No projective 16-divisible binary linear code of length 131 exists

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***Abstract*—We show that no projective 16-divisible binary linear code of length 131 exists. This implies several improved upper bounds for constant-dimension codes, used in random linear network coding, and partial spreads.**

***Index Terms*—divisible codes, projective codes, partial spreads, constant-dimension codes.**

I. INTRODUCTION

An $[n,k,d]_q$ code is a q-ary linear code with length n, dimension k, and minimum Hamming distance d. Since we will only consider binary codes, we also speak of $[n,k,d]$ codes. Linear codes have numerous applications so that constructions or non-existence results for specific parameters were the topic of many papers. One motivation was the determination of the smallest integer $n(k,d)$ for which an $[n,k,d]$ code exists. As shown in [1] for every fixed dimension k there exists an integer $D(k)$ such that $n(k,d)=g(k,d)$ for all $d \ge D(k)$, where $n(k,d) \ge g(k,d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil$, is the so-called Griesmer bound. Thus, the determination of $n(k,d)$ is a finite problem. In 2000 the determination of $n(8,d)$ was completed in [2]. Not many of the open cases for $n(9,d)$ have been resolved since then and we only refer to most recent paper [6].

The aim of this note is to to circularize a recent application of non-existence results of linear codes. In random linear network coding so-called constant-dimension codes are used. These are sets of k-dimensional subspaces of $\mathbb{F}_q^n$ with subspace distance $d_S(U,W) := \dim(U) + \dim(W) - 2\dim(U \cap W)$. By $A_q(n,d;k)$ we denote the maximum possible cardinality, where $A_q(n,d;k) = A_q(n,d;n-k)$, so that we assume $2k \le n$. In [5] the upper bounds $A_q(n,d;k) \le \left\llbracket \frac{(q^n-1)\cdot A_q(n-1,d;k-1)/(q-1)}{(q^k-1)/(q-1)} \right\rrbracket_{q^{k-1}}$ for $d > 2k$ and $A_q(n,2k;k) \le \left\llbracket \frac{(q^n-1)/(q-1)}{(q^k-1)/(q-1)} \right\rrbracket_{q^{k-1}}$ were proven. Here $\llbracket a/b \rrbracket_{q^r}$ denotes the maximal integer t such that there exists a q^r-divisible q-ary linear code of effective length $n = a - tb$ and a code is called q^r-divisible if the Hamming weights $\mathrm{wt}(c)$ of all codewords c are divisible by q^r. For integers r the possible length of q^r-divisible codes have been completely determined in [5] and except for the cases $(n,d,k,q) = (6,4,3,2)$ and $(8,4,3,2)$ no tighter bound for $A_q(n,d;k)$ with $d > 2k$ is known. For the case $d = 2k$, where the constant-dimension codes are also called partial spreads, the notion of $\llbracket a/b \rrbracket_{q^r}$ can be sharpened by requiring the existence of a projective q^r-divisible q-ary linear code of effective length $n = a - tb$. Doing so, all known upper bounds for $A_q(n,2k;k)$ follow from non-existence results of projective q^r-divisible codes, see e.g. [3]. For each field size q and each integer r there exists only a finite set $\mathcal{E}_q(r)$ such that there does not exist a projective q^r-divisible code of effective length n iff $n \in \mathcal{E}_q(r)$. We have $\mathcal{E}_2(1) = \{1,2\}$, $\mathcal{E}_2(2) = \{1,2,3,4,5,6,9,10,11,12,13\}$, and remark that the determination of $\mathcal{E}_2(3)$ was recently completed in [4] by excluding length $n = 59$.

In this paper we show the non-existence of 16-divisible binary codes of effective length $n = 131$, which e.g. implies $A_2(13,10;5) \le 259$.

II. PRELIMINARIES

Since the minimum Hamming distance is not relevant in our context, we speak of $[n,k]$ codes.

