_\bullet[\times\Omega\times\mathbb{R}^3)}$$

and

$$\begin{aligned}&\|F f^\alpha \cdot \partial_v \eta\|_{L^2([0,T_\bullet]\times\Omega\times\mathbb{R}^3)}, \|\eta F f^\alpha\|_{L^2([0,T_\bullet]\times\Omega\times\mathbb{R}^3;\mathbb{R}^3)}\\ &\leq \|F\|_{L^2([0,T_\bullet]\times\Omega;\mathbb{R}^3)}\|f^\alpha\|_{L^\infty([0,T_\bullet]\times\Omega\times\mathbb{R}^3)}\|\eta\|_{L^\infty([0,T_\bullet]\times\Omega;H^1(\mathbb{R}^3))}\end{aligned}$$

implies $\partial_t(\eta f^\alpha) + \widehat{v}_\alpha \cdot \partial_x(\eta f^\alpha) \in L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))$ and (2.1.6) because of $|F|^2 \leq 2|q_\alpha|^2\big(|E|^2 + |H|^2\big)$. □

The next lemma gives an $L^{\frac{4}{3}}$-estimate on j^{int} in view of the inequality constraints of (P) and will be useful later.

Lemma 2.1.3. *The constraints* (2.1.1) *and* (2.1.2) *yield $j^{\mathrm{int}} \in L^{\frac{4}{3}}([0,T_\bullet]\times\Omega;\mathbb{R}^3)$ with*

$$\|j^{\mathrm{int}}\|_{L^{\frac{4}{3}}([0,T_\bullet]\times\Omega;\mathbb{R}^3)} \leq \left(\sum_{\alpha=1}^N |q_\alpha|^4\left(\frac{4\pi}{3}\|\mathring{f}^\alpha\|_{L^\infty(\Omega\times\mathbb{R}^3)} + 1\right)^4\right)^{\frac{1}{4}} \mathcal{I}(u)^{\frac{3}{4}}.$$

Proof. Similarly to (1.4.31) and (1.4.32), we have

$$\int_{\mathbb{R}^3} f^\alpha(t,x,v)\,dv \leq \left(\int_{\mathbb{R}^3} v_\alpha^0 f^\alpha(t,x,v)\,dv\right)^{\frac{3}{4}}\left(\frac{4\pi}{3}\|f^\alpha\|_{L^\infty([0,T_\bullet]\times\Omega\times\mathbb{R}^3)} + 1\right)$$

$$\leq \left(\int_{\mathbb{R}^3} v_\alpha^0 f^\alpha(t,x,v)\,dv\right)^{\frac{3}{4}} \left(\frac{4\pi}{3}\left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)}+1\right) \tag{2.1.7}$$

for almost all $(t,x) \in [0,T_\bullet] \times \Omega$ by (2.1.1), whence

$$\left(\int_0^{T_\bullet}\int_\Omega \left|j^{\text{int}}\right|^{\frac{4}{3}} dxdt\right)^{\frac{3}{4}} \leq \sum_{\alpha=1}^N |q_\alpha| \left(\int_0^{T_\bullet}\int_\Omega \left|\int_{\mathbb{R}^3} f^\alpha\,dv\right|^{\frac{4}{3}} dxdt\right)^{\frac{3}{4}}$$

$$\leq \left(\sum_{\alpha=1}^N |q_\alpha|^4 \left(\frac{4\pi}{3}\left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)}+1\right)^4\right)^{\frac{1}{4}} \left(\sum_{\alpha=1}^N \int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 f^\alpha\,dvdxdt\right)^{\frac{3}{4}}, \tag{2.1.8}$$

which, together with the constraint (2.1.2), implies the assertion. □

2.2 Existence of minimizers

The usual strategy to obtain a minimizer of an optimization problem is to consider a minimizing sequence. By the structure of the objective function or the constraints, this sequence is bounded in some norm so that we can extract a weakly converging subsequence (of course, we have to work in a reflexive space for this). To pass to the limit in a nonlinear optimization problem, some compactness is needed. As for passing to the limit in a nonlinear PDE (system), usually the same tools have to be exploited that were established to be able to pass to the limit in an iteration scheme to prove existence of solutions to the PDE (system).

This general strategy also applies to our case.

Theorem 2.2.1. *There is a (not necessarily unique) minimizer of* (P).

Proof. First notice that there are feasible points to (P) by Theorem 1.4.4. Thus, we may consider a minimizing sequence $\left(\left(f_k^\alpha, f_{k,+}^\alpha\right)_\alpha, E_k, H_k, u_k\right)$ of (P). By structure of $\mathcal{J}$, the sequences $\left(f_{k,+}^\alpha\right)$ are bounded in $L^q\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)$ and the sequence (u_k) is bounded in $\mathcal{U}$. By reflexivity, we may thus assume that these sequences converge weakly, after possibly extracting suitable subsequences, in the respective spaces to some $f_+^\alpha \in L^q\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)$ and $u \in \mathcal{U}$; recall that $1 < q < \infty$.

Since $\mathcal{U}$ is compactly embedded in $L^2\left([0,T_\bullet]\times\Gamma;\mathbb{R}^3\right)$, we have

$$\|u\|_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)} = \lim_{k\to\infty}\|u_k\|_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}. \tag{2.2.1}$$

In combination with the constraints (2.1.1) and (2.1.2), this yields that the sequences $\left(f_k^\alpha\right)$ are bounded in $\left(L^1_{\alpha\text{kin}} \cap L^\infty\right)\left([0,T_\bullet]\times\Omega\times\mathbb{R}^3\right)$ and that the sequence $((E_k,H_k))$ is bounded in $L^2\left([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^6\right)$. The property of $\left(f_k^\alpha\right)$ implies the boundedness of $\left(f_k^\alpha\right)$ in any $L^p\left([0,T_\bullet]\times\Omega\times\mathbb{R}^3\right)$, $1 \leq p \leq \infty$, by interpolation. Therefore, after extracting a further subsequence, f_k^α converges weakly to some f^α in any $L^p\big([0,T_\bullet]\times\Omega\times\mathbb{R}^3\big)$, $1<p\le\infty$ (weak-* if $p=\infty$), and $((E_k,H_k))$ converges weakly to some (E,H) in $L^2\big([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^6\big)$.

By weak-* convergence in $L^\infty\big([0,T_\bullet]\times\Omega\times\mathbb{R}^3\big)$, the constraint (2.1.1) is preserved in the limit. As for the constraint (2.1.2), let $R>0$. By weak convergence of the f_k^α, weak convergence of (E_k,H_k), (2.1.2) along the minimizing sequence, and (2.2.1), we have

$$\begin{aligned}
&\sum_{\alpha=1}^N\int_0^{T_\bullet}\int_\Omega\int_{B_R} v_\alpha^0 f^\alpha\,dvdxdt+\frac{\sigma}{8\pi}\int_0^{T_\bullet}\int_{\mathbb{R}^3}\Big(|E|^2+|H|^2\Big)\,dxdt\\
&\le\liminf_{k\to\infty}\left(\sum_{\alpha=1}^N\int_0^{T_\bullet}\int_\Omega\int_{B_R} v_\alpha^0 f_k^\alpha\,dvdxdt+\frac{\sigma}{8\pi}\int_0^{T_\bullet}\int_{\mathbb{R}^3}\Big(|E_k|^2+|H_k|^2\Big)\,dxdt\right)\\
&\le 2T_\bullet\sum_{\alpha=1}^N\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0\mathring{f}^\alpha\,dvdx+\frac{T_\bullet\sigma'}{4\pi}\Big\|\Big(\mathring{E},\mathring{H}\Big)\Big\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}+2\pi T_\bullet^2\sigma^{-1}\lim_{k\to\infty}\|u_k\|^2_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}\\
&=2T_\bullet\sum_{\alpha=1}^N\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0\mathring{f}^\alpha\,dvdx+\frac{T_\bullet\sigma'}{4\pi}\Big\|\Big(\mathring{E},\mathring{H}\Big)\Big\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^6)}+2\pi T_\bullet^2\sigma^{-1}\|u\|^2_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)},
\end{aligned}$$

which, after letting $R\to\infty$, on the one hand yields $f^\alpha\in L^1_{\alpha\mathrm{kin}}\big([0,T_\bullet]\times\Omega\times\mathbb{R}^3\big)$ and on the other hand implies that the constraint (2.1.2) also holds in the limit. Here we should point out that (2.2.1) was crucial since we needed

$$\liminf_{k\to\infty}\|u_k\|_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}\le\|u\|_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}.$$

If we had chosen a cost term with the $L^2\big([0,T_\bullet]\times\Gamma;\mathbb{R}^3\big)$-norm instead of the $\mathcal{U}$-norm of u in the objective function, we would only have been able to extract a subsequence (u_k) that converges weakly in $L^2\big([0,T_\bullet]\times\Gamma;\mathbb{R}^3\big)$ rendering the above lim inf-estimate false in general.

The next step is to pass to the limit in the Vlasov–Maxwell system (VM). This is done in much the same way as in Section 1.4 but we carry out the proof for the sake of completeness. By Lemma 2.1.3 the internal currents converge weakly, after extracting a further subsequence, in $L^{\frac{4}{3}}\big([0,T_\bullet]\times\Omega;\mathbb{R}^3\big)$. The weak limit—call it $\widetilde{j}^{\mathrm{int}}$—has to be the internal current j^{int} induced by the limit functions f^α because of the following: Take $\vartheta\in C_c^\infty\big(]0,T_\bullet[\times\Omega;\mathbb{R}^3\big)$ and $s>0$. Using weak convergence of j_k^{int} and f_k^α, respectively, we deduce

$$\begin{aligned}
&\left|\int_0^{T_\bullet}\int_\Omega\Big(j^{\mathrm{int}}-\widetilde{j}^{\mathrm{int}}\Big)\cdot\vartheta\,dxdt\right|=\left|\lim_{k\to\infty}\iint\limits_{\operatorname{supp}\vartheta}\Big(j^{\mathrm{int}}-j_k^{\mathrm{int}}\Big)\cdot\vartheta\,dxdt\right|\\
&=\left|\lim_{k\to\infty}\iint\limits_{\operatorname{supp}\vartheta}\left(\sum_{\alpha=1}^N q_\alpha\int_{\mathbb{R}^3}\widehat{v}_\alpha f^\alpha\,dv-\sum_{\alpha=1}^N q_\alpha\int_{\mathbb{R}^3}\widehat{v}_\alpha f_k^\alpha\,dv\right)\cdot\vartheta\,dxdt\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{k\to\infty}\left|\sum_{\alpha=1}^{N} q_\alpha \iint\limits_{\operatorname{supp}\vartheta} \int_{B_s} \widehat{v}_\alpha\left(f^\alpha - f_k^\alpha\right)\cdot\vartheta\,dv\,dx\,dt\right| \\
&\quad + \limsup_{k\to\infty}\left|\sum_{\alpha=1}^{N} q_\alpha \iint\limits_{\operatorname{supp}\vartheta}\left(\int_{|v|\geq s}\widehat{v}_\alpha f^\alpha\,dv - \int_{|v|\geq s}\widehat{v}_\alpha f_k^\alpha\,dv\right)\cdot\vartheta\,dx\,dt\right| \\
&\leq 0 + \limsup_{k\to\infty}\frac{1}{s}\|\vartheta\|_{L^\infty(]0,T_\bullet[\times\Omega;\mathbb{R}^3)}\sum_{\alpha=1}^{N}|q_\alpha|\iint\limits_{\operatorname{supp}\vartheta}\left(\int_{\mathbb{R}^3} v_\alpha^0 f^\alpha\,dv + \int_{\mathbb{R}^3} v_\alpha^0 f_k^\alpha\,dv\right)dx\,dt \\
&\leq \frac{C}{s},
\end{aligned}
$$

where C is finite by virtue of (2.1.2) and the boundedness of (u_k) and does not depend on s. Since $s > 0$ and $\vartheta \in C_c^\infty\left(]0,T_\bullet[\times\Omega;\mathbb{R}^3\right)$ was arbitrary, we conclude $j^{\text{int}} = \widetilde{j}^{\text{int}}$ almost everywhere. Thus, we can pass to the limit in (1.1.2) and (1.1.3) easily in all terms but the nonlinear one. To handle this remaining term, we again apply Lemma 1.4.2: Let $\zeta \in C_c^\infty\left(\mathbb{R}^3\right)$ and $s > 0$ such that ζ vanishes for $|v| > s-1$. Our goal is to show that $\int_{\mathbb{R}^3}\zeta f_k^\alpha\,dv$ converges strongly (and not only weakly) to $\int_{\mathbb{R}^3}\zeta f^\alpha\,dv$ in $L^2([0,T_\bullet]\times\Omega)$. To this end, let $\eta \in C_c^\infty(]0,T_\bullet[\times\Omega\times B_s)$. We have

$$
\begin{aligned}
&\partial_t\left(\eta f_k^\alpha\right) + \widehat{v}_\alpha\cdot\partial_x\left(\eta f_k^\alpha\right) \\
&= -\operatorname{div}_v\left(q_\alpha(E_k + \widehat{v}_\alpha\times H_k)\left(\eta f_k^\alpha\right)\right) \\
&\quad + f_k^\alpha\partial_t\eta + f_k^\alpha\widehat{v}_\alpha\cdot\partial_x\eta + q_\alpha f_k^\alpha(E_k + \widehat{v}_\alpha\times H_k)\cdot\partial_v\eta \\
&=: \operatorname{div}_v g_1^k + g_0^k
\end{aligned}
$$

on $\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$ in the sense of distributions. Clearly, the $L^2(\mathbb{R}\times\mathbb{R}\times B_s)$-norms of g_0^k and g_1^k are uniformly bounded in k due to $\eta \in C_c^\infty(]0,T_\bullet[\times\Omega\times B_s)$ and the already known uniform boundedness of f_k^α in L^∞ and L^2 and E_k, H_k in L^2—the latter being a consequence of imposing (2.1.1) and (2.1.2)! Thus, applying Lemma 1.4.2 yields the uniform boundedness of

$$
\left\|\int_{B_s}\zeta(v)\left(\eta f_k^\alpha\right)(\cdot,\cdot,v)\,dv\right\|_{H^{\frac{1}{4}}(\mathbb{R}\times\mathbb{R}^3)} = \left\|\int_{B_s}\zeta(v)\left(\eta f_k^\alpha\right)(\cdot,\cdot,v)\,dv\right\|_{H^{\frac{1}{4}}(]0,T_\bullet[\times\Omega)}.
$$

By boundedness of $]0,T_\bullet[\times\Omega$, $H^{\frac{1}{4}}(]0,T_\bullet[\times\Omega)$ is compactly embedded in $L^2(]0,T_\bullet[\times\Omega)$ so that the sequence $\left(\int_{B_s}\zeta(v)\left(\eta f_k^\alpha\right)(\cdot,\cdot,v)\,dv\right)$ converges, after extracting a suitable subsequence, strongly to $\int_{B_s}\zeta(v)\left(\eta f^\alpha\right)(\cdot,\cdot,v)\,dv$ in $L^2(]0,T_\bullet[\times\Omega)$. Again by the uniform boundedness of the f_k^α in L^2 we can estimate

$$
\left\|\int_{\mathbb{R}^3}\zeta(v)\left((1-\eta)\left(f_k^\alpha - f^\alpha\right)\right)(\cdot,\cdot,v)\,dv\right\|_{L^2([0,T_\bullet]\times\Omega)} \tag{2.2.2}
$$

$$= \left\| \int_{B_s} \zeta(v)\big((1-\eta)\big(f_k^\alpha - f^\alpha\big)\big)(\cdot,\cdot,v)\,dv \right\|_{L^2([0,T_\bullet]\times\Omega)} \le C \left\|1-\eta\right\|_{L^2([0,T_\bullet]\times\Omega\times B_s)}$$

with a constant $C \ge 0$ that does not depend on k. Now let $l \in \mathbb{N}$ be arbitrary and choose $\eta = \eta_l \in C_c^\infty(]0,R^*[\times\Omega\times B_s)$ such that the right-hand side of (2.2.2) is smaller than $\frac{1}{l}$; note that $[0,T_\bullet]\times\Omega\times B_s$ is bounded. We iteratively choose subsequences $\left(f_{l,k}^\alpha\right)_{k\in\mathbb{N}}$ of $\left(f_k^\alpha\right)$ such that $\left(f_{l+1,k}^\alpha\right)_{k\in\mathbb{N}}$ is a subsequence $\left(f_{l,k}^\alpha\right)_{k\in\mathbb{N}}$ and such that

$$\lim_{k\to\infty} \int_{B_s} \zeta(v)\left(\eta_l f_{l,k}^\alpha\right)(\cdot,\cdot,v)\,dv = \int_{B_s} \zeta(v)\big(\eta_l f^\alpha\big)(\cdot,\cdot,v)\,dv$$

in $L^2(]0,T_\bullet[\times\Omega)$ for all $l \in \mathbb{N}$. Considering the diagonal sequence, now again denoted by $\left(f_k^\alpha\right)$, these considerations imply

$$\int_{\mathbb{R}^3} \zeta(v) f_k^\alpha(\cdot,\cdot,v)\,dv \to \int_{\mathbb{R}^3} \zeta(v) f^\alpha(\cdot,\cdot,v)\,dv \text{ strongly in } L^2([0,T_\bullet]\times\Omega) \text{ for } k\to\infty \tag{2.2.3}$$

because of

$$\begin{aligned} &\left\| \int_{\mathbb{R}^3} \zeta(v) f_k^\alpha(\cdot,\cdot,v)\,dv - \int_{\mathbb{R}^3} \zeta(v) f^\alpha(\cdot,\cdot,v)\,dv \right\|_{L^2([0,T_\bullet]\times\Omega)} \\ &\le \frac{1}{k} + \left\| \int_{B_s} \zeta(v)\big(\eta_k f_k^\alpha\big)(\cdot,\cdot,v)\,dv - \int_{B_s} \zeta(v)\big(\eta_k f^\alpha\big)(\cdot,\cdot,v)\,dv \right\|_{L^2([0,T_\bullet]\times\Omega)}. \end{aligned}$$

Finally take $\psi \in \Psi_{T_\bullet}$ and consider the limit of the crucial product term in (1.1.2). By a density argument (as in Section 1.4) we may assume that ψ factorizes, i.e.,

$$\psi(t,x,v) = \psi_1(t,x)\psi_2(v).$$

We have

$$\begin{aligned} \lim_{k\to\infty} \int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3} E_k\cdot\big(\partial_v\psi\big) f_k^\alpha\,dvdxdt &= \lim_{k\to\infty} \int_0^{T_\bullet}\int_\Omega E_k\psi_1\cdot\left(\int_{\mathbb{R}^3} f_k^\alpha \nabla\psi_2\,dv\right)dxdt \\ = \int_0^{T_\bullet}\int_\Omega E\psi_1\cdot\left(\int_{\mathbb{R}^3} f^\alpha\nabla\psi_2\,dv\right)dxdt &= \int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3} E\cdot\big(\partial_v\psi\big)f^\alpha\,dvdxdt \end{aligned}$$

by $\psi_1 \in L^\infty([0,T_\bullet]\times\Omega)$, $E_k \rightharpoonup E$ weakly in $L^2\big([0,T_\bullet]\times\Omega;\mathbb{R}^3\big)$, and (2.2.3) defining $\zeta := \big(\nabla\psi_2\big)_i$, $i = 1,2,3$. Similarly, we obtain

$$\lim_{k\to\infty} \int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3} (\widehat{v}_\alpha\times H_k)\cdot\big(\partial_v\psi\big) f_k^\alpha\,dvdxdt = \int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3} (\widehat{v}_\alpha\times H)\cdot\big(\partial_v\psi\big) f^\alpha\,dvdxdt.$$

Altogether, (VM) is satisfied in the limit.

By Lemma 2.1.2, we even have $f^\alpha \in \mathcal{Y}_{\mathrm{pd}}^\alpha$ and thus $y = \big((f^\alpha, f_+^\alpha)_\alpha, E, H\big) \in \mathcal{Y}$ altogether.

Finally, the objective function indeed attains its minimum at $\big(y,u\big)$ by weak lower semicontinuity of any norm. □

2.3 Weak formulation—revisited

For later reasons, it is convenient to revisit the weak formulation of Definition 1.1.1 and write the equations there as an identity

$$G\big((f^\alpha, f_+^\alpha)_\alpha, E, H, j\big) = 0$$

in the dual space of some reflexive space. Throughout this section, we fix $1 < p < 2$, $2 < q, \tilde{q} < \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{q} + \frac{1}{\tilde{q}} + \frac{1}{2}. \tag{2.3.1}$$

We will restrict ourselves to a finite time interval, i.e., $T_\bullet < \infty$, and assume $f^\alpha \in L^q\big([0, T_\bullet] \times \Omega \times \mathbb{R}^3\big)$, $f_+^\alpha \in L^q\Big(\gamma_{T_\bullet}^+, d\gamma_\alpha\Big)$, $E, H \in L^2\big([0, T_\bullet] \times \mathbb{R}^3; \mathbb{R}^3\big)$, and $j = j^{\text{int}} + u$ where $j^{\text{int}} = \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv \in L^{\frac{4}{3}}\big([0, T_\bullet] \times \Omega; \mathbb{R}^3\big)$, $u \in L^2\big([0, T_\bullet] \times \Gamma; \mathbb{R}^3\big)$. Note that for such u there is a weak solution in the sense of Definition 1.1.1 with these properties due to Theorem 1.4.4. To work in the most general setting, the g^α do not have to vanish for $\alpha = 1, \ldots, N'$ although they are assumed to be zero in the formulation of (P).

Clearly, Definition 1.1.1.(ii) and 1.1.1.(iii) are equivalent to

$$\begin{aligned}
0 = \sum_{\alpha=1}^N \Bigg(&- \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} \big(\partial_t \psi^\alpha + \widehat{v}_\alpha \cdot \partial_x \psi^\alpha + q_\alpha (E + \widehat{v}_\alpha \times H) \cdot \partial_v \psi^\alpha\big) f^\alpha \, dv dx dt \\
&+ \int_{\gamma_{T_\bullet}^+} f_+^\alpha \psi^\alpha \, d\gamma_\alpha - \int_{\gamma_{T_\bullet}^-} \big(\mathcal{K}_\alpha f_+^\alpha + g^\alpha\big) \psi^\alpha \, d\gamma_\alpha - \int_\Omega \int_{\mathbb{R}^3} \mathring{f}^\alpha \psi^\alpha(0) \, dv dx \Bigg) \\
&+ \int_0^{T_\bullet} \int_{\mathbb{R}^3} \big(\varepsilon E \cdot \partial_t \vartheta^e - H \cdot \operatorname{curl}_x \vartheta^e - 4\pi j \cdot \vartheta^e\big) \, dx dt + \int_{\mathbb{R}^3} \varepsilon \mathring{E} \cdot \vartheta^e(0) \, dx \\
&+ \int_0^{T_\bullet} \int_{\mathbb{R}^3} \Big(\mu H \cdot \partial_t \vartheta^h + E \cdot \operatorname{curl}_x \vartheta^h\Big) \, dx dt + \int_{\mathbb{R}^3} \mu \mathring{H} \cdot \vartheta^h(0) \, dx \\
=: {} & G\big((f^\alpha, f_+^\alpha)_\alpha, E, H, j\big)\Big(\big(\psi^\alpha\big)_\alpha, \vartheta^e, \vartheta^h\Big)
\end{aligned} \tag{2.3.2}$$

for all $\big(\psi^\alpha\big)_\alpha \in \Psi_{T_\bullet}^N$ and $\vartheta^e, \vartheta^h \in \Theta_{T_\bullet}$.

2.3.1 Some estimates

From now on, $\big((f^\alpha, f_+^\alpha)_\alpha, E, H, j\big)$ does not have to be a solution of (VM). All assertions are made under the assumptions mentioned above.

In the following we will estimate each summand, one by one, where we often need (2.3.1). Furthermore, C denotes various positive, finite constants that only depend on $T_\bullet$, Ω, and Γ and that may change from line to line. We have

$$\left| \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} \partial_t \psi^\alpha f^\alpha \, dv dx dt \right|$$

$$\leq \sqrt{T_\bullet \lambda(\Omega)} \int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left| f^\alpha \right|^q dxdt \right)^{\frac{1}{q}} \left(\int_0^{T_\bullet} \int_\Omega \left| \partial_t \psi^\alpha \right|^{\tilde{q}} dxdt \right)^{\frac{1}{\tilde{q}}} dv$$
$$\leq C \left\| f^\alpha \right\|_{L^q([0,T_\bullet]\times\Omega\times\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left| \partial_t \psi^\alpha \right|^{\tilde{q}} dxdt \right)^{\frac{p}{\tilde{q}}} dv \right)^{\frac{1}{p}};$$

next

$$\left| \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} \widehat{v}_\alpha \cdot \partial_x \psi^\alpha f^\alpha \, dvdxdt \right|$$
$$\leq \sqrt{T_\bullet \lambda(\Omega)} \int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left| f^\alpha \right|^q dxdt \right)^{\frac{1}{q}} \left(\int_0^{T_\bullet} \int_\Omega \left| \partial_x \psi^\alpha \right|^{\tilde{q}} dxdt \right)^{\frac{1}{\tilde{q}}} dv$$
$$\leq C \left\| f^\alpha \right\|_{L^q([0,T_\bullet]\times\Omega\times\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left| \partial_x \psi^\alpha \right|^{\tilde{q}} dxdt \right)^{\frac{p}{\tilde{q}}} dv \right)^{\frac{1}{p}};$$

then,

$$\left| \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} q_\alpha (E + \widehat{v}_\alpha \times H) \cdot \partial_v \psi^\alpha f^\alpha \, dvdxdt \right|$$
$$\leq \left| q_\alpha \right| \int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left| f^\alpha \right|^q dxdt \right)^{\frac{1}{q}} \left(\int_0^{T_\bullet} \int_\Omega \left| E + \widehat{v}_\alpha \times H \right|^2 dxdt \right)^{\frac{1}{2}}$$
$$\cdot \left(\int_0^{T_\bullet} \int_\Omega \left| \partial_v \psi^\alpha \right|^{\tilde{q}} dxdt \right)^{\frac{1}{\tilde{q}}} dv$$
$$\leq \sqrt{2} \left| q_\alpha \right| \left\| f^\alpha \right\|_{L^q([0,T_\bullet]\times\Omega\times\mathbb{R}^3)} \left\| (E,H) \right\|_{L^2([0,T_\bullet]\times\Omega;\mathbb{R}^6)} \left(\int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left| \partial_v \psi^\alpha \right|^{\tilde{q}} dxdt \right)^{\frac{p}{\tilde{q}}} dv \right)^{\frac{1}{p}}.$$

Now have in mind that there is a bounded trace operator

$$W^{1,\tilde{q}}(]0,T_\bullet[\times\Omega) \to L^{\tilde{q}}((]0,T_\bullet[\times\partial\Omega) \cup (\{0\}\times\Omega) \cup (\{T_\bullet\}\times\Omega)).$$

Thus,

$$\left| \int_{\gamma^+_{T_\bullet}} f^\alpha_+ \psi^\alpha \, d\gamma_\alpha \right| \leq \int_{\mathbb{R}^3} \int_0^{T_\bullet} \int_{\{x\in\partial\Omega | \widehat{v}_\alpha\cdot n(x)>0\}} \left| f^\alpha_+ \psi^\alpha \right| \left| \widehat{v}_\alpha \cdot n \right| dS_x dt dv$$
$$\leq C \int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_{\{x\in\partial\Omega | \widehat{v}_\alpha\cdot n(x)>0\}} \left| f^\alpha_+ \right|^q \left| \widehat{v}_\alpha \cdot n \right|^q dS_x dt \right)^{\frac{1}{q}} \left(\int_0^{T_\bullet} \int_{\partial\Omega} \left| \psi^\alpha \right|^{\tilde{q}} dS_x dt \right)^{\frac{1}{\tilde{q}}} dv$$

$$\leq C \int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_{\{x \in \partial\Omega | \widehat{v}_\alpha \cdot n(x) > 0\}} \left| f_+^\alpha \right|^q |\widehat{v}_\alpha \cdot n| \, dS_x dt \right)^{\frac{1}{q}}$$

$$\cdot \left(\int_0^{T_\bullet} \int_\Omega \left(\left| \psi^\alpha \right|^{\tilde{q}} + \left| \partial_t \psi^\alpha \right|^{\tilde{q}} + \left| \partial_x \psi^\alpha \right|^{\tilde{q}} \right) dx dt \right)^{\frac{1}{\tilde{q}}} dv$$

$$\leq C \left\| f_+^\alpha \right\|_{L^q\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)} \left(\int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left(\left| \psi^\alpha \right|^{\tilde{q}} + \left| \partial_t \psi^\alpha \right|^{\tilde{q}} + \left| \partial_x \psi^\alpha \right|^{\tilde{q}} \right) dx dt \right)^{\frac{p}{\tilde{q}}} dv \right)^{\frac{1}{p}}$$

by $|\widehat{v}_\alpha \cdot n| \leq 1$. Similarly,

$$\left| \int_{\gamma_{T_\bullet}^-} g^\alpha \psi^\alpha \, d\gamma_\alpha \right|$$

$$\leq C \left\| g^\alpha \right\|_{L^q\left(\gamma_{T_\bullet}^-, d\gamma_\alpha\right)} \left(\int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left(\left| \psi^\alpha \right|^{\tilde{q}} + \left| \partial_t \psi^\alpha \right|^{\tilde{q}} + \left| \partial_x \psi^\alpha \right|^{\tilde{q}} \right) dx dt \right)^{\frac{p}{\tilde{q}}} dv \right)^{\frac{1}{p}}$$

and

$$\left| \int_{\gamma_{T_\bullet}^-} (\mathcal{K}_\alpha f_+^\alpha) \psi^\alpha \, d\gamma_\alpha \right| = \left| \int_{\gamma_{T_\bullet}^-} a^\alpha (K f_+^\alpha) \psi^\alpha \, d\gamma_\alpha \right|$$

$$\leq C \left\| f_+^\alpha \right\|_{L^q\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)} \left(\int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left(\left| \psi^\alpha \right|^{\tilde{q}} + \left| \partial_t \psi^\alpha \right|^{\tilde{q}} + \left| \partial_x \psi^\alpha \right|^{\tilde{q}} \right) dx dt \right)^{\frac{p}{\tilde{q}}} dv \right)^{\frac{1}{p}}$$

since $|a^\alpha| \leq 1$ and $v \mapsto v - 2(v \cdot n(x))n(x)$ has Jacobian determinant -1. Analogously,

$$\left| \int_\Omega \int_{\mathbb{R}^3} \mathring{f}^\alpha \psi^\alpha(0) \, dv dx \right|$$

$$\leq C \left\| \mathring{f}^\alpha \right\|_{L^q(\Omega \times \mathbb{R}^3)} \left(\int_{\mathbb{R}^3} \left(\int_0^{T_\bullet} \int_\Omega \left(\left| \psi^\alpha \right|^{\tilde{q}} + \left| \partial_t \psi^\alpha \right|^{\tilde{q}} + \left| \partial_x \psi^\alpha \right|^{\tilde{q}} \right) dx dt \right)^{\frac{p}{\tilde{q}}} dv \right)^{\frac{1}{p}}$$

making use of the boundedness of the trace operator, now regarding the slice $\{0\} \times \Omega$ instead of $]0, T_\bullet[\times \partial\Omega$.