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The dual code of an $[n, k]$ code C is the $[n, n-k]$ code C^* consisting of the elements of $\mathbb{F}_2^n$ that are perpendicular to all codewords of C. By a_i we denote the number of codewords of C of weight i. With this, the weight enumerator is given by $W(z)=\sum_{i\ge 0} a_i z^i$. The numbers a_i^* of codewords of the dual code of weight i are related by the so-called MacWilliams identities

$$\sum_{i\ge 0} a_i^* z^i = \frac{1}{2^k}\cdot\sum_{i\ge 0} a_i(1+z)^{n-i}(1-z)^i. \quad (1)$$

Clearly we have $a_0=a_0^*=1$. In this paper we assume that all lengths are equal to the so-called effective length, i.e., $a_1^*=0$. A linear code is called projective if $a_2^*=0$. Let C be a projective $[n,k]$ code. By comparing the coefficients of z^0, z^1, z^2, and z^3 on both sides of Equation 1 we obtain:

$$\sum_{i>0} a_i = 2^k-1, \quad (2)$$

$$\sum_{i\ge 0} i a_i = 2^{k-1}n, \quad (3)$$

$$\sum_{i\ge 0} i^2 a_i = 2^{k-1}\cdot n(n+1)/2, \quad (4)$$

$$\sum_{i\ge 0} i^3 a_i = 2^{k-2}\cdot\left(\frac{n^2(n+3)}{2}-3a_3^*\right) \quad (5)$$

The weight enumerator of a linear $[n,k]$ code C can be refined to a so-called partition weight enumerator, see e.g. [7]. To this end let $r\ge 1$ be an integer and $\cup_{j=1}^r P_j$ be a partition of the coordinates $\{1,\dots,n\}$. By $I=(i_1,\dots,i_r)$ we denote a multi-index, where $0\le i_j\le p_j$ and $p_j=\#P_j$ for all $1\le j\le r$. With this, $a_I\in\mathbb{N}$ denotes the number of codewords c such that $\#\{h\in P_j : c_h\ne 0\}=i_j$ for all $1\le j\le r$, which generalizes the notion of the counts a_i. By $a_I^*\in\mathbb{N}$ we denote the corresponding counts for the dual code C^* of C. The generalized relation between the a_I^* and the a_I is given by:

$$\sum_{I=(i_1,\dots,i_r)} a_I^* \prod_{j=1^r} z_j^{i_j}$$

$$= \frac{1}{2^k}\cdot\sum_{I=(i_1,\dots,i_r)} a_I \prod_{j=1}^{r}(1+z_j)^{n-i_j}(1-z_j)^{i_j} \quad (6)$$

The support $\operatorname{supp}(c)$ of a codeword $c\in\mathbb{F}_2^n$ is the set of coordinates $\{1\le i\le n : c_i\ne 0\}$. The residual of a linear code C with respect of a codeword $c\in C$ is the restriction of the codewords of C to those coordinates that are not in the support of c, i.e., the resulting effective length is given by $n-\operatorname{wt}(c)$. If c is a codeword of a q^r-divisible q-ary code C, where $r\ge 1$, then the residual code with respect to c is q^{r-1}-divisible, see e.g. [3]. The partition weight enumerator with respect to a codeword c is given by Equation (6), where we choose $r=2$, $P_2=\operatorname{supp}(c)$, and $P_1=\{1,\dots,n\}\backslash P_2$, so that restricting to the coordinates in P_1 gives the residual code.

III. No projective 16-divisible binary linear code of length 131 exists

Assume that C is a projective 16-divisible $[131,k]$ code. Since for every codeword $c\in C$ the residual code is 8-divisible and projective, we conclude from $\{3,19,35\}\subseteq\mathcal{E}_2(3)$, see e.g. [4], that the possible non-zero weights of the codewords in C are contained in $\{16,32,48,64,80\}$. For codewords of weight 80 the weight enumerator of the corresponding residual code can be uniquely determined:

Lemma 1: ([3, Lemma 24])
The weight enumerator of a projective 8-divisible binary linear code of (effective) length $n=51$ is given by $W(z)=1+204z^{24}+51z^{32}$, i.e., it is an 8-dimensional two-weight code.

Lemma 2: Each projective 16-divisible $[131,k]$ code satisfies

$$a_{48} = -6a_{16}-3a_{32}-10+11\cdot 2^{k-9},$$

$$a_{64} = 8a_{16}+3a_{32}+15+221\cdot 2^{k-8},$$

$$a_{80} = -3a_{16}-a_{32}-6+59\cdot 2^{k-9},$$

$$a_3^* = 2^{17-k}a_{16}+2^{15-k}a_{32}-311+5\cdot 2^{16-k},$$

$k\ge 9$, and $a_{80}\ge 4+3\cdot 2^{k-5}\ge 52$.

Proof. Solving the constraints (2)-(5) for a_{48}, a_{64}, a_{80}, and a_3^* gives the stated equations for general dimension k. Since $a_{48}\in\mathbb{N}$ (or $a_{80}\in\mathbb{N}$) we have $k\ge 9$. Since $a_{48}\ge 0$, we have $6a_{16}+3a_{32}\le 11\cdot 2^{k-9}-10$, so that $a_{80}=-3a_{16}-a_{32}-6+59\cdot 2^{k-9}\ge 4+3\cdot 2^{k-5}\ge 52$. □

First we exclude the case of dimension $k=9$:

Lemma 3: No projective 16-divisible $[131,9]$ code exists.

PROOF. For $k=9$ the equations of Lemma 2 yield

$$\begin{aligned} a_{48} &= -6a_{16}-3a_{32}+1, \\ a_{64} &= 8a_{16}+3a_{32}+457, \\ a_{80} &= -3a_{16}-a_{32}+53, \text{ and} \\ a_3^* &= 256a_{16}+64a_{32}+329 \end{aligned}$$

for a projective 16-divisible $[131,9]$ code C. Since $a_{48}\ge 0$ and $a_{16},a_{32}\in\mathbb{N}$, we have $a_{16}=a_{32}=0$, so that $a_{48}=1$, $a_{64}=457$, $a_{80}=53$, and $a_3^*=329$. Now consider a codeword $c_{80}\in C$ of weight 80 and the unique codeword $c_{48}\in C$ of weight 48. In the residual code of c_{80} the restriction of c_{48} has weight 24 or 32 due to Lemma 1. In the latter case the codeword $c_{80}+c_{48}\in C$ has weight 96, which cannot occur in a projective 16-divisible binary linear code of length 131. Thus, we have that $c_{80}+c_{48}\in C$ gives another codeword of weight 80. However, since a_{80} is odd, this yields a contradiction and the code C does not exist. □

Lemma 4: A projective 16-divisible binary linear code C of length 131 does not contain a codeword of weight 16 or 32.

PROOF. Let $c\in C$ be an arbitrary codeword of weight 80 (which indeed exists, see Lemma 2) and $c'\in C$ a codeword of weight 16 or 32. We consider the residual code C' of C with respect to the codeword c. From Lemma 1 we conclude that the restriction $\tilde{c}'$ of c' in C' has weight 0, 24, or 32. Since $c+c'\in C$ has a weight of at most 80, $\tilde{c}'$ is the zero codeword of weight 0. In other words, we have $\operatorname{supp}(c')\subseteq\operatorname{supp}(c)$. If L denotes the set of codewords of weight 80 in C, then $\operatorname{supp}(c')\subseteq\cap_{l\in L}\operatorname{supp}(l)=:M$, with $M\subseteq\{1,\dots 131\}$ and $\#M\ge 16$.