As for the Maxwell part, we can easily estimate

$$\left| \int_0^{T_\bullet} \int_{\mathbb{R}^3} \varepsilon E \cdot \partial_t \vartheta^e \, dx dt \right| \leq \sigma' \|E\|_{L^2([0,T_\bullet] \times \mathbb{R}^3; \mathbb{R}^3)} \|\partial_t \vartheta^e\|_{L^2([0,T_\bullet] \times \mathbb{R}^3; \mathbb{R}^3)},$$

$$\left| \int_0^{T_\bullet} \int_{\mathbb{R}^3} H \cdot \operatorname{curl}_x \vartheta^e \, dx dt \right| \leq \|H\|_{L^2([0,T_\bullet] \times \mathbb{R}^3; \mathbb{R}^3)} \|\operatorname{curl}_x \vartheta^e\|_{L^2([0,T_\bullet] \times \mathbb{R}^3; \mathbb{R}^3)},$$

$$\left|\int_0^{T_\bullet}\int_{\mathbb{R}^3}\mu H\cdot\partial_t\vartheta^h\,dxdt\right|\le\sigma'\|H\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^3)}\left\|\partial_t\vartheta^h\right\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^3)},$$
$$\left|\int_0^{T_\bullet}\int_{\mathbb{R}^3}E\cdot\operatorname{curl}_x\vartheta^h\,dxdt\right|\le\|E\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^3)}\left\|\operatorname{curl}_x\vartheta^h\right\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^3)}.$$

Concerning the terms with the initial data, we first notice that for all $x\in\mathbb{R}^3$ we have

$$\vartheta^e(0,x)=\vartheta^e(0,x)-\vartheta^e(T_\bullet,x)=-\int_0^{T_\bullet}\partial_t\vartheta^e(t,x)\,dt;$$

consequently

$$|\vartheta^e(0,x)|^2\le T_\bullet\int_0^{T_\bullet}|\partial_t\vartheta^e(t,x)|^2\,dt,$$

and therefore

$$\left|\int_{\mathbb{R}^3}\varepsilon\mathring{E}\cdot\vartheta^e(0)\,dx\right|\le\sigma'C\left\|\mathring{E}\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\|\partial_t\vartheta^e\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^3)}.$$

Similarly, we conclude

$$\left|\int_{\mathbb{R}^3}\mu\mathring{H}\cdot\vartheta^h(0)\,dx\right|\le\sigma'C\left\|\mathring{H}\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\left\|\partial_t\vartheta^h\right\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^3)}.$$

Lastly, we turn to the term with j. By Sobolev's embedding theorem, $H^1(]0,T_\bullet[\times A)$ is continuously embedded in $L^4(]0,T_\bullet[\times A)$, $A=\Omega,\Gamma$, yielding

$$\begin{aligned}\left|\int_0^{T_\bullet}\int_{\mathbb{R}^3}j\cdot\vartheta^e\,dxdt\right|&\le\left|\int_0^{T_\bullet}\int_\Omega j^{\mathrm{int}}\cdot\vartheta^e\,dxdt\right|+\left|\int_0^{T_\bullet}\int_\Gamma u\cdot\vartheta^e\,dxdt\right|\\ &\le\left\|j^{\mathrm{int}}\right\|_{L^{\frac43}([0,T_\bullet]\times\Omega;\mathbb{R}^3)}\|\vartheta^e\|_{L^4([0,T_\bullet]\times\Omega;\mathbb{R}^3)}+\|u\|_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}\|\vartheta^e\|_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}\\ &\le C\left(\left\|j^{\mathrm{int}}\right\|_{L^{\frac43}([0,T_\bullet]\times\Omega;\mathbb{R}^3)}+\|u\|_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}\right)\|\vartheta^e\|_{H^1(]0,T_\bullet[\times\mathbb{R}^3;\mathbb{R}^3)}.\end{aligned}$$

Altogether, we conclude that $G\big((f^\alpha,f_+^\alpha)_\alpha,E,H,j\big)$ is a bounded linear operator on $\Psi_{T_\bullet}^N\times\Theta_{T_\bullet}^2$ if we equip $\Psi_{T_\bullet}$ with the norm

$$\|\psi\|_{W^{1,p,\tilde q}}:=\left(\int_{\mathbb{R}^3}\left(\int_0^{T_\bullet}\int_\Omega\left(|\psi|^{\tilde q}+|\partial_t\psi|^{\tilde q}+|\partial_x\psi|^{\tilde q}+|\partial_v\psi|^{\tilde q}\right)dxdt\right)^{\frac{p}{\tilde q}}dv\right)^{\frac1p}\tag{2.3.3}$$

and $\Theta_{T_\bullet}$ with the usual H^1-norm on $]0,T_\bullet[\times\mathbb{R}^3$.

2.3.2 The space $W^{1,p,\tilde{q}}$ and the extended functional

The choice of the norm for the test functions ψ suggests having a detailed look at the space $W^{1,p,\tilde{q}}$. This space, so to say a mixed order Sobolev space, is defined to be the space consisting of all measurable functions on $]0,T_\bullet[\times\Omega\times\mathbb{R}^3$ with values in $\mathbb{R}$ such that their derivatives of first order are locally integrable functions and additionally the right-hand side of (2.3.3) is finite.

We first consider the corresponding $L^{p,\tilde{q}}$-space, that is,

$$L^{p,\tilde{q}} := \left\{ \psi\colon\,]0,T_\bullet[\times\Omega\times\mathbb{R}^3\to\mathbb{R} \text{ measurable} \mid \right.$$
$$\left. \|\psi\|_{L^{p,\tilde{q}}} := \left(\int_{\mathbb{R}^3}\left(\int_0^{T_\bullet}\int_\Omega |\psi|^{\tilde{q}}\,dx\,dt\right)^{\frac{p}{\tilde{q}}} dv\right)^{\frac{1}{p}} < \infty \right\}.$$

Since we can identify $L^{p,\tilde{q}}$ with the Bochner space $L^p\big(\mathbb{R}^3; L^{\tilde{q}}([0,T_\bullet]\times\Omega)\big)$, we get the following basic property.

Lemma 2.3.1. *$L^{p,\tilde{q}}$ is a uniformly convex Banach space.*

Proof. This is easy to see using the identification above. The uniform convexity follows from a classical result of Day [Day41] since $1 < p, \tilde{q} < \infty$. □

The uniform convexity will be crucial later.

These properties of $L^{p,\tilde{q}}$ carry over to $W^{1,p,\tilde{q}}$ in the same natural way as such properties carry over from standard L^p-spaces to standard Sobolev spaces $W^{1,p}$: The space $W^{1,p,\tilde{q}}$ can be interpreted as a closed subspace of $\big(L^{p,\tilde{q}}\big)^7$ via the isometry

$$\psi \mapsto \big(\psi, \partial_t\psi, \partial_{x_1}\psi, \partial_{x_2}\psi, \partial_{x_3}\psi, \partial_{v_1}\psi, \partial_{v_2}\psi, \partial_{v_3}\psi\big).$$

Hence, one can argue in the same way as in the standard context to prove the following lemma.

Lemma 2.3.2. *$W^{1,p,\tilde{q}}$ is a uniformly convex, reflexive Banach space.*

Proof. Note that uniform convexity and completeness implies reflexivity by the classical Milman–Pettis theorem; see, for example, [Pet39]. □

Now we turn back to our weak formulation. Recall that we have proved

$$G\big((f^\alpha, f_+^\alpha)_\alpha, E, H, j\big) \in \left(\Psi_{T_\bullet}^N \times \Theta_{T_\bullet}^2\right)^*.$$

If we denote $\Lambda := \overline{\Psi_{T_\bullet}}^N \times \overline{\Theta_{T_\bullet}}^2$, where the closure is to be understood in $W^{1,p,\tilde{q}}$ and $H^1\big(]0,T_\bullet[\times\mathbb{R}^3;\mathbb{R}^3\big)$, respectively, we can extend $G\big((f^\alpha, f_+^\alpha)_\alpha, E, H, j\big)$ uniquely to a bounded linear operator on Λ and still the formula in (2.3.2) applies. Since $H^1\big(]0,T_\bullet[\times\mathbb{R}^3;\mathbb{R}^3\big)$ is also a uniformly convex, reflexive Banach space and since Λ is a closed subspace, we instantly conclude the following.

Lemma 2.3.3. *Λ, equipped with the norm*

$$\Big\|\Big(\big(\psi^\alpha\big)_\alpha,\vartheta^e,\vartheta^h\Big)\Big\|_\Lambda := \left(\sum_{\alpha=1}^N\big\|\psi^\alpha\big\|^2_{W^{1,p,\tilde q}} + \|\vartheta^e\|^2_{H^1(]0,T_\bullet[\times\mathbb{R}^3;\mathbb{R}^3)} + \big\|\vartheta^h\big\|^2_{H^1(]0,T_\bullet[\times\mathbb{R}^3;\mathbb{R}^3)}\right)^{\frac{1}{2}},$$

is a uniformly convex, reflexive Banach space.

Proof. By Clarkson [Cla36], a finite Cartesian product of uniformly convex spaces is again uniformly convex if one sums up the norms properly. Note that we have chosen the 2-norm on $\mathbb{R}^{N+2}$ to sum up the particular norms (any other $\tilde p$-norm, $1<\tilde p<\infty$, would work as well). Thus, Λ is uniformly convex. Again by completeness of Λ and the Milman–Pettis theorem, we conclude that Λ is additionally reflexive. ☐

Thus, we can regard $G\big((f^\alpha,f^\alpha_+)_\alpha,E,H,j\big)\in\Lambda^*$ as an element of the dual space of a uniformly convex, reflexive Banach space, and we have that, under the assumptions made in the beginning of Section 2.3, $G\big((f^\alpha,f^\alpha_+)_\alpha,E,H,j\big)=0$ is equivalent to $\big((f^\alpha,f^\alpha_+)_\alpha,E,H,j\big)$ being a weak solution of the Vlasov–Maxwell system (VM) on the time interval $[0,T_\bullet]$.

Notice that Λ is a proper subspace of $\big(W^{1,p,\tilde q}\big)^N\times\big(H^1\big(]0,T_\bullet[\times\mathbb{R}^3;\mathbb{R}^3\big)\big)^2$ since $\psi\in\Psi_{T_\bullet}$ and $\vartheta\in\Theta_{T_\bullet}$ vanish for $t=T_\bullet$.

Later, in Section 2.4, we want to derive first order optimality conditions for a (local) minimizer of (P). To this end, it will be helpful that G ($\mathcal{G}$, to be more precise; see below) is differentiable in $\big((f^\alpha,f^\alpha_+)_\alpha,E,H,u\big)$ with respect to a suitable norm; here and in the following, differentiability always means differentiability in the sense of Fréchet. As in the formulation of (P), we restrict ourselves to $\big((f^\alpha,f^\alpha_+)_\alpha,E,H,u\big)\in\mathcal{Y}\times\mathcal{U}$. Note that this yields $f^\alpha\in L^q\big([0,T_\bullet]\times\Omega\times\mathbb{R}^3\big)$ by interpolation and thus we can carry through the previous considerations of this section. We equip $\mathcal{Y}\times\mathcal{U}$ with the norm

$$\begin{aligned}\big\|(y,u)\big\|_{\mathcal{Y}\times\mathcal{U}} &= \big\|\big((f^\alpha,f^\alpha_+)_\alpha,E,H,u\big)\big\|_{\mathcal{Y}\times\mathcal{U}}\\ &:= \sum_{\alpha=1}^N\left(\big\|f^\alpha\big\|_{\mathcal{Y}^\alpha_{\mathrm{pd}}} + \big\|f^\alpha_+\big\|_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)}\right) + \|(E,H)\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^6)} + \|u\|_{\mathcal{U}},\end{aligned}$$

where

$$\big\|f\big\|_{\mathcal{Y}^\alpha_{\mathrm{pd}}} := \big\|f\big\|_{L^1_{\alpha\mathrm{kin}}([0,T_\bullet]\times\Omega\times\mathbb{R}^3)} + \big\|f\big\|_{L^\infty([0,T_\bullet]\times\Omega\times\mathbb{R}^3)} + \mathcal{N}_\alpha(f).$$

The latter indeed is a norm on $\mathcal{Y}^\alpha_{\mathrm{pd}}$ since $\mathcal{N}_\alpha$ is a seminorm on $\mathcal{Y}^\alpha_{\mathrm{pd}}$, as is easily seen. Note that the following lemma does not need the adding of $\mathcal{N}_\alpha$ as above; however, this will heavily be exploited in Section 2.4.

Lemma 2.3.4. *The map*

$$\mathcal{G}\colon \mathcal{Y}\times\mathcal{U}\to\Lambda^*,$$
$$\mathcal{G}\big((f^\alpha,f_+^\alpha)_\alpha,E,H,u\big)=G\big((f^\alpha,f_+^\alpha)_\alpha,E,H,j^{\mathrm{int}}+u\big)$$

is differentiable, and we have

$$\begin{aligned}
&\big(\mathcal{G}'(y,u)(\delta y,\delta u)\big)\Big((\psi^\alpha)_\alpha,\vartheta^e,\vartheta^h\Big)\\
&=\sum_{\alpha=1}^N\Bigg(-\int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3}\big((\partial_t\psi^\alpha+\widehat{v}_\alpha\cdot\partial_x\psi^\alpha+q_\alpha(E+\widehat{v}_\alpha\times H)\cdot\partial_v\psi^\alpha)\delta f^\alpha\\
&\qquad\qquad+q_\alpha(\delta E+\widehat{v}_\alpha\times\delta H)f^\alpha\cdot\partial_v\psi^\alpha\big)\,dvdxdt\\
&\qquad+\int_{\gamma_{T_\bullet}^+}\delta f_+^\alpha\psi^\alpha\,d\gamma_\alpha-\int_{\gamma_{T_\bullet}^-}\big(\mathcal{K}_\alpha\delta f_+^\alpha\big)\psi^\alpha\,d\gamma_\alpha\Bigg)\\
&\quad+\int_0^{T_\bullet}\int_{\mathbb{R}^3}\big(\varepsilon\delta E\cdot\partial_t\vartheta^e-\delta H\cdot\mathrm{curl}_x\,\vartheta^e-4\pi\big(\delta j^{\mathrm{int}}+\delta u\big)\cdot\vartheta^e\big)\,dxdt\\
&\quad+\int_0^{T_\bullet}\int_{\mathbb{R}^3}\Big(\mu\delta H\cdot\partial_t\vartheta^h+\delta E\cdot\mathrm{curl}_x\,\vartheta^h\Big)\,dxdt,
\end{aligned}\tag{2.3.4}$$

where, in accordance with the previous notation,

$$\delta j^{\mathrm{int}}=\sum_{\alpha=1}^N q_\alpha\int_{\mathbb{R}^3}\widehat{v}_\alpha\delta f^\alpha\,dv.$$

Proof. The candidate for the linearization at a point (y,u) in direction $(\delta y,\delta u)=\big((\delta f^\alpha,\delta f_+^\alpha)_\alpha,\delta E,\delta H,\delta u\big)$ is $\mathcal{G}'(y,u)(\delta y,\delta u)$ as stated above. Recalling the estimates of Section 2.3.1, we see that $\mathcal{G}'(y,u)(\delta y,\delta u)\in\Lambda^*$ and

$$\begin{aligned}
&\big\|\mathcal{G}'(y,u)(\delta y,\delta u)\big\|_{\Lambda^*}\\
&\le C\Bigg(\sum_{\alpha=1}^N\bigg(\big\|\delta f^\alpha\big\|_{L^q([0,T_\bullet]\times\Omega\times\mathbb{R}^3)}+\|(E,H)\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^6)}\big\|\delta f^\alpha\big\|_{L^q([0,T_\bullet]\times\Omega\times\mathbb{R}^3)}\\
&\qquad+\big\|f^\alpha\big\|_{L^q([0,T_\bullet]\times\Omega\times\mathbb{R}^3)}\|(\delta E,\delta H)\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^6)}+\big\|\delta f_+^\alpha\big\|_{L^q\left(\gamma_{T_\bullet}^+,d\gamma_\alpha\right)}\bigg)\\
&\quad+\|(\delta E,\delta H)\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^6)}+\big\|\delta j^{\mathrm{int}}\big\|_{L^{\frac43}([0,T_\bullet]\times\Omega;\mathbb{R}^3)}+\|\delta u\|_{L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)}\Bigg),
\end{aligned}\tag{2.3.5}$$

where C only depends on $T_\bullet$, Ω, Γ, σ', and the q_α.

Similarly to (2.1.7) and (2.1.8), we deduce

$$\big\|\delta j^{\mathrm{int}}\big\|_{L^{\frac43}([0,T_\bullet]\times\Omega;\mathbb{R}^3)}$$

$$\leq \left(\sum_{\alpha=1}^{N} |q_\alpha|^4 \left(\frac{4\pi}{3}\|\delta f^\alpha\|_{L^\infty([0,T_\bullet]\times\Omega\times\mathbb{R}^3)} + 1\right)^4\right)^{\frac{1}{4}} \left(\sum_{\alpha=1}^{N} \|\delta f^\alpha\|_{L^1_{\alpha\mathrm{kin}}([0,T_\bullet]\times\Omega\times\mathbb{R}^3)}\right)^{\frac{3}{4}}.$$

This and (2.3.5) yields that $\mathcal{G}'(y,u)(\delta y,\delta u) \to 0$ in Λ^* when $(\delta y,\delta u) \to 0$ in $\mathcal{Y}\times\mathcal{U}$. Therefore, $\mathcal{G}'(y,u)\colon \mathcal{Y}\times\mathcal{U} \to \Lambda^*$ is a bounded linear map; linearity is of course easy to see.

To show that $\mathcal{G}'(y,u)$ indeed is the derivative of $\mathcal{G}$ at (y,u), we consider the remainder, which only contains terms that come from the nonlinearity in the Vlasov–Maxwell system:

$$\begin{aligned}&\left(\mathcal{G}(y+\delta y, u+\delta u) - \mathcal{G}(y,u) - \mathcal{G}'(y,u)(\delta y,\delta u)\right)\left((\psi^\alpha)_\alpha, \vartheta^e, \vartheta^h\right)\\ &= -\sum_{\alpha=1}^{N} q_\alpha \int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3} (\delta E + \widehat{v}_\alpha \times \delta H)\cdot \partial_v \psi^\alpha \delta f^\alpha \, dv dx dt.\end{aligned}$$

Hence, again using the corresponding estimate of Section 2.3.1,

$$\begin{aligned}&\left\|\mathcal{G}(y+\delta y, u+\delta u) - \mathcal{G}(y,u) - \mathcal{G}'(y,u)(\delta y,\delta u)\right\|_{\Lambda^*}\\ &\leq C\sum_{\alpha=1}^{N} \|\delta f^\alpha\|_{L^q([0,T_\bullet]\times\Omega\times\mathbb{R}^3)} \|(\delta E,\delta H)\|_{L^2([0,T_\bullet]\times\Omega;\mathbb{R}^6)} = o\left(\|(\delta y,\delta u)\|_{\mathcal{Y}\times\mathcal{U}}\right)\end{aligned}$$

for $(\delta y,\delta u) \to 0$ in $\mathcal{Y}\times\mathcal{U}$, where C only depends on σ' and the q_α. This completes the proof. □

2.4 First order optimality conditions

A standard step when treating an optimization problem is to derive first order necessary optimality conditions. Typically, one exploits differentiability of the control-to-state operator. Unfortunately, we do not have such an operator at hand since we do not even have uniqueness of weak solutions for a fixed control u. Lions [Lio85] introduced a way to tackle optimization problems having a PDE (system), that (possibly) admits multiple solutions, as a constraint. The main strategy is to consider approximate optimization problems that no longer have the PDE (system) as a constraint but merely penalize points (y,u) that do not solve this PDE (system). For such approximate problems, one can show that minimizers exist and derive first order optimality conditions. Then the penalization parameter is driven to ∞, and one hopes the PDE (system) to be solved in the limit, that is to say, the limit of minimizers (in whatever sense) is a solution of the PDE (system), and moreover it is a minimizer of the original problem. Furthermore, one may show that passage to the limit in the approximate optimality conditions, in particular in the adjoint PDE (system), is possible, too.

We fix $q > 2$ and $p, \tilde{q}$ satisfying (2.3.1) so that the results of Section 2.3 can be applied.

2.4.1 An approximate optimization problem

Following the outlined strategy, we introduce a penalization parameter $s > 0$ (which will be driven to ∞ later) and consider the approximate problem

$$\left.\begin{aligned} \min_{y\in\mathcal{Y},u\in\mathcal{U}} \quad & \mathcal{J}_s(y,u), \\ \text{s.t.} \quad & (2.1.1),\ (2.1.2),\ \text{and (2.4.1) hold} \end{aligned}\right\} \tag{P$_\mathrm{s}$}$$

where the objective function is

$$\begin{aligned} \mathcal{J}_s(y,u) &= \mathcal{J}(y,u) + \frac{s}{2}\left\|\mathcal{G}(y,u)\right\|_{\Lambda^*}^2 \\ &= \frac{1}{q}\sum_{\alpha=1}^{N} w_\alpha \left\|f_+^\alpha\right\|_{L^q\left(\gamma_{T_\bullet}^+, d\gamma_\alpha\right)}^q + \frac{1}{r}\|u\|_{\mathcal{U}}^r + \frac{s}{2}\left\|\mathcal{G}(y,u)\right\|_{\Lambda^*}^2 \end{aligned}$$

and the additional constraint is

$$\mathcal{N}_\alpha(f^\alpha) \le \mathcal{L}_\alpha, \quad \alpha = 1,\dots,N. \tag{2.4.1}$$

Here,

$$\begin{aligned} \mathcal{L}_\alpha := 2\left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)} & \left(1 + \frac{4|q_\alpha|\sqrt{\pi}}{\sqrt{\sigma}}\right. \\ & \left.\cdot\left(2T_\bullet \sum_{\alpha'=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0 \mathring{f}^{\alpha'}\, dv dx + \frac{T_\bullet\sigma'}{4\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^6)}^2 + 2\pi T_\bullet^2\sigma^{-1}C_\Gamma^2 (r\min\mathcal{J})^{\frac{2}{r}}\right)^{\frac{1}{2}}\right) \end{aligned}$$

where $\min\mathcal{J} := \mathcal{J}(y_*,u_*)$, (y_*,u_*) being some minimizer of (P), and C_Γ is the (optimal) constant corresponding to the continuous embedding $\mathcal{U} \subset L^2([0,T_\bullet]\times\Gamma;\mathbb{R}^3)$. On the one hand, (2.4.1) is automatically satisfied for any minimizer (y_*,u_*) of (P)—in particular, there are feasible points for (P$_\mathrm{s}$)—which can be verified as follows: Due to (2.1.2) it holds that

$$\begin{aligned} &\|(E_*,H_*)\|_{L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^6)} \le \sqrt{\frac{8\pi}{\sigma}\mathcal{I}(u_*)} \\ &\le \sqrt{\frac{8\pi}{\sigma}}\left(2T_\bullet\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0\mathring{f}^\alpha\, dvdx + \frac{T_\bullet\sigma'}{4\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^6)}^2 + 2\pi T_\bullet^2\sigma^{-1}C_\Gamma^2\|u_*\|_{\mathcal{U}}^2\right)^{\frac{1}{2}} \\ &\le \sqrt{\frac{8\pi}{\sigma}}\left(2T_\bullet\sum_{\alpha=1}^{N}\int_\Omega\int_{\mathbb{R}^3} v_\alpha^0\mathring{f}^\alpha\, dvdx + \frac{T_\bullet\sigma'}{4\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^6)}^2 + 2\pi T_\bullet^2\sigma^{-1}C_\Gamma^2(r\min\mathcal{J})^{\frac{2}{r}}\right)^{\frac{1}{2}}, \end{aligned}$$

which yields (2.4.1) in view of (2.1.1) and (2.1.6).

On the other hand, (2.4.1) ensures a certain weak lower semicontinuity of $\|\mathcal{G}\|_{\Lambda^*}$ by the following lemma—and this is conversely the very reason why we impose (2.4.1).

Lemma 2.4.1. *Let* $\big((y_k, u_k)\big) \subset \mathcal{Y} \times \mathcal{U}$ *with* $f_k^\alpha \geq 0$ *and limit functions* $u \in \mathcal{U}$, $f^\alpha \in L^\infty\big([0, T_\bullet] \times \Omega \times \mathbb{R}^3\big)$, $f_+^\alpha \in L^q\Big(\gamma_{T_\bullet}^+, d\gamma_\alpha\Big)$, $(E, H) \in L^2\big([0, T_\bullet] \times \Omega; \mathbb{R}^6\big)$ *such that for* $k \to \infty$ *it holds that* $u_k \rightharpoonup u$ *in* $\mathcal{U}$, $f_k^\alpha \overset{*}{\rightharpoonup} f^\alpha$ *in* $L^\infty\big([0, T_\bullet] \times \Omega \times \mathbb{R}^3\big)$, $f_{k,+}^\alpha \rightharpoonup f_+^\alpha$ *in* $L^q\Big(\gamma_{T_\bullet}^+, d\gamma_\alpha\Big)$, $(E_k, H_k) \rightharpoonup (E, H)$ *in* $L^2\big([0, T_\bullet] \times \Omega; \mathbb{R}^6\big)$. *Furthermore, assume that* (2.1.2) *and* (2.4.1) *are satisfied along the sequence. Then,* $(y, u) \in \mathcal{Y} \times \mathcal{U}$, (2.1.2) *and* (2.4.1) *are preserved in the limit, and*

$$\big\|\mathcal{G}(y, u)\big\|_{\Lambda^*} \leq \liminf_{k\to\infty} \big\|\mathcal{G}(y_k, u_k)\big\|_{\Lambda^*}. \tag{2.4.2}$$

Proof. Note that (u_k) converges to u strongly in $L^2\big([0, T_\bullet] \times \Gamma; \mathbb{R}^3\big)$.

Step 1. $f^\alpha \in \mathcal{Y}_{\mathrm{pd}}^\alpha$ and (2.1.2) and (2.4.1) are preserved in the limit: Take $\eta \in C_c^\infty\big(]0, T_\bullet[\times \Omega \times \mathbb{R}^3\big)$ and consider

$$g_k := \partial_t\big(\eta f_k^\alpha\big) + \widehat{v}_\alpha \cdot \partial_x\big(\eta f_k^\alpha\big).$$

In light of (2.4.1), the sequence $\big(g_k\big)$ is bounded in $L^2\big([0, T_\bullet] \times \Omega; H^{-1}\big(\mathbb{R}^3\big)\big)$. Therefore, $\big(g_k\big)$ converges, after possibly extracting a suitable subsequence, to some g weak-* in $L^2\big([0, T_\bullet] \times \Omega; H^{-1}\big(\mathbb{R}^3\big)\big)$. Since for all $\varphi \in C_c^\infty\big(]0, T_\bullet[\times \Omega \times \mathbb{R}^3\big)$

$$\begin{aligned} g(\varphi) &= \lim_{k\to\infty} \big(\partial_t\big(\eta f_k^\alpha\big) + \widehat{v}_\alpha \cdot \partial_x\big(\eta f_k^\alpha\big)\big)(\varphi) \\ &= \lim_{k\to\infty} -\int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3} \big(\eta f_k^\alpha \partial_t\varphi + \widehat{v}_\alpha \eta f_k^\alpha \cdot \partial_x\varphi\big)\, dv dx dt \\ &= -\int_0^{T_\bullet}\int_\Omega\int_{\mathbb{R}^3} \big(\eta f^\alpha \partial_t\varphi + \widehat{v}_\alpha \eta f^\alpha \cdot \partial_x\varphi\big)\, dv dx dt = \big(\partial_t\big(\eta f^\alpha\big) + \widehat{v}_\alpha \cdot \partial_x\big(\eta f^\alpha\big)\big)(\varphi) \end{aligned}$$

and since $C_c^\infty\big(]0, T_\bullet[\times \Omega \times \mathbb{R}^3\big)$ is dense in $L^2\big([0, T_\bullet] \times \Omega; H^1\big(\mathbb{R}^3\big)\big)$, we have

$$\partial_t\big(\eta f^\alpha\big) + \widehat{v}_\alpha \cdot \partial_x\big(\eta f^\alpha\big) = g \in L^2\big([0, T_\bullet] \times \Omega; H^{-1}\big(\mathbb{R}^3\big)\big).$$

Furthermore, by weak-*-convergence it holds that

$$\begin{aligned} &\big\|\partial_t\big(\eta f^\alpha\big) + \widehat{v}_\alpha \cdot \partial_x\big(\eta f^\alpha\big)\big\|_{L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))} \\ &\leq \liminf_{k\to\infty} \big\|\partial_t\big(\eta f_k^\alpha\big) + \widehat{v}_\alpha \cdot \partial_x\big(\eta f_k^\alpha\big)\big\|_{L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))} \leq \mathcal{L}_\alpha \end{aligned}$$

if η satisfies (2.1.4). Thus, (2.4.1) is preserved in the limit. Moreover, as in the proof of Theorem 2.2.1, we also see that $f^\alpha \in \Big(L^1_{\alpha\mathrm{kin}} \cap L^\infty\Big)\big([0, T_\bullet] \times \Omega \times \mathbb{R}^3\big)$ and that (2.1.2) is preserved in the limit. Altogether, $(y, u) \in \mathcal{Y} \times \mathcal{U}$.