Now let D be the code generated by the elements in M, i.e., the codewords of weight 80. By k' we denote the dimension of D and by k the dimension of C. Since D contains all codewords of weight 80 and due to Lemma 2 we have

$$4+3\cdot 2^{k-5}\le a_{80}\le 2^{k'}-1 \qquad (7)$$

for C. Since $\#M\ge 16$ each generator matrix G of D contains a column that occurs at least 16 times, i.e., the maximum column multiplicity is at least 16. If a row is appended to G then the maximum column multiplicity can go down by a factor of at most the field size q, i.e., 2 in our situation. Thus, we have $k'\le k-4$. Since Inequality 7) gives

$$4+3\cdot 2^{k-5}\le 2^{k'}-1\le 2^{k-4}-1,$$

we obtain a contradiction. Thus, we conclude $a_{16}=a_{32}=0$. □

Theorem 5: No projective 16-divisible binary linear code of length 131 exists.

PROOF. Assume that C is a projective 16-divisible $[131,k]$ code. From Lemma 4 we conclude $a_{16}=a_{32}=0$, so that Lemma 2 yields $a_3^*=5\cdot 2^{16-k}-311$. Note that for $k\ge 11$ the non-negative integer a_3^* would be negative. The case $k=9$ is excluded in Lemma 3. In the remaining case $k=10$ we have $a_3^*=9$ and $a_{80}=112$.

Now consider the residual code C' of C with respect to a codeword c of weight 80. Plugging in the weight enumerator for C' from Lemma 1 in Equations (2)-(5) gives $a_3^*(C')=17$. Thus, we conclude $a_3^*(C)\ge 17$, which is a contradiction. □

We remark that some parts of our argument can be replaced using the partition weight enumerator from Equation (6). If we consider the partition weight enumerator with respect to a codeword c of weight 80, then we have $r=2$, $p_1=51$, and $p_2=80$. The possible indices where a_I might be positive are given by $(0,0)$, $(0,16)$, $(0,32)$, $(0,48)$, $(0,64)$, $(0,80)$, $(24,24)$, $(24,40)$, $(24,56)$, $(32,32)$, and $(32,48)$. Clearly, we have $a_{(0,0)}=1$ and $a_{(0,80)}=1$. By considering the sums of a codeword with c we conclude $a_{(0,16)}=a_{(0,64)}$, $a_{(0,32)}=a_{(0,48)}$, $a_{(24,24)}=a_{(24,56)}$, and $a_{(32,32)}=a_{(32,48)}$. From Lemma 1 we conclude $a_{(32,32)}=a_{(32,48)}=51\cdot 2^{k-9}$, $a_{(24,24)}=a_{(24,56)}=t$, and $a_{(24,40)}=204\cdot 2^{k-8}-2t$, where k is the dimension of the code and $t\in\mathbb{N}$ a free parameter. Plugging into Equation (6) this gives $a_{(0,16)}+a_{(0,32)}=2^{k-9}-1$ for the coefficients of $t_1^0t_2^0$ since $a_{(0,0)}^*=1$. Using this equation automatically gives $a_{(1,0)}^*=0$, $a_{(2,0)}^*=0$, and $a_{(3,0)}^*=17$.

Since $a^*_{(0,2)} = 0$ the coefficient of t_2^2 gives $6320 - 7344 \cdot 2^{k-9} + 1024t + 2224a_{(0,16)} + 176a_{(0,32)} = 0$. Thus, we have $a_{(0,16)} = 7 \cdot 2^{k-10} - 3 - \frac{t}{2}$ and $a_{(0,32)} = 2 - 5 \cdot 2^{k-10} + \frac{t}{2}$. The coefficient of $t_1^1 t_2^2$ then gives $a^*_{(1,2)} = 408 - 3t \cdot 2^{14-k}$. For $k = 9$ the non-negativity conditions $a_{(0,16)}, a_{(0,32)} \ge 0$ force $t = 1$, so that $a_{(0,0)} = 1$, $a_{(0,16)} = a_{(0,64)} = 0$, $a_{(0,32)} = a_{(0,48)} = 0$, $a_{(0,80)} = 1$, $a_{(24,40)} = 406$, $a_{(24,24)} = a_{(24,56)} = 1$, and $a_{(32,32)} = a_{(32,48)} = 51$. It can be checked that all coefficients on the right hand side of Equation (6) are non-negative. $a_{(0,32)} \ge 0$ implies $t \ge 5 \cdot 2^{k-9} - 4$, so that $a^*_{(1,2)}$ would be negative for $k \ge 12$.