Step 2. Proof of (2.4.2): To this end, we have to pass to the limit in the right-hand sides of (1.1.2) and (1.1.3); this procedure has already been carried out a few times in similar, yet not identical situations. As a consequence of Lemma 2.1.3, we may assume that $\Big(j_k^{\mathrm{int}}\Big)$ converges weakly to j^{int} in $L^{\frac{4}{3}}\big([0, T_\bullet] \times \Omega; \mathbb{R}^3\big)$; in order to verify that this weak limit indeed is j^{int}, we recall that an energy estimate like (2.1.2) is sufficient. Hence, we can easily pass to the limit in all terms but the nonlinear one, first for $\left((\psi^\alpha)_\alpha, \vartheta^e, \vartheta^h\right) \in \Psi_{T_\bullet}^N \times \Theta_{T_\bullet}^2$ and then for arbitrary $\left((\psi^\alpha)_\alpha, \vartheta^e, \vartheta^h\right) \in \Lambda$ with the help of Section 2.3.1. Regarding the nonlinear term, we first consider $\psi^\alpha \in \Psi_{T_\bullet}$ that factorizes, as in Section 1.4. For some $l \in \mathbb{N}$ and $\zeta \in C_c^\infty(\mathbb{R}^3)$ with supp $\zeta \subset B_R$ (for some $R > 0$), we find an $\eta_l \in C_c^\infty(]0, T_\bullet[\times \Omega \times B_R)$, similarly to (1.4.17), such that

$$\left\| \int_{\mathbb{R}^3} \zeta(v) \left((1 - \eta_l) \left(f_k^\alpha - f^\alpha \right) \right)(\cdot, \cdot, v) \, dv \right\|_{L^2([0,T_\bullet] \times \Omega)} < \frac{1}{l}; \tag{2.4.3}$$

note that the L^2-norms of the f_k^α are uniformly bounded. For this fixed η_l it holds that

$$\begin{aligned} &\left\|\partial_t\left(\eta_l f_k^\alpha\right) + \widehat{v}_\alpha \cdot \partial_x\left(\eta_l f_k^\alpha\right)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^3; H^{-1}(\mathbb{R}^3))} = \left\|\partial_t\left(\eta_l f_k^\alpha\right) + \widehat{v}_\alpha \cdot \partial_x\left(\eta_l f_k^\alpha\right)\right\|_{L^2([0,T_\bullet]\times\Omega; H^{-1}(\mathbb{R}^3))} \\ &\le \mathcal{N}_\alpha\left(f_k^\alpha\right)\left(\left\|\eta_l\right\|_{H^1(]0,T_\bullet[\times\Omega\times\mathbb{R}^3)} + \left\|\eta_l\right\|_{L^\infty([0,T_\bullet]\times\Omega; H^1(\mathbb{R}^3))}\right). \end{aligned}$$

By virtue of (2.4.1), the right-hand side is uniformly bounded in k, whence we have for a subsequence possibly depending on l,

$$\int_{\mathbb{R}^3} \zeta(v) \left(\eta_l f_{k_j}^\alpha\right)(\cdot, \cdot, v) \, dv \overset{j\to\infty}{\longrightarrow} \int_{\mathbb{R}^3} \zeta(v) \left(\eta_l f^\alpha\right)(\cdot, \cdot, v) \, dv$$

in $L^2([0, T_\bullet] \times \Omega)$ due to Lemma 1.4.2. Assuming that all $\psi^\alpha \in \Psi_{T_\bullet}$ factorize, i.e., $\psi^\alpha(t, x, v) = \psi_1^\alpha(t, x)\psi_2^\alpha(v)$, and using (2.4.3), we may now pass to the limit in all terms along a common subsequence, that is,

$$\mathcal{G}(y, u)\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right) = \lim_{j\to\infty} \mathcal{G}\left(y_{k_j}, u_{k_j}\right)\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right),$$

via the same diagonal sequence argument as in Section 1.4.2 or the proof of Theorem 2.2.1. Since the limit on the left-hand side does not depend on the extraction of this subsequence, we conclude that the equality above even holds for the full limit $k \longrightarrow \infty$ by using the standard subsubsequence argument. Thus,

$$\left|\mathcal{G}(y, u)\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right)\right| \le \liminf_{k\to\infty} \left\|\mathcal{G}(y_k, u_k)\right\|_{\Lambda^*} \left\|\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right)\right\|_\Lambda.$$

This inequality then also holds for general $\left((\psi^\alpha)_\alpha, \vartheta^e, \vartheta^h\right) \in \Lambda$ by a density argument; see Section 1.4 and the definition of Λ. Altogether, (2.4.2) is proved. □

Remark 2.4.2. It is important to understand the necessity of (2.4.1) for Lemma 2.4.1 and for later treating (P_s): In the proof of Theorem 2.2.1, we applied the momentum averaging lemma 1.4.2 to a sequence where any f_k^α *already solves* a Vlasov equation in the sense of distributions, that is,

$$\partial_t f_k^\alpha + \widehat{v}_\alpha \cdot \partial_x f_k^\alpha = -\operatorname{div}_v\left(F_k f_k^\alpha\right),$$

which gave us a direct estimate on the $L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))$-norm of some ηf_k^α by the corresponding a priori L^p-bounds on F_k and f_k^α. However, the f^α of some (y,u) that is feasible for $(\mathrm{P_s})$ do not necessarily solve a Vlasov equation as above. Thus, suitable estimates on the $L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))$-norm along some sequence cannot be obtained without imposing them a priori, that is, imposing (2.4.1). Without this, we would not be able to pass to the limit as in the proof above, and the important weak lower semicontinuity of $\|\mathcal{G}\|_{\Lambda^*}$ could not be proved.

Now we are able to prove existence of minimizers of $(\mathrm{P_s})$.

Theorem 2.4.3. *There is a (not necessarily unique) minimizer of* $(\mathrm{P_s})$.

Proof. This is proved in much the same way as Theorem 2.2.1 was proved. We no longer have to show that (VM) has to be preserved in the limit. Instead, we apply Lemma 2.4.1: The assumptions there are satisfied for a minimizing sequence (after extracting a suitable subsequence) and the respective weak limits. Thus, the new constraint (2.4.1) is also preserved in the limit, and the new objective function $\mathcal{J}_s$ indeed attains its minimum at the limit tuple (y,u). □

Later, we will need that $\mathcal{Y}\times\mathcal{U}$ is complete; this is proved in the following lemma.

Lemma 2.4.4. $\mathcal{Y}\times\mathcal{U}$ *is a Banach space.*

Proof. We only have to show completeness of $\mathcal{Y}_{\mathrm{pd}}^\alpha$: Let (f_k) be a Cauchy sequence in $\mathcal{Y}_{\mathrm{pd}}^\alpha$. Clearly, this sequence converges to some f with respect to the $L^1_{\alpha\mathrm{kin}}$- and L^∞-norm. For some $\eta\in C_c^\infty(]0,T_\bullet[\times\Omega\times\mathbb{R}^3)$, the sequence $(\partial_t(\eta f_k)+\widehat{v}_\alpha\cdot\partial_x(\eta f_k))$ converges to some g in $L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))$ since this space is complete. As in Step 1 of the proof of Lemma 2.4.1, we see that $g=\partial_t(\eta f)+\widehat{v}_\alpha\cdot\partial_x(\eta f)$. If η satisfies (2.1.4), then

$$\begin{aligned}
&\left\|\partial_t(\eta(f-f_k))+\widehat{v}_\alpha\cdot\partial_x(\eta(f-f_k))\right\|_{L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))}\\
&\le\left\|\partial_t(\eta(f-f_m))+\widehat{v}_\alpha\cdot\partial_x(\eta(f-f_m))\right\|_{L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))}\\
&\quad+\left\|\partial_t(\eta(f_m-f_k))+\widehat{v}_\alpha\cdot\partial_x(\eta(f_m-f_k))\right\|_{L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))}\\
&\le\left\|\partial_t(\eta(f-f_m))+\widehat{v}_\alpha\cdot\partial_x(\eta(f-f_m))\right\|_{L^2([0,T_\bullet]\times\Omega;H^{-1}(\mathbb{R}^3))}+\mathcal{N}_\alpha(f_m-f_k)
\end{aligned}$$

for any $k,m\in\mathbb{N}$. Here, the second summand of the right-hand side can be made arbitrarily small (uniformly in η) for large k and m because of the Cauchy property, and the first summand is arbitrarily small if $m=m(\eta)$ is large enough. Thus, (f_k) converges to f in the whole $\mathcal{Y}_{\mathrm{pd}}^\alpha$-norm altogether. □

Next, we want to derive first order optimality conditions for a minimizer of $(\mathrm{P_s})$. To this end, we consider the differentiability of the objective function $\mathcal{J}_s$. Clearly, the only difficult term is $\|\mathcal{G}(y,u)\|_{\Lambda^*}^2$. To tackle this one, we state a duality result, which links differentiability of a norm to uniform convexity of the dual space.

Proposition 2.4.5. *A Banach space X is uniformly smooth if and only if X^* is uniformly convex. In this case, for each unit vector $x \in X$ there is exactly one $x^* \in X^*$ with $\|x^*\|_{X^*} = 1$ satisfying $x^*x = 1$. Furthermore, this x^* is the derivative of the norm at x.*

Here, "uniformly smooth" means that

$$\lim_{t\to 0} \frac{\|x+ty\|_X - \|x\|_X}{t}$$

exists and is uniform in $x, y \in \{z \in X \mid \|z\|_X = 1\}$. The original work in this subject was done by Day [Day44]; see also [Lin04, Chapter 2] and [Bre11, Section 3.7, Problem 13] for an overview of different concepts of and relations between convexity and smoothness of normed spaces.

From Proposition 2.4.5 we easily get the following corollary, which we will need in the following.

Corollary 2.4.6. *Let X be a Banach space such that X^* is uniformly convex. Then the map $z\colon X \to \mathbb{R}$, $z(x) = \frac{1}{2}\|x\|_X^2$ is differentiable on X with derivative $z'(x) = x^*$ where x^* is the unique element of X^* satisfying $\|x^*\|_{X^*} = \|x\|_X$ and $x^*x = \|x\|_X^2$. (The map $z'\colon X \to X^*$ is often referred to as the duality map.)*

Proof. By Proposition 2.4.5, the norm is differentiable on the unit sphere of X. Since the norm is positive homogeneous, this holds true on X except in $x = 0$, and the derivative is x^* such that $\|x^*\|_{X^*} = 1$ and $x^*x = \|x\|_X$ (still this x^* is uniquely determined by these two properties). Applying the chain rule we see that z is differentiable on $X \setminus \{0\}$ and has the asserted derivative.

That z is differentiable in $x = 0$ and $z'(0) = 0$ is clear. □

With this corollary we see that the objective function $\mathcal{J}_s$ is differentiable.

Lemma 2.4.7. *The objective function $\mathcal{J}_s$ is differentiable, and its derivative is given by*

$$\begin{aligned}
&\mathcal{J}_s'(y,u)(\delta y, \delta u)\\
&= \sum_{\alpha=1}^N w_\alpha \int_{\gamma_{T_\bullet}^+} \operatorname{sign}(f_+^\alpha)\left|f_+^\alpha\right|^{q-1} \delta f_+^\alpha \, d\gamma_\alpha\\
&\quad + \sum_{j=1}^3 \int_0^{T_\bullet} \int_\Gamma \Bigg(\operatorname{sign}(u_j)\left|u_j\right|^{r-1} \delta u_j + \kappa_1 \operatorname{sign}(\partial_t u_j)\left|\partial_t u_j\right|^{r-1} \partial_t \delta u_j\\
&\qquad\qquad + \kappa_2 \sum_{i=1}^3 \operatorname{sign}(\partial_{x_i} u_j)\left|\partial_{x_i} u_j\right|^{r-1} \partial_{x_i} \delta u_j \Bigg)\, dx dt\\
&\quad + \sum_{\alpha=1}^N \Bigg(- \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} \big((\partial_t \psi^\alpha + \widehat{v}_\alpha \cdot \partial_x \psi^\alpha + q_\alpha (E + \widehat{v}_\alpha \times H) \cdot \partial_v \psi^\alpha) \delta f^\alpha\\
&\qquad\qquad + q_\alpha (\delta E + \widehat{v}_\alpha \times \delta H) f^\alpha \cdot \partial_v \psi^\alpha \big)\, dv dx dt
\end{aligned}$$

$$+\int_{\gamma_{T_\bullet}^+} \delta f_+^\alpha \psi^\alpha \, d\gamma_\alpha - \int_{\gamma_{T_\bullet}^-} \left(\mathcal{K}_\alpha \delta f_+^\alpha\right) \psi^\alpha \, d\gamma_\alpha \Bigg)$$

$$+ \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\varepsilon \delta E \cdot \partial_t \vartheta^e - \delta H \cdot \operatorname{curl}_x \vartheta^e - 4\pi\left(\delta j^{\mathrm{int}} + \delta u\right) \cdot \vartheta^e\right) dx dt$$

$$+ \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(\mu \delta H \cdot \partial_t \vartheta^h + \delta E \cdot \operatorname{curl}_x \vartheta^h\right) dx dt, \tag{2.4.4}$$

where $\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right) \in \Lambda$ *is the unique element in* Λ *satisfying*

$$\left\|\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right)\right\|_\Lambda = s\left\|\mathcal{G}(y,u)\right\|_{\Lambda^*}, \quad \mathcal{G}(y,u)\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right) = s\left\|\mathcal{G}(y,u)\right\|_{\Lambda^*}^2. \tag{2.4.5}$$

Proof. The only difficult term is $\frac{s}{2}\left\|\mathcal{G}(y,u)\right\|_{\Lambda^*}^2$. The other terms are easy to handle in a standard way.

Denoting $Z(y,u) = \frac{s}{2}\left\|\mathcal{G}(y,u)\right\|_{\Lambda^*}^2$ we apply Lemma 2.3.4 and Corollary 2.4.6. The latter is applicable since the dual of Λ^*, that is, $\Lambda^{**} \cong \Lambda$, is uniformly convex due to Lemma 2.3.3. At this point we should mention that this step is exactly the reason why we work with a uniformly convex, reflexive test function space. Hence, additionally using the chain rule, we see that Z is differentiable with

$$Z'(y,u)(\delta y, \delta u) = s\lambda^{**}\mathcal{G}'(y,u)(\delta y, \delta u) \tag{2.4.6}$$

where $\lambda^{**} \in \Lambda^{**}$ uniquely satisfies

$$\|\lambda^{**}\|_{\Lambda^{**}} = \left\|\mathcal{G}(y,u)\right\|_{\Lambda^*}, \quad \lambda^{**}\mathcal{G}(y,u) = \left\|\mathcal{G}(y,u)\right\|_{\Lambda^*}^2. \tag{2.4.7}$$

Since Λ is reflexive, we can regard λ^{**} as a $\lambda \in \Lambda$ via the canonical isomorphism. We define $\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right)$ by multiplying this λ with the positive number s. On the one hand, from (2.4.6) we get the remaining part of (2.4.4), that is,

$$\mathcal{G}'(y,u)(\delta y, \delta u)\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right),$$

which is given by (2.3.4). On the other hand, (2.4.7) instantly yields (2.4.5). □

Remark 2.4.8. Such a $\left(\left(\psi^\alpha\right)_\alpha, \vartheta^e, \vartheta^h\right)$ will later act as a Lagrangian multiplier with respect to the Vlasov–Maxwell system, that is, a solution of the adjoint system, if the point (y,u) is a minimizer of (P_s) or, later, of (P). In general, when one has a differentiable control-to-state operator $u \mapsto y(u)$ at hand (which we do not have in our case), computing the adjoint state as the solution of the adjoint system, which is a part of the first order optimality conditions, is an efficient way to compute the total derivative $\frac{d}{du}\mathcal{J}(y(u),u)$ when trying to find a minimizer numerically; see [Hin+09, Section 1.6.2], for example.

Next, we derive necessary first order optimality conditions for (P_s). To tackle an optimization problem with certain constraints and to prove existence of Lagrangian multipliers with respect to them, one has to verify some constraint qualification. To this end, we state a famous result of Zowe and Kurcyusz [ZK79], which is based on a fundamental work of Robinson [Rob76].

Proposition 2.4.9. *Let X, Y be Banach spaces, $S \subset X$ nonempty, closed, and convex, $Q \subset Y$ a closed convex cone (Q is a "cone" means $0 \in Q$, $x \in Q \Rightarrow \forall \lambda > 0 : \lambda x \in Q$), $\phi\colon X \to \mathbb{R}$ differentiable, and $g\colon X \to Y$ continuously differentiable. Denote for $A \subset X$ (and similarly for $A \subset Y$)*

$$A^+ = \{x^* \in X^* \mid \forall a \in A : x^* a \geq 0\}$$

and denote for $x \in X$ and $y \in Y$

$$\begin{aligned} S_x &= \{\lambda(c - x) \mid c \in S, \lambda \geq 0\}, \\ Q_y &= \left\{k - \lambda y \mid k \in Q, \lambda \geq 0\right\}. \end{aligned}$$

Let $x_* \in X$ be a local minimizer (i.e., a local minimizer of the objective function restricted to all feasible points) of the problem

$$\begin{aligned} \min_{x \in X} \quad & \phi(x) \\ \text{s.t.} \quad & x \in S,\ g(x) \in Q, \end{aligned}$$

and let the constraint qualification

$$g'(x_*) S_{x_*} - Q_{g(x_*)} = Y \tag{CQ}$$

hold.

Then there is a Lagrange multiplier $y^* \in Y^*$ at x_* for the problem above, i.e.,

(i) $y^* \in Q^+$,
(ii) $y^* g(x_*) = 0$,
(iii) $\phi'(x_*) - y^* \circ g'(x_*) \in S^+_{x_*}$.

We apply this result to our problem ($\mathrm{P_s}$). As we have shown in Lemma 2.4.7, the objective function is differentiable. In the following, let

$$\begin{aligned} S &:= \left\{ (y, u) \in \mathcal{Y} \times \mathcal{U} \mid 0 \leq f^\alpha \leq \left\| \mathring{f}^\alpha \right\|_{L^\infty(\Omega \times \mathbb{R}^3)} \text{ a.e.}, \mathcal{N}_\alpha\left(f^\alpha\right) \leq \mathcal{L}_\alpha, \alpha = 1, \ldots, N \right\} \\ &\quad \subset \mathcal{Y} \times \mathcal{U} =: X, \\ Q &:= \mathbb{R}_{\geq 0} \subset \mathbb{R} =: Y. \end{aligned}$$

Clearly, S is nonempty, closed, and convex, and Q is a closed convex cone. Furthermore, the constraints (2.1.1), (2.1.2), and (2.4.1) are equivalent to

$$(y, u) \in S,\ g(y, u) \in Q,$$

where

$$g(y,u) = \mathcal{I}(u) - \sum_{\alpha=1}^{N} \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 f^\alpha \, dv dx dt - \frac{\sigma}{8\pi} \int_0^{T_\bullet} \int_{\mathbb{R}^3} \left(|E|^2 + |H|^2\right) dx dt.$$

It is easy to see that g is continuously differentiable with

$$g'(y,u)(\delta y, \delta u) = \beta \int_0^{T_\bullet} \int_\Gamma u \cdot \delta u \, dx dt - \sum_{\alpha=1}^{N} \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \delta f^\alpha \, dv dx dt$$
$$- \frac{\sigma}{4\pi} \int_0^{T_\bullet} \int_{\mathbb{R}^3} (E \cdot \delta E + H \cdot \delta H) \, dx dt,$$

where

$$\beta := 4\pi T_\bullet^2 \sigma^{-1}.$$

We verify the constraint qualification (CQ).

Lemma 2.4.10. *Let* (y_s, u_s) *be a (global) minimizer of* (P$_\mathrm{s}$)*. Then,* (CQ) *is satisfied if* s *is sufficiently large.*

Proof. First, we exclude the possibility that some f_s^α is identically zero for s sufficiently large (since then the term $\frac{s}{2}\|\mathcal{G}(y_s, u_s)\|_{\Lambda^*}^2$ is too large for (y_s, u_s) to be a minimizer of (P$_\mathrm{s}$)): For each α, let $\psi_*^\alpha \colon [0, T_\bullet] \times \Omega \times \mathbb{R}^3 \to \mathbb{R}$, $\psi_*^\alpha(t,x,v) = \eta(t)\varphi^\alpha(x,v)$, where $\eta \in C^\infty([0, T_\bullet])$ with $\eta(0) = 1$ and $\operatorname{supp} \eta \subset [0, T_\bullet[$, and $\varphi^\alpha \in C_c^\infty(\Omega \times \mathbb{R}^3)$ with

$$\left\|\mathring{f}^\alpha - \varphi^\alpha\right\|_{L^2(\Omega\times\mathbb{R}^3)} \leq \frac{1}{2}\left\|\mathring{f}^\alpha\right\|_{L^2(\Omega\times\mathbb{R}^3)}.$$

Clearly, $\psi_*^\alpha \in \Psi_{T_\bullet}$. Now assume $f_s^{\alpha_0} = 0$ for some α_0. We have

$$\left|\mathcal{G}(y_s, u_s)\left((0, \ldots, 0, \psi_*^{\alpha_0}, 0, \ldots, 0), 0, 0\right)\right| = \left|\int_\Omega \int_{\mathbb{R}^3} \mathring{f}^{\alpha_0} \varphi^{\alpha_0} \, dv dx\right|$$
$$= \left|\left\|\mathring{f}^{\alpha_0}\right\|_{L^2(\Omega\times\mathbb{R}^3)}^2 - \int_\Omega \int_{\mathbb{R}^3} \mathring{f}^{\alpha_0}\left(\mathring{f}^{\alpha_0} - \varphi^{\alpha_0}\right) dv dx\right| \geq \frac{1}{2}\left\|\mathring{f}^{\alpha_0}\right\|_{L^2(\Omega\times\mathbb{R}^3)}^2.$$

Then,

$$\|\mathcal{G}(y_s, u_s)\|_{\Lambda^*} \geq \frac{\left\|\mathring{f}^{\alpha_0}\right\|_{L^2(\Omega\times\mathbb{R}^3)}^2}{2\left\|\psi_*^{\alpha_0}\right\|_{W^{1,p,\tilde{q}}}},$$

and

$$\mathcal{J}_s(y_s, u_s) \geq s \cdot \frac{\left\|\mathring{f}^{\alpha_0}\right\|_{L^2(\Omega\times\mathbb{R}^3)}^4}{8\left\|\psi_*^{\alpha_0}\right\|_{W^{1,p,\tilde{q}}}^2} > \mathcal{J}(y_*, u_*) = \mathcal{J}_s(y_*, u_*), \tag{2.4.8}$$

where (y_*, u_*) is a minimizer of (P) and where the strict inequality holds for s sufficiently large, i.e.,

$$s > \max_{\alpha=1,\dots,N} \frac{8\|\psi_*^\alpha\|^2_{W^{1,p,\tilde{q}}} \mathcal{J}(y_*, u_*)}{\left\|\mathring{f}^\alpha\right\|^4_{L^2(\Omega\times\mathbb{R}^3)}};$$

note that the right-hand side does not depend on s and α_0 and that no $\mathring{f}^\alpha$ is identically zero. Since (y_*, u_*) is feasible for ($\mathrm{P_s}$), (2.4.8) is a contradiction to (y_s, u_s) being a minimizer of ($\mathrm{P_s}$).

To prove the lemma, we have to show that for each $d \in \mathbb{R}$ there are $\lambda_1, \lambda_2 \geq 0$, $k \geq 0$, and $(\delta y, \delta u) \in S$ satisfying

$$\lambda_1 g'(y_s, u_s)(\delta y - y_s, \delta u - u_s) - k + \lambda_2 g(y_s, u_s) = d. \tag{2.4.9}$$

We choose $\delta f_+^\alpha = f_{s,+}^\alpha$ for all α, $\delta E = E_s$, $\delta H = H_s$, $\delta u = u_s$, and consider two cases; note that in the following it always holds that $\lambda_1, \lambda_2 \geq 0$, $k \geq 0$, and $(\delta y, \delta u) \in S$:

Case 1. $d \leq 0$: Choose $\lambda_1 = \lambda_2 = 0$, $\delta f^\alpha = f_s^\alpha$ for all α, $k = -d$.

Case 2. $d > 0$: Choose $\lambda_2 = 0$, $\delta f^1 = 0$, $\delta f^\alpha = f_s^\alpha$ for $\alpha \geq 2$, $k = 0$. Since

$$g'(y_s, u_s)(\delta y - y_s, \delta u - u_s) = \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} v_1^0 f_s^1 \, dv dx dt > 0,$$

we can choose $\lambda_1 > 0$ such that (2.4.9) is satisfied.

In all cases (2.4.9) holds; the proof is complete. □

Now, Proposition 2.4.9 gives us the following theorem.

Theorem 2.4.11. *Let s be sufficiently large and (y_s, u_s) a minimizer of* ($\mathrm{P_s}$). *Then there exist $\nu_s \geq 0$ and $\tau_s^\alpha \in \left(\mathcal{Y}_{\mathrm{pd}}^\alpha\right)^*$, $\alpha = 1, \dots, N$, such that:*

(i) $\nu_s = 0$ *or* $g(y_s, u_s) = 0$.

(ii)

$$\sum_{\alpha=1}^N \tau_s^\alpha f_s^\alpha \leq \sum_{\alpha=1}^N \tau_s^\alpha \delta f^\alpha$$

for all $\delta f^\alpha \in \mathcal{Y}_{\mathrm{pd}}^\alpha$ *satisfying* $0 \leq \delta f^\alpha \leq \left\|\mathring{f}^\alpha\right\|_{L^\infty(\Omega\times\mathbb{R}^3)}$ *a.e. and* $\mathcal{N}_\alpha(\delta f^\alpha) \leq \mathcal{L}_\alpha$.

(iii) For all $(\delta y, \delta u) \in \mathcal{Y} \times \mathcal{U}$ *it holds that*

$$0 = \sum_{\alpha=1}^N w_\alpha \int_{\gamma_{T_\bullet}^+} \mathrm{sign}(f_{s,+}^\alpha)\left|f_{s,+}^\alpha\right|^{q-1} \delta f_+^\alpha \, d\gamma_\alpha$$
$$+ \sum_{j=1}^3 \int_0^{T_\bullet} \int_\Gamma \Bigg(\mathrm{sign}(u_{s,j})\left|u_{s,j}\right|^{r-1} \delta u_j + \kappa_1 \, \mathrm{sign}(\partial_t u_{s,j})\left|\partial_t u_{s,j}\right|^{r-1} \partial_t \delta u_j$$

$$
\begin{aligned}
&\qquad\qquad +\kappa_2 \sum_{i=1}^{3} \operatorname{sign}\big(\partial_{x_i} u_{s,j}\big)\big|\partial_{x_i} u_{s,j}\big|^{r-1} \partial_{x_i} \delta u_j \Bigg) \, dxdt \\
&+ \sum_{\alpha=1}^{N} \Bigg(- \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} \big(\big(\partial_t \psi_s^\alpha + \widehat{v}_\alpha \cdot \partial_x \psi_s^\alpha + q_\alpha (E_s + \widehat{v}_\alpha \times H_s) \cdot \partial_v \psi_s^\alpha \big) \delta f^\alpha \\
&\qquad\qquad + q_\alpha (\delta E + \widehat{v}_\alpha \times \delta H) f_s^\alpha \cdot \partial_v \psi_s^\alpha \big) \, dv dx dt \\
&\qquad + \int_{\gamma_{T_\bullet}^+} \delta f_+^\alpha \psi_s^\alpha \, d\gamma_\alpha - \int_{\gamma_{T_\bullet}^-} \big(\mathcal{K}_\alpha \delta f_+^\alpha\big) \psi_s^\alpha \, d\gamma_\alpha \Bigg) \\
&+ \int_0^{T_\bullet} \int_{\mathbb{R}^3} \big(\varepsilon \delta E \cdot \partial_t \vartheta_s^e - \delta H \cdot \operatorname{curl}_x \vartheta_s^e - 4\pi \big(\delta j^{\text{int}} + \delta u\big) \cdot \vartheta_s^e \big) \, dxdt \\
&+ \int_0^{T_\bullet} \int_{\mathbb{R}^3} \Big(\mu \delta H \cdot \partial_t \vartheta_s^h + \delta E \cdot \operatorname{curl}_x \vartheta_s^h \Big) \, dxdt \\
&+ \nu_s \beta \int_0^{T_\bullet} \int_\Gamma u_s \cdot \delta u \, dxdt - \nu_s \sum_{\alpha=1}^{N} \int_0^{T_\bullet} \int_\Omega \int_{\mathbb{R}^3} v_\alpha^0 \delta f^\alpha \, dv dx dt \\
&- \frac{\nu_s \sigma}{4\pi} \int_0^{T_\bullet} \int_{\mathbb{R}^3} (E_s \cdot \delta E + H_s \cdot \delta H) \, dxdt - \sum_{\alpha=1}^{N} \tau_s^\alpha \delta f^\alpha
\end{aligned} \tag{2.4.10}
$$

where $\big(\big(\psi_s^\alpha\big)_\alpha, \vartheta_s^e, \vartheta_s^h\big) \in \Lambda$ *is, in accordance with* (2.4.5), *given by*

$$
\begin{aligned}
\Big\| \Big(\big(\psi_s^\alpha\big)_\alpha, \vartheta_s^e, \vartheta_s^h \Big) \Big\|_\Lambda &= s \big\| \mathcal{G}(y_s, u_s) \big\|_{\Lambda^*}, \\
\mathcal{G}(y_s, u_s) \Big(\big(\psi_s^\alpha\big)_\alpha, \vartheta_s^e, \vartheta_s^h \Big) &= s \big\| \mathcal{G}(y_s, u_s) \big\|_{\Lambda^*}^2.
\end{aligned}
$$

In other words, (2.4.10) *can be interpreted as* $\big(\big(\psi_s^\alpha\big)_\alpha, \vartheta_s^e, \vartheta_s^h\big)$ *being a solution of the adjoint system*

$$\partial_t \psi_s^\alpha + \widehat{v}_\alpha \cdot \partial_x \psi_s^\alpha + q_\alpha (E_s + \widehat{v}_\alpha \times H_s) \cdot \partial_v \psi_s^\alpha = 4\pi \widehat{v}_\alpha \cdot \vartheta_s^e + \nu_s v_\alpha^0 + \tau_s^\alpha \quad \text{on } [0, T_\bullet] \times \Omega \times \mathbb{R}^3, \tag{Ad$_\text{s}$.1}$$

$$K a^\alpha K \psi_{s,-}^\alpha = \psi_{s,+}^\alpha + w_\alpha \operatorname{sign}\big(f_{s,+}^\alpha\big) \big|f_{s,+}^\alpha\big|^{q-1} \quad \text{on } \gamma_{T_\bullet}^+, \tag{Ad$_\text{s}$.2}$$

$$\psi_s^\alpha(T_\bullet) = 0 \quad \text{on } \Omega \times \mathbb{R}^3, \tag{Ad$_\text{s}$.3}$$

$$\varepsilon \partial_t \vartheta_s^e + \operatorname{curl}_x \vartheta_s^h = - \sum_{\alpha=1}^{N} q_\alpha \int_{\mathbb{R}^3} f_s^\alpha \partial_v \psi_s^\alpha \, dv - \frac{\nu_s \sigma}{4\pi} E_s \quad \text{on } [0, T_\bullet] \times \mathbb{R}^3, \tag{Ad$_\text{s}$.4}$$

$$\mu \partial_t \vartheta_s^h - \operatorname{curl}_x \vartheta_s^e = - \sum_{\alpha=1}^{N} q_\alpha \int_{\mathbb{R}^3} f_s^\alpha \big(\partial_v \psi_s^\alpha \times \widehat{v}_\alpha\big) \, dv - \frac{\nu_s \sigma}{4\pi} H_s \quad \text{on } [0, T_\bullet] \times \mathbb{R}^3, \tag{Ad$_\text{s}$.5}$$

$$\vartheta_s^e(T_\bullet) = \vartheta_s^h(T_\bullet) = 0 \quad \text{on } \mathbb{R}^3, \tag{Ad$_\text{s}$.6}$$

and the stationarity condition

$$
\begin{aligned}
0 = \sum_{j=1}^{3} \int_0^{T_\bullet} \int_\Gamma \Bigg(& \operatorname{sign}(u_{s,j}) \big|u_{s,j}\big|^{r-1} \delta u_j + \kappa_1 \operatorname{sign}(\partial_t u_{s,j}) \big|\partial_t u_{s,j}\big|^{r-1} \partial_t \delta u_j \\
& + \kappa_2 \sum_{i=1}^{3} \operatorname{sign}(\partial_{x_i} u_{s,j}) \big|\partial_{x_i} u_{s,j}\big|^{r-1} \partial_{x_i} \delta u_j \Bigg) \, dx dt \\
& - \int_0^{T_\bullet} \int_\Gamma \big(4\pi \vartheta_s^e - \nu_s \beta u_s\big) \cdot \delta u \, dx dt \qquad \text{for all } \delta u \in \mathcal{U} \qquad (\mathrm{SC_s})
\end{aligned}
$$

being satisfied.

Proof. Since (CQ) holds due to Lemma 2.4.10 and $\mathcal{Y} \times \mathcal{U}$ is a Banach space due to Lemma 2.4.4, by Proposition 2.4.9 there is $\nu_s \in \mathbb{R}$ acting as a Lagrangian multiplier with respect to (2.1.2). Proposition 2.4.9.(i) implies $\nu_s \geq 0$, and Proposition 2.4.9.(ii) yields part 2.4.11.(i).

With Proposition 2.4.9.(iii) and the notation used there we see that

$$
\tau_s := \mathcal{J}_s'(y_s, u_s) - \nu_s \cdot g'(y_s, u_s) \in S^+_{(y_s, u_s)} \subset (\mathcal{Y} \times \mathcal{U})^*. \tag{2.4.12}
$$

Consequently, τ_s can be decomposed into

$$
\begin{aligned}
\tau_s &\equiv \Big((\tau_s^\alpha)_\alpha, \big(\tau_{s,+}^\alpha\big)_\alpha, \tau_s^e, \tau_s^h, \tau_s^u \Big) \\
&\in \left(\bigtimes_{\alpha=1}^{N} \Big(\mathcal{Y}_{\mathrm{pd}}^\alpha\Big)^* \times L^q\Big(\gamma_{T_\bullet}^+, d\gamma_\alpha\Big)^* \right) \times \Big(L^2\big([0, T_\bullet] \times \mathbb{R}^3; \mathbb{R}^3\big)^* \Big)^2 \times \mathcal{U}^*.
\end{aligned}
$$

Since the set $S_{(y_s, u_s)}$ only limits the directions δf^α and not the directions δf_+^α, δE, δH, and δu, the property $\tau_s \in S^+_{(y_s, u_s)}$ yields that all $\tau_{s,+}^\alpha$ and moreover τ_s^e, τ_s^h, and τ_s^u have to vanish. Thus, $\tau_s \equiv (\tau_s^\alpha)_\alpha$ via

$$
\tau_s(\delta y, \delta u) = \sum_{\alpha=1}^{N} \tau_s^\alpha \delta f^\alpha. \tag{2.4.13}
$$

On the one hand, by $\tau_s \in S^+_{(y_s, u_s)}$ and the identification (2.4.13) we have for all $\delta f^\alpha \in \mathcal{Y}_{\mathrm{pd}}^\alpha$ satisfying $0 \leq \delta f^\alpha \leq \big\|\mathring{f}^\alpha\big\|_{L^\infty(\Omega \times \mathbb{R}^3)}$ a.e. and $\mathcal{N}_\alpha(\delta f^\alpha) \leq \mathcal{L}_\alpha$,

$$
\sum_{\alpha=1}^{N} \tau_s^\alpha \big(\delta f^\alpha - f_s^\alpha\big) \geq 0,
$$

which is part 2.4.11.(ii). On the other hand, (2.4.12) and (2.4.13) instantly yields (2.4.10) recalling the formula for $\mathcal{J}_s'$ from Lemma 2.4.7.

Setting δu and all but one of the directions δf^α, δf_+^α, δE, and δH to zero and the one remaining arbitrary, we conclude that the adjoint system ($\mathrm{Ad_s}$) holds. Note that a priori the ψ_s^α, ϑ_s^e, and ϑ_s^h vanish for $t = T_\bullet$ by definition of the test function space Λ.

Finally, setting all directions but δu to zero yields ($\mathrm{SC_s}$). Thus, also the proof of part 2.4.11.(iii) is complete. □

Remark 2.4.12. If, for example, $r = 2$ and the boundary of Γ is smooth, ($\mathrm{SC_s}$) can easily be interpreted as the weak form of the second order PDE

$$\begin{aligned} \kappa_1 \partial_t^2 u_s + \kappa_2 \Delta_x u_s &= -4\pi \vartheta_s^e + \big(\nu_s \beta + 1\big) u_s && \text{on } [0, T_\bullet] \times \Gamma, \\ \partial_t u_s(0) = \partial_t u_s(T_\bullet) &= 0 && \text{on } \Gamma, \\ \partial_{n_\Gamma} u_s &= 0 && \text{on } [0, T_\bullet] \times \partial\Gamma. \end{aligned}$$

Here, ∂_{n_Γ} denotes the directional derivative in the direction of the outer unit normal n_Γ of $\partial\Gamma$.

2.4.2 Passing to the limit

There remains to pass to the limit $s \to \infty$. A natural approach is to try to pass to the limit in the optimality conditions of ($\mathrm{P_s}$). This would require boundedness of the adjoint state in a certain norm. To this end, typically one needs to exploit some compactness result for the linearized PDE (system). In many situations, such results are available, and one can then verify that the optimality conditions also hold in the limit, i.e., for a minimizer of the original problem. We refer to [Lio85] for an abundance of examples of such PDEs.

However, for the Vlasov–Maxwell system no such results are available. In the author's opinion, the most problematic terms are the source terms on the right-hand side of ($\mathrm{Ad_s}$.4) and ($\mathrm{Ad_s}$.5) which include $\partial_v \psi_s^\alpha$, i.e., a derivative of the adjoint state. This is a structural problem arising because of the Vlasov–Maxwell system. Conversely, there are artificial problems, that is, the appearance of ν_s and τ_s^α. They only appear because it is unknown whether the artificial constraints (2.1.1) and (2.1.2) in (P) (or then (2.1.1), (2.1.2), and (2.4.1) in ($\mathrm{P_s}$)) are automatically satisfied for any weak solution of (VM) (or for a minimizing sequence of ($\mathrm{P_s}$)). Especially τ_s^α is very irregular and there are no weak compactness results for the space which τ_s^α lies in. Moreover, to gain compactness via some momentum averaging lemma seems not possible since the right-hand side of ($\mathrm{Ad_s}$.1) (in particular, $4\pi \widehat{v}_\alpha \cdot \vartheta_s^e$) is not square integrable over $[0, T_\bullet] \times \Omega \times \mathbb{R}^3$.

Thus, we are not able to prove that a minimizer of (P) satisfies the desired optimality conditions, i.e., ($\mathrm{Ad_s}$) and ($\mathrm{SC_s}$) with s removed. Nevertheless, the following holds.

Theorem 2.4.13. *For each $s > 0$, let $\big(y_s, u_s\big) \in \mathcal{Y} \times \mathcal{U}$ be a minimizer of* ($\mathrm{P_s}$). *Then*

$$\big\|\mathcal{G}\big(y_s, u_s\big)\big\|_{\Lambda^*} \leq \sqrt{2 \min \mathcal{J}}\, s^{-\frac{1}{2}}, \tag{2.4.14}$$

and there is a minimizer $\big(y_*, u_*\big) \in \mathcal{Y} \times \mathcal{U}$ of the original problem (P) *such that, after choosing a suitable sequence $s_k \to \infty$, $f_{s_k}^\alpha \overset{(*)}{\rightharpoonup} f_*^\alpha$ in $L^z\big([0, T_\bullet] \times \Omega \times \mathbb{R}^3\big)$ for $1 < z \leq \infty$, $f_{s_k,+}^\alpha \to f_{*,+}^\alpha$*

in $L^q\left(\gamma^+_{T_\bullet}, d\gamma_\alpha\right)$, $(E_{s_k}, H_{s_k}) \rightharpoonup (E_*, H_*)$ *in* $L^2([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^6)$, *and* $u_{s_k} \to u_*$ *in* $\mathcal{U}$ *for* $k\to\infty$. *Furthermore,*

$$\lim_{k\to\infty} s_k \left\|\mathcal{G}(y_{s_k}, u_{s_k})\right\|^2_{\Lambda^*} = 0.$$

Proof. Let (y,u) be some minimizer of (P). Since this (y,u) is also feasible for ($\mathrm{P_s}$), $\mathcal{G}(y,u) = 0$, and since (y_s, u_s) is a minimizer of ($\mathrm{P_s}$), it holds that

$$\frac{s}{2}\left\|\mathcal{G}(y_s, u_s)\right\|^2_{\Lambda^*} \le \mathcal{J}_s(y_s, u_s) \le \mathcal{J}_s(y,u) = \mathcal{J}(y,u) = \min \mathcal{J}, \tag{2.4.15}$$

which implies (2.4.14) and that (u_s) is bounded in $\mathcal{U}$ and $\left(f^\alpha_{s,+}\right)$ in $L^q\left(\gamma^+_{T_\bullet}, d\gamma_\alpha\right)$. Thus, by (2.1.1) and (2.1.2) each $\left(f^\alpha_s\right)$ is bounded in any $L^z\left([0,T_\bullet]\times\Omega\times\mathbb{R}^3\right)$, $1\le z\le\infty$, and $((E_s, H_s))$ in $L^2\left([0,T_\bullet]\times\mathbb{R}^3;\mathbb{R}^6\right)$. Therefore, the asserted convergences hold true, at least weakly, if the sequence (s_k) is suitably chosen. Since (2.1.2) and (2.4.1) are satisfied along the sequence, we can apply Lemma 2.4.1 to obtain

$$\left\|\mathcal{G}(y_*, u_*)\right\|_{\Lambda^*} \le \liminf_{k\to\infty}\left\|\mathcal{G}(y_{s_k}, u_{s_k})\right\|_{\Lambda^*} = 0$$

because of (2.4.14). Hence, (y_*, u_*) is feasible for (P). By weak lower semicontinuity of any norm, we have

$$\mathcal{J}(y_*, u_*) \le \liminf_{k\to\infty}\mathcal{J}(y_{s_k}, u_{s_k}) \le \liminf_{k\to\infty}\mathcal{J}_{s_k}(y_{s_k}, u_{s_k}) \le \limsup_{k\to\infty}\mathcal{J}_{s_k}(y_{s_k}, u_{s_k}) \le \min\mathcal{J}, \tag{2.4.16}$$

where the last inequality is implied by (2.4.15). Consequently, (y_*, u_*) is indeed a minimizer of (P), and equality holds in (2.4.16). Thus,

$$\mathcal{J}(y_*, u_*) = \liminf_{k\to\infty}\mathcal{J}(y_{s_k}, u_{s_k}) \le \limsup_{k\to\infty}\mathcal{J}(y_{s_k}, u_{s_k}) \le \limsup_{k\to\infty}\mathcal{J}_{s_k}(y_{s_k}, u_{s_k}) = \min\mathcal{J},$$

and also equality holds everywhere. This yields

$$\begin{aligned}\frac{1}{q}\sum_{\alpha=1}^N w_\alpha\left\|f^\alpha_{*,+}\right\|^q_{L^q\left(\gamma^+_{T_\bullet}, d\gamma_\alpha\right)} + \frac{1}{r}\|u_*\|^r_{\mathcal{U}} &= \mathcal{J}(y_*, u_*) = \lim_{k\to\infty}\mathcal{J}(y_{s_k}, u_{s_k}) \\ &= \lim_{k\to\infty}\left(\frac{1}{q}\sum_{\alpha=1}^N w_\alpha\left\|f^\alpha_{s_k,+}\right\|^q_{L^q\left(\gamma^+_{T_\bullet}, d\gamma_\alpha\right)} + \frac{1}{r}\left\|u_{s_k}\right\|^r_{\mathcal{U}}\right).\end{aligned} \tag{2.4.17}$$

Combining (2.4.16) and (2.4.17) implies

$$\lim_{k\to\infty}\frac{s_k}{2}\left\|\mathcal{G}(y_{s_k}, u_{s_k})\right\|^2_{\Lambda^*} = \lim_{k\to\infty}\left(\mathcal{J}_{s_k}(y_{s_k}, u_{s_k}) - \mathcal{J}(y_{s_k}, u_{s_k})\right) = \mathcal{J}(y_*, u_*) - \mathcal{J}(y_*, u_*) = 0.$$

There remains to show that the convergences of $f^{\alpha}_{s_k,+}$ and u_{s_k} are even strong. To this end, suppose that

$$\left\|f^{\alpha_0}_{*,+}\right\|_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} < \limsup_{k\to\infty}\left\|f^{\alpha_0}_{s_k,+}\right\|_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)}$$

for some α_0. By weak lower semicontinuity of the remaining norms and by (2.4.17), this implies

$$\begin{aligned}
&\frac{1}{q}\sum_{\alpha=1}^{N} w_\alpha \left\|f^{\alpha}_{*,+}\right\|^q_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} + \frac{1}{r}\|u_*\|^r_{\mathcal{U}}\\
&< \limsup_{k\to\infty}\left\|f^{\alpha_0}_{s_k,+}\right\|^q_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} + \frac{1}{q}\sum_{\substack{\alpha=1\\ \alpha\neq\alpha_0}}^{N} w_\alpha \liminf_{k\to\infty}\left\|f^{\alpha}_{s_k,+}\right\|^q_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} + \liminf_{k\to\infty}\frac{1}{r}\left\|u_{s_k}\right\|^r_{\mathcal{U}}\\
&\le \limsup_{k\to\infty}\left(\frac{1}{q}\sum_{\alpha=1}^{N} w_\alpha \left\|f^{\alpha}_{s_k,+}\right\|^q_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} + \frac{1}{r}\left\|u_{s_k}\right\|^r_{\mathcal{U}}\right)\\
&= \frac{1}{q}\sum_{\alpha=1}^{N} w_\alpha \left\|f^{\alpha}_{*,+}\right\|^q_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} + \frac{1}{r}\|u_*\|^r_{\mathcal{U}},
\end{aligned}$$

which is a contradiction. Thus,

$$\left\|f^{\alpha}_{*,+}\right\|_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} \le \liminf_{k\to\infty}\left\|f^{\alpha}_{s_k,+}\right\|_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} \le \limsup_{k\to\infty}\left\|f^{\alpha}_{s_k,+}\right\|_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} \le \left\|f^{\alpha}_{*,+}\right\|_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)},$$

whence

$$\left\|f^{\alpha}_{*,+}\right\|_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)} = \lim_{k\to\infty}\left\|f^{\alpha}_{s_k,+}\right\|_{L^q\left(\gamma^+_{T_\bullet},d\gamma_\alpha\right)}$$

for each α. Similarly,

$$\|u_*\|_{\mathcal{U}} = \lim_{k\to\infty}\left\|u_{s_k}\right\|_{\mathcal{U}}$$

as well. Since we already have $f^{\alpha}_{s_k,+} \rightharpoonup f^{\alpha}_{*,+}$ in $L^q\left(\gamma^+_{T_\bullet}, d\gamma_\alpha\right)$ and $u_{s_k} \rightharpoonup u_*$ in $\mathcal{U}$, and since $L^q\left(\gamma^+_{T_\bullet}, d\gamma_\alpha\right)$ and $\mathcal{U}$ are uniformly convex, $f^{\alpha}_{s_k,+}$ even converges strongly to $f^{\alpha}_{*,+}$ and u_{s_k} strongly to u_*. □

Note that the convergences of $f^{\alpha}_{s_k,+}$ and u_{s_k} are strong, which is due to the fact that the original objective function $\mathcal{J}$ is an expression in f^{α}_{+} and u. Since the actual goal is to adjust u suitably and u is the only function which can be really adjusted from outside, it is no big drawback to have to consider ($\mathrm{P_s}$) instead of (P): As we have seen in Theorem 2.4.13, on the one hand $\left\|\mathcal{G}(y_s,u_s)\right\|_{\Lambda^*}$ decays with a certain rate to zero for $s\to\infty$, whence (VM) is "almost" satisfied for a minimizer (y_s,u_s) and s large;

on the other hand, first order optimality conditions for ($\mathrm{P_s}$) have been established in Theorem 2.4.11 and optimal points of ($\mathrm{P_s}$) converge (at least weakly) to an optimal point of (P) (along a suitable sequence), and the convergence of the controls is even strong. We cannot expect to get convergence for the full limit $s \to \infty$ since minimizers of (P) and ($\mathrm{P_s}$) are not known to be unique due to the lack of convexity.

2.5 Final remarks

One can consider other optimal control problems than (P), with a different objective function, for example, a problem of tracking type:

$$\tilde{\mathcal{J}}(y,u) = \mathcal{J}(y,u) + \sum_{\alpha=1}^{N} \frac{b_\alpha}{2} \|f^\alpha - f_d^\alpha\|^2_{L^2([0,T]\times\Omega\times\mathbb{R}^3)} + \frac{b_E}{2}\|E - E_d\|^2_{L^2([0,T]\times\mathbb{R}^3;\mathbb{R}^3)}$$
$$+ \frac{b_H}{2}\|H - H_d\|^2_{L^2([0,T]\times\mathbb{R}^3;\mathbb{R}^3)},$$

where $b_\alpha, b_E, b_H > 0$ are parameters and f_d^α, E_d, H_d are desired states. Since this new objective function already grants coercivity in f^α, E, and H with respect to the L^2-norm, at first sight it seems that the artificial constraint (2.1.2) can be abolished. However, without this constraint, we cannot pass to the limit in the term of (1.1.3a) with j^{int} during an analog proof of Theorem 2.2.1 since for this an $L^1_{\alpha\mathrm{kin}}$-estimate on f^α is necessary; cf. Lemma 2.1.3. Thus, imposing (2.1.1) and (2.1.2) is still necessary. Analogues of Theorems 2.2.1, 2.4.3, 2.4.11, and 2.4.13 can be proved, and in Theorem 2.4.13 the convergences of $f^\alpha_{s_k}$, E_{s_k}, and H_{s_k} are also strong in L^2 because of the tracking terms in the new objective function.

We could also consider the case that we additionally try to control the system by inserting particles from outside, that is, considering nonvanishing g^α in the right-hand side of (VM.2) and treating them as controls as well. Then we add some norm of the g^α to the objective function as a penalization term. There occur two problems: Firstly, since (2.1.1) is still necessary and since we have to include L^∞-norms of the g^α there on the right-hand side, the set of functions satisfying this new constraint is no longer convex. We can bypass this problem by imposing L^∞-bounds on the g^α a priori, for example, by imposing box constraints. Secondly, we have to add the $L^1_{\alpha\mathrm{kin}}$-norms of the g^α to the right-hand side of (2.1.2). To be then able to pass to the limit in (2.1.2), we need that the space the g^α lie in is compactly embedded in $L^1_{\alpha\mathrm{kin}}\left(\gamma^-_{T_\bullet}, d\gamma_\alpha\right)$—this is analogous to the consideration of $\mathcal{U}$ as the control space instead of simply L^2. That compact embedding is, for example, guaranteed by the restriction $g^\alpha \in H^1\left(\gamma^-_{T_\bullet} \cap \{|v| < R\}\right)$ and $g^\alpha = 0$ for $|v| > R$ with $R > 0$ fixed. Another possibility is to impose an a priori bound on the $L^1_{\alpha\mathrm{kin}}$-norms of the g^α, for example, by imposing box constraints as above and a bound on the support of the g^α with respect to v, and then adding this a priori bound to the right-hand side of (2.1.2) instead of the $L^1_{\alpha\mathrm{kin}}$-norms of the g^α.

In Theorem 2.4.13, a suitable sequence of optimal points of ($\mathrm{P_s}$) converges to an optimal point of (P), at least weakly, some components even strongly. However, we do not know if *all* minimizers of (P) can be "obtained" in this way. In [Lio85], usually an approximate problem with an adaptive objective function is considered, in order to derive first order optimality conditions for *any* given, fixed minimizer of (P). Here, this means adding norms of $f^\alpha - f^\alpha_*$, $f^\alpha_+ - f^\alpha_{*,+}$, $E - E_*$, $H - H_*$, and $u - u_*$ to $\mathcal{J}$. With an analogue of Theorem 2.4.13, one can then show that (y_s, u_s) converges strongly to (y_*, u_*) in a suitable norm, and this holds for the *full* limit $s \to \infty$. However, this method is not constructive since one has to know (y_*, u_*) a priori to consider the approximate problem, and thus in our case not reasonable; in general it is reasonable if one can pass to the limit in the first order optimality conditions.

CHAPTER 3

Confined steady states in an infinitely long cylinder

3.1 The set-up

The previous chapter was devoted to the question how to adjust the currents (and thus the external electromagnetic fields) in some external electric coils to confine the plasma as best as possible. With "good confinement" we meant that the amount of the plasma particles hitting the boundary of Ω are to be kept as small as possible, while the control costs should be not too exhaustive. However, one might ask two questions: Firstly, as they were given and fixed through these considerations, what is a reasonable choice of the initial data for the particle densities and the electromagnetic fields? Secondly, is there really a choice of initial data and external currents such that the plasma is really confined during the whole time interval $[0, T_\bullet]$, i.e., such that there are no hits on the boundary? This leads to another question, which we will deal with in this chapter: Is there a configuration, that is independent of time and where the plasma particles are away from the boundary of the fusion reactor?

Before we analyze this problem about the existence of such a configuration, which we henceforth call a "confined steady state", we first discuss the basic ideas for plasma confinement—more information on fusion plasma physics can be found in the classical book of Stacey [Sta12]. The physical basis for confinement is the fact that charged particles spiral about magnetic field lines. The so-called gyroradius, that is, the radius of such a spiral, is inversely proportional to the strength of the magnetic field. This gives rise to the idea of linear confinement devices: The fusion reactor is a long cylinder and the external magnetic field points in the direction of the symmetry axis of this cylinder. If this external magnetic field is sufficiently strong, the gyroradii of the plasma particles will be smaller than the radius of the cylinder, whence the plasma is confined in the fusion device. However, this setting cannot prevent the plasma current from having a nonvanishing component in the direction of the symmetry axis. Thus, there will be losses at the ends of the long cylinder. In practice, one can try to overcome this problem by one of the two following modifications: Firstly, so-called magnetic mirrors are added at these ends. Secondly, the long cylinder is bent into a torus. This second idea is pursued typically in modern research. Toroidal geometry has the advantage of avoiding such losses but has the disadvantage that it gives rise to drifts of the plasma particles, which finally cause the particles to move radially outwards and thus make confinement impossible. Therefore, the external magnetic field needs to have a poloidal component additional to its toroidal one. This approach then leads to Tokamak devices.

However, analyzing the problem of existence of confined steady states from a mathematics point of view in toroidal geometry seems quite hard. We discuss the difficulties in Section 3.6. As a first step towards this, we consider the set-up of a linear confinement device instead. For mathematical reasons, it will be convenient to assume that the cylinder is infinitely long (which is of course not conceivable from a practical point of view). Thus, we fix $R_0 > 0$ and let

$$\Omega := \left\{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < R_0^2\right\}.$$

In contrast to the previous chapters, Ω is no longer bounded since it extends infinitely in the x_3-direction. Because of the axial symmetry of the set-up, it is natural to work with cylindrical coordinates (r, φ, x_3). In these coordinates, we simply have $\Omega = \left\{x \in \mathbb{R}^3 \mid r < R_0\right\}$. Furthermore, we now consider purely reflecting boundary conditions for the particles and perfect conductor boundary conditions for the fields on $\partial\Omega$. Due to perfect conductor boundary conditions, Maxwell's equations are only imposed on Ω, where $\varepsilon = \mu = \mathrm{Id}$ by assumption. Hence, we no longer distinguish the E- and D-, and the H- and B-field, respectively, and use E and B for denotation of the electromagnetic fields. Moreover, we consider an external magnetic field B^{ext}, which is supposed to be divergence free, as given and thus no longer consider an external current density u (whence we neglect an external electric field). Therefore, the only charge and current densities are the internal ones, i.e.,

$$\rho = \rho^{\mathrm{int}} = \sum_{\alpha=1}^{N} q_\alpha \int_{\mathbb{R}^3} f^\alpha \, dv, \quad j = j^{\mathrm{int}} = \sum_{\alpha=1}^{N} q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv.$$

In the following, there often occur cylindrical coordinates and the corresponding local, orthonormal coordinate basis (e_r, e_φ, e_3), where

$$e_r = (\cos\varphi, \sin\varphi, 0), \quad e_\varphi = (-\sin\varphi, \cos\varphi, 0), \quad e_3 = (0, 0, 1).$$

For a vector $w \in \mathbb{R}^3$, we denote with w_r, w_φ, and w_3 the coordinates of w in this local coordinate system, i.e.,

$$w_r = w \cdot e_r, \quad w_\varphi = w \cdot e_\varphi, \quad w_3 = w \cdot e_3.$$

Altogether, the whole Vlasov–Maxwell system in this set-up reads

$$\begin{aligned} \partial_t f^\alpha + \widehat{v}_\alpha \cdot \partial_x f^\alpha + q_\alpha \left(E + \widehat{v}_\alpha \times \left(B + B^{\mathrm{ext}}\right)\right) \cdot \partial_v f^\alpha &= 0 && \text{on } I_{T_\bullet} \times \Omega \times \mathbb{R}^3, \\ f_-^\alpha &= K f_+^\alpha && \text{on } \gamma_{T_\bullet}^-, \end{aligned}$$

$$
\begin{aligned}
f^\alpha(0) &= \mathring{f}^\alpha && \text{on } \Omega \times \mathbb{R}^3, \\
\partial_t E - \operatorname{curl}_x B &= -4\pi j && \text{on } I_{T_\bullet} \times \Omega, \\
\partial_t B + \operatorname{curl}_x E &= 0 && \text{on } I_{T_\bullet} \times \Omega, \\
\operatorname{div}_x E &= 4\pi\rho && \text{on } I_{T_\bullet} \times \Omega, \\
\operatorname{div}_x B &= 0 && \text{on } I_{T_\bullet} \times \Omega, \\
E_\varphi = E_3 = B_r + B_r^{\mathrm{ext}} &= 0 && \text{on } I_{T_\bullet} \times \partial\Omega, \\
(E, B)(0) &= \left(\mathring{E}, \mathring{B}\right) && \text{on } \Omega.
\end{aligned}
$$

Note that in the following the divergence part of Maxwell's equations will be important and not neglected anymore. Furthermore, the perfect conductor boundary condition reads $E \times n = 0 = B^{\mathrm{tot}} \cdot n$ in the general case (where B^{tot} is the total magnetic field; here, $B^{\mathrm{tot}} = B + B^{\mathrm{ext}}$) and reduces to $E_\varphi = E_3 = B_r^{\mathrm{tot}} = 0$ in the case of Ω being an infinitely long cylinder since here $n = e_r$.

It is convenient to introduce electromagnetic potentials, which will be the functions we work with mostly, namely, the electric scalar potential ϕ and the magnetic vector potential $A^{\mathrm{tot}} = A + A^{\mathrm{ext}}$, which splits into the internal and external potentials A and A^{ext}. The electromagnetic fields and potentials are related via

$$E = -\partial_x \phi - \partial_t A, \quad B = \operatorname{curl}_x A, \quad B^{\mathrm{ext}} = \operatorname{curl}_x A^{\mathrm{ext}}. \tag{3.1.1}$$

Then, Gauss's law for magnetism ($\operatorname{div}_x B = 0$) and Faraday's law ($\partial_t B + \operatorname{curl}_x E = 0$) are automatically satisfied. There is some freedom to demand a certain gauge condition on the potentials. We will consider Lorenz gauge for the internal potentials

$$\partial_t \phi + \operatorname{div}_x A = 0, \tag{3.1.2}$$

which of course is the same as Coulomb gauge

$$\operatorname{div}_x A = 0$$

if the potentials are independent of time, and similarly $\operatorname{div}_x A^{\mathrm{ext}} = 0$ for the external potential. Using the gauge (3.1.2), the remaining Maxwell's equations, i.e., $\partial_t E - \operatorname{curl}_x B = -4\pi j$ and $\operatorname{div}_x E = 4\pi\rho$, become

$$\partial_t^2 \phi - \Delta_x \phi = 4\pi\rho, \quad \partial_t^2 A - \Delta_x A = 4\pi j, \tag{3.1.3}$$

where the latter equation is to be understood componentwise (in Cartesian coordinates).

Similar set-ups have already been studied earlier, for example, in [Pou92; Rei92]. The basic strategy to obtain steady states was first mentioned in [Deg90]. Closely related to our considerations is [BF93], where (among other set-ups) existence of steady states in an infinitely long cylinder without external magnetic field was proved. However, an important condition there is that there is only one particle species and thus only a fixed sign of particle charges appears. Therefore, ρ has a fixed sign and ϕ is monotone, which is crucial for the considerations in [BF93]. As opposed to this, we allow positively and negatively charged particles.

The question about existence of confined steady states for a Vlasov–Poisson plasma (that is, $B = 0$) was considered in [Sku14] and [Kno19]. The approach of the latter work is similar to ours but needs some smallness assumption on the ansatz functions, which we can avoid, and is restricted to homogeneous external magnetic fields parallel to the symmetry axis.