Theorem 5 implies a few further results.

Proposition 6: For $t \ge 0$ we have $A_2(8 + 5t, 10; 5) \le 3 + 2^8 \cdot \frac{32^t - 1}{31}$.

PROOF. Assume that $\mathcal{C}$ is a set of $4 + 2^8 \cdot \frac{32^t-1}{31}$ 5-dimensional subspaces in $\mathbb{F}_2^{8+5t}$ with pairwise trivial intersection. Then, the number of vectors in $\mathbb{F}_2^{8+5t}$ that are disjoint to the vectors of the elements of $\mathcal{C}$ is given by $\left(2^{8+5t} - 1\right) - 31 \cdot \left(4 + 2^8 \cdot \frac{32^t-1}{31}\right) = 131$. Thus, by [3, Lemma 16], there exists a projective 2^{5-1}-divisible binary linear code of length $n = 131$, which contradicts Theorem 5. □

The recursive upper bound for constant-dimension codes mentioned in the introduction implies:

Corollary 7: We have $A_2(14, 10; 6) \le 67\,349$, $A_2(15, 10; 7) \le 17\,727\,975$, and $A_2(19, 10, 6) \le 70\,329\,353$.

As an open problem we mention that the non-existence of a projective 16-divisible binary linear code of length $n = 130$ would imply $A_2(15, 12; 6) \le 514$.

Lemma 8: For $k \ge 1$, $r \ge 3$, and $j \le 2r - 1$ no projective 2^r-divisible $[3 + j \cdot 2^r, k]$ code exists.

PROOF. In [3, Theorem 12] it was proven that the length n of a projective 2^r-divisible binary linear code either satisfies $n > r2^{r+1}$ or can be written as $n = a\left(2^{r+1} - 1\right) + b2^{r+1}$ for some non-negative integers a and b. Using $r \ge 3$, we note that $3 + j \cdot 2^r \le 3 + (2r - 1) \cdot 2^r = 3 - 2^r + r2^{r+1} < r2^{r+1}$. If $a\left(2^{r+1} - 1\right) + b2^{r+1} = 3 + j \cdot 2^r$, then $3 + a$ is divisible by 2^r, so that $a \ge 2^r - 3$. However, for $r \ge 3$ we have $a\left(2^{r+1} - 1\right) + b2^{r+1} \ge (2^r - 3) \cdot \left(2^{r+1} - 1\right) > 3 + (2r - 1) \cdot 2^r \ge 3 + j \cdot 2^r$ – contradiction. □

Proposition 9: For $k \ge 1$, $r \ge 4$, and $j \le 2r$ no projective 2^r-divisible $[3 + j \cdot 2^r, k]$ code exists.

PROOF. Due to Lemma 8 it suffices to consider $j = 2r$. The case $r = 4$ is given by Theorem 5. For $r > 4$ we proof the statement by induction on r. Assuming the existence of such a code, Equation (3) minus $r2^r$ times Equation (2) yields

$$\sum_{i>0} (i - r)2^r \cdot a_{i2^r} = 3 \cdot 2^{k-1} + r \cdot 2^r > 0. \quad (8)$$

The residual code of a codeword of weight $i2^r$ is projective, 2^{r-1}-divisible, and has length $3 + (2r - i) \cdot 2^r$. If $i \ge r + 2$, then we can apply Lemma 8 to deduce $a_{i2^r} = 0$. For $i = r + 1$ the induction hypothesis gives $a_{i2^r} = 0$. Since $(i - r)2^r \cdot a_{i2^r} \le 0$ for $i \le r$ the left hand side of Inequality (8) is non-positive – contradiction. □

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