3.2 Symmetries and invariants

Due to the symmetry properties of Ω, it is natural to consider the case that the tuple $\left((f^\alpha)_\alpha, \phi, A, A^{\mathrm{ext}}\right)$ has some symmetry properties as well:

Firstly, as Ω is invariant under translations in the e_3-direction, we assume that the tuple $\left((f^\alpha)_\alpha, \phi, A, A^{\mathrm{ext}}\right)$ is independent of x_3, that is,

$$f^\alpha = f^\alpha(t, x_1, x_2, v_1, v_2, v_3),\ \phi = \phi(t, x_1, x_2),\ A = A(t, x_1, x_2),\ A^{\mathrm{ext}} = A^{\mathrm{ext}}(t, x_1, x_2).$$

Then, of course the same property also holds for E, B, and B^{ext}. With this assumption, the resulting system is also called the "two and one-half dimensional" relativistic Vlasov–Maxwell system, since an f^α as above only depends on two space and three momentum variables. Due to Glassey and Schaeffer [GS97], unique, classical solutions of the resulting system in case of $\Omega = \mathbb{R}^3$ and $B^{\mathrm{ext}} = 0$ exist globally in time under suitable assumptions about the initial data.

Secondly, as Ω is invariant under rotations about the x_3-axis, we assume that the tuple $\left((f^\alpha)_\alpha, \phi, A, A^{\mathrm{ext}}\right)$ has the following property:

$$f^\alpha(t, Rx, Rv) = f^\alpha(t, x, v), \qquad \phi(t, Rx) = \phi(t, x),$$
$$A(t, Rx) = RA(t, x), \qquad A^{\mathrm{ext}}(t, Rx) = RA^{\mathrm{ext}}(t, x)$$

for any rotation $R \in \mathbb{R}^{3\times 3}$ about the x_3-axis. With the use of cylindrical coordinates, this assumption about the potentials is equivalent to the assumption that

$$\phi = \phi(t, r, x_3)$$

and that the components of the vector potentials in the local coordinate basis $\left(e_r, e_\varphi, e_3\right)$ be independent of the angle φ, that is,

$$A_r = A_r(t, r, x_3), \qquad A_\varphi = A_\varphi(t, r, x_3), \qquad A_3 = A_3(t, r, x_3),$$
$$A_r^{\mathrm{ext}} = A_r^{\mathrm{ext}}(t, r, x_3), \qquad A_\varphi^{\mathrm{ext}} = A_\varphi^{\mathrm{ext}}(t, r, x_3), \qquad A_3^{\mathrm{ext}} = A_3^{\mathrm{ext}}(t, r, x_3).$$

With this symmetry, we can also reduce the number of variables in (x, v)-space from six to five and can write $f = f(r, x_3, \theta, u, v_3)$ where $u = \sqrt{v_1^2 + v_2^2}$ and θ is the angle between (x_1, x_2) and (v_1, v_2). However, we will not make use of the Vlasov equation written in these variables.

Additionally to these two space symmetries, we consider time symmetry, i.e., the tuple $\left((f^\alpha)_\alpha, \phi, A, A^{\mathrm{ext}}\right)$ is assumed to be independent of t, since we are interested in the existence of (confined) steady states.

In cylindrical coordinates, (for any scalar function ϕ and any vector-valued function A) it holds that

$$\begin{aligned}\partial_x\phi &= e_r\partial_r\phi + \frac{1}{r}e_\varphi\partial_\varphi\phi + e_3\partial_{x_3}\phi,\\ \mathrm{curl}_x\, A &= e_r\left(\frac{1}{r}\partial_\varphi A_3 - \partial_{x_3}A_\varphi\right) + e_\varphi\left(\partial_{x_3}A_r - \partial_r A_3\right) + \frac{1}{r}e_3\left(\partial_r\left(rA_\varphi\right) - \partial_\varphi A_r\right).\end{aligned}$$

Thus, assuming time symmetry and the two space symmetries, (3.1.1) becomes

$$\begin{gathered}E_r = -\partial_r\phi, \quad E_\varphi = E_3 = 0,\\ B_r = 0, \quad B_\varphi = -\partial_r A_3, \quad B_3 = \frac{1}{r}\partial_r\left(rA_\varphi\right),\\ B_r^{\mathrm{ext}} = 0, \quad B_\varphi^{\mathrm{ext}} = -\partial_r A_3^{\mathrm{ext}}, \quad B_3^{\mathrm{ext}} = \frac{1}{r}\partial_r\left(rA_\varphi^{\mathrm{ext}}\right).\end{gathered}$$

Hence, perfect conductor boundary conditions on $\partial\Omega$ are always satisfied in this case and we can let $A_r = 0$ without loss of generality since A_r does not affect the electromagnetic fields.

Furthermore, we have (for any scalar function ϕ and any vector-valued function A)

$$\begin{aligned}\Delta_x\phi &= \frac{1}{r}\partial_r\left(r\partial_r\phi\right) + \frac{1}{r^2}\partial_\varphi^2\phi + \partial_{x_3}^2\phi,\\ \Delta_x A &= e_r\left(\Delta_x A_r - \frac{1}{r^2}A_r - \frac{2}{r^2}\partial_\varphi A_\varphi\right) + e_\varphi\left(\Delta_x A_\varphi - \frac{1}{r^2}A_\varphi + \frac{2}{r^2}\partial_\varphi A_r\right) + e_3\Delta_x A_3.\end{aligned}$$

Thus, assuming time symmetry, the two space symmetries, and $A_r = 0$, on the one hand the gauge (3.1.2) is automatically satisfied, as

$$\mathrm{div}_x\, A = \frac{1}{r}\partial_r(rA_r) + \frac{1}{r}\partial_\varphi A_\varphi + \partial_{x_3}A_3 \tag{3.2.1}$$

in general, and on the other hand (3.1.3) becomes

$$-\frac{1}{r}\left(r\phi'\right)' = 4\pi\rho, \quad -\left(\frac{1}{r}\left(rA_\varphi\right)'\right)' = 4\pi j_\varphi, \quad -\frac{1}{r}\left(rA_3'\right)' = 4\pi j_3. \tag{3.2.2}$$

As ϕ, A_φ, and A_3 only depend on r, we denote the r-derivative with simply $'$. Note that the choice $A_r = 0$ launches the constraint

$$j_r = 0,$$

i.e., no radial currents are allowed to appear.

A basic physical principle is that to each symmetry there corresponds an invariant. As for the two space symmetries, we consider the Lagrangian formalism, where the characteristic equation

$$\dot{v} = q_\alpha \left(E + \widehat{v}_\alpha \times B^{\mathrm{tot}} \right)$$

can be recovered from the Lagrangian (without the use of any gauge)

$$\mathcal{L}^\alpha = \mathcal{L}^\alpha(t, x, \dot{x}) = -\sqrt{1 - |\dot{x}|^2} - q_\alpha \left(\phi(t, x) - \dot{x} \cdot A^{\mathrm{tot}}(t, x) \right)$$

via

$$\frac{d}{dt}(\partial_{\dot{x}} \mathcal{L}^\alpha) = \partial_x \mathcal{L}^\alpha$$

if $\dot{x} = \widehat{v}_\alpha$ is supposed. From this, for each space symmetry we can derive an invariant. As for translation invariance, we find that

$$\mathcal{G}^\alpha := \partial_{\dot{x}_3} \mathcal{L}^\alpha = v_3 + q_\alpha A_3^{\mathrm{tot}}$$

is the corresponding invariant. Similarly, the invariant corresponding to rotational symmetry is

$$\mathcal{F}^\alpha := \partial_{\dot{\varphi}} \mathcal{L}^\alpha = r \left(v_\varphi + q_\alpha A_\varphi^{\mathrm{tot}} \right).$$

Note that in the formulae for $\mathcal{F}^\alpha$ (the "canonical angular momentum") and $\mathcal{G}^\alpha$, components of the so-called "canonical momentum"

$$p_\alpha = v + q_\alpha A^{\mathrm{tot}}$$

appear. In the variables (x, p_α), the particle energy

$$\mathcal{E}^\alpha := v_\alpha^0 + q_\alpha \phi = \sqrt{m_\alpha^2 + \left| p_\alpha - q_\alpha A^{\mathrm{tot}} \right|^2} + q_\alpha \phi$$

is the (in general time-dependent) Hamiltonian governing the motion of the particles of the α-th species. Assuming that the electromagnetic potentials are independent of time, $\mathcal{E}^\alpha$ is also independent of time and thus another invariant, the one corresponding to time symmetry.

3.3 Steady states—definition and ansatz

The preceding considerations about symmetry motivate the definition of what we call a (confined) steady state in our set-up. Before that we collect our symmetry assumptions.

Definition and Remark 3.3.1. (a) A function $f\colon \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ / a function $\phi\colon \overline{\Omega} \to \mathbb{R}$ / a vector field $A\colon \overline{\Omega} \to \mathbb{R}^3$ is called

(i) independent of x_3 if $\partial_{x_3} f = 0$ / $\partial_{x_3}\phi = 0$ / $\partial_{x_3} A = 0$;

(ii) axially symmetric if $f(Rx, Rv) = f(x, v)$ for any $x \in \overline{\Omega}$, $v \in \mathbb{R}^3$, and rotation $R \in \mathbb{R}^{3\times3}$ about the x_3-axis / $\phi(Rx) = \phi(x)$ for any $x \in \overline{\Omega}$ and rotation $R \in \mathbb{R}^{3\times3}$ about the x_3-axis / $A(Rx) = RA(x)$ for any $x \in \overline{\Omega}$ and rotation $R \in \mathbb{R}^{3\times3}$ about the x_3-axis.

(b) With these two symmetries, the functions ϕ, A_r, A_φ, and A_3 only depend on r. Accordingly, we will often view them as functions on $[0, R_0]$.

(c) An axially symmetric vector field A automatically satisfies $A_1(x) = A_2(x) = 0$ if $x_1 = x_2 = 0$, i.e., if x lies on the x_3-axis.

Remark 3.3.2. From a geometric point of view, the main idea of the setting and the symmetry assumptions is the following: The confinement device Ω is a coordinate surface with respect to a suitable orthogonal curvilinear coordinate system (here, r = const. in cylindrical coordinates) and in these coordinates the potentials only depend on one variable, namely, on the coordinate which is constant on $\partial\Omega$. The symmetry assumption about the magnetic potential thus implies that the magnetic field lies in the tangent space of the submanifold $\partial\Omega$, and it carries over to the electromagnetic fields, which in particular means that the magnetic field is invariant under parallel transport around closed loops on $\partial\Omega$. Thus, with this approach confinement devices whose boundaries have nontrivial curvature (such as a ball) are a priori excluded in order to allow nontrivial magnetic fields. Conversely, an infinitely long cylinder or (the interior of) a torus are consistent with this approach since their boundaries are coordinate surfaces of a suitable orthogonal curvilinear coordinate system and are flat.

We proceed with an assumption about the external potential, which is supposed to hold henceforth.

Condition 3.3.3. The external potential $A^{\text{ext}}\colon \overline{\Omega} \to \mathbb{R}$ is independent of x_3 and axially symmetric such that $A_r^{\text{ext}} = 0$ and $A_\varphi^{\text{ext}}, A_3^{\text{ext}} \in C^1([0, R_0])$ (viewed as functions of r) with $A_\varphi^{\text{ext}}(0) = A_3^{\text{ext}}(0) = \left(A_3^{\text{ext}}\right)'(0) = 0$.

Note that $A_3^{\text{ext}}(0) = 0$ can be assumed—for simplicity—without loss of generality since adding a constant to A_3^{ext} does not affect B^{ext} because of $\operatorname{curl}_x e_3 = 0$ (as opposed to this, this invariance under adding constants does not hold for A_φ^{ext}, as $\operatorname{curl}_x e_\varphi \neq 0$).

We first prove some technicalities.

Lemma 3.3.4. *Let $\phi, A_\varphi, A_3 \in C^1([0, R_0])$ with*

$$\phi'(0) = A_\varphi(0) = A_3'(0) = 0 \tag{3.3.1}$$

and assume $A_r = 0$. Then:

(i) The potentials $\phi = \phi(x)$ and $A = A(x)$ are continuously differentiable on $\overline{\Omega}$. Thus, the electromagnetic fields

$$E = -\partial_x \phi = -\phi' e_r, \quad B = \operatorname{curl}_x A = -A_3' e_\varphi + \frac{1}{r}\big(rA_\varphi\big)' e_3 \tag{3.3.2}$$

are continuous on $\overline{\Omega}$. Moreover, $\operatorname{div}_x A = 0$ on $\overline{\Omega}$.

(ii) If $\phi, A_3 \in C^2([0,R_0])$, they are even twice continuously differentiable on $\overline{\Omega}$ with respect to x. Accordingly, E is of class C^1 on $\overline{\Omega}$. If moreover $A_\varphi \in C^2(]0,R_0])$ such that

$$A_\varphi'(r) - \frac{A_\varphi(r)}{r} = O(r), \quad A_\varphi''(r) = O(1) \quad \text{for } r \to 0, \tag{3.3.3}$$

then $A \in W^{2,\infty}(\Omega;\mathbb{R}^3) \cap C^2\Big(\overline{\Omega} \setminus \mathbb{R}e_3;\mathbb{R}^3\Big)$. Accordingly, B is of class $W^{1,\infty}$ on Ω and of class C^1 on $\overline{\Omega} \setminus \mathbb{R}e_3$.

Proof. We easily see that the maps $x \mapsto \phi(x)$ and $x \mapsto A_3(x)e_3$ are (twice) continuously differentiable on $\overline{\Omega}$ if the maps $r \mapsto \phi(r)$ and $r \mapsto A_3(r)$ are (twice) continuously differentiable on $[0,R_0]$ since $\phi'(0) = A_3'(0) = 0$. There remains to take care of $x \mapsto A_\varphi(x)e_\varphi(x)$, in particular at $r = 0$. Indeed, this map can be continuously extended to whole $\overline{\Omega}$ because of $A_\varphi(0) = 0$ and is differentiable for $r > 0$ with

$$\partial_x\big(A_\varphi e_\varphi\big)(r,\varphi) = \begin{pmatrix} -\sin\varphi\cos\varphi\Big(A_\varphi'(r) - \frac{A_\varphi(r)}{r}\Big) & -\sin^2\varphi\Big(A_\varphi'(r) - \frac{A_\varphi(r)}{r}\Big) - \frac{A_\varphi(r)}{r} & 0 \\ \cos^2\varphi\Big(A_\varphi'(r) - \frac{A_\varphi(r)}{r}\Big) + \frac{A_\varphi(r)}{r} & \sin\varphi\cos\varphi\Big(A_\varphi'(r) - \frac{A_\varphi(r)}{r}\Big) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.3.4}$$

where all entries have a limit as $r \to 0$. Hence, also $A_\varphi e_\varphi$ is continuously differentiable on $\overline{\Omega}$. Furthermore, A is divergence free with respect to x, as was already observed in Section 3.2 because of (3.2.1). Thus, part 3.3.4.(i) is proved. If moreover the assumptions about A_φ in part 3.3.4.(ii) are satisfied, all second order derivatives (with respect to x) of $A_\varphi e_\varphi$ are bounded for $r \to 0$, since we see by differentiating the entries of (3.3.4) once more that these second order derivatives are expressions in $\sin\varphi$, $\cos\varphi$, $\frac{1}{r}\Big(A_\varphi'(r) - \frac{A_\varphi(r)}{r}\Big)$, and $A_\varphi''(r)$, and thus bounded by assumption. Therefore, all second order derivatives exist on Ω in the weak sense, coincide with the classical derivatives almost everywhere, and are bounded. This proves the remaining part of 3.3.4.(ii). □

Note that this lemma yields that under Condition 3.3.3 the external potential A^{ext} is continuously differentiable on $\overline{\Omega}$ and divergence free, and that the external magnetic field $B^{\text{ext}} = \operatorname{curl}_x A^{\text{ext}}$ is continuous on $\overline{\Omega}$.

Remark 3.3.5. In Lemma 3.3.4.(ii), we cannot expect that $A \in C^2\Big(\overline{\Omega};\mathbb{R}^3\Big)$ in general if $A_\varphi \in C^2([0,R_0])$ and (3.3.3) holds, as the example $A_\varphi(r) = r^2$ shows since

$$\Delta_x\big(A_\varphi e_\varphi\big)_1 = -\Delta_x\big(r^2\sin\varphi\big) = -3\sin\varphi$$

has no limit for $r \to 0$.

We proceed with a basic definition.

Definition 3.3.6. Let Condition 3.3.3 hold.

(a) A tuple $\left((f^\alpha)_\alpha, \phi, A\right)$ is called an axially symmetric steady state of the two and one-half dimensional relativistic Vlasov–Maxwell system on $\overline{\Omega}$ with external potential A^{ext} (hereafter abbreviated as steady state) if the following conditions are satisfied:

(i) For each $\alpha = 1, \ldots, N$, the functions $f^\alpha : \overline{\Omega} \times \mathbb{R}^3 \to [0, \infty[$ are continuously differentiable satisfying $f^\alpha(x, \cdot) \in L^1(\mathbb{R}^3)$ for each $x \in \overline{\Omega}$.

(ii) The potentials satisfy

$$\phi \in C^2\left(\overline{\Omega}\right), \quad A \in C^1\left(\overline{\Omega}; \mathbb{R}^3\right) \cap C^2\left(\overline{\Omega} \setminus \mathbb{R}e_3; \mathbb{R}^3\right) \cap W^{2,\infty}(\Omega; \mathbb{R}^3).$$

(This condition is motivated in view of Lemma 3.3.4.)

(iii) Any f^α and ϕ, A are independent of x_3 and axially symmetric.

(iv) The equations

$$\widehat{v}_\alpha \cdot \partial_x f^\alpha + q_\alpha \left(E + \widehat{v}_\alpha \times B^{\text{tot}}\right) \cdot \partial_v f^\alpha = 0 \qquad \text{on } \overline{\Omega} \times \mathbb{R}^3, \tag{3.3.5a}$$

$$f^\alpha(x, v - 2v_r e_r) = f^\alpha(x, v), \qquad x \in \partial\Omega, v \in \mathbb{R}^3, v_r < 0, \tag{3.3.5b}$$

$$-\Delta_x \phi = 4\pi\rho, \quad -\Delta_x A = 4\pi j, \quad \operatorname{div}_x A = 0 \qquad \text{on } \overline{\Omega}, \tag{3.3.5c}$$

are satisfied. Here, $e_r = e_r(x)$, $v_r = v \cdot e_r$, and

$$E = -\partial_x \phi, \quad B^{\text{tot}} = \operatorname{curl}_x\left(A + A^{\text{ext}}\right),$$

$$\rho = \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} f^\alpha \, dv, \quad j = \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} \widehat{v}_\alpha f^\alpha \, dv.$$

(b) A steady state $\left((f^\alpha)_\alpha, \phi, A\right)$ is said to

(i) have finite charge if

$$\int_{B_{R_0}} \int_{\mathbb{R}^3} f^\alpha \, dv d(x_1, x_2) < \infty$$

for each $\alpha = 1, \ldots, N$;

(ii) be compactly supported with respect to v if there is $S > 0$ such that $f^\alpha(x, v) = 0$ for each $\alpha = 1, \ldots, N$, $x \in \overline{\Omega}$, $|v| \geq S$;

(iii) be nontrivial if $f^\alpha \not\equiv 0$ for each $\alpha = 1, \dots, N$;

(iv) be confined with radius at most R if $0 < R < R_0$ such that $f^\alpha(x, v) = 0$ for each $\alpha = 1, \dots, N$, $x \in \overline{\Omega}$ with $|(x_1, x_2)| \geq R$, and $v \in \mathbb{R}^3$.

Note that perfect conductor boundary conditions are automatically satisfied due to symmetry, as was already observed in Section 3.2, and are thus omitted in (3.3.5).

Remark 3.3.7. A physically reasonable steady state should have finite charge, which usually means $f^\alpha \in L^1\big(\Omega \times \mathbb{R}^3\big)$ for each $\alpha = 1, \dots, N$. However, this is impossible in our setting (unless all f^α vanish identically) by f^α being independent of x_3. Thus, here we have to modify this definition suitably as above.

According to [Deg90], the natural ansatz for f^α is that

$$f^\alpha = \eta^\alpha(\mathcal{E}^\alpha, \mathcal{F}^\alpha, \mathcal{G}^\alpha) \tag{3.3.6}$$

is a function of the three invariants obtained in Section 3.2. We collect some basic assumptions about the ansatz functions η^α.

Condition 3.3.8. For each $\alpha = 1, \dots, N$ it holds that:

(i) $\eta^\alpha \in C^1\big(\mathbb{R}^3; [0, \infty[\big)$.

(ii) There exists $\eta^\alpha_* \in L^1\big(\mathbb{R}^2\big)$ such that

$$\int_{\mathbb{R}^2} \big|\mathcal{E}\eta^\alpha_*(\mathcal{E}, \mathcal{G})\big| \, d(\mathcal{E}, \mathcal{G}) < \infty$$

and

$$\big|\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G})\big| \leq \eta^\alpha_*(\mathcal{E}, \mathcal{G})$$

for all $(\mathcal{E}, \mathcal{F}, \mathcal{G}) \in \mathbb{R}^3$.

(iii) There exists $\eta^\alpha_\# : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\forall\, d \in \mathbb{R} : \eta^\alpha_\#, |\mathcal{E}|\eta^\alpha_\# \in L^1(]d, \infty[\, \times \mathbb{R})$$

and

$$\big|\nabla\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G})\big| \leq \eta^\alpha_\#(\mathcal{E}, \mathcal{G})$$

for all $(\mathcal{E}, \mathcal{F}, \mathcal{G}) \in \mathbb{R}^3$.

We first prove that the ansatz (3.3.6) already ensures (3.3.5a) and (3.3.5b). Here and in the following, we will always write $A^{\text{tot}} = A + A^{\text{ext}}$.

Lemma 3.3.9. *Let Conditions 3.3.3 and 3.3.8.(i) hold and let $\phi, A_\varphi, A_3 \in C^1([0, R_0])$ with*

$$\phi'(0) = A_\varphi(0) = A_3'(0) = 0.$$

Then, for each $\alpha = 1, \dots, N$,

$$f^\alpha \colon \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}, \quad f^\alpha(x,v) = \eta^\alpha(\mathcal{E}^\alpha(x,v), \mathcal{F}^\alpha(x,v), \mathcal{G}^\alpha(x,v)) \\ = \eta^\alpha\Big(v_\alpha^0 + q_\alpha\phi(r), r\Big(v_\varphi + q_\alpha A_\varphi^{\mathrm{tot}}(r)\Big), v_3 + q_\alpha A_3^{\mathrm{tot}}(r)\Big) \tag{3.3.7}$$

is continuously differentiable, independent of x_3, *axially symmetric, and satisfies* (3.3.5a) *and* (3.3.5b).

Proof. We first note that f^α is continuously differentiable because of $rv_\varphi = x_1v_2 - x_2v_1$ and $\phi'(0) = \Big(rA_\varphi^{\mathrm{tot}}\Big)'(0) = \big(A_3^{\mathrm{tot}}\big)'(0) = 0$. Clearly, f^α is independent of x_3 and axially symmetric. Furthermore, it is easy to see that (3.3.5b) holds since $\mathcal{E}^\alpha$ is even in v_r and $\mathcal{F}^\alpha$, $\mathcal{G}^\alpha$ do not depend on v_r. To ensure (3.3.5a) for f^α it suffices to prove that $\mathcal{E}^\alpha$, $\mathcal{F}^\alpha$, and $\mathcal{G}^\alpha$ themselves satisfy (3.3.5a)—this clearly holds, as they are invariants of the motion; for the sake of completeness, we carry out the computation. Since they are of class C^1 on $\overline{\Omega} \times \mathbb{R}^3$, this only needs to be verified for $r > 0$. In the following, have (3.3.2) in mind. Firstly,

$$\widehat{v}_\alpha \cdot \partial_x\mathcal{E}^\alpha + q_\alpha\big(E + \widehat{v}_\alpha \times B^{\mathrm{tot}}\big) \cdot \partial_v\mathcal{E}^\alpha = -q_\alpha\widehat{v}_\alpha \cdot E + q_\alpha\big(E + \widehat{v}_\alpha \times B^{\mathrm{tot}}\big) \cdot \widehat{v}_\alpha = 0.$$

Secondly,

$$\begin{aligned}
&\widehat{v}_\alpha \cdot \partial_x\mathcal{F}^\alpha + q_\alpha\big(E + \widehat{v}_\alpha \times B^{\mathrm{tot}}\big) \cdot \partial_v\mathcal{F}^\alpha \\
&= \widehat{v}_\alpha \cdot \Big(v_\varphi + q_\alpha A_\varphi^{\mathrm{tot}}\Big)e_r - \widehat{v}_\alpha \cdot v_r e_\varphi + q_\alpha\widehat{v}_\alpha \cdot r\Big(A_\varphi^{\mathrm{tot}}\Big)' e_r + q_\alpha\big(E + \widehat{v}_\alpha \times B^{\mathrm{tot}}\big) \cdot re_\varphi \\
&= q_\alpha\widehat{v}_\alpha \cdot e_r\left(A_\varphi^{\mathrm{tot}} + r\Big(A_\varphi^{\mathrm{tot}}\Big)' - r \cdot \frac{1}{r}\Big(rA_\varphi^{\mathrm{tot}}\Big)'\right) = 0.
\end{aligned}$$

Thirdly,

$$\widehat{v}_\alpha \cdot \partial_x\mathcal{G}^\alpha + q_\alpha\big(E + \widehat{v}_\alpha \times B^{\mathrm{tot}}\big) \cdot \partial_v\mathcal{G}^\alpha = q_\alpha\widehat{v}_\alpha \cdot \big(A_3^{\mathrm{tot}}\big)' e_r + q_\alpha\big(E + \widehat{v}_\alpha \times B^{\mathrm{tot}}\big) \cdot e_3 = 0.$$

Thus, (3.3.5a) holds for f^α by the chain rule. □

The ansatz (3.3.6) in turn can be inserted into the definition of ρ and j to derive representations of these densities in terms of the potentials.

Lemma 3.3.10. *Let* $\phi\colon [0, R_0] \to \mathbb{R}$, $A\colon [0, R_0] \to \mathbb{R}^3$, *Condition 3.3.8.(ii) hold, and* f^α *be defined as in* (3.3.7) *for each* $\alpha = 1, \dots, N$. *Then,* $f^\alpha(x, \cdot) \in L^1(\mathbb{R}^3)$ *for each* $x \in \overline{\Omega}$. *Furthermore,* ρ *and* j *are independent of* x_3 *and axially symmetric, and we have*

$$4\pi\rho(r) = g_1\Big(r, \phi(r), A_\varphi^{\mathrm{tot}}(r), A_3^{\mathrm{tot}}(r)\Big), \tag{3.3.8a}$$

$$j_r(r) = 0, \tag{3.3.8b}$$

$$4\pi j_\varphi(r) = g_2\Big(r, \phi(r), A_\varphi^{\mathrm{tot}}(r), A_3^{\mathrm{tot}}(r)\Big), \tag{3.3.8c}$$

$$4\pi j_3(r) = g_3\Big(r, \phi(r), A_\varphi^{\mathrm{tot}}(r), A_3^{\mathrm{tot}}(r)\Big) \tag{3.3.8d}$$

for $r \in [0, R_0]$, *where* $g_1, g_2, g_3\colon [0,R_0]\times\mathbb{R}^3 \to \mathbb{R}$,

$$\begin{aligned}
&\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}(r,a,b,c) \\
&= 4\pi \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}} \int_{\sqrt{m_\alpha^2 + (\mathcal{G}-q_\alpha c)^2} + q_\alpha a}^{\infty} \int_0^{2\pi} \begin{pmatrix} \mathcal{E} - q_\alpha a \\ \sqrt{(\mathcal{E}-q_\alpha a)^2 - (\mathcal{G}-q_\alpha c)^2 - m_\alpha^2}\,\sin\theta \\ \mathcal{G} - q_\alpha c \end{pmatrix} \\
&\qquad \cdot \eta^\alpha\Big(\mathcal{E}, r\sqrt{(\mathcal{E}-q_\alpha a)^2 - (\mathcal{G}-q_\alpha c)^2 - m_\alpha^2}\,\sin\theta + r q_\alpha b, \mathcal{G}\Big)\, d\theta\, d\mathcal{E}\, d\mathcal{G} \qquad (3.3.9) \\
&=: \sum_{\alpha=1}^N \begin{pmatrix} g_1^\alpha \\ g_2^\alpha \\ g_3^\alpha \end{pmatrix}(r,a,b,c)
\end{aligned}$$

are continuous functions. Moreover,

$$\left|\left(g_2^\alpha, g_3^\alpha\right)\right| \le \left|g_1^\alpha\right| \tag{3.3.10}$$

on $[0,R_0]\times\mathbb{R}^3$ *for each* $\alpha = 1, \ldots, N$.

Proof. At least formally we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \widehat{v}_\alpha \cdot e_r \\ \widehat{v}_\alpha \cdot e_\varphi \\ \widehat{v}_\alpha \cdot e_3 \end{pmatrix} \eta^\alpha(\mathcal{E}^\alpha, \mathcal{F}^\alpha, \mathcal{G}^\alpha)\, dv \\
&= \int_{\mathbb{R}} \int_0^\infty \int_0^{2\pi} \frac{u}{\sqrt{m_\alpha^2 + u^2 + v_3^2}} \begin{pmatrix} \sqrt{m_\alpha^2 + u^2 + v_3^2} \\ u\cos\theta \\ u\sin\theta \\ v_3 \end{pmatrix} \\
&\qquad \cdot \eta^\alpha\Big(\sqrt{m_\alpha^2 + u^2 + v_3^2} + q_\alpha\phi(r), ru\sin\theta + r q_\alpha A_\varphi^{\mathrm{tot}}(r), v_3 + q_\alpha A_3^{\mathrm{tot}}(r)\Big)\, d\theta\, du\, dv_3 \\
&= \int_{\mathbb{R}} \int_{\sqrt{m_\alpha^2 + (\mathcal{G}-q_\alpha A_3^{\mathrm{tot}}(r))^2} + q_\alpha\phi(r)}^{\infty} \int_0^{2\pi} \begin{pmatrix} \mathcal{E} - q_\alpha\phi(r) \\ 0 \\ \sqrt{(\mathcal{E}-q_\alpha\phi(r))^2 - (\mathcal{G}-q_\alpha A_3^{\mathrm{tot}}(r))^2 - m_\alpha^2}\,\sin\theta \\ \mathcal{G} - q_\alpha A_3^{\mathrm{tot}}(r) \end{pmatrix} \\
&\qquad \cdot \eta^\alpha\Big(\mathcal{E}, r\sqrt{(\mathcal{E}-q_\alpha\phi(r))^2 - (\mathcal{G}-q_\alpha A_3^{\mathrm{tot}}(r))^2 - m_\alpha^2}\,\sin\theta + r q_\alpha A_\varphi^{\mathrm{tot}}(r), \mathcal{G}\Big)\, d\theta\, d\mathcal{E}\, d\mathcal{G},
\end{aligned}$$

where we introduced polar coordinates in the (v_1, v_2)-plane with basis (e_r, e_φ) and then substituted firstly $\mathcal{E} = \sqrt{m_\alpha^2 + u^2 + v_3^2} + q_\alpha \phi(r)$ and secondly $\mathcal{G} = v_3 + q_\alpha A_3^{\mathrm{tot}}(r)$. Note that the integral in the second line vanishes after substituting $y = \sin\theta$. Due to Condition 3.3.8.(ii), the modulus of the integrand in the first line can be estimated by

$$\left(|\mathcal{E}| + |q_\alpha||\phi(r)|\right)\eta_*^\alpha(\mathcal{E}, \mathcal{G})$$

and is hence integrable. Because of $|\widehat{v}_\alpha| < 1$ also the other integrals exist. Thus, the above calculation is justified. Multiplying these identities with q_α and summing over α yields the representation. The above estimate on the integrands also implies that g_i is continuous, $i = 1, 2, 3$. Finally, (3.3.10) is also a consequence of $|\widehat{v}_\alpha| < 1$. □

Remark 3.3.11. The proof of the preceding lemma additionally shows that any steady state obtained in the following sections has finite charge. Indeed, for this it is sufficient that $r\phi$ is integrable over $[0, R_0]$, which is of course the case when ϕ is continuous.

According to Lemma 3.3.10, after integrating (3.2.2) and using the representation (3.3.8), the problem of finding a steady state with the ansatz (3.3.6) reduces to finding $\phi, A_3 \in C^2([0, R_0])$, $A_\varphi \in C^2(]0, R_0]) \cap C^1([0, R_0])$ satisfying (3.3.1), (3.3.3), and

$$\phi(r) = -\int_0^r \frac{1}{s} \int_0^s \sigma g_1\left(\sigma, \phi(\sigma), A_\varphi^{\mathrm{tot}}(\sigma), A_3^{\mathrm{tot}}(\sigma)\right) d\sigma ds, \tag{3.3.11a}$$

$$A_\varphi(r) = -\frac{1}{r}\int_0^r s \int_0^s g_2\left(\sigma, \phi(\sigma), A_\varphi^{\mathrm{tot}}(\sigma), A_3^{\mathrm{tot}}(\sigma)\right) d\sigma ds, \tag{3.3.11b}$$

$$A_3(r) = -\int_0^r \frac{1}{s} \int_0^s \sigma g_3\left(\sigma, \phi(\sigma), A_\varphi^{\mathrm{tot}}(\sigma), A_3^{\mathrm{tot}}(\sigma)\right) d\sigma ds \tag{3.3.11c}$$

for $r > 0$ in view of Lemmas 3.3.4 and 3.3.9; note that we could prescribe arbitrary values for ϕ and A_3 at $r = 0$, and we choose both of these values to be zero. Therefore, it is convenient to introduce the map

$$\mathcal{M}\colon C\left([0, R_0]; \mathbb{R}^3\right) \to C\left([0, R_0]; \mathbb{R}^3\right),$$

$$\mathcal{M}\left(\phi, A_\varphi, A_3\right) = \left([0, R_0] \ni r \mapsto \begin{pmatrix} -\int_0^r \frac{1}{s}\int_0^s \sigma g_1\left(\sigma, \phi(\sigma), A_\varphi^{\mathrm{tot}}(\sigma), A_3^{\mathrm{tot}}(\sigma)\right) d\sigma ds \\ -\frac{1}{r}\int_0^r s \int_0^s g_2\left(\sigma, \phi(\sigma), A_\varphi^{\mathrm{tot}}(\sigma), A_3^{\mathrm{tot}}(\sigma)\right) d\sigma ds \\ -\int_0^r \frac{1}{s}\int_0^s \sigma g_3\left(\sigma, \phi(\sigma), A_\varphi^{\mathrm{tot}}(\sigma), A_3^{\mathrm{tot}}(\sigma)\right) d\sigma ds \end{pmatrix}\right).$$

The following lemma shows that indeed $\mathcal{M}$ is well-defined (with the obvious interpretation $\mathcal{M}\left(\phi, A_\varphi, A_r\right)(0) = (0, 0, 0)$) and that it suffices to search for fixed points of $\mathcal{M}$.

Lemma 3.3.12. *Assume Conditions 3.3.3, 3.3.8.(i), and 3.3.8.(ii).*

(i) For any $\left(\phi, A_\varphi, A_3\right) \in C\left([0, R_0]; \mathbb{R}^3\right)$ *we have*

$$\left(\tilde{\phi}, \tilde{A}_\varphi, \tilde{A}_3\right) := \mathcal{M}\left(\phi, A_\varphi, A_3\right) \in C^2\left([0, R_0]; \mathbb{R}^3\right).$$

Furthermore, $\left(\tilde\phi, \tilde A_\varphi, \tilde A_3\right)$ *satisfies* (3.3.1) *and* (3.3.3).

(ii) If $\left(\phi, A_\varphi, A_3\right) \in C\left([0,R_0];\mathbb{R}^3\right)$ *is a fixed point of* $\mathcal{M}$*, then* $\left(\left(f^\alpha\right)_\alpha, \phi, A\right)$ *is a steady state, where the* f^α *are defined via the ansatz* (3.3.6).

Proof. Due to Lemma 3.3.10, the functions

$$\tilde g_i\colon [0,R_0] \to \mathbb{R}, \quad \tilde g_i(\sigma) = g_i\left(\sigma, \phi(\sigma), A_\varphi^{\mathrm{tot}}(\sigma), A_3^{\mathrm{tot}}(\sigma)\right)$$

are continuous, $i = 1,2,3$, and hence bounded by some constant $C > 0$. Thus,

$$\left|\tilde\phi(r)\right|, \left|\tilde A_3(r)\right| \le C \int_0^r \frac{1}{s} \int_0^s \sigma d\sigma = \frac{C}{4} r^2, \quad \left|\tilde A_\varphi(r)\right| \le \frac{C}{r} \int_0^r s \int_0^s d\sigma = \frac{C}{3} r^2$$

for $r \in \,]0,R_0]$. Hence, $\tilde\phi$, $\tilde A_\varphi$, and $\tilde A_3$ are continuous also in $r = 0$, and $\frac{\tilde A_\varphi(r)}{r} = \mathcal{O}(r)$ for $r \to 0$. Furthermore, the 'tilde'-potentials are twice continuously differentiable on $]0,R_0]$ with

$$\begin{aligned}
\tilde\phi'(r) &= -\frac{1}{r} \int_0^r s \tilde g_1(s)\, ds, \quad \tilde\phi''(r) = \frac{1}{r^2} \int_0^r s \tilde g_1(s)\, ds - \tilde g_1(r), \\
\tilde A_\varphi'(r) &= \frac{1}{r^2} \int_0^r s \int_0^s \tilde g_2(\sigma)\, d\sigma ds - \int_0^r \tilde g_2(s)\, ds, \\
\tilde A_\varphi''(r) &= -\frac{2}{r^3} \int_0^r s \int_0^s \tilde g_2(\sigma)\, d\sigma ds + \frac{1}{r} \int_0^r \tilde g_2(s)\, ds - \tilde g_2(r), \\
\tilde A_3'(r) &= -\frac{1}{r} \int_0^r s \tilde g_3(s)\, ds, \quad \tilde A_3''(r) = \frac{1}{r^2} \int_0^r s \tilde g_3(s)\, ds - \tilde g_3(r).
\end{aligned}$$

Because of

$$\begin{aligned}
\left|\tilde\phi'(r)\right|, \left|\tilde A_3'(r)\right| &\le \frac{C}{r} \int_0^r s\, ds = \frac{C}{2} r, \\
\left|\tilde A_\varphi'(r)\right| &\le \frac{C}{r^2} \int_0^r s \int_0^s d\sigma ds + Cr = \frac{4C}{3} r
\end{aligned}$$

they are continuously differentiable on $[0,R_0]$ with vanishing derivative at $r = 0$, and moreover $\tilde A_\varphi'(r) = \mathcal{O}(r)$ for $r \to 0$. Furthermore, by l'Hôpital's rule we have

$$\begin{aligned}
\lim_{r\to 0} \tilde\phi''(r) &= \lim_{r\to 0} \frac{r \tilde g_1(r)}{2r} - \tilde g_1(0) = -\frac{\tilde g_1(0)}{2}, \\
\lim_{r\to 0} \tilde A_\varphi''(r) &= -\lim_{r\to 0} \frac{2r \int_0^r \tilde g_2(s)\, ds}{3r^2} + \tilde g_2(0) - \tilde g_2(0) = -\frac{2\tilde g_2(0)}{3}, \\
\lim_{r\to 0} \tilde A_3''(r) &= \lim_{r\to 0} \frac{r \tilde g_3(r)}{2r} - \tilde g_3(0) = -\frac{\tilde g_3(0)}{2}.
\end{aligned}$$

Therefore, $\tilde{\phi}, \tilde{A}_\varphi, \tilde{A}_3 \in C^2([0,R_0])$ and clearly $\tilde{A}''_\varphi(r) = \mathcal{O}(1)$ for $r \to 0$. Finally, from Lemmas 3.3.4, 3.3.9, and 3.3.10 it follows that $\left(\left(f^\alpha\right)_\alpha, \phi, A\right)$ is a steady state if $\left(\phi, A_\varphi, A_3\right)$ is a fixed point of $\mathcal{M}$; note that (3.3.11) implies (3.2.2) and this yields $-\Delta_x \phi = 4\pi\rho$ on $\overline{\Omega}$ and $-\Delta_x A = 4\pi j$ on $\overline{\Omega} \setminus \mathbb{R}e_3$ in the classical sense, and $-\Delta_x A = 4\pi j$ on Ω in the weak sense. □

3.4 Existence of steady states

3.4.1 A priori estimates

There only remains to find a fixed point of $\mathcal{M}$. For this, the most important tool is to derive a priori bounds for the potentials. Therefore, we assume for the time being that we already have a solution $\left(\phi, A_\varphi, A_3\right) \in C\left([0,R_0]; \mathbb{R}^3\right)$ of (3.3.11). Due to (3.3.9), we first have the following estimate on g_1^α for each $(r,a,b,c) \in [0,R_0] \times \mathbb{R}^3$:

$$\left|g_1^\alpha(r,a,b,c)\right| \le 4\pi \left|q_\alpha\right| \cdot 2\pi \int_{\mathbb{R}^2} \left(|\mathcal{E}| + \left|q_\alpha\right||a|\right) \eta_*^\alpha(\mathcal{E},\mathcal{G})\, d(\mathcal{E},\mathcal{G}).$$

Using (3.3.10) and summing over α yields

$$\left|g_i(r,a,b,c)\right| \le c_1 + c_2|a|, \quad i = 1,2,3, \tag{3.4.1}$$

where we introduced the abbreviations

$$c_1 := 8\pi^2 \sum_{\alpha=1}^N \left|q_\alpha\right| \int_{\mathbb{R}^2} |\mathcal{E}| \eta_*^\alpha(\mathcal{E},\mathcal{G})\, d(\mathcal{E},\mathcal{G}) < \infty,$$
$$c_2 := 8\pi^2 \sum_{\alpha=1}^N \left|q_\alpha\right|^2 \int_{\mathbb{R}^2} \eta_*^\alpha(\mathcal{E},\mathcal{G})\, d(\mathcal{E},\mathcal{G}) < \infty.$$

Therefore, in view of (3.3.11a) an integral inequality for ϕ follows, namely,

$$\left|\phi(r)\right| \le \int_0^r \frac{1}{s} \int_0^s \sigma\left(c_1 + c_2\left|\phi(\sigma)\right|\right) d\sigma ds = \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma \left|\phi(\sigma)\right| d\sigma ds \tag{3.4.2}$$

for $r \in [0,R_0]$. We could thus easily derive the inequality

$$\left|\phi(r)\right| \le \frac{c_1}{4} R_0^2 + c_2 R_0 \int_0^r \left|\phi(s)\right| ds \tag{3.4.3}$$

and therefore

$$\left|\phi(r)\right| \le \frac{c_1}{4} R_0^2 e^{c_2 R_0 r} \tag{3.4.4}$$

via Gronwall's lemma. However, (3.4.3) is way too crude and hence (3.4.4) is not very sharp. If we were to use this a priori estimate later to show confinement of a steady state, the needed assumption about the external potential would be quite strong. Consequently, in order to allow a wider class for external potentials ensuring confinement later, we now search for a sharper a priori estimate on ϕ.

Thus, we search for a solution of the integral equation corresponding to (3.4.2), that is,

$$\xi(r) = \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma \xi(\sigma)\, d\sigma ds. \tag{3.4.5}$$

For any $\xi \in C([0, R_0])$, the elementary identity

$$\int_0^r \frac{1}{s} \int_0^s \sigma \xi(\sigma)\, d\sigma ds = \int_0^r \int_\sigma^r \frac{1}{s} \sigma \xi(\sigma)\, ds d\sigma = \int_0^r (\ln r - \ln s) s \xi(s)\, ds \tag{3.4.6}$$

holds for any $r \in [0, R_0]$ (where the right-hand side is defined to be zero in $r = 0$). Therefore, (3.4.5) becomes a Volterra integral equation of the second kind, namely,

$$\xi(r) = \frac{c_1}{4} r^2 + c_2 \int_0^r (\ln r - \ln s) s \xi(s)\, ds \tag{3.4.7}$$

with nonnegative, square integrable Volterra kernel

$$V\colon [0, R_0]^2 \to \mathbb{R}, \quad V(r,s) = \begin{cases} c_2(\ln r - \ln s)s, & 0 < s \le r \le R_0, \\ 0, & \text{else.} \end{cases}$$

It is well known that Volterra integral equations such as (3.4.7) have a unique square integrable solution; see [Tri57, Section 1.5.]. To find this solution, we rather work with (3.4.5), which suggests a series ansatz

$$\xi(r) = \sum_{k=0}^{\infty} a_k r^k$$

for ξ. With this ansatz, at least formally we demand

$$\begin{aligned} \sum_{k=0}^{\infty} a_k r^k &\overset{!}{=} \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma \xi(\sigma)\, d\sigma ds = \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma \sum_{k=0}^{\infty} a_k \sigma^k\, d\sigma ds \\ &= \frac{c_1}{4} r^2 + c_2 \int_0^r \sum_{k=0}^{\infty} \frac{a_k}{k+2} s^{k+1}\, ds = \frac{c_1}{4} r^2 + c_2 \sum_{k=0}^{\infty} \frac{a_k}{(k+2)^2} r^{k+2} = \frac{c_1}{4} r^2 + \sum_{k=2}^{\infty} \frac{c_2 a_{k-2}}{k^2} r^k. \end{aligned} \tag{3.4.8}$$

Thus,

$$a_0 = a_1 = 0, \quad a_2 = \frac{c_1}{4} + \frac{c_2 a_0}{2^2} = \frac{c_1}{4}.$$

Therefore, $a_k = 0$ if k is odd, and

$$a_{2m} = \frac{c_2 a_{2(m-1)}}{4m^2}$$

for $m \geq 2$. Hence, we have

$$a_{2m} = \frac{c_1 c_2^{m-1}}{4^m (m!)^2}$$

for $m \in \mathbb{N}$ by induction. Consequently, we define

$$\xi \colon \mathbb{R} \to \mathbb{R}, \quad \xi(r) = \sum_{k=1}^{\infty} \frac{c_1 c_2^{k-1}}{4^k (k!)^2} r^{2k}.$$

Obviously, this series is uniformly convergent on any bounded interval, whence the calculation (3.4.8) is justified and ξ indeed is the unique square integrable solution of (3.4.7) on $[0, R_0]$ by (3.4.6). Moreover, ϕ satisfies the corresponding integral inequality

$$\left|\phi(r)\right| \leq \frac{c_1}{4} r^2 + c_2 \int_0^r (\ln r - \ln s) s \left|\phi(s)\right| ds.$$

Thus,

$$\left|\phi(r)\right| \leq \xi(r) \tag{3.4.9}$$

for all $r \in [0, R_0]$ as a consequence of the positivity of Volterra operators in the case $V \geq 0$; see [Bee69, Theorem 5]. Therefore, we have established a quite sharp a priori bound on ϕ.

In order to obtain similar estimates also for A_φ and A_3, we insert (3.4.1) and (3.4.9) into (3.3.11b) and (3.3.11c). On the one hand, we conclude

$$\begin{aligned}
\left|A_\varphi(r)\right| &\leq \frac{1}{r} \int_0^r s \int_0^s \left(c_1 + c_2 \left|\phi(\sigma)\right|\right) d\sigma ds \leq \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r s \int_0^s \xi(\sigma)\, d\sigma ds \\
&= \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r \sum_{k=1}^{\infty} \frac{c_1 c_2^{k-1}}{(2k+1) 4^k (k!)^2} s^{2k+2}\, ds \\
&= \frac{c_1}{3} r^2 + \sum_{k=1}^{\infty} \frac{c_1 c_2^k}{(2k+1)(2k+3) 4^k (k!)^2} r^{2k+2} = \sum_{k=1}^{\infty} \frac{c_1 c_2^{k-1}}{\left(1 - \frac{1}{4k^2}\right) 4^k (k!)^2} r^{2k} =: \zeta(r)
\end{aligned} \tag{3.4.10}$$

and on the other hand

$$|A_3(r)| \leq \int_0^r \frac{1}{s} \int_0^s \sigma \left(c_1 + c_2 \left|\phi(\sigma)\right|\right) d\sigma ds \leq \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma \xi(\sigma)\, d\sigma ds = \xi(r) \tag{3.4.11}$$

for $r \in [0, R_0]$. Note that the a priori bound on A_φ is slightly weaker than the bounds on ϕ and A_3 since obviously $\xi \leq \zeta$.

Thus, we have proved the following important a priori estimate.

Lemma 3.4.1. *Let $\big(\phi, A_\varphi, A_3\big) \in C\big([0,R_0];\mathbb{R}^3\big)$ be a fixed point of $\mathcal{M}$. Then it holds that*

$$\big|\phi(r)\big|, |A_3(r)| \le \xi(r), \quad \big|A_\varphi(r)\big| \le \zeta(r)$$

for $r \in [0, R_0]$.

For the sake of completeness, we remark that ξ can be written in terms of a Bessel function, which corresponds to the fact that (3.4.5) implies

$$r^2\xi'' + r\xi' - c_2 r^2 \xi = c_1 r^2,$$

whence

$$z(r) := \frac{c_2}{c_1}\xi\left(\frac{r}{\sqrt{c_2}}\right) + 1$$

solves the modified Bessel equation

$$r^2 z'' + rz' - r^2 z = 0.$$

Endowed with the initial condition $\xi(0) = \xi'(0) = 0$, this yields $z = I_0$, where I_0 is the modified Bessel function of the first kind (with parameter 0). Consequently,

$$\xi(r) = \frac{c_1}{c_2}\big(I_0\big(\sqrt{c_2}r\big) - 1\big).$$

3.4.2 Fixed point argument

We proceed with proving that steady states really do exist via some fixed point argument. Throughout the rest of this chapter, we assume that Condition 3.3.8 holds and equip the space $C\big([0,R_0];\mathbb{R}^3\big)$ with the norm

$$\big\|\big(\phi, A_\varphi, A_3\big)\big\|_{C([0,R_0];\mathbb{R}^3)} = \sup_{r\in[0,R_0]} \big|\big(\phi(r), A_\varphi(r), A_3(r)\big)\big|. \tag{3.4.12}$$

The a priori bounds obtained in the last section are an important tool to prove existence of solutions to (3.3.11). In view of Schaefer's fixed point theorem—see [Eva10, Section 9.2.2.], for example—we have to prove that $\mathcal{M}$ is continuous and compact, and we have to establish a priori bounds on possible fixed points of the operators $\lambda\mathcal{M}$ for $0 \le \lambda \le 1$. The second task is easily carried out by using the results of Section 3.4.1.

Lemma 3.4.2. *Let $\big(\phi, A_\varphi, A_3\big) \in C\big([0,R_0];\mathbb{R}^3\big)$ such that $\big(\phi, A_\varphi, A_3\big) = \lambda\mathcal{M}\big(\phi, A_\varphi, A_3\big)$ for some $0 \le \lambda \le 1$. Then it holds that*

$$\big|\phi(r)\big|, |A_3(r)| \le \xi(r), \quad \big|A_\varphi(r)\big| \le \zeta(r)$$

for $r \in [0, R_0]$. In particular, the set

$$\big\{\big(\phi, A_\varphi, A_3\big) \in C\big([0,R_0];\mathbb{R}^3\big) \mid \big(\phi, A_\varphi, A_3\big) = \lambda\mathcal{M}\big(\phi, A_\varphi, A_3\big) \text{ for some } 0 \le \lambda \le 1\big\}$$

is bounded.

Proof. By (3.4.1), we obtain

$$\left|\phi(r)\right| \le \lambda \int_0^r \frac{1}{s} \int_0^s \sigma\left(c_1 + c_2\left|\phi(\sigma)\right|\right) d\sigma ds \le \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma\left|\phi(\sigma)\right| d\sigma ds$$

similarly to (3.4.2). Hence, $\left|\phi(r)\right| \le \xi(r)$ for $r \in [0, R_0]$. Similarly to (3.4.10) and (3.4.11), we also have

$$\left|A_\varphi(r)\right| \le \frac{\lambda}{r} \int_0^r s \int_0^s \left(c_1 + c_2\left|\phi(\sigma)\right|\right) d\sigma ds \le \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r s \int_0^s \xi(\sigma)\, d\sigma ds = \zeta(r),$$

$$|A_3(r)| \le \lambda \int_0^r \frac{1}{s} \int_0^s \sigma\left(c_1 + c_2\left|\phi(\sigma)\right|\right) d\sigma ds \le \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma\xi(\sigma)\, d\sigma ds = \xi(r)$$

for $r \in [0, R_0]$. □

Thus, there remains to prove the following lemma.

Lemma 3.4.3. *The map $\mathcal{M}$ is (even locally Lipschitz) continuous and compact.*

Proof. Let $S > 0$ and $(\phi, A_\varphi, A_3), \left(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\right) \in \overline{B_S} \subset C\left([0, R_0]; \mathbb{R}^3\right)$. On the one hand, following the calculation in the proof of Lemma 3.3.10, we have for each $r \in [0, R_0]$ for some (a, b, c), possibly depending on the integration variables, in the line segment connecting $\left(\phi(r), A_\varphi(r), A_3(r)\right)$ and $\left(\overline{\phi}(r), \overline{A}_\varphi(r), \overline{A}_3(r)\right)$,

$$\begin{aligned}
&\left|(g_1, g_2, g_3)\left(r, \phi(r), A_\varphi^{\mathrm{tot}}(r), A_3^{\mathrm{tot}}(r)\right) - (g_1, g_2, g_3)\left(r, \overline{\phi}(r), \overline{A}_\varphi^{\mathrm{tot}}(r), \overline{A}_3^{\mathrm{tot}}(r)\right)\right| \\
&= \Bigg| 4\pi \sum_{\alpha=1}^N q_\alpha^2 \int_{\mathbb{R}} \int_0^\infty \int_0^{2\pi} \frac{u}{\sqrt{m_\alpha^2 + u^2 + v_3^2}} \begin{pmatrix} \sqrt{m_\alpha^2 + u^2 + v_3^2} \\ u \sin\theta \\ v_3 \end{pmatrix} \\
&\quad \cdot \Bigg[\nabla\eta^\alpha\left(\sqrt{m_\alpha^2 + u^2 + v_3^2} + q_\alpha a, ru\sin\theta + rq_\alpha b + rq_\alpha A_\varphi^{\mathrm{ext}}(r), v_3 + q_\alpha c + q_\alpha A_3^{\mathrm{ext}}(r)\right) \\
&\qquad \cdot \begin{pmatrix} \phi(r) - \overline{\phi}(r) \\ r\left(A_\varphi(r) - \overline{A}_\varphi(r)\right) \\ A_3(r) - \overline{A}_3(r) \end{pmatrix} \Bigg] d\theta du dv_3 \Bigg| \\
&= \Bigg| 4\pi \sum_{\alpha=1}^N q_\alpha^2 \int_{\mathbb{R}} \int_0^{2\pi} \int_{\sqrt{m_\alpha^2 + \left(\mathcal{G} - q_\alpha c - q_\alpha A_3^{\mathrm{ext}}(r)\right)^2} + q_\alpha a}^\infty \\
&\qquad \begin{pmatrix} \mathcal{E} - q_\alpha a \\ \sqrt{\left(\mathcal{E} - q_\alpha a\right)^2 - \left(\mathcal{G} - q_\alpha c - q_\alpha A_3^{\mathrm{ext}}(r)\right)^2 - m_\alpha^2}\, \sin\theta \\ \mathcal{G} - q_\alpha c - q_\alpha A_3^{\mathrm{ext}}(r) \end{pmatrix}
\end{aligned}$$

$$\cdot \left[\nabla\eta^\alpha\left(\mathcal{E}, r\sqrt{\left(\mathcal{E}-q_\alpha a\right)^2-\left(\mathcal{G}-q_\alpha c-q_\alpha A_3^{\mathrm{ext}}(r)\right)^2-m_\alpha^2}\sin\theta+rq_\alpha b+rq_\alpha A_\varphi^{\mathrm{ext}}(r),\mathcal{G}\right)\right.$$

$$\left.\cdot\begin{pmatrix}\phi(r)-\overline{\phi}(r)\\ r\left(A_\varphi(r)-\overline{A}_\varphi(r)\right)\\ A_3(r)-\overline{A}_3(r)\end{pmatrix}\right] d\mathcal{E}d\theta d\mathcal{G}\Bigg|$$

$$\leq 8\sqrt{3}\pi^2(1+R_0)\sum_{\alpha=1}^N\left|q_\alpha\right|^2\int_{\mathbb{R}}\int_{-\left|q_\alpha\right|S}^\infty\left(|\mathcal{E}|+\left|q_\alpha\right|S\right)\eta_\#^\alpha(\mathcal{E},\mathcal{G})\,d\mathcal{E}d\mathcal{G}$$

$$\cdot\left|(\phi,A_\varphi,A_3)(r)-\left(\overline{\phi},\overline{A}_\varphi,\overline{A}_3\right)(r)\right|$$

$$= C(S)\left|(\phi,A_\varphi,A_3)(r)-\left(\overline{\phi},\overline{A}_\varphi,\overline{A}_3\right)(r)\right|, \tag{3.4.13}$$

where the constant $C(S)$ is finite due to Condition 3.3.8.(iii) (with $d := -\left|q_\alpha\right|S$ there). Integrating this estimate, we conclude

$$\left|\mathcal{M}(\phi,A_\varphi,A_3)(r)-\mathcal{M}\left(\overline{\phi},\overline{A}_\varphi,\overline{A}_3\right)(r)\right|$$
$$\leq C(S)\left\|(\phi,A_\varphi,A_3)-\left(\overline{\phi},\overline{A}_\varphi,\overline{A}_3\right)\right\|_{C([0,R_0];\mathbb{R}^3)}$$
$$\cdot\left|\left(\int_0^r\frac{1}{s}\int_0^s\sigma\,d\sigma ds,\frac{1}{r}\int_0^r s\int_0^s d\sigma ds,\int_0^r\frac{1}{s}\int_0^s\sigma\,d\sigma ds\right)\right|$$
$$= C(S)\cdot\frac{\sqrt{34}}{12}r^2\left\|(\phi,A_\varphi,A_3)-\left(\overline{\phi},\overline{A}_\varphi,\overline{A}_3\right)\right\|_{C([0,R_0];\mathbb{R}^3)}, \tag{3.4.14}$$

whence

$$\left\|\mathcal{M}(\phi,A_\varphi,A_3)-\mathcal{M}\left(\overline{\phi},\overline{A}_\varphi,\overline{A}_3\right)\right\|_{C([0,R_0];\mathbb{R}^3)}$$
$$\leq C(S)\cdot\frac{\sqrt{34}}{12}R_0^2\left\|(\phi,A_\varphi,A_3)-\left(\overline{\phi},\overline{A}_\varphi,\overline{A}_3\right)\right\|_{C([0,R_0];\mathbb{R}^3)}. \tag{3.4.15}$$

Therefore, $\mathcal{M}$ is locally Lipschitz continuous.

On the other hand, by (3.4.1) we have

$$\left|g_i\left(r,\phi(r),A_\varphi^{\mathrm{tot}}(r),A_3^{\mathrm{tot}}(r)\right)\right|\leq c_1+c_2\left|\phi(r)\right|\leq c_1+c_2S=:\tilde{C}(S)$$

for $i=1,2,3$ and $r\in[0,R_0]$. Furthermore,

$$\left(\mathcal{M}(\phi,A_\varphi,A_3)\right)'(0)=(0,0,0)$$

by (the proof of) Lemma 3.3.12.(i), and for $0<r\leq R_0$ we have

$$\left|\left(\mathcal{M}_i(\phi,A_\varphi,A_3)\right)'(r)\right|=\left|-\frac{1}{r}\int_0^r s g_i\left(s,\phi(s),A_\varphi^{\mathrm{tot}}(s),A_3^{\mathrm{tot}}(s)\right)ds\right|\leq\frac{\tilde{C}(S)r}{2}\leq\frac{\tilde{C}(S)R_0}{2}$$

for $i = 1, 3$ and

$$\begin{aligned}
&\left|(\mathcal{M}_2(\phi, A_\varphi, A_3))'(r)\right| \\
&= \left|\frac{1}{r^2}\int_0^r s\int_0^s g_2\Big(\sigma, \phi(\sigma), A_\varphi^{\text{tot}}(\sigma), A_3^{\text{tot}}(\sigma)\Big)\, d\sigma ds - \int_0^r g_2\Big(s, \phi(s), A_\varphi^{\text{tot}}(s), A_3^{\text{tot}}(s)\Big)\, ds\right| \\
&\le \frac{\tilde{C}(S)r}{3} + \tilde{C}(S)r \le \frac{4\tilde{C}(S)R_0}{3}.
\end{aligned}$$

Therefore, for each $(\phi, A_\varphi, A_3) \in \overline{B_S}$, we have that $\mathcal{M}(\phi, A_\varphi, A_3)$ is Lipschitz continuous with a uniform Lipschitz constant, i.e., a Lipschitz constant only depending on S. By the theorem of Arzelà–Ascoli, $\mathcal{M}$ thus maps bounded sets to precompact sets, that is, $\mathcal{M}$ is compact. □

Theorem 3.4.4. *Let Conditions 3.3.3 and 3.3.8 hold. Then $\mathcal{M}$ has a unique fixed point. Thus, there exists an axially symmetric steady state $\big((f^\alpha)_\alpha, \phi, A\big)$ of the two and one-half dimensional relativistic Vlasov–Maxwell system on $\overline{\Omega}$ with external potential A^{ext}, where the f^α are written in terms of ϕ and A; cf.* (3.3.7).

Proof. Combining Lemmas 3.4.2 and 3.4.3 and invoking Schaefer's fixed point theorem we conclude that $\mathcal{M}$ has a fixed point. Due to Lemma 3.3.12, we obtain a corresponding steady state.

There remains to prove that a fixed point of $\mathcal{M}$ is unique. If we have two fixed points (ϕ, A_φ, A_3), $\Big(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\Big)$ of $\mathcal{M}$, let $S > 0$ such that $(\phi, A_\varphi, A_3), \Big(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\Big) \in \overline{B_S} \subset C\big([0, R_0]; \mathbb{R}^3\big)$. By (3.4.13) it holds that

$$\begin{aligned}
&\left|(\phi, A_\varphi, A_3)(r) - \Big(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\Big)(r)\right| = \left|(\mathcal{M}(\phi, A_\varphi, A_3))(r) - \Big(\mathcal{M}\Big(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\Big)\Big)(r)\right| \\
&\le C(S)\left|\left(\int_0^r \frac{1}{s}\int_0^s \sigma\left|(\phi, A_\varphi, A_3)(\sigma) - \Big(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\Big)(\sigma)\right| d\sigma ds,\right.\right. \\
&\qquad \frac{1}{r}\int_0^r s\int_0^s \left|(\phi, A_\varphi, A_3)(\sigma) - \Big(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\Big)(\sigma)\right| d\sigma ds, \\
&\qquad \left.\left.\int_0^r \frac{1}{s}\int_0^s \sigma\left|(\phi, A_\varphi, A_3)(\sigma) - \Big(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\Big)(\sigma)\right| d\sigma ds\right)\right| \\
&\le C(S)\cdot\sqrt{3}R_0\int_0^r \left|(\phi, A_\varphi, A_3)(s) - \Big(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\Big)(s)\right| ds
\end{aligned}$$

for each $r \in [0, R_0]$. Thus, the two fixed points coincide due to Gronwall's lemma. □

3.4.3 Direct construction

Since the above proof of existence of steady states is not constructive, in this section we provide a method to obtain steady states which is constructive. To this end, we define an approximating sequence $\left(\left(\phi^k, A_\varphi^k, A_3^k\right)\right)_{k\in\mathbb{N}_0}$ recursively via

$$\left(\phi^0, A_\varphi^0, A_3^0\right) = (0,0,0), \quad \left(\phi^{k+1}, A_\varphi^{k+1}, A_3^{k+1}\right) = \mathcal{M}\left(\phi^k, A_\varphi^k, A_3^k\right).$$

To show that this sequence indeed converges to a (and thus the) fixed point of $\mathcal{M}$, we first prove that this sequence is bounded. In fact, the a priori estimates of Section 3.4.1 carry over.

Lemma 3.4.5. *For each $k \in \mathbb{N}_0$ and $r \in [0, R_0]$ it holds that*

$$\left|\phi^k(r)\right|, \left|A_3^k(r)\right| \le \xi(r), \quad \left|A_\varphi^k(r)\right| \le \zeta(r).$$

In particular,

$$\left\|\left(\phi^k, A_\varphi^k, A_3^k\right)\right\|_{C([0,R_0];\mathbb{R}^3)} \le \sqrt{2\xi(R_0)^2 + \zeta(R_0)^2} =: S.$$

Proof. We prove

$$\left|\phi^k(r)\right|, \left|A_3^k(r)\right| \le \sum_{j=1}^k \frac{c_1 c_2^{j-1}}{4^j (j!)^2} r^{2j}, \quad \left|A_\varphi^k(r)\right| \le \sum_{j=1}^k \frac{c_1 c_2^{j-1}}{\left(1-\frac{1}{4j^2}\right) 4^j (j!)^2} r^{2j}$$

via induction, from which the assertion follows. Indeed, this obviously holds true for $k = 0$, and thanks to (3.4.1) we also have

$$\begin{aligned}
&\left|\phi^{k+1}(r)\right|, \left|A_3^{k+1}(r)\right| \le \int_0^r \frac{1}{s} \int_0^s \sigma\left(c_1 + c_2\left|\phi^k(\sigma)\right|\right) d\sigma ds \\
&\le \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma \sum_{j=1}^k \frac{c_1 c_2^{j-1}}{4^j (j!)^2} \sigma^{2j}\, d\sigma ds = \frac{c_1}{4} r^2 + c_2 \int_0^r \sum_{j=1}^k \frac{c_1 c_2^{j-1}}{4^j (j!)^2 (2j+2)} s^{2j+1}\, ds \\
&= \frac{c_1}{4} r^2 + \sum_{j=1}^k \frac{c_1 c_2^j}{4^{j+1}\left((j+1)!\right)^2} r^{2j+2} = \sum_{j=1}^{k+1} \frac{c_1 c_2^{j-1}}{4^j (j!)^2} r^{2j}
\end{aligned}$$

and

$$\begin{aligned}
\left|A_\varphi^{k+1}(r)\right| &\le \frac{1}{r} \int_0^r s \int_0^s \left(c_1 + c_2\left|\phi^k(\sigma)\right|\right) d\sigma ds \le \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r s \int_0^s \sum_{j=1}^k \frac{c_1 c_2^{j-1}}{4^j (j!)^2} \sigma^{2j}\, d\sigma ds \\
&= \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r \sum_{j=1}^k \frac{c_1 c_2^{j-1}}{4^j (j!)^2 (2j+1)} s^{2j+2}\, ds \\
&= \frac{c_1}{3} r^2 + \sum_{j=1}^k \frac{c_1 c_2^j}{\left(1 - \frac{1}{4(j+1)^2}\right) 4^{j+1}\left((j+1)!\right)^2} r^{2j+2} = \sum_{j=1}^{k+1} \frac{c_1 c_2^{j-1}}{\left(1-\frac{1}{4j^2}\right) 4^j (j!)^2} r^{2j}.
\end{aligned}$$

□

We can now prove the following result.

Theorem 3.4.6. *Let Conditions 3.3.3 and 3.3.8 hold. Then,* $\left(\left(\phi^k, A_\varphi^k, A_3^k\right)\right)_{k\in\mathbb{N}_0}$*, where*

$$\left(\phi^0, A_\varphi^0, A_3^0\right) = (0,0,0), \quad \left(\phi^{k+1}, A_\varphi^{k+1}, A_3^{k+1}\right) = \mathcal{M}\left(\phi^k, A_\varphi^k, A_3^k\right),\ k \in \mathbb{N}_0,$$

is a Cauchy sequence in $C\left([0,R_0];\mathbb{R}^3\right)$*. The limit* (ϕ, A_φ, A_3) *is the fixed point of* $\mathcal{M}$*, whence* $\left((f^\alpha)_\alpha, \phi, A\right)$ *is an axially symmetric steady state of the two and one-half dimensional relativistic Vlasov–Maxwell system on* $\overline{\Omega}$ *with external potential* A^{ext}*, where the* f^α *are written in terms of* ϕ *and* A*; cf.* (3.3.7).

Proof. We abbreviate $P^k := \left(\phi^k, A_\varphi^k, A_3^k\right)$ for $k \in \mathbb{N}_0$. By Lemma 3.4.5 and (3.4.13) we have

$$\begin{aligned}&\left|\phi^{k+1}(r) - \phi^k(r)\right|, \left|A_\varphi^{k+1}(r) - A_\varphi^k(r)\right|, \left|A_3^{k+1}(r) - A_3^k(r)\right| \\ &\leq C(S)\int_0^r \int_0^s \left|P^k(\sigma) - P^{k-1}(\sigma)\right| d\sigma ds\end{aligned}$$

and thus

$$\left|P^{k+1}(r) - P^k(r)\right| \leq \sqrt{3}C(S)\int_0^r \int_0^s \left|P^k(\sigma) - P^{k-1}(\sigma)\right| d\sigma ds$$

for $r \in [0, R_0]$, $k \in \mathbb{N}$. With $C := \sqrt{3}C(S)$ this yields

$$\left|P^{k+1}(r) - P^k(r)\right| \leq \frac{SC^k}{(2k)!} r^{2k}$$

for each $r \in [0, R_0]$, $k \in \mathbb{N}_0$ via induction: Indeed, this estimate obviously holds true for $k = 0$, and moreover we have

$$\begin{aligned}\left|P^{k+1}(r) - P^k(r)\right| &\leq C\int_0^r \int_0^s \left|P^k(\sigma) - P^{k-1}(\sigma)\right| d\sigma ds \leq C\int_0^r \int_0^s \frac{SC^{k-1}}{(2k-2)!}\sigma^{2k-2}\, d\sigma ds \\ &= \frac{SC^k}{(2k-1)!}\int_0^r s^{2k-1}\, ds = \frac{SC^k}{(2k)!} r^{2k}\end{aligned}$$

for $k \geq 1$. Therefore, for each $m \geq k$ and $r \in [0, R_0]$ it holds that

$$\left|P^m(r) - P^k(r)\right| \leq \sum_{j=k}^{m-1}\left|P^{j+1}(r) - P^j(r)\right| \leq \sum_{j=k}^{m-1}\frac{SC^j}{(2j)!}r^{2j} \leq \sum_{j=k}^{\infty}\frac{SC^j}{(2j)!}R_0^{2j}.$$

Since the series $\sum_{j=0}^{\infty}\frac{SC^j}{(2j)!}R_0^{2j}$ converges, it follows that $\left(P^k\right)$ is a Cauchy sequence in $C\left([0,R_0];\mathbb{R}^3\right)$. Passing to the limit, we easily see that

$$(\phi, A_\varphi, A_3) = \lim_{k\to\infty}\left(\phi^{k+1}, A_\varphi^{k+1}, A_3^{k+1}\right) = \lim_{k\to\infty}\mathcal{M}\left(\phi^k, A_\varphi^k, A_3^k\right) = \mathcal{M}(\phi, A_\varphi, A_3)$$

since $\mathcal{M}$ is continuous due to Lemma 3.4.3. Hence, (ϕ, A_φ, A_3) is a (and by Theorem 3.4.4 the) fixed point of $\mathcal{M}$ and the corresponding tuple $\left((f^\alpha)_\alpha, \phi, A\right)$ is a steady state. □

Remark 3.4.7. We should mention that there is yet another way to construct a fixed point of $\mathcal{M}$, which to some extent corresponds to the fixed point iteration above: Looking at (3.2.2) we see that this system of three ordinary differential equations has singular coefficients at $r = 0$. Firstly, we solve the integrated system, i.e., (3.3.11), on some small interval $[0, \delta]$ as follows: Choose some $S > \|\mathcal{M}(0,0,0)\|_{C([0,R_0];\mathbb{R}^3)}$ and let $0 < \delta \leq R_0$ such that

$$SC(S) \cdot \frac{\sqrt{34}}{12}\delta^2 + \|\mathcal{M}(0,0,0)\|_{C([0,R_0];\mathbb{R}^3)} \leq S \text{ and } C(S) \cdot \frac{\sqrt{34}}{12}\delta^2 < 1$$

where $C(S)$ is the constant from (3.4.13). Clearly, (3.4.13) also holds on $[0, \delta]$ for any $(\phi, A_\varphi, A_3), \left(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\right) \in C\left([0,\delta];\mathbb{R}^3\right)$ with $C\left([0,\delta];\mathbb{R}^3\right)$-norm (similarly defined as in (3.4.12)) less or equal S. For such potentials, proceeding as in (3.4.14) and (3.4.15) with R_0 replaced by δ, we conclude

$$\begin{aligned} &\left\|\mathcal{M}_\delta(\phi, A_\varphi, A_3) - \mathcal{M}_\delta\left(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\right)\right\|_{C([0,\delta];\mathbb{R}^3)} \\ &\leq C(S) \cdot \frac{\sqrt{34}}{12}\delta^2 \left\|(\phi, A_\varphi, A_3) - \left(\overline{\phi}, \overline{A}_\varphi, \overline{A}_3\right)\right\|_{C([0,\delta];\mathbb{R}^3)} \end{aligned}$$

where $\mathcal{M}_\delta$ is defined as $\mathcal{M}$ only R_0 replaced by δ. Thus, denoting

$$X := \left\{(\phi, A_\varphi, A_3) \in C\left([0,\delta];\mathbb{R}^3\right) \mid \left\|(\phi, A_\varphi, A_3)\right\|_{C([0,\delta];\mathbb{R}^3)} \leq S\right\},$$

the map $\mathcal{M}_\delta \colon X \to X$ is well-defined and a contraction by choice of δ, and therefore has a unique fixed point, which is the unique continuous solution of (3.3.11) on $[0, \delta]$. Secondly, we consider the system (3.2.2) of three ordinary differential equations on $[\delta, R_0]$, where all appearing coefficients are now smooth. We equip this system with the initial condition that the potentials themselves and their first derivatives at $r = \delta$ shall coincide with the values and first derivatives at $r = \delta$ of the solution on $[0, \delta]$ obtained in the first step—note that a posteriori these potentials on $[0, \delta]$ are of class C^2; cf. Lemma 3.3.12.(i). Since the right-hand sides of (3.2.2) written in terms of the potentials are continuous, locally Lipschitz continuous with respect to the potentials, and grow at most linearly in the potentials due to Lemma 3.3.10, (3.4.13), and (3.4.1), we infer from standard ODE theory that this initial value problem has a unique solution on $[\delta, R_0]$. Altogether, combining the obtained potentials on $[0, \delta]$ and $[\delta, R_0]$, we arrive at a solution of (3.3.11) on $[0, R_0]$, that is, a fixed point of $\mathcal{M}$.

3.4.4 Further properties

A desirable property of a steady state is that it is compactly supported with respect to v. It is well known in similar settings that for this there should exist a cut-off energy.

Indeed, the existence of such a cut-off energy guarantees this property also in our setting, as is shown below. Another obvious property which should hold is that the steady state is nontrivial—for example, we have not excluded the pointless possibility $\eta^\alpha = 0$ yet. We first state conditions under which a steady state indeed has these two properties and then prove the corresponding theorem.

Condition 3.4.8. For each $\alpha = 1, \dots, N$ it holds that:

(i) There exists $\mathcal{E}_0^\alpha \geq 0$ such that $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ if $\mathcal{E} \geq \mathcal{E}_0^\alpha$.
(ii) There exist $\mathcal{E}_u^\alpha > m_\alpha$, $\mathcal{G}_l^\alpha < 0$, $\mathcal{G}_u^\alpha > 0$, and
   (1) $\mathcal{F}_l^\alpha < 0$, $\mathcal{F}_u^\alpha \geq 0$ or
   (2) $\mathcal{F}_l^\alpha \leq 0$, $\mathcal{F}_u^\alpha > 0$

   such that

$$\forall (\mathcal{E}, \mathcal{F}, \mathcal{G}) \in \left]m_\alpha, \mathcal{E}_u^\alpha\right[\times \left]\mathcal{F}_l^\alpha, \mathcal{F}_u^\alpha\right[\times \left]\mathcal{G}_l^\alpha, \mathcal{G}_u^\alpha\right[: \eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) > 0.$$

Theorem 3.4.9. *Let Conditions 3.3.3 and 3.3.8 hold and let* $\left((f^\alpha)_\alpha, \phi, A\right)$ *be a steady state, where* $\left(\phi, A_\varphi, A_3\right)$ *is the fixed point of* $\mathcal{M}$ *and the* f^α *are given by* (3.3.7)*. Then we have:*

(i) If Condition 3.4.8.(i) is satisfied, then the steady state is compactly supported with respect to v*.*
(ii) If Condition 3.4.8.(ii) is satisfied, then the steady state is nontrivial.

Proof. As for part 3.4.9.(i), we find that, if

$$|v| \geq \max_{\alpha=1,\dots,N} \left(\mathcal{E}_0^\alpha + |q_\alpha| \xi(R_0)\right),$$

then for each $\alpha = 1, \dots, N$ and $x \in \overline{\Omega}$ we have

$$\mathcal{E}^\alpha(x, v) = v_\alpha^0 + q_\alpha \phi(r) \geq |v| - |q_\alpha| \xi(R_0) \geq \mathcal{E}_0^\alpha$$

due to Lemma 3.4.1 and hence $f^\alpha(x, v) = 0$.

As for part 3.4.9.(ii), we follow the idea of [Kno19]. For fixed $\alpha \in \{1, \dots, N\}$ choose $0 < r_\alpha \leq \frac{R_0}{2}$ small enough such that

$$\sqrt{m_\alpha^2 + r_\alpha} - |q_\alpha| \xi(2r_\alpha) > m_\alpha, \quad \sqrt{m_\alpha^2 + 5r_\alpha} + |q_\alpha| \xi(2r_\alpha) < \mathcal{E}_u^\alpha,$$
$$\sqrt{r_\alpha} + |q_\alpha| \xi(2r_\alpha) + |q_\alpha| \sup_{0 \leq r \leq 2r_\alpha} \left|A_3^{\mathrm{ext}}(r)\right| < \min\{-\mathcal{G}_l^\alpha, \mathcal{G}_u^\alpha\}$$

and

$$4 r_\alpha^{\frac{3}{2}} + 2|q_\alpha| r_\alpha \zeta(2r_\alpha) + 2|q_\alpha| r_\alpha \sup_{0 \leq r \leq 2r_\alpha} \left|A_\varphi^{\mathrm{ext}}(r)\right| < -\mathcal{F}_l^\alpha,$$
$$-\frac{1}{\sqrt{2}} r_\alpha^{\frac{3}{2}} + 2|q_\alpha| r_\alpha \zeta(2r_\alpha) + 2|q_\alpha| r_\alpha \sup_{0 \leq r \leq 2r_\alpha} \left|A_\varphi^{\mathrm{ext}}(r)\right| < 0$$

in case 3.4.8.(ii).(1) and

$$4r_\alpha^{\frac{3}{2}} + 2\left|q_\alpha\right| r_\alpha \zeta(2r_\alpha) + 2\left|q_\alpha\right| r_\alpha \sup_{0\le r\le 2r_\alpha}\left|A_\varphi^{\mathrm{ext}}(r)\right| < \mathcal{F}_u^\alpha,$$

$$\frac{1}{\sqrt{2}} r_\alpha^{\frac{3}{2}} - 2\left|q_\alpha\right| r_\alpha \zeta(2r_\alpha) - 2\left|q_\alpha\right| r_\alpha \sup_{0\le r\le 2r_\alpha}\left|A_\varphi^{\mathrm{ext}}(r)\right| > 0$$

in case 3.4.8.(ii).(2), respectively. Indeed, this choice of r_α is possible since $\xi(r), \zeta(r), rA_\varphi^{\mathrm{ext}}(r) = O\left(r^2\right)$ for $r \to 0$, $A_3^{\mathrm{ext}}(0) = 0$, and $\frac{1}{2}, \frac{3}{2} \in]0,2[$. Next, let $\theta_\alpha := \frac{3\pi}{2}$ in case 3.4.8.(ii).(1) and $\theta_\alpha := \frac{\pi}{2}$ in case 3.4.8.(ii).(2), respectively, and let

$$S_\alpha := \Big\{(r,u,\theta,v_3) \in [0,R_0] \times [0,\infty[\times [0,2\pi] \times \mathbb{R} \mid r_\alpha < r < 2r_\alpha, \sqrt{r_\alpha} < u < 2\sqrt{r_\alpha}, \\ \theta_\alpha - \frac{\pi}{4} < \theta < \theta_\alpha + \frac{\pi}{4}, -\sqrt{r_\alpha} < v_3 < \sqrt{r_\alpha}\Big\}.$$

In (r,u,θ,v_3)-coordinates, where $u = \sqrt{v_1^2 + v_2^2}$ and θ is the polar angle in the (v_1,v_2)-plane with basis $\left(e_r, e_\varphi\right)$, it holds that

$$\mathcal{E}^\alpha(r,u,\theta,v_3) = \sqrt{m_\alpha^2 + u^2 + v_3^2} + q_\alpha \phi(r),$$

$$\mathcal{F}^\alpha(r,u,\theta,v_3) = r\Big(u \sin\theta + q_\alpha A_\varphi(r) + q_\alpha A_\varphi^{\mathrm{ext}}(r)\Big),$$

$$\mathcal{G}^\alpha(r,u,\theta,v_3) = v_3 + q_\alpha A_3(r) + q_\alpha A_3^{\mathrm{ext}}(r).$$

For each $(r,u,\theta,v_3) \in S_\alpha$, we have by Lemma 3.4.1

$$\mathcal{E}^\alpha(r,u,\theta,v_3) \ge \sqrt{m_\alpha^2 + r_\alpha} - \left|q_\alpha\right|\xi(2r_\alpha) > m_\alpha,$$

$$\mathcal{E}^\alpha(r,u,\theta,v_3) \le \sqrt{m_\alpha^2 + 5r_\alpha} + \left|q_\alpha\right|\xi(2r_\alpha) < \mathcal{E}_u^\alpha,$$

$$\mathcal{G}^\alpha(r,u,\theta,v_3) \ge -\sqrt{r_\alpha} - \left|q_\alpha\right|\xi(2r_\alpha) - \left|q_\alpha\right| \sup_{0\le r\le 2r_\alpha}\left|A_3^{\mathrm{ext}}(r)\right| > \mathcal{G}_l^\alpha,$$

$$\mathcal{G}^\alpha(r,u,\theta,v_3) \le \sqrt{r_\alpha} + \left|q_\alpha\right|\xi(2r_\alpha) + \left|q_\alpha\right| \sup_{0\le r\le 2r_\alpha}\left|A_3^{\mathrm{ext}}(r)\right| < \mathcal{G}_u^\alpha$$

and

$$\mathcal{F}^\alpha(r,u,\theta,v_3) \ge -4r_\alpha^{\frac{3}{2}} - 2\left|q_\alpha\right| r_\alpha \zeta(2r_\alpha) - 2\left|q_\alpha\right| r_\alpha \sup_{0\le r\le 2r_\alpha}\left|A_\varphi^{\mathrm{ext}}(r)\right| > \mathcal{F}_l^\alpha,$$

$$\mathcal{F}^\alpha(r,u,\theta,v_3) \le -\frac{1}{\sqrt{2}} r_\alpha^{\frac{3}{2}} + 2\left|q_\alpha\right| r_\alpha \zeta(2r_\alpha) + 2\left|q_\alpha\right| r_\alpha \sup_{0\le r\le 2r_\alpha}\left|A_\varphi^{\mathrm{ext}}(r)\right| < 0 \le \mathcal{F}_u^\alpha$$

in case 3.4.8.(ii).(1) and

$$\mathcal{F}^\alpha(r,u,\theta,v_3) \le 4r_\alpha^{\frac{3}{2}} + 2\left|q_\alpha\right| r_\alpha \zeta(2r_\alpha) + 2\left|q_\alpha\right| r_\alpha \sup_{0\le r\le 2r_\alpha}\left|A_\varphi^{\mathrm{ext}}(r)\right| < \mathcal{F}_u^\alpha,$$

$$\mathcal{F}^\alpha(r,u,\theta,v_3) \geq \frac{1}{\sqrt{2}} r_\alpha^{\frac{3}{2}} - 2|q_\alpha| r_\alpha \zeta(2r_\alpha) - 2|q_\alpha| r_\alpha \sup_{0\leq r\leq 2r_\alpha} \left|A_\varphi^{\text{ext}}(r)\right| > 0 \geq \mathcal{F}_l^\alpha$$

in case 3.4.8.(ii).(2), respectively. Therefore,

$$ru\eta^\alpha(\mathcal{E}^\alpha(r,u,\theta,v_3), \mathcal{F}^\alpha(r,u,\theta,v_3), \mathcal{G}^\alpha(r,u,\theta,v_3)) > 0.$$

Thus, we have

$$\begin{aligned}\int_{B_{R_0}} \int_{\mathbb{R}^3} f^\alpha \, dv d(x_1,x_2) &= 2\pi \int_0^{R_0} r \int_{\mathbb{R}^3} \eta^\alpha(\mathcal{E}^\alpha, \mathcal{F}^\alpha, \mathcal{G}^\alpha) \, dv dr \\ &= 2\pi \int_0^{R_0} \int_{\mathbb{R}} \int_0^\infty \int_0^{2\pi} ru\eta^\alpha(\mathcal{E}^\alpha, \mathcal{F}^\alpha, \mathcal{G}^\alpha) \, d\theta du dv_3 dr \\ &\geq \int_{S_\alpha} ru\eta^\alpha(\mathcal{E}^\alpha, \mathcal{F}^\alpha, \mathcal{G}^\alpha) \, d(r,u,\theta,v_3) > 0\end{aligned}$$

since S_α has positive Lebesgue measure. In particular, $f^\alpha \not\equiv 0$. □

Remark 3.4.10. Intuitively, the proof of Theorem 3.4.9.(ii) shows that, for each species, there are some particles near the symmetry axis with small momentum. Moreover, it was proved that in case 3.4.8.(ii).(1) (or 3.4.8.(ii).(2), respectively) there are some particles with negative (or positive, respectively) canonical angular momentum.

3.5 Confined steady states

There remains to find conditions on the external potential A^{ext} and the ansatz functions η^α under which a corresponding steady state is confined. We consider two possibilities:

- A suitable A_φ^{ext} (corresponding to an external magnetic field in the e_3-direction) ensures confinement. This configuration is often called "θ-pinch".
- A suitable A_3^{ext} (corresponding to an external magnetic field in the e_φ-direction) ensures confinement. This configuration is often called "z-pinch".

A combination of these two—often called "screw-pinch"—would of course also be possible, whence the following options are not exhaustive.

Theorem 3.5.1. *Let Conditions 3.3.3, 3.3.8, and 3.4.8 hold and let* $\left((f^\alpha)_\alpha, \phi, A\right)$ *be a steady state, where* (ϕ, A_φ, A_3) *is the fixed point of* $\mathcal{M}$ *and the* f^α *are given by* (3.3.7)*. We define*

$$\mathcal{N} := \left\{\alpha \in \{1,\ldots,N\} \mid q_\alpha < 0\right\}, \quad \mathcal{P} := \left\{\alpha \in \{1,\ldots,N\} \mid q_\alpha > 0\right\}.$$

Furthermore, let $0 < R < R_0$ *and one of the following four options hold:*

(i) (θ-pinch)

(a) *For each $\alpha \in \mathcal{N}$, case 3.4.8.(ii).(1) is satisfied and we have $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ whenever $\mathcal{F} \geq 0$ (thus, necessarily $\mathcal{F}_u^\alpha = 0$). For each $\alpha \in \mathcal{P}$, case 3.4.8.(ii).(2) is satisfied and we have $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ whenever $\mathcal{F} \leq 0$ (thus, necessarily $\mathcal{F}_l^\alpha = 0$). Moreover, assume*

$$A_\varphi^{\mathrm{ext}}(r) \leq -a_\varphi(r), \quad R \leq r \leq R_0.$$

(b) *For each $\alpha \in \mathcal{N}$, case 3.4.8.(ii).(2) is satisfied and we have $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ whenever $\mathcal{F} \leq 0$ (thus, necessarily $\mathcal{F}_l^\alpha = 0$). For each $\alpha \in \mathcal{P}$, case 3.4.8.(ii).(1) is satisfied and we have $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ whenever $\mathcal{F} \geq 0$ (thus, necessarily $\mathcal{F}_u^\alpha = 0$). Moreover, assume*

$$A_\varphi^{\mathrm{ext}}(r) \geq a_\varphi(r), \quad R \leq r \leq R_0.$$

Here,

$$a_\varphi(r) := \max_{\alpha=1,\dots,N} \frac{\sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2}}{|q_\alpha|} + \zeta(r).$$

(ii) *(z-pinch)*

(a) *For each $\alpha \in \mathcal{N}$, there exists $\mathcal{G}_0^\alpha < 0$ such that $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ whenever $\mathcal{G} \leq \mathcal{G}_0^\alpha$. For each $\alpha \in \mathcal{P}$, there exists $\mathcal{G}_0^\alpha > 0$ such that $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ whenever $\mathcal{G} \geq \mathcal{G}_0^\alpha$. Moreover, assume*

$$A_3^{\mathrm{ext}}(r) \geq a_3(r), \quad R \leq r \leq R_0.$$

(b) *For each $\alpha \in \mathcal{N}$, there exists $\mathcal{G}_0^\alpha > 0$ such that $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ whenever $\mathcal{G} \geq \mathcal{G}_0^\alpha$. For each $\alpha \in \mathcal{P}$, there exists $\mathcal{G}_0^\alpha < 0$ such that $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ whenever $\mathcal{G} \leq \mathcal{G}_0^\alpha$. Moreover, assume*

$$A_3^{\mathrm{ext}}(r) \leq -a_3(r), \quad R \leq r \leq R_0.$$

Here,

$$a_3(r) := \max_{\alpha=1,\dots,N} \frac{|\mathcal{G}_0^\alpha| + \sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2}}{|q_\alpha|} + \xi(r).$$

Then the steady state is confined with radius at most R, compactly supported with respect to v, and nontrivial.

Proof. First note that for each $(x, v) \in \overline{\Omega} \times \mathbb{R}^3$ and $\alpha = 1, \dots, N$ we have $f^\alpha(x, v) = 0$ if

$$|v| \geq \sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2}$$

since then

$$\mathcal{E}^\alpha(x,v) \geq \sqrt{m_\alpha^2 + |v|^2} - |q_\alpha|\xi(r) \geq \mathcal{E}_0^\alpha$$

by Lemma 3.4.1. Thus, for each $\alpha = 1, \dots, N$ it suffices to consider $v \in \mathbb{R}^3$ with

$$|v| < \sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2}.$$

In the following, always let $r \in [R, R_0]$, $\alpha \in \mathcal{N}$, $\beta \in \mathcal{P}$, and v as above.

If option 3.5.1.(i).(a) is satisfied, we have

$$\begin{aligned}
\mathcal{F}^\alpha(x,v) &\geq r\left(-|v| + q_\alpha\zeta(r) + q_\alpha A_\varphi^{\mathrm{ext}}(r)\right) \geq r\left(-|v| + q_\alpha\zeta(r) - q_\alpha a_\varphi(r)\right) \\
&\geq r\left(-\sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2} + q_\alpha\zeta(r) - q_\alpha\left(\frac{\sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2}}{-q_\alpha} + \zeta(r)\right)\right) = 0, \\
\mathcal{F}^\beta(x,v) &\leq r\left(|v| + q_\beta\zeta(r) + q_\beta A_\varphi^{\mathrm{ext}}(r)\right) \leq r\left(|v| + q_\beta\zeta(r) - q_\beta a_\varphi(r)\right) \\
&\leq r\left(\sqrt{\left(\mathcal{E}_0^\beta + |q_\beta|\xi(r)\right)^2 - m_\beta^2} + q_\beta\zeta(r) - q_\beta\left(\frac{\sqrt{\left(\mathcal{E}_0^\beta + |q_\beta|\xi(r)\right)^2 - m_\beta^2}}{q_\beta} + \zeta(r)\right)\right) = 0
\end{aligned}$$

and thus $f^\alpha(x,v) = f^\beta(x,v) = 0$.

If option 3.5.1.(i).(b) is satisfied, we have

$$\begin{aligned}
\mathcal{F}^\alpha(x,v) &\leq r\left(|v| - q_\alpha\zeta(r) + q_\alpha A_\varphi^{\mathrm{ext}}(r)\right) \leq r\left(|v| - q_\alpha\zeta(r) + q_\alpha a_\varphi(r)\right) \\
&\leq r\left(\sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2} - q_\alpha\zeta(r) + q_\alpha\left(\frac{\sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2}}{-q_\alpha} + \zeta(r)\right)\right) = 0, \\
\mathcal{F}^\beta(x,v) &\geq r\left(-|v| - q_\beta\zeta(r) + q_\beta A_\varphi^{\mathrm{ext}}(r)\right) \geq r\left(-|v| - q_\beta\zeta(r) + q_\beta a_\varphi(r)\right) \\
&\geq r\left(-\sqrt{\left(\mathcal{E}_0^\beta + |q_\beta|\xi(r)\right)^2 - m_\beta^2} - q_\beta\zeta(r) + q_\beta\left(\frac{\sqrt{\left(\mathcal{E}_0^\beta + |q_\beta|\xi(r)\right)^2 - m_\beta^2}}{q_\beta} + \zeta(r)\right)\right) = 0
\end{aligned}$$

and thus $f^\alpha(x,v) = f^\beta(x,v) = 0$.

If option 3.5.1.(ii).(a) is satisfied, we have

$$\mathcal{G}^\alpha(x,v) \leq |v| - q_\alpha\xi(r) + q_\alpha A_3^{\mathrm{ext}}(r) \leq |v| - q_\alpha\xi(r) + q_\alpha a_3(r)$$

$$\leq \sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2} - q_\alpha\xi(r) + q_\alpha\left(\frac{-\mathcal{G}_0^\alpha + \sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2}}{-q_\alpha} + \xi(r)\right)$$

$$= \mathcal{G}_0^\alpha,$$

$$\mathcal{G}^\beta(x,v) \geq -|v| - q_\beta\xi(r) + q_\beta A_3^{\mathrm{ext}}(r) \geq -|v| - q_\beta\xi(r) + q_\beta a_3(r)$$

$$\geq -\sqrt{\left(\mathcal{E}_0^\beta + |q_\beta|\xi(r)\right)^2 - m_\beta^2} - q_\beta\xi(r) + q_\beta\left(\frac{\mathcal{G}_0^\beta + \sqrt{\left(\mathcal{E}_0^\beta + |q_\beta|\xi(r)\right)^2 - m_\beta^2}}{q_\beta} + \xi(r)\right) = \mathcal{G}_0^\beta$$

and thus $f^\alpha(x,v) = f^\beta(x,v) = 0$.

If option 3.5.1.(ii).(b) is satisfied, we have

$$\mathcal{G}^\alpha(x,v) \geq -|v| + q_\alpha\xi(r) + q_\alpha A_3^{\mathrm{ext}}(r) \geq -|v| + q_\alpha\xi(r) - q_\alpha a_3(r)$$

$$\geq -\sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2} + q_\alpha\xi(r) - q_\alpha\left(\frac{\mathcal{G}_0^\alpha + \sqrt{\left(\mathcal{E}_0^\alpha + |q_\alpha|\xi(r)\right)^2 - m_\alpha^2}}{-q_\alpha} + \xi(r)\right)$$

$$= \mathcal{G}_0^\alpha,$$

$$\mathcal{G}^\beta(x,v) \leq |v| + q_\beta\xi(r) + q_\beta A_3^{\mathrm{ext}}(r) \leq |v| + q_\beta\xi(r) - q_\beta a_3(r)$$

$$\leq \sqrt{\left(\mathcal{E}_0^\beta + |q_\beta|\xi(r)\right)^2 - m_\beta^2} + q_\beta\xi(r) - q_\beta\left(\frac{-\mathcal{G}_0^\beta + \sqrt{\left(\mathcal{E}_0^\beta + |q_\beta|\xi(r)\right)^2 - m_\beta^2}}{q_\beta} + \xi(r)\right) = \mathcal{G}_0^\beta$$

and thus $f^\alpha(x,v) = f^\beta(x,v) = 0$.

Hence, in all four cases the steady state is confined with radius at most R. That the steady state is compactly supported with respect to v and nontrivial has already been proved in Theorem 3.4.9. □

We point out that ξ and ζ—and thus a_φ and a_3—*do not depend* on A_φ^{ext} and A_3^{ext}, whence the above inequality conditions on A_φ^{ext} or A_3^{ext}, respectively, are *explicit*.

Intuitively, for example, option 3.5.1.(i).(a) means that all negatively (positively) charged particles have negative (positive) canonical angular momentum thanks to the ansatz function and that, however, for $R \leq r \leq R_0$ a sufficiently small negative A_φ^{ext} would cause a positive (negative) canonical angular momentum of negatively (positively) charged particles possibly located there. Similarly, for example, option 3.5.1.(ii).(a) says that there cannot exist negatively (positively) charged particles with too small (large) third component of the canonical momentum thanks to the ansatz function and that, however, for $R \leq r \leq R_0$ a sufficiently large positive A_3^{ext} would cause a too small (large) third component of the canonical momentum of negatively (positively) charged particles possibly located there.

Since $A_\varphi^{\mathrm{ext}}(0) = A_3^{\mathrm{ext}}(0) = 0$ due to Condition 3.3.3 and $a_\varphi(0) \neq 0 \neq a_3(0)$ due to Condition 3.4.8, $\left|A_\varphi^{\mathrm{ext}}\right|$ or $\left|A_3^{\mathrm{ext}}\right|$, respectively, has to increase sufficiently fast on $[0, R]$ to satisfy the respective condition on $[R, R_0]$. Moreover, a_φ and a_3 increase when the ansatz functions η^α (and hence ξ, ζ) increase. Thus, a larger external magnetic field is necessary to confine a larger amount of particles (as one would expect).

To obtain a specific example for an external magnetic field ensuring confinement, we consider a θ-pinch configuration and a homogeneous external magnetic field parallel to the symmetry axis, i.e., $B^{\mathrm{ext}} = B_3^{\mathrm{ext}} e_3$ and $B_3^{\mathrm{ext}} \equiv b$ for some constant $b \in \mathbb{R}$. As $B_3^{\mathrm{ext}}(r) = \frac{1}{r}\left(r A_\varphi^{\mathrm{ext}}(r)\right)'$ and $A_\varphi^{\mathrm{ext}}(0) = 0$, it has to holds that $A_\varphi^{\mathrm{ext}}(r) = \frac{b}{2} r$. Therefore, the steady state is confined for a sufficiently strong external magnetic field, that is to say, if

$$|b| \geq 2 \sup_{r \in [R, R_0]} \frac{a_\varphi(r)}{r}$$

and $b < 0$ (if option 3.5.1.(i).(a) is satisfied) or $b > 0$ (if option 3.5.1.(i).(b) is satisfied), respectively. As opposed to this, no configuration can exist where the φ-component of the external magnetic field is constant (and nontrivial) since in this case A_3^{ext} would have to be a linear function of r because of $B_\varphi^{\mathrm{ext}} = -\left(A_3^{\mathrm{ext}}\right)'$, which contradicts the necessary condition $\left(A_3^{\mathrm{ext}}\right)'(0) = 0$.

We finish this section with an important remark.

Remark 3.5.2. Another interesting setting is that there is no confinement device and thus no boundary at $r = R_0$ in the first place. In this case, $\Omega = \mathbb{R}^3$ and no boundary conditions at $r = R_0$ have to be imposed. Moreover, Definition 3.3.6 can be suitably adapted to this new setting by abolishing (3.3.5b) and setting $R_0 = \infty$. If we seek a steady state of this new setting that is confined with radius at most $R > 0$, we firstly choose a (slightly) larger $R_0 > R$, secondly consider the confinement problem as before with boundary at $r = R_0$ and choose A_φ^{ext} or A_3^{ext} suitably to ensure confinement of the obtained steady state with radius at most R, and thirdly "glue" this steady state defined on $[0, R_0]$ and the vacuum solution on $[R_0, \infty[$ together, i.e., extend each f^α by zero and the potentials by their respective integral formula, that is,

$$\begin{aligned}
\phi(r) &= -4\pi \int_0^r \frac{1}{s} \int_0^s \sigma\rho(\sigma)\, d\sigma ds \\
&= -4\pi \int_0^R \frac{1}{s} \int_0^s \sigma\rho(\sigma)\, d\sigma ds - 4\pi \int_R^r \frac{1}{s} \int_0^R \sigma\rho(\sigma)\, d\sigma ds \\
&= -4\pi \int_0^R \frac{1}{s} \int_0^s \sigma\rho(\sigma)\, d\sigma ds - 4\pi \int_0^R s\rho(s)\, ds \cdot (\ln r - \ln R), \\
A_\varphi(r) &= -\frac{4\pi}{r} \int_0^r s \int_0^s j_\varphi(\sigma)\, d\sigma ds \\
&= -\frac{4\pi}{r} \int_0^R s \int_0^s j_\varphi(\sigma)\, d\sigma ds - \frac{4\pi}{r} \int_R^r s \int_0^R j_\varphi(\sigma)\, d\sigma ds
\end{aligned}$$

$$= -\frac{4\pi}{r}\int_0^R s\int_0^s j_\varphi(\sigma)\,d\sigma ds - 2\pi\int_0^R j_\varphi(s)\,ds\cdot\left(r-\frac{R^2}{r}\right),$$

$$A_3(r) = -4\pi\int_0^r \frac{1}{s}\int_0^s \sigma j_3(\sigma)\,d\sigma ds$$

$$= -4\pi\int_0^R \frac{1}{s}\int_0^s \sigma j_3(\sigma)\,d\sigma ds - 4\pi\int_R^r \frac{1}{s}\int_0^R \sigma j_3(\sigma)\,d\sigma ds$$

$$= -4\pi\int_0^R \frac{1}{s}\int_0^s \sigma j_3(\sigma)\,d\sigma ds - 4\pi\int_0^R s j_3(s)\,ds\cdot(\ln r - \ln R)$$

for $r \geq R$. Note that for this procedure it is important that the f^α already vanish on $[R, R_0]$ so that the composite f^α have no jumps at $r = R_0$. With the identities above we can furthermore determine the asymptotics of the potentials for $r \to \infty$. In particular,

$$\phi(r) = -2a\ln r + \text{const.},\quad A_3(r) = -2b\ln r + \text{const.},\quad r \geq R,$$
$$A_\varphi(r) + cr = O\big(r^{-1}\big)\quad \text{for } r \to \infty,$$

where

$$a = 2\pi\int_0^R s\rho(s)\,ds,\quad b = 2\pi\int_0^R s j_3(s)\,ds,\quad c = 2\pi\int_0^R j_\varphi(s)\,ds.$$

Here, a and b can be interpreted as the total charge and the third component of the total current on each slice perpendicular to the symmetry axis.

3.6 Final remarks

From a fusion plasma physics point of view, a very interesting case is that Ω is a torus instead of an infinitely long cylinder. In accordance with Remark 3.3.2, we choose an orthogonal curvilinear coordinate system for which tori are coordinate surfaces. A canonical choice are the so-called "toroidal coordinates" (ξ, η, φ) from the range $0 \leq \xi \leq 1$, $0 \leq \eta < 2\pi$, $0 \leq \varphi < 2\pi$. Here and in the following, we adopt the notation of [Bat97] for the coordinates (ξ or η, respectively, are now coordinates and no longer a function describing an a priori bound for the electric potential or an ansatz function, respectively, as above). Note that there are also other coordinates commonly called toroidal coordinates, for example, using $\tilde{\xi}$ instead of ξ, where $\xi^{-1} = \cosh\tilde{\xi}$. These toroidal coordinates are related to Cartesian coordinates via

$$x_1 = \frac{a_0\sqrt{1-\xi^2}\cos\varphi}{1-\xi\cos\eta},\quad x_2 = \frac{a_0\sqrt{1-\xi^2}\sin\varphi}{1-\xi\cos\eta},\quad x_3 = \frac{a_0\xi\sin\eta}{1-\xi\cos\eta}.$$

Toroidal coordinates result from rotating the two-dimensional bipolar coordinate system

$$x_1 = \frac{a_0\sqrt{1-\xi^2}}{1-\xi\cos\eta},\quad x_2 = \frac{a_0\xi\sin\eta}{1-\xi\cos\eta}$$

about the x_3-axis. The number $a_0 > 0$ yields the two foci $(a_0, 0)$ and $(-a_0, 0)$, which become a focal ring after rotation. Note that the coordinate surfaces $\xi = \text{const.}$ are tori, whence it seems a natural idea for an approach that the role of r in cylindrical coordinates should now be played by ξ in toroidal coordinates.

The main advantage of Ω being an infinitely long cylinder and thus assuming corresponding symmetries was that two variables (φ and x_3) of the Lagrangian $\mathcal{L}^\alpha$ written in cylindrical coordinates were cyclic. Thus, r was left as the only variable and the equations were reduced to three ordinary differential equations, which could be integrated explicitly. In other words, it was very important that Poisson's equation reduces to an ODE since under those symmetry assumptions the Laplacian

$$\Delta = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\varphi^2 + \partial_{x_3}^2 \equiv \frac{1}{r}\partial_r(r\partial_r)$$

is in fact an ordinary differential operator.

However, in toroidal coordinates the same strategy fails as the Laplace equation $\Delta\phi = 0$ is not fully separable in toroidal coordinates. Yet it is "R-separable", i.e., it admits a complete set of separable solutions of the form

$$\phi(\xi,\eta,\varphi) = R(\xi,\eta,\varphi)\Xi(\xi)H(\eta)\Phi(\varphi)$$

where

$$R(\xi,\eta,\varphi) \equiv R(\xi,\eta) = \sqrt{1 - \xi\cos\eta}.$$

In particular,

$$\Xi(\xi) \equiv \Xi_{mn}(\xi) = \begin{cases} \xi^{-\frac{1}{2}} P^n_{m-\frac{1}{2}}\left(\xi^{-1}\right) =: S_{mn}(\xi) & \text{or} \\ \xi^{-\frac{1}{2}} Q^n_{m-\frac{1}{2}}\left(\xi^{-1}\right) =: T_{mn}(\xi), \end{cases}$$

$$H(\eta) \equiv H_m(\eta) = \begin{cases} \cos(m\eta) & \text{or} \\ \sin(m\eta), \end{cases}$$

$$\Phi(\varphi) \equiv \Phi_n(\varphi) = \begin{cases} \cos(n\varphi) & \text{or} \\ \sin(n\varphi), \end{cases}$$

for parameters $m, n \in \mathbb{N}_0$. Here, P^μ_λ and Q^μ_λ are associated Legendre functions of the first and second kind. Note that S_{mn} and T_{mn} are singular at the focal ring, where $\xi = 0$. From this, a Green's function for a torus $\{\xi = \xi_0\}$ can be derived, namely,

$$\begin{aligned} &G\big((\xi,\eta,\varphi),(\xi',\eta',\varphi')\big) \\ &= \frac{1}{\pi a_0}\sqrt{1-\xi\cos\eta}\sqrt{1-\xi'\cos\eta'}\sum_{n=0}^\infty\sum_{m=0}^\infty (-1)^n \varepsilon_n \varepsilon_m \frac{\Gamma\left(m-n+\frac{1}{2}\right)}{\Gamma\left(m+n+\frac{1}{2}\right)} \frac{T_{mn}(\min\{\xi,\xi'\})}{T_{mn}(\xi_0)} \\ &\qquad \cdot \left[T_{mn}(\xi_0)S_{mn}(\max\{\xi,\xi'\}) - T_{mn}(\max\{\xi,\xi'\})S_{mn}(\xi_0)\right] \end{aligned}$$

$$\cdot \cos\big(m(\eta - \eta')\big) \cos\big(n(\varphi - \varphi')\big); \tag{3.6.1}$$

see [Bat97]. Here, $\varepsilon_0 = 1$, $\varepsilon_n = 2$ ($n \geq 2$), and Γ is the Gamma function.

Thus, a strategy to construct steady states confined in a torus based on our previous strategy could be the following:

1. Consider two free variables (ξ, η) instead of one (r) as before.
2. Thus, the number of invariants corresponding to space symmetry is reduced from two $(\mathcal{F}^\alpha, \mathcal{G}^\alpha)$ to only one $(\mathcal{F}^\alpha)$. Therefore, only A_φ^{ext} (and no longer A_3^{ext}) is important and may ensure confinement.
3. Since the current density j now has only a φ-component, only differential equations for ϕ and A_φ have to be considered; the other components of A can be set to zero without loss of generality.
4. Write down representations for ρ and j_φ and derive estimates in terms of the potentials. This will be clearly different to our previous setting since we only have two invariants instead of three as before and the same changes of variables as in the proof of Lemma 3.3.10 are not applicable anymore.
5. Solve the differential equations for ϕ and A_φ formally. As for ϕ, the Green's function G, see (3.6.1) (where only $n = 0$ remains due to symmetry in φ), should be used. For the determination of A_φ, however, a "torsional" Green's function, which incorporates the impact of the basis vector e_φ in the equation for $A_\varphi = A \cdot e_\varphi$, provides a solution formula; cf. [Bat97].
6. Derive suitable a priori estimates for ϕ and A_φ using the above solution formulae and prove existence of steady states via a fixed point argument or applying the method of sub- and supersolutions as in [BF93].
7. Try to adjust A_φ^{ext} suitably to ensure confinement via imposing a condition on A_φ^{ext} in the region $\xi_c \leq \xi \leq \xi_0$ such that the plasma is confined within $\{\xi \leq \xi_c\}$ which is a proper subset of the fusion reactor $\Omega = \{\xi < \xi_0\}$. The external magnetic potential inside the confinement region $\{\xi \leq \xi_c\}$, however, cannot be arbitrary and is "influenced" by this condition since A_φ^{ext} should, for example, vanish at $\{\xi = 0\}$ (the focal ring) to ensure nontriviality of the steady state.

Such a configuration with only an external magnetic potential in the φ-direction that is independent on φ is in fact a z-pinch configuration (the role played by x_3 before in the case of a linear confinement device is now played by φ as the cylinder is bent into a torus). Thus, the corresponding magnetic field has no φ-component, i.e., lies in the cross-section of the torus. However, a main concept of a Tokamak is to supply a large toroidal magnetic field to ensure confinement. This is due to the empirical observation that z-pinches are subject to powerful instabilities, for example, the kink instability. To overcome (some of) these instabilities, a toroidal magnetic field should be added. These considerations lead to very interesting questions about the stability of steady states, which have not been addressed in this work. Firstly, in the case of an infinitely long cylinder as confinement device, it would be desirable to verify observations—in particular, z-pinches tend to be unstable and θ-pinches tend to be stable—analytically. Secondly, similar questions are interesting in the practice-oriented case of a toroidal confinement device, i.e., can pure z-pinches proved to be unstable and can an additional, suitably adjusted toroidal magnetic field ensure stability of (confined) steady states? For example, a criterion for linear stability without the presence of external magnetic fields was given in [NS14]. Maybe a suitable external magnetic field ensures this criterion and/or prevents (or reduces) possible drifts in the ξ-direction, i.e., preventing the plasma particles from getting closer to the boundary of their container. Here, it would also be interesting to investigate whether toroidal coordinates (ξ, η, φ)—instead of the coordinates (s, θ, φ), where

$$x_1 = (\tilde{a} + s\cos\theta)\cos\varphi, \quad x_2 = (\tilde{a} + s\cos\theta)\sin\varphi, \quad x_3 = s\sin\theta,$$

that were used in [NS14] but do not allow R-separation of Laplace's equation—turn out to be advantageous.

Bibliography

[Bat97] J. W. Bates. "On toroidal Green's functions." In: *Journal of Mathematical Physics* 38.7 (1997), pp. 3679–3691.

[Bee69] P. R. Beesack. "Comparison theorems and integral inequalities for Volterra integral equations." In: *Proceedings of the American Mathematical Society* 20.1 (1969), pp. 61–66.

[BF93] J. Batt and K. Fabian. "Stationary solutions of the relativistic Vlasov–Maxwell system of plasma physics." In: *Chinese Annals of Mathematics Series B* 14.3 (1993), pp. 253–278.

[BP87] R. Beals and V. Protopopescu. "Abstract time-dependent transport equations." In: *Journal of Mathematical Analysis and Applications* 121.2 (1987), pp. 370–405.

[Bre11] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer New York, 2011.

[CCM12] S. Caprino, G. Cavallaro, and C. Marchioro. "Time evolution of a Vlasov–Poisson plasma with magnetic confinement." In: *Kinetic and Related Models* 5.4 (2012), pp. 729–742.

[CCM14] S. Caprino, G. Cavallaro, and C. Marchioro. "On a magnetically confined plasma with infinite charge." In: *SIAM Journal on Mathematical Analysis* 46.1 (2014), pp. 133–164.

[CCM15] S. Caprino, G. Cavallaro, and C. Marchioro. "On a Vlasov–Poisson plasma confined in a torus by a magnetic mirror." In: *Journal of Mathematical Analysis and Applications* 427.1 (2015), pp. 31–46.

[CCM16] S. Caprino, G. Cavallaro, and C. Marchioro. "A Vlasov–Poisson plasma with unbounded mass and velocities confined in a cylinder by a magnetic mirror." In: *Kinetic and Related Models* 9.4 (2016), pp. 657–686.

[Cla36] J. A. Clarkson. "Uniformly convex spaces." In: *Transactions of the American Mathematical Society* 40.3 (1936), pp. 396–414.

[Day41] M. M. Day. "Some more uniformly convex spaces." In: *Bulletin of the American Mathematical Society* 47.6 (1941), pp. 504–507.

[Day44] M. M. Day. "Uniform convexity in factor and conjugate spaces." In: *Annals of Mathematics* 45.2 (1944), pp. 375–385.

[Deg90] P. Degond. "Solutions stationnaires explicites du système de Vlasov–Maxwell relativiste." In: *Comptes Rendus de l'Académie des Sciences Paris, Série I* 310 (1990), pp. 607–612.

[DL89] R. J. Di Perna and P.-L. Lions. "Global weak solutions of Vlasov–Maxwell systems." In: *Communications on Pure and Applied Mathematics* 42.6 (1989), pp. 729–757.

[Dra03] S. S. Dragomir. *Some Gronwall Type Inequalities and Applications*. Nova Science Publishers, 2003.

[Eva10] L. C. Evans. *Partial Differential Equations*. 2nd ed. Vol. 19. Graduate studies in mathematics. American Mathematical Society, 2010.

[GH15] O. Glass and D. Han-Kwan. "On the controllability of the relativistic Vlasov–Maxwell system." In: *Journal de Mathématiques Pures et Appliquées* 103.3 (2015), pp. 695–740.

[Gla96] R. T. Glassey. *The Cauchy Problem in Kinetic Theory*. Other Titles in Applied Mathematics. Society for Industrial and Applied Mathematics, 1996.

[GMP87] W. Greenberg, C. van der Mee, and V. Protopopescu. *Boundary Value Problems in Abstract Kinetic Theory*. Vol. 23. Operator Theory: Advances and Applications. Birkhäuser Basel, 1987.

[GS86] R. T. Glassey and W. A. Strauss. "Singularity formation in a collisionless plasma could occur only at high velocities." In: *Archive for Rational Mechanics and Analysis* 92.1 (1986), pp. 59–90.

[GS90] R. T. Glassey and J. Schaeffer. "On the 'one and one-half dimensional' relativistic Vlasov–Maxwell system." In: *Mathematical Methods in the Applied Sciences* 13.2 (1990), pp. 169–179.

[GS97] R. T. Glassey and J. Schaeffer. "The "two and one-half dimensional" relativistic Vlasov Maxwell system." In: *Communications in Mathematical Physics* 185.2 (1997), pp. 257–284.

[GS98a] R. T. Glassey and J. Schaeffer. "The relativistic Vlasov–Maxwell system in two space dimensions: part I." In: *Archive for Rational Mechanics and Analysis* 141.4 (1998), pp. 331–354.

[GS98b] R. T. Glassey and J. Schaeffer. "The relativistic Vlasov–Maxwell system in two space dimensions: part II." In: *Archive for Rational Mechanics and Analysis* 141.4 (1998), pp. 355–374.

[Guo93] Y. Guo. "Global weak solutions of the Vlasov–Maxwell system with boundary conditions." In: *Communications in Mathematical Physics* 154.2 (1993), pp. 245–263.

[Guo95] Y. Guo. "Singular solutions of the Vlasov–Maxwell system on a half line." In: *Archive for Rational Mechanics and Analysis* 131.3 (1995), pp. 241–304.

[Han10] D. Han-Kwan. "On the confinement of a Tokamak plasma." In: *SIAM Journal on Mathematical Analysis* 42.6 (2010), pp. 2337–2367.

[Hin+09] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE Constraints*. Vol. 23. Mathematical Modelling: Theory and Applications. Springer Netherlands, 2009.

[Kat75] T. Kato. "The Cauchy problem for quasi-linear symmetric hyperbolic systems." In: *Archive for Rational Mechanics and Analysis* 58 (1975), pp. 181–205.

[Kno18] P. Knopf. "Optimal control of a Vlasov–Poisson plasma by an external magnetic field." In: *Calculus of Variations and Partial Differential Equations* 57.5 (2018), p. 134.

[Kno19] P. Knopf. "Confined steady states of a Vlasov–Poisson plasma in an infinitely long cylinder." In: *Mathematical Methods in the Applied Sciences* (2019), pp. 1–16.

[KW18] P. Knopf and J. Weber. "Optimal control of a Vlasov–Poisson plasma by fixed magnetic field coils." In: *Applied Mathematics & Optimization* (2018).

[Lin04] P.-K. Lin. *Köthe-Bochner Function Spaces*. Birkhäuser Boston, 2004.

[Lio85] J.-L. Lions. *Control of Distributed Singular Systems*. Gauthier-Villars, 1985.

[NNS15] T. T. Nguyen, T. V. Nguyen, and W. A. Strauss. "Global magnetic confinement for the 1.5D Vlasov–Maxwell system." In: *Kinetic and Related Models* 8.1 (2015), pp. 153–168.

[NS14] T. T. Nguyen and W. A. Strauss. "Linear stability analysis of a hot plasma in a solid torus." In: *Archive for Rational Mechanics and Analysis* 211.2 (2014), pp. 619–672.

[Pet39] B. J. Pettis. "A proof that every uniformly convex space is reflexive." In: *Duke Mathematical Journal* 5.2 (1939), pp. 249–253.

[Pfa92] K. Pfaffelmoser. "Global classical solutions of the Vlasov–Poisson system in three dimensions for general initial data." In: *Journal of Differential Equations* 95.2 (1992), pp. 281–303.

[Pou92] F. Poupaud. "Boundary value problems for the stationary Vlasov–Maxwell system." In: *Forum mathematicum* 4.5 (1992), pp. 499–527.

[Rei04] G. Rein. "Global weak solutions to the relativistic Vlasov–Maxwell system revisited." In: *Communications in Mathematical Sciences* 2.2 (2004), pp. 145–158.

[Rei92] G. Rein. "Existence of stationary, collisionless plasmas in bounded domains." In: *Mathematical Methods in the Applied Sciences* 15.5 (1992), pp. 365–374.

[Rob76] S. M. Robinson. "Stability theory for systems of inequalities, part II: differentiable nonlinear systems." In: *SIAM Journal on Numerical Analysis* 13.4 (1976), pp. 497–513.

[Sch91] J. Schaeffer. "Global existence of smooth solutions to the Vlasov Poisson system in three dimensions." In: *Communications in Partial Differential Equations* 16.8-9 (1991), pp. 1313–1335.

[Sku14] A. L. Skubachevskii. "Vlasov–Poisson equations for a two-component plasma in a homogeneous magnetic field." In: *Russian Mathematical Surveys* 69.2 (2014), pp. 291–330.

[Sta12] W. M. Stacey. *Fusion plasma physics*. 2nd ed. Physics textbook. Wiley-VCH, 2012.

[Tri57] F. G. Tricomi. *Integral Equations*. Vol. 5. Pure and applied mathematics. Interscience Publishers, 1957.

[Trö10] F. Tröltzsch. *Optimal Control of Partial Differential Equations: Theory, Methods, and Applications*. Vol. 112. Graduate studies in mathematics. American Mathematical Society, 2010.

[Wal02] W. Walter. *Analysis 2*. 5th ed. Springer Berlin Heidelberg, 2002.

[Web18] J. Weber. "Optimal control of the two-dimensional Vlasov–Maxwell system." In: *arXiv e-prints*, arXiv:1809.10016 (2018).

[Zha19] K. Z. Zhang. "Linear stability analysis of the relativistic Vlasov–Maxwell system in an axisymmetric domain." In: *SIAM Journal on Mathematical Analysis* 51.6 (2019), pp. 4683–4723.

[ZHR15] D. Zhelyazov, D. Han-Kwan, and J. D. Rademacher. "Global stability and local bifurcations in a two-fluid model for Tokamak plasma." In: *SIAM Journal on Applied Dynamical Systems* 14.2 (2015), pp. 730–763.

[ZK79] J. Zowe and S. Kurcyusz. "Regularity and stability for the mathematical programming problem in Banach spaces." In: *Applied Mathematics and Optimization* 5.1 (1979), pp. 49–62.

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